Part III

Appendix

Appendix A

Numerical Issues and Computer Programs

A.1 Global versus Local Methods

Many problems in hydrodynamic stability theory result in eigenvalue problems. The asymptotic growth/decay rates are the imaginary part of the eigenvalues of the underlying stability operator, for an analysis of the transient behavior we found it advantageous to expand in eigenfunctions, the computation of the critical energy Reynolds number requires the solution of an eigenvalue problem, and the secondary growth rates for breakdown of vortices, Tollmien-Schlichting waves or streaks are the result of an eigenvalue computation.

From a numerical point of view, there are two general concepts of finding the eigenvalues of a discretized stability operator. The first method, the local method, starts with an initial guess for the eigenvalue and solves the original eigenvalue problem as an initial value problem. The initial guess is then adjusted until the boundary conditions are satisfied. The iteration procedure can be accelerated by common root finding methods. The second method, the global method, uses the fully discretized stability operator and supplies it to a matrix eigenvalue solver which results in the spectrum.

The advantage of the local method lies in its high accuracy, whereas the advantage of the global method lies in the fact that a large part of the spectrum rather than one eigenvalue is computed.

Below we will describe algorithms that have proven useful in the computation of eigenvalues for hydrodynamic stability problems.

A.2 Runge-Kutta Methods

A numerical method for the solution of the two-dimensional Orr-Sommerfeld equation will now be presented. With the use of Squire's transformation this can also be used to solve the three-dimensional version. For simplicity the assumption that the flow is contained between two parallel plates located at $y = \pm 1$ will be made. The equation together with the suitable boundary condition can be written

$$\Phi' = A(y)\Phi \tag{A.1}$$

$$\Phi_1(\pm 1) = \Phi_2(\pm 1) = 0 \tag{A.2}$$

where

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{pmatrix} = \begin{pmatrix} \hat{v} \\ \hat{v}' \\ \hat{v}'' \\ \hat{v}''' \end{pmatrix} \tag{A.3}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & 0 & b & 0 \end{pmatrix} \tag{A.4}$$

$$a = -i\alpha\mathrm{Re}(\alpha^2(U - c) + U'') - \alpha^4 \tag{A.5}$$

$$b = i\alpha\mathrm{Re}(U - c) + 2\alpha^2. \tag{A.6}$$

The general solution to the equation can be written

$$\Phi(y) = \gamma_1\Phi^1 + \gamma_2\Phi^2 + \gamma_3\Phi^3 + \gamma_4\Phi^4 \tag{A.7}$$

where $\Phi^1, \Phi^2, \Phi^3, \Phi^4$ are the four linearly independent solutions and γ_i are the integration constants. Φ^i are chosen to satisfy

$$[\Phi^1(-1), \Phi^2(-1), \Phi^3(-1), \Phi^4(-1)] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{A.8}$$

Applying the boundary conditions at $y = -1$ and using the above conditions on the linearly independent parts of the solution we find that $\gamma_3 = \gamma_4 = 0$. From the boundary conditions at $y = 1$ we find

$$\begin{pmatrix} \Phi_1^1(1) & \Phi_1^2(1) \\ \Phi_2^1(1) & \Phi_2^2(1) \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = 0. \tag{A.9}$$

This means that the determinant of the matrix has to be zero for solutions to exist, i.e.,

$$d = \Phi_1^1(1)\Phi_2^2(1) - \Phi_1^2(1)\Phi_2^1(1) = 0. \tag{A.10}$$

c must be chosen so that the dispersion relation above is satisfied.
The eigenvalue c can be found by a shooting procedure. A particular value of c is chosen, c_1 say, Φ^1 and Φ^2 are found by integration over y and the determinant is evaluated, d_1 say. This is done again for another value of c, c_2 say, resulting in another value, d_2, of the determinant. If these values are sufficiently close to a correct eigenvalue, a better approximation can be found by linear extrapolation:

$$c = c_2 - d_2 \frac{c_1 - c_2}{d_1 - d_2}. \tag{A.11}$$

This process can be repeated until the desired accuracy is obtained.
The integration method used is a fourth-order Runge-Kutta method which is defined as follows

$$\begin{aligned} g_1 &= hA(y)\Phi(y) && \text{(A.12)} \\ g_2 &= hA(y + h/2)(\Phi(y) + g_1/2) && \text{(A.13)} \\ g_3 &= hA(y + h/2)(\Phi(y) + g_2/2) && \text{(A.14)} \\ g_4 &= hA(y + h)(\Phi(y) + g_3) && \text{(A.15)} \\ \Phi(y + h) &= \Phi(y) + (g_1 + 2g_2 + 2g_3 + g_4)/6 && \text{(A.16)} \end{aligned}$$

where $h = h(y)$ is the discretization step. For increased accuracy the step size h is not a constant; instead the grid points can be uniformly spaced in terms of the stretched coordinate $z = y \exp(y^2 - 1)$.
Because the matrix elements vary over several orders of magnitude the two linearly independent solutions Φ^1 and Φ^2 become increasingly collinear which will result in a loss of accuracy when the upper boundary conditions are applied. By orthonormalizing Φ^1 and Φ^2 this problem can be overcome. We will follow Conte (1966) in the outline of the orthonormalization process.

First we define the inner product in the usual manner

$$(\Phi^1, \Phi^2) = \sum_{j=1}^{4} \Phi_j^1 \Phi_j^{2*} \tag{A.17}$$

where the superscript * denotes the complex conjugate operation. The orthonormalization is done using the Gram-Schmidt procedure

$$\Psi^1 = \Phi^1 / \sqrt{(\Phi^1, \Phi^1)} \tag{A.18}$$
$$t = \Phi^2 - (\Phi^2, \Psi^1)\Psi^1 \tag{A.19}$$
$$\Psi^2 = t / \sqrt{(t, t)} \tag{A.20}$$

where Ψ^j $(j = 1, 2)$ are the new orthonormalized solution vectors. The above operation can be written in matrix form in the following way

$$[\Psi^1, \Psi^2] = [\Phi^1, \Phi^2]P \tag{A.21}$$

where

$$P = \begin{pmatrix} p_{11} & p_{12} \\ 0 & p_{22} \end{pmatrix} \tag{A.22}$$

$$p_{11} = 1 / \sqrt{(\Phi^1, \Phi^2)} \tag{A.23}$$
$$p_{12} = -(\Phi^2, \Psi^1) / (\sqrt{(\Phi^1, \Phi^1)} \sqrt{(t, t)}) \tag{A.24}$$
$$p_{22} = 1 / \sqrt{(t, t)} \tag{A.25}$$

After the orthonormalization the new values Ψ^j are used as initial conditions. The above procedure can also be used when a forcing term is present. In the last step for this case the homogeneous part will be subtracted from the particular solution. However, the new particular solution should not be normalized since the forcing term itself is not rescaled. The equation for the horizontal disturbance velocity parallel to the wave front is a second-order inhomogeneous differential equation and will thus also have two solution vectors, the particular solution Φ^0 and the homogeneous solution vector Φ^1, (the other solution has been set to zero using the boundary condition at $y = -1$). If orthonormalization of these solutions is necessary Φ^0 can be substituted for Φ^2 in the expressions above; in this case Ψ^0 should not be normalized.

The eigenvalue will be independent of the orthonormalization process since we are only changing the initial conditions to another set of linearly independent vectors. However, to obtain the eigenfunctions we have to make use of the matrices P. Let us call the current discretized solution vectors u^1 and u^2. Let y_n denote the points used in the orthonormalizations, let P_n stand for the corresponding matrices and call the total number of orthonormalizations N. We start by orthonormalizing at the endpoints which results in

$$[\Psi^1(1), \Psi^2(1)] = [u^1(1), u^2(1)]P_{N+1}. \tag{A.26}$$

Next, we apply the upper boundary condition which gives

$$\Psi(1) = [\Psi^1(1), \Psi^2(1)]\gamma \tag{A.27}$$

where γ is the two component column vector determined from (A.9). For the solution at locations between the endpoint and the previous orthonormalization (i.e., $y : y_N \leq y \leq y_{N+1}$) we apply the same transformation and obtain

$$\begin{aligned} \Psi(y) &= [u^1(y), u^2(y)]P_{N+1}\gamma, \qquad y_N \leq y \leq y_{N+1} && \text{(A.28)} \\ &= [u^1, u^2]\gamma_N && \text{(A.29)} \end{aligned}$$

with $\gamma_N = P_{N+1}\gamma$. The recovered solution between the two orthonormalizations, y_{n-1} and y_n say, will thus be

$$\begin{aligned} \Psi(y) &= [u^1, u^2]\gamma_{n-1}, \qquad y_{n-1} \leq y \leq y_n && \text{(A.30)} \\ \gamma_{n-1} &= P_n P_{n-1} \cdots P_{N+1}\gamma = P_n \gamma_n. && \text{(A.31)} \end{aligned}$$

By iterating backwards in this manner the entire solution can be recovered. The angle between the solution vectors could be used as a criterion for orthonormalization, but the following somewhat simpler criterion presents a viable alternative

$$\max\left(\sqrt{(u^1, u^1)}, \sqrt{(u^2, u^2)}\right) > M \tag{A.32}$$

where M in our calculations has been set to 500.

A.3 Chebyshev Expansions

Spectral methods have had a significant impact on the accurate discretization of both initial value and eigenvalue problems. Especially in a bounded domain, the use of Chebyshev polynomials has been advantageous. Most of the stability calculations shown in this book have been obtained by a Chebyshev discretization of the inhomogeneous coordinate direction. Below we will present the most useful relations for the discretization of derivatives and integrals.

Chebyshev polynomials can be defined in many ways, for example, in terms of trigonometric functions

$$T_n(y) = \cos(n \cos^{-1}(y)), \tag{A.33}$$

as solutions of the singular Sturm-Liouville problem

$$\frac{d}{dy}\left(\sqrt{1-y^2}\frac{d}{dy}T_n(y)\right) + \frac{n^2}{\sqrt{1-y^2}}T_n(y) = 0, \tag{A.34}$$

in terms of a recurrence relation

$$T_0(y) = 1, \quad T_1(y) = y, \quad T_{n+1}(y) = 2yT_n(y) - T_{n-1}(y), \tag{A.35}$$

by a direct formula

$$T_n(y) = \frac{1}{2}\left[(y+\sqrt{y^2-1})^n + (y-\sqrt{y^2-1})^n\right], \tag{A.36}$$

or by the so-called Rodrigues' formula

$$T_n(y) = \frac{(-1)^n 2^n n!}{(2n)!}\sqrt{1-y^2}\frac{d^n}{dy^n}\left[(1-y^2)^{n-(1/2)}\right]. \tag{A.37}$$

For numerical purposes the definition in terms of trigonometric functions is most practical. Chebyshev polynomials satisfy an orthogonality condition of the form

$$\int_{-1}^{1} \frac{T_n(y)T_m(y)}{\sqrt{1-y^2}}\, dy = C_n \delta_{nm} \qquad C_0 = \pi, \quad C_n = \frac{\pi}{2} \quad (n \neq 0). \tag{A.38}$$

We will approximate the dependent variables by a Chebyshev expansion

$$f(y) = \sum_{n=0}^{N} a_n T_n(y) \tag{A.39}$$

and evaluate the Chebyshev polynomials at the extrema of the N-th Chebyshev polynomial given as

$$y_j = \cos(\frac{j\pi}{N}). \tag{A.40}$$

These locations are also known as the Gauss-Lobatto points.
When discretizing ordinary or partial differential equations, derivatives of the solution are needed as well. These derivatives have to be expressed in terms of Chebyshev polynomials and the following recurrence relation between Chebyshev polynomials and their derivatives is used.

$$T_0^{(k)}(y_j) = 0, \tag{A.41}$$

$$T_1^{(k)}(y_j) = T_0^{(k-1)}(y_j), \tag{A.42}$$

$$T_2^{(k)}(y_j) = 4T_1^{(k-1)}(y_j), \tag{A.43}$$

$$T_n^{(k)}(y_j) = 2nT_{n-1}^{(k-1)}(y_j) + \frac{n}{n-1}T_{n-1}^{(k)}(y_j) \quad n = 3, 4, \ldots . \tag{A.44}$$

with the superscript $k \geq 1$ denoting the order of differentiation.

The evaluation of scalar products requires formulas for the integration of functions that are represented as Chebyshev expansions on a Gauss-Lobatto grid. These formulas ensure spectral accuracy. We have

$$\int_{-1}^{1} f(y)\, dy = \sum_{j=0}^{N} f(y_j)W(y_j), \tag{A.45}$$

with $W(y_j)$ as the Chebyshev integration weight function which can be found as follows

$$f(y) = \sum_{n=0}^{N} a_n T_n(y) = \int_{n=0}^{N} c_n T_n(y) \sum_{j=0}^{N} \frac{b_j}{N} f(y_j)T_n(y_j). \tag{A.46}$$

Integrating this expression with respect to y yields

$$\int_{-1}^{1} f(y)\, dy = \frac{1}{N}\sum_{j=0}^{N} b_j f(y_j) \sum_{n=0}^{N} c_n T_n(y_j) \int_{-1}^{1} T_n(y)\, dy. \tag{A.47}$$

We have

$$\int_{-1}^{1} T_n(y)\, dy = \frac{2}{1-n^2} \qquad n \text{ even} \tag{A.48}$$

which results in

$$W(y_j) = \frac{b_j}{N}\left\{2 + \sum_{n=2}^{N} c_n \frac{1+(-1)^n}{1-n^2}\cos(\frac{nj\pi}{N})\right\}. \tag{A.49}$$

If a mapping $y = y(\hat{y})$ is used to transform the physical domain into the Chebyshev interval, the formula gets slightly modified

$$W(y_j) = \frac{b_j}{N}\sum_{n=0}^{N} c_n \cos\left(\frac{nj\pi}{N}\right)\int_{-1}^{1} T_n(y)\frac{d\hat{y}}{dy}\, dy. \tag{A.50}$$

A.4 Infinite Domain and Continuous Spectrum

When solving stability problems on a semi-infinite or infinite domain, a mapping of the domain onto a finite domain is necessary before Chebyshev polynomials can be applied. The most frequently used mappings of a semi-infinite to a finite domain are algebraic

$$x = \frac{1+\xi}{1-\xi} \qquad \xi = \frac{x-1}{x+1} \tag{A.51}$$

or exponential

$$x = -\ln(\frac{1-\xi}{2}) \qquad \xi = 1 - 2e^{-x}. \tag{A.52}$$

Investigations of different mappings have shown that algebraic maps are, in practice, more accurate and more robust than exponential ones.
To achieve the highest possible accuracy, it is sometimes important to control the number of points in a specified subsection of the physical domain. A rational map for the semi-infinite domain has proven successful in distributing grid points such that the near-wall region is sufficiently resolved. The map

$$y = a\frac{1+\hat{y}}{b-\hat{y}} \tag{A.53}$$

with

$$a = \frac{y_i y_{\max}}{y_{\max} - 2y_i} \qquad b = 1 + \frac{2a}{y_{\max}} \tag{A.54}$$

clusters grid points near the wall resulting in the accurate resolution of near-wall shear layers, and distributes half of the grid points in the interval $0 \le y \le y_i$.

Infinite intervals can be mapped to a finite domain using the map

$$x = -\cot(\xi/2) \tag{A.55}$$

in conjunction with a Fourier series.

In conjunction with Chebyshev polynomials the mappings

$$x = \frac{\xi}{\sqrt{1-\xi^2}} \tag{A.56}$$

or

$$x = \tanh^{-1} \xi \tag{A.57}$$

are most common for infinite domains..

A.5 Chebyshev Discretization of the Orr-Sommerfeld Equation

In this section we will present a spectral collocation method based on Chebyshev polynomials and apply it to the Orr-Sommerfeld equation. This method has been used extensively to compute the stability characteristics of shear flows presented throughout this book. The method is highly accurate and easy to implement.

The Orr-Sommerfeld equation reads

$$\left(-Uk^2 - U'' - \frac{k^4}{i\alpha\mathrm{Re}}\right)\tilde{v} + \left(U + \frac{2k^2}{i\alpha\mathrm{Re}}\right)\mathcal{D}^2\tilde{v} - \frac{1}{i\alpha\mathrm{Re}}\mathcal{D}^4\tilde{v} = c\left(\mathcal{D}^2\tilde{v} - k^2\tilde{v}\right) \tag{A.58}$$

with the boundary conditions

$$\tilde{v}(\pm 1) = \mathcal{D}\tilde{v}(\pm 1) = 0. \tag{A.59}$$

We will expand the eigenfunctions in a Chebyshev series

$$\tilde{v}(y) = \sum_{n=0}^{N} a_n T_n(y). \tag{A.60}$$

The derivatives of the eigenfunctions are obtained by differentiating the expansion above. We obtain for the second derivative, for example,

$$\mathcal{D}^2 \tilde{v}(y) = \sum_{n=0}^{N} a_n T_n''(y) \tag{A.61}$$

and similarly for the fourth derivative. Upon substitution into the Orr-Sommerfeld equation we get

$$\left(U(y)k^2 - U''(y) - \frac{k^4}{i\alpha \mathrm{Re}} \right) \sum_{n=0}^{N} a_n T_n(y) + \left(U(y) + \frac{2k^2}{i\alpha \mathrm{Re}} \right) \sum_{n=0}^{N} a_n T''(y)$$
$$- \frac{1}{i\alpha \mathrm{Re}} \sum_{n=0}^{N} a_n T''''(y) = c \left(\sum_{n=0}^{N} a_n T_n''(y) - k^2 \sum_{n=0}^{N} a_n T_n(y) \right). \tag{A.62}$$

We then require this equation to be satisfied at the Gauss-Lobatto collocation points $y_j = \cos(\pi j/N)$. This allows us to use the recurrence relations (A.41-A.44) to evaluate the derivatives of the Chebyshev polynomials.

The discretized boundary conditions read

$$\sum_{n=0}^{N} a_n T_n(1) = 0 \qquad \sum_{n=0}^{N} a_n T_n(-1) = 0 \tag{A.63}$$

$$\sum_{n=0}^{N} a_n T_n'(1) = 0 \qquad \sum_{n=0}^{N} a_n T_n'(-1) = 0. \tag{A.64}$$

The final result is a generalized eigenvalue problem of the form

$$\mathbf{A}\,\mathbf{a} = c\,\mathbf{B}\,\mathbf{a} \tag{A.65}$$

with the right-hand side

$c\,\mathbf{B}\,\mathbf{a} =$

$$
c \begin{pmatrix}
T_0(1) & T_1(1) & \cdots \\
T_0'(1) & T_1'(1) & \cdots \\
T_0''(y_2) - k^2 T_0(y_2) & T_1''(y_2) - k^2 T_1(y_2) & \cdots \\
\vdots & \vdots & \vdots \\
T_0''(y_{N-2}) - k^2 T_0(y_{N-2}) & T_1''(y_{N-2}) - k^2 T_1(y_{N-2}) & \cdots \\
T_0'(-1) & T_1'(-1) & \cdots \\
T_0(-1) & T_1(-1) & \cdots
\end{pmatrix}
\begin{pmatrix}
a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N
\end{pmatrix}
\tag{A.66}
$$

and similarly for the left-hand side $\mathbf{A}\,\mathbf{a}$. We have chosen to use the first, second, last and next-to-last row of $\mathbf{B}$ to implement the four boundary conditions. The same rows in the matrix $\mathbf{A}$ can be chosen as a complex multiple of the corresponding rows in $\mathbf{B}$. By carefully selecting this complex multiple, the spurious modes associated with the boundary conditions can be mapped to an arbitrary location in the complex plane.

