Reduction of simple mechanical systems

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Abstract

An overview is first given of reduction for simple mechanical systems (i.e., those whose Lagrangians are kinetic energy minus potential energy) with symmetry in the case when the action is free. Both Lagrangian and Hamiltonian perspectives are given.

1. Outline

These notes are intended to be a sketchy, broad discussion of some of what is known about reduction for so-called simple mechanical systems. Some of what we say can be interpreted for general systems on tangent or cotangent bundles without the additional assumption about a corresponding simple Lagrangian or Hamiltonian. However, to be concrete, let us stick to the true simple case.

Our point of view is one which, in the Hamiltonian framework, is associated with Poisson reduction. That is to say, our Lagrangian reduction strategy is an analogue of Poisson reduction when viewed in a Hamiltonian context.

The Hamiltonian content here may be found in the dissertation of Montgomery [1986]. The Lagrangian perspective for free actions is currently being worked out by [Cendra, Marsden, and Ratiu 2001], and also see the paper of Marsden and Scheurle [1993].

2. Simple mechanical systems with symmetry

The basic data with which these notes will concern themselves is

1. an n-dimensional manifold Q (the configuration manifold),
2. a Riemannian metric k on Q (the kinetic energy metric),
3. a function V on Q (the potential energy function), and
4. an r-dimensional Lie group G which acts on (Q, k) by isometries and which leaves V invariant.

Let us denote by $\Phi\colon G \times Q \to Q$ the action and $\Phi_g\colon q \mapsto \Phi(g, q)$. We shall also write $g.q = \Phi(g, q)$. If $\xi \in \mathfrak{g}$ then we let ξ_Q denote the associated infinitesimal generator:

$$\xi_Q(q) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi(\exp(t\xi), q).$$

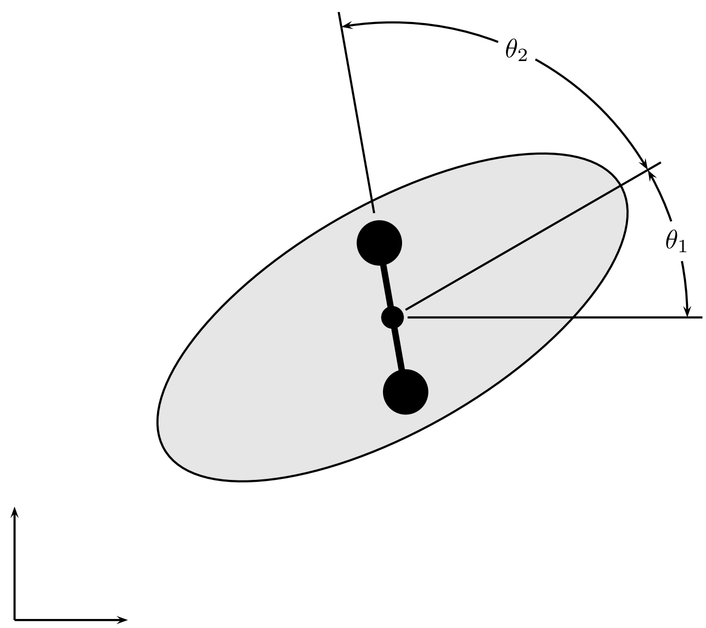

Figure 1.

2.1 Example: As an *extremely* simple example, we consider two rigid bodies in the plane which are pinned so each body rotates about the same point (see Figure 1). This example is sometimes called *Elroy's beanie*. The configuration manifold is $Q = \mathbb{S}^1 \times \mathbb{S}^1$ for which we use coordinates (θ_1, θ_2) as indicated in the figure. With this choice of coordinates the kinetic energy is

$$\frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2(\dot{\theta}_1 + \dot{\theta}_2)^2.$$

Thus the Riemannian metric we consider is

$$k = (I_1 + I_2)\mathrm{d}\theta_1 \otimes \mathrm{d}\theta_1 + I_2\mathrm{d}\theta_1 \otimes \mathrm{d}\theta_2 + I_2\mathrm{d}\theta_2 \otimes \mathrm{d}\theta_1 + I_2\mathrm{d}\theta_2 \otimes \mathrm{d}\theta_2$$

where I_1 is the inertia of the body measured by angle θ_1 and I_2 is the inertia of the other body, both inertias being measured about the rotation point. In terms of its matrix

$$k \text{ ``}=\text{''} \begin{bmatrix} I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}.$$

We put here for the only time quotes around the equals sign. Of course k is not a matrix. But we shall find it convenient when working with examples to simply write certain objects as if they were matrices. For this example we can also consider a potential function of the form $V(\theta_2)$.

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The symmetry group we consider is $G = SO(2)$. Let us parameterise the group by θ in the usual manner. We also identify $\mathfrak{so}(2)$ with $\mathbb{R}$ in the usual manner so that the basis vector e_1 corresponds to a counterclockwise ('scuse me, *anti*clockwise) rotation. Obviously the Lie bracket is trivial. The exponential map is $\exp(\omega) = \omega \mod 2\pi$. The adjoint action of $SO(2)$ on $\mathfrak{so}(2)$ is $\mathrm{Ad}_\theta = \mathrm{id}_{\mathbb{R}}$. We also have the derivative of left translation given by $T_0 L_g(\omega) = \omega \frac{\partial}{\partial\theta}$.

The action of $\mathfrak{so}(2)$ we consider on Q is $(\theta, (\theta_1, \theta_2)) \mapsto (\theta_1 + \theta, \theta_2)$. The infinitesimal generator is then $e_{1,Q} = \frac{\partial}{\partial\theta_1}$. This action obviously leaves the metric and the assumed potential function invariant. •

2.2 Example: This example will be a bit more complicated to show that we can work things out in cases which are not completely trivial. The example is a pair of coupled planar rigid bodies. If we choose an inertial reference frame $\mathcal{F}_I$ and attach a reference frame $\mathcal{F}_B$ to the centre of gravity of one of the bodies, the configuration space is seen to be $Q = SE(2) \times \mathbb{S}^1$. We shall use the convention of measuring the angle of the second body relative (rather than absolutely) to the body whose reference frame we are keeping track of. This is depicted in Figure 2. We shall use coordinates $(x, y, \theta_1, \theta_2)$ for Q where the first three coordinates

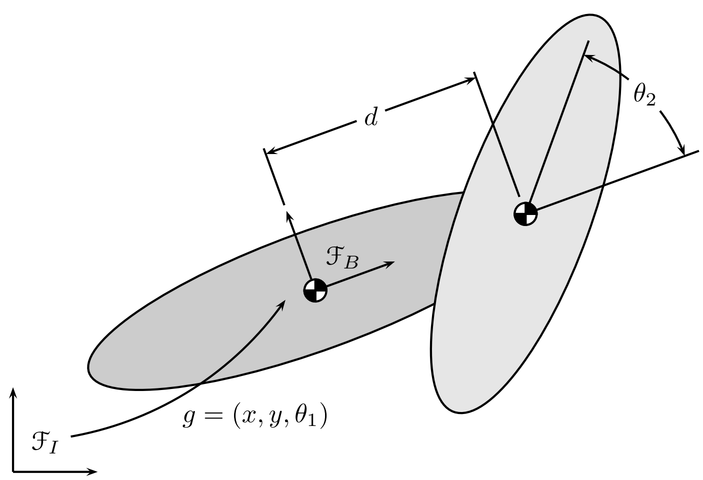

Figure 2.

parameterise $SE(2)$ by

$$(x, y, \theta_1) \mapsto \begin{bmatrix} R(\theta_1) & p \\ 0 & 1 \end{bmatrix}$$

where $R(\theta) = \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$ and $p = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. It will also sometimes be advantageous to write elements of $SE(2)$ simply by g, especially when we come to symmetry operations.

