LIE BRACKETS AND THE MAXIMUM PRINCIPLE

Héctor J. Sussmann — December 6, 2000

CONTENTS

1. COORDINATE INVARIANCE, LIE BRACKETS, AND THE MAXIMUM PRINCIPLE

In this note we show how the concept of a **Lie bracket** plays an essential role in the coordinate-free formulation of the Maximum Principle. We will explain how the important discovery by Lagrange of the **invariance of the Euler-Lagrange system under arbitrary coordinate changes** has an optimal control analgue, and why this analogue must involve Lie brackets.

1.1. Covariance, invariance, and coordinate democracy. While Euler's work was full of "geometric" arguments, and contained many pictures, Lagrange used no pictures. (Newton's 1687 *Principia* had 250 figures. Lagrange proudly stated in the preface of his 1788 *Mécanique Analytique* that "no diagrams will be found in this work.")

Lagrange was very proud of having freed mechanics from geometry by doing everything analytically. Little did he know that one of his discoveries anticipated Riemann's revolutionary creation of differential geometry.

Lagrange allowed the configuration of a collection of N particles to be represented in terms of completely general systems of $3N$ coordinates. This was a decisive step in the process that would free geometry from the constraint of having to deal with 3-dimensional Euclidean space. "To appreciate how all was new in this work [the *Mécanique Analytique*], let us point out that, when Lagrange described a motion in terms of a system of Cartesian coordinates in space, this was such a non-trivial step that he devoted two pages to trying to persuade the reader" (Lochak, [**6**], p. 49).

He then showed that the Euler-Lagrange system has the same form in "curvilinear" or "generalized" coordinates, so that the system is invariant under *arbitrary*, nonlinear coordinate changes. This was the first example of what would later become the cornerstone of differential geometry, namely, the principle of "coordinate democracy," according to which all coordinate systems, "rectangular" as well as "curvilinear," have to be treated equally.

Lagrange saw the significance of this invariance property:

> It is perhaps one of the principal advantages of our method that it expresses the equations of every problem in the most simple form relative to each set of variables and that it enables us to see beforehand which variables one should use in order to facilitate the integration as much as possible. [Quoted from [**9**], p. 33.]

REMARK 1.1.1. For a striking illustration of what Lagrange had in mind here, the reader should think of how much simpler it is to derive the equations of motion of a body under an inverse-square attracting force if one first proves that the motion must take place in a plane, and then does the whole calculation in polar coordinates, using the invariance of the Euler-Lagrange equations. $\diamondsuit$

In 1854, sixty-six years after the publication of Lagrange's treatise, Riemann gave his famous lecture, sketching the brilliant ideas that would become the basis of modern differential geometry. It turns out that one of these ideas was precisely that of *invariance under general coordinate changes*, whose importance Lagrange had presciently anticipated.

1.2. Differentiable manifolds.

DEFINITION 1.2.1. Let Q be a set and let k be a nonnegative integer. An *atlas of class C^k on Q* is a set $\mathcal{A}$ of coordinate charts on Q such that:

1. Any two charts $\mathbf{x}$, $\mathbf{X} \in \mathcal{A}$ are C^k-related.
2. $\bigcup \left\{ D^{\mathbf{x}} : \mathbf{x} \in \mathcal{A} \right\} = Q$.

If n is a nonnegative integer such that all the charts belonging to $\mathcal{A}$ are n-dimensional then $\mathcal{A}$ is said to be an *n-dimensional atlas* on Q. $\diamondsuit$

DEFINITION 1.2.2. Let k be a nonnegative integer. A *differentiable manifold of class C^k* is a pair $Q = (Q_0, \mathcal{A})$ such that Q_0 is a set and $\mathcal{A}$ is an atlas of class C^k on Q_0.

If $Q = (Q_0, \mathcal{A})$ is a differentiable manifold of class C^k, we will refer to $\mathcal{A}$ as the *structural atlas*, or the *differentiable structure* of Q. The set Q_0 is the *set of points of Q*. Often we will use the same notation Q to refer to the set Q_0, so we shall talk about "a point $q \in Q$" rather than "a point $q \in Q_0$." $\diamondsuit$

If k is a nonnegative integer, $Q = (Q_0, \mathcal{A})$ is a differentiable manifold of class C^k, and $q \in Q$, we use $\mathcal{A}(q)$ to denote the set of all the charts $\mathbf{x} \in \mathcal{A}$ such that $q \in D^{\mathbf{x}}$. It is easy to show that all the charts $\mathbf{x} \in \mathcal{A}(q)$ have the dimension. (This follows from a simple rank argument if $k > 0$, and from the theorem on invariance of domains if $k = 0$.) This common dimension of all the charts $\mathbf{x} \in \mathcal{A}(q)$ is called the *dimension of Q at q*.

