ALE-DG scheme for special relativistic Euler equations

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1 Conservation laws

The special relativistic Euler equations in an inertial frame can be written as a system of conservation laws of the form

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = 0 \tag{1}$$

where

$$\boldsymbol{U} = \begin{bmatrix} D \\ m_1 \\ m_2 \\ E \end{bmatrix}, \qquad \boldsymbol{F} = \begin{bmatrix} Du \\ m_1 u_1 + p \\ m_2 u_1 \\ (E+p)u_1 \end{bmatrix}$$

The conserved quantities $\boldsymbol{U}$ are related to the primitive variables $\boldsymbol{V} = [\rho, u_1, u_2, p]^\top$ by

$$D = \beta\rho \tag{2}$$
$$m_i = \beta^2(\varepsilon + p)u_i, \quad i = 1, 2 \tag{3}$$
$$E = \beta^2(\varepsilon + p) - p \tag{4}$$

where ρ, p, ε are the rest frame density, pressure, internal energy, and u is velocity in the laboratory frame, while

$$\beta = (1 - u^2)^{-\frac{1}{2}}, \qquad u = |\boldsymbol{u}| = \sqrt{u_1^2 + u_2^2}$$

is the Lorentz factor. The pressure is given by an equation of state $p = p(\rho, \varepsilon)$. The ideal gas model corresponds to

$$p = (\gamma - 1)(\varepsilon - \rho) \qquad \text{i.e.} \qquad \varepsilon = \rho + \frac{p}{\gamma - 1}$$

This can also be written in terms of the enthalpy h as

$$\rho h = \varepsilon + p = \rho + \frac{\gamma p}{\gamma - 1}, \qquad 1 < \gamma \leq 2 \tag{5}$$

The sound speed is defined as

$$c = \sqrt{\frac{\gamma p}{\rho h}} = \sqrt{\frac{(\gamma - 1)(h - 1)}{h}}$$

Since $h > 1$, the sound speed is bounded by $0 < c \leq 1$. The set of admissible states for the conserved variables is

$$\mathcal{U}_{\text{ad}} = \{\boldsymbol{U} \in \mathbb{R}^4 : \rho(\boldsymbol{U}) > 0,\ p(\boldsymbol{U}) > 0, u < 1\}$$

Moreover, it is shown in (Mignone/Bodo 2005) that $\mathcal{U}_{\text{ad}}$ is convex and can also be written as

$$\mathcal{U}_{\text{ad}} = \{\boldsymbol{U} \in \mathbb{R}^4 : D > 0,\ E > \sqrt{D^2 + m_1^2 + m_2^2}\}$$

For later use, define

$$\alpha = \frac{\gamma}{\gamma - 1} \in [2, \infty), \qquad m = |\boldsymbol{m}| = \sqrt{m_1^2 + m_2^2}$$

and note that $\boldsymbol{m} \cdot \boldsymbol{u} = mu$.

2 Taylor Predictor

A usual form of conservation law looks like

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x} = 0 \tag{6}$$

For simple Euler equations, $\boldsymbol{U}$ and $\boldsymbol{F}(\boldsymbol{U})$ are given by

$$\boldsymbol{U} = \begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix}, \qquad \boldsymbol{F}(\boldsymbol{U}) = \begin{bmatrix} \rho v \\ p + \rho v^2 \\ \rho h v \end{bmatrix}$$

The DG scheme applied to Euler equations leads to an implicit scheme in its integral form.

$$\begin{aligned} h_j^{n+1} \boldsymbol{U}_{j,l}^{n+1} &= h_j^n \boldsymbol{U}_{j,l}^n + \int_{t_n}^{t_{n+1}} \int_{x_{j-\frac{1}{2}}(t)}^{x_{j+\frac{1}{2}}(t)} \boldsymbol{g}(\boldsymbol{U}_h, w) \frac{\partial}{\partial x} \varphi_l(x,t) \mathrm{d}x \mathrm{d}t \\ &\quad + \int_{t_n}^{t_{n+1}} [\hat{\boldsymbol{g}}_{j-\frac{1}{2}}(t) \varphi_l(x^+_{j-\frac{1}{2}}, t) - \hat{\boldsymbol{g}}_{j+\frac{1}{2}}(t) \varphi_l(x^-_{j+\frac{1}{2}}, t)] \mathrm{d}t \end{aligned}$$

