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Multivariate Generalizations of Cumulative Sum Quality-Control Schemes

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This article presents the design procedures and average run lengths for two multivariate cumulative sum (CUSUM) quality-control procedures. The first CUSUM procedure reduces each multivariate observation to a scalar and then forms a CUSUM of the scalars. The second CUSUM procedure forms a CUSUM vector directly from the observations. These two procedures are compared with each other and with the multivariate Shewhart chart. Other multivariate quality-control procedures are mentioned. Robustness, the fast initial response feature for CUSUM schemes, and combined Shewhart-CUSUM schemes are discussed.

KEY WORDS: Average run length; CUSUM; Fast initial response; Robustness; Shewhart chart.

1. INTRODUCTION

Cumulative sum (CUSUM) schemes are used in industry to detect a change in the quality of a manufactured product. The most frequent application of CUSUM schemes is the detection of a change in the mean of a normally distributed variable. It is often helpful to view CUSUM schemes as a sequence of sequential probability ratio tests. For a mean-shift application, an algebraic rearrangement of the logarithm of the probability ratio yields a simple, recursive test statistic. There are two major problems that can arise in deriving CUSUM schemes from the theory of sequential tests. First, the theory requires specification or two *simple* hypotheses [as opposed to composite hypotheses—see, e.g., Mood, Graybill, and Boes (1974, p. 402)] to be tested; but the most common application of quality-control schemes requires testing the simple hypothesis that the mean is at its desired level versus the composite hypothesis that the mean has shifted from its target value. Second, the logarithm of the probability ratio may be too complex to yield a simple, practical scheme. Both of these problems have had an impact on the development of a multivariate CUSUM scheme.

The specification of both an aim or target value $\mathbf{a}$ and a specific alternative $\mathbf{d}$ for the mean vector $\boldsymbol{\mu}$ of a multivariate normal distribution with known covariance matrix $\boldsymbol{\Sigma}$ does yield a CUSUM scheme (see Healy 1987). In many applications, however, there is no basis for selecting a specific point $\mathbf{d}$ for the alternative hypothesis. In the univariate case this problem is not so severe; there are only two directions for the mean to shift—either higher or lower. The exact amount of shift is not a problem. To paraphrase Tukey (1986), detecting a shift of 5 standard deviations is nearly trivial, whereas detecting a shift of .05 standard deviations is nearly impossible. CUSUM schemes designed to detect a shift of 1 standard deviation are widely used and, of course, they also quickly detect shifts of more than 1 standard deviation. The choice between detecting a shift to the high side and detecting a shift to the low side is resolved by doing both—that is, operating two schemes simultaneously. The two schemes are considered to be a single two-sided CUSUM scheme and the performance of the procedure is determined accordingly.

Woodall and Ncube (1985) suggested extending the univariate CUSUM procedure to the multivariate case: To detect a shift in the mean vector of a p-variate normal distribution, operate p one-sided or two-sided CUSUM schemes simultaneously and evaluate the performance of the collection of schemes. The performance of CUSUM schemes is usually measured by the *average run length* (ARL), which is the average number of samples required for the scheme to signal that the mean has changed. In the univariate case, the ARL is often plotted as a function of the standardized deviation of the mean from its target value; the graph is referred to as an ARL curve. The ARL of the Woodall and Ncube (1985) procedure depends on the direction that the mean vector shifts. The dependency of the ARL on the direction of the shift is alleviated but not removed by using principal components rather than the original variables.

In contrast, the ARL of the multivariate Shewhart

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chart [due to Hotelling (1947, pp. 111–184) and also known as a T^2 chart or as a χ^2 chart] depends on the mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ only through the noncentrality parameter

$$d = +[(\boldsymbol{\mu} - \mathbf{a})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{a})]^{1/2}. \quad (1)$$

Hence it is possible to consider the ARL as a function of d and construct an ARL curve. The multivariate Shewhart chart signals that the mean has shifted at the first observation $\mathbf{x}_n$ such that

$$T_n = +[(\mathbf{x}_n - \mathbf{a})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}_n - \mathbf{a})]^{1/2} > \mathrm{SCL}, \quad (2)$$

where SCL is the Shewhart control limit. Hotelling (1947, p. 123) suggested the plotting of T^2, rather than T, to avoid "the labor of extracting square roots." That is not a serious concern now because of electronic computing, and I will take the positive square root of all quadratic forms to put the quantities on a more meaningful scale.

A natural method to develop a multivariate CUSUM procedure that has ARL's dependent only on the noncentrality parameter is to base the CUSUM on the noncentrality parameter itself—that is, test for a change in the noncentrality parameter of a chi-square distribution. Unfortunately, this produces a mathematically intractable probability ratio, and it is not clear what the CUSUM test statistic should be. Transforming from the chi-square distribution to the chi distribution does not help, as the Jacobian of the transformation appears in both the numerator and denominator of the probability ratio—thus canceling out and yielding the same complex expression.

Previously (Crosier 1986), I proposed two heuristically derived multivariate CUSUM procedures that have ARL's that depend only on the noncentrality parameter and hence are easily compared with multivariate Shewhart charts. Similar but distinct procedures were independently developed by Pignatiello and Kasunic (1985) and by Pignatiello, Runger, and Korpela (1986).

Section 2 describes the CUSUM schemes examined in this article and gives a numerical example of their operation. Sections 3, 4, and 5 present the design procedures and ARL's of multivariate Shewhart charts, a CUSUM of T (COT) statistics scheme, and a CUSUM vector scheme, respectively. (I will send a set of tables corresponding to Figs. 1–10, which present the ARL's of the multivariate quality control methods discussed in this article, on request. The tables also include the cases $p = 1$, 3, and 4.) Section 6 examines two applications of multivariate quality control schemes. Section 7 summarizes the results. The methods used to calculate the ARL's of the schemes are given in Appendix A.

2. DESCRIPTION OF SCHEMES

The most direct and obvious method of replacing the multivariate Shewhart chart by a CUSUM procedure is to form a CUSUM of the scalars T_n ($n = 1, 2, 3, \ldots$). This will be referred to as a COT scheme; the CUSUM is given by

$$S_n = \max(0, S_{(n-1)} + T_n - k), \quad (3)$$

where $S_0 \geq 0$ and $k > 0$. The COT scheme signals when $S_n > h$. Although multivariate Shewhart charts based on T^2 and T are equivalent, CUSUM schemes based on T^2 and T are not. I chose to form a COT, as this cumulates distance rather than squared distance. Healy (1987) showed that a CUSUM of T^2 statistics is the appropriate sequential theory test for an inflation (multiplication by a scalar constant) of the covariance matrix $\boldsymbol{\Sigma}$.

A vector-valued CUSUM scheme can be "derived" by replacing the scalar quantities of a univariate CUSUM scheme by vectors. The univariate CUSUM scheme to detect an increase in the mean is $S_n = \max(0, S_{n-1} + (X_n - a) - k\sigma)$, where a is the aim point or target value for the mean, σ is the standard deviation of the X's, $k > 0$, and $S_0 = 0$. Replacing the scalars by vectors gives $\mathbf{s}_n = \max(\mathbf{0}, \mathbf{s}_{n-1} + (\mathbf{x}_n - \mathbf{a}) - \mathbf{k})$. The problems are how to find $\mathbf{k}$ and how to interpret taking the maximum of a vector and the null vector. In the univariate case, the quantity $S_{n-1} + X_n - a$ is shrunk toward 0 by k standard deviations. If this is to hold for the multivariate case, $\mathbf{k}$ must satisfy $\mathbf{k}'\boldsymbol{\Sigma}^{-1}\mathbf{k} = k^2$—that is, $\mathbf{k}$ must be of length k, where length is defined using the covariance matrix $\boldsymbol{\Sigma}$. If the subtraction of $\mathbf{k}$ is to shrink $\mathbf{s}_{n-1} + \mathbf{x}_n - \mathbf{a}$ toward $\mathbf{0}$, then $\mathbf{k}$ must be in the same direction as $\mathbf{s}_{n-1} + \mathbf{x}_n - \mathbf{a}$. Hence $\mathbf{k} = (k/C_n)(\mathbf{s}_{n-1} + \mathbf{x}_n - \mathbf{a})$, where C_n is the length of $\mathbf{s}_{n-1} + \mathbf{x}_n - \mathbf{a}$. The maximum with the null vector can be interpreted as setting $\mathbf{s}_n = \mathbf{0}$, rather than subtracting $\mathbf{k}$, whenever the length of $\mathbf{k}$ is greater than the length of $\mathbf{s}_{n-1} + \mathbf{x}_n - \mathbf{a}$,—that is, whenever $k > C_n$. Rather than calculate the vector $\mathbf{k}$ and subtract it from $\mathbf{s}_{n-1} + \mathbf{x}_n - \mathbf{a}$, it is simpler to just contract the vector $\mathbf{s}_{n-1} + \mathbf{x}_n - \mathbf{a}$ by $(1 - k/C_n)$, provided that $k < C_n$. Hence the multivariate CUSUM scheme may be expressed as follows: Let