A manifold Q such that the dimension of Q at q is the same for all $q \in Q$ is said to be *of pure dimension.* In that case the integer n such that the dimension of Q at q equals n for all $q \in Q$ is called *the dimension of* Q, and Q is said to be n*-dimensional.*

REMARK 1.2.3. If $Q = (Q_0, \mathcal{A})$ is a differentiable manifold, then one can define a topology on Q by declaring a subset S of Q to be *open* if $\mathbf{x}(S \cap D^{\mathbf{x}})$ is open in $R^{\mathbf{x}}$ for every chart $\mathbf{x} \in \mathcal{A}$.

One can then show easily that *every connected component of a manifold is a manifold of pure dimension.* $\diamondsuit$

DEFINITION 1.2.4. Let k be a strictly positive integer, and let $Q = (Q_0, \mathcal{A})$ be a differentiable manifold of class C^k. Let $q \in Q$ and let n be the dimension of Q at q. A *tangent vector for* Q *at* q is a map $\mathcal{A}(q) \ni \mathbf{x} \mapsto v^{\mathbf{x}} \in \mathbb{R}^n$ that assigns to each chart $\mathbf{x} \in \mathcal{A}(q)$ a column vector

$$v^{\mathbf{x}} = \begin{bmatrix} v^{\mathbf{x},1} \\ v^{\mathbf{x},2} \\ \vdots \\ v^{\mathbf{x},n} \end{bmatrix} \in \mathbb{R}^n ,$$

in such a way that, if $\mathbf{x}, \mathbf{X}$ are any two charts in $\mathcal{A}(q)$, then

$$v^{\mathbf{X}} = \frac{\partial \Phi^{\mathbf{x},\mathbf{X}}}{\partial x}(x) \cdot v^{\mathbf{x}} \tag{1.1}$$

or, equivalently,

$$v^{\mathbf{X},i} = \sum_{j=1}^{n} \frac{\partial \Phi^{\mathbf{x},\mathbf{X},i}}{\partial x^j}(x) v^{\mathbf{x},j} \ \text{ for } \ i = 1, \ldots, n , \tag{1.2}$$

where $x = \mathbf{x}(q)$ and

$$\Phi^{\mathbf{x},\mathbf{X}} = \begin{bmatrix} \Phi^{\mathbf{x},\mathbf{X},1} \\ \Phi^{\mathbf{x},\mathbf{X},2} \\ \vdots \\ \Phi^{\mathbf{x},\mathbf{X},n} \end{bmatrix} \in \mathbb{R}^n$$

is the change of coordinates map from $\mathbf{x}$ to $\mathbf{X}$. $\diamondsuit$

Equations (1.1), (1.2) are the *transformation formulas for vectors.*

We use T_qQ to denote the set of all tangent vectors for Q at q, and refer to T_qQ as the *tangent space of* Q *at* q.

To define a tangent vector $v \in T_qQ$, it suffices to specify its component representation $v^{\mathbf{x}}$ relative to some coordinate chart $\mathbf{x}$ belonging to the structural atlas of Q, because once this is done the representation $v^{\mathbf{X}}$ relative to any other chart $\mathbf{X}$ is determined by (1.1).

DEFINITION 1.2.5. Let k be a strictly positive integer, and let $Q = (Q_0, \mathcal{A})$ be a differentiable manifold of class C^k. Let $q \in Q$ and let n be the dimension of Q at q. A *cotangent vector*—or *covector*—*for* Q *at* q is a map $\mathcal{A}(q) \ni \mathbf{x} \mapsto w^{\mathbf{x}} \in \mathbb{R}^n$ that assigns to each chart $\mathbf{x} \in \mathcal{A}(q)$ a row vector

$$w^{\mathbf{x}} = \begin{bmatrix} w^{\mathbf{x}}_1 \\ w^{\mathbf{x}}_2 \\ \vdots \\ w^{\mathbf{x}}_n \end{bmatrix} \in \mathbb{R}_n ,$$

in such a way that, if $\mathbf{x}, \mathbf{X}$ are any two charts in $\mathcal{A}(q)$, then

$$w^{\mathbf{x}} = w^{\mathbf{X}} \cdot \frac{\partial \Phi^{\mathbf{x},\mathbf{X}}}{\partial x}(x) \tag{1.3}$$

or, equivalently,

$$w^{\mathbf{x}}_j = \sum_{i=1}^{n} w^{\mathbf{X}}_i \frac{\partial \Phi^{\mathbf{x},\mathbf{X},i}}{\partial x^j}(x) \ \text{ for } \ j = 1, \ldots, n\,, \tag{1.4}$$

where $x = \mathbf{x}(q)$ and

$$\Phi^{\mathbf{x},\mathbf{X}} = \begin{bmatrix} \Phi^{\mathbf{x},\mathbf{X},1} \\ \Phi^{\mathbf{x},\mathbf{X},2} \\ \vdots \\ \Phi^{\mathbf{x},\mathbf{X},n} \end{bmatrix} \in \mathbb{R}^n$$

is the change of coordinates map from $\mathbf{x}$ to $\mathbf{X}$. $\diamondsuit$

Equations (1.3), (1.4), are the *transformation formulas for covectors.*

We use $T^*_q Q$ to denote the set of all covectors of Q at q, and refer to $T^*_q Q$ as the *cotangent space of Q at q*.