In determining the kinetic energy for the system, we make the assumption that the second body is joined to the first at its centre of mass. With this assumption, the kinetic energy is

$$\frac{1}{2}m_1(\dot{x}^2+\dot{y}^2)+\frac{1}{2}I_1\dot{\theta}_1^2+\\ \frac{1}{2}m_2(\dot{x}^2+\dot{y}^2+d^2\dot{\theta}_1^2+2d\cos\theta_1\dot{y}\dot{\theta}_1-2d\sin\theta_1\dot{x}\dot{\theta}_1)+\frac{1}{2}I_2(\dot{\theta}_1+\dot{\theta}_2)^2.$$

Here m_i is the mass of the ith body, I_i is the moment of inertia of the ith body about its centre of mass, and d is the distance from the centre of mass of the first body to the pivot point. The associated Riemannian metric is

$$k=(m_1+m_2)\mathrm{d}x\otimes\mathrm{d}x-m_2d\sin\theta_1\mathrm{d}x\otimes\mathrm{d}\theta_1+(m_1+m_2)\mathrm{d}y\otimes\mathrm{d}y+m_2d\cos\theta_1\mathrm{d}y\otimes\mathrm{d}\theta_1-\\ m_2d\sin\theta_1\mathrm{d}\theta_1\otimes\mathrm{d}x+m_2d\cos\theta_1\mathrm{d}\theta_1\otimes\mathrm{d}y+(I_1+I_2+m_2d^2)\mathrm{d}\theta_1\otimes\mathrm{d}\theta_1+I_2\mathrm{d}\theta_1\otimes\mathrm{d}\theta_2+\\ I_2\mathrm{d}\theta_2\otimes\mathrm{d}\theta_1+I_2\mathrm{d}\theta_2\otimes\mathrm{d}\theta_2$$

or, if you prefer matrices,

$$k=\begin{bmatrix} m_1+m_2 & 0 & -m_2d\sin\theta_1 & 0\\ 0 & m_1+m_2 & m_2d\cos\theta_1 & 0\\ -m_2d\sin\theta_1 & m_2d\cos\theta_1 & I_1+I_2+m_2d^2 & I_2\\ 0 & 0 & I_2 & I_2\end{bmatrix}. \tag{2.1}$$

If one wished, one could add a potential function of the form $V(\theta_2)$ which would model some type of force (e.g., spring force) at the joint.

The symmetry group we consider is $SE(2)$. Let us record some basic facts about this Lie group for future reference. We shall write an element of $SE(2)$ as above:

$$\begin{bmatrix} R(\theta) & p\\ 0 & 1\end{bmatrix},\qquad R(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix},\ p=\begin{pmatrix}p_1\\ p_2\end{pmatrix}.$$

Written in this manner, the group operation on $SE(2)$ is simply matrix multiplication (thus we consider $SE(2)$ as a subgroup of $GL(3;\mathbb{R})$). The Lie algebra of $SE(2)$, $\mathfrak{se}(2)$, we represent by matrices of the form

$$\begin{bmatrix} A(\omega) & v\\ 0 & 0\end{bmatrix},\qquad A(\omega)=\begin{bmatrix}0 & -\omega\\ \omega & 0\end{bmatrix},\ v=\begin{pmatrix}v_1\\ v_2\end{pmatrix}.$$

Thus $\mathfrak{se}(2)$ is parameterised by the triple (v_1,v_2,ω). The exponential map $\exp\colon\mathfrak{se}(2)\to SE(2)$ is given by

$$\exp(v_1,v_2,\omega)=\begin{bmatrix} R(\omega) & R(\omega)\frac{\hat{v}}{\omega}-\frac{\hat{v}}{\omega}\\ 0 & 1\end{bmatrix},\qquad \hat{v}=\begin{pmatrix}v_2\\ -v_1\end{pmatrix}.$$

We will consider the action of the symmetry group $G=SE(2)$ given by

$$\Phi(h,(g,\theta_2))=(hg,\theta_2).$$

For this action, let us compute the infinitesimal generators. We use the basis

$$e_1=\begin{bmatrix}A(0) & \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\\ 0 & 1\end{bmatrix},\quad e_1=\begin{bmatrix}A(0) & \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\\ 0 & 1\end{bmatrix},\quad e_3=\begin{bmatrix}A(1) & 0\\ 0 & 1\end{bmatrix}$$

for $\mathfrak{se}(2)$. The corresponding infinitesimal generators are

$$e_{1,Q} = \frac{\partial}{\partial x}, \quad e_{2,Q} = \frac{\partial}{\partial y}, \quad e_{3,Q} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{\partial}{\partial \theta_1}.$$

The adjoint action of $SE(2)$ on $\mathfrak{se}(2)$ in the given basis is, writing $g = (x, y, \theta)$,

$$\mathrm{Ad}_g(\xi) = \begin{bmatrix} R(\theta) & \hat{p} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix}$$

where $p = (x, y)$. Also,

$$T_e L_g(\xi) = \begin{bmatrix} R(\theta) & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix}.$$

With this action, one readily verifies that k is G-invariant. Further, if we assume a potential function of the form indicated above, then it too will be invariant. •

Let us for the moment suppose that G acts freely and properly on Q so that $\pi\colon Q \to B \triangleq Q/G$ defines a principal fibre bundle. Note that a point in the base space B is a set of configurations which differ by some group translation. One may thus think of B as being the set of "shapes" or "internal configurations" of the system. Thus B is often called the *shape space*.

2.3 Example: (Example 2.1 cont'd) For Elroy's beanie $B = (\mathbb{S}^1 \times \mathbb{S}^1)/\mathfrak{so}(2)$ which we naturally parameterise by θ_2. This simply describes the relative orientation of the two bodies. •

2.4 Example: (Example 2.2 cont'd) For the coupled planar rigid bodies we are considering, the shape space is $B = (SE(2) \times \mathbb{S}^1)/SE(2) \simeq \mathbb{S}^1$. Thus, as expected, B describes the relative orientations of the bodies, or their shapes. •

In this case $VQ \triangleq \ker(T\pi)$ defines a subbundle of TQ called the *vertical subbundle*. A *connection* on Q is a subbundle HQ so that

1. $TQ = VQ \oplus HQ$, and
2. $T_q\Phi_g(H_qQ) = H_{g.q}Q$.

