Computing Reduced Equations for Robotic Systems with Constraints and Symmetries

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Abstract—This paper develops easily computable methods for deriving the reduced equations for mechanical systems with Lie group symmetries. These types of systems occur frequently in robotics, and are found generically in robotic locomotion, wheeled mobile robots, and satellites or underwater vehicles with robotic arms. Results are presented for two important cases:

1) the unconstrained case, for both body and spatial representations; and

2) the constrained (mixed kinematic and dynamic) case.

In each case, the dynamic equations for these nonholonomic mechanical systems are given, and illustrated by the appropriate calculations for an example system. A primary result of this paper is to show that the spectrum of possible constraints—ranging from no constraints to fully constrained systems—can be expressed within a single unifying principle for calculating the reduced equations. In this process, the structure of the reduced Lagrangian directly reveals two useful components in the reduction process, namely the local forms of the locked inertia tensor and the mechanical connection. Finally, it is shown that the reduced dynamics decouple from any explicit dependence on the group configuration variables.

Index Terms—Lie group symmetries, nonholonomic reduction, robotic locomotion.

I. INTRODUCTION

AN important first step in working with the dynamics and control of complex mechanical systems, such as are found in robotics, is to simplify the governing equations. At the same time, however, one would like to gain insight into the geometric structure of the equations, since this knowledge can be useful in areas such as design, control, and motion planning. In this paper, we describe a process called *reduction* that combines both of these goals. The method described here relies on factoring out Lie group symmetries from mechanical systems in order to simplify the resulting equations.

Classical reduction utilizes Noether's theorem to isolate invariants of the dynamics, such as momenta or energy, that arise due to symmetries. Since the invariants are conserved along trajectories, they can effectively be factored out of the analysis for a given problem. More recently, the results in reduction have been extended to include external nonholonomic constraints, such as those found in wheeled vehicles or other types of robotic systems. This paper develops methods to generate explicitly the *reduced* equations of motion for mechanical systems subject to symmetries and/or nonholonomic constraints. We show that there is a surprisingly simple structure to the reduced formulation of unconstrained systems with symmetries, and that this structure can be obtained directly from the kinetic energy metric. We also derive results for modifying this reduced structure in the presence of invariant nonholonomic constraints.

These methods lead to simplified equations that are useful for studying the mechanics of a variety of robotic systems, including reorienting satellites with robotic arms, wheeled mobile robots, and numerous biologically motivated robotic locomotion systems. Underlying this wide spectrum of applications lies a common geometric framework that can be used to simplify the governing equations for these systems. Our goal here is to use recent results [1], [2] to synthesize the explicit calculations and simplified relationships needed to make practical use of this geometric framework.

The motivation for the main technical content of this paper can be understood as follows. Assume a mechanical system with configuration space, Q, and Lagrangian function, $L(q, \dot{q})$, on TQ. We also assume that this system might interact with its environment via k constraints that are linear in the velocities

$$\omega^i_j(q)\dot{q}^j = 0, \quad \text{for } i = 1 \cdots k, j = 1, \cdots, n \qquad (1)$$

with $q = (q^1, \cdots, q^n) \in Q$ (and an implied summation over repeated indices). This class of constraints includes most commonly investigated *nonholonomic* constraints. In conventional engineering mechanics, the constraints are incorporated into Lagrange's equations of motion through the use of Lagrange multipliers, λ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} + \lambda_j \omega^j_i - \tau_i = 0$$
$$\text{for } i = 1, \cdots, n; j = 1, \cdots, k \qquad (2)$$

where τ is a forcing function. This defines a set of n second order equations in $2n + k$ unknowns. In order to solve this system, we must add the k constraint equations given in (1).

However, the motivating applications discussed below have two additional characteristics that can be used to simplify the equations of motion. First, the configuration spaces of these systems can be decomposed into the structure

$$Q = G \times M \qquad (3)$$

where G is a Lie group (often used to describe position and orientation), and M is any manifold (often called the *shape*

space, as it is used to describe the internal shape of a robot). In many cases, the shape space is assumed to be controlled by suitable inputs. The calculation of the reduced equations presented here is important in justifying this assumption and designing appropriate controls. In other cases, such as the wobblestone [32] and bicycle [3], the reduced equations play a central role in determining the dynamics of the system.

Second, the Lagrangians of these systems are *invariant* with respect to the group action of G. That is, these systems exhibit a *symmetry*. In these cases, it can be shown that the equations of motion can be put in the following simplified form

$$g^{-1}\dot{g} = -\mathbb{A}(r)\dot{r} + \tilde{I}^{-1}(r)p \tag{4}$$

$$\dot{p} = \tfrac{1}{2}\dot{r}^T\sigma_{\dot{r}\dot{r}}(r)\dot{r} + p^T\sigma_{p\dot{r}}(r)\dot{r} + \tfrac{1}{2}p^T\sigma_{pp}(r)p \tag{5}$$

$$\tilde{M}\ddot{r} + \dot{r}^T\tilde{C}(r)\dot{r} + \tilde{N}(r,\dot{r},p) = T(r)\tau \tag{6}$$

Equations (4) and (5) describe the dynamics of the system along the group directions. A very important point to notice is that the group variables, g, decouple from the rest of the equations—they appear only once, in (4). This decoupling motivates the use of such a reduced form of the equations, since all subsequent calculations, for example, for the purpose of analysis or control, can be made without reference to the group (position and orientation) variables. This is a powerful idea that can be particularly useful when working with groups such as $SO(3)$ that often require difficult choices for coordinate representations.

The coordinate p in the above equations represents the generalized momentum, interpreted as the momentum along the constrained group directions. The final equation, (6), is a second order ODE that governs the motion in the shape space, M. The focus of this paper is to explore some of the geometric structure of these equations, and in particular to make explicit calculations of this final equation.

Previous work [1], [2], [4] has shown how the symmetries and nonholonomic constraints lead to the special form of these equations. In particular, the work in [1] laid the foundation for many of the results presented here. The prior work, however, did not actually consider how to generate these equations of motion in a structured or automated way. In this paper, we develop straightforward methods to compute the equations of motion in an efficient manner for this class of systems. These methods have been implemented in a Mathematica software package that is available for public use [5].

This paper is further motivated by the author's research on the mechanics and control of robotic and biological locomotion. For mechanisms that locomote one can generally divide the mechanism's configuration variables into *shape* ($r \in M$) and *position* ($g \in G$) variables. In fact, this division is quite natural. Additionally, the position of the robot will always be an element of the Lie group, $SE(3)$, or one of its subgroups, such as $SE(2)$ or $SO(3)$. Hence, the configuration space of a locomoting mechanism always has the product structure of (3). The basic concept of locomotion is to generate changes in position by using cyclic changes in shape—whether it be the wriggling of a snake to slither along the ground or the rhythmic oscillation of the cilia of a paramecium swimming through water.

A second point of commonality found in all locomotion systems studied to date is that the dynamics of the system (and in most cases the external ground constraints, as well) are independent of the inertial position of the body. Thus, the dynamics and constraints are *invariant* (unchanged) under a change of position coordinates (e.g., a Lie group motion such as rigid body or matrix group transformations). This explains our focus on systems whose Lagrangians are invariant with respect to a group action. Physically speaking, this independence implies that the net motion resulting from these internal shape changes is independent of the initial position. This is an obvious physical principal which implies, for example, that a snake wriggling in a particular manner will generate the same net motion, relative to its starting point, regardless of the initial position and orientation of the snake.

There are three primary contributions which we hope to make in this paper. First, we introduce to the robotics community a novel and useful framework within which problems of locomotion and rigid body motion can be analyzed (the reader is also referred to [6]). With this goal in mind, we provide several examples of a tutorial nature. Second, we highlight some very interesting results on the formulation of the dynamics of unconstrained systems with symmetries/invariances. These results apply to a variety of systems, including those involving free floating rigid body motion, e.g., satellites with robotic arms. They provide a quick and easy means (as compared, for example, to using Lagrange's equations) of writing down the reduced dynamics describing the overall motion of unconstrained systems, given just the terms in the mass-inertia matrix. And finally, we derive the details necessary to extend these ideas to include external nonholonomic constraints. This allows one to apply these relationships to a much larger class of systems, including, for example, many forms of locomotion that include some interaction with the terrain.

The layout of this paper is as follows. In Section II, we review some of the relevant literature, focusing on work done in reduction and in understanding robotic locomotion. Section III introduces many of the mathematical concepts and notation underlying this work. In Section IV, we derive new results on the formulation of the reduced equations based on the kinetic energy metric. This section gives details both on the generic structure of an invariant Lagrangian function, as well as the form of the mass-inertia matrix formed during the process of reduction. Finally, in Section V these results for unconstrained systems are extended to include nonholonomic constraints. We show that the reduced equations in the constrained case have a very similar structure to those found in the unconstrained case.

