Lecture 11: Self Gravitating Equilibria

10^{th} Sept, 2018

Overview

1 Self-gravitating equilibrium for a spherical planet

The gravitational potential Ψ_G due to a small mass dm is

$$d\Psi_G = -\frac{Gdm}{|\underline{x} - \underline{x}_0|}$$

And hence for a body of volume V, the potential is

$$\Psi_G = -\int_V \frac{G\rho_0 dV}{|\underline{x} - \underline{x}_0|}$$

For the potential in the exterior of a spherical body, this integral is trivial

$$\begin{aligned}
\Psi_G &= -\int_V \frac{G\rho_0 dV}{|\underline{x} - \underline{x}_0|} \\
&= -\frac{G\rho_0}{r}\int_V dV \\
&= -\frac{GM}{r}
\end{aligned}$$

This gives us the familiar result that potential (and field) at an external point can be calculated by assuming the entire mass M to be concentrated at the center of the body.

In order to calculate the potential in the interior region of a spherical body, we must perform the integral $\int_V \frac{dV}{r} = \int_V \frac{dV}{|\underline{x}-\underline{x}_0|}$ in spherical coordinates. The result we get is

$$\Psi_G = -2\pi\rho_0 G(R^2 - \frac{r^2}{3})$$

At $r = R$ both interior and exterior regions give the same potential, so the result seems consistent.

Since potential in the interior region is quadratic in r, and force is gradient of the potential, the force will be linear in r. Since $r = |\underline{x} - \underline{x}_0|$, it implies that the force is only a function of $\underline{x}$ (since the center $\underline{x}_0$ is fixed). The force is given by

$$F_G = -\frac{4\pi}{3}\rho_0 G\underline{x}$$

This is the integral approach for obtaining the force due to a gravitating sphere. There is also a differential approach that utilises the idea of Green's function of a Laplacian. Recall that the Green's Function of a Laplacian in 3D is given by $G(\underline{x}) = -\frac{1}{4\pi|\underline{x}-\underline{x}_0|}$. Using the equation of the potenial, taking the Laplacian and using the Green's Function, we can obtain the same result as follows:

$$\begin{aligned}
\Psi_G &= -\int_V \frac{G\rho_0 dV}{|\underline{x}-\underline{x}_0|} \\
\nabla^2\Psi_G &= 4\pi G\rho_0 \int_V \nabla^2\left(\frac{-1}{4\pi|\underline{x}-\underline{x}_0|}\right)dV \\
\nabla^2\Psi_G &= 4\pi G\rho_0 \int_V \delta(\underline{x}-\underline{x}_0)dV && \text{(by defintion of Green's Function)} \\
\nabla^2\Psi_G &= 4\pi G\rho_0 && \text{(by defintion of Delta Function)} \\
\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\Psi_G}{dr} &= 4\pi G\rho_0 \\
r^2\frac{d\Psi_G}{dr} &= \frac{4\pi G\rho_0 r^3}{3} \\
\frac{d\Psi_G}{dr} &= \frac{4\pi G\rho_0 r}{3} \\
-\underline{F} &= \frac{4\pi}{3}G\rho_0 r\underline{1}_r \\
-\underline{F} &= \frac{4\pi}{3}G\rho_0(\underline{x}-\underline{x}_0)
\end{aligned}$$

This is a radial(directed towards the center) harmonic force. The harmonic nature can be easily shown

$$\begin{aligned}
\nabla^2\Psi_G &= 4\pi G\rho_0 \\
\nabla(\nabla^2\Psi_G) &= 0 && \text{(taking } \nabla \text{ on both sides)} \\
\nabla^2\nabla\Psi_G &= 0 && (\nabla \text{ and } \nabla^2 \text{ commute}) \\
\nabla^2\underline{F} &= 0 && (F = \nabla\Psi_G)
\end{aligned}$$

To summarize,

- Interior: $\underline{F} \propto r \quad V \propto r^2$
- Exterior: $\underline{F} \propto 1/r^2 \quad V \propto 1/r$