$$C_n = [(\mathbf{s}_{n-1} + \mathbf{x}_n - \mathbf{a})'\boldsymbol{\Sigma}^{-1}(\mathbf{s}_{n-1} + \mathbf{x}_n - \mathbf{a})]^{1/2};$$

then

$$\begin{aligned} \mathbf{s}_n &= \mathbf{0} && \text{if } C_n \leq k \\ \mathbf{s}_n &= (\mathbf{s}_{n-1} + \mathbf{x}_n - \mathbf{a})(1 - k/C_n) && \text{if } C_n > k, \end{aligned} \quad (4)$$

where $\mathbf{s}_0 = \mathbf{0}$ and $k > 0$. Let

$$Y_n = [\mathbf{s}_n'\boldsymbol{\Sigma}^{-1}\mathbf{s}_n]^{1/2}. \quad (5)$$

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Table 1. Numerical Example of Bivariate Quality-Control Schemes

	Observations			CUSUM of T		CUSUM Vector		
n	X_1	X_2	T^2	T	S	S_1	S_2	Y
1	−1.19	.59	3.29	1.81	.40	−.86	.43	1.31
2	.12	.90	.96	.98	.00	−.56	1.01	1.60
3	−1.69	.40	4.92	2.22	.81	−1.95	1.22	3.20
4	.30	.46	.22	.47	.00	−1.40	1.43	2.83
5	.89	−.75	2.70	1.64	.23	−.30	.39	.69
6	.82	.98	1.11	1.05	.00	.33	.88	.89
7	−.30	2.28	7.96	2.82	1.41	.03	2.72	3.13
8	.63	1.75	3.14	1.77	1.77	.59	4.01	4.33
9	1.56	1.58	3.29	1.81	2.18	1.96	5.09	5.14
10	1.46	3.05	9.31	3.05	3.82	3.21	7.65	7.68[a]
Signal criteria				3.26[b]	4.04[c]			5.50[c]

[a] Off-aim signal; the other two schemes have not yet signaled.
[b] SCL value.
[c] h value.

The multivariate CUSUM scheme signals when $Y_n > h$.

Table 1 gives a numerical example of these schemes for a bivariate normal distribution with unit variances and correlation .5. The process mean is (0, 0) for the first five observations and (1, 2) for the last five observations. The first three columns of Table 1 give the sample number (n) and the observations of the random variables X_1 and X_2. The columns headed S_1 and S_2 are the elements of the multivariate CUSUM vector [Eq. (4)], and the last column is the multivariate CUSUM test statistic Y_n from (5). The k values for the CUSUM of T and multivariate CUSUM schemes are 1.41 and .5, respectively. These k values are optimal for detecting a shift of one generalized standard deviation (in the sense that the ARL at $d = 1$ is a minimum, given that the on-target ARL is held constant). The signal criteria (values of h and SCL) yield an on-target ARL of 200 for each scheme. Note that when the multivariate CUSUM scheme signals, the CUSUM vector elements are both positive and approximately in the 1:2 ratio of the individual component means. This gives an indication of the direction that the mean has shifted; it is not intended as a formal mechanism for estimating the process mean.

3. MULTIVARIATE SHEWHART CHARTS

Figures 1 and 2 give ARL's of Shewhart charts as a function of d for $p = 2$, 5, 10, and 20 variables. The SCL's are the 99.5 and 99.8 percentiles of the (central) chi distribution with p df. This choice of SCL's gives on-target ARL's of 200 and 500, respectively. The off-target ($d > 0$) ARL's were found using the numerical approximations described in Appendix A. One characteristic of multivariate Shewhart charts is that the SCL must be increased as the number of variables increases to obtain the same on-target ARL. (Interpreting the multivariate Shewhart charts as multivariate CUSUM schemes with $h = 0$ and $k =$ SCL indicates that multivariate Shewhart charts are "designed" to detect larger shifts in the noncentrality parameter as p, the number of variables, increases.)

A practical problem with multivariate Shewhart charts is their lack of robustness. They are sensitive to *multivariate outliers*. Multivariate outliers are observations $\mathbf{x}$ that have large T statistics, yet no element of $\mathbf{x}$ would be considered an outlier by a univariate outlier test. I have examined hundreds of multivariate outliers in industrial plant data. Many (if not most) of the multivariate outliers were caused by clerical or keypunching errors (I often found the correct numbers on the original data sheets). Further, in my experience, when the process mean does shift, it often does so along the major axis of the multivariate probability ellipsoid. These experiences lead me to conclude that robustness is an essential feature for a multivariate quality-control scheme and, in many applications, it may be helpful to have outlier criteria that are sensitive to the direction of the observation as well as its distance from the target value.

4. CUMULATIVE SUM OF T

The COT schemes offer several advantages over multivariate Shewhart charts. First, COT schemes may be designed to detect a specific shift in the process mean (i.e., in the noncentrality parameter). Second, recent enhancements for CUSUM schemes may be applied to COT schemes. A robustness procedure, such as the two-in-a-row rule (Lucas and Crosier 1982a), may be applied to the sequence of T

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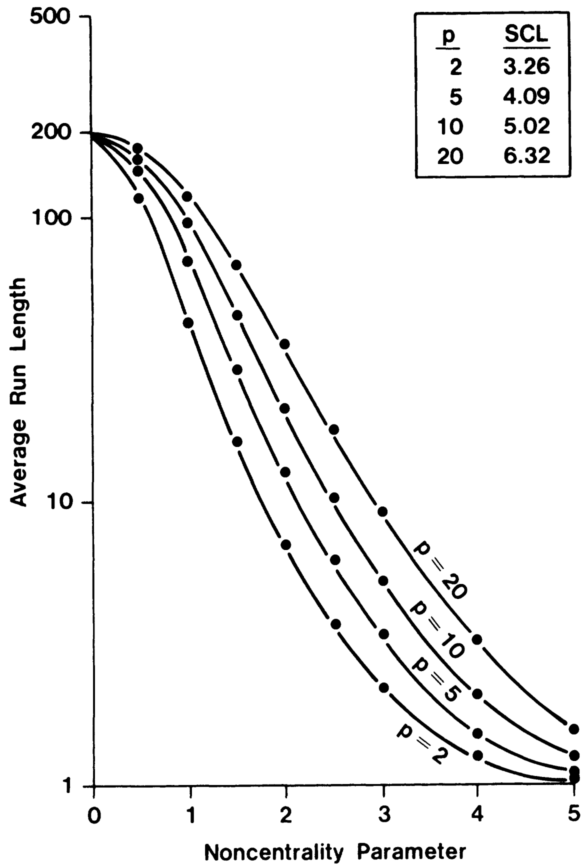

Figure 1. ARL Curves for Multivariate Shewhart Charts With On-Target ARL's of 200.

statistics. The two-in-a-row rule states that outliers should be ignored (i.e., not used in the CUSUM calculations) but the occurrence of two outliers in succession is taken as an off-aim signal. The fast initial response (FIR) feature for CUSUM schemes (Lucas and Crosier 1982b) is easily extended to COT schemes. Quick detection of an initial off-aim condition is obtained by starting the scheme with S_0 equal to $h/2$ rather than 0. If the process is off-aim, the CUSUM will signal quickly because of the head start. If the process is on target at startup, the head start will likely be removed by the subtraction of k at each observation. Finally, a combined multivariate Shewhart-COT scheme, analogous to the combined Shewhart-CUSUM schemes of Lucas (1982), is readily obtained using the T statistics. A combined multivariate Shewhart-COT quality control scheme is implemented by operating a COT scheme and a multivariate Shewhart chart simultaneously; the combined scheme signals if either component scheme signals. Combined Shewhart-COT schemes lack robustness, but they may be used to monitor laboratory measurement error (Lucas 1982) in which any extreme value must be examined.

