ENPM667

Final Project

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Introduction

The given system is a crane that moves along an one-dimensional track. It behaves as a friction-less cart with mass M actuated by an external force F that constitutes the input of the system. There are two loads suspended from cables attached to the crane. The loads have mass m_1 and m_2, and the lengths of the cables are l_1 and l_2, respectively. Figure 1 depicts the crane and the associated variables: x, θ_1 and θ_2.

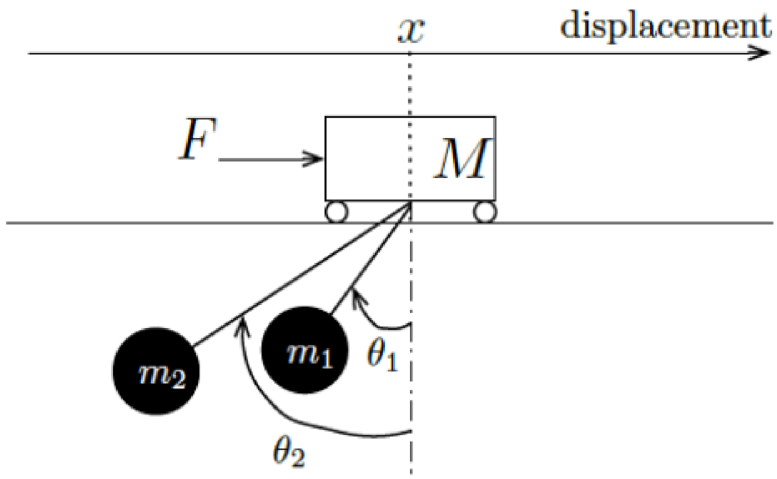

Fig. 1: Crane system

Assumptions:

- The payload is regarded as a material particle.
- The rope is considered as an inflexible rod.
- Compared with the mass m1 and m2, the mass of the rope is ignored.
- The trolley moves in the x-direction.
- The masses m1 and m2 move on the x–y plane.
- Friction is ignored.

Part A

Equations of motion

The equations of motion of the given system can be obtained based on the above assumptions. Euler-Lagrange method can be used to formulate the equations:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q} = f \tag{1}$$

where, $L = T - V$ is the Lagrange operator, q is the generalized coordinate, and T and V denote the kinetic and potential energy of the system and f is the acting force on the system.

The generalized coordinates in the given system are x, θ_1 and θ_2. Using Fig. 1, we get,

For mass 1:

$$x_1 = x - l_1 sin(\theta_1) \tag{2}$$

$$y_1 = -l_1 cos(\theta_1) \tag{3}$$

Thus the position vector for mass 1 is given by:

$$\vec{r_1} = (x - l_1 sin(\theta_1))\hat{i} - l_1 cos(\theta_1)\hat{j} \tag{4}$$

For mass 2:

$$x_2 = x - l_2 sin(\theta_2) \tag{5}$$

$$y_2 = l_2 cos(\theta_1) \tag{6}$$

Thus the position vector for mass 2 is given by:

$$\vec{r_2} = (x - l_2 sin(\theta_2))\hat{i} - l_2 cos(\theta_2)\hat{j} \tag{7}$$

To compute the velocities associated with each pendulum, we compute derivatives of the above equation, thus,

$$\dot{x}_1 = \dot{x} - l_1 cos(\theta_1)\dot{\theta}_1 \tag{8}$$

$$\dot{y}_1 = -l_1 sin(\theta_1)\dot{\theta}_1 \tag{9}$$

$$\dot{x}_2 = \dot{x} - l_2 cos(\theta_2)\dot{\theta}_2 \tag{10}$$

$$\dot{y}_2 = -l_2 sin(\theta_2)\dot{\theta}_2 \tag{11}$$

The magnitude of the velocity vector associated with each pendulum can be calculated as:

$$v_1 = \sqrt{(\dot{x} - l_1 cos(\theta_1)\dot{\theta}_1)^2 + (l_1 sin(\theta_1)\dot{\theta}_1)^2} \tag{12}$$
$$v_1 = \sqrt{(\dot{x}^2 - 2l_1 \dot{x} cos(\theta_1)\dot{\theta}_1 + l_1^2 \dot{\theta}_1^2}$$

$$v_2 = \sqrt{(\dot{x} - l_2 cos(\theta_2)\dot{\theta}_2)^2 + (l_2 sin(\theta_2)\dot{\theta}_2)^2} \tag{13}$$
$$v_2 = \sqrt{\dot{x}^2 - 2l_2 \dot{x} cos(\theta_2)\dot{\theta}_2 + l_2^2 \dot{\theta}_2^2}$$

Kinetic energy of the system can be written as,

$$T = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m_1 \ v_1^2 + \frac{1}{2} m_2 \ v_2^2 \tag{14}$$

$$T = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m_1 (\dot{x}^2 - 2l_1 \dot{x} cos(\theta_1)\dot{\theta}_1 + l_1^2 \dot{\theta}_1^2) + \frac{1}{2} m_2 (\dot{x}^2 - 2l_1 \dot{x} cos(\theta_1)\dot{\theta}_1 + l_1^2 \dot{\theta}_1^2)$$

$$= \frac{1}{2}(M + m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 - m_1 l_1 \dot{x} cos(\theta_1)\dot{\theta}_1 - m_2 l_2 \dot{x} cos(\theta_2)\dot{\theta}_2 \tag{15}$$

Potential energy of the system,

$$V = -m_1 g l_1 cos(\theta_1) - m_2 g l_2 cos(\theta_2) \tag{16}$$

The Lagrange of the system is computed as,

$$L = T - V \tag{17}$$

$$L = \frac{1}{2}(M + m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2$$
$$-m_1 l_1 \dot{x} cos(\theta_1)\dot{\theta}_1 - m_2 l_2 \dot{x} cos(\theta_2)\dot{\theta}_2 + m_1 g l_1 cos(\theta_1) + m_2 g l_2 cos(\theta_2) \tag{18}$$

The Lagrange equations are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F \tag{19}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = 0 \tag{20}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = 0 \tag{21}$$

Calculating the terms in the equations (19), (20) and (21)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = (M + m_1 + m_2)\ddot{x} - m_1 l_1 cos(\theta_1)\ddot{\theta}_1 + m_1 l_1 sin(\theta_1)\dot{\theta}_1^2 - m_2 l_2 cos(\theta_2)\ddot{\theta}_2 + m_2 l_2 sin(\theta_2)\ddot{\theta}_2^2 \tag{22}$$

$$\frac{\partial L}{\partial x} = 0 \tag{23}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1 cos(\theta_1) + m_1 l_1 sin(\theta_1)\dot{\theta}_1 \tag{24}$$

$$\frac{\partial L}{\partial \theta_1} = m_1 l_1 \dot{x} sin(\theta_1)\dot{\theta}_1 - m_1 g l_1 sin(\theta_1) \tag{25}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 cos(\theta_2) + m_2 l_2 sin(\theta_2)\dot{\theta}_2 \tag{26}$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_2 \dot{x} sin(\theta_2)\dot{\theta}_2 - m_2 g l_2 sin(\theta_2) \tag{27}$$

Thus, substituting the above in (19), (20) and (21), we get

$$(M + m_1 + m_2)\ddot{x} - m_1 l_1 cos(\theta_1)\ddot{\theta}_1 + m_1 l_1 sin(\theta_1)\dot{\theta}_1^2 - m_2 l_2 cos(\theta_2)\ddot{\theta}_2 + m_2 l_2 sin(\theta_2)\ddot{\theta}_2^2 = F \tag{28}$$

$$m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1 cos(\theta_1) + m_1 g l_1 sin(\theta_1) = 0 \tag{29}$$

$$m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 cos(\theta_2) + m_2 g l_2 sin(\theta_2) = 0 \tag{30}$$

Hence,

$$\ddot{x} = \frac{F + m_1 l_1 (cos(\theta_1)\ddot{\theta}_1 - sin(\theta_1)\dot{\theta}_1^2) + m_2 l_2 (cos(\theta_2)\ddot{\theta}_2 - sin(\theta_2)\dot{\theta}_2^2)}{(M + m_1 + m_2)} \tag{31}$$

$$\ddot{\theta}_1 = \frac{cos(\theta_1)\ddot{x} - g sin(\theta_1)}{l_1} \tag{32}$$

$$\ddot{\theta}_2 = \frac{cos(\theta_2)\ddot{x} - g sin(\theta_2)}{l_2} \tag{33}$$

Substituting (32) and (33) in (31),

$$\ddot{x} = \frac{F - m_1 (g sin(\theta_1) cos(\theta_1) + l_1 sin(\theta_1)\dot{\theta}_1^2) - m_2 (g sin(\theta_2) cos(\theta_2) + l_2 sin(\theta_2)\dot{\theta}_2^2)}{(M + m_1 sin^2(\theta_1) + m_2 sin^2(\theta_2))} \tag{34}$$

Substituting (34) in (32) and (33),

$$\ddot{\theta}_1 = \frac{cos(\theta_1)}{l_1}\frac{(F - m_1 (g sin(\theta_1) cos(\theta_1) + l_1 sin(\theta_1)\dot{\theta}_1^2) - m_2 (g sin(\theta_2) cos(\theta_2) + l_2 sin(\theta_2)\dot{\theta}_2^2))}{(M + m_1 sin^2(\theta_1) + m_2 sin^2(\theta_2))} - \frac{g sin(\theta_1)}{l_1} \tag{35}$$

$$\ddot{\theta}_2 = \frac{cos(\theta_2)}{l_2}\frac{(F - m_1(gsin(\theta_1)cos(\theta_1) + l_1 sin(\theta_1)\dot{\theta}_1^2) - m_2(gsin(\theta_2)cos(\theta_2) + l_2 sin(\theta_2)\dot{\theta}_2^2))}{(M + m_1 sin^2(\theta_1) + m_2 sin^2(\theta_2))} - \frac{gsin(\theta_2)}{l_2} \tag{36}$$

Thus, these are the final equations obtained.