The generalized eigenvalue problem can now be solved using standard software, e.g., LAPACK.

A.6 MATLAB Codes for Hydrodynamic Stability Calculations

Below are a suite of MATLAB routines (written by S. Reddy) to compute transient growth and spectral characteristics for plane Poiseuille and plane Couette flow.

MATLAB Driver Program

```
% osmat.m
%
% Program to compute the Orr-Sommerfeld matrix for three-
% dimensional Poiseuille or Couette flows and to compute
% energy matrix
%
%
```

```
% INPUT
%
% nosmod     = number of Orr-Sommerfeld modes
% R          = Reynolds number
% alp        = alpha (streamwise wave number)
% beta       = beta  (spanwise wave number)
% iflow      = type of flow  (Poiseuille=1, Couette=2)
% nosmod     = total number of modes for normal velocity
% iflag      = flag
%              iflag = 1: compute the maximum growth and
%                         initial condition in time
%                         interval [0,T]
%              iflag = 2: compute the initial disturbance
%                         yielding maximum growth at time T
%
% OUTPUT
% d          =   3D Orr-Sommerfeld matrix
% M          =   energy matrix
%
    zi=sqrt(-1);

% input data

    iflow=input('Poiseuille (1) or Couette flow (2) ');
    nosmod=input('Enter N the number of OS modes: ');
    R=input('Enter the Reynolds number: ');
    alp=input('Enter alpha: ');
    beta=input('Enter beta: ');
    iflag=input(...
    '(1) Max growth in [Tmin,Tmax] (2) Max growth at T ');

    if iflag==1,
      Tmin=input('Enter Tmin: ');
      Tmax=input('Enter Tmax: ');
      T=[Tmin Tmax];
    else
      T=input('Enter T: ');
    end;

% generate Chebyshev differentiation matrices

    [D0,D1,D2,D4]=Dmat(nosmod);

% set up Orr-Sommerfeld matrices A and B
```

```
    if iflow==1,
      [A,B]=pois(nosmod,alp,beta,R,D0,D1,D2,D4);
    else
      [A,B]=couet(nosmod,alp,beta,R,D0,D1,D2,D4);
    end;

% generate energy weight matrix

    ak2=alp^2+beta^2;
    M=energy(nosmod+1,nosmod+1,ak2);

% compute the Orr-Sommerfeld matrix (by inverting B)

    d=inv(B)*A;

% compute the optimal

    [flowin,flowot,gg]=optimal(d,T,M,ak2,iflag);
    figure(1); semilogy(gg(:,1),gg(:,2));grid on
```

Computing Chebyshev Differentiation Matrices

```
    function [D0,D1,D2,D4]=Dmat(N);

%
% Function to create differentiation matrices
%
% N       = number of modes
% D0      = zero'th derivative matrix
% D1      = first derivative matrix
% D2      = second derivative matrix
% D4      = fourth derivative matrix

% initialize

    num=round(abs(N));

% create D0

    D0=[];
    vec=(0:1:num)';
    for j=0:1:num
```

```
    D0=[D0 cos(j*pi*vec/num)];
  end;

% create higher derivative matrices

  lv=length(vec);
  D1=[zeros(lv,1) D0(:,1) 4*D0(:,2)];
  D2=[zeros(lv,1) zeros(lv,1) 4*D0(:,1)];
  D3=[zeros(lv,1) zeros(lv,1) zeros(lv,1)];
  D4=[zeros(lv,1) zeros(lv,1) zeros(lv,1)];
  for j=3:num
    D1=[D1 2*j*D0(:,j)+j*D1(:,j-1)/(j-2)];
    D2=[D2 2*j*D1(:,j)+j*D2(:,j-1)/(j-2)];
    D3=[D3 2*j*D2(:,j)+j*D3(:,j-1)/(j-2)];
    D4=[D4 2*j*D3(:,j)+j*D4(:,j-1)/(j-2)];
  end;
```

Computing Orr-Sommerfeld Matrix for Poiseuille Flow

```
  function [A,B]=pois(nosmod,alp,beta,R,D0,D1,D2,D4);
%
% Function to create Orr-Sommerfeld matrices using Chebyshev
% pseudospectral discretization for plane Poiseuille flow
% profile
%
% nosmod  = number of modes
% alp     = alpha
% beta    = beta
% R       = Reynolds number
% D0      = zero'th derivative matrix
% D1      = first derivative matrix
% D2      = second derivative matrix
% D3      = third derivative matrix
% D4      = fourth derivative matrix

  zi=sqrt(-1);

% mean velocity

  ak2=alp^2+beta^2;
  Nos=nosmod+1;
  Nsq=nosmod+1;
```

```
    vec=(0:1:nosmod)';
    u=(ones(length(vec),1)-cos(pi*vec/nosmod).^2);
    du=-2*cos(pi*vec/nosmod);

% set up Orr-Sommerfeld matrix

    B11=D2-ak2*D0;
    A11=-(D4-2*ak2*D2+(ak2^2)*D0)/(zi*R);
    A11=A11+alp*(u*ones(1,length(u))).*B11+alp*2*D0;
    er=-200*zi;
    A11=[er*D0(1,:); er*D1(1,:); A11(3:Nos-2,:); ...
         er*D1(Nos,:); er*D0(Nos,:) ];
    B11=[D0(1,:); D1(1,:); B11(3:Nos-2,:); ...
         D1(Nos,:); D0(Nos,:)];

% set up Squire matrix and cross-term matrix

    A21=beta*(du*ones(1,length(u))).*D0(1:Nos,:);
    A22=alp*(u*ones(1,length(u))).*D0-(D2-ak2*D0)/(zi*R);
    B22=D0;
    A22=[er*D0(1,:); A22(2:Nsq-1,:); er*D0(Nsq,:)];
    A21=[zeros(1,Nos); A21(2:Nsq-1,:); zeros(1,Nos)];

% combine all the blocks

    A=[A11 zeros(Nos,Nsq); A21 A22];
    B=[B11 zeros(Nos,Nsq); zeros(Nsq,Nos) B22];
```

Computing Orr-Sommerfeld Matrix for Couette Flow

```
    function [A,B]=couet(nosmod,alp,beta,R,D0,D1,D2,D4);
%
% Function to create Orr-Sommerfeld matrices using Chebyshev
% pseudospectral discretization for plane Couette flow
% profile
%
% nosmod  = number of even or odd modes
% alp     = alpha
% R       = Reynolds number
% D0      = zero'th derivative matrix
% D1      = first derivative matrix
% D2      = second derivative matrix
```

```
% D4       = fourth derivative matrix

    zi=sqrt(-1);

% mean velocity

    ak2=alp^2+beta^2;
    Nos=nosmod+1;
    Nsq=nosmod+1;
    u=cos(pi*(0:1:Nos-1)'/(Nos-1));
    du=ones(length(u),1);

% set up Orr-Sommerfeld matrix

    B11=D2-ak2*D0;
    A11=-(D4-2*ak2*D2+(ak2^2)*D0)/(zi*R);
    A11=A11+alp*(u*ones(1,length(u))).*B11;
    er=-200*zi;
    A11=[er*[D0(1,:); D1(1,:)]; A11(3:Nos-2,:); ...
         er*[D1(Nos,:); D0(Nos,:)]];
    B11=[D0(1,:); D1(1,:); B11(3:Nos-2,:); ...
         D1(Nos,:); D0(Nos,:)];

% set up Squire matrix and cross-term matrix

    A21=beta*(du*ones(1,length(u))).*D0;
    A22=alp*(u*ones(1,length(u))).*D0-(D2-ak2*D0)/(zi*R);
    B22=D0;
    A22=[er*D0(1,:); A22(2:Nsq-1,:); er*D0(Nsq,:)];

% combine all the blocks

    A=[A11 zeros(Nos,Nsq); A21 A22];
    B=[B11 zeros(Nos,Nsq); zeros(Nsq,Nos) B22];
```

Computing Energy Weight Matrix

```
    function M=energy(Nos,Nsq,ak2);
%
% Program to compute the energy weight matrix for three-
% dimensional Poiseuille and Couette flows
%
```

```
% INPUT
% Nos   = Number of normal velocity modes
% Nsq   = Number of normal vorticity modes
%
% OUTPUT
% M     = weight matrix

    M=eye(Nos+Nsq,Nos+Nsq);
    Cos=two(Nos);
    Dos=deven(Nos);
    Wos=Dos'*Cos*Dos+ak2*Cos;
    Wsq=two(Nsq);

    [u,s,v]=svd(Wos); s=sqrt(diag(s));
    Mos=diag(s)*u';

    [u,s,v]=svd(Wsq); s=sqrt(diag(s));
    Msq=diag(s)*u';

    M=[Mos zeros(Nos,Nsq); zeros(Nsq,Nos) Msq];
```

Computing Two Norm Weight Matrix

```
    function c=two(N);
%
% This program determines the two norm weight matrix c for
% Chebyshev polynomials. The matrix c is defined by
%
% c_{ij}= int_{-1}^{1} T_{i}(x) T_{j}(x) dx,
%
% where  T_k is a Chebyshev polynomial. The above product
% satisfies
%
% c_{ij}= 1/(1-(i+j)^2)+1/(1-(i-j)^2)   for i+j even
%       = 0                             for i+j odd
%
% Input
%
% N     = number of modes
%
% The maximum degree M of polynomials c_i,c_j satisfies
% M = N-1
```

```
%

    num=round(abs(N));
    c=zeros(num,num);

    for i=(0:num-1)
      for j=(0:num-1)
        if rem(i+j,2)==0,
          p=1/(1-(i+j)^2)+1/(1-(i-j)^2);
          c(i+1,j+1)=p;
        else
          c(i+1,j+1)=0;
        end;
      end;
    end;
```

Computing Chebyshev Conversion Matrix

```
    function d1=deven(N);
%
% Compute matrix which converts Chebyshev coefficients of a
% polynomial to coefficients of derivative
%
% Reference:
% Gottlieb and Orszag, Numerical Analysis of Spectral
% Methods: Theory and Applications, SIAM, Philadelphia,
% 1977.
%
% d1  = derivative matrix (N,N)
% N   = number of coefficients
%
    num=round(abs(N));
    d1=zeros(num,num);

    for i=0:(num-1)
      for j=(i+1):2:(num-1)
        d1(i+1,j+1)=2*j;
      end;
    end;

    d1(1,:)=d1(1,:)/2;
```

Computing Transient Growth

```
   function [flowin,flowot,gg]=optimal(d,T,M,ak2,iflag);
%
% This function computes the initial flow structure which
% achieves the maximum transient energy growth
%
% INPUT
% d       = 3D Orr-Sommerfeld operator
% T       = time
% M       = energy weight matrix
% ak2     = alpha^2+beta^2
% iflag   = flag
%           iflag = 1:  compute the maximum growth and
%                       initial condition in time
%                       interval [0,T]
%           iflag = 2:  compute the initial disturbance
%                       yielding maximum growth at time T
% OUTPUT
% flowin  = coefficients of optimal disturbance
%           flowin(1:Nos)         = normal velocity
%                                   coefficients
%           flowin(Nos+1:Nos+Nsq) = normal vorticity
%                                   coefficients
% flowot  = coefficients of field at optimal time
%           flowot(1:Nos)         = normal velocity
%                                   coefficients
%           flowot(Nos+1:Nos+Nsq) = normal vorticity
%                                   coefficients

    global qb;

% Phase 1: Compute eigenvalues and eigenfunctions of
% Orr-Sommerfeld matrix and sort in order of descending
% imaginary part. The function nlize normalizes the
% eigenfunctions with respect to the weight matrix M.

    [xs,es]=iord2(d);
    xs=nlize(xs,M);

% Phase 2: Choose which modes are to be used in optimal
```

```
% calculation. Modes with imaginary part > 1 are neglected.
% Modes with imaginary part < imin are neglected as well.

    ishift=1;
    imin=-1.5;

    while imag(es(ishift))>1, ishift=ishift+1; end;

    [n1,n2]=n9(es,imin);

    cols=(ishift:n2);
    xu=xs(:,cols);
    eu=es(cols);
    ncols=length(cols);
    fprintf('Number of modes used: %1.0f \n',ncols);

% Phase 3: Compute the reduced Orr-Sommerfeld operator

    [qb,invF]=qbmat(M,xu,eu);

% Phase 4: Compute the time for the maximum growth using
% the built-in Matlab routine 'fmin'

    if iflag==1,
      gcheck=maxer(1/100);
      gcheck=gcheck^2;
      if gcheck<1,
        tformax=0;
        mgrowth=1;
      else
        ts=T(1);
        tf=T(2);
        options=[0 1e-3 1e-3];
        tformax=fmy(ts,tf,options);
        mgrowth=maxer(tformax);
        mgrowth=mgrowth^2;
      end;
      fprintf('Time for maximum growth:  %e \n',tformax);
    else
      tformax=T;
    end;

% Phase 5: Compute the initial condition that yields the
% maximum growth. This is obtained by
% (1) computing the matrix exponential evaluated at the
```

```
%     optimal time;
% (2) computing the SVD of the matrix exponential
%     exp(-i*A*t)=USV.
% The initial condition that yields the maximum growth is
% the first column of V. To convert the initial condition
% to a vector of coefficients in the eigenfunction basis
% multiply by the matrix of eigenfunctions and inv(F)

    evol=expm(tformax*qb);
    [U,S,V]=svd(evol);
    mgrowth=S(1,1)^2;
    fprintf('Maximum growth in energy:  %e \n',mgrowth);

    flowin=sqrt(2*ak2)*xu*invF*V(:,1);
    flowot=sqrt(2*ak2)*xu*invF*U(:,1);

    for i=1:100,
      tid = ts + (tf-ts)/99*(i-1);
      gg(i,2) = norm(expm(tid*qb))^2;
      gg(i,1) = tid;
    end
```

Computing Ordered Eigenvalues

```
    function [xs,es]=iord2(d);
%
% This function computes the eigenvalues of a matrix d and
% orders the eigenvalues so that the imaginary parts are
% decreasing.
%
% INPUT
% d    = input matrix
%
% OUTPUT
% es   = ordered eigenvalues
% xs   = eigenvectors

    [v,e]=eig(d);
    e=diag(e);
    [eimag,is]=sort(-imag(e));
    xs=v(:,is);
    es=e(is);
```

Normalizing Matrix Columns

```
    function x=nlize(x,M);
%
% This function normalizes the columns of x such that
%
%  || M x_i ||_2 = 1
%
    nc=size(x); nc=nc(2);
    for i=1:nc
      x(:,i)=x(:,i)/norm(M*x(:,i));
    end;
```

Selecting Eigenvalues

```
   function [n1,n2]=n9(e,a);
%
% This function computes the number of eigenvalues
% satisfying
%
% a <= Imag(lambda) <= .5
%
% INPUT
% e   = eigenvalues ordered with decreasing imaginary
%        part
%
% OUTPUT
% n1  = position of first eigenvalue in the interval
% n2  = position of last eigenvalue in the interval

   n1=1;

   while imag(e(n1))>.5,
    n1=n1+1;
   end;

   n2=n1;
```

```
while imag(e(n2))>a,
  n2=n2+1;
end;

n2=n2-1;
```

Computing Transient Growth Matrix

```
    function [qb,invF]=qbmat(M,xu,e)
%
% This function computes the matrix  Q=-i*F*diag(e)*inv(F)
% which is used to compute the maximum transient growth
% (see Reddy and Henningson, "Energy Growth in Viscous
% Channel Flows", JFM 252, page 209, 1993).
%
% INPUT
% M      =  energy weight matrix
% xu     =  matrix of eigenfunctions (expansion coefficients)
% e      =  vector of eigenvalues of the stability matrix
%
% OUTPUT
% qb     =  output matrix Q
% invF   =  inverse of F
%

% Phase 1: compute inner product of the eigenfunctions
%          in energy norm

    work=M*xu;
    A=work'*work;

% Phase 2: compute decomposition A=F^*F

    [U,S,V]=svd(A);
    s=sqrt(diag(S));
    F=diag(s)*U';
    invF=U*diag(ones(size(s))./s);

% Phase 3: compute Q=-i*F*diag(e)*inv(F)

    qb=-sqrt(-1)*F*diag(e)*invF;
```

Computing Norm of Matrix Exponential

```
    function a=maxer(t);
%
% This function computes the norm of the matrix exponential
% of qb*t

    global qb;
    a=-norm(expm(t*qb));
```

Maximizing Transient Growth

```
    function t=fmy(t1,t2,options);
%
% This function uses the built-in function 'fmin' to find
% the maximum value of a function on the interval [t1,t2].
% The function is in the file maxer.m
%
% INPUT:
% t1,t2   = lower and upper bounds of interval
% options = input parameters for minimization routine
%
% OUTPUT
% t       = value at which function maxer(t) is minimized

    f1=maxer(t1);
    f2=maxer(t2);

    tt=fmin('maxer',t1,t2,options);

    f3=maxer(tt);
    f=[f1 f2 f3];
    tm=[t1 t2 tt];
    [y,is]=sort(f);
    t=tm(is(1));
```