Figures 3 and 4 present ARL's of COT schemes for $p = 2$, 5, 10, and 20 variables. These schemes are designed to detect any shift in the mean vector yielding $d = 1$. The h values were chosen to provide an on-target ARL of 200 or 500. The ARL curves in Figures 3 and 4 indicate that COT schemes give improved detection of small shifts (e.g., $d = 1$) over multivariate Shewhart charts. ARL curves for the COT schemes of Figures 3 and 4 with the FIR feature implemented are given in Figures 5 and 6. The head start decreases the on-target ARL more for larger values of p than for smaller values of p. For COT schemes with an on-target ARL of 200, the FIR feature decreased the on-target ARL by 9% for the $p = 2$ case, and by 15% for the $p = 20$ case. As the off-target ($d = 1$) ARL's decreased by 28% and 27%, respectively, the FIR feature is less advantageous at $p = 20$ than at $p = 2$.

I wished to compare the ARL curves for COT schemes to the ARL curves for the multivariate Shewhart charts in Section 2, which have on-target ARL's of 200 or 500. For this purpose, I considered a k value optimal to detect a shift to $d = 1$ if the ARL at $d = 1$ was a minimum given that the on-target ARL was 200 or 500. (For any value of k, an on-target ARL of 200 or 500 can be obtained by varying

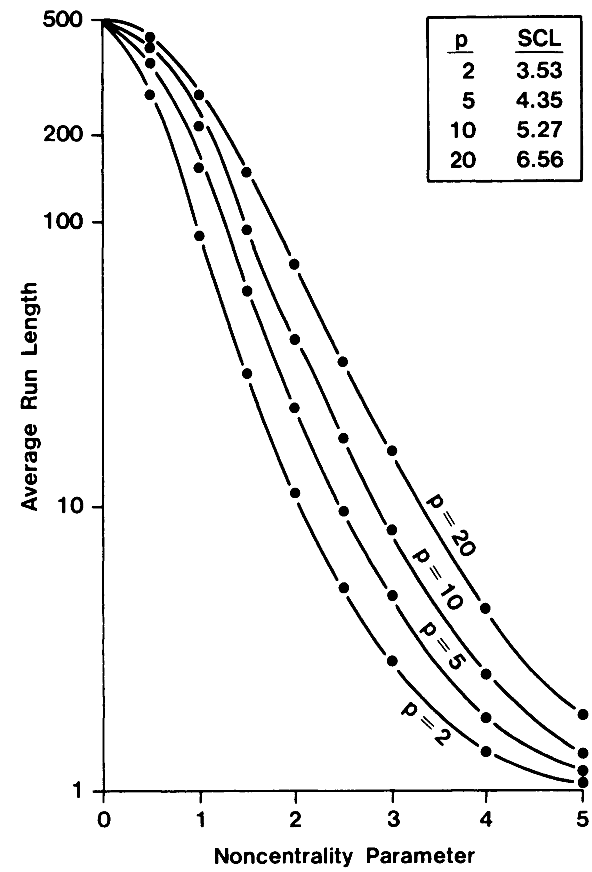

Figure 2. ARL Curves for Multivariate Shewhart Charts With On-Target ARL's of 500.

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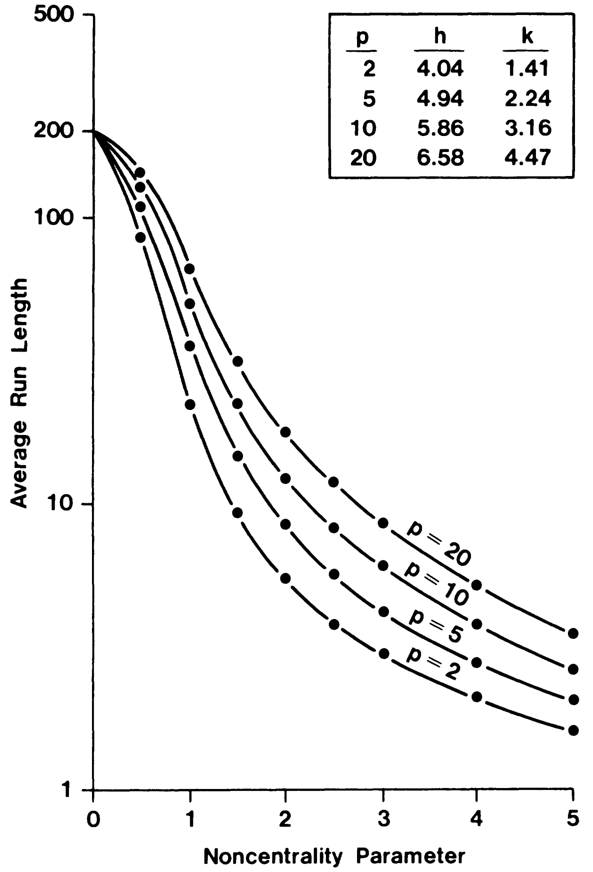

Figure 3. ARL Curves for CUSUM of T Schemes With On-Target ARL's of 200.

h.) A search for the optimal *k*'s produced a sequence that closely resembled the square root of the number of variables. Hence the *k*'s used in Figures 3–6 were found, to about two and one-half significant figures, by a simple search; the square root of *p* sequence determined the last digit.

5. MULTIVARIATE CUSUM SCHEMES

The ARL's of multivariate Shewhart charts and COT schemes depend on the mean vector and covariance matrix only through the noncentrality parameter, because these procedures are based on Hotelling's T statistic. It is not as clear that multivariate CUSUM schemes have this property, even though both parameters, h and k, are used to define ellipsoidal regions. Initially, the dependence of the ARL on only the noncentrality parameter was checked by using simulation results for the bivariate case. John Healy has provided a detailed proof of the dependence of the ARL on only the noncentrality parameter. The proof and some mathematical properties of the multivariate CUSUM procedure are discussed in Appendix B.

Multivariate CUSUM schemes, like COT schemes, offer the advantages of a CUSUM scheme over a Shewhart chart—ability to design the scheme to detect a specific shift in the mean vector and ability to implement recent enhancements for CUSUM schemes, such as the FIR feature. In addition, the directional nature of multivariate CUSUM offers some advantages over the directionless COT schemes. Because multivariate CUSUM schemes allow observations in opposite directions from the target value to cancel each other, one suspects that multivariate CUSUM schemes will have better ARL properties than COT schemes, that is, the cancellation will occur more often when the process mean is on target, thus raising the on-target ARL relative to the off-target ARL. Figures 7 and 8 give ARL curves for multivariate CUSUM schemes that can be compared with the ARL curves for multivariate Shewhart charts (Figs. 1 and 2) and COT schemes (Figs. 3 and 4). The ARL curves in Figures 7 and 8 show that multivariate CUSUM schemes give quicker detection of small shifts than both multivariate Shewart charts and COT schemes. Another advantage to the vector-valued CUSUM scheme is that it provides at least some indication of where the mean has shifted when a signal is given.

The design procedure of multivariate CUSUM

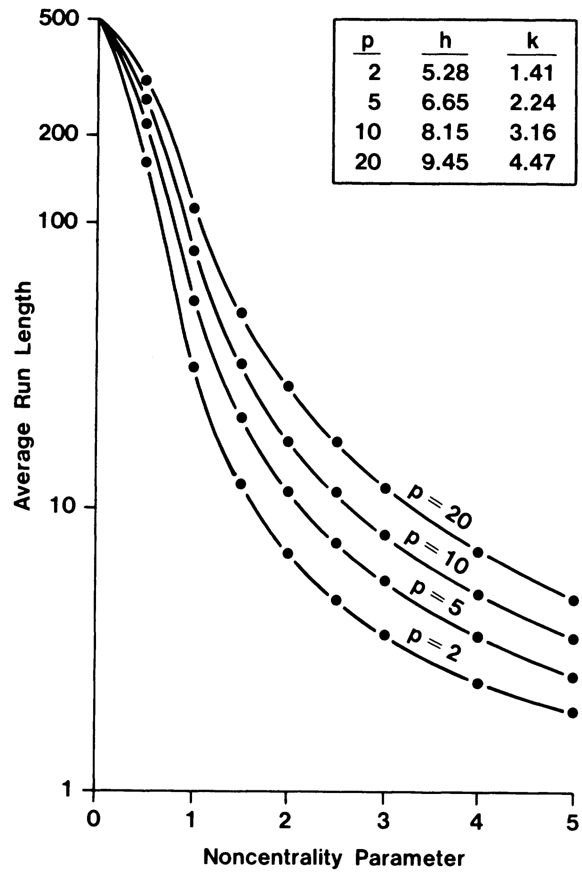

Figure 4. ARL Curves for CUSUM of T Schemes With On-Target ARL's of 500.