We write $v_q = \mathrm{hor}(v_q) + \mathrm{ver}(v_q)$ where $v_q \in T_qQ$, $\mathrm{hor}(v_q) \in H_qQ$, and $\mathrm{ver}(v_q) = V_qQ$. If $x \in B$ and if $q \in \pi^{-1}(x)$ then there is an isomorphism from T_xB to H_qQ which we denote by hlft_q.

The corresponding *connection one-form* is the $\mathfrak{g}$-valued one-form α on Q defined by

$$\alpha(v_q) = \{\xi \in \mathfrak{g} \mid \ \xi_Q(q) = \mathrm{ver}(v_q)\}.$$

If $U \times G$ is a local trivialisation of π with coordinates (x^α, g^a) then the local form of α is

$$\alpha(x, \dot{x}, g, \dot{g}) = \mathrm{Ad}_g(A(x)\dot{x} + T_g L_{g^{-1}}\dot{g}) \tag{2.2}$$

for some map $x \mapsto A(x) \in \mathrm{L}(T_xU; \mathfrak{g})$ called the *local connection form*. The *curvature form* of α is the $\mathfrak{g}$-valued two-form β defined by

$$\beta(u_q, v_q) = \boldsymbol{d}\alpha(\mathrm{hor}(u_q), \mathrm{hor}(v_q)).$$

The local form of the curvature form is

$$\beta(x, g)((\dot{x}_1, \dot{g}_1), (\dot{x}_1, \dot{g}_1)) = \mathrm{Ad}_g\big(\boldsymbol{d}A(x)(\dot{x}_1, \dot{x}_2) - [A(x)(\dot{x}_1), A(x)(\dot{x}_2)]\big).$$

2.5 Remark: If and only if $\beta = 0$ the horizontal distribution HQ is integrable. In this case there is an isomorphism of the fibre bundles $\pi\colon Q \to B$ and $\mathrm{pr}_1\colon B \times G \to B$ which maps the connection HQ on π to the flat connection on pr_1. If we choose a natural trivialisation for pr_1, then the local connection form for the flat connection is zero. •

Simple mechanical systems endow $\pi\colon Q \to Q/G$ with a *natural* connection called the mechanical connection. This is defined by choosing $H_qQ = V_qQ^{\perp}$. We define a q-dependent inner product $\mathbb{I}$ on $\mathfrak{g}$ by

$$\mathbb{I}(q)(\xi, \eta) = k_q(\xi(q), \eta(q))$$

which we call the *locked inertia tensor*. For systems for which the statement makes sense, this is the inertia of the system whose shape is $\pi(q)$.

Let us see how one may easily determine the mechanical connection and the locked inertia tensor by looking at the local form of k. Simply by G-invariance of k, in a local trivialisation we may write the matrix of k as

$$\begin{bmatrix} \tilde{M}(x) & \tilde{A}^*(x) \\ \tilde{A}(x) & I(x) \end{bmatrix}$$

where $\tilde{M}(x)$ is an inner product on T_xU, $I(x)$ is an inner product on $\mathfrak{g}$, and $\tilde{A}(x)$ is a linear map from T_xU to $\mathfrak{g}^*$. One may readily verify that the local form of the locked inertia tensor is

$$\mathbb{I}(x, g)(\xi, \eta) = I(x)(\mathrm{Ad}_{g^{-1}}\xi, \mathrm{Ad}_{g^{-1}}\eta)$$

which gives meaning to the lower right block of the matrix for k. If we define $A(x) \in \mathrm{L}(T_xU; \mathfrak{g})$ by $A(x)(\dot{x}) = I^{\sharp}(x)(\tilde{A}(x)(\dot{x}))$ then we may show that A is the local connection form for the mechanical connection. Thus, to summarise, the local form of the matrix for k is

$$\begin{bmatrix} \tilde{M}(x) & A^*(x) \circ I^{\flat}(x) \\ I^{\flat}(x) \circ A(x) & I(x) \end{bmatrix}.$$

In this representation we have erased the G-dependence of k by, instead of acting on a vector $(\dot{x}, \dot{g})$, acting on $(\dot{x}, \xi)$ where $\xi = g^{-1}\dot{g}$.

2.6 Remark: If the curvature of the mechanical connection is zero we may look at Remark 2.5 to assert that there is a diffeomorphism of Q with $B \times G$ which takes the metric on Q to a product metric on $B \times G$. •

What is the meaning of $\tilde{M}(x)$? It is tempting to regard it as a Riemannian metric on U, as indeed it is. But it has no intrinsic meaning in this capacity. One *can* define an intrinsic Riemannian metric on B, however. To do this, proceed as follows. Let $x \in B$ and $u_x, v_x \in T_xB$. We define k_B by

$$k_{B,x}(u_x, v_x) = k_q(\mathrm{hlft}_q(u_x), \mathrm{hlft}_q(v_x))$$

where $q \in \pi^{-1}(x)$. Locally we have

$$k_{B,x}(\dot{x}_1, \dot{x}_2) = \tilde{M}(x)(\dot{x}_1, \dot{x}_2) - I(x)(A(x)(\dot{x}_1), A(x)(\dot{x}_2)) \triangleq M(x)(\dot{x}_1, \dot{x}_2)$$

so defining $M(x)$. In matrix form we have

$$M(x) = \tilde{M}(x) - A^*(x) \circ I(x) \circ A(x).$$

Not at all surprisingly, if we write the local matrix of k with respect to a basis of vector fields which are horizontal and vertical then the matrix decouples to give

$$\begin{bmatrix} M(x) & 0 \\ 0 & I(x) \end{bmatrix}.$$

2.7 Example: (Example 2.1 cont'd) Let's do the easy example first. The vertical subbundle for Elroy's beanie is $VQ = \frac{\partial}{\partial\theta_1}$. The orthogonal complement to this subbundle is generated by the vector field $-I_2\frac{\partial}{\partial\theta_1} + (I_1 + I_2)\frac{\partial}{\partial\theta_2}$. The connection form is then directly computed to be

$$\alpha = \begin{bmatrix} 1 & \frac{I_2}{I_1+I_2} \end{bmatrix}.$$

One should bear in mind that this is the matrix of a $\mathfrak{so}(2)$ valued one-form on Q. Let us write this in a form where we can pick off the local connection form:

$$\alpha = \mathrm{Ad}_\theta \begin{bmatrix} 1T_e^*L_\theta & \frac{I_2}{I_1+I_2} \end{bmatrix}$$

so that the local connection form is simply $A(\theta_2) = \left[\, \frac{I_2}{I_1+I_2} \,\right]$. The curvature of this connection is zero for multiple reasons, starting with the fact that the base space is one-dimensional.

The locked inertia tensor is readily computed to have the matrix $[\, I_1+I_2 \,]$ from which we determine the local locked inertia tensor to be

$$I(\theta_2) = \mathrm{Ad}_\theta^* \mathbb{I}(\theta, \theta_2) \,\mathrm{Ad}_g = \begin{bmatrix} I_1 + I_2 \end{bmatrix}$$

which is clearly the inertia of the two bodies when locked.