Here w is the mesh velocity which is zero for our static mesh case. In order to make this scheme explicit, we need to evaluate the time integrals, which requires the knowledge of $\boldsymbol{U}$ at the next time step. To overcome this problem we use a predictor. Time integration is done using just a one point gaussian quadrature. The predictor serves to give us a predicted solution at the next gauss point for time integration τ_r(which is the midpoint

between t_n and t_{n+1}). When our solution polynomial is of degree 1 or less we use a truncated Taylor series to predict the solution.

$$\begin{aligned} \boldsymbol{U}(x_q, \tau_r) &= \boldsymbol{U}(X_q, t_n) + (\tau_r - t_n)\frac{\partial \boldsymbol{U}}{\partial t}(X_q, t_n) + (x_q - X_q)\frac{\partial \boldsymbol{U}}{\partial x}(X_q, t_n) \\ &\quad + O(\tau_r - t_n)^2 + O(x_q - X_q)^2 \end{aligned}$$

and hence the predicted solution is

$$\boldsymbol{U}(x_q, \tau_r) = \boldsymbol{U}_h(X_q, t_n) + (\tau_r - t_n)\frac{\partial \boldsymbol{U}_h}{\partial t}(X_q, t_n) + (x_q - X_q)\frac{\partial \boldsymbol{U}_h}{\partial x}(X_q, t_n)$$

HereX_q represents the location of gauss point used for spatial integration in the cell at the n^{th} time step. Then the spatial location of the gauss point changes from X_q to x_q in time t_n to τ_r.

But we do not know how the solution evolves in time, and therefore cannot calculate the $\frac{\partial \boldsymbol{U}}{\partial t}$ term. Do get around this problem, we use the conservation law and write the time derivative as -

$$\frac{\partial \boldsymbol{U}}{\partial t} = -\frac{\partial \boldsymbol{F}}{\partial x} = -A\frac{\partial \boldsymbol{U}}{\partial x}$$

Then the predictor becomes -

$$\boldsymbol{U}(x_q, \tau_r) = \boldsymbol{U}_h(X_q, t_n) - (\tau_r - t_n)A\frac{\partial \boldsymbol{U}_h}{\partial x}(X_q, t_n) + (x_q - X_q)\frac{\partial \boldsymbol{U}_h}{\partial x}(X_q, t_n)$$

This is the equation used in the final code for simple euler equations. Here we have evaluated the Jacobian A directly, since for simple euler equations the flux matrix $\boldsymbol{F}$ can be easily constructed using the elements of the matrix of conserved quatities $\boldsymbol{U}$. A code snippet demonstrating this is

```
p = (GAMMA - 1.0) * (U[2] - 0.5 * U[1] * U[1] / U[0]);

flux[0] = U[1] - w * U[0];
flux[1] = p + U[1] * U[1] / U[0] - w * U[1];
flux[2] = (U[2] + p) * U[1] / U[0] - w * U[2];
```

Then direct differentiation of $\boldsymbol{F}$ elements with $\boldsymbol{U}$ elements is possible and we obtain A as $A = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}$. This is not feasible for Relativistic Euler equations and that makes the key difference between computational methods for simple and relativistic Euler equations.

2.1 Taylor Predictor for relativistic Euler

For the relativistic equations our primitive, conserved and flux matrices are given by

$$\boldsymbol{V} = \begin{bmatrix} \rho \\ u_1 \\ u_2 \\ p \end{bmatrix}, \qquad \boldsymbol{U} = \begin{bmatrix} D \\ m_1 \\ m_2 \\ E \end{bmatrix}, \qquad \boldsymbol{F} = \begin{bmatrix} Du \\ m_1 u_1 + p \\ m_2 u_1 \\ (E + p)u \end{bmatrix}$$

Where, the conserved quantities $\boldsymbol{U}$ are related to the primitive variables $\boldsymbol{V} = [\rho, u_1, u_2, p]^\top$ by

$$D = \beta\rho \tag{7}$$
$$m_i = \beta^2(\varepsilon + p)u_i, \quad i = 1, 2 \tag{8}$$
$$E = \beta^2(\varepsilon + p) - p \tag{9}$$