To define a tangent vector $w \in T^*_q Q$, it suffices to specify its component representation $w^{\mathbf{x}}$ relative to some coordinate chart $\mathbf{x}$ belonging to the structural atlas of Q, because once this is done the representation $w^{\mathbf{X}}$ relative to any other chart $\mathbf{X}$ is determined by (1.3).

Proposition 1.2.6. *Let k be a strictly positive integer, and let $Q = (Q_0, \mathcal{A})$ be a differentiable manifold of class C^k. Let $q \in Q$, and let $v \in T_q Q$, $w \in T^*_q Q$. Define the inner product*

$$w \cdot v \stackrel{\text{def}}{=} w^{\mathbf{x}} \cdot v^{\mathbf{x}}\,,$$

where $\mathbf{x}$ is any chart belonging to $\mathcal{A}(q)$. Then $w \cdot v$ does not depend on the choice of $\mathbf{x}$.

Proof. Let $\mathbf{x}$, $\mathbf{X}$ be two charts belonging to $\mathcal{A}(q)$. Then

$$\begin{aligned} w^{\mathbf{X}} \cdot v^{\mathbf{X}} &= \left(w^{\mathbf{x}} \cdot \Big(\frac{\partial \Phi^{\mathbf{x},\mathbf{X}}}{\partial x}(x)\Big)^{-1} \right) \cdot \left(\frac{\partial \Phi^{\mathbf{x},\mathbf{X}}}{\partial x}(x) v^{\mathbf{x}} \right) \\ &= w^{\mathbf{x}} \cdot v^{\mathbf{x}}\,, \end{aligned}$$

using (1.1), (1.3), and the fact that the square matrix $\frac{\partial \Phi^{\mathbf{x},\mathbf{X}}}{\partial x}(x)$ is invertible. (The invertibility of $\frac{\partial \Phi^{\mathbf{x},\mathbf{X}}}{\partial x}(x)$ follows from (1.1), because (1.1) implies that

$$\begin{aligned} v^{\mathbf{X}} &= \frac{\partial \Phi^{\mathbf{x},\mathbf{X}}}{\partial x}(x) v^{\mathbf{x}} v^{\mathbf{x}} \\ &= \frac{\partial \Phi^{\mathbf{x},\mathbf{X}}}{\partial x}(x) \frac{\partial \Phi^{\mathbf{X},\mathbf{x}}}{\partial x}(x) v^{\mathbf{X}} \end{aligned}$$

for every $v \in T_q Q$, so

$$v = \frac{\partial \Phi^{\mathbf{x},\mathbf{X}}}{\partial x}(x) \frac{\partial \Phi^{\mathbf{X},\mathbf{x}}}{\partial x}(x) v$$

for every $v \in \mathbb{R}^n$. This implies that

$$\frac{\partial \Phi^{\mathbf{x},\mathbf{X}}}{\partial x}(x) \frac{\partial \Phi^{\mathbf{X},\mathbf{x}}}{\partial x}(x) = \text{identity matrix}\,,$$

and the conclusion follows.) □

It follows from Proposition 1.2.6 that *the cotangent space T_q^*Q can be regarded in a natural way as the dual of the tangent space T_qQ.*

1.3. Lie brackets. What do we need to make the Maximum Principle completely coordinate-free? To answer this question we recall that a control system is, basically, a family of vector fields, so it is natural to ask oneself what are the "natural" coordinate-free operations among vector fields.

An obvious example of such an operation is **addition**. It turns out that, if f and g are two vector fields, then there is another natural covariant object derived from them, namely, the **Lie bracket** $[f, g]$. This is defined in coordinates by the formula

$$[f,g](x) = \frac{\partial g}{\partial x}(x)\cdot f(x) - \frac{\partial f}{\partial x}(x)\cdot g(x)\,. \tag{1.5}$$

Here we are writing f and g as columns of functions, that is,

$$f(x) = \begin{bmatrix} f^1(x) \\ f^2(x) \\ \vdots \\ f^n(x) \end{bmatrix}, \qquad g(x) = \begin{bmatrix} g^1(x) \\ g^2(x) \\ \vdots \\ g^n(x) \end{bmatrix}. \tag{1.6}$$

In addition, if h is a vector field, written as a column of functions as above, then $\frac{\partial h}{\partial x}$ is the Jacobian matrix of h, which is a square matrix of functions:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \frac{\partial f^2}{\partial x^1} & \cdots & \frac{\partial f^2}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^n}{\partial x^1} & \cdots & \frac{\partial f^n}{\partial x^n} \end{bmatrix}. \tag{1.7}$$