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{F - m_1(gsin(\theta_1)cos(\theta_1) + l_1 sin(\theta_1)\dot{\theta}_1^2) - m_2(gsin(\theta_2)cos(\theta_2) + l_2 sin(\theta_2)\dot{\theta}_2^2)}{(M + m_1 sin^2(\theta_1) + m_2 sin^2(\theta_2))} \\ \dot{\theta}_1 \\ \frac{cos(\theta_1)}{l_1}\frac{(F - m_1(gsin(\theta_1)cos(\theta_1) + l_1 sin(\theta_1)\dot{\theta}_1^2) - m_2(gsin(\theta_2)cos(\theta_2) + l_2 sin(\theta_2)\dot{\theta}_2^2))}{(M + m_1 sin^2(\theta_1) + m_2 sin^2(\theta_2))} - \frac{gsin(\theta_1)}{l_1} \\ \dot{\theta}_2 \\ \frac{cos(\theta_2)}{l_2}\frac{(F - m_1(gsin(\theta_1)cos(\theta_1) + l_1 sin(\theta_1)\dot{\theta}_1^2) - m_2(gsin(\theta_2)cos(\theta_2) + l_2 sin(\theta_2)\dot{\theta}_2^2))}{(M + m_1 sin^2(\theta_1) + m_2 sin^2(\theta_2))} - \frac{gsin(\theta_2)}{l_2} \end{bmatrix} \tag{37}$$

Part B

Linearized system

The given non-linear system can be linearized using two methods as given below:

1. Small angle approximation

   $sin(\theta) = \theta \qquad \cos(\theta) = 1 \qquad \theta^2 = 0$

   Substituting the values in (38)

   $$\ddot{x} = \frac{F - m_1 g\theta_1 - m_2 g\theta_2}{M} \tag{39}$$

   $$\ddot{\theta}_1 = \frac{F - m_1 g\theta_1 - m_2 g\theta_2}{Ml_1} - \frac{g\theta_1}{l_1} \tag{40}$$

   $$\ddot{\theta}_1 = \frac{F - m_1 g\theta_1 - m_2 g\theta_2}{Ml_2} - \frac{g\theta_2}{l_2} \tag{41}$$

2. Jacobi Linearization

   For the system given by (37), we can calculate:

$$A = J|_{x=0,\theta_1=\theta_2=0} = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial \dot{x}} & \frac{\partial \dot{x}}{\partial \theta_1} & \frac{\partial \dot{x}}{\partial \dot{\theta}_1} & \frac{\partial \dot{x}}{\partial \theta_2} & \frac{\partial \dot{x}}{\partial \dot{\theta}_2} \\ \frac{\partial \ddot{x}}{\partial x} & \frac{\partial \ddot{x}}{\partial \dot{x}} & \frac{\partial \ddot{x}}{\partial \theta_1} & \frac{\partial \ddot{x}}{\partial \dot{\theta}_1} & \frac{\partial \ddot{x}}{\partial \theta_2} & \frac{\partial \ddot{x}}{\partial \dot{\theta}_2} \\ \frac{\partial \dot{\theta}_1}{\partial x} & \frac{\partial \dot{\theta}_1}{\partial \dot{x}} & \frac{\partial \dot{\theta}_1}{\partial \theta_1} & \frac{\partial \dot{\theta}_1}{\partial \dot{\theta}_1} & \frac{\partial \dot{\theta}_1}{\partial \theta_2} & \frac{\partial \dot{\theta}_1}{\partial \dot{\theta}_2} \\ \frac{\partial \ddot{\theta}_1}{\partial x} & \frac{\partial \ddot{\theta}_1}{\partial \dot{x}} & \frac{\partial \ddot{\theta}_1}{\partial \theta_1} & \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_1} & \frac{\partial \ddot{\theta}_1}{\partial \theta_2} & \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_2} \\ \frac{\partial \dot{\theta}_2}{\partial x} & \frac{\partial \dot{\theta}_2}{\partial \dot{x}} & \frac{\partial \dot{\theta}_2}{\partial \theta_1} & \frac{\partial \dot{\theta}_2}{\partial \dot{\theta}_1} & \frac{\partial \dot{\theta}_2}{\partial \theta_2} & \frac{\partial \dot{\theta}_2}{\partial \dot{\theta}_2} \\ \frac{\partial \ddot{\theta}_2}{\partial x} & \frac{\partial \ddot{\theta}_2}{\partial \dot{x}} & \frac{\partial \ddot{\theta}_2}{\partial \theta_1} & \frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_1} & \frac{\partial \ddot{\theta}_2}{\partial \theta_2} & \frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_2} \end{bmatrix}_{|x=0,\theta_1=\theta_2=0} \tag{42}$$

Now, we know the equilibrium points as $x = 0, \theta_1 = \theta_2 = 0$. Thus substituting in the Jacobian matrix, we get,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm_1}{M} & 0 & \frac{-gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(M+m_1)}{Ml_1} & 0 & \frac{-gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-gm_1}{Ml_2} & 0 & \frac{-g(M+m_2)}{Ml_2} & 0 \end{bmatrix} \tag{43}$$

$$B = J|_{x=0,\theta_1=\theta_2=0} = \begin{bmatrix} \frac{\partial \dot{x}}{\partial F} \\ \frac{\partial \ddot{x}}{\partial F} \\ \frac{\partial \dot{\theta}_1}{\partial F} \\ \frac{\partial \ddot{\theta}_1}{\partial F} \\ \frac{\partial \dot{\theta}_2}{\partial F} \\ \frac{\partial \ddot{\theta}_2}{\partial F} \end{bmatrix}_{|x=0,\theta_1=\theta_2=0} \tag{44}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} \tag{45}$$

Thus, the state-space representation of the linearized system is expressed as below:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm_1}{M} & 0 & \frac{-gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(M+m_1)}{Ml_1} & 0 & \frac{-gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-gm_1}{Ml_2} & 0 & \frac{-g(M+m_2)}{Ml_2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} F \tag{46}$$

Part C

Controllability

A Linear Time Invariant system is controllable if the controllability matrix C is a full rank matrix, i.e., $rank(C) = n$ for given system of $A_{n\times n}, B_{n\times 1}$,where n is the number of state variables. The matrix can be calculated as:

$$rank(C) = rank\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{47}$$

The controllability matrix computed for the given system is:

$$C = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix} \tag{48}$$

$$C = \begin{bmatrix} 0 & \frac{1}{M} & 0 & C_{14} & 0 & C_{16} \\ \frac{1}{M} & 0 & C_{23} & 0 & C_{25} & 0 \\ 0 & \frac{1}{ML_1} & 0 & C_{34} & 0 & C_{36} \\ \frac{1}{ML_1} & 0 & C_{43} & 0 & C_{45} & 0 \\ 0 & \frac{1}{ML_2} & 0 & C_{54} & 0 & C_{56} \\ \frac{1}{ML_2} & 0 & C_{63} & 0 & C_{65} & 0 \end{bmatrix}$$

$$C_{14} = C_{23} = -\frac{m_1 g}{M^2 l_1} - \frac{m_2 g}{M^2 l_2}$$

$$C_{16} = C_{25} = \frac{gm_1(Mg + gm_1)}{l_1 M^2} + \frac{g^2 m_1 m_2}{l_2 M^3 l_1} + \frac{gm_2(Mg + gm_2)}{l_2 M^2} + \frac{g^2 m_1 m_2}{l_1 M^3 l_2}$$

$$C_{34} = C_{43} = -\frac{Mg + gm_1}{l_1^2 M^2} - \frac{gm_2}{l_1 l_2 M^2}$$

$$C_{36} = C_{45} = \frac{gm_2(Mg + gm_1}{l_1^2 M^2} + \frac{gm_2(Mg + gm_2)}{l_1 l_2 M^3 l_2} + \frac{(Mg + gm_1)^2}{l_1^2 M^2} + \frac{g^2 m_1 m_2}{l_1 l_2 M^3 l_1}$$

$$C_{54} = C_{63} = -\frac{Mg + gm_2}{l2^2 M^2} - \frac{gm_1}{l_1 l_2 M^2}$$

$$C_{56} = C_{65} = \frac{gm_1(Mg + gm_2)}{l_2^2 M^2} + \frac{gm_1(Mg + gm_1)}{l_1 l_2 M^3 l_1} + \frac{Mg + gm_2)^2}{l_2^2 M^2} + \frac{g^2 m_1 m_2}{l_1 l_2 M^3 l_2}$$

Thus,

$$rank(C) = 6$$

Conditions on M, m_1, m_2, l_1, l_2 :

The determinant of the controllability matrix should never be zero for a system to be controllable, i.e., $\det(C) \neq 0$. The determinant for the above matrix is given by:

$$\det C = \frac{-g^6 (L_1 - L_2)^2}{M_6 L_1^6 L_2^6} \tag{49}$$

Thus, $L_1 - L_2 = 0 \Rightarrow L_1 = L_2$ makes the system uncontrollable. For the system to be controllable, the only condition is $L_1 \neq L_2$.