Here $\beta = \sqrt{1 - u^2}$ is the Lorentz factor. The presence of β makes it difficult to write $\boldsymbol{F}$ in terms of $\boldsymbol{U}$. Since we cannot write $\boldsymbol{F}$ in terms of components of $\boldsymbol{U}$, we cannot evaluate $\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}$ and construct the A as before. Hence we take another approach. As both $\boldsymbol{U}$ and $\boldsymbol{F}$ are functions of $\boldsymbol{V}$, we evaluate a new Jacobian for the primitives as

$$\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{V}}\frac{\partial \boldsymbol{V}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{V}}\frac{\partial \boldsymbol{V}}{\partial x} = 0 \tag{10}$$

Left multiplying by the inverse of $\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{V}}$, we get

$$\frac{\partial \boldsymbol{V}}{\partial t} + \left[\left(\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{V}}\right)^{-1}\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{V}}\right]\frac{\partial \boldsymbol{V}}{\partial x} = 0 \tag{11}$$

And we define the primitive variable Jacobian as

$$\widetilde{A} = \left[\left(\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{V}}\right)^{-1}\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{V}}\right]$$

We can differentiate both $\boldsymbol{U}$ and $\boldsymbol{F}$ with respect to $\boldsymbol{V}$ and evaluate $\widetilde{A}$ by direct multiplication of the two required matrices. However, with this Jacobian, we can only find the time derivative for primitive variables $\frac{\partial \boldsymbol{V}}{\partial t} = -\widetilde{A}\frac{\partial \boldsymbol{V}}{\partial x}$ and not for the conserved variables $\frac{\partial \boldsymbol{U}}{\partial t}$ as required by the predictor. This forces us to redesign our predictor. We now write a predictor which predicts the solution at the next gauss point for time integration in terms of primitive variables. As we use Taylor predictor only for linear polynomials, we just have one spatial gauss integration point- the midpoint of the cell. We have the solution at time t_n and we need the prediction at the midpoint of the cell at the next temporal gauss point τ_r. Denoting the midpoint at time t_n with X_j and that at τ_r with x_j, we write

$$V_p(x_j, \tau_r) = V_p(X_j, t_n) + (\tau_r - t_n)\frac{\partial \boldsymbol{V}}{\partial t} + (x_j - X_j)\frac{\partial \boldsymbol{V}}{\partial x} \tag{12}$$
$$V_p(x_j, \tau_r) = V_p(X_j, t_n) - \widetilde{A}\frac{\partial \boldsymbol{V}}{\partial x}(\tau_r - t_n) + (x_j - X_j)\frac{\partial \boldsymbol{V}}{\partial x} \tag{13}$$

3 Eigenvalues and Eigenvectors

But the numerical fluxes like the Lax-Friedrichs, Rusanov and Roe fluxes used require the conserved quantities and hence we modify them to take in the predicted primitive

solution and use it to construct the $\boldsymbol{F}$ quantities. (Note here that numerical fluxes are different from Euler flux $\boldsymbol{F}$. Numerical fluxes provide a consistent approximation to the average flux value at the cell boundaries.) Apart from the conserved quantities the numerical fluxes also need the Eigenvectors and Eigenvalues of the original Jacobian $\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}$. This is a problem, since we don't have the original Jacobian. So we must derive the Eigenvalues and eigenvectors for the original Jacobian using the new Jacobian $\widetilde{A}$ that we evaluated. We now show that $\widetilde{A}$ is similar to A

$$\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{V}} \widetilde{A} \left(\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{V}} \right)^{-1} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{V}} \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{U}} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}} = A \tag{14}$$

Then, if λ_i's, L_i's and R_i's represent eigenvalues, left eigenvectors and right eigenvectors respectively of A and $\widetilde{\lambda_i}$'s, $\widetilde{L_i}$'s and $\widetilde{R_i}$'s are those of $\widetilde{A}$, we can show that

- $\lambda_i = \widetilde{\lambda_i}$
- $R_i = \frac{\partial \boldsymbol{U}}{\partial \boldsymbol{V}} \widetilde{R_i}$
- $L_i = \widetilde{L_i} \left(\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{V}} \right)^{-1}$

The equality of eigenvalues is a standard result regarding similar matrices. The other two have been proved in the appendix. Hence we can calculate the eigenvalues and eigenvectors and use them for numerical fluxes in the code.