The reduced metric on $B \simeq \mathbb{S}^1$ is then readily computed to give

$$k_B = \begin{bmatrix} \frac{I_2^2}{I_1+I_2} \end{bmatrix}$$

Of course, we may simply determine the local locked inertia tensor by looking at the matrix for k and picking off the top right block. The local mechanical connection form is then the inverse of the locked inertia tensor multiplied by the top left block of k. •

2.8 Example: (Example 2.2 cont'd) Okay, let's carry out these constructions for our planar rigid body. The vertical subspace is spanned by the infinitesimal generators:

$$VQ = \operatorname{span}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}).$$

A computation gives the horizontal distribution for the mechanical connection as

$$HQ = \operatorname{span}\Big\{ I_2 m_2 d \sin\theta_1 \frac{\partial}{\partial x} + I_2 m_2 d \cos\theta_1 \frac{\partial}{\partial y} - (m_1+m_2) I_2 \frac{\partial}{\partial \theta_1} + \\ -((m_1+m_2)(I_1+I_2) + d^2)\frac{\partial}{\partial \theta_2} \Big\}.$$

Using this information we may directly compute the connection form to be represented by the matrix

$$\alpha = \begin{bmatrix} 1 & 0 & y & \frac{I_2((m_1+m_2)y+m_2 d\sin\theta_1)}{\Delta} \\ 0 & 1 & -x & -\frac{I_2((m_1+m_2)x+m_2 d\cos\theta_1)}{\Delta} \\ 0 & 0 & 1 & \frac{I_2(m_1+m_2)}{\Delta} \end{bmatrix}$$

where $\Delta = (m_1+m_2)(I_1+I_2)+m_1 m_2 d^2$. In writing α in this manner, we are thinking of α as a $\mathfrak{g}$-valued one-form on Q. Thus the ath row of the matrix is a one-form giving the ath component of α in the basis $\{e_1, e_2, e_3\}$ for $\mathfrak{se}(2)$. Let's write this in a more illuminating fashion; one where we may see the local form of (2.2):

$$\operatorname{Ad}_g \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} T_e^* L_g \quad \begin{pmatrix} 0 \\ -\frac{I_2 d}{\Delta} \\ \frac{I_2(m_1+m_2)}{\Delta} \end{pmatrix} \right]$$

from which we immediately derive the local connection form to be

$$A(\theta_2) = \begin{pmatrix} 0 \\ -\frac{I_2 m_2 d}{\Delta} \\ \frac{I_2(m_1+m_2)}{\Delta} \end{pmatrix}.$$

Since the base space is one-dimensional, the curvature of this connection is automatically zero. Therefore, by Remark 2.6 we may find coordinates in which the coordinate representation of the inertia metric decouples (i.e., the local connection form is zero). Go ahead and find these coordinates, and let me know what they are...

In this same basis for $\mathfrak{se}(2)$ the locked inertia tensor is represented by the matrix

$$\begin{bmatrix} m_1+m_2 & 0 & -(m_1+m_2)y - m_2 d\sin\theta_1 \\ 0 & m_1+m_2 & (m_1+m_2)x + m_2 d\cos\theta_1 \\ -(m_1+m_2)y - m_2 d\sin\theta_1 & (m_1+m_2)x + m_2 d\cos\theta_1 & I_1 + I_2 + m_2 d^2 (m_1+m_2)(x^2+y^2) + 2m_2 dx\cos\theta_1 + 2m_2 dy\sin\theta_1 \end{bmatrix}$$

If we pull out the $SE(2)$ part of this we determine the local locked inertia tensor to be represented by the matrix

$$I(\theta_2) = \mathrm{Ad}_g^* \, \mathbb{I}(g, \theta_2) \, \mathrm{Ad}_g = \begin{bmatrix} m_1 + m_2 & 0 & 0 \\ 0 & m_1 + m_2 & m_2 d \\ 0 & m_2 d & I_1 + I_2 + m_2 d^2 \end{bmatrix}.$$

We recognise this as the inertia of the two bodies at $(x, y, \theta_1) = (0, 0, 0)$ when locked in the configuration θ_2 (although this does not depend on θ_2 in this example...)

With the local connection form and the local locked inertia tensor, we may easily compute the reduced metric on $B \simeq \mathbb{S}^1$ to have the matrix

$$\left[\tfrac{I_1 I_2 (m_1 + m_2) + I_2 m_1 m_2 d^2}{\Delta} \right].$$

Rather than computing the mechanical connection and locked inertia tensors directly as we did above, it is also possible to simply read them off by looking at the matrix representation (2.1) of k. Just set $(x, y, \theta_1) = (0, 0, 0)$ and then the top left 3×3 corner will be the local locked inertia tensor $I(\theta_2)$. The local connection form is then $I^{-1}(\theta_2)$ multiplied by the 3×1 matrix occupying the top right corner. Now that's a lot easier, isn't it? *Note that this all works in this example because the coordinates* (x, y, θ_1) *have the property that their coordinate vector fields are exactly* (e_1, e_2, e_3) *when evaluated at* $(0, 0, 0)$. •

3. The reduced Euler-Lagrange equations

In this section we investigate what happens when we are given the problem data stated in Section 2 and we try to simplify the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

We will be concerned specifically with the situation when $L(v_q) = \frac{1}{2} k_q(v_q, v_q) - V(q)$.

3.1. The Euler-Poincaré equations. Let us first examine the case when $Q = G$ and when k is a left-invariant Riemannian metric. In this case we may as well take $V = 0$. We shall denote by $\mathbb{I}$ the inner product induced by k on $\mathfrak{g}$. We let ∇ denote the Levi-Civita connection associated with k and recall that ∇ is defined by the relation

$$k(\nabla_X Y, Z) = \frac{1}{2}\big(\mathscr{L}_X (k(Y, Z)) + \mathscr{L}_Y (k(Z, X)) - \mathscr{L}_Z (k(X, Y)) + \\ k([X, Y], Z) - k([X, Z], Y) - k([Y, Z], X) \big)$$

for vector fields X, Y, and Z on G (see [Kobayashi and Nomizu 1963], for example). If we let X, Y, and Z be left-invariant extensions of $\xi, \eta, \zeta \in \mathfrak{g}$, then this gives

$$2k(\nabla_X Y, Z)(g) = k_g(T_e L_g([\xi, \eta]), T_e L_g \zeta) - \\ k_g(T_e L_g([\xi, \zeta], T_e L_g \eta) - k_g(T_e L_g([\eta, \zeta]), T_e L_g \xi).$$

This shows that ∇ is left-invariant (meaning $L_g^*(\nabla_X Y) = \nabla_X Y$ for $g \in G$ and left-invariant vector fields X and Y) and that

$$(\nabla_X Y)(g) = T_e L_g \left(\frac{1}{2}[\xi, \eta] - \frac{1}{2}(\mathrm{ad}_\xi^* \eta^\flat)^\sharp - \frac{1}{2}(\mathrm{ad}_\eta^* \xi^\flat)^\sharp\right)$$

where X and Y are the left-invariant extensions of ξ and η, respectively. Here $\flat\colon \mathfrak{g} \to \mathfrak{g}^*$ and $\sharp\colon \mathfrak{g}^* \to \mathfrak{g}$ are the musical isomorphisms associated with the inner product $\mathbb{I}$. Let us then define a product $\overset{\mathfrak{g}}{\nabla}$ on $\mathfrak{g}$ by

$$\overset{\mathfrak{g}}{\nabla}_\xi \eta = \frac{1}{2}[\xi, \eta] - \frac{1}{2}(\mathrm{ad}_\xi^* \eta^\flat)^\sharp - \frac{1}{2}(\mathrm{ad}_\eta^* \xi^\flat)^\sharp. \tag{3.1}$$

We have the following result which describes geodesics of ∇.