II. PREVIOUS WORK

A. The Theory of Reduction

The study of reduction has a long history, dating as far back as Routh (around 1860) [7], who investigated unconstrained Lagrangian systems with Abelian (e.g., translational) symmetries. In this process, one identifies cyclic variables (those variables that appear only as velocities in the Lagrangian), for

which there are associated momenta. From this, one can equate momentum conservation laws with invariances (symmetries) of the Lagrangian function. Reduction in this case consists of removing the effect of the cyclic variables from the rest of the system (since the motion in the cyclic variables is essentially determined by the conservation laws)—hence analysis of the system is reduced to a smaller dimensional space.

The reduction method for the non-Abelian case (e.g., rigid body rotations) was developed much more recently, with its beginnings in symplectic and Poisson reduction traced to Smale [8], Marsden and Weinstein [9], and Meyer [10]; and for Lagrangian reduction due to Marsden and Scheurle [11], and Marsden, Montgomery, and Ratiu [12].

The reduction process involves moving from the total configuration space to the reduced space using a map that relates tangent vectors in the reduced space to those in the full space. This map is called a *connection*, which arises due to the invariances of the Lagrangian function. These invariances allow for reduction to a lower dimensional space. Along with playing an integral role in the reduction process, the connection also encodes the necessary information to reconstruct the full system, given only the dynamics in the reduced space. Work by Smale [8], Abraham and Marsden [13], and Kummer [14] has led to an understanding of the reduction process, building what is known as the *mechanical connection*. A discussion of the use of connections is given in [15].

In this paper, we show a very interesting result—that *the mechanical connection, and hence the reduced equations, can be taken almost directly from the kinetic energy metric*. That this could be done was originally suggested by Murray [16], and later refined by Ostrowski in [2]. Bloch, Krishnaprasad, Marsden, and Murray [1] have extended the theory of reduction to show how nonholonomic constraints can be incorporated into the reduction process. In particular, they showed that all systems with symmetries (with or without constraints) can be studied within the same unifying framework. These results have led to many simplifications in the study of robotic systems, as discussed below [7], [17], [18].

B. Robotic Systems and Locomotion

As mentioned above, this geometric framework has been particularly useful for studying locomotion systems. One of the earliest relevant works is that of Shapere and Wilczek [19] who studied the movement of small organisms through a highly viscous fluid. Using this same formalism, Montgomery [20] analyzed the dynamics and (optimal) control of a falling cat and investigated the geometric properties involved when a cat, dropped from an inverted position, reorients itself in order to land on its feet. In this situation, there are no external constraint forces, only momentum conservation laws that can be used to build a connection for reducing the equations.

Kelly and Murray [21], [22] have modeled a number of locomotive systems, such as idealized inch-worms, walking insects, and low Reynolds' number swimmers, using kinematic constraints. They show how these systems can be modeled using a *principal kinematic connection*. In this case, the connection (the mapping between shape changes and position

changes) is fully provided for by the external constraints. They also provide results on controllability, and give an interpretation of movement in terms of geometric phases. Similar results for snake robots were derived by Ostrowski and Burdick [23]. Tsakiris and Krishnaprasad [27] have investigated a snake-like mechanism based on Variable Geometry Truss (VGT) modules.

Certainly, it is true that many robotic locomoting mechanisms use wheels to provide nonholonomic kinematic constraints on the robot's motion. These nonholonomic systems, however, have largely been treated as purely *kinematic* systems, i.e., their dynamics are constrained in a manner such that only *velocities* (as opposed to accelerations) need be considered. This assumption has led to some excellent progress in areas such as controllability [24], stabilization [25], and trajectory generation [26]. However, dynamic effects are *essential* to the motion of some systems. Examples mentioned above include rigid body reorientation, such as the spinning satellite [27]–[29] and the falling cat [20]. Researchers have also studied autonomous underwater vehicles [18] and the motion of underactuated and floating mechanisms [28], [30], [38]. These systems do not possess external nonholonomic constraints, but instead are governed by internal types of "constraints" in the form of angular momentum laws.

There are a number of systems, however, such as the snakeboard [31] (described below), the wobblestone [33], and the roller racer [32], where *both* kinematic constraints and symmetry (dynamic) constraints come into play. These *mixed* nonholonomic systems have rarely been treated in the literature. Of notable exception is [24], where control results were established under the restrictive assumption that the unconstrained directions are fully actuated, and [1], [2], [4].

III. BACKGROUND

A. Matrix Calculations of Lagrange's Equations

One of the central foci of this paper is to discuss the effect of the reduction process on the structure of the equations, particularly the reduced, or shape, equations. We show that the process of reduction, both for unconstrained and constrained systems, leads to a simple shift in the mass-inertia matrix for the reduced variables. We review briefly the use of matrix quantities in formulating Lagrange's equations.

Given a Lagrangian function $L = L(q, \dot{q})$, with coordinates q on a configuration manifold Q, the mass-inertia matrix can be defined as

$$M_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}.$$

Applying Lagrange's equations results in the following set of equations governing q:

$$M\ddot{q} + \dot{q}^T C \dot{q} + N(q, \dot{q}) = \tau \tag{7}$$

where M is as defined above, N includes conservative forces and any additional external forces such as friction, τ represents forces and torques applied to the joints, and C is defined by

$$C_{ijk}\dot{q}^j\dot{q}^k = \frac{1}{2}\left(\frac{\partial M_{ij}}{\partial q^k} + \frac{\partial M_{ji}}{\partial q^k} - \frac{\partial M_{jk}}{\partial q^i}\right)\dot{q}^j\dot{q}^k \tag{8}$$

and contains Coriolis and centripetal force terms. The structure of (7) is well-known [34], and will be used below as we derive reduced equations for systems with symmetries and constraints.

B. Lie Groups/Homogeneous Representations

One point of commonality among the various problems mentioned above is that the motion of the body evolves on a simple spatial manifold, such as $SE(2)$ or $SO(3)$. Each of these manifolds can be described using a homogeneous matrix representation to describe rigid body rotations and translations. Rigid body motion also has a natural geometric interpretation in terms of Lie groups, and so we would like to make use of this inherent structure in developing the results in this paper.[1]

Let G be a differentiable $(\mathcal{C}^\infty)$ manifold which is at the same time a group. For $g, h \in G$, let hg denote the product of g and h.

Definition 3.1: The manifold G is said to be a *Lie group* if the product mapping, hg: $G \times G \to G$ and the inverse mapping, g^{-1}: $G \to G$ are both $\mathcal{C}^\infty$ mappings. A Lie group possesses a unique *identity element*, e, such that $eg = ge = g$.

Example 3.2: As an example, consider $SE(2)$, the group of rotations and translations in the plane. A point $g = (x, y, \theta) \in SE(2)$ can be represented using homogeneous coordinates

$$g = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}.$$

In doing so, the product of two elements in $SE(2)$ is given simply by matrix multiplication. Thus, the element $hg \in SE(2)$, for $h = (a^1, a^2, \alpha)$, is given by $hg = (a^1 + x\cos\alpha - y\sin\alpha, a^2 + x\sin\alpha - y\cos\alpha, \theta + \alpha)$.

Note that matrix multiplication in general does not commute, and so multiplying on the right differs from multiplying on the left. Lie groups for which this is true are called *non-Abelian* and come naturally equipped with two maps, L_g: $G \to G$: $h \mapsto gh$ and R_g: $G \to G$: $h \mapsto hg$, called, respectively, *left* and *right translation* (or *action*) of G. The terms "left" and "right" apply obviously to non-Abelian matrix groups such as $SE(2)$ or $SO(3)$, as multiplication on the left and multiplication on the right, respectively. We will primarily be concerned here with left actions, since left actions naturally arise when considering invariances of mechanical systems with nonholonomic constraints.

C. Lie Algebras/Screw Representations

Just as velocities associated with rigid body motion can be represented in terms of screw motions, there is associated to each Lie group, G, an invariant velocity representation in terms of a Lie algebra, $\mathfrak{g}$.

Definition 3.3: A vector space $\mathfrak{g}$ over $\mathbb{R}$ is called a *Lie algebra* if it possesses a product, $[\cdot, \cdot]$: $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket*, which is bilinear, show-communicative, and satisfies the Jacobi identity. Beyond this formal definition, there are some important points to note about the Lie algebra associated with a Lie group, G. It can be shown that the Lie

algebra is isomorphic to the tangent space of G at the identity, i.e., $\mathfrak{g} \simeq T_e G$. For systems that are invariant with respect to the Lie group, we can represent velocities in the group directions as Lie algebra elements, by pulling the velocities back to $T_e G$. Thus, we write $\xi = T_g L_g^{-1} \dot{g}$, where $\xi \in \mathfrak{g}$ and "T" represents the tangent or derivative map of L_g (often written DL_g or $(L_g)_*$). The Lie group structure allows us to represent group velocities (i.e., vectors in $T_g G$) in terms of Lie algebra elements in a manner which is independent of the actual group element, $g \in G$ (thus, the description "invariant"). Important note: we will generally write the lifted action of $T_g L_{g^{-1}}$ on a velocity vector $\dot{g}$ as simply $g^{-1}\dot{g}$. The reader should be aware that whenever a group element is used to multiply a velocity, it is done through the lifted action. This is standard in the literature and is motivated by the fact that for homogeneous matrix representations the action L_g is linear, and so $TL_g = L_g$.