It follows that, if f, g are vector fields, and we let $h = [f, g]$, then the components h^i of h are given by

$$h^i = \sum_{j=1}^{n} \frac{\partial g^i}{\partial x^j} f^j - \sum_{j=1}^{n} \frac{\partial f^i}{\partial x^j} g^j\,.$$

Remark 1.3.1. *If f, g are vector fields, then $[f, g]$ is a vector field.* The precise meaning of this is that "$[f, g]$ transforms like a vector field under a change of coordinates." To prove this, let us recall the way vector fields transform. Suppose h is a vector field, given with respect to a coordinate chart $\mathbf{x} : D^{\mathbf{x}} \mapsto \mathbb{R}^n$ by functions $h^{\mathbf{x},1}, \ldots, h^{\mathbf{x},n}$. Suppose $\mathbf{X}$ is another chart, and h is given in this new chart by functions $h^{\mathbf{X},1}, \ldots, h^{\mathbf{X},n}$. Suppose

$$\Phi^{\mathbf{x},\mathbf{X}} : \mathbf{x}(D^{\mathbf{x}} \cap D^{\mathbf{X}}) \mapsto \mathbf{X}(D^{\mathbf{x}} \cap D^{\mathbf{X}})$$

and

$$\Phi^{\mathbf{X},\mathbf{x}} : \mathbf{X}(D^{\mathbf{x}} \cap D^{\mathbf{X}}) \mapsto \mathbf{x}(D^{\mathbf{x}} \cap D^{\mathbf{X}})$$

be the coordinate change maps.

The fact that h is a vector field says that, at each point $q \in D^{\mathbf{x}} \cap D^{\mathbf{X}}$, if we take a curve ξ such that $\xi(0) = q$, whose tangent vector at time 0 is $h(q)$, and let $\xi^{\mathbf{x}}$, $\xi^{\mathbf{X}}$, be the coordinate representations, so that

$$\xi^{\mathbf{x}}(t) = \mathbf{x}(\xi(t)) \quad \text{and} \quad \xi^{\mathbf{X}}(t) = \mathbf{X}(\xi(t))\,,$$

then, if we let $x = \mathbf{x}(q)$, $X = \mathbf{X}(q)$, we have

$$\begin{aligned} h^{\mathbf{X}}(X) &= \frac{d}{dt}\Big|_{t=0}\Big(\xi^{\mathbf{X}}(t)\Big) \\ &= \frac{d}{dt}\Big|_{t=0}\Big(\mathbf{X}(\xi(t))\Big) \\ &= \frac{d}{dt}\Big|_{t=0}\Big(\Phi^{\mathbf{x},\mathbf{X}}\Big(\mathbf{x}(\xi(t))\Big)\Big) \\ &= \frac{d}{dt}\Big|_{t=0}\Big(\Phi^{\mathbf{x},\mathbf{X}}\Big(\xi^{\mathbf{x}}(t)\Big)\Big) \\ &= \frac{\partial \Phi^{\mathbf{x},\mathbf{X}}}{\partial x}(x)\cdot h^{\mathbf{x}}(x)\,, \end{aligned}$$

so that the components $h^{\mathbf{X},i}$, $h^{\mathbf{x},j}$ are related by

$$\begin{aligned} h^{\mathbf{X},i}(X) &= \sum_{j=1}^{n} \frac{\partial \Phi^i}{\partial x^j}(x) h^{\mathbf{x},j}(x) \\ &= \sum_{j=1}^{n} \frac{\partial \Phi^i}{\partial x^j}(\Psi(X)) h^{\mathbf{x},j}(\Psi(X)) \end{aligned}$$

for $i = 1, \ldots, n$, if we write

$$\Phi = \Phi^{\mathbf{x},\mathbf{X}}\,, \quad \Psi = \Phi^{\mathbf{X},\mathbf{x}}\,.$$

Now suppose f, g are two vector fields, and q is a point belonging to $D^{\mathbf{x}} \cap D^{\mathbf{X}}$. Then

$$f^{\mathbf{X},i}(X) = \sum_{j=1}^{n} \frac{\partial \Phi^i}{\partial x^j}(\Psi(X)) f^{\mathbf{x},j}(\Psi(X)) = \sum_{j=1}^{n} \left(\frac{\partial \Phi^i}{\partial x^j} f^{\mathbf{x},j}\right)(\Psi(X))\,,$$

so

$$\frac{\partial f^{\mathbf{X},i}}{\partial X^k}(X) = \sum_{j=1}^{n}\sum_{\ell=1}^{n} \left(\frac{\partial^2 \Phi^i}{\partial x^j \partial x^\ell} f^{\mathbf{x},j} + \frac{\partial \Phi^i}{\partial x^j}\frac{\partial f^{\mathbf{x},j}}{\partial x^\ell}\right)(\Psi(X)) \cdot \frac{\partial \Psi^\ell}{\partial x^k}(X)\,.$$