A.7 Eigenvalues of Parallel Shear Flows

Plane Poiseuille flow, Re = 2000							
$\alpha = 1, \beta = 0$		$\alpha = 0.5, \beta = 1$		$\alpha = 0.25, \beta = 3$		$\alpha = 0, \beta = 2$
c_r	c_i	c_r	c_i	c_r	c_i	ω_i
Orr-Sommerfeld		Orr-Sommerfeld		Orr-Sommerfeld		Orr-Sommerfeld
0.31210030	−0.01979866	0.37226932	−0.03737398	0.56329537	−0.08548514	−0.00507754
0.42418427	−0.07671992	0.49935557	−0.09920592	0.83796079	−0.14010066	−0.01107002
0.92078667	−0.07804706	0.88770220	−0.10945538	0.84492959	−0.14965217	−0.01982549
0.92091806	−0.07820060	0.88808805	−0.10962449	0.58717755	−0.15289251	−0.03078177
0.85717055	−0.13990151	0.79534673	−0.19331077	0.72738833	−0.22753394	−0.04449131
0.85758968	−0.14031674	0.79830962	−0.19657914	0.73333862	−0.28917720	−0.06038365
0.79399812	−0.20190508	0.72648153	−0.26096201	0.63548529	−0.36625533	−0.07903219
0.79413424	−0.20232063	0.64779065	−0.26971348	0.62259513	−0.41560943	−0.09985929
0.63912513	−0.22134137	0.70474692	−0.29872825	0.64672215	−0.44988239	−0.12344417
0.53442105	−0.22356175	0.43320720	−0.30659209	0.66265886	−0.56491518	−0.14920592
Squire		Squire		Squire		Squire
0.98418861	−0.01631139	0.97763932	−0.02361068	0.96837728	−0.03974775	−0.00323370
0.95256584	−0.04793417	0.93291797	−0.06833204	0.90513428	−0.10299426	−0.00693480
0.92094306	−0.07955694	0.88819669	−0.11305351	0.84190728	−0.16629312	−0.01310331
0.88932028	−0.11117972	0.84347472	−0.15777756	0.39061440	−0.21452290	−0.02173921
0.24936056	−0.13725811	0.31232252	−0.16986946	0.39061296	−0.21452788	−0.03284251
0.24936056	−0.13725811	0.31232252	−0.16986946	0.77824142	−0.23018702	−0.04641322
0.85769752	−0.14280249	0.79871747	−0.20251584	0.70571029	−0.29246161	−0.06245133
0.82607494	−0.17442537	0.75360282	−0.24701616	0.65305016	−0.31673071	−0.08095684
0.79445264	−0.20605114	0.53467243	−0.27140532	0.65688684	−0.35866635	−0.10192975
0.42863639	−0.22466515	0.53470480	−0.27146759	0.67269250	−0.46026968	−0.12537006

TABLE A.1. PLANE POISEUILLE FLOW eigenvalues

Couette flow, Re = 800						
$\alpha = 1, \beta = 0$		$\alpha = 0.5, \beta = 1$		$\alpha = 0.25, \beta = 3$		$\alpha = 0, \beta = 2$
c_r	c_i	c_r	c_i	c_r	c_i	ω_i
Orr-Sommerfeld		Orr-Sommerfeld		Orr-Sommerfeld		Orr-Sommerfeld
±0.57647380	−0.12952206	±0.47711120	−0.17057387	±0.39990294	−0.25329463	−0.01269385
±0.33837303	−0.28699889	±0.18033226	−0.36896022	±0.37283853	−0.48343132	−0.02767504
±0.65474385	−0.31845690	±0.55316723	−0.39709328	0.00000000	−0.48561430	−0.04956371
±0.13853225	−0.41451617	0.00000000	−0.52654272	0.00000000	−0.56306665	−0.07695442
±0.39155287	−0.44983177	±0.21826018	−0.56176351	0.00000000	−0.72239155	−0.11122827
0.00000000	−0.51543904	0.00000000	−0.68690207	0.00000000	−0.92956625	−0.15095912
±0.17496627	−0.56239996	0.00000000	−0.80374929	0.00000000	−1.17658449	−0.19758048
0.00000000	−0.64277951	0.00000000	−0.96841110	0.00000000	−1.44426340	−0.24964822
0.00000000	−0.69476878	0.00000000	−1.14407304	0.00000000	−1.73554371	−0.30861043
0.00000000	−0.80564720	0.00000000	−1.32986473	0.00000000	−2.04876750	−0.37301480
Squire		Squire		Squire		Squire
±0.78187852	−0.12718249	±0.72518416	−0.16179000	±0.65375374	−0.22021787	−0.00808425
±0.61863618	−0.22143050	±0.51951170	−0.28053505	±0.39462265	−0.36982719	−0.01733701
±0.48498831	−0.29859214	±0.35112593	−0.37775262	±0.18249973	−0.49241970	−0.03275826
±0.36686964	−0.36678798	±0.20230357	−0.46367578	0.00000000	−0.57627651	−0.05434802
±0.25889370	−0.42912792	±0.06726756	−0.54144268	0.00000000	−0.71401555	−0.08210628
±0.15827968	−0.48721835	0.00000000	−0.65186131	0.00000000	−0.94254293	−0.11603305
±0.06311251	−0.54178178	0.00000000	−0.80563552	0.00000000	−1.19022327	−0.15612832
0.00000000	−0.59624095	0.00000000	−0.97104292	0.00000000	−1.45959937	−0.20239209
0.00000000	−0.69585432	0.00000000	−1.14689443	0.00000000	−1.75143991	−0.25482436
0.00000000	−0.80692602	0.00000000	−1.33355240	0.00000000	−2.06630407	−0.31342514

TABLE A.2. COUETTE FLOW eigenvalues

Pipe Poiseuille flow, Re = 2000						
$\alpha = 1, n = 0$		$\alpha = 0.5, n = 1$		$\alpha = 0.25, n = 2$		$\alpha = 0, n = 1$
c_r	c_i	c_r	c_i	c_r	c_i	ω_i
0.93675536	−0.06374551	0.84646970	−0.07176332	0.72551688	−0.14895301	−0.00734099
0.93675445	−0.06374555	0.42653865	−0.10120107	0.37381075	−0.17973957	−0.01318731
0.87350890	−0.12699110	0.30753390	−0.14961348	0.51310797	−0.19545604	−0.02460923
0.87353385	−0.12701919	0.92769296	−0.15332927	0.85236191	−0.27489963	−0.03542500
0.46474928	−0.12733652	0.74827192	−0.15401221	0.59574165	−0.29082257	−0.05174973
0.24930527	−0.13770406	0.66105308	−0.23357367	0.60783255	−0.40189549	−0.06751036
0.81026386	−0.19023731	0.84409924	−0.25072952	0.72720893	−0.41549788	−0.08876038
0.81012530	−0.19098283	0.53491162	−0.28781712	0.64896591	−0.50429278	−0.10946009
0.42850340	−0.22516774	0.46768245	−0.32736073	0.66139917	−0.61313884	−0.13564083
0.74691455	−0.25352712	0.63139623	−0.33464114	0.66664316	−0.73870170	−0.16127756
0.70634286	−0.25379497	0.75997902	−0.34520452	0.66017652	−0.86358022	−0.19239095
0.73511224	−0.26390239	0.67575624	−0.43243735	0.66968104	−1.00995818	−0.22296378
0.57089335	−0.28598470	0.67491966	−0.46947547	0.65952947	−1.15557527	−0.25901072
0.68508153	−0.30991044	0.67065768	−0.56007978	0.67077599	−1.31762599	−0.29451918
0.40118595	−0.33427144	0.66958056	−0.63844170	0.65971050	−1.48607071	−0.33550011
0.68879517	−0.36815020	0.66947927	−0.72799068	0.67093631	−1.66406005	−0.37594393
0.67317200	−0.37669040	0.66800877	−0.81713158	0.66029737	−1.85486324	−0.42185912
0.66423205	−0.43620004	0.66915390	−0.91515764	0.67069161	−2.04976452	−0.46723813
0.67232230	−0.47977566	0.66709014	−1.01401073	0.66099154	−2.26220681	−0.51808775
0.67131307	−0.53467710	0.66899996	−1.12083979	0.67030895	−2.47485248	−0.56840184

TABLE A.3. PIPE POISEUILLE FLOW eigenvalues

Blasius boundary layer flow, $\mathrm{Re}_{\delta^*} = 800$							
$\alpha = 1, \beta = 0$		$\alpha = 0.5, \beta = 0.1$		$\alpha = 0.25, \beta = 0.2$		$\alpha = 0.125, \beta = 0.3$	
c_r	c_i	c_r	c_i	c_r	c_i	c_r	c_i
Orr-Sommerfeld		Orr-Sommerfeld		Orr-Sommerfeld		Orr-Sommerfeld	
0.29440241	−0.08240950	0.39192905	−0.04349817	0.39065421	+0.00287572	0.42986402	−0.01526073
0.46408909	−0.16979273	0.48131508	−0.13904767	0.54772364	−0.23434181	0.67108511	−0.31529294
0.58341130	−0.21355653	0.28194476	−0.26456053	0.33866341	−0.31005379	0.43734404	−0.36626806
0.23752687	−0.21441674	0.64194579	−0.29009419	0.79181869	−0.37872068	0.86076659	−0.46712319
0.67030439	−0.28694526	0.51869001	−0.35381783	0.65749147	−0.40505900		
0.42182040	−0.29556202	0.81507495	−0.37808042				
0.78475538	−0.35409567	0.72900959	−0.42004769				
0.57920596	−0.35864989						
0.72486516	−0.40824449						
Squire		Squire		Squire		Squire	
0.15023935	−0.08789091	0.18934427	−0.10971644	0.23869653	−0.13769021	0.30108694	−0.17329180
0.26294369	−0.15227682	0.33172068	−0.19019408	0.41904327	−0.23747142	0.53079487	−0.29497981
0.35574122	−0.20409837	0.44962930	−0.25384742	0.57017612	−0.31360121	0.72806936	−0.38097627
0.43849974	−0.24866890	0.55577060	−0.30704908	0.70889059	−0.37342899	0.91630206	−0.43964252
0.51510271	−0.28789626	0.65531838	−0.35196883	0.84255313	−0.41937502		
0.58764090	−0.32268836	0.75120476	−0.38960568				
0.65744431	−0.35354522	0.84540756	−0.42054153				
0.72545592	−0.38077281						
0.79239966	−0.40457786						

TABLE A.4. BLASIUS BOUNDARY LAYER FLOW eigenvalues

Appendix B
Resonances and Degeneracies

B.1 Resonances and Degeneracies

The case of degenerate eigenvalues will be considered in this appendix.

Let us consider one double eigenvalue ω_d. In the degenerate case a generalized eigenfunction must be added to complete the set of expansion functions. This function $\partial\tilde{\mathbf{q}}_d$ and its adjoint satisfy the following equations (DiPrima & Habetler, 1969)

$$\left(\mathbf{L} - i\omega_d\mathbf{M}\right)\partial\tilde{\mathbf{q}}_d = i\mathbf{M}\tilde{\mathbf{q}}_d \tag{B.1}$$

$$\left(\mathbf{L}^+ + i\omega_d^*\mathbf{M}\right)\partial\tilde{\mathbf{q}}_d^+ = -i\mathbf{M}\tilde{\mathbf{q}}_d^+ \tag{B.2}$$

with boundary conditions analogous to the regular eigenvalue problem. The symbol ∂ indicates that the generalized eigenfunction can be considered as the derivative of the regular eigenfunction with respect to the eigenvalue. For a degeneracy between two Orr-Sommerfeld modes we obtain the problem studied by Shantini (1990).

The bi-orthogonality conditions for the case of degenerate eigenvalues are somewhat more involved but can be derived in a manner similar to the regular eigenvalue problem. After proper normalization we find the following relations.

$$\begin{aligned}
\left(\mathbf{M}\tilde{\mathbf{q}}_d, \tilde{\mathbf{q}}_p^+\right) &= \left(\tilde{\mathbf{q}}_d, \mathbf{M}\tilde{\mathbf{q}}_p^+\right) = \left(\mathbf{M}\tilde{\mathbf{q}}_p, \tilde{\mathbf{q}}_d^+\right) \\
&= \left(\tilde{\mathbf{q}}_p, \mathbf{M}\tilde{\mathbf{q}}_d^+\right) = 0 \qquad\qquad (B.3) \\
\left(\mathbf{M}\partial\tilde{\mathbf{q}}_d, \tilde{\mathbf{q}}_p^+\right) &= \left(\partial\tilde{\mathbf{q}}_d, \mathbf{M}\tilde{\mathbf{q}}_p^+\right) = \left(\mathbf{M}\tilde{\mathbf{q}}_p, \partial\tilde{\mathbf{q}}_d^+\right) \\
&= \left(\tilde{\mathbf{q}}_p, \mathbf{M}\partial\tilde{\mathbf{q}}_d^+\right) = \delta_{pd} \qquad\qquad (B.4) \\
\left(\mathbf{L}\partial\tilde{\mathbf{q}}_d, \tilde{\mathbf{q}}_p^+\right) &= \left(\partial\tilde{\mathbf{q}}_d, \mathbf{L}^+\tilde{\mathbf{q}}_p^+\right) = \left(\mathbf{L}\tilde{\mathbf{q}}_p, \partial\tilde{\mathbf{q}}_d^+\right) \\
&= \left(\tilde{\mathbf{q}}_p, \mathbf{L}^+\partial\tilde{\mathbf{q}}_d^+\right) = i\omega_d\delta_{pd} \qquad\qquad (B.5) \\
\left(\partial\tilde{\mathbf{q}}_d, \mathbf{L}^+\partial\tilde{\mathbf{q}}_d^+\right) &- i\omega_d\left(\partial\tilde{\mathbf{q}}_d, \mathbf{M}\partial\tilde{\mathbf{q}}_d^+\right) \\
&= \left(\mathbf{L}\partial\tilde{\mathbf{q}}_d, \partial\tilde{\mathbf{q}}_d^+\right) - i\omega_d\left(\mathbf{M}\partial\tilde{\mathbf{q}}_d, \partial\tilde{\mathbf{q}}_d^+\right) = i \qquad\qquad (B.6)
\end{aligned}$$

The inner products in (B.3) are zero also for the case $p = d$, and the subscript $_d$ refers only to the eigenmode associated with the degenerate eigenvalue.

The solution to the linear initial value problem, assuming one degenerate eigenvalue ω_d, can be written in the form

$$\hat{\mathbf{q}} = K_d(t)\tilde{\mathbf{q}}_d + K_{d+1}(t)\partial\tilde{\mathbf{q}}_d + \sum_p K_p(t)\tilde{\mathbf{q}}_p \qquad (B.7)$$

where it is assumed that $p \neq d$, $p \neq d+1$. We introduce the expansion into the linear initial value problem, multiply by the adjoint eigenfunctions (including the generalized one) and integrate across the channel. Using the bi-orthogonality conditions and (B.3-B.6) we obtain a system of uncoupled equations for each K_p together with two coupled equations for K_d and K_{d+1}. The solution to this system is

$$\begin{aligned}
K_p &= K_p^0 e^{-i\omega_p t} \qquad (B.8) \\
K_d &= (K_d^0 - iK_{d+1}^0 t)e^{-i\omega_d t} \qquad (B.9) \\
K_{d+1} &= K_{d+1}^0 e^{-i\omega_d t}. \qquad (B.10)
\end{aligned}$$

The coefficients K_p^0 are identical to the coefficients found in Chapter 3, while the initial coefficients associated with the degenerate eigenvalue become

$$\begin{aligned}
K_d^0 &= \left(\mathbf{M}\hat{\mathbf{q}}^0, \partial\tilde{\mathbf{q}}_d^+\right) - K_{d+1}^0\left(\mathbf{M}\partial\tilde{\mathbf{q}}_d, \partial\tilde{\mathbf{q}}_d^+\right) \qquad (B.11) \\
K_{d+1}^0 &= \left(\mathbf{M}\hat{\mathbf{q}}^0, \tilde{\mathbf{q}}_d^+\right). \qquad (B.12)
\end{aligned}$$

The complete solution to the degenerate case thus can be written

$$\begin{aligned}\hat{\mathbf{q}} &= \sum_p \left(\mathbf{M}\hat{\mathbf{q}}^0, \tilde{\mathbf{q}}_p^+\right) \tilde{\mathbf{q}}_p e^{-i\omega_p t} + \left(\mathbf{M}\hat{\mathbf{q}}^0, \tilde{\mathbf{q}}_d^+\right) \partial\tilde{\mathbf{q}}_d e^{-i\omega_d t} + \\ & \left[\left(\mathbf{M}\hat{\mathbf{q}}^0, \partial\tilde{\mathbf{q}}_d^+\right) - \left(\mathbf{M}\hat{\mathbf{q}}^0, \tilde{\mathbf{q}}_d^+\right)\left(\mathbf{M}\partial\tilde{\mathbf{q}}_d, \partial\tilde{\mathbf{q}}_d^+\right) \right. \\ & \left. - it\left(\mathbf{M}\hat{\mathbf{q}}^0, \tilde{\mathbf{q}}_d^+\right)\right] \tilde{\mathbf{q}}_d e^{-i\omega_d t}. \qquad \text{(B.13)}\end{aligned}$$

Shantini (1990), using a Fourier-Laplace method, found the same expressions for a degeneracy between Orr-Sommerfeld modes. It is easy to generalize the result to vector eigenfunctions. Since they include both the Orr-Sommerfeld and Squire modes, the normal velocity - normal vorticity resonance studied by Benney & Gustavsson (1981) reduces to a degeneracy between vector modes of the linear system. The details regarding this type of degeneracy will be presented in the subsequent section. We close this section by presenting the special case of (B.13) when the initial condition only includes the generalized eigenfunction, i.e. $\hat{\mathbf{q}}^0 = \partial\tilde{\mathbf{q}}_d$, in which case

$$\hat{\mathbf{q}} = (\partial\tilde{\mathbf{q}}_d - it\tilde{\mathbf{q}}_d)e^{-i\omega_d t}. \qquad \text{(B.14)}$$

Reddy & Henningson (1993) assessed the importance of including the generalized eigenfunction when calculating optimal disturbances and found that the optimal disturbances varied smoothly across the degeneracy. Thus, it is sufficient to consider the regular eigenvalues close to the degeneracy to obtain an accurate representation of the solution; see Figure B.1.