Notice, however, that there are two possible ways to pull a velocity back to the Lie algebra ($T_e G$), using the left *or* right action. In robotics this is the well known distinction which arises when choosing between body and spatial representations of *screw* velocities [34]. Thus, we will call $\xi^b = T_g L_{g^{-1}} \dot{g} = g^{-1}\dot{g}$ the *body* representation and $\xi^s = T_g R_{g^{-1}} \dot{g} = \dot{g}g^{-1}$ the *spatial* representation. The relationship between spatial and body velocities is determined by the *adjoint action* of G on $\mathfrak{g}$, given by Ad_g: $\mathfrak{g} \to \mathfrak{g}$: $\xi \mapsto T_g R_g^{-1} T_e L_g \xi$. Notice that Ad_g acts like a shift from the left action representation to a right action representation (i.e., from body coordinates to spatial coordinates) $\xi^s = \mathrm{Ad}_g \xi^b = g(g^{-1}\dot{g})g^{-1} = \dot{g}g^{-1}$. Traditionally, reduction methods have been formulated in terms of spatial coordinates, but we will see that the use of body coordinates can be quite valuable in certain examples, particularly when external nonholonomic constraints are added.

The equivalence between left-invariant vector fields on G and elements of the Lie algebra also provides a natural way to calculate the Lie bracket of two Lie algebra elements, by using the standard Lie bracket on the associated vector fields. Thus, for $\xi, \eta \in \mathfrak{g}$, let

$$[\xi, \eta] = T_g L_{g^{-1}}[T_e L_g \xi, T_e L_g \eta]. \tag{9}$$

In particular, we will make use of the coordinate version of the bracket

$$[\xi, \eta]^a = c_{bc}^a \xi^b \eta^c \tag{10}$$

where the coefficients c_{bc}^a are called the *structure constants* of the Lie algebra. The structure constants of $\mathfrak{g}$ are quite important for these calculations, as they define the basic characteristics of the Lie group. In fact, we will see that the only influence of the group variables in the reduced equations will be through the presence of the structure constants of G.

D. Fiber Bundles

In all of the problems mentioned in Section I, there is a natural division between position and shape variables. In the same way in which we can divide a snake's motion into its position and orientation on the ground and its internal configuration (shape), so can we split the configuration variables

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[1] For more details on the use of Lie groups in robotics, see [34] and [35].

of a satellite with rotors into its orientation (as an element of $SO(3)$) and the relative angles of the rotors with respect to the body (its "internal shape"). In such cases, the position variables are described by a Lie group, G, and the total configuration space is just the product of G with a shape space, denoted M, giving $Q = G \times M$. Mathematically, this product space is called a *trivial principal fiber bundle*, where the configuration space, Q, is said to be composed of fibers, G, over a base manifold (shape space), M. IMotion in the base space, M, corresponds to internal changes in shape which do not affect the overall position or orientation of the system. Likewise, motion along the fibers, G, represents translations and rotations of the system, with no internal motion. Real motion of a system is a composition of these two types of motions, and it is exactly the interaction between the internal shape changes and net movement that we seek to model using this formulation.

As a natural extension of the action of G on itself, we can trivially define the action of G on Q. To do so, define the left action of G on Q as Φ_h: $Q \to Q$: $(g, r) \mapsto (L_h g, r) = (hg, r)$.

Definition 3.4: The *lifted action* is the map $T\Phi_g$: $TQ \to TQ$: $(q, v) \mapsto (\Phi_g(q), T_q\Phi_g(v))$ for all $g \in G$ and $q \in Q$. Again, this is pointwise just the derivative map of Φ_g, often denoted $D\Phi_g$ or $(\Phi_g)_*$.

Notationally, we will often write a vector at a point $q \in Q$ as $v_q \in T_qQ$. Also, we will often use the following shorthand: gq to denote $\Phi_g q$, $h\dot{g}$ for $T_gL_h\dot{g}$, and $h\dot{q}$ for $T_q\Phi_h\dot{q}$.

When we speak of *symmetries*, we mean that certain quantities, e.g., the Lagrangian and the constraints, remain invariant under the action of the Lie group, G. From a differential geometric viewpoint, this manifests itself as invariance under the pull-back of the lifted action (see Marsden, Montgomery, and Ratiu [12] for more details).

Definition 3.5: A Lagrangian function, L, is said to be G-*invariant* if it is invariant with respect to the lifted action, $T\Phi_g$, i.e., if $L(\Phi_g q, T_q\Phi_g v_q) = L(q, v_q)$, $\forall g \in G$ and $v_q \in T_qQ$.

Throughout this paper, it will be assumed that L is G-invariant.

E. Nonholonomic Constraints

When there are no external constraints, the Lagrangian fully describes the motion of the system, and so invariance of L is all that is required. In order to incorporate constraints, we will additionally need to require that the constraints be G-invariant. In this paper, we address the effect of *Pfaffian* constraints, i.e., autonomous, linear velocity constraints motivated by our interest in no-slip wheel constraints.

Given k linear constraints, we can write them as a vector-valued set of k equations

$$\omega_j^a(q)\dot{q}^j = 0, \quad \text{for } a = 1, \cdots, k \quad (11)$$

where $\omega^1, \cdots, \omega^k$ have a natural interpretation as one-forms over Q and can be written as $\omega^a = \omega_j^a\, dq^j$. Then, we can define the *constraint distribution*, $\mathcal{D}$, as the set of vectors over Q which are annihilated by the constraints

$$\mathcal{D}_q = \{\dot{q} \in T_qQ | \langle \omega^a; \dot{q} \rangle = \omega_j^a\dot{q}^j = 0, \text{ for } a = 1, \cdots, k\}.$$

There are two equivalent ways to define G-invariance for the constraints. First, we can require that (11) be invariant with respect to the lifted action, i.e., that

$$\omega^a(\Phi_g q)T_q\Phi_g\dot{q} = \omega^a(q)\dot{q} = 0, \quad \text{for } a = 1, \cdots, k, \forall g \in G. \quad (12)$$

Alternatively, we could work directly with $\mathcal{D}$ and require that there exist a basis for $\mathcal{D}$ that is G-invariant. In other words, if $\mathcal{D} = \text{span}\{X_1, \cdots, X_{n-k}\}$, then we require that

$$T_{gq}\Phi_{g^{-1}}X_i(gq) = X_i(q), \quad \forall i = 1, \cdots, n - k. \quad (13)$$

The advantage of this perspective is based on the fact that there is an equivalence between left-invariant vector fields and Lie algebra elements. We will return to this issue in Section V.

Finally, we can write down the equations of motion for a nonholonomically constrained system. Define coordinates for TQ as $(q^i, \dot{q}^i)$, for $i = 1, \cdots, n$. Then we can write down Lagrange's equations for a mechanical system by making use of Lagrange multipliers. Namely, a valid trajectory must lie in $\mathcal{D}$ for all $q \in Q$, and satisfy

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}(q(t), \dot{q}(t))\right) - \frac{\partial L}{\partial q^i}(q(t), \dot{q}(t)) + \lambda_a\omega_i^a = \tau_i \quad (14)$$

where the λ_a's are chosen in order that the constraints be satisfied, i.e., $\langle \omega^a(q(t)); \dot{q}(t) \rangle = 0$, for $a = 1, \cdots, k$, and the τ_i's are the components of the applied forces.

One purpose of this paper is to present a method for writing down the reduced equations for a system without reference to Lagrange multipliers. In order to contrast the two methods, we will briefly present here the use of Lagrange multipliers on a simple problem of a two-wheeled mobile robot. We will return to this problem in Section V in order to recast it in a framework that makes use of the intrinsic structure present for nonholonomic mechanical systems with symmetries.