Similarly,

$$\frac{\partial g^{\mathbf{X},i}}{\partial X^k}(X) = \sum_{j=1}^{n}\sum_{\ell=1}^{n} \left(\frac{\partial^2 \Phi^i}{\partial x^j \partial x^\ell} g^{\mathbf{x},j} + \frac{\partial \Phi^i}{\partial x^j}\frac{\partial g^{\mathbf{x},j}}{\partial x^\ell}\right)(\Psi(X)) \cdot \frac{\partial \Psi^\ell}{\partial x^k}(X)\,.$$

Therefore

$$\begin{aligned}
&\sum_{k=1}^{n}\frac{\partial g^{\mathbf{X},i}}{\partial X^k}f^{\mathbf{X},k}(X)\\
&=\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{\ell=1}^{n}\left(\frac{\partial^2\Phi^i}{\partial x^j\partial x^\ell}g^{\mathbf{x},j}+\frac{\partial\Phi^i}{\partial x^j}\frac{\partial g^{\mathbf{x},j}}{\partial x^\ell}\right)(\Psi(X))\cdot\frac{\partial\Psi^\ell}{\partial x^k}(X)\cdot f^{\mathbf{X},k}(X)\\
&=\sum_{j=1}^{n}\sum_{\ell=1}^{n}\left(\frac{\partial^2\Phi^i}{\partial x^j\partial x^\ell}g^{\mathbf{x},j}+\frac{\partial\Phi^i}{\partial x^j}\frac{\partial g^{\mathbf{x},j}}{\partial x^\ell}\right)(\Psi(X))\cdot\left(\sum_{k=1}^{n}\frac{\partial\Psi^\ell}{\partial x^k}(X)\cdot f^{\mathbf{X},k}(X)\right)\\
&=\sum_{j=1}^{n}\sum_{\ell=1}^{n}\left(\frac{\partial^2\Phi^i}{\partial x^j\partial x^\ell}g^{\mathbf{x},j}+\frac{\partial\Phi^i}{\partial x^j}\frac{\partial g^{\mathbf{x},j}}{\partial x^\ell}\right)(x)\cdot f^{\mathbf{x},\ell}(x)\\
&=\sum_{j=1}^{n}\sum_{\ell=1}^{n}\left(\frac{\partial^2\Phi^i}{\partial x^j\partial x^\ell}g^{\mathbf{x},j}f^{\mathbf{x},\ell}\right)(x)+\sum_{j=1}^{n}\sum_{\ell=1}^{n}\left(\frac{\partial\Phi^i}{\partial x^j}\frac{\partial g^{\mathbf{x},j}}{\partial x^\ell}f^{\mathbf{x},\ell}\right)(x)\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{k=1}^{n}\frac{\partial f^{\mathbf{X},i}}{\partial X^k}g^{\mathbf{X},k}(X)\\
&=\sum_{j=1}^{n}\sum_{\ell=1}^{n}\left(\frac{\partial^2\Phi^i}{\partial x^j\partial x^\ell}f^{\mathbf{x},j}g^{\mathbf{x},\ell}\right)(x)+\sum_{j=1}^{n}\sum_{\ell=1}^{n}\left(\frac{\partial\Phi^i}{\partial x^j}\frac{\partial f^{\mathbf{x},j}}{\partial x^\ell}g^{\mathbf{x},\ell}\right)(x)\\
&=\sum_{j=1}^{n}\sum_{\ell=1}^{n}\left(\frac{\partial^2\Phi^i}{\partial x^\ell\partial x^j}f^{\mathbf{x},\ell}g^{\mathbf{x},j}\right)(x)+\sum_{j=1}^{n}\sum_{\ell=1}^{n}\left(\frac{\partial\Phi^i}{\partial x^\ell}\frac{\partial f^{\mathbf{x},\ell}}{\partial x^\ell}g^{\mathbf{x},j}\right)(x)\\
&=\sum_{j=1}^{n}\sum_{\ell=1}^{n}\left(\frac{\partial^2\Phi^i}{\partial x^j\partial x^\ell}f^{\mathbf{x},\ell}g^{\mathbf{x},j}\right)(x)+\sum_{j=1}^{n}\sum_{\ell=1}^{n}\left(\frac{\partial\Phi^i}{\partial x^\ell}\frac{\partial f^{\mathbf{x},\ell}}{\partial x^\ell}g^{\mathbf{x},j}\right)(x)\,,
\end{aligned}$$

if we assume that $\mathbf{x}$ and $\mathbf{X}$ are C^2-related, so that

$$\frac{\partial^2\Phi^i}{\partial x^\ell\partial x^j}=\frac{\partial^2\Phi^i}{\partial x^j\partial x^\ell}\,.$$