B.2 Orr-Sommerfeld-Squire Resonance

If one of the degenerate modes is of Orr-Sommerfeld type and the other is a Squire mode, the degeneracy is identical to the resonance studied by Benney & Gustavsson (1981). It is interesting to see how this special case appears from a degeneracy of the vector eigenfunctions. We will follow Henningson & Schmid (1992) in this derivation. First we have to solve (3.28) to obtain the regular eigenfunction at the degeneracy. When ω is an eigenvalue of the homogeneous parts of both equations, the normal velocity has to be zero; otherwise the compatibility condition requiring the driving term of the normal vorticity equation to be orthogonal to the adjoint eigenfunction is not satisfied. Similarly, a solution to the equation governing the regular adjoint eigenfunction only exists when its normal vorticity is zero. We have the following equations governing the regular eigenfunctions at the degeneracy

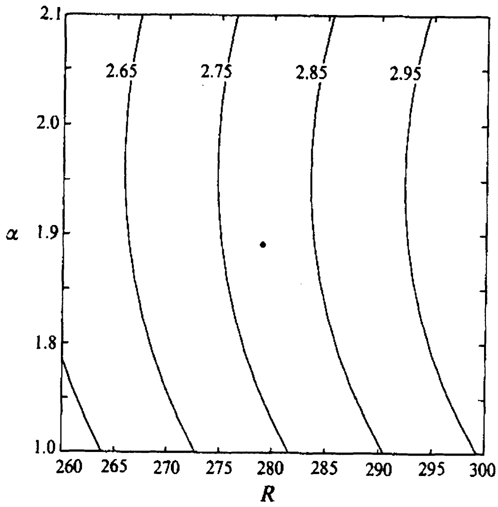

FIGURE B.1. Contours of G_{max} for plane Poiseuille flow in the neighborhood of the degeneracy at Re $\approx$ 279 and $\alpha \approx 1.89$ (the dot), where the maximum growth is ≈ 2.80. From Reddy & Henningson (1993).

$$\tilde{v}_d = 0 \tag{B.15}$$

$$L_{SQ}\tilde{\eta}_d - i\omega_d\tilde{\eta}_d = 0 \tag{B.16}$$

and

$$L_{OS}^{+}\tilde{v}_d^{+} + i\omega_d^{*}M\tilde{v}_d^{+} = 0 \tag{B.17}$$

$$\tilde{\eta}_d^{+} = 0. \tag{B.18}$$

In component form the equations (B.1, B.2) for the generalized eigenfunctions at an Orr-Sommerfeld-Squire degeneracy now become

$$L_{OS}\partial\tilde{v}_d - i\omega_d M\partial\tilde{v}_d = 0 \tag{B.19}$$

$$L_{SQ}\partial\tilde{\eta}_d - i\omega_d\partial\tilde{\eta}_d = -i\beta U'\partial\tilde{v}_d + i\tilde{\eta}_d \tag{B.20}$$

and

$$L_{OS}^{+}\partial\tilde{v}_d^{+} + i\omega_d^{*}M\partial\tilde{v}_d^{+} = i\beta U'\partial\tilde{\eta}_d^{+} - iM\tilde{v}_d^{+} \tag{B.21}$$

$$L_{SQ}^{+}\partial\tilde{\eta}_d^{+} + i\omega_d^{*}\partial\tilde{\eta}_d^{+} = 0 \tag{B.22}$$

where we have used that $\tilde{v}_d = 0$ and $\tilde{\eta}_d^{+} = 0$. This result implies that $\partial\tilde{v}_d$ is the solution to the Orr-Sommerfeld equation and $\partial\tilde{\eta}_d^{+}$ is the solution to the adjoint Squire equation. The bi-orthogonality conditions governing the degenerate modes can now be evaluated in component form. It is easily verified that equation (B.3) is satisfied, while equation (B.4) becomes

$$\int_{-1}^{1} \tilde{v}_p^{+*} M \partial \tilde{v}_d dy = \delta_{pd} \tag{B.23}$$

$$\int_{-1}^{1} \partial \tilde{\eta}_d^{+*} \tilde{\eta}_p dy = \delta_{pd} \tag{B.24}$$

$$\int_{-1}^{1} \left(\tilde{\zeta}_p^{+*} M \partial \tilde{v}_d + \tilde{\eta}_p^{+*} \partial \tilde{\eta}_d \right) dy = 0 \tag{B.25}$$

$$\int_{-1}^{1} \left(\partial \tilde{v}_d^{+*} M \tilde{v}_p + \partial \tilde{\eta}_d^{+*} \tilde{\xi}_p \right) dy = 0. \tag{B.26}$$

The two last equations are identically zero when $p = d$. Equation (B.5), along with the above relations, can be used to derive

$$\begin{aligned} \beta \int_{-1}^{1} U' \tilde{\eta}_p^{+*} \partial \tilde{v}_d \, dy &= -(\omega_d - \omega_p) \int_{-1}^{1} \tilde{\zeta}_p^{+*} M \partial \tilde{v}_d \, dy \\ &= (\omega_d - \omega_p) \int_{-1}^{1} \tilde{\eta}_p^{+*} \partial \tilde{\eta}_d \, dy \end{aligned} \tag{B.27}$$

$$\begin{aligned} \beta \int_{-1}^{1} U' \partial \tilde{\eta}_d^{+*} \tilde{v}_p \, dy &= (\omega_d - \omega_p) \int_{-1}^{1} \partial \tilde{v}_d^{+*} M \tilde{v}_p \, dy \\ &= -(\omega_d - \omega_p) \int_{-1}^{1} \partial \tilde{\eta}_d^{+*} \tilde{\xi}_p \, dy. \end{aligned} \tag{B.28}$$

Finally, equation (B.6) yields

$$\beta \int_{-1}^{1} U' \partial \tilde{\eta}_d^{+*} \partial \tilde{v}_d dy = 1. \tag{B.29}$$

The last equation implies that the inhomogeneous terms in equations (B.20) and (B.21) are orthogonal to the solutions of the respective homogeneous operators. Thus, the compatibility condition is satisfied and solutions for both $\partial \tilde{\eta}_d$ and $\partial \tilde{v}_d^+$ can be found. To find $\partial \tilde{\eta}_d$, for example, we assume that it has an expansion in eigenmodes of the homogeneous operator (excluding the resonant term), and calculate the expansion coefficients using the bi-orthogonality condition for the normal vorticity. We find

$$
\begin{aligned}
\partial\tilde{\eta}_d &= \sum_{p\neq d} \int_{-1}^{1} \tilde{\eta}_p^{+*} \partial\tilde{\eta}_d dy \, \tilde{\eta}_p \\
&= \sum_{p\neq d} \frac{\beta \int_{-1}^{1} U' \tilde{\eta}_p^{+*} \partial\tilde{v}_d dy}{\omega_d - \omega_p} \tilde{\eta}_p
\end{aligned}
\tag{B.30}
$$

where equation (B.27) has been used to rewrite the expansion coefficient. An arbitrary amount of the eigenvector $\tilde{\eta}_d$ can be added to this solution. We have chosen to add zero, any other amount would only change the relative importance of the expansion coefficients multiplying the $\tilde{\eta}_d$ and $\partial\tilde{\eta}_d$ modes.

Using the above results the solution (B.14) at a v-η degeneracy can be written

$$
\begin{pmatrix} \hat{v} \\ \hat{\eta} \end{pmatrix} = \left[\begin{pmatrix} \partial\tilde{v}_d \\ \partial\tilde{\eta}_d \end{pmatrix} - it \begin{pmatrix} 0 \\ \tilde{\eta}_d \end{pmatrix} \right] e^{-i\omega_d t}. \tag{B.31}
$$

Finally, we will show that the expression for the v-η degeneracy is recovered as the Orr-Sommerfeld and Squire eigenvalues approach each other. Let us consider two degenerate modes and expand (4.19) in a series with $\omega_r^{OS} - \omega_p^{SQ}$ as the small parameter. The algebraically growing term reads

$$
\hat{\eta} = -i\beta \int_{-1}^{1} U' \tilde{\eta}_p^{+*} \tilde{v}_r \, dy \left\{ t - \mathcal{O}\left[(\omega_r^{OS} - \omega_p^{SQ})^2 \right] \right\} \, \tilde{\eta}_p \, e^{-i\omega_d t} \tag{B.32}
$$

where $\omega_d = (\omega_r^{OS} + \omega_p^{SQ})/2$ and the expansion coefficients are set to one. As the degeneracy is approached

$$
\begin{aligned}
\tilde{\eta}_p &\to \tilde{\eta}_d \\
\tilde{\eta}_p^{+*} &\to \partial\tilde{\eta}_d^{+*} \\
\tilde{v}_r &\to \partial\tilde{v}_d.
\end{aligned}
\tag{B.33}
$$

The algebraically growing term in expression (B.31) can be recovered from equation (B.32), since (B.29) shows that β times the integral in (B.32) equals unity at the degeneracy.

Appendix C
Adjoint of the Linearized Boundary Layer Equation

C.1 Adjoint of the Linearized Boundary Layer Equation

In general, the adjoint of any bounded linear operator $\mathcal{A}$ between two inner-product (Hilbert) spaces is defined through the relation

$$(\boldsymbol{\Psi}, \mathcal{A}\mathbf{u}) = (\mathcal{A}^{+}\boldsymbol{\Psi}, \mathbf{u}) \tag{C.1}$$

where $\mathbf{u}$ is an element in the domain of $\mathcal{A}$, and $\boldsymbol{\Psi}$ is an element in the space associated with the inner product in (C.1).

The operator $\bar{\mathcal{A}}^{+}$ is the adjoint of the operator $\bar{\mathcal{A}}$ with respect to the given inner products associated with the norms (7.274). Let $\psi_1(y)$, $\phi_2(y)$ and $\phi_3(y)$ be square-integrable functions, the definition of the adjoint implies

$$(\psi_1, \bar{\mathcal{A}}\mathbf{q}) = (\bar{\mathcal{A}}^{+}\psi_1, \mathbf{q}) = (\boldsymbol{\Phi}, \mathbf{q}) \tag{C.2}$$

where

$$(\psi_1, \bar{\mathcal{A}}\mathbf{q}) = \int_0^\infty \psi_1(y) u_1(y)\ dy \qquad (\boldsymbol{\Phi}, \mathbf{q}) = \int_0^\infty (\phi_2 v_0 + \phi_3 w_0)\ dy. \tag{C.3}$$

Thus, the action of the adjoint $\bar{\mathcal{A}}^{+}$ on ψ_1 is the vector $\boldsymbol{\Phi}$. We have used the notation $u_1(y) = u(x_f, y)$.

We will derive an expression for the action of $\bar{\mathcal{A}}^{+}$ by integration by parts of the governing equations. Introducing compact notation, we start by writing equations (7.259)-(7.262) as a system

$$(\mathbf{Af})_x = \mathbf{B}_0\mathbf{f} + \mathbf{B}_1\mathbf{f}_y + \mathbf{B}_2\mathbf{f}_{yy} \tag{C.4}$$

where

$$\mathbf{f} = (u, v, w, p)^T = \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ U & 0 & 0 & 0 \\ V & U & 0 & 0 \\ 0 & 0 & U & 0 \end{pmatrix} \qquad \mathbf{B}_0 = \begin{pmatrix} 0 & 0 & -\beta & 0 \\ -\beta^2 & -U_y & 0 & 0 \\ 0 & -2V_y - \beta^2 & -\beta V & 0 \\ 0 & 0 & -V_y - \beta^2 & \beta \end{pmatrix}$$

$$\mathbf{B}_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -V & 0 & 0 & 0 \\ 0 & -2V & 0 & -1 \\ 0 & 0 & -V & 0 \end{pmatrix} \qquad \mathbf{B}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The initial condition for system (C.4) is

$$\mathbf{f}(x_0, y) = \begin{pmatrix} 0 \\ 0 \\ v_0 \\ w_0 \end{pmatrix}.$$

A remark on notation: Most variables in this section, $\mathbf{f}$, $\mathbf{A}$, and $\mathbf{B}_1$ for instance, are vector- or matrix functions of both the streamwise and wall-normal coordinates x and y, and we will sometimes use $\mathbf{f}(x)$ as shorthand for the function $y \mapsto \mathbf{f}(x, y)$ and $\mathbf{f}(y)$ for the function $x \mapsto \mathbf{f}(x, y)$.

Let

$$\mathbf{g} = (p^+, u^+, v^+, w^+)^T = (p^+(x,y), u^+(x,y), v^+(x,y), w^+(x,y))^T = \begin{pmatrix} p^+ \\ \mathbf{u}^+ \end{pmatrix}$$

be a smooth vector function defined on $x \geq x_0$, $y \geq 0$. We will impose further restrictions on $\mathbf{g}$ as we proceed, allowing the identification of the components of $\mathbf{g}$ as pressure- and velocity-like 'adjoint' variables, as the notation suggests. For now, however, $\mathbf{g}$ could be any suitable function.

Taking the scalar product of the vector $\mathbf{g}$ with equation (C.4), integrating over the domain $[x_0, x_f] \times [0, y_{\max}]$, and applying integration by parts yields

$$
\begin{aligned}
0 &= \int_0^\infty \int_{x_0}^{x_f} \mathbf{g}^T \left[(\mathbf{A}\mathbf{f})_x - \mathbf{B}_0\mathbf{f} - \mathbf{B}_1\mathbf{f}_y - \mathbf{B}_2\mathbf{f}_{yy} \right] dx\, dy \\
&= \int_0^\infty \int_{x_0}^{x_f} \mathbf{f}^T \left[-\mathbf{A}^T\mathbf{g}_x - \mathbf{B}_0^T\mathbf{g} + (\mathbf{B}_1^T\mathbf{g})_y - \mathbf{B}_2^T\mathbf{g}_{yy} \right] dx\, dy \\
&\quad + \int_0^\infty \mathbf{f}^T(x_f)\mathbf{A}^T(x_f)\mathbf{g}(x_f)\, dy - \int_0^\infty \mathbf{f}^T(x_0)\mathbf{A}^T(x_0)\mathbf{g}(x_0)\, dy \\
&\quad - \int_{x_0}^{x_f} \left[\mathbf{f}^T(y)\mathbf{B}_1^T(y)\mathbf{g}(y) + \mathbf{f}_y^T(y)\mathbf{B}_2^T\mathbf{g}(y) - \mathbf{f}^T(y)\mathbf{B}_2^T\mathbf{g}_y(y) \right] dx \Bigg|_{y=0}^{y=y_{\max}} .
\end{aligned}
\tag{C.5}
$$

Now we require that $\mathbf{g}$ satisfies the *adjoint equation*

$$
-\mathbf{A}^T\mathbf{g}_x = \mathbf{B}_0^T\mathbf{g} - (\mathbf{B}_1^T\mathbf{g})_y + \mathbf{B}_2^T\mathbf{g}_{yy}
$$

whose components are given in equation (7.281)-(7.284), i.e.,

$$
\begin{aligned}
&v_y^+ + \beta w^+ = 0 \\
&-p_x^+ - Uu_x^+ - Vv_x^+ - Vu_y^+ = u_{yy}^+ + (V_y - \beta^2)u^+ \\
&-Uv_x^+ - 2Vv_y^+ + U_y u^+ - p_y^+ = v_{yy}^+ - \beta^2 v^+ \\
&-Uw_x^+ - Vw_y^+ + \beta V v^+ + \beta p^+ = w_{yy}^+ - \beta^2 w^+ .
\end{aligned}
\tag{C.6}
$$

Appropriate boundary conditions for the adjoint equations can be deduced by expanding the last integral in (C.5),

$$
\begin{aligned}
&- \int_{x_0}^{x_f} \left[\mathbf{f}^T(y)\mathbf{B}_1^T(y)\mathbf{g}(y) + \mathbf{f}_y^T(y)\mathbf{B}_2^T\mathbf{g}(y) - \mathbf{f}^T(y)\mathbf{B}_2^T\mathbf{g}_y(y) \right] dx \Bigg|_{y=0}^{y=y_{\max}} \\
&= \int_{x_0}^{x_f} \Big[v(y_{\max}) \left(v_y^+(y_{\max}) + 2V(y_{\max})v^+(y_{\max}) + p^+(y_{\max}) \right) \\
&\qquad - u_y(y_{\max})u^+(y_{\max}) - w_y(y_{\max})w^+(y_{\max}) \Big]\, dx \\
&\quad + \int_{x_0}^{x_f} \Big[u_y(0)u^+(x,0) + v_y(0)v^+(0) + w_y(0)w^+(0) \\
&\qquad - p(0)v^+(0) \Big]\, dx
\end{aligned}
\tag{C.7}
$$

where the boundary conditions (7.264) have been used. Expression (C.7) vanishes if the boundary conditions (7.285), i.e.,

$$
u^+ = v^+ = w^+ = 0 \qquad \text{at} \quad y = 0 \tag{C.8}
$$

$$
p^+ + 2Vv^+ + v_y^+ = u^+ = w^+ = 0 \qquad \text{at} \quad y = y_{\max}. \tag{C.9}
$$

are enforced. If $\mathbf{g}$ satisfies equation (C.6) with the wall-normal boundary conditions (C.8), expression (C.5) reduces to

$$\int_0^\infty \mathbf{f}^T(x_f)\mathbf{A}^T(x_f)\mathbf{g}(x_f)\,dy = \int_0^\infty \mathbf{f}^T(x_0)\mathbf{A}^T(x_0)\mathbf{g}(x_0)\,dy. \tag{C.10}$$

Choosing the initial conditions on the adjoint equations given in (7.286), i.e.,

$$\begin{aligned} U(x_f,y)u^+(x_f,y) + V(x_f,y)v^+(x_f,y) + p^+(x_f,y) &= \psi_1(y) \\ v^+(x_f,y) &= 0 \\ w^+(x_f,y) &= 0 \end{aligned} \tag{C.11}$$

the left hand side of equation (C.10) reduces to

$$\int_0^\infty \mathbf{f}^T(x_f)\mathbf{A}^T(x_f)\mathbf{g}(x_f)\,dy = \int_0^\infty u_1\psi_1\,dy. \tag{C.12}$$

Using the initial conditions on the right-hand side of (C.10) yields

$$\int_0^\infty \mathbf{f}^T(x_0)\mathbf{A}^T(x_0)\mathbf{g}(x_0)\,dy = \int_0^\infty \left[U(x_0)v_0v_0^+ + U(x_0)w_0w_0^+\right]\,dy. \tag{C.13}$$

Matching this expression with (C.3) we obtain the action of the adjoint operator

$$\begin{aligned} \phi_2(y) &= U(x_0,y)v^+(x_0,y) \\ \phi_3(y) &= U(x_0,y)w^+(x_0,y). \end{aligned} \tag{C.14}$$

Appendix D
Selected Problems on Part I

Chapter 2

1. Piecewise linear channel flow

   Consider piecewise linear channel flow given by

$$U(y) = \begin{cases} 1 - y & 0 \le y \le 1 \\ 1 + y & -1 \le y \le 0 \end{cases} \tag{D.1}$$

   and determine the dispersion relation for waves in a inviscid fluid. What is the group velocity of the waves? Use the group velocity to show that short waves are found in the front of the dispersive disturbance and that long waves are in the back.

2. Numerical solution of the Rayleigh equation

   Solve the Rayleigh equation numerically for the following velocity profiles:

   (a) Bickley jet, $U = 1/\cosh^2 y$, $-\infty < y < \infty$.
   i) Verify that $v = 1/\cosh^2 y$, $\alpha = 2$, $c = 2/3$ represents a neutral symmetric solution and that $v = \tanh y/\cosh y$, $\alpha = 1$, $c = 2/3$ is a neutral antisymmetric solution.
   ii) For parameter combinations that result in stable solutions the integration contour has to be extended into the complex plane,

around the critical layer singularity. Compute stable solutions of the Rayleigh equation for the Bickley jet.

(b) Poiseuille flow, $U = 1 - y^2$, $-1 < y < 1$.

Plot the phase speed and growth rate as a function of the wavenumber for the odd and the even mode. Compare your results to the analytic dispersion relation for piecewise linear Poiseuille flow.

Hint: Write a local eigenvalue solver for the Rayleigh equation and declare all quantities complex. Then choose an appropriate integration path in the complex plane around the critical layer singularity.

Chapter 3

1. Stability of streamwise vortices

Show that the normal modes in plane Poiseuille flow for $\alpha = 0$ have zero phase speed and a decay rate inversely proportional to the Reynolds number. Note that these disturbances are slowly decaying streamwise vortices or streaks.