Example 3.6: Consider the two wheeled planar mobile robot presented by Kelly and Murray in [21] and shown in Fig. 1. The robot's position, $(x, y, \theta) \in SE(2)$, is measured via a frame located at the center of the wheel base. The position of each wheel is measured relative to vertical and is denoted (ϕ_1, ϕ_2). Each wheel is controlled independently and is assumed to roll without slipping. The configuration space is then $Q = G \times M = SE(2) \times (\mathbb{S} \times \mathbb{S})$. Let m denote the mass of the robot, J its inertia about the center of mass, and J_w the inertia about the wheel pivot point. Then the Lagrangian is simply the kinetic energy

$$L(q, \dot{q}) = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}J\dot{\theta}^2 + \tfrac{1}{2}J_w(\dot{\phi}_1^2 + \dot{\phi}_2^2).$$

The group action is that of $G = SE(2)$ on Q, similar to Example 3.2 above, but now extended to the trivial bundle $Q = SE(2) \times \mathbb{S} \times \mathbb{S}$

$$\Phi_g(q) = (a^1 + x\cos\alpha - y\sin\alpha, a^2 + x\sin\alpha - y\cos\alpha$$
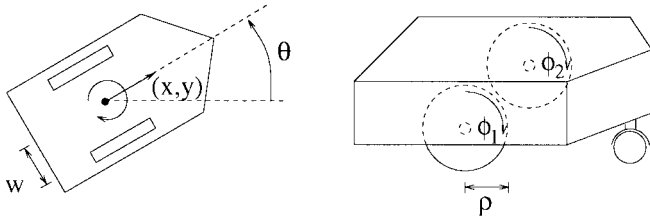$$\alpha + \theta, \phi_1, \phi_2) \quad (15)$$

Fig. 1.   Two wheeled planar mobile robot.

and the lifted action is $T_q \Phi_g(\dot{q})$ where $g = (a^1, a^2, \alpha) \in SE(2)$. A straightforward calculation shows that the Lagrangian is invariant, i.e., that $L(\Phi_g q, T_q \Phi_g \dot{q})$.

The constraints defining the no-slip condition are linear and can be written as in (11)

$$\dot{x} \cos\theta + \dot{y} \sin\theta - \frac{\rho}{2}(\dot{\phi}_1 + \dot{\phi}_2) = 0$$
$$-\dot{x} \sin\theta + \dot{y} \cos\theta = 0$$
$$\dot{\theta} - \frac{\rho}{2w}(\dot{\phi}_1 - \dot{\phi}_2) = 0 \qquad (17)$$

and similar calculations readily show that the constraints are also G-invariant. We can write the dynamical equations using Lagrange multipliers as in (14). In order to solve for the unknown multipliers, we must first differentiate (16) and then substitute in both for the acceleration terms (14) and for the variables we wish to eliminate [using (16)]. After solving for the Lagrange multipliers, we then return to (14) in order to write out explicitly the reduced base equations by substituting in for the Lagrange multipliers

$$\left(J_w + \frac{m\rho^2}{4} + \frac{J\rho^2}{4w^2}\right)\ddot{\phi}_1 + \left(\frac{m\rho^2}{4} - \frac{J\rho^2}{4w^2}\right)\ddot{\phi}_2 = \tau_1$$
$$\left(\frac{m\rho^2}{4} - \frac{J\rho^2}{4w^2}\right)\ddot{\phi}_1 + \left(J_w + \frac{m\rho^2}{4} + \frac{J\rho^2}{4w^2}\right)\ddot{\phi}_2 = \tau_2.$$

The mass matrix for the reduced base variables has transformed

$$\begin{pmatrix} J_w & 0 \\ 0 & J_w \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} J_w + \dfrac{m\rho^2}{4} + \dfrac{J\rho^2}{4w^2} & \dfrac{m\rho^2}{4} - \dfrac{J\rho^2}{4w^2} \\ \dfrac{m\rho^2}{4} - \dfrac{J\rho^2}{4w^2} & J_w + \dfrac{m\rho^2}{4} + \dfrac{J\rho^2}{4w^2} \end{pmatrix}.$$

Note that the reduced equations are independent of the group variables (x, y, θ) and that the symmetry of the reduced mass-inertia matrix is preserved. The methods presented below provide a systematic means for deriving the equations of motion with these two properties.

IV. UNCONSTRAINED MECHANICAL SYSTEMS WITH SYMMETRIES

We begin by exploring mechanical systems with symmetries, but in the absence of constraints. Doing so will help to build some intuition as to the structure of the resulting equations and the goals of this process. The use of the body representation leads us to some very interesting results regarding the kinetic energy metric.

A. Body Representation

Given a Lagrangian function as described above, $L(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} - V(g, r)$, if L is invariant with respect to the action of G, then we split the configuration variables according to the trivial fiber bundle structure. For $q = (g, r) \in G \times M = Q$ we can rewrite $L(q, \dot{q})$ as

$$L(q, \dot{q}) = L(g, r, \dot{g}, \dot{r})$$
$$= \frac{1}{2}(\dot{g}^T \quad \dot{r}^T) M(g, r) \begin{pmatrix} \dot{g} \\ \dot{r} \end{pmatrix} - V(g, r). \qquad (17)$$

The invariance of the Lagrangian allows us to work directly on the Lie algebra $\mathfrak{g}$. Let $\xi = g^{-1}\dot{g}$ and write down a *reduced* Lagrangian l as

$$l(r, \xi, \dot{r}) = L(g^{-1}g, r, g^{-1}\dot{g}, \dot{r})$$
$$= \frac{1}{2}(\xi^T \quad \dot{r}^T)\hat{M}(r)\begin{pmatrix}\xi \\ \dot{r}\end{pmatrix} - V(r).$$

We can further break down the structure of the reduced Lagrangian, as shown in the following proposition. Proofs for this proposition and those below can be found in [2].

Proposition 4.1 [2]: Given an unconstrained mechanical system with symmetries, the reduced Lagrangian can be written as

$$\boxed{l(r, \xi, \dot{r}) = \frac{1}{2}(\xi^T \quad \dot{r}^T)\begin{pmatrix} I & I\mathrm{A} \\ \mathrm{A}^T I & m(r) \end{pmatrix}\begin{pmatrix}\xi \\ \dot{r}\end{pmatrix} - V(r)}$$

(18)

where I and A are the local forms of the *locked inertia tensor* and the *mechanical connection*, respectively, and are functions of the base variables, r, only.

We have used the fact that L is invariant to pull out a great deal of the structure for the problem. The local form of the locked inertia tensor is exactly the inertia of the system with respect to the body-fixed frame when all of the joints are locked. What is very nice about the form found in (20) is that it is quite easy to write down the matrix $\hat{M}(r)$ (given the problem data, just evaluate $M(g, r)$ from (17) at the identity, $g = e$), and from this we can directly pull out expressions for $m(r)$ and $I(r)$. Further, $I(r)$ will in general be invertible, and so we can use its inverse to determine $\mathrm{A}(r)$. To the authors' knowledge, the existence of this simple, yet revealing, structure for the reduced Lagrangian has not yet been explored in the literature. A great deal of research, however, has been spent studying systems of this form, e.g., much research in space-based robotics, often going through complex manipulations to achieve this same net result.

The most important term in (18) is the local form of the connection, A (we call it "local" to distinguish it from the traditional definition of a connection one-form as a one-form on Q [36]). For the purposes of robotic locomotion systems, the primary role of the connection is to *relate internal shape changes* $(\dot{r})$ *to position changes* $(\dot{g})$.

Definition 4.2: A *connection* on a trivial principal fiber bundle $Q = G \times M$ is a Lie algebra-valued one-form on M. That is, it can be written pointwise as a map $\mathrm{A}(r): T_r M \mapsto \mathfrak{g}$.

Thus, it is clear that the connection encodes for locomotion the desired relationship of shape changes to position changes.

For those that are familiar with the connections, we can relate this definition to the more formal (and slightly more general) definitions of a connection in the following manner. To define a connection as a Lie algebra-valued one-form on Q, we simply define $\mathcal{A}(g,r) \cdot (\dot{g}, \dot{r}) = \text{Ad}_g(g^{-1}\dot{g} + \text{A}(r)\dot{r})$. Alternatively, one can define the connection as an invariant splitting of the tangent space T_qQ into horizontal and vertical vectors. In order to do so using the local connection, A, one decomposes any velocity vector $(\dot{g}, \dot{r})$ into two components: *vertical* vectors, tangent to the group (they will look like $(\dot{g} - g\text{A}(r)\dot{r}, 0)$), and *horizontal* vectors, $(g\text{A}(r)\dot{r}, \dot{r})$, that are complementary to vertical vectors. For many systems, horizontal vectors will be used to characterize allowable motions (for example, under a zero angular momentum or no-slip wheel constraint).

The fact that the connection can be written down almost directly by inspection of the reduced Lagrangian is a very useful and interesting result in terms of the reduction process. The goal here, however, is to examine the resulting reduced equations, and to look at how this process is simplified in terms of the quantities defined above. In doing so, we first define the *generalized momentum* corresponding to l as

$$p = \frac{\partial l}{\partial \xi} = I\xi + I\text{A}\dot{r}. \tag{19}$$

The generalized momentum has a natural representation as an element of $\mathfrak{g}^*$, the dual space of $\mathfrak{g}$ (if we represent an element $\eta \in \mathfrak{g}$ as a column vector, then a dual element $p \in \mathfrak{g}^*$ can be thought of as a row vector). The notation $\langle p; \eta \rangle$ denotes the *natural pairing* of $p \in \mathfrak{g}^*$ with $\eta \in \mathfrak{g}$ (which for row and column vectors is just vector multiplication).