Part D

LQR Controller

The system will be controlled using linear state feedback. The state of the plant measured at the output and is transformed by a linear gain matrix -K before being added to the plant input. This forms a closed loop system to stabilize the system. The gain matrix K is optimized by a convex quadratic cost function that balances speed of response (Q) with input magnitude(R). The importance attributed to these aspects of cost are chosen by the designer with values in Q assigning cost to the different state variables and size of R penalizing input effort.

The given values of masses and lengths are:

$M = 1000kg$ $\quad$ $\mathrm{m}_1 = m_2 = 100kg$ $\quad$ $\mathrm{l}_1 = 20m$ $\quad$ $\mathrm{l}_2 = 10m$

For the linearized system,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm_1}{M} & 0 & \frac{-gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(M+m_1)}{Ml_1} & 0 & \frac{-gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-gm_1}{Ml_2} & 0 & \frac{-g(M+m_2)}{Ml_2} & 0 \end{bmatrix} \tag{50}$$

Substituting the given values, we get

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.981 & 0 & -0.981 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -0.534 & 0 & -0.049 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.098 & 0 & -1.079 & 0 \end{bmatrix} \tag{51}$$

Eigenvalues of A:

$\lambda_1 = \ 0 + 0i; \lambda_2 = 0 + 0i; \lambda_3 = 0 + 0.7285i; \lambda_4 = 0 - 0.7285i; \lambda_5 = 0 + 1.0430i; \lambda_6 = 0 - 1.0430i$

The state equation can be written as:

$$\dot{X} = AX + B_K U \qquad \text{U= -KX} \tag{52}$$

Thus, from (44), we get,

$$\dot{X} = (A - B_K K)X \tag{53}$$

The aim is the minimize the cost function, J, given by,

$$J(K, X(0)) = \int_0^{\infty} X^T(t)QX(t) + {U_K}^T(t)RU_K(t)dt \tag{54}$$

where Q and R are symmetric positive definite matrices.

The optimal solution of (46) is given by the following controller:

$$K = -R^{-1}{B_K}^T P \tag{55}$$

where, P is the symmetric positive solution of the following stationary Riccati equation:

$$A^T P + PA - PB_K R^{-1}{B_K}^T P = -Q \tag{56}$$

The cost $J(K, X(0))$ can be expressed as:

$$J(K, X(0)) = X(0)^T PX(0) \tag{57}$$

Eigenvalues of the closed loop observer, i.e., $[A - BK]$:

$\lambda_1 = -0.12 + 0.10i; \lambda_2 = -0.120.10i; \lambda_3 = -0.10 + 1.04i;$

$\lambda_4 = -0.10 - 1.04i; \lambda_5 = -0.04 + 0.72i; \lambda_6 = -0.04 - 0.72i$

It can be observed that the real parts of the eigenvalues shifted to the left half plane after the controller was applied. Thus, according to Lyapunov's indirect method for stability, we can certify that the system is stable.

Step response of linearized system:

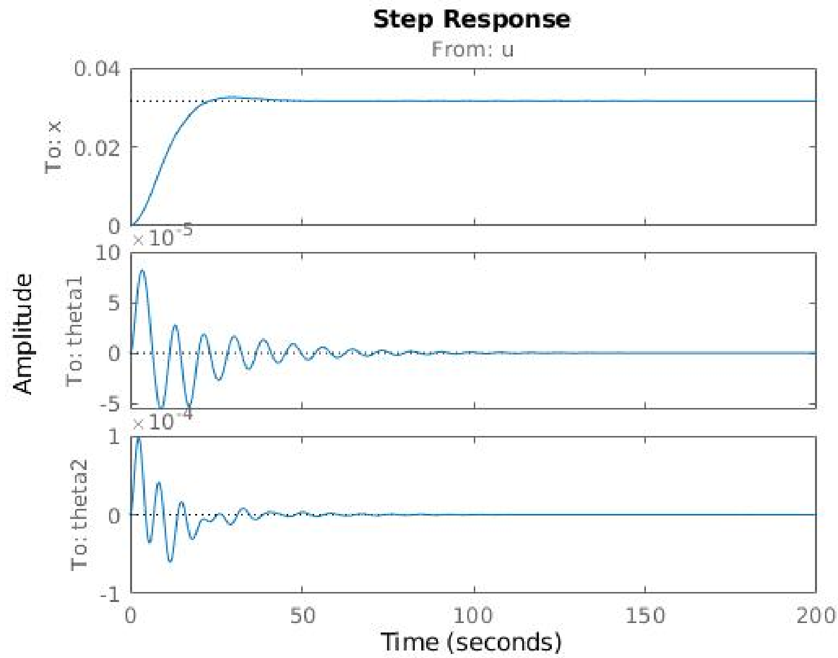

Fig. 2

Initial state response of the linearized system with initial conditions:

$x = 1 \qquad \theta_1 = 10 \qquad \theta_2 = 20$

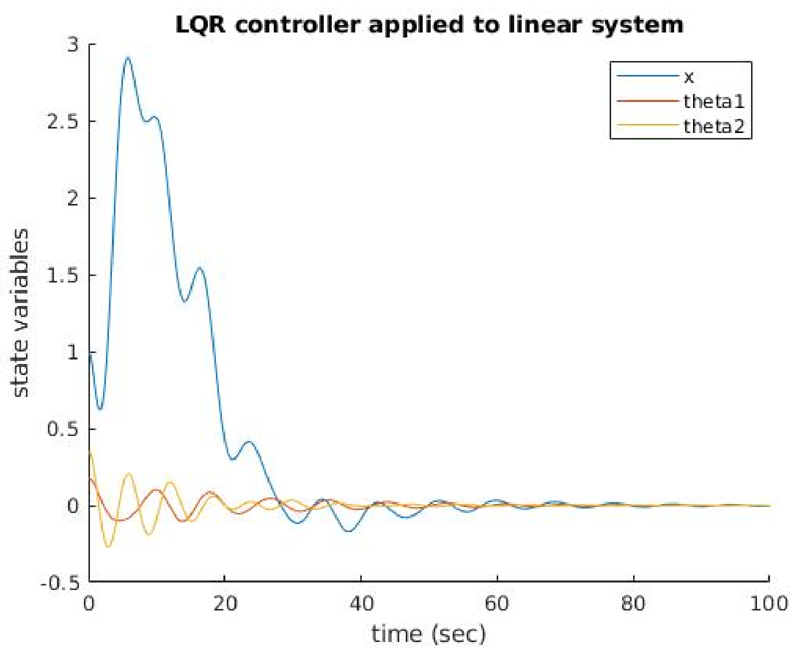

Fig. 3

Initial state response of the original nonlinear system with initial conditions:

$x = 1 \qquad \theta_1 = 10 \qquad \theta_2 = 20$

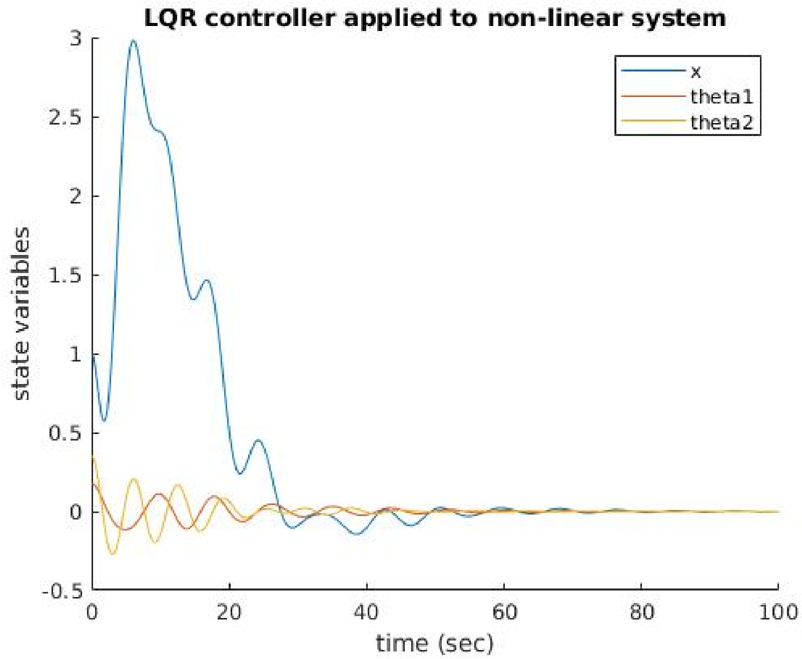

Fig. 4

Part E

Observable vectors:

A Linear Time Invariant system is observable if the observability matrix is a full rank matrix for given system of $A_{n\times n}, C_{p\times n}$, where, n is the number of state variables and p is the number of outputs. The observability matrix satisfies:

$$rank \begin{bmatrix} C \\ CA \\ CA^2 \\ \cdots \\ CA^{n-1} \end{bmatrix} = n \tag{58}$$

which can be written as:

$$rank \begin{bmatrix} C^T & (A^T)C^T & (A^T)^2 C^T & \cdots & (A^T)^{n-1} C^T \end{bmatrix} = n \tag{59}$$

Then the complete state of the variables can be determined from a certain limited number of measurements of the output.