2. Three-dimensional mean velocity profile

(a) Derive the Orr-Sommerfeld equation for a three-dimensional parallel mean flow, such as a Falkner-Skan-Cooke boundary layer. Assume that the mean velocity profile has the form

$$U = U(y) \qquad\qquad W = W(y). \tag{D.2}$$

Also derive the generalization of Squire's transformation for this mean flow.

(b) Consider three-dimensional channel flow of the form

$$U = 1 - y^2 \qquad\qquad W = (1 - y^2)y^2. \tag{D.3}$$

Using the inflection point criterion, determine the region in wave number space (α, β) where the inviscid flow must be stable. Plot your results.

3. Benard convection I

The linearized equations for Benard convection between two infinite plates located at $y = 0$ and $y = 1$ are given as

$$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} + \text{Pr Ra } \theta\delta_{i2} + \text{Pr } \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{D.4}$$

$$\frac{\partial \theta}{\partial t} = v + \frac{\partial^2 \theta}{\partial x_j \partial x_j} \tag{D.5}$$

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{D.6}$$

The boundary conditions are

$$\theta = v = \frac{\partial v}{\partial y} = 0 \qquad \text{at} \quad y = 0, 1 \tag{D.7}$$

for the rigid-rigid problem and

$$\theta = v = \frac{\partial^2 v}{\partial y^2} = 0 \qquad \text{at} \quad y = 0, 1 \tag{D.8}$$

for the free-free problem.

Derive a stability equation for the normal velocity v and compute solutions for both the rigid-rigid and the free-free problem.

4. Benard convection II

Show that the solution for the free-free Benard problem derived in the previous exercise

$$v = \text{Real}\{\tilde{v}(y) e^{i(\alpha x + \beta z)} e^{st}\} \tag{D.9}$$

takes the form of counter rotating rolls in physical space. Recall that the horizontal velocities can be determined from continuity and the definition of the normal vorticity.

5. Stability of inviscid, stratified flow

Consider the stability of a parallel flow of an inviscid, stratified fluid. The governing, non-dimensionalized equations can be written,

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} \tag{D.10}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y}-\frac{\rho}{F^2} \tag{D.11}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \tag{D.12}$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y} = 0. \tag{D.13}$$

We assume that there exists a basic state $U(y), P(y), \overline{\rho}(y)$ which is an exact solution to the above equations. Introduce wave-like disturbances and derive the linear stability equation for the normal velocity v,

$$(U-c)(D^2-\alpha^2)\hat{v}-U''\hat{v}+\frac{N^2\hat{v}}{(U-c)}=0 \tag{D.14}$$

where we have assumed that

$$\frac{\overline{\rho}'}{\overline{\rho}} \ll \frac{\overline{\rho}'}{\overline{\rho}F^2}. \tag{D.15}$$

The parameter $N^2 = -\overline{\rho}'/\overline{\rho}F^2$ is known as the overall Richardson number.

Solve the equation for the simplified case $U=0, N=$ constant and find the dispersion relation for internal gravity waves.

Chapter 4

1. Model problem for transient analysis

   We will investigate a linear evolution equation of the form

$$\frac{d}{dt}\begin{pmatrix} v \\ \eta \end{pmatrix} = \begin{pmatrix} -\frac{1}{\mathrm{Re}} & 0 \\ 1 & -\frac{2}{\mathrm{Re}} \end{pmatrix}\begin{pmatrix} v \\ \eta \end{pmatrix} \tag{D.16}$$

where the variables v and η resemble the normal velocity and normal vorticity, respectively, and Re denotes an equivalent Reynolds number. The influence of the velocity on the vorticity is accounted for by the off-diagonal term in the evolution operator.

Let the system matrix of the linear system (D.16) be called $\mathbf{A}$ and $\mathbf{q} = (v, \eta)^T$, $\|\mathbf{q}\|^2 = v^2 + \eta^2$, and $G(t) = \max_{\mathbf{q}_0} \frac{\|\mathbf{q}(t)\|}{\|\mathbf{q}_0\|}$.

(a) Show that the eigenvalues and normalized eigenvectors of the model equation are

$$\lambda_1 = -\frac{1}{\mathrm{Re}} \qquad \Phi_1 = \frac{1}{\sqrt{1+\mathrm{Re}^2}} \begin{pmatrix} 1 \\ \mathrm{Re} \end{pmatrix} \tag{D.17}$$

$$\lambda_2 = -\frac{2}{\mathrm{Re}} \qquad \Phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{D.18}$$

and that the complete solution can be written as

$$\begin{pmatrix} v \\ \eta \end{pmatrix} = \frac{A}{\sqrt{1+\mathrm{Re}^2}} \begin{pmatrix} 1 \\ \mathrm{Re} \end{pmatrix} e^{-t/Re} + B \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t/Re}. \tag{D.19}$$

How are the constants A and B determined?

(b) The formal solution of the initial value problem can be represented

$$\mathbf{q}(t) = e^{t\mathbf{A}} \mathbf{q}_0 \tag{D.20}$$

with $\mathbf{q}_0$ as the vector consisting of the initial velocity and vorticity. Show that

$$G(t) = \|e^{t\mathbf{A}}\| \tag{D.21}$$

where

$$e^{t\mathbf{A}} = \begin{pmatrix} e^{-t/Re} & 0 \\ -(e^{-2t/Re} - e^{-t/Re})\mathrm{Re} & e^{-2t/Re} \end{pmatrix} \tag{D.22}$$

Verify the results in Figure D.1(b) numerically.

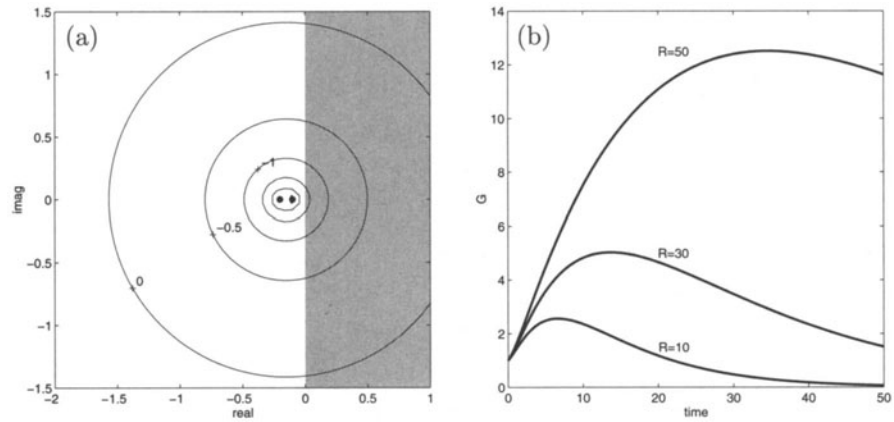

FIGURE D.1. Resolvent norm and growth curves for model problem. (a) Resolvent plot. (b) Maximum amplification for selected Reynolds numbers.

(c) A number $z \in \mathbf{C}$ lies in the ε-pseudospectrum of a matrix $\mathbf{A}$, if $\|(z\mathbf{I}-\mathbf{A})^{-1}\| \geq \varepsilon^{-1}$. Show that the quantity $(z\mathbf{I}-\mathbf{A})^{-1}$, known as the reslovent, is given as

$$(z\mathbf{I}-\mathbf{A})^{-1} = \begin{pmatrix} \dfrac{\mathrm{Re}}{z\mathrm{Re}+1} & 0 \\ \dfrac{\mathrm{Re}^2}{(z\mathrm{Re}+1)(z\mathrm{Re}+2)} & \dfrac{\mathrm{Re}}{z\mathrm{Re}+2} \end{pmatrix}. \tag{D.23}$$

Verify the results in Figure D.1(a) numerically.

(d) Recall that the reason for the large transient growth of the disturbance norm lies in the nonnormal nature of the evolution operator $\mathbf{A}$, show that $\mathbf{A}^T\mathbf{A} \neq \mathbf{A}\mathbf{A}^T$, and that the angle between the non-orthogonal eigenvectors are

$$\phi = \arccos\left(\frac{\mathrm{Re}}{\sqrt{1+\mathrm{Re}^2}}\right) \tag{D.24}$$

(e) Let $\|\mathbf{q}\| = 1$ and show that the numerical range $\mathcal{F}$ satisfies

$$\frac{d}{dt}\|\mathbf{q}\|^2 = -\frac{2(2v^2+\eta^2)}{\mathrm{Re}} + 2v\eta = \mathcal{F}, \tag{D.25}$$

and that $\mathcal{F}_{\max}$ satisfies

$$\mathcal{F}_{\max} = -\frac{1}{2\mathrm{Re}}(3 - \frac{1}{\sqrt{1+\mathrm{Re}^2}}) + \frac{\mathrm{Re}}{2\sqrt{1+\mathrm{Re}^2}} = \left.\frac{dG}{dt}\right|_{t=0} \tag{D.26}$$

(f) Let Re_G be the Reynolds number for which $\mathcal{F}_{\max}$ is equal to zero, or, equivalently, the Reynolds number below which no transient growth can be expected. Verify that

$$\mathrm{Re}_g = \sqrt{8} \qquad \text{and thus} \qquad \frac{dG}{dt} < 0 \quad \text{for Re} < \sqrt{8}. \tag{D.27}$$

2. Algebraic growth of vorticity

An expression for the Fourier transformed normal vorticity ($\hat{\eta}$) for $\alpha = 0$ can be derived as

$$\hat{\eta} = \sum_m \tilde{\eta}_m \sum_j \frac{i\mathrm{Re}\beta K_j}{\mu_m - \nu_j} \int_{-1}^{1} U' \tilde{v}_j \tilde{\eta}_m dy \left(e^{-\mu_m t/Re} - e^{-\nu_j t/Re} \right) \tag{D.28}$$

where $\alpha c_m = -i\mu_m/\mathrm{Re}$ are the Squire eigenvalues and $\alpha c_j = -i\nu_j/\mathrm{Re}$ are the Orr-Sommerfeld eigenvalues, both for $\alpha = 0$.

In Chapter 4 the first term in the expansion of the above expression for $t/\mathrm{Re} \to 0$ has been calculated. We found that growth proportional to t resulted. Calculate the next term in the expansion and show that the maximum growth is proportional to Re and that this maximum occurs for times proportional to Re. Simplify your expressions as much as possible.

3. Algebraic instability

Derive the algebraic instability directly from the governing linearized inviscid disturbance equations in primitive variables. Assume a streamwise independent flow, i.e., $U_i = U(y)\delta_{i1}$ and $\partial/\partial x = 0$, but do not use Fourier-transformed variables.

Chapter 5

1. Nonlinear model equation

To investigate the effects of nonlinearities on transient growth we will consider the nonlinear system of equations for a disturbance described by $(v \;\; \eta)^T$

$$\frac{d}{dt}\begin{pmatrix} v \\ \eta \end{pmatrix} = \begin{pmatrix} -1/\mathrm{Re} & 0 \\ 1 & -2/\mathrm{Re} \end{pmatrix}\begin{pmatrix} v \\ \eta \end{pmatrix} + \sqrt{v^2+\eta^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} v \\ \eta \end{pmatrix} \tag{D.29}$$

The parameter Re plays the role of the Reynolds number. The linear operator in the above system of equations will be referred to as **A**.

(a) Derive an equation for the 2-norm of $(v \;\; \eta)^T$ and interpret your result.

(b) Use the expression for the resolvent of the linear operator **A** given in (D.23) and plot the resolvent norm for various values of Re. What can be said about possible energy growth of the solutions to the disturbance equations? How does the resolvent norm vary with Re?

(c) Use the expression for the matrix exponential $e^{\mathbf{A}t}$ given in (D.22) and plot the growth function for various values of Re. How good is the bound on the growth derived above? How does $e^{\mathbf{A}t}$ vary with Re? Calculate the initial conditions giving the optimal growth for a few selected parameter combinations.
Hint: $e^{\mathbf{A}t} = Ve^{\Lambda t}V^{-1}$ where $\mathbf{A} = V\Lambda V^{-1}$ is the eigendecomposition of **A** with V as the eigenvector matrix and Λ as the diagonal eigenvalue matrix.

(d) Solve the nonlinear disturbance equations numerically and determine the threshold amplitude for transition as a function of Re. How does the threshold amplitude scale with Re? Transition has occurred when the solution does not decay as t tends to infinity. Use optimal disturbances as initial conditions.

2. Energy method for Benard convection

(a) Show that the energy equation for Benard convection reads

$$\frac{dE}{dt} = -\int_V \left[u_i u_j \frac{\partial U_i}{\partial x_j} - \mathrm{Pr}\left(\frac{\partial u_i}{\partial x_j}\right)^2 + 2\mathrm{Pr}\;\mathrm{Ra}\; v\theta - \mathrm{Pr}\;\mathrm{Ra}\left(\frac{\partial \theta}{\partial x_j}\right)^2 \right] dV \tag{D.30}$$

(b) Show that the Euler-Lagrange equations for this problem are identical to the linear stability equations for marginal stability of Benard convection.

3. Energy method for plane Couette flow

   (a) Derive the Reynolds-Orr equation for general shear flows.

   (b) Show that the corresponding Euler-Lagrange equations are

$$\frac{1}{2}u_j\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)=-\frac{\partial \lambda}{\partial x_i}+\frac{1}{\mathrm{Re}}\nabla^2 u_i \tag{D.31}$$

   where λ is the Lagrange multiplier associated with the constraint from the continuity equation.

   (c) Let $U_i = y\delta_{i1}$, $\partial/\partial x = 0$, $u_i = \hat{u}_i(y)\exp[i\beta z]$ and show that the resulting equations can be identified with the Benard rigid-rigid problem. Deduce the energy Reynolds number Re_E for plane Couette flow for this case.

4. Comment on the differences in Ra_E and Ra_L for Benard convection and Re_E and Re_L for plane Couette flow.

Bibliography

ABRAMOWITZ, M. & STEGUN, I. A. 1973 *Handbook of Mathematical Functions*. Dover Publications.

ABU-GHANAMM, B. J. & SHAW, R. 1980 Natural transition of boundary layers – the effects of turbulence, pressure gradient, and flow history. *J. Mech. Eng. Sci.* **22**, 213–228.

ACARLAR, M. S. & SMITH, C. R. 1987 A study of hairpin vortices in a laminar boundary layer. Part 2. Hairpin vortices generated by fluid injection. *J. Fluid Mech.* **175**, 43–48.

AIHARA, Y., TOMITA, Y. & ITO, A. 1985 Generation, development and distortion of longitudinal vortices in the boundary layer along concave and flat plates. In *Laminar-Turbulent Transition* (ed. V. V. Kozlov), pp. 447–454. Springer-Verlag.

ALFREDSSON, P. H. & MATSUBARA, M. 1996 Streaky structures in transition. In *Transitional Boundary Layers in Aeronautics.* (ed. R. Henkes & J. van Ingen), pp. 374–386. Elsevier Science Publishers.

ALFREDSSON, P. H. & MATSUBARA, M. 2000 Freestream turbulence, streaky structures and transition in boundary layer flows. *AIAA* **2000-2534**.

ANDERSSON, P., BERGGREN, M. & HENNINGSON, D. 1997 Optimal disturbances in boundary layers. In *Progress in Systems Control.* Birkhäuser.

ANDERSSON, P., BERGGREN, M. & HENNINGSON, D. 1999 Optimal disturbances and bypass transition in boundary layers. *Phys. Fluids* **11** (1), 134–150.

ANDERSSON, P., HENNINGSON, D. & HANIFI, A. 1998 On a stabilization procedure for the parabolic stability equations. *J. Eng. Math.* **33**, 311–332.

ARNAL, D. 1987 Transition description and prediction. In *Numerical simulation of unsteady flows and transition to turbulence* (ed. O. Pironneau, W. Rodi, I. Ryhming, A. Savill & T. Truong), pp. 303–316. Cambridge University Press.

ARNAL, D. 1999 Transition prediction in industrial applications. In *Transition, Turbulence and Combustion Modelling* (ed. A. Hanifi, P. H. Alfredsson, A. V. Johansson & D. S. Henningson). ERCOFTAC Series, Vol. 6, Kluwer Academic Publishers, Dordrecht.

ASAI, M. & NISHIOKA, M. 1995 Boundary-layer transition triggered by hairpin eddies at subcritical Reynolds numbers. *J. Fluid Mech.* **297**, 101–122.

ASHPIS, D. E. & RESHOTKO, E. 1990 The vibrating ribbon problem revisited. *J. Fluid Mech.* **213**, 531–547.

BAGGETT, J. S. & TREFETHEN, L. N. 1997 Low-dimensional models of subcritical transition to turbulence. *Phys. Fluids* **9**, 1043–1053.

BAKE, S., FERNHOLZ, H. H. & KACHANOV, Y. S. 2000 Resemblance of K- and N-regimes of boundary-layer transition at late stages. *Europ. J. Mech. B* **19**, 1–22.

BAMIEH, B. & DAHLEH, M. 1999 Disturbance energy amplification in three-dimensional channel flows. In *Proceedings of the 1999 American Control Conference, IEEE, Piscataway, NJ*, pp. 4532–4537.

BAYLY, B. J. 1986 Three-dimensional instability of elliptical flow. *Phys. Lett. Rev.* **57**, 2160.

BAYLY, B. J., ORSZAG, S. A. & HERBERT, T. 1988 Instability mechanisms in shear flow transition. *Ann. Rev. Fluid Mech.* **20**, 359–91.

BECK, K. H., HENNINGSON, D. S. & HENKES, R. A. W. M. 1998 Linear and nonlinear development of localized disturbances in zero and adverse pressure gradient boundary layers. *Phys. Fluids.* **10**, 1405–1418.

BENJAMIN, T. B. 1961 The development of three-dimensional disturbances in an unstable film of liquid flowing down an inclined plane. *J. Fluid Mech.* **10**, 401–419.

BENJAMIN, T. B. & FEIR, J. 1967 The disintegration of wave trains on deep water. Part 1. Theory. *J. Fluid Mech.* **27**, 417–430.

BENNEY, D. J. & GUSTAVSSON, L. H. 1981 A new mechanism for linear and nonlinear hydrodynamic instability. *Stud. Appl. Math.* **64**, 185–209.

BERGSTRÖM, L. 1993 Optimal growth of small disturbances in pipe Poiseuille flow. *Phys. Fluids A* **5**, 2710–2720.

BERGSTRÖM, L. 1999 Interactions of three components and subcritical self-sustained amplification of disturbances in plane Poiseuille flow. *Phys. Fluids* **11**, 590–601.

BERLIN, S. 1998 Oblique waves in boundary layer transition. PhD thesis, Royal Institute of Technology, Stockholm, Sweden.

BERLIN, S., HANIFI, A. & HENNINGSON, D. S. 1998 The neutral stability curve for non-parallel boundary layer flow. In: Berlin, S.: Oblique waves in boundary layer transition, Dissertation, Royal Institute of Technology, Stockholm.

