PDE for the final project

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We wish to solve the following PDE via a Fourier-Galerkin method

$$\frac{\partial \rho}{\partial t} = \frac{1}{Pe}\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^6 \rho}{\partial x^6} + 2\rho\frac{\partial^4 \rho}{\partial x^4} + 4\frac{\partial \rho}{\partial x}\frac{\partial^3 \rho}{\partial x^3} + 1 - \mu\rho \tag{1}$$

In order to map the Fourier domain $(0, 2\pi)$ to a physical domain $(0, L)$, we incorporate a scaling factor $s = 2\pi/L$, such that the computational domain

$$x = s\, x_p \tag{2}$$

Here x is the Fourier domain and x_p is the physical domain. Then we have,

$$\frac{\partial \rho}{\partial x_p} = s\frac{\partial \rho}{\partial x} \tag{3}$$

Since the equation is originally written in the physical domain, converting it to the Fourier domain gives

$$\frac{\partial \rho}{\partial t} = s^2\frac{1}{Pe}\frac{\partial^2 \rho}{\partial x^2} + s^6\frac{\partial^6 \rho}{\partial x^6} + 2s^4\rho\frac{\partial^4 \rho}{\partial x^4} + 4s^4\frac{\partial \rho}{\partial x}\frac{\partial^3 \rho}{\partial x^3} + 1 - \mu\rho \tag{4}$$

Let

$$\rho = \sum_{k=-\infty}^{\infty} a_k(t)e^{ikx} \tag{5}$$

$$\rho\frac{\partial^4 \rho}{\partial x^4} = \sum_{k=-\infty}^{\infty} b_k(t)e^{ikx} \tag{6}$$

$$\frac{\partial \rho}{\partial x}\frac{\partial^3 \rho}{\partial x^3} = \sum_{k=-\infty}^{\infty} c_k(t)e^{ikx} \tag{7}$$

$$1 = \sum_{k=-\infty}^{\infty} d_k(t)e^{ikx} \tag{8}$$

where we determine $b_k(t)$ and $c_k(t)$ by evaluating first $\rho\frac{\partial^4 \rho}{\partial x^4}$ and $\frac{\partial \rho}{\partial x}\frac{\partial^3 \rho}{\partial x^3}$ and then writing their Fourier series (or, in the program, taking their FFTs). Substituting these expressions in (4) we get

$$a_k'(t) = -\nu s^2 k^2 a_k(t) - s^6\ k^6\ a_k(t) + 2\ s^4\ b_k(t) + 4\ s^4\ c_k(t) + d_k(t) - \mu a_k(t) \tag{9}$$

where $\nu = 1/Pe$. We will use Adams-Moulton two step scheme for all terms linear in ρ and the Adams-Bashforth two-step scheme for the two non-linear terms. This gives (dropping the subscript k)

$$\begin{aligned}
a^{n+2} - a^{n+1} = & \\
& - \frac{\nu k^2 s^2 \Delta t}{2}(a^{n+2} + a^{n+1}) \\
& - \frac{k^6 s^6 \Delta t}{2}(a^{n+2} + a^{n+1}) \\
& + 2\ \Delta t\ s^4\ \frac{1}{2}(3b^{n+1} - b^n) \\
& + 4\ \Delta t\ s^4\ \frac{1}{2}(3c^{n+1} - c^n) \\
& - \mu\ \Delta t\ \frac{1}{2}(a^{n+2} + a^{n+1}) \\
& + d^n \Delta t
\end{aligned} \tag{10}$$

$\Longrightarrow$

$$\begin{aligned}
a^{n+2}\left[1 + \frac{\nu k^2 s^2 \Delta t}{2} + \frac{k^6 s^6 \Delta t}{2} + \frac{\mu \Delta t}{2}\right] = & \\
& a^{n+1}\left[1 - \frac{\nu k^2 s^2 \Delta t}{2} - \frac{k^6 s^6 \Delta t}{2} - \frac{\mu \Delta t}{2}\right] \\
& + b^{n+1}(3\ s^4\ \Delta t) - b^n(s^4\ \Delta t) \\
& + c^{n+1}(6\ s^4\ \Delta t) - c^n(2\ s^4\ \Delta t) \\
& + d^n \Delta t
\end{aligned} \tag{11}$$

which is the final system of equations to be solved.