We see that the differential equation governing g (and hence ξ) is determined by multiplying (19) by I^{-1}, yielding

$$g^{-1}\dot{g} = \xi = -\text{A}\dot{r} + I^{-1}p. \tag{20}$$

This equation should look familiar—see (4). The coordinates for translational and rotational motion of the center of mass are then expressed in terms of ξ. In this reference frame, the momentum is not constant, but is instead governed by a momentum equation. The momentum equation in body coordinates for an unconstrained system is given by

$$\dot{p} = \text{ad}_\xi^* p \quad \text{or} \quad \langle \dot{p}; \cdot \rangle = p([\xi, \cdot]). \tag{21}$$

The computation of this equation is straightforward, as can be seen by looking at the coordinate version

$$\dot{p}_c = c_{bc}^a \xi^b p_a \tag{22}$$

where recall (10) the c_{bc}^a's are the structure constants of the Lie algebra, $\mathfrak{g}$. Equations (20) and (22) describe the motion in the group direction [(20) can be used to replace ξ in (22) to achieve the same form as originally seen in (4)]. This leaves only the base equations in terms of r and $\dot{r}$. For this we assume that the forcing function is G-invariant [when thought of as a one-form—see (12)], and hence can be divided into $\tau = (\tau_a, \tau_i)$ representing fiber and base directions, respectively. For general forcing, see [2].

Proposition 4.3: The reduced equations for an unconstrained system with symmetries takes the following form:

$$g^{-1}\dot{g} = \xi = -\text{A}\dot{r} + I^{-1}p \tag{23}$$

$$\dot{p} = \text{ad}_\xi^* p + \overline{\tau} \tag{24}$$

$$\tilde{M}\ddot{r} + \dot{r}^T \tilde{C}(r)\dot{r} + \tilde{N} = T(r)\tau \tag{25}$$

where $\tilde{M}$ is the *reduced mass matrix* given by

$$\boxed{\tilde{M}(r) = m(r) - \text{A}^T(r)I(r)\text{A}(r)} \tag{26}$$

$\tilde{C}(r)$ represents reduced Coriolis and centrifugal terms

$$\dot{r}^T \tilde{C}(r)\dot{r} = \tilde{C}_{ijk}\dot{r}^j \dot{r}^k$$

$$= \frac{1}{2}\left(\frac{\partial \tilde{M}_{ij}}{\partial r^k} + \frac{\partial \tilde{M}_{ik}}{r^j} - \frac{\partial \tilde{M}_{kj}}{r^i}\right)\dot{r}^j \dot{r}^k \tag{27}$$

and the remaining terms are

$$\langle \tilde{N}; \delta r \rangle = \left\langle \frac{\partial l}{\partial \xi}; d\text{A}(\dot{r}, \delta r) + [\xi, \text{A}(\delta r)] + \frac{1}{2}\frac{\partial I^{-1}p}{\partial r}(\delta r) \right\rangle$$

$$+ \frac{\partial V}{\partial r} \tag{28}$$

$$= \left\langle p; d\text{A}(\dot{r}), \delta r \rangle - [\text{A}(\dot{r}, \text{A}(\delta r)] + [I^{-1}p, \text{A}(\delta r)] \right.$$

$$\left. + \frac{1}{2}\frac{\partial I^{-1}p}{\partial r}(\delta r) \right\rangle + \frac{\partial V}{\partial r} \tag{29}$$

and

$$(T(r)\tau)_i = \tau_i - \tau_a \text{A}_i^a \quad \text{and} \quad \overline{\tau}_a = \tau_a. \tag{30}$$

Remarks:

1) Equations (23)–(25) are simply the reduced Euler–Lagrange equations, written in a form that exposes the structure of the reduced mass matrix. These equations are really just the Euler–Poincaré equations extended to the case of a principal fiber bundle—see [1, Section V-C] for a more thorough exposition of this relationship.

2) The form of the reduced mass-inertia matrix, $\tilde{M}$, notably parallels the modified mass-inertia matrix found in closed-end linkages, for example, in problems of dual-arm grasping where the constraint of grasping can be thought of as an internal constraint on the system.[2]

3) Notice that the Coriolis terms retain the same structural relationship to the reduced mass matrix as in the standard form given in Section III-A.

4) The term $d\text{A}(\dot{r}, \delta r)$ is the exterior derivative given by

$$d\text{A}(\dot{r}, \delta r) = \left(\frac{\partial \text{A}_i}{\partial r^j} - \frac{\partial \text{A}_j}{\partial r^i}\right)\dot{r}^i \delta r^j. \tag{31}$$

5) The momentum, p, in the body frame is *not* conserved. As we will see below, momentum is conserved when viewed from a spatial frame. The equations given above, however, can be quite useful when the forces are fixed relative to a body frame (thrusters on a satellite or an eel swimming, for example).
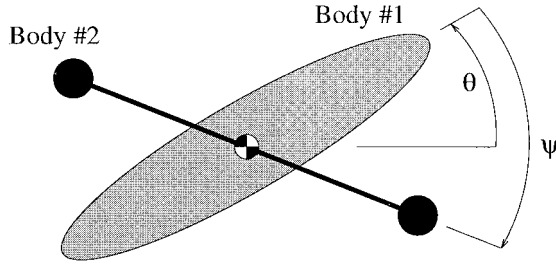
[2] Thanks to S. Agrawal for pointing out this parallel.

Fig. 2. Elroy's beanie.

6) For an Abelian group, such as that found in translational invariances, the structure constants c_{bc}^a are all zero, and we find that $\dot{p} = 0$ (a momentum conservation law—linear momentum, for example).

Example 4.4: Elroy's Beanie: To illustrate the ideas presented above, we will examine a system consisting of two planar rigid bodies attached at their centers of mass, as shown in Fig. 2. This is perhaps the simplest example of a dynamical system with non-Abelian Lie group symmetries in which the configuration space is not just the group itself. We will allow the rigid bodies to move freely in the plane, and assume the presence of control torques, $\tau = (\tau_x, \tau_y, \tau_\theta, \tau_\psi)$ (for a fixed center of mass, this problem is often referred to as Elroy's beanie [12]).

Again, let $(x, y, \theta) \in SE(2)$ be the position and orientation of the center of mass of body #1, and let $\psi \in \mathbb{S}$ be the relative angle between body #1 and body #2. Also, denote by m the total mass of the system, and J and J_ψ the inertias of body #1 and body #2, respectively. The fiber bundle is $Q = SE(2) \times \mathbb{S} = G \times M$, with base coordinates, $r = \psi$ and fiber coordinates, $g = (x, y, \theta)$ (and hence $q = (x, y, \theta, \psi)$). The Lagrangian is given by a simple kinetic energy term

$$L(q, \dot{q}) = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2} J\dot{\theta}^2 + \tfrac{1}{2} J_\psi(\dot{\theta} + \dot{\psi})^2.$$

Notice that the Lagrangian is actually independent of the configuration variables, in such cases where the Lagrangian is solely a function of velocities, the position variables $g = (x, y, \theta)$ are called *cyclic* variables (here, ψ is also a cyclic variable). In the case of unforced cyclic variables, reduction is always possible.

First, notice that as with the mobile robot above, the Lagrangian is G-invariant. We can write down the reduced Lagrangian as

$$l(r, \xi, \dot{r}) = \tfrac{1}{2} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \dot{\psi} \end{pmatrix}^T \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & J + J_\psi & J_\psi \\ 0 & 0 & J_\psi & J_\psi \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \dot{\psi} \end{pmatrix}$$

where recall that $\xi = g^{-1}\dot{g}$. The action is again $G = SE(2)$ on itself, as described in (15). Then, we can pull off immediately that the base space inertia is $m(r) = J_\psi$, and

$$I = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J + J_\psi \end{pmatrix}$$

and, using $\mathrm{A} = I^{-1}(I\mathrm{A})$

$$\mathrm{A} = \begin{pmatrix} \dfrac{1}{m} & 0 & 0 \\ 0 & \dfrac{1}{m} & 0 \\ 0 & 0 & \dfrac{1}{J + J_\psi} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ J_\psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{J_\psi}{J + J_\psi} \end{pmatrix}.$$

Then

$$\tilde{M} = m - \mathrm{A}^T I\mathrm{A} = J_\psi - \begin{pmatrix} 0 & 0 & \dfrac{J_\psi}{J + J_\psi} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ J_\psi \end{pmatrix}$$

$$= \frac{JJ_\psi}{J + J_\psi}.$$

The motion in the group variables can then be written down in local coordinates as

$$g^{-1}\dot{g} = - \begin{pmatrix} 0 \\ 0 \\ \dfrac{J_\psi}{J + J_\psi} \end{pmatrix} \dot{\psi} + \begin{pmatrix} \dfrac{1}{m} & 0 & 0 \\ 0 & \dfrac{1}{m} & 0 \\ 0 & 0 & \dfrac{1}{J + J_\psi} \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

The momenta, $p = (p_1, p_2, p_3) = \partial l/\partial \xi$ are now used as coordinates and are governed by (22). In order to compute this equation, we will need to determine the structure constants of the Lie algebra. For $\xi, \eta \in \mathfrak{g}$

$$\begin{aligned} [\xi, \eta] &= T_g L_{g^{-1}}[T_e L_g \xi, T_e L_g \eta] \\ &= \begin{pmatrix} -(\eta_2 \xi_3 - \eta_3 \xi_2) \\ \eta_1 \xi_3 - \eta_3 \xi_1 \\ 0 \end{pmatrix}. \end{aligned}$$

From this, we can write down the structure constants as

$$c_{23}^1 = c_{31}^2 = 1, c_{32}^1 = c_{13}^2 = -1$$

and all others zero.