BERLIN, S. & HENNINGSON, D. S. 1999 A nonlinear mechanism for receptivity of free-stream disturbances. *Phys. Fluids* **11**, 3749–3760.

BERLIN, S., LUNDBLADH, A. & HENNINGSON, D. S. 1994 Spatial simulations of oblique transition. *Phys. Fluids* **6**, 1949–1951.

BERLIN, S., WIEGEL, M. & HENNINGSON, D. S. 1999 Numerical and experimental investigations of oblique boundary layer transition. *J. Fluid Mech.* **393**, 23–57.

BERTOLOTTI, F. P. 1991 Linear and nonlinear stability of boundary layers with streamwise varying properties. PhD thesis, Ohio State University, Department of Mechanical Engineering, Columbus, Ohio.

BERTOLOTTI, F. P. 1997 Response of the blasius boundary layer to free-stream vorticity. *Phys. Fluids* **9** (8), 2286–2299.

BERTOLOTTI, F. P., HERBERT, T. & SPALART, P. R. 1992 Linear and nonlinear stability of the Blasius boundary layer. *J. Fluid Mech.* **242**, 441–474.

BESTEK, H., GRUBER, K. & FASEL, H. 1989 Self-excited unsteadiness of laminar separation bubbles by natural transition. In *The Prediction and Exploitation of Separated Flow*, pp. 14.1–14.16. Proceedings of the Royal Aeronautical Society.

BETCHOV, R. & CRIMINALE, W. O. 1966 Spatial instability of the inviscid jet and wake. *Phys. Fluids* **9**, 359–362.

BIPPES, H. 1978 Experiments on Görtler vortices. Proceedings of the Royal Aeronautical Society Conference on Boundary-Layer Tranisiton and Control, Cambridge University, England.

BOBERG, L. & BROSA, U. 1988 Onset of turbulence in a pipe. *Z. Naturforsch.* **43a**, 697–726.

BODONYI, R. J. 1990 Nonlinear triple-deck studies in boundary layer receptivity. *Appl. Mech. Rev.* **43** **(5)**, 158–165.

BODONYI, R. J. & SMITH, F. T. 1981 The upper branch stability of the Blasius boundary layer, including non-parallel flow effects. *Proc. Roy. Soc. Lond. Ser. A* **375**, 65–92.

BOTTARO, A. & KLINGMANN, B. G. B. 1996 On the linear breakdown of Görtler vortices. *Europ. J. Mech. B* **15** **(3)**, 301–330.

BOTTARO, A. & LUCHINI, P. 1999 Görtler vortices: are they amenable to local eigenvalue analysis? *Europ. J. Mech. B* **18** **(1)**, 47–65.

BOYCE, W. & DIPRIMA, R. 1997 *Elementary Differential Equations and Boundary Value Problems*. John Wiley & Sons.

BREUER, K. S., COHEN, J. & HARITONIDIS, J. H. 1997 The late stages of transition induced by a low-amplitude wave packet in a laminar boundary layer. *J. Fluid Mech.* **340**, 395–411.

BREUER, K. S. & HARITONIDIS, J. H. 1990 The evolution of a localized disturbance in a laminar boundary layer. Part I: Weak disturbances. *J. Fluid Mech.* **220**, 569–594.

BREUER, K. S. & KURAISHI, T. 1994 Transient growth in two- and three-dimensional boundary layers. *Phys. Fluids* **6**, 1983–1993.

BRIGGS, R. J. 1964 *Electron-Stream Interaction with Plasmas*. MIT Press, Cambridge.

BROWN, W. B. 1961 A stability criterion for three-dimensional laminar boundary layers. In *Boundary Layer and Flow Control* (ed. G. Lachmann), , vol. 2, pp. 313–329. Pergamon Press, London.

BROWN, W. B. 1962 Exact numerical solution of the complete linearized equations for the stability of compressible boundary layers. *Tech. Rep.* NOR-62-15. NorAir Report.

BURRIDGE, D. & DRAZIN, P. 1969 Comments on 'Stability of pipe Poiseuille flow'. *Phys. Fluids* **12** **(1)**, 264–265.

BUSSE, F. H. & WHITEHEAD, J. A. 1971 Instabilities of convection rolls in a high Prandtl number fluid. *J. Fluid Mech.* **47**, 305–320.

BUTLER, K. M. & FARRELL, B. F. 1992 Three-dimensional optimal perturbations in viscous shear flow. *Phys. Fluids A* **4**, 1637–1650.

CANTWELL, B. J., COLES, D. & DIMOTAKIS, P. E. 1978 Structure and entrainment in the plane of symmetry of a turbulent spot. *J. Fluid Mech.* **87**, 641–672.

CARLSON, D. R., WIDNALL, S. E. & PEETERS, M. F. 1982 A flow visualization study of transition in plane Poiseuille flow. *J. Fluid Mech.* **121**, 487–505.

CARR, J. 1981 *Applications of Centre Manifold Theory*. Springer-Verlag.

CASE, K. M. 1960 Stability of inviscid plane Couette flow. *Phys. Fluids* **3**, 143–148.

CAULFIELD, C. P. & KERSWELL, R. R. 2000 The nonlinear development of three-dimensional disturbances at hyperbolic stagnation points: a model of the braid region in mixing layers. *Phys. Fluids* **12**, 1032–1043.

CHAMBERS, F. W. & THOMAS, A. S. W. 1983 Turbulent spots, wave packets and growth. *Phys. Fluids* **26**, 1160–1162.

CHANG, C.-L. & MALIK, M. R. 1994 Oblique-mode breakdown and secondary instability in supersonic boundary layers. *J. Fluid Mech.* **273**, 323–360.

CHIN, R. W.-Y. 1981 Stability of flows down an inclined plane. PhD thesis, Harvard University, Division of Applied Sciences, Cambridge, Mass.

CHOMAZ, J.-M., HUERRE, P. & REDEKOPP, L. G. 1991 A frequency selection criterion in spatially developing flows. *Stud. Appl. Math.* **84**, 119–144.

CHOUDHARI, M. 1996 Boundary-layer receptivity to three-dimensional unsteady vortical disturbances in free stream. AIAA Paper 96-0181.

COHEN, J., BREUER, K. S. & HARITONIDS, J. H. 1991 On the evolution of a wave packet in a laminar boundary layer. *J. Fluid Mech.* **225**, 575–606.

CONTE, S. D. 1966 The numerical solution of linear boundary value problems. *Stud. Appl. Math.* **8** (3), 309–321.

COOKE, J. C. 1950 The boundary layer of a class of infinite yawed cylinders. *Proc. Camb. Phil. Soc.* **46**, 645–648.

CORBETT, P. & BOTTARO, A. 2000*a* Optimal linear growth in three-dimensional boundary layers. (submitted).

CORBETT, P. & BOTTARO, A. 2000*b* Optimal perturbations for boundary layers subject to streamwise pressure gradient. *Phys. Fluids* **12** (1), 120–130.

COSSU, C. & CHOMAZ, J.-M. 1997 Global measures of local convective instabilities. *Phys. Rev. Lett.* **78 (23)**, 4387–4390.

CRAIK, A. D. D. 1971 Nonlinear resonant instability in boundary layers. *J. Fluid Mech.* **50**, 393–413.

CRAIK, A. D. D. 1985 *Wave interactions and fluid flows*. Cambridge University Press.

CROUCH, J. D. 1992*a* Localized receptivity of boundary layers. *Phys. Fluids* **4 (7)**, 1408–1414.

CROUCH, J. D. 1992*b* Non-localized receptivity of boundary layers. *J. Fluid Mech.* **244**, 567–581.

CROUCH, J. D. 1997 Excitation of secondary instabilities in boundary layers. *J. Fluid Mech.* **336**, 245–266.

CROUCH, J. D. & NG, L. L. 2000 Variable N-factor method for transition prediction in three-dimensional boundary layers. *AIAA J.* **38**, 211–216.

CROUCH, J. D. & SPALART, P. R. 1995 A study of non-parallel and nonlinear effects on the localized receptivity of boundary layers. *J. Fluid Mech.* **290**, 29–37.

DAVIS, S. H. & REID, W. J. 1977 On the stability of stratified viscous plane Couette flow. *J. Fluid Mech.* **80**, 509.

DEYHLE, H. & BIPPES, H. 1996 Disturbance growth in an unstable three-dimensional boundary layer and its dependence on environmental conditions. *J. Fluid Mech.* **316**, 73–113.

DIKII, L. A. 1960 The stability of plane-parallel flows of an ideal fluid. *Sov. Phys. Doc.* **135**, 1179–1182.

DIPRIMA, R. C. 1967 Vector eigenfunction expansions for the growth of Taylor vortices in the flow between rotating cylinders. In *Nonlinear Partial Differential Equations* (ed. W. F. Ames), pp. 19–42. Academic Press.

DIPRIMA, R. C. & HABETLER, G. J. 1969 A completeness theorem for non-selfadjoint eigenvalue problems in hydrodynamic stability. *Arch. Rat. Mech. Anal.* **32**, 218–227.

DOLPH, C. L. & LEWIS, D. C. 1958 On the application of infinite systems of ordinary differential equations to perturbations to plane Poiseuille flow. *Q. Appl. Math.* **16**, 97–110.

DOVGAL, A. V., KOZLOV, V. V. & MICHALKE, A. 1994 Laminar boundary layer separation: instability and associated phenomena. *Prog. Aerospace Sci.* **30**, 61–94.

DRAZIN, P. G. & HOWARD, L. N. 1962 The instability to long waves of unbounded parallel shear flow. *J. Fluid Mech.* **14**, 257–283.

DRAZIN, P. G. & REID, W. H. 1981 *Hydrodynamic Stability*. Cambridge University Press.

VAN DRIEST, E. R. & BLUMER, C. B. 1963 Boundary layer transition: freestream turbulence and pressure gradient effects. *AIAA J.* **1**, 1303–1306.

DUNN, D. W. & LIN, C. C. 1955 On the stability of the laminar boundary layer in a compressible fluid. *J. Aero. Sci.* **22**, 455–477.

VAN DYKE, M. 1975 *Perturbation Methods in Fluid Mechanics*. Parabolic Press, Stanford.

ECKHAUS, W. 1965 *Studies in Non-Linear Stability*. Springer-Verlag.

EHRENSTEIN, U. & KOCH, W. 1991 Three-dimensional wavelike equilibrium states in plane Poiseuille flow. *J. Fluid Mech.* **228**, 111–148.

EL-HADY, N. M. 1991 Nonparallel instability of supersonic and hypersonic boundary layers. *Phys. Fluids* **3 (9)**, 2164–2178.

ELLINGSEN, T. & PALM, E. 1975 Stability of linear flow. *Phys. Fluids* **18**, 487–488.

ELOFSSON, P. A. 1998 Experiment on oblique transition in wall bounded shear flows. PhD thesis, Royal Institute of Technology, Stockholm, Sweden.

ELOFSSON, P. A. & ALFREDSSON, P. H. 1998 An experimental study of oblique transition in plane Poiseuille flow. *J. Fluid Mech.* **358**, 177–202.

ELOFSSON, P. A., KAWAKAMI, M. & ALFREDSSON, P. H. 1999 Experiments on the stability of streamwise streaks in plane Poiseuille flow. *Phys. Fluids* **11 (4)**, 915–930.

FARRELL, B. F. & IOANNOU, P. J. 1993 Stochastic forcing of the linearized Navier-Stokes equations. *Phys. Fluids A* **5 (11)**, 2600–2609.

FARRELL, B. F. & IOANNOU, P. J. 1996 Generalized stabilty theory. II. Nonautonomous operators. *J. Atm. Sci.* **53**, 2041–2053.

FASEL, H., THUMM, A. & BESTEK, H. 1993 Direct numerical simulation of transition in supersonic boundary layers: oblique breakdown. In *Transitional and Turbulent Compressible Flows* (ed. L. D. Kral & T. A. Zang), pp. 77–92. FED-Vol. 151, ASME.

FINLAY, W. H., KELLER, J. B. & FERZIGER, J. H. 1988 Instabilities and transition in curved channel flow. *J. Fluid Mech.* **194**, 417–456.

FISCHER, T. M. & DALLMANN, U. 1991 Primary and secondary stability analysis of a three-dimensional boundary layer flow. *Phys. Fluids A* **3**, 2378–2391.

FJØRTOFT, R. 1950 Application of integral theorems in deriving criteria for instability for laminar flows and for the baroclinic circular vortex. *Geofys. Publ., Oslo* **17** (6), 1–52.

FLORYAN, J. M., DAVIS, S. H. & KELLY, R. E. 1987 Instabilities of a liquid film flowing down an inclined plane. *Phys. Fluids* **30 (4)**, 983–989.

FLORYAN, J. M. & SARIC, W. 1982 Stability of Görtler vortices in boundary layers. *AIAA J.* **20 (3)**, 316–324.

FUJIMURA, K. 1989 The equivalence between two perturbation methods in weakly nonlinear stability theory for parallel shear flows. *Proc. Roy. Soc. Lond. A* **424**, 373–392.

FUJIMURA, K. 1991 Methods of centre manifold and multiple scales in the theory of weakly nonlinear stability for fluid motions. *Proc. Roy. Soc. Lond. A* **434**, 719–733.

GANTMACHER, F. R. 1959 *The Theory of Matrices*. Chelsea Pub. Co., New York.

GASTER, M. 1962 A note on the relation between temporally-increasing and spatially-increasing disturbances in hydrodynamic stability. *J. Fluid Mech.* **14**, 222–224.

GASTER, M. 1975 A theoretical model for the development of a wave packet in a laminar boundary layer. *Proc. Roy. Soc. Lond. Ser. A* **347**, 271–289.

GASTER, M. 1982*a* The development of a two-dimensional wave packet in a growing boundary layer. *Proc. Roy. Soc. Lond. Ser. A* **384**, 317–332.

GASTER, M. 1982*b* Estimates of the errors incurred in various asymptotic representations of wave packets. *J. Fluid Mech.* **121**, 365–377.

GILL, A. E. 1965 Instability of 'top-hat' jets and wakes in compressible fluids. *Phys. Fluids* **8**, 1428–1430.

GOLDSTEIN, M. E. 1983 The evolution of Tollmien-Schlichting waves near a leading edge. *J. Fluid Mech.* **127**, 59–81.

GOLDSTEIN, M. E. & HULTGREN, L. S. 1989 Boundary-layer receptivity to long-wave freestream disturbances. *Ann. Rev. Fluid Mech.* **21**, 137–166.

GOLDSTEIN, M. E. & WUNDROW, D. W. 1998 On the environmental realizability of algebraically growing disturbances and their relation to Klebanoff modes. *Theoret. Comput. Fluid Dynamics* **10**, 171–186.

GÖRTLER, H. 1940 Über eine dreidimensionale instabilität laminarer grenzschichten an konkaven wänden. *Nachr. Ges. Wiss. Göttingen, N.F.* **2**, 1–26, translated in *Tech. Memor. Nat. Adv. Comm. Aero., Wash.* **1375**, (1954).

GREGORY, N., STUART, J. T. & WALKER, W. S. 1955 On the stability of three-dimensional boundary layers with applications to flow due to a rotating disc. *Phil. Trans. R. Soc. Lond. A* **248**, 155–199.

GREY, W. E. 1952 The effect of wing sweep on laminar flow. RAE TM Aero 255.

GROSCH, C. E. & SALWEN, H. 1978 The continuous spectrum of the Orr-Sommerfeld equation. Part 1. The spectrum and the eigenfunctions. *J. Fluid Mech.* **87**, 33–54.

GUO, Y. & FINLAY, W. H. 1991 Splitting, merging and wavelength selection of vortices in curved and/or rotating channel flow due to Eckhaus instability. *J. Fluid Mech.* **228**, 661–691.

GUO, Y. & FINLAY, W. H. 1994 Wavenumber selection and irregularity of spatially developing nonlinear Dean and Görtler vortices. *J. Fluid Mech.* **264**, 1–40.

GUSTAVSSON, L. H. 1978 On the evolution of disturbances in boundary layer flows. PhD thesis, Royal Institute of Technology, Stockholm, Sweden, (TRITA-MEK 78-02).

GUSTAVSSON, L. H. 1979 Initial value problem for boundary layer flows. *Phys. Fluids* **22**, 1602–1605.

GUSTAVSSON, L. H. 1982 Dynamics of three-dimensional disturbances in watertable flow. TECHNICAL REPORT 1982:036 T, Department of Fluid Mechanics, Luleå University of Technology, Sweden.

GUSTAVSSON, L. H. 1986 Excitation of direct resonances in plane Poiseuille flow. *Stud. Appl. Math.* **75**, 227–248.

GUSTAVSSON, L. H. 1991 Energy growth of three-dimensional disturbances in plane Poiseuille flow. *J. Fluid Mech.* **224**, 241–260.

HAJ-HARIRI, H. 1988 Transformations reducing the order of the parameter in differential eigenvalue problems. *J. Comp. Phys.* **77**, 472–484.

HALL, P. 1983 The linear development of Görtler vortices in growing boundary layers. *J. Fluid Mech.* **130**, 41–58.

HALL, P. & HORSEMAN, N. J. 1991 The linear inviscid secondary instability of longitudinal vortex structures in boundary layers. *J. Fluid Mech.* **232**, 357–375.

HAMMERTON, P. W. & KERSCHEN, E. J. 1997 Boundary layer receptivity for a parabolic leading edge. Part 2. The small-Strouhal-number limit. *J. Fluid Mech.* **353**, 205–220.

HANIFI, A. & HENNINGSON, D. S. 1998 The compressible inviscid algebraic instability for streamwise independent disturbances. *Phys. Fluids* **10**, 1784–1786.

HANIFI, A., HENNINGSON, D. S., HEIN, S., BERTOLOTTI, F. & SIMEN, M. 1994 Linear nonlocal instability analysis – the linear NOLOT code. FFA TN 1994-54, Technical Report from the Aeronautical Research Institute of Sweden (FFA), Bromma.

HANIFI, A., SCHMID, P. J. & HENNINGSON, D. S. 1996 Transient growth in compressible boundary layer flow. *Phys. Fluids* **8**, 51–65.

HEALEY, J. 1995 On the neutral curve of the flat-plate boundary layer: comparison between experiment, Orr-Sommerfeld theory and asymptotic theory. *J. Fluid Mech.* **288**, 59–73.

HEALEY, J. 1998 Private communication.

HEIN, S., STOLTE, A. & DALLMANN, U. C. 1999 Identification and analysis of nonlinear transition scenarios using NOLOT/PSE. *Z. Angew. Math. Mech.* **79**, S109–S112.

HENNINGSON, D. S. 1987 Stability of parallel inviscid shear flow with mean spanwise variation. FFA-TN 1987-57, Aeronautical Research Institute of Sweden, Bromma.

HENNINGSON, D. S. 1989 Wave growth and spreading of a turbulent spot in plane Poiseuille flow. *Phys. Fluids A* **1**, 1876–1882.