Thus for the given system, we have to compute:

$$rank \begin{bmatrix} C^T & (A^T)C^T & (A^T)^2 C^T & (A^T)^3 C^T & (A^T)^4 C^T & (A^T)^5 C^T \end{bmatrix} \tag{60}$$

We have the choice of selecting the following combination of output vectors:

1. $x(t)$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{61}$$

$$rank \begin{bmatrix} C_1^T & (A^T)C_1^T & (A^T)^2C_1^T & (A^T)^3C_1^T & (A^T)^4C_1^T & (A^T)^5C_1^T \end{bmatrix} = 6$$

2. $\theta_1(t), \theta_2(t)$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{62}$$

$$rank \begin{bmatrix} C_2^T & (A^T)C_2^T & (A^T)^2C_2^T & (A^T)^3C_2^T & (A^T)^4C_2^T & (A^T)^5C_2^T \end{bmatrix} = 4$$

3. $x(t), \theta_2(t)$

$$C_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{63}$$

$$rank \begin{bmatrix} C_3^T & (A^T)C_3^T & (A^T)^2C_3^T & (A^T)^3C_3^T & (A^T)^4C_3^T & (A^T)^5C_3^T \end{bmatrix} = 6$$

4. $x(t), \theta_1(t), \theta_2(t)$

$$C_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{64}$$

$$rank \begin{bmatrix} C_4^T & (A^T)C_4^T & (A^T)^2C_4^T & (A^T)^3C_4^T & (A^T)^4C_4^T & (A^T)^5C_4^T \end{bmatrix} = 6$$

For Case (3), the observability matrix is not a full rank matrix. Thus, the system is not observable for the choice of output vectors $\theta_1(t), \theta_2(t)$.

Part F

Observer Design

The Luenberger Observer is a non-optimized state estimator which is represented as dynamical system. It has the following state-space representation:

$$\dot{\hat{X}}(t) = A\hat{X}(t) + B_K U_K(t) + L[Y(t) - C\hat{X}(t)], \hat{X}(0) = 0 \tag{65}$$

$$\dot{\hat{X}} = (A - LC)\hat{X} + \begin{bmatrix} B & L \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \tag{66}$$

Here:

L is the observer gain matrix and

$Y(t) - C\hat{X}(t)$ is the correction term needed so as to correct the error which may be unbounded if A is unstable. The correction term is used to update the estimate of the state and if the pair (A, C) is observable, the estimated state will converge on the true state.

The estimation error $X_e(t) = X(t) - \hat{X}(t)$ has the following state-space representation:

$$\dot{X}_e(t) = \dot{X}(t) - \dot{\hat{X}}(t) = AX_e(t) - L[Y(t) - C\hat{X}(t)] + B_D U_D(t) \tag{67}$$

Now, we assume $D = 0, Y = CX(t)$, indicate that only a specific set of measurements are being made with which to estimate the state.

Therefore (66) could be written as:

$$\dot{X}_e(t) = [A - LC]X_e(t) + B_D U_D(t) \tag{68}$$

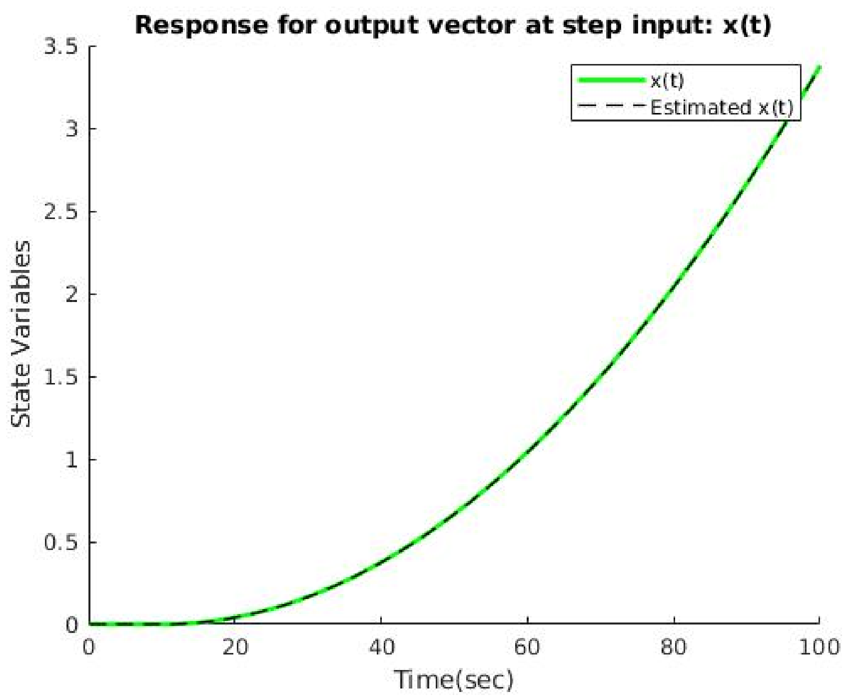

Fig. 5

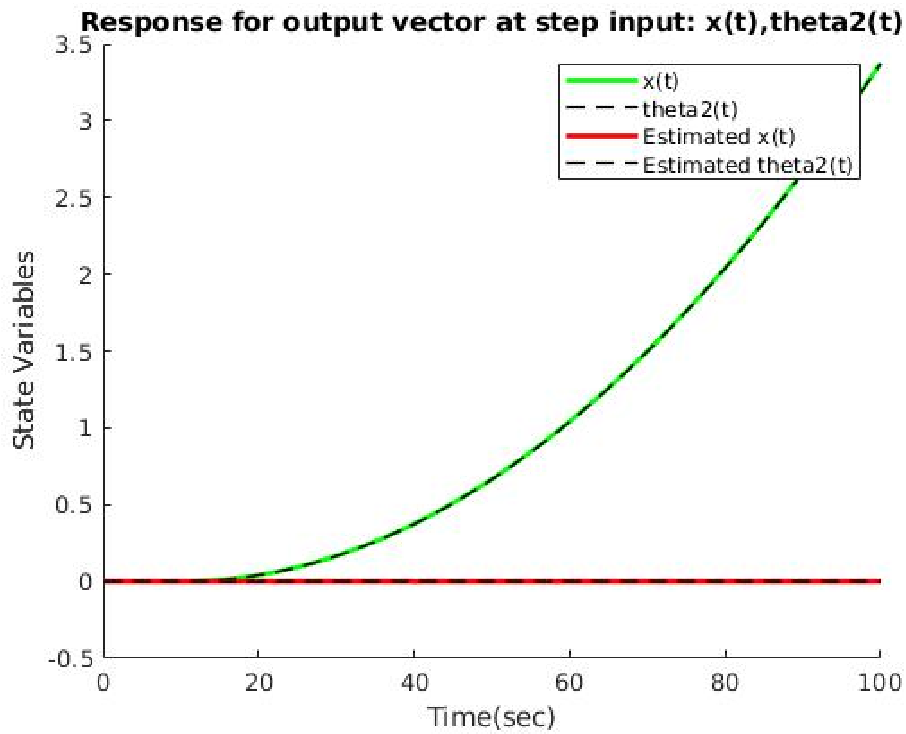
Response for output vector at step input: x(t),theta2(t)
x(t)
theta2(t)
Estimated x(t)
Estimated theta2(t)
State Variables
Time(sec)
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
0
20
40
60
80
100

Fig. 6

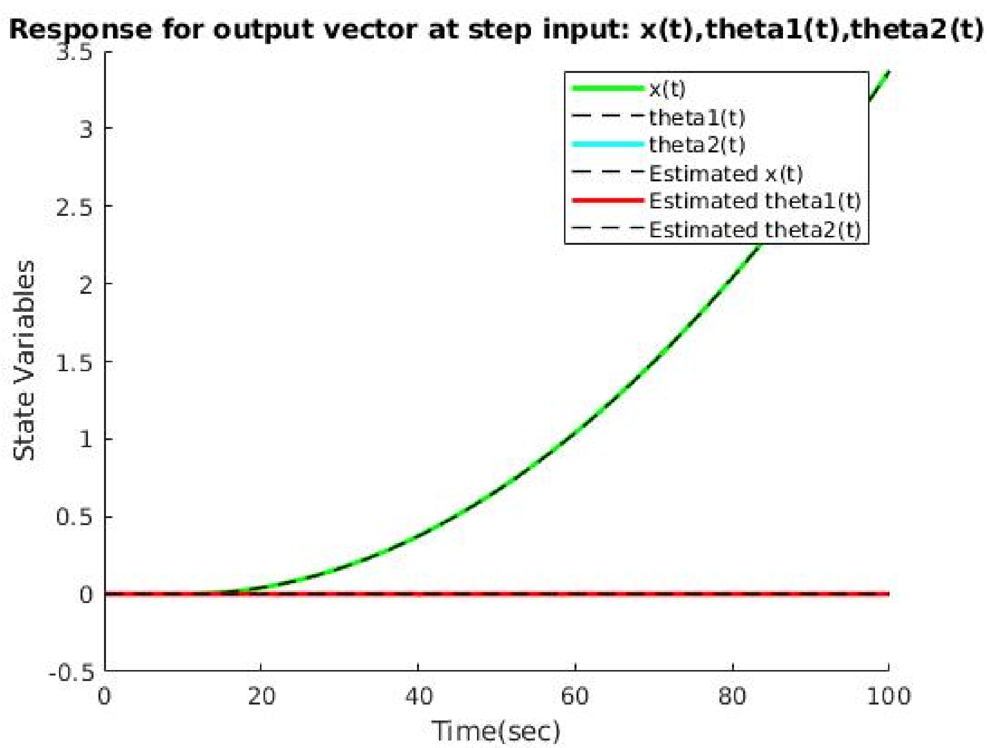
Response for output vector at step input: x(t),theta1(t),theta2(t)
x(t)
theta1(t)
theta2(t)
Estimated x(t)
Estimated theta1(t)
Estimated theta2(t)
State Variables
Time(sec)
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
0
20
40
60
80
100

