A geometric approach to modeling and controlof a Variable Speed Control Moment Gyro (VSCMG)

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Abstract

The Variable Speed Control Moment Gyro (VSCMG) has been much studied in the aerospace and control community. There have been two distinct schools of work - one from the aerospace community and the other from the geometric mechanics community. The former adopts the Newtonian approach to derive the equations of motion and then goes on to study singularity issues and control law synthesis. The latter adopt tools from geometric mechanics - principle fiber bundles, energy-Casimir notions - to derive control laws, though restricted to spin stabilization and not full attitude control. This paper attempts to make connections with the two approaches.

1 Modelling in a geometric framework

We first model a single variable-speed CMG using the geometric framework. We then compare the equations obtained with those in SRJ which are highly cited in the aerospace literature.

1.1 Configuration space and moments of inertia

With reference to the single CMG model as seen in figure, we explain our notation as:
Viewed as a rigid body with independent of the The significant differences in our approach as compared to the one by Schaub al are:

1. We use the Riemannian metric to express the kinetic energy and then the momentum map to express the equations of motion

2. The

The configuration space is $Q = \mathbb{SO}(3) \times \mathbb{S}^1 \times \mathbb{S}^1$. The first element denotes the attitude R_s of the spacecraft with respect to fixed frame, the second denotes the degree of freedom β of the gimbal frame, the third denotes the degree of freedom γ of the rotor about its spin axis. For the purpose of later geoemtrical interpretation, we club $x \triangleq (\beta, \gamma)$.

There are three rigid bodies involved here, each having relative motion (rotation) about the other. Three frames of reference are chosen - the first is the spacecraft, denoted by the subscript s, the second is the gimbal frame, denoted by the subscript g, the third is the rotor frame, denoted by the subscript r. The moments of inertia of the homogeneous rotor and the gimbal in their respective body frames are assumed to be

$$\mathbb{I}_r = \begin{pmatrix} J_x & 0 & 0 \\ 0 & J_x & 0 \\ 0 & 0 & J_z \end{pmatrix} \quad \mathbb{I}_g = \begin{pmatrix} I_t & 0 & 0 \\ 0 & I_g & 0 \\ 0 & 0 & I_s \end{pmatrix} \tag{1}$$

Since the rotor is assumed to be homogeneous and symmetric, its inertia is represented in the gimbal frame and the combined gimbal-rotor inertia is rewritten as

$$\mathbb{I}_{gr} = \begin{pmatrix} (J_x + I_t) & 0 & 0 \\ 0 & (J_x + I_g) & 0 \\ 0 & 0 & (J_z + I_s) \end{pmatrix} \tag{2}$$

If the rotational transformation that relates the gimbal frame to the spacecraft frame is given by R_β, where β denotes the gimballing angle, then the gimbal-rotor inertia reflected in the spacecraft frame is

$$(\mathbb{I}_{gr})_s \triangleq R_\beta \mathbb{I}_{gr} R_\beta^T \tag{3}$$

Here the subscript s denotes the spacecraft frame.

1.2 Group action and kinetic energy

The action of the $\mathbb{SO}(3)$ group on Q is given by

$$\mathbb{SO}(3) \times Q \to Q \qquad (M, (R_s, \beta, \gamma)) \to (MR_s, \beta, \gamma) \tag{4}$$

and the tangent lifted action is given by

$$T\,\mathbb{SO}(3)\times TQ\to TQ \quad (v_{R_s},v_\beta,v_\gamma)\to(Mv_{R_s},v_\beta,v_\gamma) \tag{5}$$

If $\Omega_s = R_s^T\dot{R}_s$ denotes the angular velocity of the satellite in its body frame, then the kinetic energy is given by

$$\frac{1}{2}\left\langle \Omega_s, \mathbb{I}_s\Omega_s \right\rangle \tag{6}$$

and the kinetic energy of the gimbal-rotor unit is given by

$$\frac{1}{2}\left\langle R_\beta^T\Omega_s + \begin{pmatrix}0\\ \dot\beta\\ \dot\gamma\end{pmatrix}, \mathbb{I}_{gr}[R_\beta^T\Omega_s + \begin{pmatrix}0\\ \dot\beta\\ \dot\gamma\end{pmatrix}]\right\rangle \tag{7}$$

The total kinetic energy is

$$\frac{1}{2}\left\langle \begin{pmatrix}\Omega_s\\ 0\\ \dot\beta\\ \dot\gamma\end{pmatrix}, \mathbb{I}_{total}\begin{pmatrix}\Omega_s\\ 0\\ \dot\beta\\ \dot\gamma\end{pmatrix}\right\rangle \tag{8}$$

where

$$\mathbb{I}_{total}(\beta) = \begin{pmatrix}(R_\beta\mathbb{I}_{gr}R_\beta^T + \mathbb{I}_s) & R_\beta\mathbb{I}_{gr}\\ \mathbb{I}_{gr}R_\beta^T & \mathbb{I}_{gr}\end{pmatrix} \tag{9}$$

claimnder the defined group action, the kinetic energy of the total system remains invariant.