3.1 Proposition: *Let k be a left-invariant Riemannian metric on G with ∇ the associated Levi-Civita connection. A curve c on G is a geodesic for ∇ if and only if the curve $t \mapsto \xi(t) \triangleq T_{c(t)} L_{c(t)^{-1}}(\dot{c}(t))$ satisfies the **Euler-Poincaré** equations*

$$\dot{\xi}(t) - (\mathrm{ad}_{\xi(t)}^* \xi^\flat(t))^\sharp = 0.$$

Proof: Let $\{e_1, \dots, e_r\}$ be a basis for $\mathfrak{g}$ and write $\xi(t) = \xi^i(t) e_i$. We then have

$$\begin{aligned}
\nabla_{\dot{c}(t)} \dot{c}(t) &= \nabla_{\xi^i(t) T_e L_{c(t)} e_i} \xi^j(t) T_e L_{c(t)} e_j \\
&= \mathscr{L}_{\dot{c}(t)} \xi^j(t) T_e L_{c(t)} e_j + \xi^i(t) \xi^j(t) \nabla_{T_e L_{c(t)} e_i} T_e L_{c(t)} e_j \\
&= T_e L_{c(t)} (\dot{\xi} + \overset{\mathfrak{g}}{\nabla}_{\xi(t)} \xi(t)).
\end{aligned}$$

This completes the proof if we substitute the expression (3.1) for $\overset{\mathfrak{g}}{\nabla}$. ■

One way to view this result is as follows. If Z denotes the geodesic spray associated with ∇ (thus Z is a second-order vector field on TG), then the representation of Z under the bundle isomorphism $\lambda\colon TG \to G \times \mathfrak{g}\colon v_g \mapsto (g, T_g L_{g^{-1}}(v_g))$ has the form

$$\lambda_* Z(g, \xi) = (g, \xi, T_e L_g \xi, (\mathrm{ad}_\xi^* \xi^\flat)^\sharp).$$

3.2 Remark: It is possible to talk about the Euler-Poincaré equations for general left-invariant Lagrangians on TG. If L is such a Lagrangian let ℓ be the associated function on $\mathfrak{g}$. The Euler-Poincaré equations are then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \ell}{\partial \xi}\right) = \mathrm{ad}_\xi^* \frac{\partial \ell}{\partial \xi}.$$

Here we think of $\frac{\partial \ell}{\partial \xi}$ as being an element of $\mathfrak{g}^*$. These are then implicit equations for $\xi(t)$ and a sufficient condition for solutions to exist is for $\frac{\partial^2 \ell}{\partial \xi \partial \xi}$ to be nondegenerate at each point in $\mathfrak{g}$. •

The Euler-Poincaré equations are rather uninteresting for the locked inertia tensor corresponding to Elroy's beanie. However, our other example is more interesting.

3.3 Example: (Example 2.2 cont'd) Let us look at the case where $G = SE(2)$ and the Riemannian metric k restricts to the Lie algebra of $\mathfrak{se}(2)$ giving the inner product represented by, say,

$$\begin{bmatrix} m_1 + m_2 & 0 & 0 \\ 0 & m_1 + m_2 & m_2 d \\ 0 & m_2 d & I_1 + I_2 + m_2 d^2 \end{bmatrix}.$$

The Lie algebra structure on $\mathfrak{se}(2)$ is defined by

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = -e_1.$$

We then readily compute the associated Euler-Poincaré equations to be

$$\begin{aligned} &\dot{\xi}^1 + \xi^2\xi^3 + \frac{m_2 d}{m_1 + m_2}(\xi^3)^2 = 0 \\ &\dot{\xi}^2 - \xi^1\xi^3 = 0 \\ &\dot{\xi}^3 = 0. \end{aligned}$$

•

3.4 Example: Let's break a bit from our to now ongoing example and look at the rigid body equations on $\mathfrak{so}(3)$. We use the basis

$$e_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which satisfies the commutator relations $[e_1, e_2] = e_3$, $[e_1, e_3] = -e_2$, and $[e_2, e_3] = e_1$. Let us use the Riemannian metric on $SO(3)$ which restricts to the inner product on $\mathfrak{so}(3)$ whose matrix in the given basis is

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}.$$

The Euler-Poincaré equations are then

$$\begin{aligned} \dot{\xi}^1 &= \frac{I_2 - I_3}{I_1}\xi^2\xi^3 \\ \dot{\xi}^2 &= \frac{I_3 - I_1}{I_2}\xi^1\xi^3 \\ \dot{\xi}^3 &= \frac{I_1 - I_2}{I_3}\xi^1\xi^2. \end{aligned}$$

•

3.2. A model for the reduced space TQ/G. Since the function L is G-invariant, by general principles it drops to a function ℓ on TQ/G. Furthermore, the Euler-Lagrange vector field on TQ drops to a vector field on TQ/G. In this section we present a model for TQ/G is the case when $\pi \colon Q \to B$ comes equipped with a connection — for example, in the case of simple mechanical systems where we use the mechanical connection.

First of all, note that TQ/G is a vector bundle over Q/G whose fibre over $[q]_G$ is isomorphic to T_qQ. Indeed, if $v_q \in TQ$ and if $g_{q'} \in G$ is the unique group element with the property that $g_{q'}.q = q'$ for $q' \in [q]_G$ then

$$u_{q'} \mapsto (q', (T_{q'}\Phi_{g_{q'}})^{-1}(u_{q'}))$$

is a diffeomorphism of $[v_q]_G$ with $[q]_G \times T_qQ$. We essentially will use a connection to make a further refinement of this by making the decomposition $T_qQ = V_qQ \oplus H_qQ$ and making the observation that H_qQ is naturally isomorphic to $T_{[q]_G}(Q/G)$ and V_qQ is naturally isomorphic to $\mathfrak{g}$. So we expect to somehow be able to make a diffeomorphism from TQ/G to "$TB \times \mathfrak{g}$."

To make sense out of this, we introduce the *adjoint bundle* which is a vector bundle over B with typical fibre $\mathfrak{g}$. Let Ad be the adjoint representation of G in $\mathfrak{g}$. The adjoint bundle, which we denote by $\tilde{\mathfrak{g}}$, is defined to be the bundle associated with $\pi\colon Q \to Q/G$ via this representation. Thus

$$\tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G$$

where we take the action $(g, (q, \xi)) \mapsto (g.q, \mathrm{Ad}_g\,\xi)$ on $Q \times \mathfrak{g}$. We denote a typical point in $\tilde{\mathfrak{g}}$ by $[(q, \xi)]_G$, and note that $\chi\colon \tilde{\mathfrak{g}} \to B$ defined by $\chi([(q, \xi)]_G) = [q]_G$ makes $\tilde{\mathfrak{g}}$ a vector bundle over B.

