Lecture 4: Introduction to Cartesian tensors

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The key ideas are

- Index notation for basic operations
- Coordinate transformations for vector and tensors
- Isotropic tensors

1 Index notation

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$$

$$\boldsymbol{a} \wedge \boldsymbol{b} = \epsilon_{ijk} a_j b_k$$

$$\boldsymbol{ab} = a_i b_j$$

Using these notations, we can briefly define some quantities of interest in transport phenomena.

- The gradient vector
$$\nabla T = \frac{\partial T}{\partial x_i}$$
which points in the direction of fastest rate of change of the scalar field T. Its inner product with a direction gives the rate of change of T in that direction.
- The divergence of velocity field
$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_i}{\partial x_i}$$
which gives the normalized volumetric rate of change of a local volume element ($\frac{1}{V}\frac{\partial V}{\partial t}$). It is zero for an incompressible flow.

- The curl of velocity field (vorticity)

$$\nabla \wedge \boldsymbol{u} = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

which is a measure of the local rotation rate of the fluid.

- The velocity gradient tensor

$$\nabla \boldsymbol{u} = \frac{\partial u_i}{\partial x_j}$$

which has information about the volumetric change, rotation as well as the local shear rate of the flow. A second order tensor packs much more information than a vector.

As examples of their usage, we can write down few equations in index notation

- Convection-diffusion equation

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \kappa \nabla^2 T \qquad \text{(Gibbs notation)}$$

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \kappa \frac{\partial^2 T}{\partial x_i \partial x_i} \qquad \text{(Index notation)}$$

- Navier-Stokes equation

$$\rho(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) = -\nabla P + \mu \nabla^2 \boldsymbol{u} \qquad \text{(Gibbs notation)}$$

$$\rho(\frac{\partial u_i}{\partial t} + u_j \cdot \frac{\partial u_i}{\partial x_j}) = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \qquad \text{(Index notation)}$$

We will henceforth be denoting second order tensors with a double underbar as $\underline{\underline{A}}$ and vectors with a single underbar as $\underline{A}$. Some identities for products of tensors are given below

$$\underline{\underline{B}} \cdot \underline{a} = B_{ij} a_j$$
$$\underline{\underline{B}} \cdot \underline{\underline{C}} = B_{ik} C_{kj}$$
$$\underline{\underline{B}} : \underline{\underline{C}} = B_{ij} C_{ji}$$

The last of these is used in calculation of viscous dissipation which is given by $2\mu\underline{\underline{E}} \colon \underline{\underline{E}}$, where $\underline{\underline{E}}$ is a second order tensor given by

$$
\begin{aligned}
E_{ij} &= \frac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T) \\
&= \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})
\end{aligned}
$$

where the form highlights the symmetry of the strain-rate tensor. Two more important relations are

- Scalar triple product

  $$\underline{a} \cdot (\underline{b} \wedge \underline{c}) = \epsilon_{ijk} a_i b_j c_k$$

  which is the determinant of the matrix with $\underline{a}, \underline{b}$ and $\underline{c}$ as its columns.

- Trace of $\underline{\underline{B}} = B_{ii}$

2 Coordinate Transformation

2.1 Vectors

We want to find out how does the representation of a vector changes when the coordinate system is transformed into another coordinate system. We will mainly be concered with Orthogonal transformations. Mathematically, this means that, the transformation matrix L is such that

$$LL^T = I \Rightarrow L = L^{-1}$$

Physically, these transformations preserve angles between any two lines. Specifically, if two vectors were orthogonal before the transformation, then their transformed counterparts will also be orthogonal. Orthogonal transformations comprise of rotations and reflections of the coordinate system. The group of all possible rotations of a n dimenisonal coordinate system is known as the SO_n group (Special Orthogonal group in n dimenisons).

Note that, in 3D, if the transformation is a rotation, then the matrix representing such a transformation has one and only one real eigenvalue, which is equal to 1. This is because the rotation matrix modifies every vector in the domain except the vector along its axis which is left unrotated (hence it's an eigenvector) and unstretched (hence the eigenvalue corresponding to it is 1). Other eigenvalues are imaginary. These transformations preserve the length of the vectors and the angle between any two vectors. In 2D, all vectors are transformed and hence both the eigenvalues are imaginary.

Let L be such a tranformation in 3D that acts on the *orthonormal unit vectors* $[\underline{1}_1, \underline{1}_2, \underline{1}_3]$ and yields the transformed *orthonormal unit vectors* $[\underline{1'}_1, \underline{1'}_2, \underline{1'}_3]$. Then, we can write

$$\underline{1}'_i = l_{ij}\underline{1}_j$$

The second order tensor l_{ij} has nine elements, but they are not all indepenedent. The constraint of orthonormality of the transformed vectors gives six relations between the elements of l_{ij}

$$\begin{aligned} \underline{1}'_i \cdot \underline{1}'_j &= 0 \quad \text{if } i \neq j \text{ (ensures orthogonality)} \\ &= 1 \quad \text{if } i = j \text{ (ensures normality)} \end{aligned}$$

and hence a second order tensor in 3D, that represents an orthogonal transformation has only 3 independent parameters. For rotations, two of these are required to fix the axis and one more to specify the angle of rotation. For reflections, all three are needed to specify the plane of reflection.

