

Lecture 6: Irreducible tensors and the Green's Function for the Laplacian

September 24, 2018

This lecture includes 3 key ideas-

- Few applications of isotropy of tensors
- Symmetric irreducible tensors
- Green's function for the Laplace equation and spherical harmonics

1 Isotropic Tensors

The moment of inertia tensor is a symmetric second order tensor. For a spheroid, it is given by $\underline{I}_{ij} = c_1\delta_{ij} + c_2p_ip_j$ while for an ellipsoid it is given by $\underline{I}_{ij} = c_1p_ip_j + c_2q_iq_j + c_3r_ir_j$. It seems that the fore-aft symmetry of the spheroid or ellipsoid rules out the cross terms i.e. since inverting \underline{p} to $-\underline{p}$ does not change the orientation of the spheroid (or ellipsoid) hence we get the term p_ip_j for spheroid (and do not get terms like q_ir_j for ellipsoid). However, note that the moment of inertia tensor is itself invariant to inversions of the principal directions. Therefore, even in cases when changing the direction of anisotropy *does indeed change* the orientation of the body, the moment of inertia tensor will not contain terms that can reflect this change.

Consider an example- an elliptic cone pointing towards the positive x-axis, will point towards negative x-axis if \underline{p} is changed to $-\underline{p}$ but still our ansatz for the moment of inertia of the cone will look exactly like the one for the ellipsoid. The ellipsoid is fore-aft symmetric whereas the cone isn't but this difference cannot lead to appearance of cross-terms in the moment of inertia tensor because δ_{ij} and x_ix_j (and therefore I_{ij}) remain invariant if \underline{p} is changed to $-\underline{p}$.

On the other hand, integrals like

$$\int x_i x_j x_k dV$$

will indeed change if one of the principal directions is reversed. These integrals are like moments of the mass distribution also known as virial integrals.

A second key idea about the moment of inertia tensor is as follows. If a second order tensor has three orthogonal directions where its behaviour is identical, then it must be proportional to the isotropic tensor δ_{ij} . As an example, consider the integral $\int x_i x_j dV$ taken over a sphere. It is immediately clear that it should be proportional to δ_{ij} . But the idea is that, this integral will be proportional to δ_{ij} even if it is carried over a cube because a cube has three independent identical directions.

1.1 Example: Particle sedimentation

Assume that a small sphere (radius = a) is sedimenting slowly ($Re = 0$) in a fluid with viscosity μ . What will be the drag force F on the sphere? Stokes law states that this force is given by $F = 6\pi\mu a u$. In vector form, for a general solid, this law takes the form

$$F_i = R_{ij} u_j$$

where R_{ij} is called the resistance tensor and it depends on the geometry of the body. For a sphere it is ofcourse isotropic but in line with our previous argument, it is isotropic for a cube as well. The scalar pre-factor relevant to a particular orientation can be extracted from R_{ij} by contracting it with $p_i p_j$ where \underline{p} is a unit vector denoting the orientation of the sedimenting cube. But the isotropic tensor δ_{ij} treats every p_i identically and hence the scalar pre-factor will be the same for any orientation. Therefore, the drag on the cube is independent of its orientation.

However, for a spheroid, orientation matters and the pre-factor differs with the orientation.

1.2 Example: Newton's law of viscosity

Assume a suspension where the collidal particles are spherical. Then the fourth order viscosity tensor in Newton's law of viscosity involves the integral

$$\int x_i x_j x_k x_l dV = c(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

There is only one constant and hence only one corresponding viscosity. However for a cube, consider δ_{ij} to be constructed through three mutually orthogonal unit vectors $\delta_{ij} = p_i p_j + q_i q_j + r_i r_j$. Constructing the fourth order isotropic tensor $(\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$ using this form of δ_{ij} results in two types of terms - single face terms $(q_i q_j q_k q_l)$ and two-faced terms $(q_i q_j r_k r_l)$. The constant multiplying the single-faced terms is different from the constant for the two-faced terms and hence a suspension of cubic particles would exhibit two distinct viscosities.

2 Symmetric irreducible tensors

These are symmetric tensors which vanish upon contracting with δ_{ij} . A symmetric irreducible tensor can be constructed from a general second order tensor by first symmetrizing it ($B_{ij} \rightarrow \frac{B_{ij} + B_{ji}}{2}$) and then enforcing that $\widetilde{B} = \frac{B_{ij} + B_{ji}}{2} + b \delta_{ij}$ goes to zero when contracted with δ_{ij} . This gives us the value of b for which \widetilde{B} becomes irreducible. As an example, $\delta_{ij} \widetilde{B}_{ij} = \widetilde{B}_{ii} = B_{ii} + 3b = 0 \Rightarrow b = -B_{ii}/3$. Therefore the corresponding symmetric irreducible tensor for B_{ij} is

$$\widetilde{B}_{ij} = \frac{B_{ij} + B_{ji}}{2} - \frac{B_{kk}}{3} \delta_{ij}$$

Similar procedure can also be adopted for higher order tensors.