Proof
Straightforward.
$\square$.

1.3 A Riemannian structure

The kinetic energy induced a metric on the configuration space Q of the system, which enables us to impart a Riemannian structure to the system. The Riemannian metric $\mathbb{G}$, is expressed as

$$\langle v_q, w_q\rangle_{\mathbb{G}} = \mathbb{G}(R_s,(\beta,\gamma))((R_s\hat\Omega_1,(v_1,v_2)),((R_s\hat\eta_2,(w_1,w_2)) \tag{10}$$

$$= \frac{1}{2}\left\langle \begin{pmatrix} R_s\hat{\Omega}_1 \\ 0 \\ v_1 \\ v_2 \end{pmatrix}, \mathbb{I}_{total} \begin{pmatrix} R_s\hat{\eta}_2 \\ 0 \\ w_1 \\ w_2 \end{pmatrix} \right\rangle \tag{11}$$

where $q = (R_s, (\beta, \gamma)) \in Q$ and $v_q = (R_s\hat{\Omega}_1, (v_1, v_2)), w_q = (R_s\hat{\Omega}_2, (w_1, w_2)) \in T_qQ$. Here $\hat{\Omega}_1$ and $\hat{\Omega}_2$ belong to the Lie algebra $\mathfrak{so}(3)$

1.4 Principle fiber bundle

More geometric structure is present in the problem. The gimbal angle and the rotor angle could be viewed as variables in a *shape space (or base space)* and the rigid body orientation could be viewed as a variable in a *fiber space*, and with a few additional requirements, the model is amenable to a principal fiber bundle description. See [] for more details on describing mechanical systems in a fiber bundle framework. Based on the above model description, we identify the principle fiber bundle (Q, B, π, G), where $Q = \mathbb{SO}(3) \times \mathbb{S}^1 \times \mathbb{S}^1$, $B = \mathbb{S}^1 \times \mathbb{S}^1$ and $\pi : Q \to B$ is the bundle projection. We now define a few mechanical quantities on this fiber bundle.

- The *infinitesimal generator* of the Lie algebraic element $\hat{\eta} \in \mathfrak{so}(3)$ under the group action is the vector field

$$\hat{\eta}_Q(q) = \frac{d}{dt}|_{t=0}(\exp(\hat{\eta}t)R_s, (\beta, \gamma)) = (\hat{\eta}R_s, (0, 0)) \tag{12}$$

- The *momentum map* $J : TQ \to \mathfrak{so}(3)^*$ is given by

$$[J(q, v_q), \xi] = \left\langle v_q, \hat{\xi}_Q(q) \right\rangle_{\mathbb{G}} \tag{13}$$

  and since the kinetic energy is invariant under the action of the $\mathbb{SO}(3)$ group, we have

$$\left\langle v_q, \hat{\xi}_Q(q) \right\rangle_{\mathbb{G}} = \left\langle v_{(e,(\beta,\gamma))}, (R_s^T \hat{\xi} R_s, (0, 0)) \right\rangle_{\mathbb{G}} \tag{14}$$

  which yields

$$[J(q, v_q), \hat{\xi}] = \left\langle Ad^*_{R_s^T}[(\mathbb{I}_{gr})_s + \mathbb{I}_s)\Omega_s + R_\beta \mathbb{I}_{gr} \begin{pmatrix} 0 \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix}], \xi \right\rangle \tag{15}$$