For 2D, we get 3 constraints and hence the angle of rotation is the only parameter. What about 4 and higher dimensions? In a D dimensional space, we will have D constraints due to orthogonality and $\binom{D}{2}$ due to normality. This results in $D(D-1)/2$ independent parameters.

Once the transformation rule for the unit vectors is clear, the transformation for any given vector $\underline{a} = a_i\underline{1}_i$ can be written as

$$\underline{a}'_i = l_{ij}\underline{a}_j$$

$$\underline{a}_j = l_{ji}\underline{a}'_i$$

2.2 Second order tensors

Let there be a second order tensor $\underline{\underline{B}}$ with eigenvectors $[\underline{X}_1, \underline{X}_2, \underline{X}_3]$. We assume a symmetric second order tensor which allows us to use the fact that three orthogonal eigenvectors will exist. Then, such a tensor can be written as

$$\underline{\underline{B}} = [\lambda_1\underline{X}_1\underline{X}_1 + \lambda_2\underline{X}_2\underline{X}_2 + \lambda_3\underline{X}_3\underline{X}_3]$$

Let the transformed version of $\underline{B}$ be $\underline{B}'$ with eigenvectors $[\underline{X}'_1, \underline{X}'_2, \underline{X}'_3]$. Then,

$$\underline{\underline{B}}' = [\lambda_1\underline{X}'_1\underline{X}'_1 + \lambda_2\underline{X}'_2\underline{X}'_2 + \lambda_3\underline{X}'_3\underline{X}'_3]$$

Now we can use the trasformation rule for vectors $\underline{a}'_i = l_{ij}\underline{a}_j$ and transform the eigenvectors. This allows us to write down the final transformation rule in Gibbs notation as

$$\underline{\underline{B}}' = \underline{\underline{L}} \cdot \underline{\underline{B}} \cdot \underline{\underline{L^T}}$$

Or in index notation

$$\begin{aligned} B'_{ij} &= l_{ik}B_{km}l^T_{mj} \\ \Rightarrow B'_{ij} &= l_{ik}B_{km}l_{jm} \\ \Rightarrow B'_{ij} &= l_{ik}l_{jm}B_{km} \end{aligned}$$

The final expression is highly sugggestive of a pattern which will allow us to generalize this expression to higher order tensors

$$B'_{i_1 i_2 i_3 \ldots i_n} = l_{i_1 j_1} l_{i_2 j_2} l_{i_3 j_3} \ldots l_{i_n j_n} B_{j_1 j_2 j_3 \ldots j_n} \qquad \text{For the case of true tensors}$$

$$B'_{i_1 i_2 i_3 \ldots i_n} = Det(\underline{\underline{L}}) l_{i_1 j_1} l_{i_2 j_2} l_{i_3 j_3} \ldots l_{i_n j_n} B_{j_1 j_2 j_3 \ldots j_n} \qquad \text{For pseudo tensors}$$

Pseudo-tensors and pseudo-vectors are tensors which require a convention for their definition (such as the right hand screw rule). Their sign can change based on the convention. They do not directly represent physical quantities. E.g. vorticity, angular velocity and magnetic field. Another way to distinguish pseudo tensors is that they do not change direction upon reflection of the physical apparatus. A ball travelling to the right will be travelling to the left in the mirror; but a disc rotating clockwise is still rotating clockwise in the mirror. Its angular velocity vector doesn't flip sign.

2.3 Quotient Rule

A transformation or a physical law relating a vector to another vector must be mediated via a second order tensor; a law relating two second order tensors be mediated via a fourth order tensor and so on.

A law relating a vector to a second order tensor must be mediated by a third order tensor, so that two of the indices of the second and third order tensors can contract and result in a vector.

3 Isotropic Tensors

- Zeroth order
  All scalars are isotropic except psedo-scalars. Pseudo-scalars form upon contraction of a true vector with a pseudo-vector and their sign changes with convention.

- First order (Vectors)
  The Null vector is the only isotropic vector. We are looking for a vector that is impervious to any orthogonal transformation. Since every vector suffers a change of direction under some or the other rotation, the only vector that remains unaffected under all rotations is the vector without a direction.

- Second order tensors
  δ_{ij} and its scalar multiples are the only second order isotropic vector. Since any vector is an eigenvector of the identity matrix, it implies that there is no special direction for such a matrix. Hence any direction changing transformation such as a rotation will leave it unchanged.