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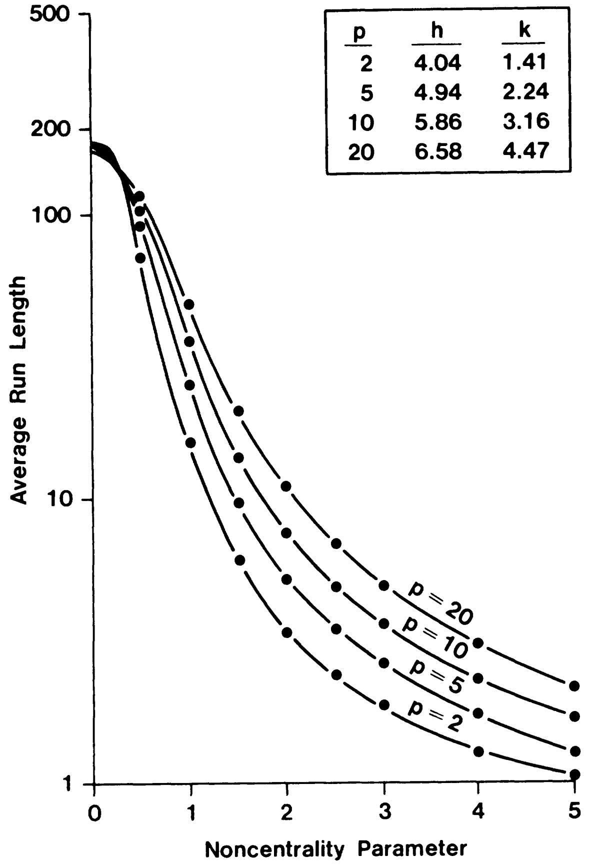

Figure 5. ARL Curves for the COT Schemes of Figure 3, but With the Fast Initial Response Feature Implemented.

schemes is straightforward. To detect any shift in the mean vector yielding noncentrality parameter d, choose $k = d/2$. Table 2 compares the ARL's of bivariate schemes with $k = .5$, $k = 1$, and $k = 1.5$. The choice of $k = d/2$ appears to minimize the ARL at d for a given on-target ARL. The decision interval h is chosen to provide an acceptable on-target ARL. Details of the method used to select the h values in Figures 7 and 8 are given in Appendix A. (A Markov chain approximation may be used to determine the on-target ARL's. See App. C.)

Development of a combined multivariate Shewhart-CUSUM quality-control scheme is straightforward—a multivariate CUSUM scheme and a multivariate Shewhart chart are operated simultaneously; a signal by either scheme indicates that the process mean has shifted. Robustness for the multivariate CUSUM scheme may be obtained by applying the two-in-a-row rule to the T statistics for the observations. Special outlier criteria, rather than T statistics, could be used to tailor the robustness procedure to specific applications.

The FIR feature for multivariate CUSUM schemes is implemented by changing the value of h at the start of the scheme. Let $h_0 = h/2$ and

$$h_n = \min[h, h_{n-1} + \max(0, k^* - T_n)], \qquad n = 1, 2, 3, \ldots, \quad (6)$$

where T_n is Hotelling's T statistic for the nth observation, and k^* is the k value for a COT scheme designed to detect the same deviation as the multivariate CUSUM scheme. Figures 9 and 10 give ARL curves of the multivariate CUSUM schemes in Figures 7 and 8, but with the FIR feature implemented. The FIR feature decreases the on-target ARL's more for larger p than for smaller p, but the effect is not as deleterious as for COT schemes because the off-target ($d = 1$) ARL's are also decreased more for larger p than for smaller p. For multivariate CUSUM schemes with an on-target ARL of 200, the FIR feature decreases the on-target ARL by 9% for $p = 2$ and by 22% for $p = 20$, but the off-target ($d = 1$) ARL's are decreased by 29% and 49%, respectively. (The results for multivariate CUSUM schemes with an on-target ARL of 500 are very similar.)

Multivariate CUSUM schemes compare favorably

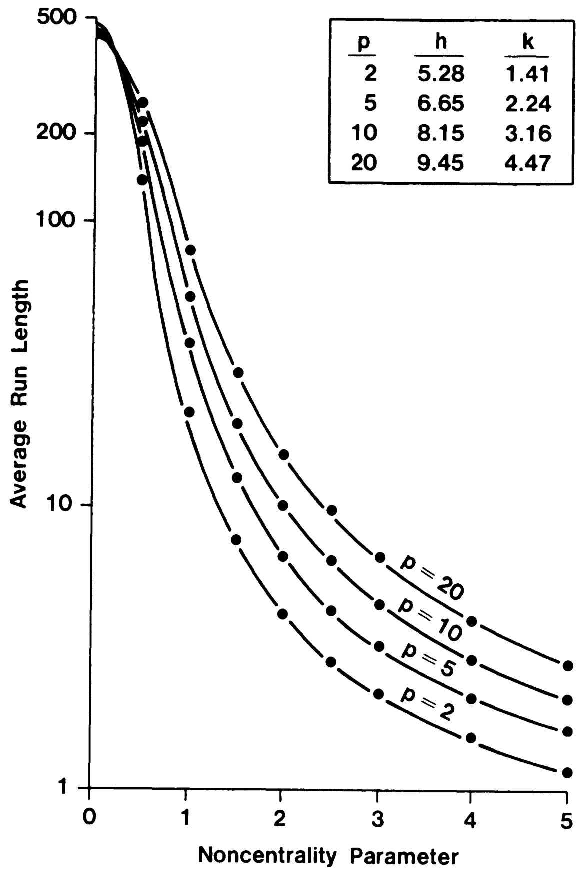

Figure 6. ARL Curves for the COT Schemes of Figure 4, but With the Fast Initial Response Feature Implemented.

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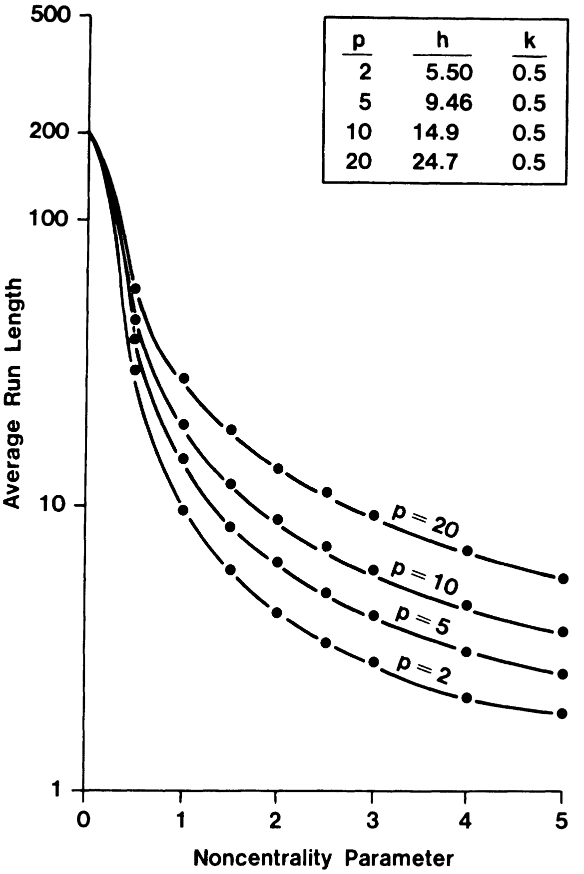

Figure 7. ARL Curves for Multivariate CUSUM Schemes With On-Target ARL's of 200.