The fact that force has a linear dependence on $\underline{x}$ can also be seen using the fact that $\underline{F}$ is harmonic. Since $\nabla^2 \underline{F} = 0$, it implies that $\underline{F}$ can be represented using the solution of the Laplacian. We know that if $\nabla^2 \Phi = 0$, then

$$\Phi = \underbrace{F(r)}_{\propto r^n, \frac{1}{r^{n+1}}} \quad \underbrace{G(\cos(\theta))}_{P_n^m(\cos(\theta))} \quad \underbrace{H(\phi)}_{e^{\pm im\phi}}$$

where n is the order of tensor involved. The solution constructed using r^n are growing harmonics and the ones constructed using $1/r^{n+1}$ are the decaying harmonics. Since the (θ, ϕ) dependence is same, we can always convert a decaying harmonic to a growing harmonic by multiplying it with r^{2n+1}.

Now we know that the vector solution to the Laplace equation is proportional to $\underline{x}/r^3$. This is a decaying harmonic. So we multiply it by $r^{2(1)+1} = r^3$ to get the growing harmonic $\underline{x}$. Therefore the force $\underline{F}$ will be in general proportional to a linear combination of the two solutions. But we want the field inside the gravitating body, so the the force at the center must not be singular. Therefore $\underline{x}/r^3$ is rejected and we get $\underline{F} \propto \underline{x}$. So $\underline{F}$ must be linear in $\underline{x}$.

1.1 Gravitation pressure

The gravitational field of a planet pulls it inwards. This causes a pressure which keeps it from shrinking to a point. This pressure can be calculated as follows

$$\begin{aligned}
&\nabla p = -\rho_0 \nabla \Psi_G \\
&\Rightarrow p = p_0 - \rho_0 \Psi_G \\
&\Rightarrow p = p_0 + 2\pi\rho_0^2 G(R^2 - r^2/3) \\
&\Rightarrow p = \hat{p_0} - 2\pi\rho_0^2 G r^2/3
\end{aligned}$$

Pressure at the surface $(r = R)$ will be extremely small (only due to the atmosphere) compared to pressure in layers inside the planet. We can assume it to be zero. Therefore, $p|_{r=R} = 0$. Using this we get,

$$\begin{aligned}
\hat{p_0} &= 2\pi\rho_0^2 G R^2/3 \\
\Rightarrow p &= \frac{2\pi\rho_0^2 G}{3}(R^2 - r^2)
\end{aligned}$$

This is the pressure which keeps a self-gravitating body from shrinking to a point.

2 Self-gravitating equilibrium with rotation

The transformation between two first order tensors must be mediated by a second order tensor. For a sphere, we obtained the relation $\underline{F} = -\nabla\Psi = C\delta_{ij}\underline{x}$. For a spheroid with axis along $\underline{p}$, this becomes $\nabla\Psi = (C_1 p_i p_j + C_2\delta_{ij})x_j$.

For the case of a rotating self-gravitating body, the eccentricity (e) is a function of ratio of rate of rotation and gravity. Therefore, our aim is to find

$$\frac{\Omega^2 R}{g} = f(e)$$

If we put $g = GM/R^2$ and $M = \rho_0 R^3$, then the equation becomes

$$\frac{\Omega^2}{\rho_0 G} = f(e)$$

This function turns out to be

$$\frac{\Omega^2}{2\pi\rho_0 G} = e^2(1-e^2)^{1/2}\int_0^\infty \frac{\lambda d\lambda}{(1+\lambda)^2(1-e^2+\lambda)}$$

which indicates that for any given rate of rotation, there are two possible equilibrium eccentricities. For small rates of rotation (as for most planets), one of these eccentricities $e \sim 1$ which signifies a nearly flat pancake like body, while the other eccentricity $e \sim 0$ which is indicative of a nearly spherical planet.

Beyond a critical rate of rotation, there do not exist any real roots i.e. centrifugal forces dominate and an equilibrium between centrifugal and gravitational forces is never established.

3 Appendix

3.1 Constant field inside a gouged out sphere

(figures required)

3.2 Derivation for Self-gravitating equilibrium with rotation