Fig. 7

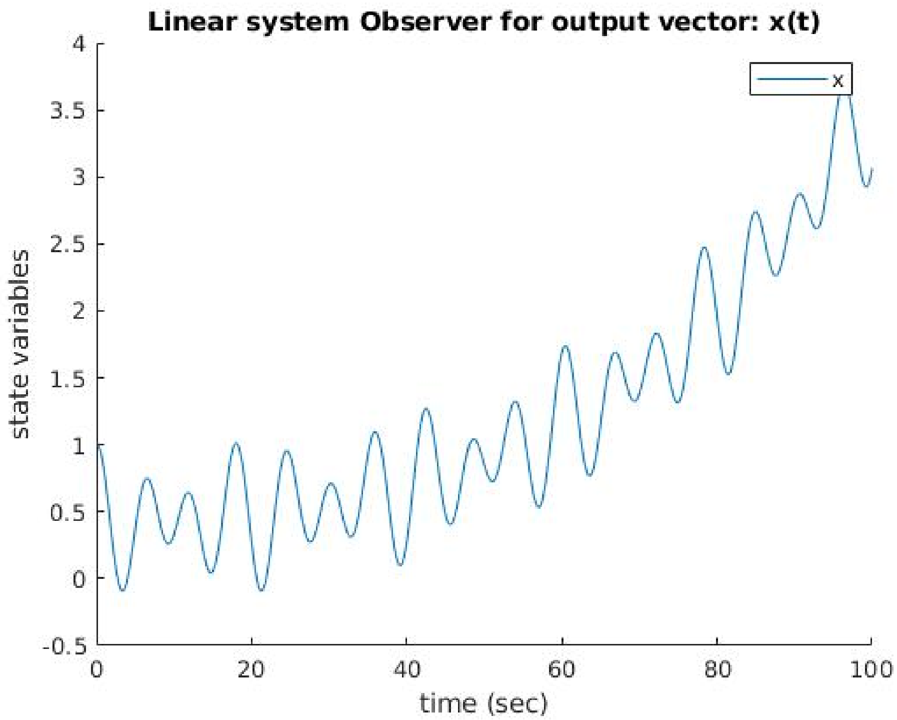
Linear system Observer for output vector: x(t)
x
state variables
time (sec)
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
0
20
40
60
80
100

Fig. 8

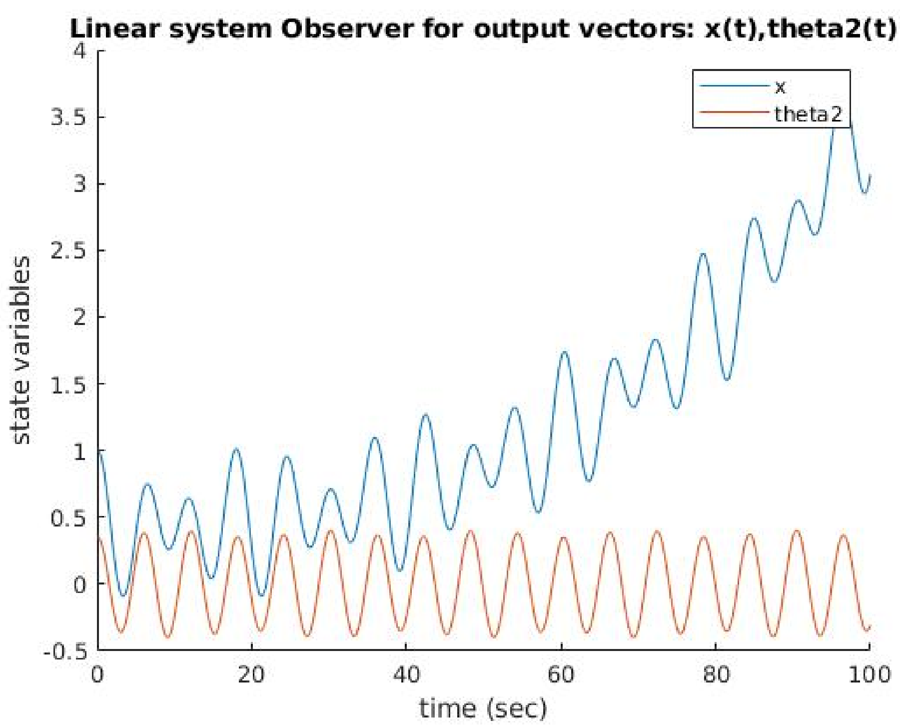
Linear system Observer for output vectors: x(t),theta2(t)
x
theta2
state variables
time (sec)
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
0
20
40
60
80
100

Fig. 9

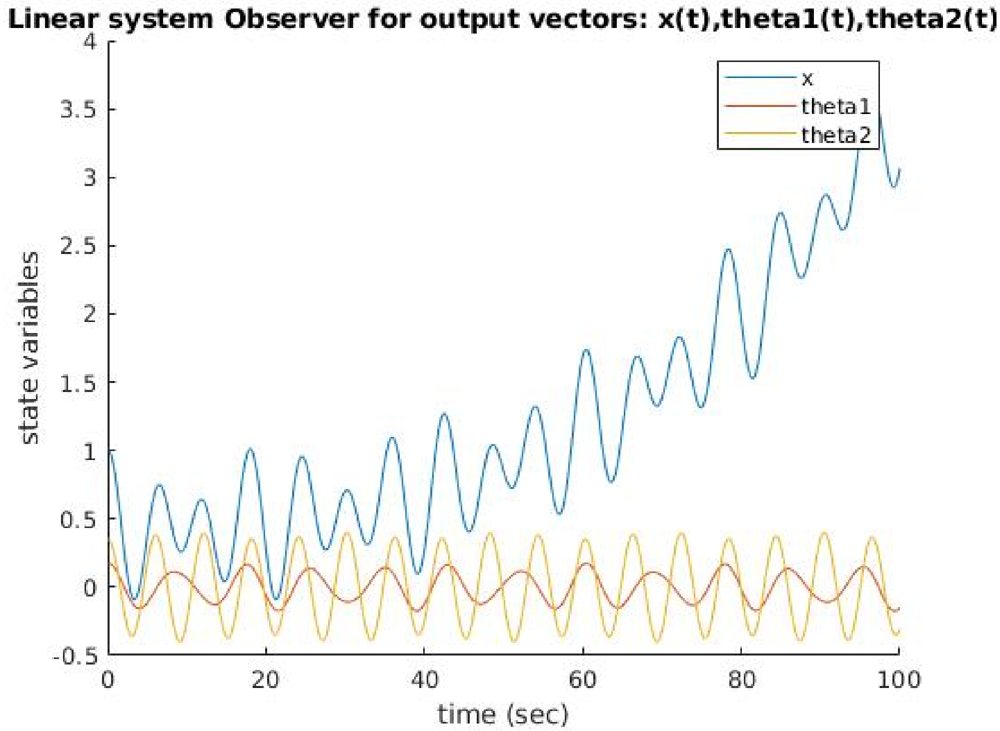
Linear system Observer for output vectors: x(t),theta1(t),theta2(t)
x
theta1
theta2
state variables
time (sec)

Fig. 10

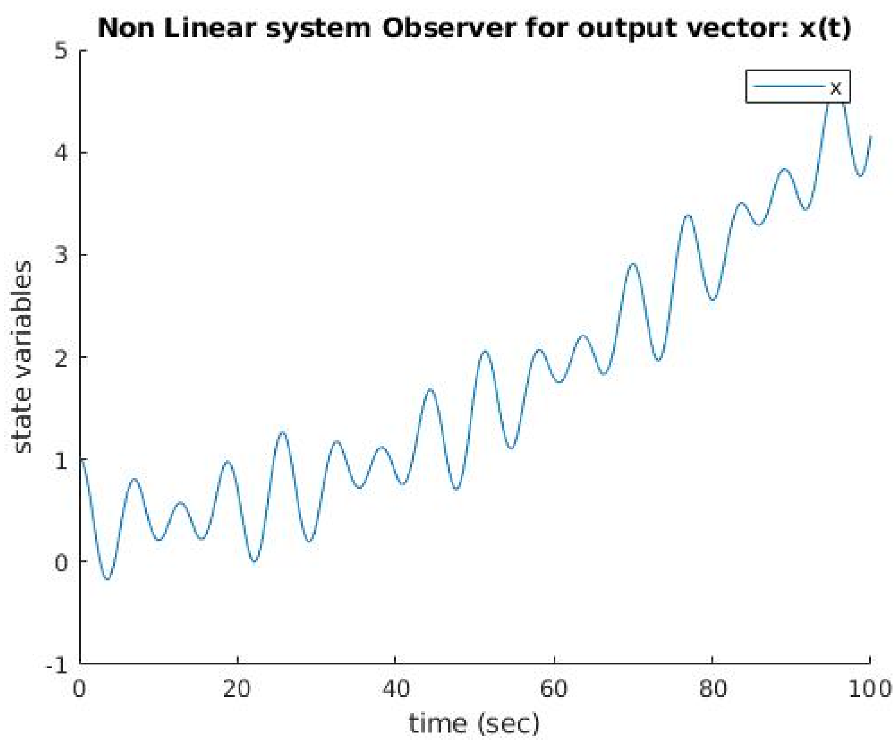
Non Linear system Observer for output vector: x(t)
x
state variables
time (sec)