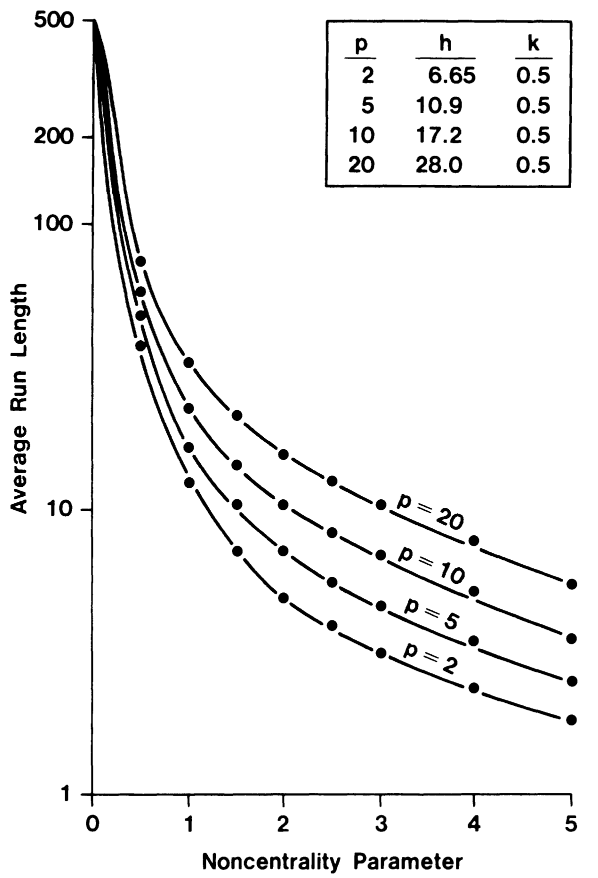

Figure 8. ARL Curves for Multivariate CUSUM Schemes With On-Target ARL's of 500.

to the Woodall and Ncube (1985) procedure. Table 3 gives the ARL's for this procedure applied to principal components (that are scaled to have variance 1) and the ARL's (with standard errors) of a comparable multivariate CUSUM scheme. (Notice that, for the same value of d, the ARL's of the Woodall and Ncube procedure are usually larger when the direction of the shift is along one of the original axes than when the direction of the shift is along one of the principal component axes.) In Table 3, the off-target ARL's of the multivariate CUSUM scheme are less than or equal to the ARL's of the Woodall and Ncube procedure. The difference between the ARL's of the two procedures is usually several times the standard error of the estimated ARL of the multivariate CUSUM scheme. There is, however, another source of error that must be considered. The value of h required to give the multivariate CUSUM scheme an on-target ARL of 125 is an estimate, obtained by the method given in Appendix A. This method requires regressing log (ARL) on h, where the ARL's for different h's are obtained by simulation. The estimated h for an on-target ARL of 125 is 4.95, the value used for Table 3. A 95% fiducial interval (see Draper and Smith 1981, pp. 47–51) for the value of h yielding an on-target ARL of 125 is (4.84, 5.09). Using an h value of 5.09, the ARL's of the multivariate CUSUM scheme, for $d = .5$, 1, 2, and 4, are 27.1, 9.3, 3.9, and 2.0, with standard errors of .95, .24, .06, and .02, respectively. Except for $d = 4$, where the ARL's of the two procedures are equal, these ARL's are still significantly smaller than the ARL's of the Woodall and Ncube procedure. (The Markov chain approach described in App. C gives an on-target ARL of 126 for the bivariate CUSUM scheme $h = 4.95$, $k = .5$.)

Table 2. ARL Curves for Bivariate Schemes With $k = .5$, 1, and 1.5

a	$h = 5.50$, $k = .5$	$h = 2.99$, $k = 1$	$h = .1.87$, $k = 1.5$
0	200	200	200
.5	28.8	48.0	78.7
1.0	9.35	11.0	18.4
1.5	5.94	5.08	7.14
2.0	4.20	3.48	3.72
2.5	3.26	2.51	2.36
3.0	2.78	2.08	1.69

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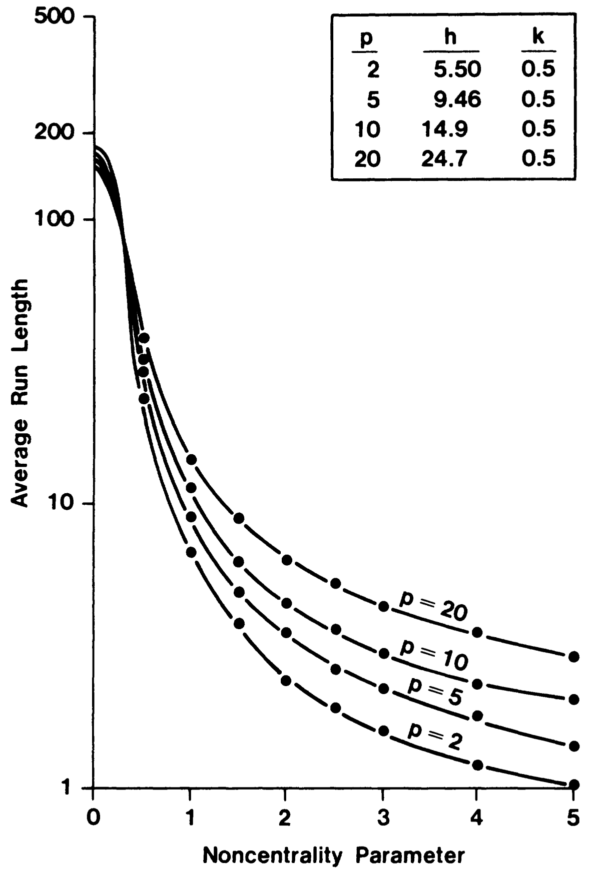

Figure 9. ARL Curves for the Multivariate CUSUM Schemes of Figure 7, but With the Fast Initial Response Feature Implemented.

6. APPLICATIONS

In the making of synthetic yarn, the fibers are stretched as they are made (Lucas 1973). This stretching of the fiber orients the molecules and makes the fiber stronger and more brittle. A measure of this stretching is the draw ratio, which is defined as the stretched length divided by the original length. The finished fiber is tested for acceptable strength by a tensile machine, which pulls a fiber until it breaks. Both the force required to break the fiber (called break strength) and the amount the fiber stretched while being pulled (called elongation) are reported by the tensile machine. Break strength and elongation are positively correlated because the fibers are all made at the same draw ratio. Only when the draw ratio is varied do break strength and elongation show a negative relationship. A multivariate quality control scheme for this example should be based on the positive correlation between break strength and elongation.

Lucas (1973) mentioned that a multivariate Shewhart chart was used to obtain tighter control of this process than is possible using simultaneous univariate CUSUM schemes. Either a COT scheme or a multivariate CUSUM scheme could be used for even tighter control of the process. Univariate CUSUM's of the principal components would also be an excellent procedure for this application. First, a rectangle based on principal components fits around the tilted ellipsoidal region much more closely than a rectangle based on the original components. Second, the principal components are interpretable; the minor component corresponds to problems with the draw ratio, whereas the major component corresponds to the usual variation in the process. A multivariate Shewhart chart or a COT scheme will not indicate what the problem is when the scheme signals. A multivariate CUSUM scheme, however, will provide an indication of the direction that the mean has shifted and hence what the problem is.

Principal components allow adoption of an outlier rule specific to this application. In my own experience with the testing of synthetic fibers by a tensile machine, unusual values of the minor principal component are much more likely to indicate data errors than an actual process shift, whereas unusual values of the major principal component are much more likely to indicate a process shift than a data error.

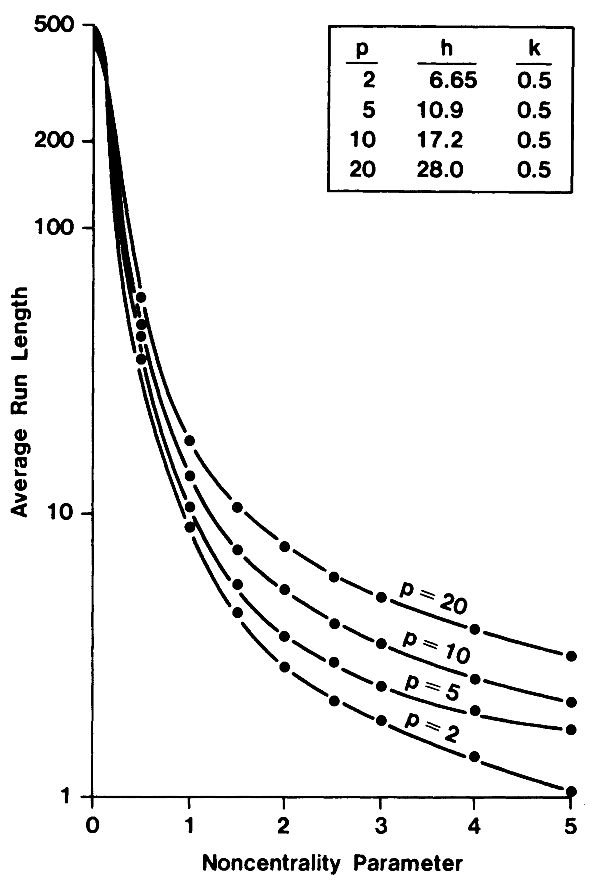

Figure 10. ARL Curves for the Multivariate CUSUM Schemes of Figure 8, but With the Fast Initial Response Feature Implemented.