Now, if α is *any* connection on π, we define the following vector bundle isomorphism:

$$\begin{aligned} \rho_\alpha\colon\ TQ/G &\to TB \oplus \tilde{\mathfrak{g}} \\ [v_q]_G &\mapsto T\pi(v_q) \oplus [(q, \alpha(v_q))]_G \end{aligned}$$

which has inverse

$$u_x \oplus [(q, \xi)]_G \mapsto [\mathrm{hlft}_q(u_x) + \xi_Q(q)]_G.$$

Let us see what this looks like in coordinates. Make a trivialisation of π with coordinates (x, g) as we have done above. The associated coordinates for TQ are denoted $(x, g, \dot{x}, \dot{g})$. The action of G in the trivialisation we write by $(h, (x, g, \dot{x}, \dot{g})) \mapsto (x, hg, \dot{x}, h.\dot{g})$ so we may use $(x, \dot{x}, \xi)$ to represent $[(x, g, \dot{x}, \dot{g})]_G$ where ξ is defined by $\dot{g} = T_eL_g(\xi)$. Now, with respect to this same trivialisation, coordinates for $TB \oplus \tilde{\mathfrak{g}}$ are defined as follows. A point in $TB \oplus \tilde{\mathfrak{g}}$ is written as $(x, \dot{x}) \oplus [((x, g), \xi)]_G$. We choose our chart for $TB \oplus \tilde{\mathfrak{g}}$ so that this point is represented by $(x, \dot{x}, \xi)$. Now that we have fixed our coordinatisation of TQ/G and $TB \oplus \tilde{\mathfrak{g}}$, let us write ρ_α. We have

$$\rho_\alpha(x, \dot{x}, \xi) = (x, \dot{x}, \xi + A(x)\dot{x})$$

What we are doing in this coordinate expression is writing the vector $(\dot{x}, \xi)$, which represents a vector over x in the local model of TQ/G, in terms of its horizontal and vertical parts with respect to the given connection.

3.3. The reduced equations. Now we use the above representation of $TQ/G \simeq TB \oplus \tilde{\mathfrak{g}}$ to write the reduced differential equations. Cendra, Marsden, and Ratiu [2001] use a variational derivation which is in many ways interesting. Rather than go through the entire construction, let us simply illustrate how it works in the case when $Q = G$. We prove the following result.

3.5 Proposition: (Theorem 13.8.3 in [Marsden and Ratiu 1999]) *Let L be a left-invariant Lagrangian on G with restriction ℓ to $\mathfrak{g}$. Let $g(t)$, $t \in [t_1, t_2]$, be a curve in G and define a curve $\xi(t) = T_{g(t)} L_{g^{-1}(t)}(\dot{g}(t))$. The following are equivalent:*

(i) $g(t)$ is a solution of the Euler-Lagrange equations with Lagrangian L;

(ii) $g(t)$ is an extremal for the variational principle

$$\delta \int_{t_1}^{t_2} L(g(t), \dot{g}(t))\,\mathrm{d}t$$

where the endpoints are fixed;

(iii) $\xi(t)$ satisfies the Euler-Poincaré equations;

(iv) $\xi(t)$ is an extremal for the variational principle

$$\delta \int_{t_1}^{t_2} \ell(\xi(t))\,\mathrm{d}t$$

where the variations are of the form $\delta\xi = \dot{\eta} + [\xi, \eta]$ where η vanishes at the endpoints.

Proof: We shall establish the correspondences (ii) $\Longleftrightarrow$ (iv) and (iv) $\Longleftrightarrow$ (iii).

(ii) $\Longleftrightarrow$ (iv) We give a proof for matrix Lie groups. Let $g(s,t)$ be a variation of a curve $g(t)$ and denote by "$\dot{}$" differentiation with respect to t at $(t,0)$ and by "δ" differentiation with respect to s at $(t,0)$. We need only show that a variation $\delta g(t)$ of a curve $g(t)$ gives rise to a variation $\delta\xi(t) = \dot{\eta}(t) + [\xi, \eta]$. For a variation $\delta g(t)$ of $g(t)$ define $\eta(t) = g^{-1}(t)\delta g(t)$. We then have

$$\dot{\eta}(t) = -g^{-1}(t)\dot{g}(t)g^{-1}(t)\delta g(t) + g^{-1}(t)\delta\dot{g}(t)$$

from which we compute

$$\begin{aligned}
\delta\xi(t) = &\ -g^{-1}(t)\delta g(t) g^{-1}(t)\dot{g}(t) + g^{-1}(t)\delta\dot{g}(t) \\
&= \dot{\eta} + (g^{-1}(t)\dot{g}(t))(g^{-1}(t)\delta g(t)) - (g^{-1}(t)\delta g(t))(g^{-1}(t)\dot{g}(t)) \\
&= \dot{\eta} + [\xi, \eta].
\end{aligned}$$

(iv) $\Longleftrightarrow$ (iii) So suppose that $\delta\xi = \dot{\eta} + [\xi, \eta]$ and compute

$$\begin{aligned}
\delta \int_{t_1}^{t_2} \ell(\xi(t))\,\mathrm{d}t &= \int_{t_1}^{t_2} \frac{\partial \ell}{\partial \xi}\delta\xi\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} \frac{\partial \ell}{\partial \xi}(\dot{\eta} + [\xi, \eta])\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \ell}{\partial \xi} + \mathrm{ad}_\xi^* \frac{\partial \ell}{\partial \xi}\right)\eta\,\mathrm{d}t
\end{aligned}$$

from which the result follows. ■

For the general situation one needs to resolve variations on Q into horizontal and vertical variations. We shall not go through the details as they require a few definitions which are neither here nor there with regard to what we wish to do. Rather, let us write down the equations induced on $TB\oplus\tilde{\mathfrak{g}}$ in a local trivialisation. First of all, the Lagrangian on $TB\oplus\tilde{\mathfrak{g}}$ is

$$\ell(v_x,[(q,\xi)]_G)=\frac{1}{2}k_B(v_x,v_x)-V_B(x)+\frac{1}{2}\mathbb{I}(q)(\xi,\xi)$$

where $q\in\pi^{-1}(x)$ and where V_B is the function obtained by dropping V to B. The last term does not depend on the choice of q. In coordinates

$$\ell(x,\dot{x},\xi)=\frac{1}{2}M(x)(\dot{x},\dot{x})-V_B(x)+\frac{1}{2}I(x)(\xi,\xi).$$

The Euler-Lagrange equations themselves drop to the differential equations

$$\begin{aligned}M(x)\overset{B}{\nabla}_{\dot{x}}\dot{x}+\frac{1}{2}\boldsymbol{d}I(x)(\xi,\xi)-\boldsymbol{d}V(x)&=B^*(x)(\dot{x},I^\flat(x)(\xi))-A^*(x)(\mathrm{ad}^*_\xi I^\flat(x)(\xi))\\ I^\flat(x)\dot{\xi}+\frac{1}{2}(\dot{x}\,\lrcorner\,\boldsymbol{d}I(x))^\flat(\xi)&=-\mathrm{ad}^*_{A(x)(\dot{x})}I^\flat(x)(\xi)+\mathrm{ad}^*_\xi I^\flat(x)(\xi)\end{aligned}$$

where $B^*(x)\colon T_xU\times\mathfrak{g}^*\to T^*_xU$ is defined by

$$\langle B^*(x)(\dot{x}_1,\nu);\dot{x}_2\rangle=\langle\nu;B(x)(\dot{x}_1,\dot{x}_2)\rangle.$$

Also, $\overset{B}{\nabla}$ is the Levi-Civita connection on B induced by the Riemannian metric k_B.

