Velocity Field From Dilatation And Vorticity Fields

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Overview

The aim of this lecture is to construct the velocity field when the rate of dilatation and the vorticity at each point in the domain has been specified.

1 Problem setup

We wish to construct the velocity field $\underline{u}$, when the rate of dilation and the vorticity has been specified at each point in the domain. Therefore, we have

$$\begin{aligned} \nabla \wedge \underline{u} &= \Delta(\underline{X}) \\ \nabla \cdot \underline{u} &= \omega(\underline{X}) \end{aligned}$$

We begin by dividing the problem into three subproblems. We write

$$\underline{u} = \underline{u}_e + \underline{u}_v + \underline{v}$$

where the velocity field $\underline{u}_e$ is such that

$$\begin{aligned} \nabla \wedge \underline{u}_e &= 0 \\ \nabla \cdot \underline{u}_e &= \Delta(\underline{X}) \end{aligned}$$

whereas $\underline{u}_v$ is such that

$$\begin{aligned} \nabla \wedge \underline{u}_v &= \omega(\underline{X}) \\ \nabla \cdot \underline{u}_v &= 0 \end{aligned}$$

and finally $\underline{v}$ follows

$$\begin{aligned} \nabla \wedge \underline{v} &= 0 \\ \nabla \cdot \underline{v} &= 0 \end{aligned}$$

Therefore, $\underline{u}_e$ is irrotational, $\underline{u}_v$ is solenoidal and $\underline{v}$ is irrotational incompressible (i.e. potential flow).

2 Velocity field due to dilatation

Since

$$\begin{aligned}
&\nabla \wedge \underline{u}_e = 0 \\
\Rightarrow &\underline{u}_e = \nabla \phi_e \\
\Rightarrow &\nabla^2 \phi_e = \Delta(\underline{X}) && \big(\text{using } \nabla \cdot \underline{u}_e = \Delta(\underline{X})\big) \\
\Rightarrow &\phi_e(\underline{X}) = \int \frac{-\Delta(\underline{X}')dX'}{4\pi|\underline{X} - \underline{X}'|} && \big(\text{using Green's function for the Laplacian}\big) \\
\Rightarrow &\underline{u}_e = \nabla \phi_e = \int \frac{(\underline{X} - \underline{X}')}{4\pi|\underline{X} - \underline{X}'|^3}\Delta(\underline{X}')dX' && \big(\text{using } \nabla 1/r = -x_i/r^3\big)
\end{aligned}$$

which just states that velocity at any point($\underline{X}$) is the sum of velocities induced at that point due to all other points($\underline{X}'$) in the domain. Note that $\underline{X}$ denotes the position at which velocity is being specified whereas $\underline{X}'$ denotes the position of the source which is inducing velocity at $\underline{X}$.

Now assume that $r = |\underline{X} - \underline{X}'|$ is much larger than the length scale over which sources are distributed. Let's say the sources are distributed over a length scale $L \ll r$. The vector charactersing the position of source is $\underline{X}'$ and hence it follows that $|\underline{X}'| \ll |\underline{X}|$. Therefore $\underline{X} - \underline{X}' \sim \underline{X}$, which gives us

$$\begin{aligned}
&\int \frac{(\underline{X} - \underline{X}')}{4\pi|\underline{X} - \underline{X}'|^3}\Delta(\underline{X}')dX' \sim \frac{\underline{X}}{4\pi|\underline{X}|^3}\int \Delta(\underline{X}')dX' \\
\Rightarrow &\underline{u}_e = \frac{\underline{X}}{4\pi|\underline{X}|^3}\int \Delta(\underline{X}')dX'
\end{aligned}$$

This states that, when seen from sufficiently far away, velocity due to the source distribution is not dependent upon the specific distribution but only on the integral $\int \Delta(\underline{X}')d\underline{X}'$ over the entire distribution. This integral signifies the net strength of the source distribution. But what if the source is such that it has equal positive and negative contributions, such as a dipole or quadrupole? In such cases $\int \Delta(\underline{X}')d\underline{X}' = 0$. Then assuming $\underline{X} - \underline{X}' \sim \underline{X}$ is not a useful approximation and we must expand to higher orders as follows:

$$\frac{\underline{X} - \underline{X}'}{|\underline{X} - \underline{X}'|^3} = \frac{\underline{X}}{|\underline{X}|^3} - \underline{X}' \cdot \nabla \frac{\underline{X}}{|\underline{X}|^3} + \frac{\underline{X}'\underline{X}'}{2!} : \nabla\nabla \frac{\underline{X}}{|\underline{X}|^3} - ...$$