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Table 3. Comparison to Woodall and Ncube's Procedure on Principal Components: Bivariate Procedures With On-Target ARL's of 125

r	μ_1	μ_2	d	Woodall and Ncube ARL	Section 5 ARL	Section 5 SE
.0	.5	0	.5	33.6	26.8	1.02
−.5	.5	0	.58	28.7	20.7	.76
−.5	.5	−.5	.58	26.7	20.6	.79
.5	.5	0	.58	28.7	18.9	.62
.5	.5	.5	.58	26.7	19.3	.68
.0	1	0	1.0	10.9	8.9	.22
−.5	.5	.5	1.0	10.9	9.2	.21
.5	.5	−.5	1.0	10.9	9.1	.23
−.5	1	0	1.15	9.8	7.1	.15
−.5	1	−1	1.15	8.8	7.2	.14
.5	1	0	1.15	9.8	7.3	.17
.5	1	1	1.15	8.8	7.6	.17
.0	2	0	2.0	6.0	3.8	.06
−.5	1	1	2.0	4.1	3.8	.06
.5	1	−1	2.0	4.1	3.7	.06
−.5	2	0	2.31	3.9	3.2	.05
−.5	2	−2	2.31	3.5	3.3	.05
.5	2	0	2.31	3.9	3.3	.05
.5	2	2	2.31	3.5	3.2	.05
−.5	2	2	4.0	2.0	2.0	.02
.5	2	−2	4.0	2.0	2.0	.02

NOTE: The Woodall and Ncube (1985) procedure has $h = 5$ and $k = .5$; the Section 5 CUSUM procedure has $h = 4.95$ and $k = .5$. SE represents standard error.

Hence the minor principal component, rather than a T statistic, could be used to judge the acceptability of an observation. This outlier criterion could be used with any multivariate quality control procedure, not just CUSUM's of the principal components.

Philpot and Ranney (1985) discussed the use of multivariate Shewhart charts for a rolling operation. A roller is used to control the thickness of a paper, plastic, or metal sheet product. If three measurements of thickness are made across the sheet (left side, middle, and right side), these measurements will be highly (positively) correlated. Two types of problems encountered in rolling operations are tilt of the roller (which makes the measurement on one side too high and the measurement on the other side too low) and build-up of material on the roller (which makes one measurement too low). Philpot and Ranney (1985) demonstrated the superiority of the multivariate Shewhart chart over simultaneous univariate Shewhart charts to detect these problems in a rolling operation. They examined the individual components of the observation vector $\mathbf{x}_n$ to determine whether the problem is tilt or build-up on the roller. Both COT schemes and multivariate CUSUM schemes would give improved detection of these types of problems over multivariate Shewhart charts. The CUSUM vector of a multivariate CUSUM scheme could be examined to determine the nature of the process problem.

The covariance matrix of thickness measurements across a sheet product made by a rolling operation will have all diagonal elements equal and all off-diagonal elements equal. This assumes that the covariance matrix is estimated from data collected during a period of stable operation—that is, a period during which the process mean is constant. If build-up on the roller occurs during the period of data collection, the covariances in one particular row and column will be too low. An unstable tilt of the roller during the data-collection period would make the covariances decrease with increasing distance between the measurements. Of course, even in a univariate case, one does not estimate the variance from data taken during a period of process instability. Sampling variability will prevent the estimated variances and covariances from being exactly equal; a constrained estimation procedure may be useful.

Rolling operations provide an example in which neither CUSUM's of the original variables nor CUSUM's of principal components are a satisfactory procedure. Only the first principal component of the equal-variance, equal-covariance matrix is uniquely defined (Morrison 1967, pp. 244–245); it is proportional to the sum of the measurements. A CUSUM of the first principal component would detect changes in the overall thickness of the sheet. A linear trend could be used for the second principal component and this would correspond to problems

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with the tilt of the roller. But the build-up problem cannot be reduced to $p - 2$ meaningful principal components. The $p - 2$ additional principal components do not indicate where the build-up has occurred, and there may be more than three measurements across the sheet—thus producing many such uninformative principal components.

7. SUMMARY

This article has presented two multivariate CUSUM quality-control procedures. The first procedure reduces each observation to a scalar—Hotelling's T statistic—and forms a CUSUM of the T statistics. This procedure is referred to as COT. The second procedure, referred to as multivariate CUSUM, forms a CUSUM vector directly from the observations. Both procedures allow use of recent enhancements for CUSUM schemes, such as robustness and the FIR feature. Both of these CUSUM procedures reduce to multivariate Shewhart charts when the CUSUM scheme parameter h is 0. (The analogous relationship holds for univariate CUSUM schemes and Shewhart charts.) The ARL of the two multivariate CUSUM procedures depends on the mean vector and correlation structure of the data only through the noncentrality parameter d. This property also holds for multivariate Shewhart charts and allows the three multivariate quality-control procedures to be compared easily. Both multivariate CUSUM schemes and COT schemes give faster detection of small shifts in the mean vector than multivariate Shewhart charts, with multivariate CUSUM schemes giving faster detection than COT schemes. Multivariate CUSUM schemes may also be preferred to COT schemes because the CUSUM vector gives an indication of the direction that the mean has shifted.

The nature of multivariate data is briefly discussed from an applied point of view. Robustness is practically required in a multivariate quality control scheme. In some applications, the correlation structure of multivariate data may allow development of a unique outlier rule for the specific application.

ACKNOWLEDGMENTS

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APPENDIX A: NUMERICAL METHODS

The calculation of ARL's for multivariate Shewhart charts is relatively straightforward: ARL $= 1/q$, where $q = \Pr(T > \text{SCL})$. Further, the probability that T exceeds SCL is the probability that a (central or noncentral) chi-square variable with p df exceeds the Shewhart control limit squared. These probabilities were found using the IMSL routine MDCHN (IMSL, Inc. 1982).

The calculation of ARL's for COT schemes was based on the Markov chain approach of Brook and Evans (1972). Their development will not be repeated here except as necessary to document the procedures used for this work. The transition probabilities were found using the techniques given by Brook and Evans (1972) but with the (central or noncentral) chi distribution replacing the Gaussian distribution. The cumulative distribution function of the chi distribution at a point x was obtained by squaring x and referring to the chi-square distribution, as calculated by the IMSL routine MDCHN. Thirty-two transient states were used in the Markov chain approximation; the ARL starting from the first state was used as the ARL when the continuous CUSUM scheme starts at 0 (Brook and Evans 1972). For the FIR feature, the ARL was calculated by quadratic interpolation (least squares fit) using the four states closest to the head-start value $h/2$.

The ARL's for the multivariate CUSUM schemes were obtained by simulation. The only difficulty was finding the value of h, call it h', that gives the required on-target ARL. To find h' for an on-target ARL of 200, an ARL was calculated from a simulation of 50 run lengths; if the ARL was less than 200, an increment of .25 was added to h, but if the ARL was greater than 200, the increment of .25 was subtracted from h. Even with a close initial guess, eight iterations of this technique (for a total of 400 run lengths) did not yield a good estimate of h'. Hence the procedure was repeated, using an increment of .1 and 20 iterations. All 28 ARL's (each based on 50 run lengths) were used in a linear regression of log (ARL) on h. The regression equation was used to estimate h'. The values of h for all other multivariate CUSUM schemes (those with on-target ARL's of 500 or 125, or with $k = 1$ or 1.5) were obtained by the same procedure, but used only 20 iterations and an increment of .1. The on-target ARL's were therefore assumed to be the required value. The ARL's for $d > 0$ were found from simulations of 400 run lengths.

The ARL's for multivariate CUSUM schemes with the FIR feature were also obtained by simulation of 400 run lengths. The ARL's at $d = 0$, however, were quite variable and were smoothed by regressing log (ARL) on p. The predicted ARL's from this regression were used in Figures 9 and 10 for the ARL's at $d = 0$.

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APPENDIX B: PROOF

This proof that the distribution of the multivariate CUSUM test statistic Y_n depends only on the value of the noncentrality parameter is based on the proof provided by John Healy.