3.6 Remark: Note that in this local form, $\xi=0$ defines an invariant submanifold. This reflects the fact that in the decomposition $TB\oplus\tilde{\mathfrak{g}}$, T^*B is invariant under the reduced dynamics. This means that if one starts with zero "group velocity" the dynamics are just the Lagrangian dynamics of the system on the base space with Lagrangian $\frac{1}{2}k_B(v_x,v_x)-V_B(x)$. This has the following interpretation in terms of dynamics on TQ: The horizontal subbundle HQ is an invariant manifold. The solution curve $q(t)$ with initial velocity $\dot{q}(0)\in H_{q(0)}Q$ is the horizontal lift of a curve $x(t)$ on B which is a solution of the Euler-Lagrange equations with the Lagrangian $\frac{1}{2}k_B(v_x,v_x)-V_B(x)$. Furthermore, if one were to be able to actuate the system on B, then the induced dynamics on Q would be horizontal lifts of the motion on B. This is a topic of some interest in control theory. •

3.7 Example: (Example 2.1 cont'd) We can immediately write the reduced equations for Elroy's beanie:

$$\begin{aligned}\ddot{\theta}_2+\frac{I_1+I_2}{I_2^2}V'(\theta_2)&=0\\ \ddot{\theta}_1&=0.\end{aligned}$$

We have divided the reduced equations by the inertia tensor. •

3.8 Example: (Example 2.2 cont'd) If we do this for the planar rigid bodies example we have been going through, we get the reduced equations to be

$$\ddot{\theta}_2=0$$

$$\begin{gathered}\dot{\xi}^1+\xi^2\xi^3+\frac{m_2d}{m_1+m_2}(\xi^3)^2+\frac{I_2(m_1+m_2)}{\Delta}\dot{\theta}_2\xi^2+\frac{I_2m_2d}{\Delta}\dot{\theta}_2\xi^3=0\\ \dot{\xi}^2-\xi^1\xi^3-\frac{I_2(m_1+m_2)}{\Delta}\dot{\theta}_2\xi^1\\ \dot{\xi}^3=0.\end{gathered}$$

In these equations we have multiplied by the inverse of the inertia matrix for the system so the highest order derivatives decouple. •

4. The Hamiltonian version of the above

Let's take a quick look at what this all looks like in a Hamiltonian setting. Thus we deal with T^*Q rather than TQ and we consider the canonical symplectic structure ω_0 on T^*Q. This induces a Poisson structure on T^*Q defined by $\{f, g\} = \omega(X_f, X_g)$ where X_f is the Hamiltonian vector field defined by $X_f \lrcorner\, \omega_0 = \boldsymbol{d}f$. If G acts on Q then its lifted action on T^*Q is defined by

$$(g, \alpha_q) \mapsto (T_{\Phi_g(q)}\Phi_{g^{-1}})^*(\alpha_q)$$

which preserves the symplectic structure ω_0. If f and g are two smooth G-invariant functions on T^*Q then they drop to smooth functions $\tilde{f}$ and $\tilde{g}$ on T^*Q/G. Conversely, any smooth functions on T^*Q/G are obtained in this manner. Because G acts symplectically on T^*Q, $\{f, g\}$ will be G-invariant if f and g are. Thus we define a Poisson structure on T^*Q/G by $\{\tilde{f}, \tilde{g}\} = \widetilde{\{f, g\}}$. Note that this is all true as long as T^*Q/G is a manifold. In this section we investigate the case when $\pi\colon Q \to Q/G$ is a principal fibre bundle.

4.1. A model for the reduced space $\boldsymbol{T^*Q/G}$. Here we proceed along lines similar to those in Section 3.2. The coadjoint representation of G on $\mathfrak{g}^*$ we denote by Ad^* and we write the image of $g \in G$ in $GL(\mathfrak{g}^*)$ as $\mathrm{Ad}^*_{g^{-1}}$. The *coadjoint bundle* is then the associated vector bundle over B defined by $\tilde{\mathfrak{g}}^* = (Q \times \mathfrak{g}^*)/G$ with the action being $(g, (q, \mu)) \mapsto (g.q, \mathrm{Ad}^*_{g^{-1}}\,\mu)$.

Now suppose that α is a connection on π. This allows us to define a splitting $T^*Q = H^*Q \oplus V^*Q$ where $H^*Q = \operatorname{ann} VQ$ and $V^*Q = \operatorname{ann} HQ$. Note that V^*Q is connection dependent, and H^*Q is not. For $\alpha_q \in T^*_qQ$ we shall write $\alpha_q = \mathrm{hor}(\alpha_q) + \mathrm{ver}(\alpha_q)$ where $\mathrm{hor}(\alpha_q) \in H^*_qQ$ and $\mathrm{ver}(\alpha_q) \in V^*_qQ$.

For each $q \in Q$ and $x = \pi(q)$, $T^*_q\pi\colon T^*_xB \to T^*_qQ$ is an isomorphism onto its image, and its image is exactly H^*_qQ. Thusly we define $\mathrm{hlft}^*_q\colon T^*_xB \to H^*_qQ$. We also define the *momentum map* of the G-action as the map $\boldsymbol{J}\colon T^*Q \to \mathfrak{g}^*$ defined by

$$\langle \boldsymbol{J}(\alpha_q); \xi \rangle = \langle \alpha_q; \xi_Q(q) \rangle.$$

Using this relation, one may directly verify that if we regard $\boldsymbol{J}$ as a $\mathfrak{g}^*$-valued vector field (i.e., a one-form on one-forms) then its kernel is exactly H^*Q. Thus, since the action is free, $\boldsymbol{J}|V^*_qQ$ is an isomorphism onto $\mathfrak{g}^*$ for each $q \in Q$. Thus we have made a pointwise isomorphism of T^*_qQ with $T^*_xB \oplus \mathfrak{g}^*$.