4 Results

The code works for simple test cases under the following conditions -

- Flux: Lax Friedrichs
- Limiters: No limiters
- CFL: < 0.9
- adaptation: No adaptation
- PORD: 1 - Both continuous and disconinuous test cases
- PORD: 2 - Only works for smooth test cases as of now

The variable PORD used in the code is an integer one more than the degree of polynomial used to construct the solution. So PORD = 2 denotes a linear polynomial while PORD = 1 denotes a constant polynomial. This essentially reduces the DG method to FVM.

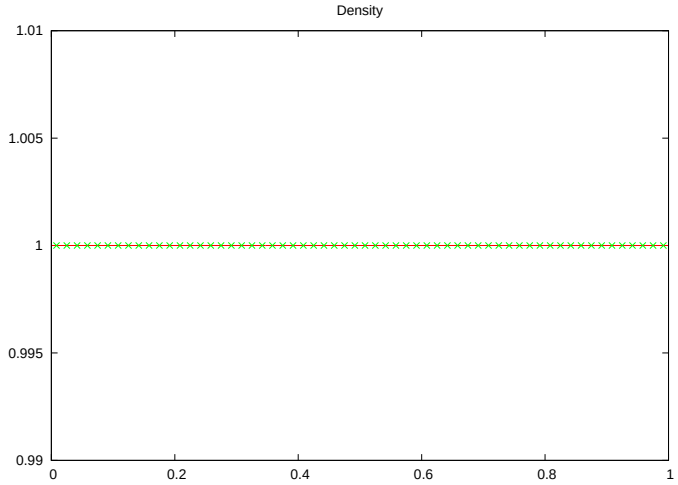

Figure 1: Test case with constant conditions. Works with pord 1 and 2.

4.1 Test Cases

4.1.1 Constant conditions test case

- $\rho = 1$
- $u_1 = 0.9$
- $p = 1$

We give constant initial conditions and expect the code to preserve them over time. Its a constant density flow at constant velocity with no pressure gradient. The code works well for this trivial test case.

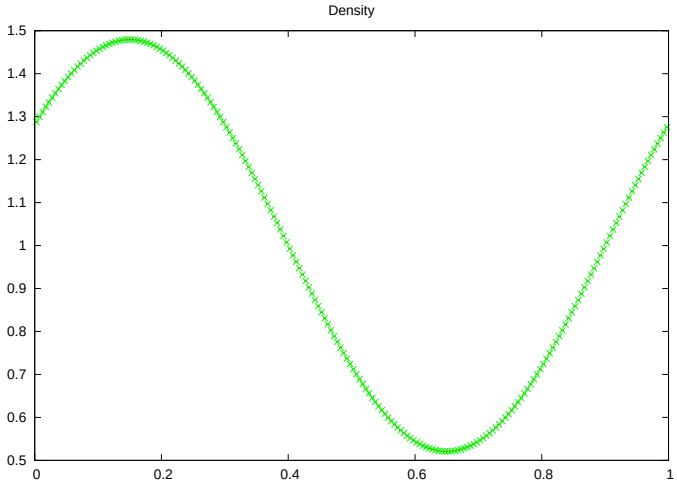

Figure 2: Sinusoidal density test case with pord = 1

4.1.2 Smooth solution test case with PORD = 1

- $\rho = 1 + 0.5sin(2\pi x)$
- $u_1 = 0.9$
- $u_2 = 0.0$
- $p = 1$
- CFL = 0.6
- NC = 200

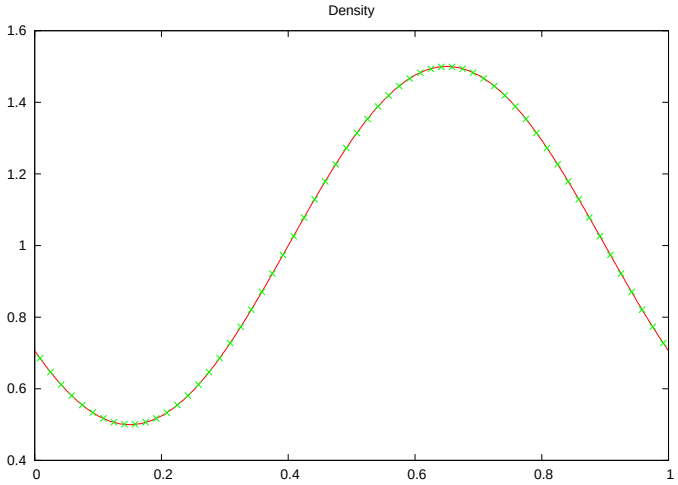

Figure 3: sinusoidal density test case with pord = 2

4.1.3 Smooth solution test case with PORD = 2

- $\rho = 1 + 0.5sin(2\pi x)$
- $u_1 = 0.9$
- $u_2 = 0.0$
- $p = 1$
- CFL $= 0.1$
- NC $= 60$