Without loss of generality, assume that $\mathbf{a} = \mathbf{0}$ and let $\mathbf{u} = E(\mathbf{x})$. Then the multivariate CUSUM becomes

$$C_n = [(\mathbf{s}_{n-1} + \mathbf{x}_n)' \boldsymbol{\Sigma}^{-1}(\mathbf{s}_{n-1} + \mathbf{x}_n)]^{1/2},$$

$$\begin{aligned} \mathbf{s}_n &= 0 && \text{if } C_n \leq k \\ &= (\mathbf{s}_{n-1} + \mathbf{x}_n)(1 - k/C_n) && \text{if } C_n > k. \end{aligned}$$

Let $\mathbf{u}'\boldsymbol{\Sigma}^{-1}\mathbf{u}$ denote the squared noncentrality parameter. I show that, if $\mathbf{u}_1'\boldsymbol{\Sigma}^{-1}\mathbf{u}_1 = \mathbf{u}_2'\boldsymbol{\Sigma}^{-1}\mathbf{u}_2$, then the distribution of Y_n is the same. First, here are some lemmas.

Lemma 1. If $\mathbf{M}$ is a $p \times p$ full rank matrix and $\mathbf{x}^* = \mathbf{Mx}$, then the statistics C_n and Y_n have the same values when calculated from $\mathbf{x}^*$ as when calculated from $\mathbf{x}$. Further, $\mathbf{s}^*$, the multivariate CUSUM vector for $\mathbf{x}^*$, satisfies $\mathbf{s}^* = \mathbf{Ms}$.

Proof. The proof is by induction. Note that $E(\mathbf{x}^*) = \mathbf{Mu}$ and $\text{var}(\mathbf{x}^*) = \mathbf{M\Sigma M}'$. For $n = 1$,

$$\begin{aligned} C_1^* &= +[\mathbf{x}_1^{*\prime}(\mathbf{M}'^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{M}^{-1})\mathbf{x}_1^*]^{1/2} \\ &= [\mathbf{x}_1'\mathbf{M}'(\mathbf{M}'^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{M}^{-1})\mathbf{M}\mathbf{x}_1]^{1/2} \\ &= +[\mathbf{x}_1'\boldsymbol{\Sigma}^{-1}\mathbf{x}_1] = C_1 \end{aligned}$$

and

$$\begin{aligned} \mathbf{s}_1^* &= \mathbf{x}_1^*(1 - k/C_1^*) = \mathbf{x}_1^*(1 - k/C_1) = \mathbf{M}\mathbf{x}_1(1 - k/C_1) \\ &= \mathbf{M}\mathbf{s}_1. \end{aligned}$$

Now assume for $n - 1$ and prove for n:

$$\begin{aligned} C_n^* &= [(\mathbf{s}_{n-1}^* + \mathbf{x}_n^*)'(\mathbf{M}'^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{M}^{-1})(\mathbf{s}_{n-1}^* + \mathbf{x}_n^*)]^{1/2} \\ &= [(\mathbf{M}\mathbf{s}_{n-1} + \mathbf{M}\mathbf{x}_n)'(\mathbf{M}'^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{M}^{-1}) \\ &\quad \times (\mathbf{M}\mathbf{s}_{n-1} + \mathbf{M}\mathbf{x}_n)]^{1/2} \\ &= [(\mathbf{s}_{n-1} + \mathbf{x}_n)'\mathbf{M}'(\mathbf{M}'^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{M}^{-1})\mathbf{M}(\mathbf{s}_{n-1} + \mathbf{x}_n)]^{1/2} \\ &= [(\mathbf{s}_{n-1} + \mathbf{x}_n)'\textstyle\sum^{-1}(\mathbf{s}_{n-1} + \mathbf{x}_n)]^{1/2} = C_n \end{aligned}$$

and

$$\begin{aligned} \mathbf{s}_n^* &= (\mathbf{s}_{n-1}^* + \mathbf{x}_n^*)(1 - k/C_n^*) \\ &= (\mathbf{s}_{n-1}^* + \mathbf{x}_n^*)(1 - k/C_n) \\ &= (\mathbf{M}\mathbf{s}_{n-1} + \mathbf{M}\mathbf{x}_n)(1 - k/C_n) \\ &= \mathbf{M}(\mathbf{s}_{n-1} + \mathbf{x}_n)(1 - k/C_n) \\ &= \mathbf{M}\mathbf{s}_n. \end{aligned}$$

Therefore,

$$\begin{aligned} Y_n^* &= [\mathbf{s}_n^{*\prime}(\mathbf{M}'^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{M}^{-1})\mathbf{s}_n^*]^{1/2} \\ &= [\mathbf{s}_n'\mathbf{M}'(\mathbf{M}'^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{M}^{-1})\mathbf{M}\mathbf{s}_n]^{1/2} = Y_n; \end{aligned}$$

that is, the test statistic Y_n is identical when $\mathbf{x}^*$ is used instead of $\mathbf{x}$.

Apply Lemma 1 with $\mathbf{M} = \mathbf{P}$, where $\mathbf{P}$ is an orthogonal matrix ($\mathbf{P}'\mathbf{P} = \mathbf{PP}' = \mathbf{I}$) that diagonalizes $\boldsymbol{\Sigma}$. The principal components are $\mathbf{w} = \mathbf{Px}$, and $\mathbf{P\Sigma P}' = \mathbf{D}$, a diagonal matrix of the eigenvalues of $\boldsymbol{\Sigma}$. This shows that the multivariate CUSUM test statistics Y_n are identical when calculated from the principal components instead of from the original variables. Lemma 1 can then be applied with $\mathbf{M} = \mathbf{D}^{-1/2}$ to show that principal components scaled to have variance 1, $\mathbf{z} = \mathbf{D}^{-1/2}\mathbf{w}$, also give the same test statistics.

Lemma 2. If $\mathbf{x}^* = \mathbf{Mx}$, where $\mathbf{M}$ is a $p \times p$ matrix of full rank, then

$$\begin{aligned} \mathbf{x}^{*\prime}(\mathbf{M}'^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{M}^{-1})\mathbf{x}^* &= \mathbf{x}'\mathbf{M}'(\mathbf{M}'^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{M}^{-1})\mathbf{Mx} \\ &= \mathbf{x}'\boldsymbol{\Sigma}^{-1}\mathbf{x}. \end{aligned}$$

This lemma implies that the noncentrality parameter has the same value whether computed from the original dependent variables, principal components, or principal components scaled to have variance 1.

Lemma 3. If $\mathbf{u}_1'\boldsymbol{\Sigma}^{-1}\mathbf{u}_1 = \mathbf{u}_2'\boldsymbol{\Sigma}^{-1}\mathbf{u}_2$, there exists a nonsingular matrix $\mathbf{M}$ such that $\mathbf{u}_1 = \mathbf{M}\mathbf{u}_2$.

Proof. $\mathbf{M}$ can be found by the following procedure. As it is a product of invertible matrices, it is nonsingular.

The first step is to transform $\mathbf{x}$ to $\mathbf{z}$, the principal components scaled to have variance 1. Let $E(\mathbf{z}) = \mathbf{v}$. By Lemma 2, $\mathbf{v}_1'\mathbf{v}_1 = \mathbf{v}_2'\mathbf{v}_2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the images of $\mathbf{u}_1$ and $\mathbf{u}_2$ under the transformation from $\mathbf{x}$ to $\mathbf{z}$. Note that there exists an orthogonal matrix $\mathbf{Q}$ such that $\mathbf{v}_1 = \mathbf{Q}\mathbf{v}_2$. The transformation defined by the matrix $\mathbf{Q}$ changes from one basis to another in a vector space. Orthogonal transformations preserve length in the sense that if $\mathbf{v}_1 = \mathbf{Q}\mathbf{v}_2$, then $\mathbf{v}_1'\mathbf{v}_1 = \mathbf{v}_2'\mathbf{Q}'\mathbf{Q}\mathbf{v}_2 = \mathbf{v}_2'\mathbf{v}_2$. To find $\mathbf{M}$, substitute $\mathbf{D}^{-1/2}\mathbf{Pu}$ for $\mathbf{v}$ in $\mathbf{v}_1 = \mathbf{Q}\mathbf{v}_2$: $\mathbf{D}^{-1/2}\ \mathbf{P}\mathbf{u}_1 = \mathbf{Q}(\mathbf{D}^{-1/2}\mathbf{P}\mathbf{u}_2)$, or $\mathbf{u}_1 = \mathbf{P}^{-1}\mathbf{D}^{-1/2}\mathbf{Q}\mathbf{D}^{-1/2}\mathbf{P}\mathbf{u}_2$, so that $\mathbf{M} = \mathbf{P}'\mathbf{D}^{1/2}\mathbf{Q}\mathbf{D}^{-1/2}\mathbf{P}$.

Lemma 4. $Y_n = \max[0, C_n - k]$.