HENNINGSON, D. S. 1996 Comment on "Transition in shear flows. Nonlinear normality versus non-normal linearity" [phys. fluids 7 3060 (1995)]. *Phys. Fluids* **8**, 2257–2258.

HENNINGSON, D. S. & ALFREDSSON, P. H. 1987 The wave structure of turbulent spots in plane Poiseuille flow. *J. Fluid Mech.* **178**, 405.

HENNINGSON, D. S., JOHANSSON, A. V. & ALFREDSSON, P. H. 1994 Turbulent spots in channel flows. *J. Eng. Mech.* **28**, 21–42.

HENNINGSON, D. S., LUNDBLADH, A. & JOHANSSON, A. V. 1993 A mechanism for bypass transition from localized disturbances in wall bounded shear flows. *J. Fluid Mech.* **250**, 169–207.

HENNINGSON, D. S. & SCHMID, P. J. 1992 Vector eigenfunction expansions for plane channel flows. *Stud. Appl. Math.* **87**, 15–43.

HENNINGSON, D. S., SPALART, P. & KIM, J. 1987 Numerical simulations of turbulent spots in plane Poiseuille and boundary layer flows. *Phys. Fluids* **30**, 2914–2917.

HERBERT, T. 1977 Die Neutrale Fläche der Ebenen Poiseuille-Strömung. Habilitationsschrift, Universität Stuttgart.

HERBERT, T. 1983 On perturbation methods in nonlinear stability theory. *J. Fluid Mech.* **126**, 167–186.

HERBERT, T. 1988 Secondary instability of boundary layers. *Ann. Rev. Fluid Mech.* **20**, 487–526.

HERBERT, T. 1991 Boundary-layer transition - analysis and prediction revisited. *AIAA Pap. No.* **91-0737**.

HERBERT, T. & BERTOLOTTI, F. 1987 Stability analysis of nonparallel boundary layers. *Bull. Am. Phys. Soc.* **32**, 2079.

HILL, D. C. 1995 Adjoint systems and their role in the receptivity problem for boundary layers. *J. Fluid Mech.* **292**, 183–204.

HÖGBERG, M. & HENNINGSON, D. S. 1998 Secondary instability of cross-flow vortices in Falkner-Skan-Cooke boundary layers. *J. Fluid Mech.* **368**, 339–357.

HORN, R. & JOHNSON, J. 1991 *Topics in Matrix Analysis*. Cambridge University Press.

HOWARD, L. 1961 Note on a paper of John W. Miles. *J. Fluid Mech.* **10**, 509–512.

HUAI, X., JOSLIN, R. D. & PIOMELLI, U. 1997 Large-eddy simulations of transition to turbulence in boundary layers. *Theoret. Comput. Fluid. Dyn.* **9**, 149–163.

HUERRE, P. & MONKEWITZ, P. A. 1990 Local and global instabilities in spatially developing flows. *Ann. Rev. Fluid Mech.* **22**, 473–537.

HUERRE, P. & ROSSI, M. 1998 Hydrodynamic instabilities in open flow. In *Hydrodynamics and Nonlinear Instabilities* (ed. C. Godreche & P. Manneville). Cambridge University Press.

HULTGREN, L. S. & AGGARWAL, A. K. 1987 Absolute instability of the Gaussian wake profile. *Phys. Fluids* **30 (11)**, 3383–3387.

HULTGREN, L. S. & GUSTAVSSON, L. H. 1981 Algebraic growth of disturbances in a laminar boundary layer. *Phys. Fluids* **24**, 1000–1004.

HUNT, R. E. & CRIGHTON, D. G. 1991 Instability of flows in spatially developing media. *Proc. Roy. Soc. Lond. A* **435**, 109–128.

VAN INGEN, J. L. 1956 A suggested semi-empirical method for the calculation of the boundary layer transition region. *Tech. Rep.* VTH-74. Department of Aeronautical Engineering, University of Delft.

ITOH, N. 1986 The origin and subsequent development in space of Tollmien-Schlichting waves in a boundary layer. *Fluid Dyn. Research* **1**, 119–130.

JACOBS, R. G. & HENNINGSON, D. S. 1999 Evaluation of data from direct numerical simulations of transition due to freestream turbulence. Annual Research Briefs, Center for Turbulence Research.

JOSEPH, D. D. 1968 Eigenvalue bounds for the Orr-Sommerfeld equation. *J. Fluid Mech.* **33**, 617–621.

JOSEPH, D. D. 1976 *Stability of Fluid Motions I*. Springer-Verlag.

JOSEPH, D. D. & CARMI, S. 1969 Stability of Poiseuille flow in pipes, annuli and channels. *Quart. Appl. Math.* **26**, 575–599.

KACHANOV, Y., KOZLOV, V., LEVCHENKO, V. & MAKSIMOV, V. 1979 Transformation of external disturbances into the boundary layer waves. In *Proc. Sixth. Intl. Conf. on Numerical Methods in Fluid Dyn.*, pp. 299–307. Springer-Verlag.

KACHANOV, Y. S. 1994 Physical mechanisms of laminar boundary-layer transition. *Ann. Rev. Fluid Mech.* **26**, 411–482.

KACHANOV, Y. S. & LEVCHENKO, V. Y. 1984 The resonant interaction of disturbances at laminar turbulent transition in a boundary layer. *J. Fluid Mech.* **138**, 209–247.

KARMAN, T. V. 1921 Über laminare und turbulente Reibung. *Z. Angew. Math. Mech.* **1**, 233–252.

KATO, T. 1976 *Perturbation Theory for Linear Operators*. Springer-Verlag, Berlin.

KAWAKAMI, M., KOHAMA, Y. & OKUTSU, M. 1999 Stability characteristics of stationary crossflow vortices in a three-dimensional boundary layer. *AIAA Paper* **99-0811**.

KELLER, H. B. 1977 Numerical solution of bifurcation and nonlinear eigenvalue problems. In *Applications of Bifurcation Theory* (ed. P. H. Rabinowitz), pp. 359–384. Academic Press.

KENDALL, J. M. 1985 Experimental study of disturbances produced in a pre-transitional laminar boundary layer by weak freestream turbulence. *AIAA Paper* **85-1695**.

KENDALL, J. M. 1990 Boundary layer receptivity to freestream turbulence. *AIAA Paper* **90-1504**.

KERCZEK, C. H. V. 1982 The stability of oscillatory plane Poiseuille flow. *J. Fluid Mech.* **116**, 91–114.

KHORRAMI, M. R., MALIK, M. R. & ASH, R. L. 1989 Application of spectral collocation techniques to the stability of swirling flows. *J. Comp. Phys.* **81**, 206–229.

KLEBANOFF, P. S. 1971 Effect of freestream turbulence on the laminar boundary layer. *Bull. Am. Phys. Soc.* **10**, 1323.

KLEBANOFF, P. S., TIDSTROM, K. D. & SARGENT, L. M. 1962 The three-dimensional nature of boundary layer instability. *J. Fluid Mech.* **12**, 1–34.

KLINGMANN, B. G. B., BOIKO, A., WESTIN, K., KOZLOV, V. & ALFREDSSON, P. 1993 Experiments on the stability of Tollmien-Schlichting waves. *Europ. J. Mech. B* **12**, 493–514.

KOBAYASHI, R., KOHAMA, Y. & KUROSAWA, M. 1983 Boundary layer transition on a rotating cone in axial flow. *J. Fluid Mech.* **127**, 341–352.

KOCH, W. 2000 Private communication.

KOCH, W., BERTOLOTTI, F. P., STOLTE, A. & HEIN, S. 2000 Nonlinear equilibrium solutions in a three-dimensional boundary layer and their secondary instability. *J. Fluid Mech.* **406**, 131–174.

KOHAMA, Y., SARIC, W. S. & HOOS, J. A. 1991 A high-frequency, secondary instability of crossflow vorticies that leads to transition. Proceedings of the Royal Aeronautical Society Conference on Boundary-Layer Transition and Control, Cambridge University, England.

KOVASZNAY, L. S. G. 1949 Hot-wire investigation of the wake behind cylinders at low Reynolds numbers. *Proc. Roy. Soc. Lond. A* **198**, 174–190.

KREISS, G., LUNDBLADH, A. & HENNINGSON, D. S. 1993 Bounds for threshold amplitudes in subcritical shear flows. TRITA-NA 9307, Royal Institute of Technology, Stockholm, Sweden.

KREISS, G., LUNDBLADH, A. & HENNINGSON, D. S. 1994 Bounds for threshold amplitudes in subcritical shear flows. *J. Fluid Mech.* **270**, 175–198.

KUPFER, K., BERS, A. & RAM, A. K. 1987 The cusp map in the complex-frequency plane for absolute instabilities. *Phys. Fluids* **30**, 3075–3082.

LANDAHL, M. T. 1975 Wave breakdown and turbulence. *SIAM J. Appl. Math* **28**, 735–756.

LANDAHL, M. T. 1980 A note on an algebraic instability of inviscid parallel shear flows. *J. Fluid Mech.* **98**, 243–251.

LANDAHL, M. T. 1983 Theoretical modeling of coherent structures in wall bounded shear flows. Eighth Biennial Symposium on Turbulence, University of Missouri-Rolla.

LANDAHL, M. T. 1993 Model for the wall-layer structure of a turbulent shear flow. *Europ. J. Mech. B* **12**, 85–96.

LANDAU, L. D. & LIFSHITZ, E. M. 1959 *Fluid Mechanics*. Addison-Wesley.

LANDMAN, M. J. & SAFFMAN, P. G. 1987 The three-dimensional instability of strained vortices in a viscous fluid. *Phys. Fluids* **30 (8)**, 2339–2342.

LEES, L. & LIN, C. C. 1946 Investigation of the stability of the laminar boundary layer in a compressible fluid. NACA TN 1115.

LEES, L. & RESHOTKO, E. 1962 Stability of the compressible laminar boundary layer. *J. Fluid Mech.* **12**, 555–590.

LEIB, S. J., WUNDROW, D. W. & GOLDSTEIN, M. E. 1999 Effect of free-stream turbulence and other vortical disturbances on a laminar boundary layer. *J. Fluid Mech.* **380**, 169–203.

LERCHE, T. 1997 Experimentelle Untersuchung nichtlinearer Strukturbildung im Transitionsprozess einer instabilen dreidimensionalen Grenzschicht. Fortschritt-Bericht VDI Reihe 7, no. 310.

LEVINSKI, V. & COHEN, J. 1995 The evolution of a localized disturbance in external shear flows. I. Theoretical considerations and preliminary experimental results. *J. Fluid Mech.* **289**, 159–177.

LI, F. & MALIK, M. R. 1994 Mathematical nature of parabolized stability equations. In *Laminar-Turbulent Transition* (ed. R. Kobayashi), pp. 205–212. Springer-Verlag.

LI, F. & MALIK, M. R. 1995 Fundamental and subharmonic secondary instability of Görtler vortices. *J. Fluid Mech.* **297**, 77–100.

LIBBY, P. A. & FOX, H. 1964 Some perturbation solutions in laminar boundary-layer theory. *J. Fluid Mech.* **17**, 433–449.

LIN, C. C. 1946 On the stability of two-dimensional parallel flows. Part III. Stability in a viscous fluid. *Quart. Appl. Math.* **3** (4), 277–301.

LIN, C. C. 1961 Some mathematical problems in the theory of the stability of parallel flows. *J. Fluid Mech.* **10**, 430–438.

LINGWOOD, R. J. 1995 Absolute instability of the boundary layer on a rotating disk. *J. Fluid Mech.* **299**, 17–33.

LINGWOOD, R. J. 1996 An experimental study of absolute instability of a rotating-disk boundary-layer. *J. Fluid Mech.* **314**, 373–405.

LINGWOOD, R. J. 1997 On the application of the Briggs' and steepest-descent methods to a boundary-layer flow. *Stud. Appl. Math.* **98**, 213–254.

LUCHINI, P. 1996 Reynolds number independent instability of the Blasius boundary layer over a flat surface. *J. Fluid Mech.* **327**, 101–115.

LUCHINI, P. 2000 Reynolds number independent instability of the boundary layer over a flat surface: optimal perturbations. *J. Fluid Mech.* **404**, 289–309.

LUCHINI, P. & BOTTARO, A. 1998 Görtler vortices: a backward-in-time approach to the receptivity problem. *J. Fluid Mech.* **363**, 1–23.

LUNDBLADH, A. 1993*a* Private communication.

LUNDBLADH, A. 1993*b* Simulation of bypass transition to turbulence in wall bounded shear flows. PhD thesis, Royal Institute of Technology, Stockholm, Sweden.

MA, B., VAN DOORNE, C. W. H., ZHANG, Z. & NIEUWSTADT, F. T. M. 1999 On the spatial evolution of a wall-imposed periodic disturbance in pipe Poiseuille flow at Re = 3000. Part 1. Subcritical disturbance. *J. Fluid Mech.* **398**, 181–224.

MACK, L. M. 1963 The inviscid stability of the compressible laminar boundary layer. In *Space Programs Summary*, , vol. 37-23, p. 297.

MACK, L. M. 1964 The inviscid stability of the compressible laminar boundary layer. Part II. In *Space Programs Summary*, , vol. 4, p. 165.

MACK, L. M. 1965 Stability of the laminar compressible boundary layer according to a direct numerical simulation. AGARDograph I (97).

MACK, L. M. 1969 Boundary layer stability theory. Jet Propulsion Laboratory Report 900-277.

MACK, L. M. 1975 Linear stability theory and the problem of supersonic boundary layer transition. *AIAA Journal* **13**, 278.

MACK, L. M. 1976 A numerical study of the temporal eigenvalue spectrum of the Blasius boundary layer. *J. Fluid Mech.* **73**, 497–520.

MACK, L. M. 1984 Boundary layer stability theory: Special course on stability and transition of laminar flow. AGARD Report 709, Paris, France.

MALIK, M. R. 1989 Prediction and control of transition in supersonic and hypersonic boundary layers. *AIAA Journal* **27**, 1487–1493.

MALIK, M. R. & CHANG, C.-L. 2000 Nonparallel and nonlinear stability of supersonic jet flow. *Comp. Fluids* **29**, 327–365.

MALIK, M. R., LI, F. & CHANG, C.-L. 1994 Crossflow disturbances in three-dimensional boundary layers: non-linear development, wave interaction and secondary instability. *J. Fluid Mech.* **268**, 1–36.

MALIK, M. R., LI, F., CHOUDHARI, M. M. & CHANG, C.-L. 1999 Secondary instability of crossflow vortices and swept-wing boundary-layer transition. *J. Fluid Mech.* **399**, 85–115.

MALIK, M. R., WILKINSON, S. P. & ORSZAG, S. A. 1981 Instability and transition in rotating disk flow. *AIAA J.* **19**, 1131–1138.

MASAD, J. A., NAYFEH, A. H. & AL-MAAITAH, A. A. 1992 Effect of heat transfer on the stability of compressible boundary layers. *Comp. Fluids* **21**, 43–61.

MASLOWE, S. 1986 Critical layers in shear flows. *Ann. Rev. Fluid Mech.* **18**, 405–432.

MATSSON, O. J. E. & ALFREDSSON, P. H. 1990 Curvature and rotation induced instabilities in channel flow. *J. Fluid Mech.* **210**, 537–563.

MATSUBARA, M. 1997 Private communication.

MATTINGLY, G. E. & CRIMINALE, W. O. 1972 The stability of an incompressible two-dimensional wake. *J. Fluid Mech.* **51**, 233–272.

METCALFE, R. W., ORSZAG, S. A., BRACHET, M. E., MENON, S. & RILEY, J. J. 1987 Secondary instability of a temporally growing mixing layer. *J. Fluid Mech.* **184**, 207–243.

MICHEL, R. 1952 ONERA Rep. 1/1578-A.

MOORE, A. M. & FARRELL, B. F. 1993 Rapid perturbation growth on spatially and temporally varying oceanic flows determined using an adjoint method: application to the Gulf Stream. *J. Phys. Oceanography* **23**, 1682–1702.

MORKOVIN, M. V. 1969 The many faces of transition. In *Viscous Drag Reduction* (ed. C. S. Wells). Plenum Press.

MOSER, R. D. & ROGERS, M. M. 1993 The three-dimensional evolution of a plane mixing layer: pairing and transition to turbulence. *J. Fluid Mech.* **247**, 275–320.

NAGATA, M. 1990 Three-dimensional finite-amplitude solutions in plane Couette flow: bifurcation from infinity. *J. Fluid Mech.* **217**, 519–527.

NITSCHKE-KOWSKY, P. & BIPPES, H. 1988 Instability and transition of a three-dimensional boundary layer on a swept flat plate. *Phys. Fluids* **31**, 786–795.

OLSSON, P. J. & HENNINGSON, D. S. 1994 Optimal disturbances in watertable flow. *Stud. Appl. Math.* **94**, 183–210.

ORR, W. M. F. 1907 The stability or instability of the steady motions of a perfect liquid and of a viscous liquid. Part I: A perfect liquid. Part II: A viscous liquid. *Proc. R. Irish Acad. A* **27**, 9–138.

ORSZAG, S. A. & PATERA, A. T. 1983 Secondary instability of wall-bounded shear flows. *J. Fluid Mech.* **128**, 347–385.

O'SULLIVAN, P. L. & BREUER, K. S. 1994 Transient growth in circular pipe flow. Part I: Linear disturbances. *Phys. Fluids* **6**, 3643–3651.

PEXIEDER, A. 1996 Effects of system rotation on the centrifugal instability of the boundary layer on a curved wall: an experimental study. PhD thesis, Ecole Polytechnique Federale de Lausanne, Lausanne, Switzerland.

PIERREHUMBERT, R. T. 1986 Universal short-wave instability of two-dimensional eddies in an inviscid fluid. *Phys. Rev. Lett.* **57**, 2157.

PIERREHUMBERT, R. T. & WIDNALL, S. E. 1982 The two- and three-dimensional instabilities of a spatially periodic shear layer. *J. Fluid Mech.* **114**, 59–82.

POLL, D. I. A. 1985 Some observations on the transition process on the windward face of a long yawed cylinder. *J. Fluid Mech.* **150**, 329–356.

PRUETT, C. D., CHANG, C. L. & STREETT, C. L. 2000 Simulation of crossflow instability on a supersonic highly swept wing. *Comp. Fluids* **29**, 33–62.

RAYLEIGH, L. 1880 On the stability of certain fluid motions. *Proc. Math. Soc. Lond.* **11**, 57–70.

RAYLEIGH, L. 1887 On the stability of certain fluid motions. *Proc. Math. Soc. Lond.* **19**, 67–74.

REDDY, S. C. & HENNINGSON, D. S. 1993 Energy growth in viscous channel flows. *J. Fluid Mech.* **252**, 209–238.

REDDY, S. C., SCHMID, P. J., BAGGETT, P. & HENNINGSON, D. S. 1998 On stability of streamwise streaks and transition thresholds in plane channel flows. *J. Fluid Mech.* **365**, 269–303.