Fig. 11

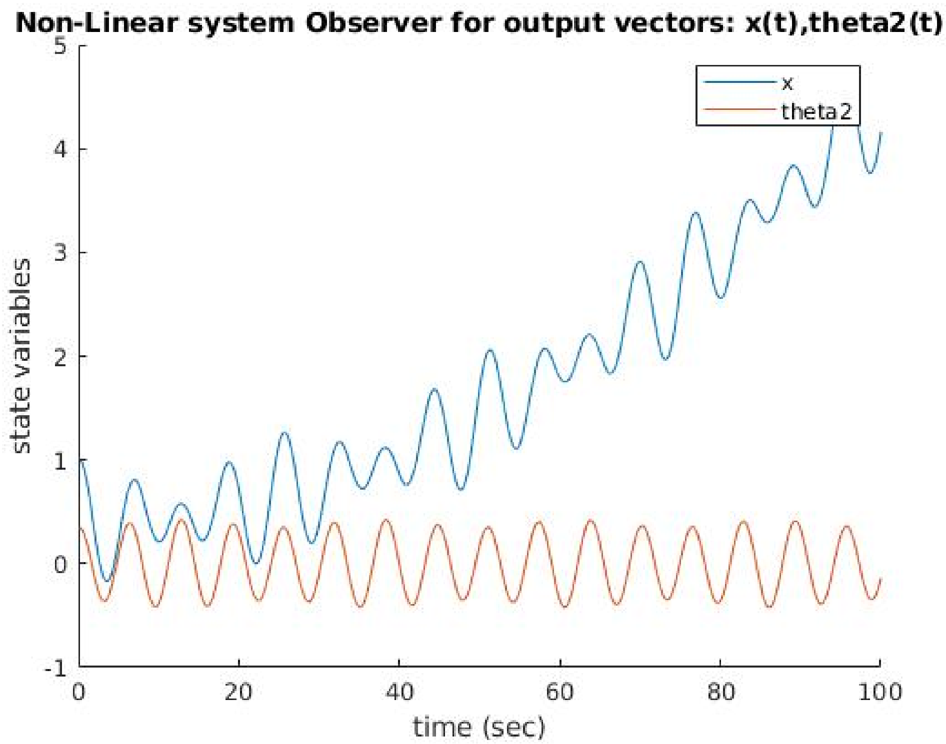
Non-Linear system Observer for output vectors: x(t),theta2(t)
x
theta2
state variables
time (sec)
5
4
3
2
1
0
-1
0
20
40
60
80
100

Fig. 12

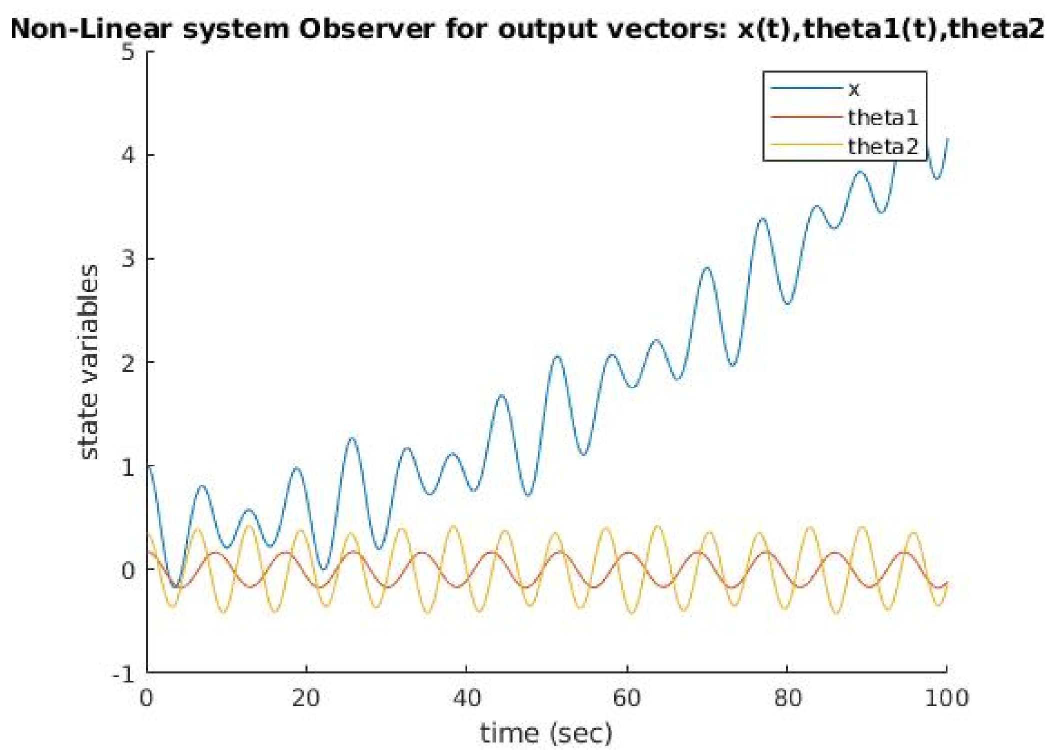
Non-Linear system Observer for output vectors: x(t),theta1(t),theta2
x
theta1
theta2
state variables
time (sec)
5
4
3
2
1
0
-1
0
20
40
60
80
100

Fig. 13

Part G

Linear Quadratic Gaussian Method

The structure of the optimal solution is given by the standard output feedback configuration with the Luenberger observer and the optimal K and L are computed separately using the LQR and the Kalman-Bucy methods. By the separation principle, the controller and estimator can be optimized separately and when combined, they still result in the optimal system.

The state-space representation can be written as:

$$\dot{X}(t) = AX(t) + B_K U_K(t) + B_D U_D(t) \tag{69}$$

$$Y(t) = CX(t) + V(t) \tag{70}$$

where, $U_D(t)$ represents the process noise and $V(t)$ represents the measurement noise and they both are independent zero mean white Gaussian processes with covariances Σ_D and Σ_V respectively.

The cost we want to minimize is given by:

$$\lim_{t\to\infty} E[X^T(t)QX(t) + U^T(t)RU(t)] \tag{71}$$

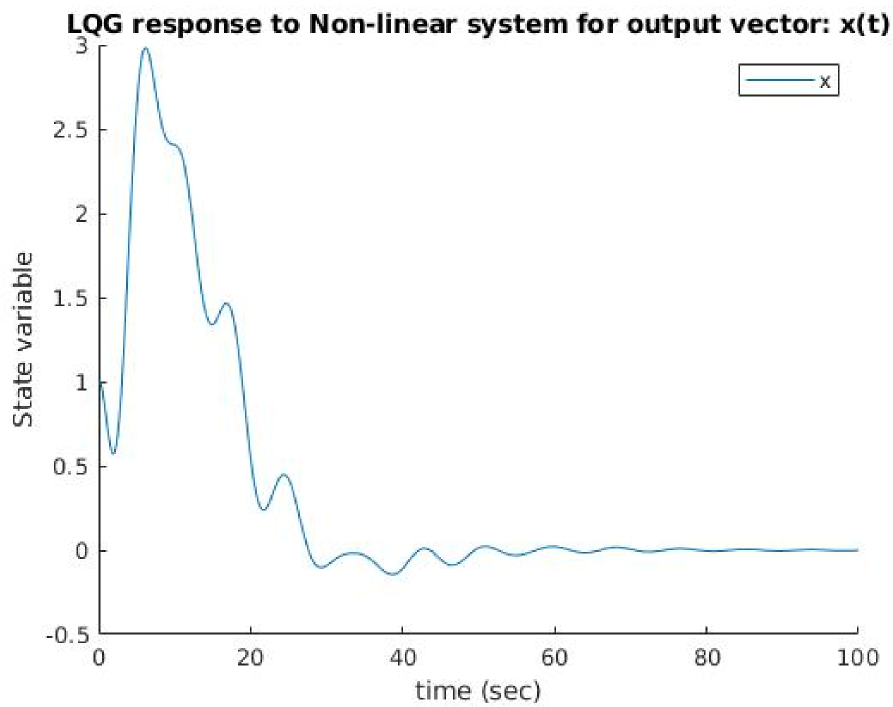

Fig. 14

Reference Tracking

To perform reference tracking, the state equation will be modified with the following terms:

$$\dot{\tilde{X}} = AX + B_K U_K + B_D U_D - AX_d - B_K U_\infty \tag{72}$$

With the new term that captures the desired state X_d in the state equation, the cost can be adjusted to minimize the error between the actual state and the desired state. This will bring the error between the state and the tracked reference X_d to zero if stabilized with the appropriate gain K.

Disturbance Rejection

A disturbance term is introduced by modifying the disturbance input matrix B_D. Values of 1, 10 and 1000 were applied as a disturbance and there was no appreciable change to the performance of the controller. The controller can reject such disturbance under certain conditions. Since the control law K was using the linearized system, it starts to break down when the system is moved from this point. If the force or disturbance applied to the system is large enough to do so, higher order terms of the Taylor series of e^{At} that were ignored as an assumption, start becoming relevant. It is also known that the LQG controller can suffer from brittleness. This results from heightened sensitivity of the observer to disturbance terms that distort the expected stability.