Since T^*Q/G is a vector bundle over B with typical fibre over $x \in B$ given by T^*_qQ for $q \in \pi^{-1}(x)$, we expect the constructions in the preceding paragraph to yield an isomorphism from T^*Q/G to $T^*Q \oplus \tilde{\mathfrak{g}}^*$. Indeed, just such an isomorphism is

$$\begin{aligned} \sigma_\alpha\colon\ & T^*Q/G \to\ T^*B \oplus \tilde{\mathfrak{g}}^* \\ & [\alpha_q]_G \mapsto\ (\mathrm{hlft}^*_q)^{-1}(\mathrm{hor}(\alpha_q)) \oplus [(q, \boldsymbol{J}(\alpha_q))]_G. \end{aligned}$$

The inverse of σ_α is

$$\alpha_x \oplus [(q, \mu)]_G \mapsto [\mathrm{hlft}^*_q(\alpha_x) + (\boldsymbol{J}|V^*_qQ)^{-1}(\mu)]_G.$$

Let us take a look at these constructions in a local trivialisation with coordinates (x, g). Coordinates for T^*Q in this trivialisation we shall denote by (x, g, p, π) (forgive me for using π here as it also stands for the projection from Q to Q/G). The G action then looks like $(h, (x, g, p, \pi)) \mapsto (x, hg, p, h.\pi)$. Thus we use (x, p, μ) to represent the point $[(x, p, g, \pi)]_G$ where $\pi = (T_g L_{g^{-1}})^* \mu$. With respect to the same trivialisation a point in $T^*B \oplus \tilde{\mathfrak{g}}^*$ is written as $(x, p) \oplus [((x, g), \mu)]_G$ which we represent by (x, p, μ). The map σ_α is then given by

$$(x, p, \mu) \mapsto (x, p - A^*(x)\mu, \mu)$$

where A is the local connection form. One may think of this as writing the vector (p, μ) in its vertical and horizontal decomposition.

4.2. The Poisson structure on T^*Q/G in a local trivialisation. We use the coordinates (x, p, μ) as constructed above for T^*Q/G. The Poisson structure is defined by the Poisson bracket between coordinates functions. We have

$$\begin{aligned} \{x^\alpha, p_\beta\} &= \delta^\alpha_\beta, \quad \alpha, \beta = 1, \dots, n-r \\ \{\mu_a, \mu_b\} &= -C^d_{ab}\mu_d, \quad a, b = 1, \dots, r \end{aligned}$$

and all other brackets are zero. Here C^a_{bd} are the structure constants for the Lie algebra. We recognise the second term as the usual Poisson structure on $\mathfrak{g}^*$ associated with the left action.

4.3. The Poisson structure on $T^*B \oplus \mathfrak{g}^*$ induced by a connection. To compute the Poisson structure on $T^*B \oplus \tilde{\mathfrak{g}}^*$ we *ask* that σ_α be a Poisson mapping. That is, we require $\sigma^*_\alpha\{f, g\} = \{\sigma^*_\alpha f, \sigma^*_\alpha g\}$ for functions f and g on $T^*B \oplus \tilde{\mathfrak{g}}^*$. Do the computations to get

$$\begin{aligned} \{x^\alpha, p_\beta\} &= \delta^\alpha_\beta, \quad \alpha, \beta = 1, \dots, n-r \\ \{p_\alpha, p_\beta\} &= B^a_{\alpha\beta}\mu_a, \quad \alpha, \beta = 1, \dots, n-r \\ \{\mu_a, \mu_b\} &= -C^d_{ab}\mu_d, \quad a, b = 1, \dots, r \\ \{p_\alpha, \mu_a\} &= -C^d_{ab}A^b_\alpha\mu_d, \quad \alpha = 1, \dots, n-r, \;\; a = 1, \dots, r \end{aligned}$$

and the remaining brackets are zero.

4.1 Remark: Since our Poisson structure on $T^*B \oplus \tilde{\mathfrak{g}}^*$ is connection dependent, it raises the question as to why we should want to work with this bundle at all, rather than with T^*Q/G which has a natural Poisson structure. The answer is, to some extent, that it is a matter of taste. What one gains by working with the coadjoint bundle description is a global decomposition involving the cotangent bundle of the shape space. To gain this, one needs a connection. For simple mechanical systems, it makes sense to use a connection since one is given to you for free. •

4.4. The reduced equations on $T^*B \oplus \mathfrak{g}^*$. Thus far in this Hamiltonian section, our presentation has been a bit abstract. Let us return to the case when the Lagrangian is simple and the connection defining the morphism σ_α is the mechanical connection. The reduced Hamiltonian on $T^*Q \oplus \tilde{\mathfrak{g}}^*$ is given by

$$h(\alpha_x, [(q, \mu)]_G) = \frac{1}{2}k_B^{-1}(\alpha_x, \alpha_x) - V_B(x) + \frac{1}{2}\mathbb{I}^{-1}(q)(\mu, \mu).$$

In a trivialisation we write

$$h(x,p,\mu) = \frac{1}{2} M^{-1}(x)(p,p) - V_B(x) + \frac{1}{2} I^{-1}(x)(\mu,\mu).$$

Using the Poisson structure on $T^*B \oplus \tilde{\mathfrak{g}}^*$ we derive the equations of motion to be

$$\begin{aligned}\dot{x} &= M^\sharp(x)(p)\\ \dot{p} &= -\boldsymbol{d}M^{-1}(x)(p,p) - B^*(M^\sharp(x)(p),\mu) - \boldsymbol{d}V_B(x) - \boldsymbol{d}I^{-1}(x)(\mu,\mu) + A^*(x)(\mathrm{ad}^*_{I^\sharp(x)(\mu)}\,\mu)\\ \dot{\mu} &= -\mathrm{ad}^*_{A(x)(M^\sharp(x)(p))}\,\mu + \mathrm{ad}^*_{I^\sharp(x)(\mu)}\,\mu.\end{aligned}$$

References

Cendra, H., Marsden, J. E., and Ratiu, T. S. [2001]. “Lagrangian reduction by stages”. *Memoirs of the American Mathematical Society* 152(722). ISSN: 0065-9266.

Kobayashi, S. and Nomizu, K. [1963]. *Foundations of Differential Geometry*. Volume 1. Interscience Tracts in Pure and Applied Mathematics 15. Reprint: [Kobayashi and Nomizu 1996]. Interscience Publishers: New York, NY.

— [1996]. *Foundations of Differential Geometry*. Volume 1. Wiley Classics Library. Original: [Kobayashi and Nomizu 1963]. John Wiley and Sons: NewYork, NY. ISBN: 9780471157335.

Marsden, J. E. and Ratiu, T. S. [1999]. *Introduction to Mechanics and Symmetry. A Basic Exposition of Classical Mechanical Systems*. 2nd edition. Texts in Applied Mathematics 17. Springer-Verlag: New York/Heidelberg/Berlin. ISBN: 9780387986432.

Marsden, J. E. and Scheurle, J. [1993]. “Lagrangian reduction and the double spherical pendulum”. *Zeitschrift für Angewandte Mathematik und Physik. ZAMP. Journal of Applied Mathematics and Physics. Journal de Mathématiques et de Physiques Appliquées* 44(1), pages 17–43. ISSN: 0044-2275. DOI: 10.1007/BF00914351.

Montgomery, R. [1986]. “The Bundle Picture in Mechanics”. Doctoral thesis. Berkeley, CA, USA: University of California, Berkeley.