REDDY, S. C., SCHMID, P. J. & HENNINGSON, D. S. 1993 Pseudospectra of the Orr-Sommerfeld operator. *SIAM J. Appl. Math.* **53**, 15–47.

REED, H. L. & SARIC, W. S. 1989 Stability of three-dimensional boundary layers. *Ann. Rev. Fluid Mech.* **21**, 235–284.

RILEY, J. J. & GAD-EL HAK, M. 1985 The dynamics of turbulent spots. In *Frontiers of Fluid Mechanics*, pp. 123–155. Springer.

RIST, U. & FASEL, H. 1995 Direct numerical simulation of controlled transition in a flat-plate boundary layer. *J. Fluid Mech.* **298**, 211–248.

RIST, U. & MAUCHER, U. 1994 Direct numerical simulation of 2-d and 3-d instability waves in a laminar separation bubble. Proceedings of the AGARD Symposium on Application of Direct and Large Eddy Simulation to Transition and Turbulence, AGARD-CP-551.

ROACH, P. E. & BRIERLEY, D. H. 1992 The influence of a turbulent free stream on zero pressure gradient transitional boundary layer development. In *Nummerical Simulations of Unsteady Flows and Transition to Turbulence*, pp. 319–347. Cambridge University Press.

ROGERS, M. M. & MOSER, R. D. 1992 The three-dimensional evolution of a plane mixing layer: the Kelvin-Helmholtz rollup. *J. Fluid Mech.* **243**, 183–226.

ROMANOV, V. A. 1973 Stability of plane-parallel Couette flow. *Funkcional Anal. i Proložen* **7** (2), 62–73, Translated in *Functional Anal. & Its Applics* **7**, 137 – 146 (1973).

SALWEN, H. & GROSCH, C. E. 1981 The continuous spectrum of the Orr-Sommerfeld equation. Part 2. Eigenfunction expansions. *J. Fluid Mech.* **104**, 445–465.

SARIC, W., REED, H. & KERSCHEN, E. 1994 Leading edge receptivity to sound: experiments, DNS, and theory. AIAA Paper 94-2222.

SARIC, W. S., HOOS, J. A. & RADEZTSKY, R. H. 1991 Boundary-layer receptivity of sound with roughness. In *Boundary Layer Stability and Transition to Turbulence*, pp. 17–22. FED-Vol. 114.

SAVILL, A. M. 1996 One-point closures applied to transition. In *Turbulence and Transition Modelling* (ed. M. Hallbäck, D. S. Henningson, A. V. Johansson & P. H. Alfredsson), pp. 233–268. Kluwer.

SCHENSTED, I. V. 1961 Contribution to the theory of hydrodynamic stability. PhD thesis, University of Michigan, Ann Arbor.

SCHLICHTING, H. 1933 Berechnung der Anfachung kleiner Störungen bei der Plattenströmung. *ZAMM* **13**, 171–174.

SCHMID, P. J. 2000 Linear stability theory and bypass transition in shear flows. *Phys. Plasmas* **7**, 1788–1794.

SCHMID, P. J. & HENNINGSON, D. S. 1994 Optimal energy density growth in Hagen-Poiseuille flow. *J. Fluid Mech.* **277**, 197–225.

SCHMID, P. J. & KYTÖMAA, H. K. 1994 Transient and asymptotic stability of granular flow. *J. Fluid Mech.* **264**, 255–275.

SCHMID, P. J., REDDY, S. C. & HENNINGSON, D. S. 1996 Transition thresholds in boundary layer and channel flow. In *Advances in Turbulence VI* (ed. S. Gavrilakis, L. Machiels & P. A. Monkewitz), pp. 381–384. Kluwer Academic Publishers.

SCHUBAUER, G. B. & SKRAMSTAD, H. F. 1947 Laminar boundary layer oscillations and the stability of laminar flow. *J. Aero. Sci.* **14**, 69–78.

SEN, P. K. & VENKATESWARLU, D. 1983 On the stability of plane Poiseuille flow to finite-amplitude disturbances, considering the higher-order Landau coefficients. *J. Fluid Mech.* **133**, 179–206.

SHAN, H., MA, B., ZHANG, Z. & NIEUWSTADT, F. T. M. 1999 Direct numerical simulation of a puff and a slug in transitional cylindrical pipe flow. *J. Fluid Mech.* **387**, 39–60.

SHANTINI, R. 1990 Degenerating Orr-Sommerfeld eigenmodes and development of three-dimensional perturbations. PhD thesis, Luleå University of Technology, Luleå, Sweden.

SIMEN, M. 1992 Local and nonlocal stability theory of spatially varying flows. In *Instability, Transition and Turbulence*, pp. 181–201. Springer-Verlag.

SIMEN, M. & DALLMANN, U. 1992 On the instability of hypersonic flow past a pointed cone – comparison of theoretical and experimental results at Mach 8. Deutscher Luft- und Raumfahrtkongress / DGLR-Jahrestagung, (29 September – 2 October, 1992, Bremen).

SINGER, B. & JOSLIN, R. 1994 Metamorphosis of a hairpin vortex into a young turbulent spot. *Phys. Fluids* **6**, 3724–3736.

SMITH, A. M. O. & GAMBERONI, N. 1956 Transition, pressure gradient and stability theory. *Tech. Rep.* ES 26388. Douglas Aircraft Co.

SMITH, F. T. 1979*a* Nonlinear stability of boundary layers for disturbances of various sizes. *Proc. Roy. Soc. Lond. A* **368**, 573–589.

SMITH, F. T. 1979*b* On the nonparallel flow stability of the Blasius boundary layer. *Proc. Roy. Soc. Lond. A* **366**, 91–109.

SMYTH, W. D. 1992 Spectral energy transfers in two-dimensional anisotropic flow. *Phys. Fluids A* **4**, 340–349.

SOIBELMAN, I. & MEIRON, D. I. 1991 Finite-amplitude bifurcations in plane Poiseuille flow: two-dimensional Hopf bifurcation. *J. Fluid Mech.* **229**, 389–416.

SOMMERFELD, A. 1908 Ein Beitrag zur hydrodynamischen Erklärung der turbulenten Flüssigkeitbewegungen. In *Atti. del 4. Congr. Internat. dei Mat. III*, pp. 116–124. Roma.

SPALART, P. R. & STRELETS, M. K. 2000 Mechanisms of transition and heat transfer in a separation bubble. *J. Fluid Mech.* **403**, 329–349.

SPALART, P. R. & YANG, K. 1987 Numerical study of ribbon induced transition in Blasius flow. *J. Fluid Mech.* **178**, 345–365.

SQUIRE, H. B. 1933 On the stability for three-dimensional disturbances of viscous fluid flow between parallel walls. *Proc. Roy. Soc. Lond. Ser. A* **142**, 621–628.

STEWARTSON, K. 1957 On asymptotic expansion in the theory of boundary layer. *J. Math. Phys.* **36**, 137.

STEWARTSON, K. & STUART, J. T. 1971 A nonlinear instability theory for a wave system in plane Poiseuille flow. *J. Fluid Mech.* **48**, 529–545.

STUART, J. T. 1960 On the nonlinear mechanics of wave disturbances in stable and unstable parallel flows. Part 1. The basic behaviour in plane Poiseuille flow. *J. Fluid Mech.* **9**, 353–370.

STUART, J. T. & DIPRIMA, R. C. 1978 The Eckhaus and Benjamin-Feir resonance mechanism. *Proc. Roy. Soc. Lond. Ser. A* **362**, 27–41.

SWEARINGEN, J. D. & BLACKWELDER, R. F. 1987 The growth and breakdown of streamwise vortices in the presence of a wall. *J. Fluid Mech.* **182**, 255–290.

SYNGE, J. 1938 Hydrodynamic stability. *Semicentenn. Publ. Amer. Math. Soc.* **2**, 227–269.

TOLLMIEN, W. 1929 Über die Entstehung der Turbulenz. Nachr. Ges. Wiss. Göttingen, 21–44, (English translation NACA TM 609, 1931).

TOLLMIEN, W. 1935 Ein allgemeines Kriterium der Instabilität laminarer Geschwindigkeitsverteilungen. Nachr. Wiss. Fachgruppe, Göttingen Math.-phys. Kl. 1 79–114, (English translation as 'General instability criterion of laminar velocity distributions', in *Tech. Memor. Nat. Adv. Comm. Aero.* Wash. No. 792 (1936)).

TREFETHEN, A. E., TREFETHEN, L. N. & SCHMID, P. J. 1999 Spectra and pseudospectra for pipe Poiseuille flow. *Comput. Meth. Appl. Mech. Eng.* **175**, 413–420.

TREFETHEN, L. N. 1992 Pseudospectra of matrices. In *Numerical Analysis 1991*, pp. 234–266. Longman.

TREFETHEN, L. N., TREFETHEN, A. E., REDDY, S. C. & DRISCOLL, T. A. 1993 Hydrodynamic stability without eigenvalues. *Science* **261**, 578–584.

TRIANTAFYLLOU, G. S., TRIANTAFYLLOU, M. S. & CHRYSSOSTOMIDIS, C. 1986 On the formation of vortex streets behind stationary cylinders. *J. Fluid Mech.* **170**, 461–477.

TRITTON, D. & DAVIES, P. 1985 Instabilities in geophysical fluid dynamics. In *Hydrodynamic Instabilities and the Transition to Turbulence, 2ed* (ed. H. Swinney & J. Gollub), pp. 229–270. Springer-Verlag.

TUMIN, A. 1996 Receptivity of pipe Poiseuille flow. *J. Fluid Mech.* **315**, 119–137.

USHER, J. R. & CRAIK, A. D. D. 1975 Nonlinear wave interactions in shear flows. Part 2. Third-order theory. *J. Fluid Mech.* **70**, 437–461.

WALEFFE, F. 1990 On the three-dimensional instability of strained vortices. *Phys. Fluids A* **2(1)**, 76–80.

WALEFFE, F. 1995 Transition in shear flows. Nonlinear normality versus nonnormal linearity. *Phys. Fluids* **7**, 3060–3066.

WATMUFF, J. H. 1999 Evolution of a wave packet into vortex loops in a laminar separation bubble. *J. Fluid Mech.* **397**, 119–169.

WATSON, J. 1960 On the nonlinear mechanics of wave disturbances in stable and unstable parallel flows. Part 2. The development of a solution for plane Poiseuille and for plane Couette flow. *J. Fluid Mech.* **9**, 371–389.

WESTIN, K. J. & HENKES, R. A. W. M. 1997 Applications of turbulence models to bypass transition. *J. Fluids Eng.* **119**, 859–866.

WESTIN, K. J. A., BAKCHINOV, A. A., KOZLOV, V. V. & ALFREDSSON, P. H. 1998 Experiments on localized disturbances in a flat plate boundary layer. Part 1. The receptivity and evolution of a localized freestream disturbance. *Europ. J. Mech. B* **17**, 823–846.

WESTIN, K. J. A., BOIKO, A. V., KLINGMANN, B. G. B., KOZLOV, V. V. & ALFREDSSON, P. H. 1994 Experiments in a boundary layer subject to free-stream turbulence. Part I: Boundary layer structure and receptivity. *J. Fluid Mech.* **281**, 193–218.

WHITE, F. M. 1974 *Viscous Fluid Flow.* McGraw-Hill, New York.

YANG, Z. Y. & VOKE, P. R. 1991 Numerical simulation of transition to turbulence. *Tech. Rep.* ME-FD/91.01. University of Surrey, Dept. Mech. Eng.

YANG, Z. Y. & VOKE, P. R. 1993 Large-eddy simulation of transition under turbulence. *Tech. Rep.* ME-FD/93.12. University of Surrey, Dept. Mech. Eng.

YIH, C. S. 1963 Stability of flow down an inclined plane. *Phys. Fluids* **6**, 321–334.

ZAHN, J. P., TOOMRE, J., SPIEGEL, E. A. & GOUGH, D. O. 1974 Nonlinear cellular motions in Poiseuille channel flow. *J. Fluid Mech.* **64**, 319–345.

Index

Applied Mathematical Sciences

(continued from page ii)

61. *Sattinger/Weaver:* Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics.
62. *LaSalle:* The Stability and Control of Discrete Processes.
63. *Grasman:* Asymptotic Methods of Relaxation Oscillations and Applications.
64. *Hsu:* Cell-to-Cell Mapping: A Method of Global Analysis for Nonlinear Systems.
65. *Rand/Armbruster:* Perturbation Methods, Bifurcation Theory and Computer Algebra.
66. *Hlavácek/Haslinger/Necasl/Lovisek:* Solution of Variational Inequalities in Mechanics.
67. *Cercignani:* The Boltzmann Equation and Its Applications.
68. *Temam:* Infinite-Dimensional Dynamical Systems in Mechanics and Physics, 2nd ed.
69. *Golubitsky/Stewart/Schaeffer:* Singularities and Groups in Bifurcation Theory, Vol. II.
70. *Constantin/Foias/Nicolaenko/Temam:* Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations.
71. *Catlin:* Estimation, Control, and the Discrete Kalman Filter.
72. *Lochak/Meunier:* Multiphase Averaging for Classical Systems.
73. *Wiggins:* Global Bifurcations and Chaos.
74. *Mawhin/Willem:* Critical Point Theory and Hamiltonian Systems.
75. *Abraham/Marsden/Ratiu:* Manifolds, Tensor Analysis, and Applications, 2nd ed.
76. *Lagerstrom:* Matched Asymptotic Expansions: Ideas and Techniques.
77. *Aldous:* Probability Approximations via the Poisson Clumping Heuristic.
78. *Dacorogna:* Direct Methods in the Calculus of Variations.
79. *Hernández-Lerma:* Adaptive Markov Processes.
80. *Lawden:* Elliptic Functions and Applications.
81. *Bluman/Kumei:* Symmetries and Differential Equations.
82. *Kress:* Linear Integral Equations, 2nd ed.
83. *Bebernes/Eberly:* Mathematical Problems from Combustion Theory.
84. *Joseph:* Fluid Dynamics of Viscoelastic Fluids.
85. *Yang:* Wave Packets and Their Bifurcations in Geophysical Fluid Dynamics.
86. *Dendrinos/Sonis:* Chaos and Socio-Spatial Dynamics.
87. *Weder:* Spectral and Scattering Theory for Wave Propagation in Perturbed Stratified Media.
88. *Bogaevski/Povzner:* Algebraic Methods in Nonlinear Perturbation Theory.
89. *O'Malley:* Singular Perturbation Methods for Ordinary Differential Equations.
90. *Meyer/Hall:* Introduction to Hamiltonian Dynamical Systems and the N-body Problem.
91. *Straughan:* The Energy Method, Stability, and Nonlinear Convection.
92. *Naber:* The Geometry of Minkowski Spacetime.
93. *Colton/Kress:* Inverse Acoustic and Electromagnetic Scattering Theory, 2nd ed.
94. *Hoppensteadt:* Analysis and Simulation of Chaotic Systems, 2nd ed.
95. *Hackbusch:* Iterative Solution of Large Sparse Systems of Equations.
96. *Marchioro/Pulvirenti:* Mathematical Theory of Incompressible Nonviscous Fluids.
97. *Lasota/Mackey:* Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics, 2nd ed.
98. *de Boor/Höllig/Riemenschneider:* Box Splines.
99. *Hale/Lunel:* Introduction to Functional Differential Equations.
100. *Sirovich (ed):* Trends and Perspectives in Applied Mathematics.
101. *Nusse/Yorke:* Dynamics: Numerical Explorations, 2nd ed.
102. *Chossat/Iooss:* The Couette-Taylor Problem.
103. *Chorin:* Vorticity and Turbulence.
104. *Farkas:* Periodic Motions.
105. *Wiggins:* Normally Hyperbolic Invariant Manifolds in Dynamical Systems.
106. *Cercignani/Illner/Pulvirenti:* The Mathematical Theory of Dilute Gases.
107. *Antman:* Nonlinear Problems of Elasticity.
108. *Zeidler:* Applied Functional Analysis: Applications to Mathematical Physics.
109. *Zeidler:* Applied Functional Analysis: Main Principles and Their Applications.
110. *Diekmann/van Gils/Verduyn Lunel/Walther:* Delay Equations: Functional-, Complex-, and Nonlinear Analysis.
111. *Visintin:* Differential Models of Hysteresis.
112. *Kuznetsov:* Elements of Applied Bifurcation Theory, 2nd ed.
113. *Hislop/Sigal:* Introduction to Spectral Theory: With Applications to Schrödinger Operators.
114. *Kevorkian/Cole:* Multiple Scale and Singular Perturbation Methods.
115. *Taylor:* Partial Differential Equations I, Basic Theory.
116. *Taylor:* Partial Differential Equations II, Qualitative Studies of Linear Equations.
117. *Taylor:* Partial Differential Equations III, Nonlinear Equations.

(continued on next page)

Applied Mathematical Sciences

(continued from previous page)

118. *Godlewski/Raviart:* Numerical Approximation of Hyperbolic Systems of Conservation Laws.
119. *Wu:* Theory and Applications of Partial Functional Differential Equations.
120. *Kirsch:* An Introduction to the Mathematical Theory of Inverse Problems.
121. *Brokate/Sprekels:* Hysteresis and Phase Transitions.
122. *Gliklikh:* Global Analysis in Mathematical Physics: Geometric and Stochastic Methods.
123. *Le/Schmitt:* Global Bifurcation in Variational Inequalities: Applications to Obstacle and Unilateral Problems.
124. *Polak:* Optimization: Algorithms and Consistent Approximations.
125. *Arnold/Khesin:* Topological Methods in Hydrodynamics.
126. *Hoppensteadt/Izhikevich:* Weakly Connected Neural Networks.
127. *Isakov:* Inverse Problems for Partial Differential Equations.
128. *Li/Wiggins:* Invariant Manifolds and Fibrations for Perturbed Nonlinear Schrödinger Equations.
129. *Müller:* Analysis of Spherical Symmetries in Euclidean Spaces.
130. *Feintuch:* Robust Control Theory in Hilbert Space.
131. *Ericksen:* Introduction to the Thermodynamics of Solids, Revised ed.
132. *Ihlenburg:* Finite Element Analysis of Acoustic Scattering.
133. *Vorovich:* Nonlinear Theory of Shallow Shells.
134. *Vein/Dale:* Determinants and Their Applications in Mathematical Physics.
135. *Drew/Passman:* Theory of Multicomponent Fluids.
136. *Cioranescu/Saint Jean Paulin:* Homogenization of Reticulated Structures.
137. *Gurtin:* Configurational Forces as Basic Concepts of Continuum Physics.
138. *Haller:* Chaos Near Resonance.
139. *Sulem/Sulem:* The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse.
140. *Cherkaev:* Variational Methods for Structural Optimization.
141. *Naber:* Topology, Geometry, and Gauge Fields: Interactions.
142. *Schmid/Henningson:* Stability and Transition in Shear Flows.