Then

$$\begin{aligned}
h^{\mathbf{X},i}(X) &= \sum_{k=1}^{n}\frac{\partial g^{\mathbf{X},i}}{\partial X^k}f^{\mathbf{X},k}(X)\sum_{k=1}^{n}-\frac{\partial f^{\mathbf{X},i}}{\partial X^k}g^{\mathbf{X},k}(X)\\
&= \sum_{j=1}^{n}\sum_{\ell=1}^{n}\left(\frac{\partial\Phi^i}{\partial x^j}\frac{\partial g^{\mathbf{x},j}}{\partial x^\ell}f^{\mathbf{x},\ell}\right)(x)-\sum_{j=1}^{n}\sum_{\ell=1}^{n}\left(\frac{\partial\Phi^i}{\partial x^\ell}\frac{\partial f^{\mathbf{x},\ell}}{\partial x^\ell}g^{\mathbf{x},j}\right)(x)\\
&= \sum_{j=1}^{n}\frac{\partial\Phi^i}{\partial x^j}(x)\left(\sum_{\ell=1}^{n}\frac{\partial g^{\mathbf{x},j}}{\partial x^\ell}(x)f^{\mathbf{x},\ell}(x)-\frac{\partial f^{\mathbf{x},\ell}}{\partial x^\ell}(x)g^{\mathbf{x},j}(x)\right)(x)\\
&= \sum_{j=1}^{n}\frac{\partial\Phi^i}{\partial x^j}(x)h^{\mathbf{x},j}(x)\,,
\end{aligned}$$

which is the desired transformation formula. $\diamondsuit$

The fact that $[f,g]$ is a vector field suggests that it should have an intrinsic, coordinate-free characterization. In other words: **rather than define what** $[f,g]$

means using coordinates, and then prove that when we change coordinates $[f,g]$ transforms as it should, it ought to be possible to say directly what $[f,g]$ without invoking coordinates at all.

This is indeed true. The precise **coordinate-free characterization of the Lie bracket** is:

$$[f,g](q) = \lim_{t\to 0} \frac{\delta(t)-q}{t^2}\,, \tag{1.8}$$

where

$$\delta(t) = qe^{tf}e^{tg}e^{-tf}e^{-tg}\,. \tag{1.9}$$

(Here $t \mapsto qe^{th}$ denotes, for a vector field h, the integral curve of h that goes through q when $t=0$.)

1.4. Control and Lie brackets. Lie brackets play a crucial role in optimal control theory in many ways and, more generally, they are the key objects in the theory of nonlinear finite-dimensional control. They play a central role in nonlinear controllability, feedback linearization, and system equivalence.

Just to give a simple illustration having nothing to with optimality, let us consider a control theory problem totally analogous to that of the characterization of flatness for a metric. In control theory, "linear control systems"—i.e., systems of the form $\dot q = Aq + Bu$, evolving in $\mathbb{R}^n$, with u taking values in $\mathbb{R}^m$, and A, B matrices of the appropriate sizes—play a crucial role, comparable in importance to that of the flat metrics within the class of general Riemannian metrics. So it is important to know when a system that looks nonlinear because it is presented in "curvilinear coordinates," is in fact linear, in the sense that one can make it linear by changing coordinates. The answer turns out to be a criterion involving Lie brackets.

Precisely, suppose we are studying control systems

$$\dot q = f(q) + u_1g_1(q) + \ldots + u_mg_m(q)\,, \quad q \in \mathbb{R}^n\,,$$

with C^∞ vector fields f, $g_1,\ldots,g_m$, and a point $\bar q$ in $\mathbb{R}^n$.

Is there a way to know if such a system is "a linear system in disguise," that is, a system obtained from a linear system by changing coordinates?

For example, look at the two systems

$$\begin{aligned} \dot x_1 &= x_2 + 2x_2x_3\,, \\ \dot x_2 &= x_3 - x_1x_2 + x_2^3\,, \\ \dot x_3 &= u + x_2^2 + x_1x_3 - x_1^2x_2 + 2x_1x_2^3\,, \end{aligned} \tag{1.10}$$

$$\begin{aligned} \dot x_1 &= u_1\,, \\ \dot x_2 &= u_2\,, \\ \dot x_3 &= u_1x_2 - u_2x_1\,. \end{aligned} \tag{1.11}$$

Which one is "more nonlinear"? We will see later that (1.10), which superficially looks very nonlinear, is in fact a linear system in disguise, whereas (1.11), whose nonlinearity appears to be much milder, is in fact highly nonlinear. And the answer will be obtained by studying the Lie bracket structures associated to both systems. It will turn out that a necessary condition for the system to be equivalent near $\bar q$ to a linear system is that all the iterated brackets of members of the set $\{f, g_1, \ldots, g_m\}$ involving two g_i's should vanish near $\bar{q}$. Under an extra technical condition ("strong accessibility," that is, the requirement that the Lie brackets $\mathrm{ad}_f^k(g_i)(\bar{q})$, $k \geq 0$, $i = 1, \ldots, m$ span the whole space) this condition is sufficient as well.