2.1 Example: The velocity gradient tensor

$$\begin{aligned} \tau_{ij} &= A_{ijkl} e_{kl} \\ &= \underbrace{c_1 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)}_{\text{volume and shape change}} + \underbrace{c_2 \frac{\partial u_k}{\partial x_k}}_{\text{only volume change}} \end{aligned}$$

The first part contains information about shape change as well as the volumetric change of a fluid parcel. Since the trace of the tensor $\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$ is not zero, it allows for a volumetric change. This can easily be converted to a symmetric irreducible tensor as shown above to obtain the following

expression

$$\begin{aligned}\tau_{ij} &= c_1 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + c_2 \frac{\partial u_k}{\partial x_k} \delta_{ij} \\ &= \underbrace{c_1 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right)}_{\text{only shape change}} + \underbrace{\left(c_2 + \frac{2c_1}{3} \right) \frac{\partial u_k}{\partial x_k} \delta_{ij}}_{\text{only volume change}}\end{aligned}$$

Now the traceless component associated with c_1 only deals with shape changes of a fluid element while that with $c_2 + \frac{2c_1}{3}$ deals with volumetric changes. This leads to the definition of the two types of viscosities - shear viscosity (c_1) and bulk viscosity ($c_2 + \frac{2c_1}{3}$)

3 Green's function for the Laplacian

The Green's function for an operator is its response to a point source. Hence, Green's function of the Laplacian is the solution to the differential equation

$$\nabla^2 \phi = \delta(\underline{x} - \underline{x}_0)$$

Using basic ideas from electrostatics, we can immediately determine the Green's function for Laplacian. Consider the statement for Gauss' Law

$$\nabla \cdot \underline{E} = \frac{\rho}{\epsilon_0}$$

But $\underline{E} = -\nabla \phi$

$$\Rightarrow \nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

Here ρ is the charge density. Since we want to find the response due to a point source, we will consider the charge density due to a point charge q placed at \underline{x}_0 . (The scalar delta function has units of length inverse and hence for vectors it will be 'per-unit-volume')

$$\begin{aligned}\Rightarrow \nabla^2 \phi &= -\frac{q \delta(\underline{x} - \underline{x}_0)}{\epsilon_0} \\ \Rightarrow \nabla^2 \left(\frac{\epsilon_0 \phi}{q} \right) &= -\delta(\underline{x} - \underline{x}_0)\end{aligned}$$

But we already know the potential due to point charge which is the solution to this equation. Hence,

$$\begin{aligned}\Rightarrow \phi &= -\frac{q}{4\pi\epsilon_0 r} \\ \Rightarrow \frac{\epsilon_0 \phi}{q} &= G(\underline{x}|\underline{x}_0) = -\frac{1}{4\pi|\underline{x} - \underline{x}_0|}\end{aligned}$$

which is the Green's function for the three dimensional Laplacian. We can also assume a line charge with field $\propto 1/r$ (instead of a point charge) and similarly evaluate the 2D Green's function for Laplacian as $\propto \ln(r)$

3.1 Green's function for higher order tensors

Assume that instead of a point charge, we wish to calculate the potential due to a dipole (charges q and $-q$ separated by a distance ϵ about the center \underline{x}_0). Then charge density is given by

$$\rho = q\delta(\underline{x} - (\underline{x}_0 + \underline{p}\frac{\epsilon}{2})) - q\delta(\underline{x} - (\underline{x}_0 - \underline{p}\frac{\epsilon}{2}))$$

Assuming that separation between charges ϵ is much less than the distances at which we are observing the potential due to the dipole, we can linearize the above expression. The leading order terms cancel because at leading order q and $-q$ are not separated at all and the net effect is zero. Hence the forcing term appears at first order and we get a gradient of the delta function. The expression is

$$\nabla^2\phi_d = -Dp_i \frac{\partial}{\partial x_i} \delta(\underline{x} - \underline{x}_0)$$

where D is the dipole strength and ϕ_d is the potential due to the dipole. This equation is different from the one for the point charge because we get a gradient and a dot product with \underline{p} on the RHS. How can we solve this equation? We know the solution to the equation $\nabla^2\phi_m = \delta(\underline{x})$ where ϕ_m is the potential due to a monopole (point charge). Let's take the gradient of this equation and then dot it with $-D\underline{p}$. We get

$$\frac{\partial^2}{\partial x_j^2} \underbrace{(-Dp_i \frac{\partial \phi_m}{\partial x_i})}_{\phi_d} = -Dp_i \frac{\partial}{\partial x_i} \delta(\underline{x} - \underline{x}_0)$$

This equation is identical to the one for dipole which implies that

$$\phi_d = -Dp_i \frac{\partial \phi_m}{\partial x_i} = -Dp_i \frac{\partial}{\partial x_i} \frac{1}{4\pi r} = \frac{D}{4\pi\epsilon_0} \frac{p_i x_i}{r^3}$$

Which gives us the vector solution to the Laplace equation. Similarly we can construct second and higher order tensor solutions. The key idea here is that *the Gradient commutes with the Laplacian*. Therefore, if ϕ is a solution to Laplace equation, then so is $\nabla\phi$, $\nabla\nabla\phi$ and so on. Three such solutions are written below

- Scalar solution: $\frac{1}{r}$
- Vector solution: $\frac{x_i}{r^3}$
- Second order tensor: $\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}$

In hindsight then, we can say that since the equation for potential for a dipole involved a vector on the RHS, hence ϕ_d must have been proportional to $\frac{x_i}{r^3}$. Since potential is a scalar, this must be contracted with a vector. The only relevant vector here is the orientation vector of the dipole p_i and hence we must have $\phi_d \propto \frac{p_i x_i}{r^3}$. Similarly scalar potential due to quadrupole can be constructed using the second order solution.

All these solutions to the Laplacian are in fact symmetric irreducible tensors. To see this, note that $\nabla^2(1/r) = 0$ and that the gradient commutes with the Laplacian. Suppose we contract $\frac{\partial^3}{\partial x_i \partial x_j \partial x_k} \frac{1}{r}$ with δ_{ij} , then it immediately results in a Laplacian and hence proves the irreducibility of the tensor solution.

Using these tensor solutions, we can write a general solution to the Laplace equation as

$$\phi = \frac{a_0}{r} + a_i \frac{\partial}{\partial x_i} \frac{x_i}{r^3} + a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r} + \dots$$

4 Appendix