Equation (22) then implies

$$\dot{p} = \begin{pmatrix} p_2 \xi^3 & -p_1 \xi^3 & p_1 \xi^2 - p_2 \xi^1 \end{pmatrix}$$

which can be rewritten using the connection to give

$$\dot{p} = \begin{pmatrix} \dfrac{p_2 p_3 - J_\psi \dot{\psi} p_2}{J + J_\psi} & \dfrac{-p_1 p_3 + J_\psi \dot{\psi} p_1}{J + J_\psi} & 0 \end{pmatrix}.$$

Notice that the body momentum is not conserved. In order to have this be the case, we must formulate the problem in terms of the spatial angular momentum. This is the topic of Section IV-B.

Finally, we can write down the reduced equations, which for this problem is an easy task, since $\tilde{C} = \tilde{N} = 0$. The shifted mass matrix for this problem is just

$$\hat{M} = \frac{JJ_\psi}{J + J_\psi}$$

which gives

$$\frac{JJ_\psi}{J + J_\psi} \ddot{\psi} = \tau_\psi - \frac{J_\psi}{J + J_\psi} \tau_\theta. \tag{32}$$

While this problem is quite simple, it serves to illustrate the ease with which we can calculate the local forms of the mechanical connection and locked inertia tensor. Note also the effects of the reduction process: the analysis of the problem has been taken from a system of four second order equations of the full configuration space to a single second order equation on the base space. Equation (32) contains all of the information necessary to describe the motion of the system given a particular set of initial conditions and control torques.

B. Spatial Representation

Traditional methods in reduction make use of the spatial representation for group velocities (Lie algebra elements). As noted in Section III-C, the adjoint action maps between these two representations: $\xi^s = \mathrm{Ad}_g \xi^b$. In this case, the reduced Lagrangian is no longer independent of g, but is instead given by

$$l^s(g, r, \xi^s, \dot{r}) = L(g^{-1}g, r, \mathrm{Ad}_g(g^{-1}\dot{g}), \dot{r}). \tag{33}$$

The structure of the kinetic energy metric, however, is still easily broken down into components corresponding to the connection and the locked inertia tensor, as given in the following proposition.

Proposition 4.6 [2]: Given an unconstrained mechanical system with symmetries, the reduced Lagrangian can be written as

$$\boxed{\begin{aligned} l^s(g, r, \xi^s, \dot{r}) &= \tfrac{1}{2} \left((\xi^s)^T \quad \dot{r}^T \right) \begin{pmatrix} \mathbb{I} & \mathbb{I}\mathrm{Ad}_g\mathrm{A} \\ \mathrm{A}^T\mathrm{Ad}_g^*\mathbb{I} & m(r) \end{pmatrix} \\ &\cdot \begin{pmatrix} \xi^s \\ \dot{r} \end{pmatrix} - V(r) \end{aligned}}$$

$$\tag{34}$$

where $\mathbb{I}$ is the classically defined *locked inertia tensor*, and $\mathrm{A}(r)$ is the local form of the mechanical connection as above.

Likewise, we can define the generalized momentum, $\mu = \partial l/\partial \xi^s$, which we can use to write the spatial velocities in terms of the momenta and base velocities as

$$\xi^s = \mathrm{Ad}_g \xi^b = -\mathrm{Ad}_g\mathrm{A}(r)\dot{r} + \mathbb{I}^{-1}\mu. \tag{35}$$

In this case, the momentum is indeed constant, as given by Noether's theorem [1], [37]

$$\dot{\mu} = 0. \tag{36}$$

To complete the picture, we give the base equations, which can be reduced in the same manner. Notice that we write the reduced mass matrix in terms of the *local forms* of the connection and locked inertia tensor.

Proposition 4.7: The *spatial representation* for an unconstrained system with symmetries can be written as

$$\dot{g}g^{-1} = \xi^s = -\mathrm{Ad}_g\mathrm{A}(r)\dot{r} + \mathbb{I}^{-1}\mu \tag{37}$$

$$\dot{\mu} = \mathrm{Ad}_g^*\overline{\tau} \tag{38}$$

$$\tilde{M}\ddot{r} + \dot{r}^T\tilde{C}(r)\dot{r} + \tilde{N} = T(r)\tau \tag{39}$$

where $\tilde{M}$ is the *reduced mass matrix* as given in (26), $\tilde{C}(r)$ represents reduced Coriolis and centrifugal terms as in (27),

and

$$\begin{aligned} \langle \tilde{N}; \delta r \rangle &= \Big\langle \mu; \mathrm{Ad}_{g}\mathrm{d}\mathrm{A}(\dot{r}, \delta r) + \mathrm{Ad}_g[\xi, \mathrm{A}(\delta r)] \\ &\quad + \frac{1}{2}\frac{\partial \mathbb{I}^{-1}\mu}{\partial r}(\delta r) \Big\rangle + \frac{\partial V}{\partial r}. \end{aligned} \tag{40}$$

Remark: Notice that the group variables are present throughout these equations, but that the momentum is conserved ($\dot{\mu} = 0$). This is the trade-off between body and spatial reference frames—in the body fixed frame, the equations of motion simplify greatly, allowing for the group variables to be isolated, but the momentum varies; in the spatial frame, the momentum is fixed (for no forcing in the fiber direction), but the shape (base) equations are no longer decoupled from the group variables. Each representation can be useful, depending on the context. For unconstrained systems, the spatial reference frame is generally used, since conservation of momentum implies that certain variables can be wholly eliminated. In the presence of nonholonomic constraints (with symmetries), however, the frame is best chosen to be compatible with the invariances of the constraints. For mechanical systems, this is most often the body-fixed frame.

V. NONHOLONOMIC CONSTRAINTS: THE MIXED DYNAMIC AND KINEMATIC CASE

Having worked through these results for the unconstrained (dynamic) case, we move to the problem of adding nonholonomic constraints to systems with symmetries. We call this the *mixed* (kinematic and dynamic) case, denoting the presence of both momentum-like terms, called *nonholonomic momenta*, and kinematic constraints. The interaction between these quantities is quite complex, and we strive here only to describe the tools necessary to make useful calculations for these systems. At the conclusion of this section, we will note the ways in which this general formulation reduces at its limiting cases. The mixed case result is very interesting, though, since we can now consider these two types of nonholonomic constraints under a single unified framework.

The theoretical framework for formulating the dynamics in the case of mixed constraints was developed by Bloch *et al.* [1], and further refined for locomotion systems by Burdick and Ostrowski [2], [6].

Given an invariant constraint distribution, $\mathcal{D}$, we select a basis for $\mathcal{D}$ composed of invariant vector fields, $X_1, \cdots, X_k$. We are interested, however, only in those directions tangent to the group directions, so we choose $X_1, \cdots, X_s \in \mathcal{D} \cap VQ$, where VQ is the vertical subbundle of TQ given by all vectors whose components lie only in the group direction. In other words, for $v \in VQ, v = (v_g, 0) \in TG \times TM$. We can then use this to define the *constrained fiber distribution*, $\mathcal{S} = \mathrm{span}X_1, \cdots, X_s$. Since each $X_\alpha \in \mathcal{S}$ is by definition G-invariant *and* lies only in the group direction, we can, at each point $q \in Q$, identify the vector $X_\alpha(q)$ with a Lie algebra element by pulling it back to the identity using the left action, $\Phi_{g^{-1}}$. In this way, we can associate with each $X_\alpha \in \mathcal{S}$ a Lie

algebra-valued function on M

$$f_\alpha(r) = g^{-1} X_\alpha(r, g), \qquad f_\alpha \in \mathfrak{g}.$$

From this we can define the generalized momenta corresponding to this basis as

$$p_\alpha = \left\langle \frac{\partial l}{\partial \xi}; f_\alpha \right\rangle. \tag{41}$$

The results of [1] and [2] involve realizing that (41), along with the original constraint equations, defines the correct number of constraints (they will be linear in the velocities) needed to specify a connection for this system, which we will call a *nonholonomic connection*. That is, if there are k constraints on an m dimensional Lie group, then there will be $s = m - k$ generalized momenta. The generalized momenta *plus* the external constraints will then effectively define $s + k = m$ constraints on the motion and hence define a connection for this type of system. If we define a *constrained* locked inertia tensor

$$\tilde{I}_{\alpha\beta} = I_{cd} f_\alpha^c f_\beta^d$$

then the connection can be written as

$$\xi = -\mathbb{A}(r)\dot{r} + \tilde{I}^{-1}(r)p \tag{42}$$

where we have written $p = (p_1, \cdots, p_s)$ in vector form.