4.1.4 Moderate Blast Wave Test Case

Domain: $[0, 1]$

- for $x < 0.5$
  - $\rho = 10$
  - $u_1 = 0.0$
  - $u_2 = 0.0$
  - $p = 13.33$
- for $x > 0.5$
  - $\rho = 1.0$
  - $u_1 = 0.0$
  - $u_2 = 0.0$
  - $p = 0.0$
- CFL $= 0.1$
- NC $= 5000$

We see rarefaction wave to the left and a shock wave travelling to the right.

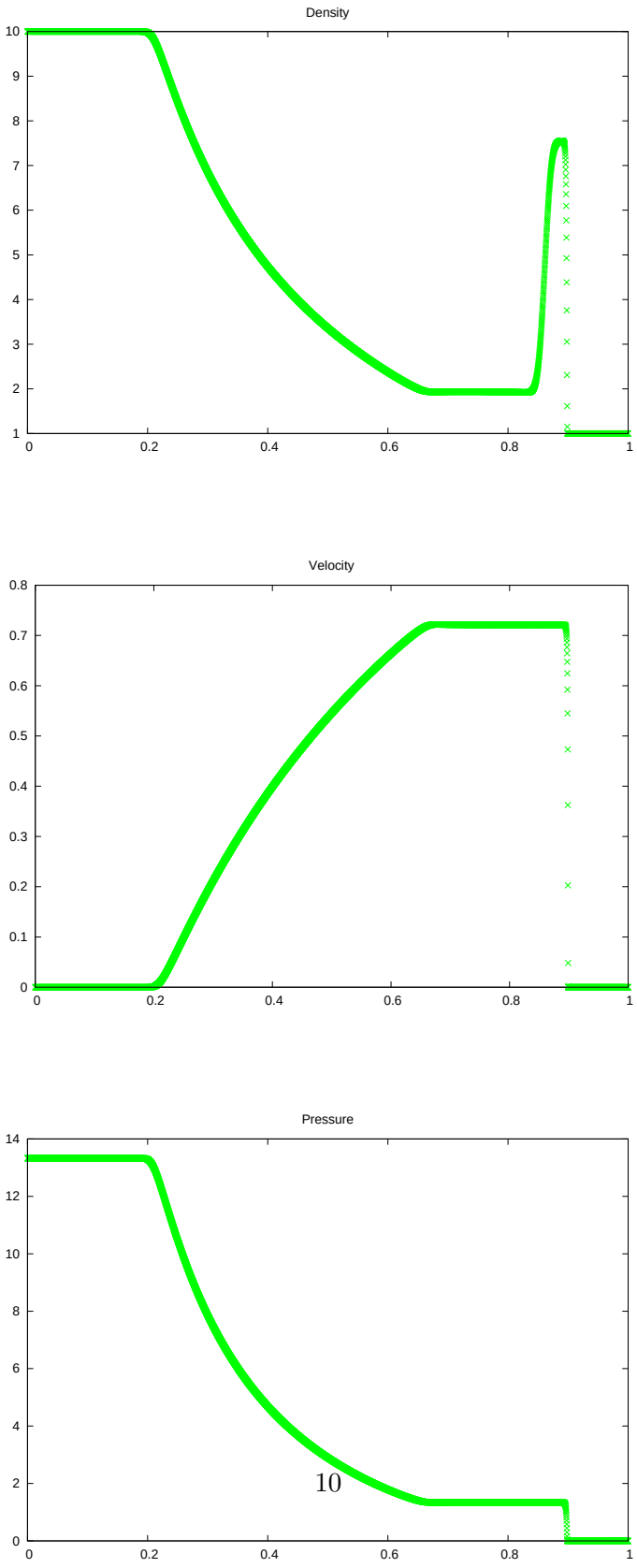

Figure 4: Moderate relativistic blast wave test case

4.1.5 Strong Blast Wave Test Case

Domain: $[0, 1]$

- for $x < 0.5$
  - $\rho = 1.0$
  - $u_1 = 0.0$
  - $u_2 = 0.0$
  - $p = 1000.0$
- for $x > 0.5$
  - $\rho = 1.0$
  - $u_1 = 0.0$
  - $u_2 = 0.0$
  - $p = 0.01$
- CFL $= 0.1$
- NC $= 5000$

We see that an extremely high density region preceeds the shock wave. This region becomes thinner if we increase the pressure jump. Also the rarefaction fan becomes more curved.