Appendix

Matlab Code

Part C

```
clear all, close all, clc

%% Controllability

syms m1 g m2 M l1 l2

A = [0 1 0 0 0 0; 0 0 -m1*g/M 0 -m2*g/M 0; 0 0 0 1 0 0; 0 0 -((M*g)+(m1*g))/(M*l1) 0
-g*m2/(M*l1) 0; 0 0 0 0 0 1; 0 0 -m1*g/(M*l2) 0 -((M*g)+(m2*g))/(M*l2) 0];

B = [0; 1/M; 0; 1/(l1*M); 0; 1/(l2*M)];

C = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];

D = [0;0;0];

Co = [B A*B (A^2)*B (A^3)*B (A^4)*B (A^5)*B];

rank = rank(Co);

fprintf('Rank of controllability matrix: %d\n',rank)

det_Co = det(Co);

disp('Determinant of controllability matrix:')

disp(det_Co)
```

Part D

```
clear all, close all, clc
```

```
% Variable definitions
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.81;
timespan = 0:0.05:100;
% Initial conditions
x_0 = 1;
theta1_0 = deg2rad(10);
theta2_0 = deg2rad(20);
X_0 = [1 0 theta1_0 0 theta2_0 0];

%% Linearized model
A = [0 1 0 0 0 0; 0 0 -m1*g/M 0 -m2*g/M 0; 0 0 0 1 0 0; 0 0 -((M*g)+(m1*g))/(M*l1) 0
-g*m2/(M*l1) 0; 0 0 0 0 0 1; 0 0 -m1*g/(M*l2) 0 -((M*g)+(m2*g))/(M*l2) 0];
B = [0; 1/M; 0; 1/(l1*M); 0; 1/(l2*M)];
C = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];
D = [0;0;0];
Co = ctrb(A,B);
rank = rank(Co);
fprintf('Rank of controllability matrix: %d\n',rank)

%% LQR Controller
Q = [1 0 0 0 0 0; 0 10 0 0 0 0; 0 0 2000 0 0 0; 0 0 0 2000 0 0; 0 0 0 0 2000 0; 0 0 0 0 0
2000];
R = 0.001;
K = lqr(A,B,Q,R);
Ac = [(A-B*K)];
Bc = [B];
Cc = [C];
Dc = [D];

states = {'x' 'x_dot' 'theta1' 'theta1_dot' 'theta2' 'theta2_dot'};
inputs = {'u'};
outputs = {'x'; 'theta1'; 'theta2'};
```

```
sys_cl = ss(Ac,Bc,Cc,Dc,'statename',states,'inputname',inputs,'outputname',outputs);
figure(1)
step(sys_cl,200);
eig_Ac = eig(Ac);
disp('Eigenvalues of A-BK:')
display(eig_Ac)

%% Linear Model response to LQR
[t,l_out] = ode45(@(t,x)linear(t,x,-K*x),timespan,X_0);
figure(2);
hold on
plot(t,l_out(:,1))
plot(t,l_out(:,3))
plot(t,l_out(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('LQR controller applied to linear system')
legclear all, close all, clc

% Variable definitions
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.81;
timespan = 0:0.05:100;
% Initial conditions
x_0 = 1;
theta1_0 = deg2rad(10);
theta2_0 = deg2rad(20);
X_0 = [1 0 theta1_0 0 theta2_0 0];

%% Linearized model
A = [0 1 0 0 0 0; 0 0 -m1*g/M 0 -m2*g/M 0; 0 0 0 1 0 0; 0 0 -((M*g)+(m1*g))/(M*l1) 0
-g*m2/(M*l1) 0; 0 0 0 0 0 1; 0 0 -m1*g/(M*l2) 0 -((M*g)+(m2*g))/(M*l2) 0];
```

```
B = [0; 1/M; 0; 1/(l1*M); 0; 1/(l2*M)];

C = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];

D = [0;0;0];

Co = ctrb(A,B);

rank = rank(Co);

fprintf('Rank of controllability matrix: %d\n',rank)


%% LQR Controller

Q = [1 0 0 0 0 0; 0 10 0 0 0 0; 0 0 2000 0 0 0; 0 0 0 2000 0 0; 0 0 0 0 2000 0; 0 0 0 0 0
2000];

R = 0.001;

K = lqr(A,B,Q,R);

Ac = [(A-B*K)];

Bc = [B];

Cc = [C];

Dc = [D];


states = {'x' 'x_dot' 'theta1' 'theta1_dot' 'theta2' 'theta2_dot'};

inputs = {'u'};

outputs = {'x'; 'theta1'; 'theta2'};


sys_cl = ss(Ac,Bc,Cc,Dc,'statename',states,'inputname',inputs,'outputname',outputs);

figure(1)

step(sys_cl,200);

eig_Ac = eig(Ac);

disp('Eigenvalues of A-BK:')

display(eig_Ac)


%% Linear Model response to LQR

[t,l_out] = ode45(@(t,x)linear(t,x,-K*x),timespan,X_0);

figure(2);

hold on

plot(t,l_out(:,1))

plot(t,l_out(:,3))

plot(t,l_out(:,5))

ylabel('state variables')

xlabel('time (sec)')
```

```
title('LQR controller applied to linear system')
legend('x','theta1','theta2')

%% Non Linear Model response to LQR
[t,nl_out] = ode45(@(t,x)nonlinear(t,x,-K*x),timespan,X_0);
figure(3);
hold on
plot(t,nl_out(:,1))
plot(t,nl_out(:,3))
plot(t,nl_out(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('LQR controller applied to non-linear system')
legend('x','theta1','theta2')
end('x','theta1','theta2')

%% Non Linear Model response to LQR
[t,nl_out] = ode45(@(t,x)nonlinear(t,x,-K*x),timespan,X_0);
figure(3);
hold on
plot(t,nl_out(:,1))
plot(t,nl_out(:,3))
plot(t,nl_out(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('LQR controller applied to non-linear system')
legend('x','theta1','theta2')
```

Part E

```
clear all, close all, clc

% Variable definitions
m1 = 100;
m2 = 100;
M = 1000;
```

```
l1 = 20;
l2 = 10;
g = 9.81;

%% Observability Check
A = [0 1 0 0 0 0; 0 0 -m1*g/M 0 -m2*g/M 0; 0 0 0 1 0 0; 0 0 -((M*g)+(m1*g))/(M*l1) 0
-g*m2/(M*l1) 0; 0 0 0 0 0 1; 0 0 -m1*g/(M*l2) 0 -((M*g)+(m2*g))/(M*l2) 0];
B = [0; 1/M; 0; 1/(l1*M); 0; 1/(l2*M)];
D = [0;0;0];
% Case 1: Output vector: x
C1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
Obs1 = [C1' (C1*A)' (C1*A^2)' (C1*A^3)' (C1*A^4)' (C1*A^5)'];
rank_Obs1 = rank(Obs1);
fprintf('Rank of controllability matrix with output vector as x: %d\n',rank_Obs1)

% Case 2 Output vectors: theta1, theta2
C2 = [0 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];
Obs2 = [C2' (C2*A)' (C2*A^2)' (C2*A^3)' (C2*A^4)' (C2*A^5)'];
rank_Obs2 = rank(Obs2);
fprintf('Rank of controllability matrix with output vectors as theta1 and theta2: %d\
n',rank_Obs2)

% Case 3 Output vectors: x, theta2
C3 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 1 0];
Obs3 = [C3' (C3*A)' (C3*A^2)' (C3*A^3)' (C3*A^4)' (C3*A^5)'];
rank_Obs3 = rank(Obs3);
fprintf('Rank of controllability matrix with output vector as x and theta2: %d\n',rank_Obs3)

% Case 2 Output vectors: x, theta1, theta2
C4 = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];
Obs4 = [C4' (C4*A)' (C4*A^2)' (C4*A^3)' (C4*A^4)' (C4*A^5)'];
rank_Obs4 = rank(Obs4);
fprintf('Rank of controllability matrix with output vector as x, theta1 and theta2: %d\
n',rank_Obs4)
```