- Third order tensors
  There is no true third order isotropic tensor. ϵ_{ijk} and its scalar multiples give a pseudo third order isotropic tensor.

- Third order tensors
  All higher order isotropic tensors are constructed using ϵ_{ijk} and δ_{ij}. E.g.
  $$D_{ijkl} = c_1\delta_{ij}\delta_{kl} + c_2\delta_{ik}\delta_{jl} + c_3\delta_{il}\delta_{jk}$$

Some examples of the use of isotropy in tensors are given below

- Fourier's law
  Fourier's law relates heat flux vector $\underline{q}$ to the temperature gradient vector ∇T
  $$q_i = K_{ij}\frac{\partial T}{\partial x_j}$$

  Here the thermal conductivity tensor K_{ij} is a second order tensor with 3 independent elements. However, for an isotropic material, the tensor will also be isotropic because it should not prefer any one direction over the other. Therefore, we can write $K_{ij} = \kappa\delta_{ij}$ and there remains only one independent parameter to be determined experimentally.

- Newton's law of viscosity

  This is a relation between the stress tensor τ_{ij} and the strain-rate tensor $\nabla \underline{u}$. Both of these are second order and hence the relation must be mediated by a fourth order tensor representing viscosity in some sense.

$$\tau_{ij} = D_{ijkl} \frac{\partial u_l}{\partial x_k}$$

  If viscosity doesn't change with direction in a medium, then $\underline{\underline{D}}$ must be an isotropic tensor. Hence we can write

$$D_{ijkl} = c_1 \delta_{ij} \delta_{kl} + c_2 \delta_{ik} \delta_{jl} + c_3 \delta_{il} \delta_{jk}$$

  But τ_{ij} is a symmetric tensor. Therefore, terms formed by interchange of i and j must be equal and hence c_2 must be equal to c_3. This gives us

$$D_{ijkl} = \left(c_1 \delta_{ij} \delta_{kl} + c_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \right) \frac{\partial u_l}{\partial x_k}$$
$$D_{ijkl} = c_1 \frac{\partial u_l}{\partial x_l} \delta_{ij} + c_2 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

  Here, c_1 is associated with the divergence and hence informs us of the resistance of the fluid to change in its volume. This is the bulk viscosity of the fluid.

  c_2 is associated with the strain-rate and tells about the resistance of the fluid to a shearing deformation (and also volume). This leads to shear viscosity.

Appendix

3.1 Expanded form for some index notation results

$$\begin{aligned}
\boldsymbol{a}\cdot\boldsymbol{b} &= (a_1\mathbf{1}_1 + a_2\mathbf{1}_2 + a_3\mathbf{1}_3)\cdot(b_1\mathbf{1}_1 + b_2\mathbf{1}_2 + b_3\mathbf{1}_3)\\
&= \sum_{i=1}^{3} a_i\mathbf{1}_i \cdot \sum_{j=1}^{3} b_j\mathbf{1}_j\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \mathbf{1}_i\cdot\mathbf{1}_j\\
&= \sum_{i=1}^{3} a_i b_i\\
&= a_i b_i
\end{aligned}$$

In the last step, we have used Einstein's summation convention which mandates that repeated indices are to be summed over. Henceforth, repeated indices will imply a summation over that index.

$$\begin{aligned}
\boldsymbol{a}\wedge\boldsymbol{b} &= (a_1\mathbf{1}_1 + a_2\mathbf{1}_2 + a_3\mathbf{1}_3)\wedge(b_1\mathbf{1}_1 + b_2\mathbf{1}_2 + b_3\mathbf{1}_3)\\
&= \sum_{i=1}^{3} a_i\mathbf{1}_i \wedge \sum_{j=1}^{3} b_j\mathbf{1}_j\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \mathbf{1}_i\wedge\mathbf{1}_j\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \epsilon_{ijk}\mathbf{1}_k\\
&= \epsilon_{ijk} a_i b_j\\
&= \epsilon_{ijk} a_j b_k
\end{aligned}$$

Where we have just relabelled the indices in the last step so that the first index i remains the direction specifying index.

3.2 Area under a Gaussian

The area under a suitably defined Gaussian is independent of its standard deviation.

3.3 Gradient vector in cylindrical and spherical coordinates

In any coordinate system, the gradient vector can be written immidiately if we incorporate the metric factors relevant to the coordinate system. Including the metric factor converts a small change in a coordinate variable to a small change in length along that coordinate (E.g.$d\theta$ to $rd\theta$).

3.4 Proof for vector transformations

$$\underline{a}_i' = l_{ij}\underline{a}_j$$

$$\underline{a}_j = l_{ji}\underline{a}_i'$$

3.5 Symmetric matrices have orthogonal eigenvectors

3.6 Proof for the quotient rule

3.7 Proof that δ_{ij} and ϵ_{ijk} are isotropic