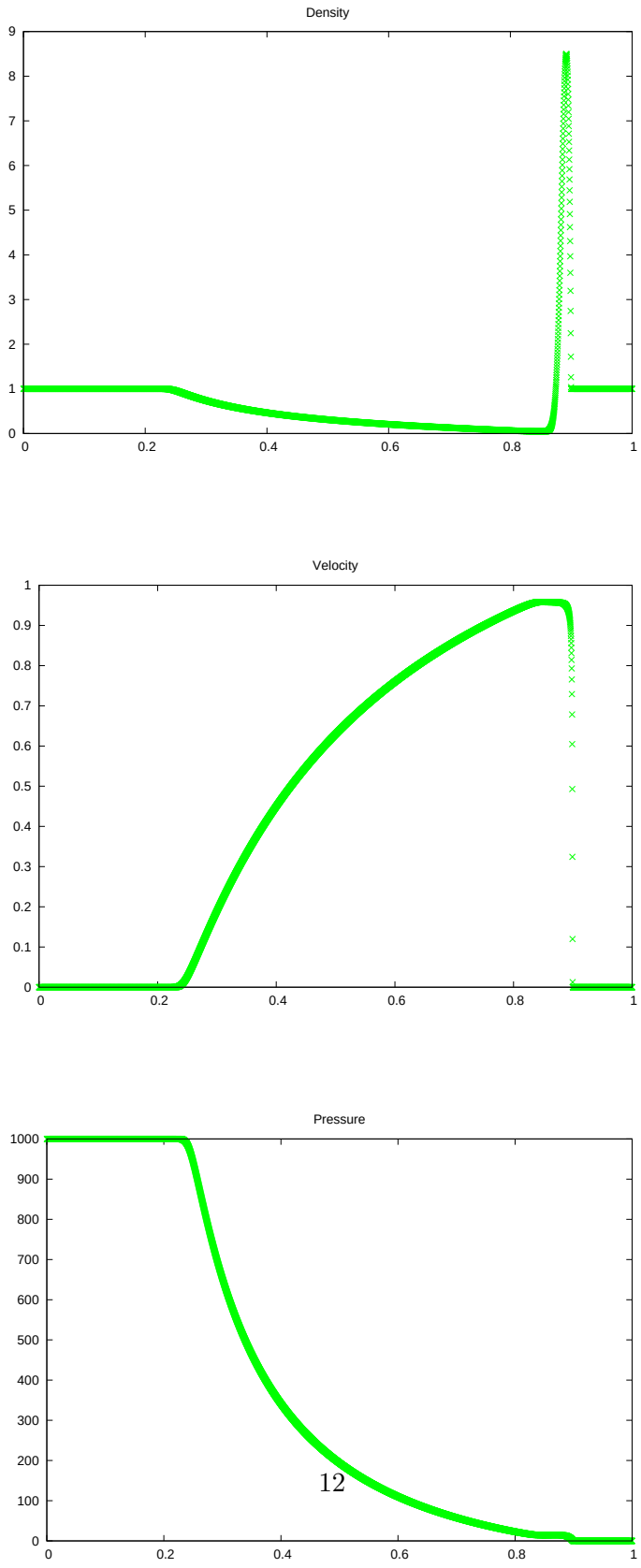

Figure 5: Strong relativistic blast wave test case

bottom

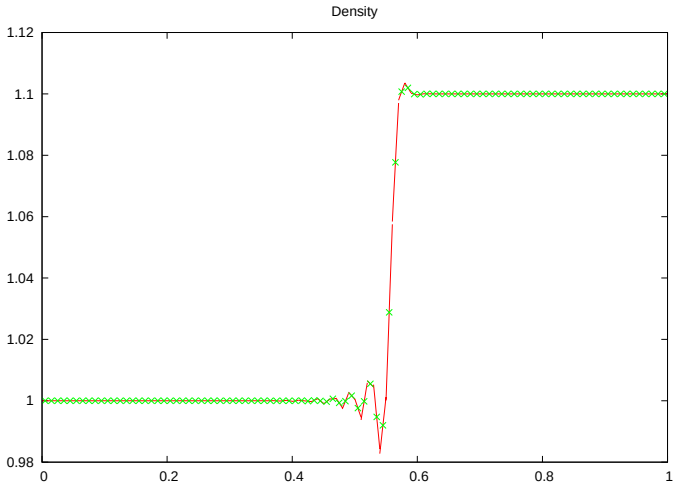

Figure 6: Oscillations with PORD = 2 for discontinuous solutions

4.2 To-Do's

- Get exact solution for Blast-Wave test case and compare.
- Get conserved eigenvectors and redesign the limiters.
- Design a method to choose boundary conditions.
- Design positivity checks.
- Design test cases.
- Design other fluxes.

A Miscellaneous identities and properties

1. Show that $\varepsilon = E - mu$

$$E - mu = \beta^2(\varepsilon + p) - p - \beta^2(\varepsilon + p)u^2 = \beta^2(\varepsilon + p)(1 - u^2) - p = \varepsilon + p - p = \varepsilon$$

2. Show that $E^2 - m^2 = \beta^2(\varepsilon^2 - u^2p^2)$

$$\begin{aligned} E^2 - m^2 &= [\beta^4(\varepsilon + p)^2 + p^2 - 2\beta^2 p(\varepsilon + p)] - \beta^4(\varepsilon + p)^2u^2 \\ &= \beta^4(\varepsilon + p)^2(1 - u^2) + p^2 - 2\beta^2 p(\varepsilon + p) \\ &= \beta^2(\varepsilon + p)^2 + p^2 - 2\beta^2 p(\varepsilon + p) \\ &= \beta^2(\varepsilon^2 + p^2 + 2\varepsilon p - 2\varepsilon p - 2p^2) + p^2 \\ &= \beta^2(\varepsilon^2 - p^2) + p^2 \\ &= \beta^2\varepsilon^2 + (1 - \beta^2)p^2 \\ &= \beta^2(\varepsilon^2 - u^2p^2) \end{aligned}$$

3. For the ideal gas model $\varepsilon > \frac{p}{\gamma - 1} \geq p$ since $\gamma \in (1, 2]$, so that