  Please note that $\langle \cdot, \cdot \rangle$ denotes the inner product, while $[\cdot, \cdot]$ denotes the primal-dual action of the vector spaces $\mathfrak{so}(3)$ and $\mathfrak{so}(3)^*$. Since the total spatial angular momentum of the system is constant, say μ, the above expression yields

$$\mu = Ad^*_{R_s^T}[(\mathbb{I}_{gr})_s + \mathbb{I}_s)\Omega_s + R_\beta \mathbb{I}_{gr} \begin{pmatrix} 0 \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix}] \tag{16}$$

$$= \tag{17}$$

- The *locked inertia tensor* at each point $q \in Q$ is the mapping

  $$\mathbb{I}(q) : \mathfrak{so}(3) \to \mathfrak{so}(3)^* \tag{18}$$

  and is defined as

  $$[\mathbb{I}(q)\eta, \xi] = \mathbb{G}(q))((\hat{\eta}R_s, (0,0)), (\hat{\xi}R_s, (0,0)) \tag{19}$$

- The *mechanical connection* is then defined as the $\mathfrak{so}(3)$-valued one-form

  $$\alpha : TQ \to \mathfrak{so}(3) \quad (q, v_q) \to \alpha(q, v_q) = \mathbb{I}(q)^{-1} J(q, v_q) \tag{20}$$

With the state-space as $X \stackrel{\triangle}{=} (R_s, (\beta, \gamma)) = (R_s, x)$, where $x \stackrel{\triangle}{=} (\beta, \gamma)$ and defining $\tilde{\mathbb{I}}(x) = (R_\beta \mathbb{I}_{gr} R_\beta^T + \mathbb{I}_s)$, the control inputs (gimbal velocity and rotor spin) at the kinematic level as $u \stackrel{\triangle}{=} \dot{x} = (\dot{\beta}, \dot{\gamma}) = \begin{pmatrix} u_\beta \\ u_\gamma \end{pmatrix}$, the affine-in-the-control system model is

$$\dot{X} = f(X) + g(X)u \tag{21}$$

where the drift and control vector fields are given by

$$f(X) = \begin{pmatrix} R_s \mathcal{S}((\tilde{\mathbb{I}}(x)^{-1}(Ad^*_{R_s}\mu))) \\ 0 \end{pmatrix} \tag{22}$$

$$g_\beta(X) = \begin{pmatrix} -R_s \mathcal{S}((\tilde{\mathbb{I}}(x)^{-1}(Ad^*_{R_\beta^T}(\mathbb{I}_{gr} i_2))) \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \qquad g_\gamma(X) = \begin{pmatrix} -R_s \mathcal{S}((\tilde{\mathbb{I}}(x)^{-1}(Ad^*_{R_\beta^T}(\mathbb{I}_{gr} i_3))) \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \tag{23}$$

Here $\mathcal{S}(\cdot) : \mathbb{R}^3 \to \mathfrak{so}(3)$ is given by

$$\mathcal{S}((\psi_1, \psi_2, \psi_3)) \stackrel{\triangle}{=} \begin{pmatrix} 0 & -\psi_3 & \psi_2 \\ \psi_3 & 0 & -\psi_1 \\ -\psi_2 & \psi_1 & 0 \end{pmatrix} \tag{24}$$

2 The dynamic model

To arrive at the dynamic model we proceed as follows. From the expression for the total momentum

$$\mu = Ad^*_{R_s^T}[((\mathbb{I}_{gr})_s + \mathbb{I}_s)\Omega_s + R_\beta \mathbb{I}_{gr} \begin{pmatrix} 0 \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix}] \tag{25}$$

$$= \tag{26}$$

we have

$$\mu = R_s \begin{pmatrix} \tilde{\mathbb{I}}(x) & R_\beta \mathbb{I}_{gr} \end{pmatrix} \begin{pmatrix} \Omega_s \\ \begin{pmatrix} 0 \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} \end{pmatrix} \tag{27}$$

We split the momentum in to two components - one due to the gimbal-rotor unit and the other due to the rigid spacecraft. Further, we assume an internal torque τ_b (in the gimbal-rotor fame) acting on the gimbal and rotor unit. We then have, due to the principle of action and reaction

$$\frac{d}{dt}(R_s \mathbb{I}_s \Omega_s) = \underbrace{-R_s \tau_b}_{reaction} \qquad \frac{d}{dt}(R_s[R_\beta \mathbb{I}_{gr} R_\beta^T \Omega_s + R_\beta \mathbb{I}_{gr} \dot{x}]) = \underbrace{R_s \tau_b}_{action} \tag{28}$$

We now simplify this expression to obtain more explicit equations, which we then compare with certain other results.

$$[\hat{\Omega}_s \tilde{\mathbb{I}}(x) \Omega_s + \tilde{\mathbb{I}}(x) \dot{\Omega}_s]$$

$$= \hat{\Omega}_s R_\beta \mathbb{I}_{gr} \dot{x} - \dot{\beta} R_\beta \mathcal{U} R_\beta^T \Omega_s - \dot{\beta} R_\beta \hat{i}_2 \mathbb{I}_{gr} \dot{x} - R_\beta \mathbb{I}_{gr} \ddot{x} \tag{29}$$

where $\mathcal{U} \stackrel{\triangle}{=} \hat{i}_2 \mathbb{I}_{gr} - \mathbb{I}_{gr} \hat{i}_2$ is a symmetric matrix.