A second result of [1] is the development of the governing equations for the generalized momenta, termed the *generalized momentum equations*. For the present setup the generalized momentum equations can be written succinctly as

$$\dot{p}_\alpha = \left\langle \frac{\partial l}{\partial \xi}; [\xi, f_\alpha] + \dot{f}_\alpha \right\rangle. \tag{43}$$

Notice that since all terms on the right hand side are dependent only on ξ (and not directly on g), we can use the connection defined in (42) above to write the generalized momentum equation in terms involving *only* the base and momentum variables, thereby decoupling the dynamics from the group motion.

Equations (42) and (43) define the equations of motion in unconstrained group directions. All that remains is to derive the dynamics on the reduced base space. It is possible, with slight modification, to write these equations in a form similar to those given above, using the constrained Lagrangian.

With the connection and the momentum equation, we have the same data as was used in the reduction for unconstrained systems above. Following this, we can define the *constrained Lagrangian*, l_c, to be the reduced Lagrangian with the constraints substituted in from (42)

$$l_c(r, \dot{r}, p) = l(\xi, r, \dot{r})|_{\xi = \tilde{I}^{-1}p - \mathbb{A}(r)\dot{r}}.$$

Proposition 5.1: The constrained Lagrangian can be written as

$$l_c = \tfrac{1}{2} \dot{r}^T \tilde{M} \dot{r} + \tfrac{1}{2} \langle p; \tilde{I}^{-1}p \rangle - V(r) \tag{44}$$

where

$$\boxed{\tilde{M}(r) = m - \mathrm{A}^T I \mathrm{A} + (\mathrm{A} - \mathbb{A})^T I (\mathrm{A} - \mathbb{A}).} \tag{45}$$

Having established the form of the reduced mass-inertia matrix, we return to the task of calculating the equations governing the reduced dynamics.

Proposition 5.2: The reduced equations of a system with symmetries and external constraints can be written in the form

$$\tilde{M}\ddot{r} + \dot{r}^T \tilde{C}\dot{r} + \tilde{N} = \tau$$

where $\tilde{M}$ and $\tilde{C}$ are as defined in Proposition 5.1 [and (27)], and

$$\tilde{N} = \left\langle \frac{\partial l}{\partial \xi}; d\mathbb{A}(\dot{r}, \delta r) + [\xi, \mathbb{A}(\delta r)] + \frac{\partial(\tilde{I}^{-1}p)}{\partial r}(\delta r) \right\rangle$$
$$- \tfrac{1}{2} \left\langle (\tilde{I}^{-1}p)^T I; \frac{\partial(\tilde{I}^{-1}p)}{\partial r}(\delta r) \right\rangle.$$

Remark: Notice the ways in which this formulation reduces to the limiting cases.

1) In the unconstrained case, $\mathbb{A} \equiv \mathrm{A}$, and so $\tilde{M} = m - \mathrm{A}^T I \mathrm{A}$, and the terms involving $\partial(\tilde{I}^{-1}p)/\partial r$ combine to form $\tfrac{1}{2}\langle \partial l/\partial \xi; \partial \tilde{I}^{-1}p/\partial r(\delta r)\rangle$ (since $\tilde{I} = I$).
2) For the principal kinematic case, $p = \dot{p} \equiv 0$ and

$$\tilde{N} = \left\langle \frac{\partial l}{\partial \xi}; d\mathbb{A}(\dot{r}, \delta r) + [\xi, \mathbb{A}(\delta r)] \right\rangle.$$

3) When $A \equiv 0$ (e.g., the mobile cart), $\tilde{M} = m + \mathbb{A}^T I \mathbb{A}$.

Example 5.3: Mobile robot (cont'd) We return now to reformulate the problem of the two-wheeled mobile robot given in Example 3.6. Notice that the constraint equations (16) are both G-invariant, and sufficient to define a connection (i.e., there are three independent constraints for a group of dimension 3). Thus, we can rewrite them in the form of (42)

$$\xi = - \begin{pmatrix} -\dfrac{\rho}{2} & -\dfrac{\rho}{2} \\ 0 & 0 \\ -\dfrac{\rho}{2w} & \dfrac{\rho}{2w} \end{pmatrix} \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}.$$

Similar to the example of Elroy's beanie above, we can use the reduced Lagrangian to determine the locked inertia tensor $I = diag\{m, m, J\}$, and the mechanical connection, A, which for this problem is zero.

The fact that $A \equiv 0$ greatly simplifies the problem, so that we can write down the reduced equations directly, using the
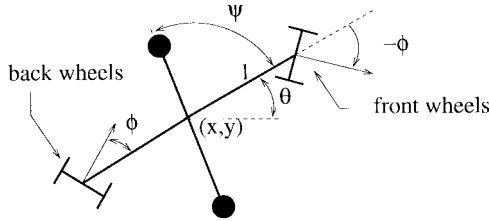
Fig. 3. The simplified model of the *Snakeboard*.

shifted mass-inertia matrix

$$\tilde{M} = m + \mathbb{A}^T I \mathbb{A}$$

$$= \begin{pmatrix} J_w & 0 \\ 0 & J_w \end{pmatrix} + \begin{pmatrix} -\dfrac{\rho}{2} & 0 & -\dfrac{\rho}{2w} \\ -\dfrac{\rho}{2} & 0 & \dfrac{\rho}{2w} \end{pmatrix} \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{pmatrix}$$

$$\cdot \begin{pmatrix} -\dfrac{\rho}{2} & -\dfrac{\rho}{2} \\ 0 & 0 \\ -\dfrac{\rho}{2w} & \dfrac{\rho}{2w} \end{pmatrix}$$

$$= \begin{pmatrix} J_w + \dfrac{m\rho^2}{4} + \dfrac{J\rho^2}{4w^2} & \dfrac{m\rho^2}{4} - \dfrac{J\rho^2}{4w^2} \\ \dfrac{m\rho^2}{4} - \dfrac{J\rho^2}{4w^2} & J_w + \dfrac{m\rho^2}{4} + \dfrac{J\rho^2}{4w^2} \end{pmatrix}$$

which yields the same reduced equations as in Example 3.6 above.

Example 5.4: The Snakeboard Next we formulate the snakeboard problem in terms of the relationships derived above. We briefly recall the description of the snakeboard as given in [31]. This example, along with other closely related examples, such as the roller racer [17], has been widely studied recently because it provides a basic, yet sufficiently complex, example of the mixed kinematic and dynamic constraint case.

The simplified model of the *Snakeboard* is shown in Fig. 3. As a mechanical system the snakeboard has a configuration manifold given by $Q = SE(2) \times \mathbb{S} \times \mathbb{S}$. The wheels are assumed to rotate through equal and opposite rotations. As coordinates for Q we shall use $(x, y, \theta, \psi, \phi)$ where (x, y, θ) describes the position of the board with respect to a reference frame, ψ is the angle of the rotor with respect to the board, and ϕ is the angle of the back wheels with respect to the board (and $-\phi$ is the angle of the front wheels with respect to the board).

Notice that the configuration space easily splits into a trivial bundle structure, with $q = (g, r)$ given by $g = (x, y, \theta) \in G = SE(2)$ and $r = (\psi, \phi) \in M = \mathbb{S} \times \mathbb{S}$. The group action is the same as that found in Example 3.6, but with the shape variables now given by (ψ, ϕ).

The parameters for this problem will be the same as for the beanie, with the addition of J_w as the inertia for each of the wheels, and l as the length from the center of mass to the wheel base.

For the snakeboard, the unconstrained Lagrangian is given simply by kinetic energy terms as

$$L = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2} J\dot{\theta}^2 + \tfrac{1}{2} J_r(\dot{\psi} + \dot{\theta})^2 + J_w(\dot{\phi}^2 + \dot{\theta}^2).$$

As above, this is easily seen to be G-invariant, and so we can write down the reduced Lagrangian as

$$l = \frac{1}{2} \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \\ \dot{\psi} \\ \dot{\phi} \end{pmatrix}^T \begin{pmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & J + J_r + 2J_w & J_r & 0 \\ 0 & 0 & J_r & J_r & 0 \\ 0 & 0 & 0 & 0 & J_w \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \\ \dot{\psi} \\ \dot{\phi} \end{pmatrix}.$$

(We will try to distinguish the reduced Lagrangian, $l(r, \dot{r}, \xi)$, from the board length, l, by including the dependencies of the Lagrangian whenever there might be confusion.)