$$E^2 - m^2 > \beta^2(1 - u^2)p^2 = p^2 > 0$$

and hence $E > m$.

4. Show that $\lambda_i = \widetilde{\lambda}_i$ let $\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{V}} = P$. Then we show that A and $\widetilde{A}$ have same characterstic polynomial

$$\begin{aligned} \chi_A(\lambda) &= det(A - \lambda I) \\ &= det(P\widetilde{A}P^{-1} - \lambda I) \\ &= det(P\widetilde{A}P^{-1} - \lambda PIP^{-1}) \\ &= det(P(\widetilde{A} - \lambda I)P^{-1}) \\ &= det(\widetilde{A} - \lambda I) \\ &= \chi_B(\lambda) \end{aligned}$$

5. Show that $R_i = \frac{\partial \boldsymbol{U}}{\partial \boldsymbol{V}}\widetilde{R_i}$

$$\begin{aligned} \boldsymbol{A}R &= \lambda R \\ \widetilde{A}\widetilde{R} &= \widetilde{\lambda}\widetilde{R} \\ &\Rightarrow P\widetilde{A}P^{-1}R = \lambda R \\ &\Rightarrow \widetilde{A}P^{-1}R = \lambda P^{-1}R \\ &\Rightarrow \widetilde{R} = P^{-1}R \\ &\Rightarrow R = P\widetilde{R} \end{aligned}$$

6. Show that $L_i = \widetilde{L}_i \left(\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{V}}\right)^{-1}$

$$\begin{aligned} L\boldsymbol{A} &= \lambda L \\ \widetilde{L}\widetilde{A} &= \widetilde{\lambda}\widetilde{L} \\ &\Rightarrow LP\widetilde{A}P^{-1} = \lambda L \\ &\Rightarrow LP\widetilde{A} = \lambda LP \\ &\Rightarrow \widetilde{L} = LP \\ &\Rightarrow L = \widetilde{L}P^{-1} \end{aligned}$$