3 The classical CMG modeling and analysis

We now draw connections between the approach outlined in the previous sections with that of the classical CMG modeling and analysis done in the Newtonian framework in [**?**], which is cited in much of the aerospace literature. We shall refer to this paper as the SRJ paper henceforth. We first relate the notation and then establish a connection with the main equations of the SRJ paper.

The two primary variables in the SRJ paper and ours are related as follows:

Gimbal angle	Our notation - β	SRJ notation - γ
Rotor spin magnitude	Our notation - $\dot{\gamma}$	SRJ notation - Ω
Satellite angular velocity	Our notation - Ω_s	SRJ notation - ω

The rotation matrix in the SRJ paper, relating the gimbal and spacecraft-body frame, is described in terms of three column vectors of unit norm, $\{\hat{g}_s, \hat{g}_t, \hat{g}_g\}$, where the subscripts s, t and g correspond to the *spin, transverse and gimbal* axes, as

$$\begin{pmatrix} | & | & | \\ \hat{g}_s & \hat{g}_t & \hat{g}_g \\ | & | & | \end{pmatrix} \tag{30}$$

and further,

$$\begin{pmatrix} \langle \hat{g}_s, \omega \rangle \\ \langle \hat{g}_t, \omega \rangle \\ \langle \hat{g}_g, \omega \rangle \end{pmatrix} = \begin{pmatrix} \omega_s \\ \omega_t \\ \omega_g \end{pmatrix} \tag{31}$$

In our convention, the following correspondence holds:

$$R_\beta \longrightarrow \begin{pmatrix} | & | & | \\ \hat{g}_t & \hat{g}_g & \hat{g}_s \\ | & | & | \end{pmatrix} \tag{32}$$

and

$$R_\beta^T \Omega_s \longrightarrow \begin{pmatrix} \omega_t \\ \omega_g \\ \omega_s \end{pmatrix} \tag{33}$$

The SRJ equation of motion (eqn. xxx) written partially in terms of our notation is

$$\tilde{\mathbb{I}}(x)\dot{\Omega}_s + \hat{\Omega}_s \tilde{\mathbb{I}}(x)\Omega_s = \tag{34}$$

$$-\hat{g}_s[J_s(\ddot{\gamma} + \dot{\beta}\omega_t) - (J_t - J_g)\omega_t\dot{\beta}] - \hat{g}_t[J_s(\dot{\gamma} + \omega_s)\dot{\beta} - (J_t + J_g)\omega_s\dot{\beta} + J_s\dot{\gamma}\omega_g] - \hat{g}_g[J_g\ddot{\beta} - J_s\dot{\gamma}\omega_t] \tag{35}$$

while the RHS of the same equation in our notation is

$$-[\hat{\Omega}_s + \dot{\beta}\hat{i}_2]R_\beta \mathbb{I}_{gr}\dot{x} - \dot{\beta}R_\beta(\hat{i}_2\mathbb{I}_{gr} - \mathbb{I}_{gr}\hat{i}_2)R_\beta^T\Omega_s - R_\beta\mathbb{I}_{gr}\ddot{x}$$

$$\begin{aligned}
= -\hat{g}_t[(J_z + I_s)\dot{\gamma}\dot{\beta} - (J_x + I_g)\dot{\beta}\omega_s + (J_z + I_s)\dot{\gamma}\omega_g + (J_z + I_s) - (J_x + I_t)\dot{\beta}\omega_s] \\
-\hat{g}_g[(J_x + I_g)\ddot{\beta} - (J_z + I_s)\dot{\gamma}\omega_t] \\
-\hat{g}_s[(J_z + I_s)\ddot{\gamma} + (J_x + I_g)\dot{\beta}\omega_t + ((J_z + I_s) - (J_x + I_t))\dot{\beta}\omega_t]
\end{aligned}$$