Proof. By definition, $Y_n = [\mathbf{s}_n'\boldsymbol{\Sigma}^{-1}\mathbf{s}_n]^{1/2}$. Recall that

$$\begin{aligned} \mathbf{s}_n &= \mathbf{0} && \text{if } C_n \leq k \\ &= (\mathbf{s}_{n-1} + \mathbf{x}_n)(1 - k/C_n) && \text{if } C_n > k. \end{aligned}$$

Therefore, for $C_n > k$,

$$\begin{aligned} Y_n^2 &= \mathbf{s}_n'\boldsymbol{\Sigma}^{-1}\mathbf{s}_n \\ &= (\mathbf{s}_{n-1} + \mathbf{x}_n)'(1 - k/C_n)\boldsymbol{\Sigma}^{-1}(\mathbf{s}_{n-1} + \mathbf{x}_n)(1 - k/C_n) \\ &= (1 - k/C_n)^2(\mathbf{s}_{n-1} + \mathbf{x}_n)'\boldsymbol{\Sigma}^{-1}(\mathbf{s}_{n-1} + \mathbf{x}_n) \\ &= (1 - k/C_n)^2 C_n^2 \\ &= (C_n - k)^2 \end{aligned}$$

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or $Y_n = C_n - k$.

Let $f[C_n \mid E(\mathbf{x}) = \mathbf{u}_1]$ denote the distribution function of C_n calculated from the random variable $\mathbf{x}$ given $E(\mathbf{x}) = \mathbf{u}_1$.

Theorem. If $\mathbf{u}_1'\boldsymbol{\Sigma}^{-1}\mathbf{u}_1 = \mathbf{u}_2'\boldsymbol{\Sigma}^{-1}\mathbf{u}_2$, then

$$f[C_n \mid E(\mathbf{x}) = \mathbf{u}_1] = f[C_n \mid E(\mathbf{x}) = \mathbf{u}_2].$$

The theorem equates the distributions of C_n when the data vectors have one of two alternative probability densities. Let pdf1 refer to the density specified by $E(\mathbf{x}) = \mathbf{u}_1$ and pdf2 refer to the density specified by $E(\mathbf{x}) = \mathbf{u}_2$. To prove the theorem, express pdf2 in transformed variates $\mathbf{t} = \mathbf{M}\mathbf{x}$, where $\mathbf{M}$ is given in Lemma 3. This gives $E(\mathbf{t}) = \mathbf{M}\mathbf{u}_2 = \mathbf{u}_1$ and $\mathrm{var}(\mathbf{t}) = \mathbf{M}\boldsymbol{\Sigma}\mathbf{M}' = \boldsymbol{\Sigma}$. To show $\mathrm{var}(\mathbf{t}) = \boldsymbol{\Sigma}$, write out

$$\mathbf{M}\boldsymbol{\Sigma}\mathbf{M}' = \mathbf{P}'\mathbf{D}^{1/2}\mathbf{Q}\mathbf{D}^{-1/2}\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\mathbf{D}^{-1/2}\mathbf{Q}'\mathbf{D}^{1/2}\mathbf{P}.$$

As $\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' = \mathbf{D}$, this becomes

$$\mathbf{M}\boldsymbol{\Sigma}\mathbf{M}' = \mathbf{P}'\mathbf{D}^{1/2}\mathbf{Q}(\mathbf{D}^{-1/2}\mathbf{D}\mathbf{D}^{-1/2})\mathbf{Q}'\mathbf{D}^{1/2}\mathbf{P}$$
$$= \mathbf{P}'\mathbf{D}^{1/2}(\mathbf{Q}\mathbf{Q}')\mathbf{D}^{1/2}\mathbf{P} = \mathbf{P}'\mathbf{D}\mathbf{P}.$$

Substituting $\mathbf{D} = \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'$ into $\mathbf{P}'\mathbf{D}\mathbf{P}$ gives

$$\mathbf{M}\boldsymbol{\Sigma}\mathbf{M}' = \mathbf{P}'(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')\mathbf{P} = \boldsymbol{\Sigma}.$$

Hence pdf2, expressed in the transformed coordinates $\mathbf{t}$, is the same as pdf1 expressed in $\mathbf{x}$ coordinates. The value of C_n is invariant with respect to the transformation from $\mathbf{x}$ to $\mathbf{t}$, so

$$f[C_n \mid E(\mathbf{x}) = \mathbf{u}_2] = f[C_n \mid E(\mathbf{t}) = \mathbf{M}\mathbf{u}_2 = \mathbf{u}_1]. \quad \text{(B.1)}$$

Because the data vectors for pdf1 (in $\mathbf{x}$ coordinates) and pdf2 (in $\mathbf{t}$ coordinates) have multivariate normal distributions with the same mean vector and covariance matrix,

$$f[C_n \mid E(\mathbf{x}) = \mathbf{u}_1] = f[C_n \mid E(\mathbf{t}) = \mathbf{u}_1]. \quad \text{(B.2)}$$

Combining (B.1) and (B.2) gives

$$f[C_n \mid E(\mathbf{x}) = \mathbf{u}_1] = f[C_n \mid E(\mathbf{x}) = \mathbf{u}_2].$$

By Lemma 4, $Y_n = \max[0, C_n - k]$, so the distribution of Y_n given $\mathbf{u}_1$ is the same as the distribution of Y_n given $\mathbf{u}_2$ when $\mathbf{u}_1\boldsymbol{\Sigma}^{-1}\mathbf{u}_1 = \mathbf{u}_2\boldsymbol{\Sigma}^{-1}\mathbf{u}_2$.

APPENDIX C: MARKOV-CHAIN REPRESENTATION

For the on-aim case, the ARL's of the multivariate CUSUM procedure can be approximated by using a discrete Markov chain. Following Brook and Evans (1972), the possible values of Y are represented by $t + 1$ states. One state is an absorbing state representing $Y > h$. The t transient states are numbered 0, 1, 2, ..., $(t - 1)$ and represent values of Y between 0 and h. It is helpful to think of the Markov chain in terms of a discrete random variable—call it Y'—that takes on values 0, w, $2w$, ..., tw, where $w = 2h/(2t - 1)$. The transition probabilities among the transient states are needed to find the ARL. The transition probabilities are

$$\Pr(Y_n' = jw \mid Y_{n-1}' = iw), \quad i, j \in \{0, 1, 2, \ldots, (t - 1)\}.$$

To find the transition probabilities, note that $Y_n = \max[0, C_n - k]$, where

$$C_n = [(\mathbf{s}_{n-1} + \mathbf{x}_n - \mathbf{a})'\boldsymbol{\Sigma}^{-1}(\mathbf{s}_{n-1} + \mathbf{x}_n - \mathbf{a})]^{1/2}.$$

The transition probabilities are conditional probabilities, so $\mathbf{s}_{n-1}$ is considered a constant rather than a random variable. Hence, for the on-aim case, $E(\mathbf{s}_{n-1} + \mathbf{x}_n - \mathbf{a}) = \mathbf{s}_{n-1}$ and $\mathrm{var}(\mathbf{s}_{n-1} + \mathbf{x}_n - \mathbf{a}) = \boldsymbol{\Sigma}$. Under the assumption of a multivariate normal distribution for $\mathbf{x}_n$, C_n has a chi distribution with noncentrality parameter $[\mathbf{s}_{n-1}'\boldsymbol{\Sigma}^{-1}\mathbf{s}_{n-1}]^{1/2} = Y_{n-1}$. Therefore,

$$\Pr(Y_n' = 0 \mid Y_{n-1}' = iw) = \Pr(C_n \le k + w/2)$$

and, for $j > 0$,

$$\Pr(Y_n' = jw \mid Y_{n-1}' = iw)$$
$$= \Pr[k + (j - .5)w < C_n \le k + (j + .5)w],$$

where C_n has a chi distribution with noncentrality parameter iw.

Brook and Evans (1972) suggested obtaining the ARL for several different sizes of Markov chains and extrapolating to the continuous case by the formula $\mathrm{ARL}(t) = \text{asymptotic ARL} + B/t + C/t^2$.

For most of the schemes discussed in this article the use of $t = 12$, 15, 18, 21, and 24 was adequate to verify the on-aim ARL's. Larger values of t are needed for schemes with large values of h. For the scheme $p = 20$, $h = 28$, and $k = .5$, the use of $t = 12$, 15, 18, 21, and 24 gave an asymptotic ARL of 634, whereas the use of $t = 12$, 24, 36, 48, and 60 gave an asymptotic ARL of 505.

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