So *Lie brackets help us organize our knowledge of nonlinear systems in terms of their deep structural properties.* This way of classifying systems is as superior to the naïve approach based on a superficial counting of the number of nonlinearities as the animal taxonomy of modern zoology is to the classifications found in medieval bestiaries. Exactly as a dolphin is structurally closer to a zebra than to a shark—in spite of the very visible ways, such as shape and habitat, in which a dolphin resembles a shark more than a zebra—System (1.10) is much closer to a linear system than System (1.11). Lie brackets enable us to discover this, and then one can find abundant confirmation by looking at the properties of both systems. (For example, (1.10) is globally stabilizable by a smooth state feedback, but (1.11) is not even locally stabilizable near any point.)

1.5. Using Lie brackets to understand optimal trajectories. For optimal control problems, the effort to understand the properties of optimal trajectories in terms of reasonable invariants has begun only recently, and most of the work remains to be done.

Just to give the flavor of some of these results, we concentrate on one problem that has attracted a lot of attention in recent years, namely, that of minimum time control. Suppose, for example, that we are looking at minimum time control of systems in $\mathbb{R}^n$ of the form $\dot{x} = f(x) + ug(x)$, with $|u| \leq 1$, where f and g are smooth vector fields. Here the control space U is the interval $[-1, 1]$. The Hamiltonian is $H = \varphi + u\psi - p_0$, where $\varphi = \langle p, f(x)\rangle$ and $\psi = \langle p, g(x)\rangle$. Then u will equal 1 when $\psi > 0$, and -1 when $\psi < 0$. A naïve genericity argument may suggest the wrong conclusion that $\psi = 0$ is a "rare" event, which will happen at most at isolated points, so that in fact the optimal trajectories will be "bang-bang," i.e., such that u is piecewise constant with values 1 and -1 and finitely many jumps. This, however, is quite wrong.

EXAMPLE 1.5.1. Let $Q = \mathbb{R}^2$, and assume that $q = (x, y) \in Q$ evolves according to $\dot{x} = 1 - y^2$, $\dot{y} = u$, $|u| \leq 1$, so f, g are the vector fields $(x, y) \mapsto (1 - y^2, 0)$, $(x, y) \mapsto (0, 1)$, respectively. (Equivalently, $f = (1-y^2)\frac{\partial}{\partial x}$ and $g = \frac{\partial}{\partial y}$.) Suppose we want a trajectory that goes from $(0, 0)$ to $(1, 0)$ in minimum time. It is obvious that the solution is given by $u(t) \equiv 0$, $y(t) \equiv 0$, $x(t) \equiv t$, $0 \leq t \leq 1$, and the optimal time is 1. So the optimal control is "singular"—i.e., takes values in the open interval $]-1, 1[$ —and therefore drastically fails to be bang-bang. And one can easily verify that this example is "stable under perturbations" in any reasonable sense of the word. So it is not true that optimal controls are always, or even generically, bang-bang. $\diamondsuit$

In view of our previous remarks, it is to be expected that the *Lie brackets* of the vector fields f and g will play a crucial role in determining the properties of a solution. Not surprisingly, a close analysis of the problem, applying (MP), shows that this is indeed so.

For example, the "singular" control of Example 1.5.1 is closely related to a property of the Lie brackets of the vector fields f and g. Precisely, *the singular trajectory exists and is contained in the x axis because $[f, g]$ vanishes on the x axis.*

Since the computation of Example 1.5.1, as presented, does not explicitly display any Lie brackets, we will now make the Lie brackets stand out by looking, more generally, at systems of the form $\dot{x} = f(x) + ug(x)$, with $|u| \leq 1$, without specializing to a particular pair of vector fields f, g.

If one computes the derivative of ψ along an extremal γ, the result is ρ evaluated along γ, where $\rho = \langle p, [f, g]\rangle$. So, if n happens to be 2, and g and $[f, g]$ are linearly independent at each point, then ψ and $\dot{\psi}$ cannot vanish simultaneously, for otherwise p would be orthogonal to $g(x)$ and $[f, g](x)$, so p would vanish, which is a contradiction. (Recall that the Maximum Principle for variable time-interval problems has the extra condition that H must vanish. Since $H = \langle p, f(x){+}u\, g(x)\rangle - p_0$, it is clear that $p = 0$ implies $p_0 = 0$, contradicting nontriviality.) Now, if a function and its derivative do not vanish simultaneously, it follows that the zeros of the function are isolated. In our case, this implies that the optimal controls must be bang-bang. So we have proved a simple theorem on the structure of optimal trajectories: *for a minimum time problem in $\mathbb{R}^2$ arising from a control system $\dot{x} = f(x) + ug(x)$, with $|u| \leq 1$, if g and $[f, g]$ are linearly independent at each point, then all optimal trajectories are bang-bang.* The reader can verify that in Example 1.5.1 the Lie bracket $[f, g]$ vanishes along the x axis, so the trajectory of that example is contained in the "singular set" of points where g and $[f, g]$ fail to be linearly independent, in perfect agreement with our theorem.