B Jacobian and Eigenvalues

The Jacobian was computed as

$$\widetilde{A} = \left[\left(\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{V}}\right)^{-1} \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{V}}\right]$$

After simplification we get

$$\widetilde{A} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}} = \begin{bmatrix} u1 & \frac{(\alpha-1)\rho^2 h}{H} & 0 & \frac{-(\alpha-1)\rho u_1}{\beta^2 H} \\ 0 & \frac{\alpha^2 p - \rho + \alpha(-2p+\rho)u_1}{H} & 0 & \frac{-\rho+\rho u_1^2 - \alpha^2 p(u_1^2-1) - \alpha(\rho(u-1^2-1)+p(1-u_1^2+u_2^2}{\beta^2 \rho h H} \\ 0 & \frac{-\alpha p u_2}{\beta^2 H} & u1 & \frac{-(\alpha^2 p - \rho + \alpha(-2p+\rho)u_1 u_2}{\beta^2 \rho h H} \\ 0 & \frac{\alpha p \rho h}{H} & 0 & \frac{\alpha^2 p - \rho + \alpha(-2p+\rho)u_1}{H} \end{bmatrix}$$

where $H = \alpha^2 p - \rho - \alpha(-\rho + p(1+u^2))$ and $\alpha = \frac{\gamma}{\gamma - 1}$
The eigenvalues are given by

$$\lambda_0 = u_1 \qquad \text{(double)} \tag{15}$$

$$\lambda_\pm = \frac{1}{1-u^2c^2}\left\{u_1(1-c^2) \pm c\sqrt{(1-u^2)(1-u_1^2-u_2^2c^2)}\right\} \tag{16}$$

C Transformations

To convert from $\boldsymbol{U}$ to $\boldsymbol{V}$, we first compute the velocity u as follows. Using $\varepsilon = E - mu$ and $\rho = D\sqrt{1-u^2}$, the pressure is given by $p = p(D\sqrt{1-u^2}, E - mu)$. Using (8), (9) we get

$$u = \frac{m}{E + p(D\sqrt{1-u^2}, E - mu)}$$

where the only unknown is the velocity u. For the ideal gas model, we can write the above equation as

$$\gamma u(E - mu) - (1-u^2)m = (\gamma - 1)Du\sqrt{1-u^2}$$

Let us square this equation on both sides, and define the function

$$\phi(\xi) = [\gamma\xi(E - m\xi) - (1 - \xi^2)m]^2 - (\gamma - 1)^2 D^2 \xi^2 (1 - \xi^2)$$

Schneider has shown that this equation has two roots in $[0, 1)$ and the correct root lies in the interval (ξ_l, ξ_u) where

$$\xi_l = \frac{1}{2(\gamma - 1)m}(\gamma E - \sqrt{\gamma^2 E^2 - 4(\gamma - 1)m^2}), \qquad \xi_u = \frac{m}{E}$$

These bounds are obtained by setting $D = 0$ and $p = 0$ respectively. Scneider suggests an initial guess for Newton method is taken as

$$\xi^0 = \frac{1}{2}(\xi_l + \xi_u) + z$$

where

$$z = \begin{cases} \frac{1}{2}(1 - D/E)(\xi_l - \xi_u) & \xi_l > 10^{-9} \\ 0 & \text{otherwise} \end{cases}$$

Now the derivative is

$$\phi'(\xi) =$$

The Newton method is given by

$$\xi^{n+1} = \xi^n - \frac{\phi(\xi^n)}{\phi'(\xi^n)}, \quad n = 0, 1, \ldots$$

During the Newton iterations, we must ensure that all iterates strictly satisfy $0 \leq \xi^n < 1$. This can be achieved by reducing the step size in Newton method. Once the velocity magnitude u is obtained, the remaining quantities are given by

$$\rho = D\sqrt{1 - u^2}, \qquad p = (\gamma - 1)(\varepsilon - \rho) = (\gamma - 1)(E - mu - \rho), \qquad u_i = \frac{m_i}{m} u$$

D Literature

XTROEM-FV: a new code for computational astrophysics based on very high order finite-volume methods - II. Relativistic hydro- and magnetohydrodynamics Jonatan Nunez-De La Rosa, Claus-Dieter Munz