Substituting this in the expression for velocity, we get

$$\begin{aligned}
&\underline{u}_e = \int \frac{(\underline{X} - \underline{X}')}{4\pi|\underline{X} - \underline{X}'|^3}\Delta(\underline{X}')dX' \\
\Rightarrow &\underline{u}_e = \frac{1}{4\pi}\Bigg[\int \Delta(\underline{X}')\left(\frac{\underline{X}}{|\underline{X}'|^3}\right)d\underline{X}' - \int \Delta(\underline{X}')\left(\underline{X}' \cdot \nabla \frac{\underline{X}}{|\underline{X}|^3}\right)d\underline{X}' + \\
&\int \Delta(\underline{X}')\left(\frac{\underline{X}'\underline{X}'}{2!} : \nabla\nabla \frac{\underline{X}}{|\underline{X}|^3}\right)d\underline{X}' + ...\Bigg]
\end{aligned}$$

where the first term represents field due to a monopole, second is due to a dipole, third due to quadrupole and so on. This is known as the Multipole Expansion. Each of the terms in the above expansion individually satisfy the governing equations $\nabla^2\phi_e = \Delta(\underline{X})$ giving us the *singular solutions* to the problem i.e. the fields due to monopole, dipole and so on.

3 Velocity field due to vorticity

The problem statement for this case is

$$\begin{aligned}\nabla \cdot \underline{u}_v &= 0\\ \nabla \wedge \underline{u}_v &= \underline{\omega}(\underline{x})\end{aligned}$$

The first of these equations allows us to introduce the vector potential $\underline{B}$ as follows

$$\begin{aligned}&\nabla \wedge \underline{B} = \underline{u}_v\\ \Rightarrow &\nabla \wedge \nabla \wedge \underline{B} = \underline{\omega}(\underline{x})\\ \Rightarrow &\nabla^2 \underline{B} = -\underline{\omega}(\underline{x})\\ \Rightarrow &\underline{B} = \int \frac{\underline{\omega}(\underline{x}')d\underline{x}'}{4\pi|\underline{x}-\underline{x}'|} && \text{(Using Green's function)}\\ \Rightarrow &\underline{u} = \nabla \wedge \underline{B} = \frac{1}{4\pi}\int \frac{\underline{\omega}(\underline{x}') \wedge (\underline{x}-\underline{x}')d\underline{x}'}{|\underline{x}-\underline{x}'|^3}\end{aligned}$$

It must be noted that in order to arrive at this result, we have imposed $\nabla \cdot \underline{B} = 0$. This implies

$$\begin{aligned}\nabla \cdot \underline{B} &= \nabla \cdot \int \frac{\underline{\omega}(\underline{x}')d\underline{x}'}{4\pi|\underline{x}-\underline{x}'|}\\ &= \int_{\partial S} \frac{dS(\underline{x}')\underline{\omega}\cdot\underline{n}}{4\pi|\underline{x}-\underline{x}'|}\end{aligned}$$

which is zero iff

$$\underline{\omega}\cdot\underline{n} = 0 \quad \forall \quad \underline{x}' \in \partial S$$

where ∂S is the surface which bounds the domain. Therefore, the normal component of vorticity on the surface must be zero. This implies that vortex lines must either close on to themselves or continue to infinity without crossing the boundary of the domain.

Another important consequence of this result is that it precludes vortex monopoles. A vortex existing at just one in the domain will not satisfy $\underline{\omega}\cdot\underline{n} = 0$ through a surface passing through that point. Hence vorticity cannot exist at just one point in the domain.

3.1 Line Vortex

A line vortex is a singular structure where entire vorticity is concentrated in an infinitesimal radius around a line. The Biot-Savart law can be specialized for the line vortex by considering the limit of $\omega \to \infty$, $dA \to 0$ while keeping the circulation $\kappa = \int \underline{\omega} \cdot \underline{n} dA$ a constant (where $d\underline{A} \perp \underline{\omega}$ is the infinitesimal area perpendicular to local vorticity). This gives us the velocity field due to the line vortex to be

$$\begin{aligned} \underline{u} = \nabla \wedge \underline{B} &= \frac{1}{4\pi} \int \frac{\underline{\omega}(\underline{x}') \wedge (\underline{x} - \underline{x}')dV}{|\underline{x} - \underline{x}'|^3} \\ \Rightarrow \underline{u} &= \frac{1}{4\pi} \int dA(\underline{x}')dl(\underline{x}') \frac{|\omega(\underline{x}')|\underline{t} \wedge (\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3} \\ \Rightarrow \underline{u} &= \frac{\kappa}{4\pi} \int \frac{d\underline{l}(\underline{x}') \wedge (\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3} \end{aligned}$$

Therefore, velocity due to a line vortex, at a point $\underline{x}$, due to a point $\underline{x}'$ on the line vortex, is given by

$$\boxed{\underline{u} = \frac{\kappa}{4\pi} \int \frac{d\underline{l}(\underline{x}') \wedge (\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3}}$$

4 Appendix

4.1 Constructing solution to the inhomogeneous problem using the Green's function

5 References

- An Introduction to Fluid Dynamics by G. K. Batchelor, section 2.4