1.6. Implementing Lagrange's covariance idea in optimal control. Our elementary arguments about Lie brackets and minimum time control provide a good illustration of how Lagrange's principle of "coordinate democracy" applies in the optimal control setting, and why a truly intrinsic formulation of the maximum principle, on manifolds, is needed.

To understand optimal trajectories, one needs to study certain functions, such as the "switching function" ψ. We can think of ψ as a "generalized component" of the (co)vector p, in the sense that the ordinary components p_i of p are the inner products of p with the coordinate vector fields $\frac{\partial}{\partial q^i}$, and ψ is the product of p with the vector field g. The adjoint equation gives us an explicit formula for the functions $\dot{p}_i$. A truly "democratic" version of the maximum principle should treat all generalized components equally, that is, should give us in one swoop the derivative of *every* function $\langle p, X(q)\rangle$, for *every* vector field X.

With such a version one could, in each case, work with the generalized components of p that are most natural for the problem under study, such as, for example, the "switching function" $\langle p, g(q)\rangle$ for a system $\dot{q} = f(q) + ug(q)$. This would be, we believe, true in spirit to Lagrange's idea that it is good to have a method that "***expresses the equations of every problem in the most simple form relative to each set of variables and ... enables us to see beforehand which variables one should use in order to facilitate the integration as much as possible.***"

An intrinsic version, treating all "generalized components" (sometimes known as "momentum functions") equally, can be formulated in a number of ways, e.g.

using Poisson brackets. For a general version of the maximum principle on manifolds, using generalized components as above (and also for an alternative version involving connections along curves), we refer the reader to [**8**]. The crucial point of the version given in [**8**] is—considering, for simplicity, the case of the minimum time problem for a control system of the form $\dot{q} = f(q,u)$—that the adjoint equation is rephrased as the following statement (the "intrinsic adjoint equation"):

(IAE) *for every smooth vector field X, the derivative with respect to t of the function*

$$t \mapsto \langle \pi(t), X(\xi_*(t)) \rangle$$

is the function

$$t \mapsto \langle \pi(t), [f_{\eta_*(t)}, X](\xi_*(t)) \rangle$$

where we define

$$f_u(q) = f(q,u)\,.$$

The point of using (IAE) is that it treats the inner product of with X—i.e., "the X component of $\pi(t)$"—equally for all vector fields X, thus enabling us to follow Lagrange's prescription (cf. Page 2) that one should express "the equations of every problem in the most simple form relative to each set of variables and ... enables us to see beforehand which variables one should use in order to facilitate the integration as much as possible," by working in each case with the functions of the form $\langle p, X\rangle$ that are most suited to the problem.

1.7. Why the language of Lie brackets is natural for optimal control. The preceding discussion, based on rather trivial examples, is meant to illustrate what we have in mind when we talk about "understanding the qualitative properties of trajectories" and why we say that the appropriate language for discussing this question and formulating conditions is that of Lie brackets of vector fields.

The reader should think of the following analogy. When one tries to solve a simple 2×2 system of linear equations such as $2x + 3y = 5$, $6x - 2y = 4$, one may fail to notice that the number $2.(-2) - 3.6$ is important. If, however, one "does it with letters" (as a Calculus student would say), i.e., sets out to solve a *general* system $a_{11}x + a_{12}y = b_1$, $a_{21}x + a_{22}y = b_2$, then one discovers immediately that the number $a_{11}a_{22} - a_{12}a_{21}$ determines the properties of the solutions, so it becomes natural to gives this number a name such as "determinant," and to look for a generalization of the concept to higher dimensions. Similarly, when one applies the maximum principle to study a particular problem such as the one of Example 1.5.1, one is in fact computing a Lie bracket, although this fact may go unnoticed. But when one "does it with letters," e.g. by working with a general problem of the form $\dot{x} = f(x) + ug(x)$, leaving f and g unspecified, then the Lie bracket $[f,g]$ stands out immediately as the object that "determines" what happens, exactly as the expression $a_{11}a_{22} - a_{12}a_{21}$ did in the linear algebra example.

It is a general fact that, when we apply the maximum principle to minimum time problems, we get conclusions of the form "if such and such thing is true of the Lie brackets of the vector fields involved, then the optimal trajectories have such and such properties." This can be ascertained in all kinds of examples, provided that one makes sure that the computations are always done "with letters." But the occurrence of the brackets is due to a profound *a priori* reason, namely, Statement (INV) of the previous section.

In recent years, ever since the advent of "differential geometric control theory," much has been done to determine, in a systematic way, how structural properties of a system, embodied in their Lie bracket relations, relate to properties of the optimal trajectories. For example, it has been known for a long time that for linear minimum time optimal control the "bang-bang property" holds. From a general nonlinear perspective, one can show that the bang-bang property holds when certain Lie brackets B_i can be expressed as linear combinations of some other brackets. Linear systems just happen to be those for which the B_i vanish. This provides an explanation of the linear bang-bang theorem from the nonlinear perspective, as well as a generalization.

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