At this point we make an additional simplifying assumption. Let $J + J_r + 2J_w = ml^2$, which roughly corresponds to having a rotor with mass concentrated at a distance l from the center of mass.[3] Then, we use $l(r, \xi, \dot{r})$ to write down the mass matrix and locked inertia tensor as

$$m = \begin{pmatrix} J_r & 0 \\ 0 & J_w \end{pmatrix} \text{ and } I = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & ml^2 \end{pmatrix}.$$

From this, we can determine the local form of the mechanical connection via

$$A = \begin{pmatrix} \dfrac{1}{m} & 0 & 0 \\ 0 & \dfrac{1}{m} & 0 \\ 0 & 0 & \dfrac{1}{ml^2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ J_r & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{J_r}{ml^2} & 0 \end{pmatrix}.$$

The wheels of the snakeboard are assumed to roll without lateral sliding. At the back wheels, the nonholonomic constraint assumes the form

$$-\sin(\theta + \phi)\dot{x} + \cos(\theta + \phi)\dot{y} - l\cos\phi\dot{\theta} = 0. \quad (46)$$

Similarly at the front wheels the constraint appears as

$$-\sin(\theta - \phi)\dot{x} + \cos(\theta - \phi)\dot{y} + l\cos\phi\dot{\theta} = 0. \quad (47)$$

It can also be shown that the constraints are G-invariant. Notice, however, that they do not by themselves define a connection. The third constraint needed to build a connection is provided by the generalized momentum itself, since the motion of the system is required to flow along the unconstrained fiber directions in a manner consistent with this momentum. The first step in synthesizing the connection in this manner is to choose a G-invariant basis for S, which will be used to define the generalized momentum (which for this problem is one-dimensional). Let

$$X_1(g, r) = -2l\cos^2\phi\cos\theta \frac{\partial}{\partial x} - 2l\cos^2\phi\sin\theta \frac{\partial}{\partial y}$$
$$+ \sin(2\phi) \frac{\partial}{\partial \theta}.$$

Then, we define

$$f_1(r) = g^{-1}X_1(g, r) = \begin{pmatrix} -2l\cos^2\phi \\ 0 \\ \sin(2\phi) \end{pmatrix} \quad (48)$$

---

[3] Thanks to A. Ruina for making this observation.

and

$$
p = p_1 = \left\langle \frac{\partial l}{\partial \xi}; f_1 \right\rangle
$$
$$
= (m\xi^1 \quad m\xi^2 \quad ml^2\xi^3 + J_r\dot{\psi}) \begin{pmatrix} -2l\cos^2\phi \\ 0 \\ \sin(2\phi) \end{pmatrix}
$$
$$
= -2ml\xi^1\cos^2\phi + (ml^2\xi^3 + J_r\dot{\psi})\sin(2\phi). \tag{49}
$$

We can use this last equation as a third constraint in order to synthesize a connection for the snakeboard problem. We do this by first rewriting (46) and (47) in a reduced form

$$
-\xi^1\sin\phi + \xi^2\cos\phi - l\xi^3\cos\phi = 0
$$
$$
\xi^1\sin\phi + \xi^2\cos\phi + l\xi^3\cos\phi = 0.
$$

These two equations, along with (49), allow us to solve for ξ as

$$
\xi = -\begin{pmatrix} -\dfrac{J_r}{2ml}\sin 2\phi & 0 \\ 0 & 0 \\ \dfrac{J_r}{ml^2}\sin^2\phi & 0 \end{pmatrix}\begin{pmatrix} \dot{\psi} \\ \dot{\phi} \end{pmatrix} + \begin{pmatrix} -\dfrac{1}{2ml} \\ 0 \\ \dfrac{1}{2ml^2}\tan\phi \end{pmatrix}\phi
$$
$$
= -\mathbb{A}\begin{pmatrix} \dot{\psi} \\ \dot{\phi} \end{pmatrix} + \tilde{I}^{-1}p. \tag{50}
$$

Note that this choice of momentum roughly corresponds to choosing the momentum of the snakeboard along the constrained fiber distribution, or instantaneously around the center of rotation defined by the wheel constraints, though the momentum chosen differs from the actual instantaneous angular momentum by a scaling factor of $\sin 2\phi$, due to our choice of basis for $\mathcal{S}$ [f_1 in (48)].

In order to fully specify the motion along the fiber, we need to add an equation governing the momentum, given by the generalized momentum equation

$$
\dot{p} = \left\langle \frac{\partial l}{\partial \xi}; [\xi, f_1] + \dot{f}_1 \right\rangle
$$
$$
= (m\xi^1 \quad m\xi^2 \quad ml^2\xi^3 + J_r\dot{\psi}) \begin{pmatrix} 2l\sin 2\phi\dot{\phi} \\ 0 \\ 2\cos 2\phi\dot{\phi} \end{pmatrix}
$$
$$
= 2J_r\cos^2\phi\dot{\phi}\dot{\psi} - \tan\phi\dot{\phi}p.
$$

Finally, we calculate the reduced equations. For reasons derived in [2], the reduced mass matrix simplifies even further to

$$
\tilde{M} = m - \mathbb{A}^T I \mathbb{A}
$$
$$
= \begin{pmatrix} J_r - \dfrac{J_r^2}{ml^2}\sin^2\phi & 0 \\ 0 & J_w \end{pmatrix}.
$$

Computing $\tilde{N}$ is more involved, but reduces to

$$
\langle \tilde{N}; \delta r \rangle
$$
$$
= (m\xi^1 \quad m\xi^2 \quad ml^2\xi^3 + J_r\dot{\psi})
$$
$$
\cdot \begin{pmatrix} -\dfrac{J_r}{ml}\cos 2\phi(\dot{\phi}\delta\psi - \delta\phi\dot{\psi}) \\ 0 \\ \dfrac{J_r}{ml^2}\sin 2\phi(\dot{\phi}\delta\psi - \delta\phi\dot{\psi}) + \dfrac{p}{2ml^2}\sec^2\phi\delta\phi \end{pmatrix}
$$
$$
= \delta\psi\left(\frac{J_r^2}{2ml^2}\sin 2\phi\dot{\psi}\dot{\phi} + \frac{J_r}{2ml^2}\dot{\phi}p \right)
$$
$$
+ \delta\phi\left(-\frac{J_r^2}{2ml^2}\sin 2\phi\dot{\psi}^2 + \frac{1}{4ml^2}\tan\phi\sec^2\phi p^2 \right).
$$

With this, we have

$$
R = -\frac{1}{2}\frac{\partial}{\partial r}((\tilde{I}^{-1}p)^T I \tilde{I}^{-1}p) = -\frac{p^2}{4ml^2}\tan\phi\sec^2\phi
$$

and

$$
\tilde{C} = \begin{pmatrix} -\dfrac{J_r^2}{ml^2}\sin 2\phi\dot{\psi}\dot{\phi} \\ \dfrac{J_r^2}{2ml^2}\sin 2\phi\dot{\psi}^2 \end{pmatrix}.
$$

Putting all of this together gives the reduced equations, which simplify to

$$
\left(J_r - \frac{J_r^2}{ml^2}\sin^2\phi \right)\ddot{\psi} - \frac{J_r^2}{2ml^2}\sin 2\phi\dot{\psi}\dot{\phi} + \frac{J_r}{2ml^2}\dot{\phi}p = \tau_\psi
$$
$$
2J_w\ddot{\phi} = \tau_\phi.
$$

Alternatively, the dynamic equations governing the base variables for this problem can be solved for directly, by substituting the constraint equations given by the connection into the full set of dynamical equations. There are many situations, however, in which it is much easier to work with the variables on the reduced space. One example of this is the ball rolling on a rotating plate, in which reduction allows one to bypass finding an amenable parameterization for $SO(3)$, and instead work with just the Lie algebra, $so(3)$, and the reduced space of the plane ($\mathbb{R}^2$).

VI. CONCLUSION

In this paper, we have presented easily computable methods for deriving the reduced equations for mechanical systems with Lie group symmetries. The material has been presented so as to highlight the results in two important cases: the unconstrained case, for both body and spatial representations, and the constrained (mixed kinematic and dynamic) case. A primary result of this paper (given by Proposition 5.2 and the accompanying equations) demonstrates that the spectrum of possible constraints—stretching from no constraints to principal kinematic constraints—fits quite nicely within a single unifying principle for calculating the reduced equations. Also, it was shown that for each of these systems, but most importantly for the unconstrained systems, the structure of the reduced Lagrangian almost transparently reveals the local forms of the locked inertia tensor and mechanical connection. This result, in itself, allows one to skip a large number of steps required in the reduction process.

The calculation of the reduced equations for nonholonomic mechanical systems with symmetries and the utility of working within this reduced framework is very often hidden by the mathematical framework in which it is derived. This paper has attempted to recast these quantities in terms that are familiar to engineers and roboticists, but which have been greatly simplified using the inherent power of the differential geometric formalism. The process of reduction can be used to drastically reduce the number of variables needed to do computations, such as finding optimal control inputs or in studying reorientation of rigid bodies with appendages. It is hoped that the results derived herein will help to simplify the calculations and aid in the continued study of robotic locomotion.

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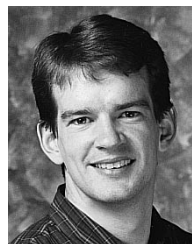
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