Part F

```
clear all, close all, clc
```

```
% Variable definitions
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.81;
dt = 0.1;
tspan = dt:dt:100;
% Initial conditions
x_0 = 1;
theta1_0 = deg2rad(10);
theta2_0 = deg2rad(20);
X_0 = [1 0 theta1_0 0 theta2_0 0];

%% Observability Check
A = [0 1 0 0 0 0; 0 0 -m1*g/M 0 -m2*g/M 0; 0 0 0 1 0 0; 0 0 -((M*g)+(m1*g))/(M*l1) 0
-g*m2/(M*l1) 0; 0 0 0 0 0 1; 0 0 -m1*g/(M*l2) 0 -((M*g)+(m2*g))/(M*l2) 0];
B = [0; 1/M; 0; 1/(l1*M); 0; 1/(l2*M)];

% Case 1: Output vector: x
C1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
Obs1 = obsv(A,C1);
rank_Obs1 = rank(Obs1);
fprintf('Rank of controllability matrix with output vector as x: %d\n',rank_Obs1)

% Case 2 Output vectors: theta1, theta2
C2 = [0 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];
Obs2 = obsv(A,C2);
rank_Obs2 = rank(Obs2);
fprintf('Rank of controllability matrix with output vectors as theta1 and theta2: %d\
n',rank_Obs2)

% Case 3 Output vectors: x, theta2
C3 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 1 0];
Obs3 = obsv(A,C3);
```

```
rank_Obs3 = rank(Obs3);

fprintf('Rank of controllability matrix with output vector as x and theta2: %d\n',rank_Obs3)


% Case 2 Output vectors: x, theta1, theta2

C4 = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];

Obs4 = obsv(A,C4);

rank_Obs4 = rank(Obs4);

fprintf('Rank of controllability matrix with output vector as x, theta1 and theta2: %d\
n',rank_Obs4)


D = zeros(size(C1,1),size(B,2));

sys1 = ss(A,B,C1,D);

sys3 = ss(A,B,C3,D);

sys4 = ss(A,B,C4,D);


%% Luenberger Observer

Bd = 0.1*eye(6);

Vn = 0.1;


% [l1,P,E] = lqe(A,Bd,C1,Bd,Vn*eye(3));

l1 = (lqr(A',C1',Bd,Vn))';

sysL1 = ss(A-(l1*C1),[B l1],C1,0);

l3 = (lqr(A',C3',Bd,Vn))';

sysL3 = ss(A-(l3*C3),[B l3],C3,0);

l4 = (lqr(A',C4',Bd,Vn))';

sysL4 = ss(A-(l4*C4),[B l4],C4,0);


u = 0*tspan;

u(100:length(tspan)) = 1;


[y1,t] = lsim(sys1,u,tspan);

[x1,t] = lsim(sysL1,[u;y1'],tspan);


[y3,t] = lsim(sys3,u,tspan);

[x3,t] = lsim(sysL3,[u;y3'],tspan);


[y4,t] = lsim(sys4,u,tspan);
```

```
[x4,t] = lsim(sysL4,[u;y4'],tspan);

figure(1)
hold on
plot(t,y1(:,1),'g',LineWidth=2);
plot(t,x1(:,1),'k--','LineWidth',1.0)
ylabel('State Variables')
xlabel('Time(sec)')
legend('x(t)','Estimated x(t)')
title('Response for output vector at step input: x(t)')
hold off

figure(2)
hold on
plot(t,y3(:,1),'g',LineWidth=2);
plot(t,x3(:,1),'k--','LineWidth',1.0)
plot(t,y3(:,3),'r',LineWidth=2);
plot(t,x3(:,3),'k--','LineWidth',1.0)
ylabel('State Variables')
xlabel('Time(sec)')
legend('x(t)','theta2(t)','Estimated x(t)','Estimated theta2(t)')
title('Response for output vector at step input: x(t),theta2(t)')
hold off

figure(3)
hold on
plot(t,y3(:,1),'g',LineWidth=2);
plot(t,x3(:,1),'k--','LineWidth',1.0)
plot(t,y3(:,2),'c',LineWidth=2);
plot(t,x3(:,2),'k--','LineWidth',1.0)
plot(t,y3(:,3),'r',LineWidth=2);
plot(t,x3(:,3),'k--','LineWidth',1.0)
ylabel('State Variables')
xlabel('Time(sec)')
legend('x(t)','theta1(t)','theta2(t)','Estimated x(t)','Estimated theta1(t)','Estimated
theta2(t)')
title('Response for output vector at step input: x(t),theta1(t),theta2(t)')
```

```
hold off

%% Linear Model response to Observer

[t,l_out1] = ode45(@(t,x)linearObserver(t,x,C1,l1),tspan,X_0);
figure(4)
hold on
plot(t,l_out1(:,1))
ylabel('state variables')
xlabel('time (sec)')
title('Linear system Observer for output vector: x(t)')
legend('x')
hold off

[t,l_out3] = ode45(@(t,x)linearObserver(t,x,C3,l3),tspan,X_0);
figure(5)
hold on
plot(t,l_out3(:,1))
plot(t,l_out3(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Linear system Observer for output vectors: x(t),theta2(t)')
legend('x','theta2')
hold off

[t,l_out4] = ode45(@(t,x)linearObserver(t,x,C4,l4),tspan,X_0);
figure(6)
hold on
plot(t,l_out4(:,1))
plot(t,l_out4(:,3))
plot(t,l_out4(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Linear system Observer for output vectors: x(t),theta1(t),theta2(t)')
legend('x','theta1','theta2')
hold off
```

```
%% Non-Linear Model response to Observer

[t,nl_out1] = ode45(@(t,x)nonlinearObserver(t,x,C1,1,l1),tspan,X_0);
figure(7)
hold on
plot(t,nl_out1(:,1))
ylabel('state variables')
xlabel('time (sec)')
title('Non Linear system Observer for output vector: x(t)')
legend('x')
hold off

[t,nl_out3] = ode45(@(t,x)nonlinearObserver(t,x,C3,1,l3),tspan,X_0);
figure(8)
hold on
plot(t,nl_out3(:,1))
plot(t,nl_out3(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear system Observer for output vectors: x(t),theta2(t)')
legend('x','theta2')
hold off

[t,nl_out4] = ode45(@(t,x)nonlinearObserver(t,x,C4,1,l4),tspan,X_0);
figure(9)
hold on
plot(t,nl_out4(:,1))
plot(t,nl_out4(:,3))
plot(t,nl_out4(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear system Observer for output vectors: x(t),theta1(t),theta2(t)')
legend('x','theta1','theta2')
hold off
```

Part G

```
clear all, close all, clc

% Variable definitions

m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.81;
dt = 0.1;
t = dt:dt:100;

% Initial conditions
x_0 = 1;
theta1_0 = deg2rad(10);
theta2_0 = deg2rad(20);
X_0 = [1 0 theta1_0 0 theta2_0 0];


%% Observability Check
A = [0 1 0 0 0 0; 0 0 -m1*g/M 0 -m2*g/M 0; 0 0 0 1 0 0; 0 0 -((M*g)+(m1*g))/(M*l1) 0
-g*m2/(M*l1) 0; 0 0 0 0 0 1; 0 0 -m1*g/(M*l2) 0 -((M*g)+(m2*g))/(M*l2) 0];
B = [0; 1/M; 0; 1/(l1*M); 0; 1/(l2*M)];

% Case 1: Output vector: x
C1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
Obs1 = obsv(A,C1);
rank_Obs1 = rank(Obs1);
fprintf('Rank of controllability matrix with output vector as x: %d\n',rank_Obs1)
D = zeros(size(C1,1),size(B,2));
sys1 = ss(A,B,C1,D);

%% LQR Controller
Q = [1 0 0 0 0 0; 0 10 0 0 0 0; 0 0 2000 0 0 0; 0 0 0 2000 0 0; 0 0 0 0 2000 0; 0 0 0 0 0
2000];
```

```
R = 0.001;
K = lqr(A,B,Q,R);
Ac = [(A-B*K)];
Bc = [B];
Cc = [C1];
Dc = [D];

sys_cl = ss(Ac,Bc,Cc,Dc);

%% Luenberger Observer

Bd = 0.1*eye(6);
Vn = 0.1;
l1 = (lqr(A',C1',Bd,Vn))';
sysL1 = ss(A-(l1*C1),[B l1],C1,0);

%% LQG applied to Non-linear system

[t,out] = ode45(@(t,x)nonlinearObserver(t,x,C1,-K*x,l1),t,X_0);
figure()
hold on
plot(t,out(:,1))
ylabel('State variable')
xlabel('time (sec)')
title('LQG response to Non-linear system for output vector: x(t)')
legend('x')
hold off
```

Functions

Linear State Space System

```
function dX = linear(t,X,U)
m1 = 100;
m2 = 100;
M = 1000;
L1 = 20;
```

```
L2 = 10;

g = 9.81;

A = [0 1 0 0 0 0; 0 0 -m1*g/M 0 -m2*g/M 0; 0 0 0 1 0 0; 0 0 -((M*g)+(m1*g))/(M*L1) 0
-g*m2/(M*L1) 0; 0 0 0 0 0 1; 0 0 -m1*g/(M*L2) 0 -((M*g)+(m2*g))/(M*L2) 0];

B = [0; 1/M; 0; 1/(L1*M); 0; 1/(L2*M)];

dX = A*X + B*U;

end
```

Non-Linear State Space System

```
function dX = nonlinear(t,X,F)

m1 = 100;

m2 = 100;

M = 1000;

l1 = 20;

l2 = 10;

g = 9.81;

x = X(1);

dx = X(2);

th1 = X(3);

dth1 = X(4);

th2 = X(5);

dth2 = X(6);

dX = zeros(6,1);

dX(1) = dx;

dX(2) = (F-m1*(g*sin(th1)*cos(th1)+l1*sin(th1)*(dth1)^2)-
m2*(g*sin(th2)*cos(th2)+l2*sin(th2)*(dth2)^2))/(M+m1*(sin(th1))^2+m2*(sin(th2))^2);

dX(3) = dth1;

dX(4) = ((cos(th1))/l1)*(F-m1*(g*sin(th1)*cos(th1)+l1*sin(th1)*(dth1)^2)-
m2*sin(th2)*cos(th2)+l2*sin(th2)*(dth2)^2)/(M+m1*(sin(th1))^2+m2*(sin(th2))^2)-(g*sin(th1))/
l1;

dX(5) = dth2;

dX(6) = ((cos(th2))/l2)*(F-m1*(g*sin(th1)*cos(th1)+l1*sin(th1)*(dth1)^2)-
m2*sin(th2)*cos(th2)+l2*sin(th2)*(dth2)^2)/(M+m1*(sin(th1))^2+m2*(sin(th2))^2)-(g*sin(th2))/
l2;

end
```

Linear Observer

```
function dX = linearObserver(t,X,C,L)

m1 = 100;

m2 = 100;

M = 1000;

L1 = 20;

L2 = 10;

g = 9.81;

A = [0 1 0 0 0 0; 0 0 -m1*g/M 0 -m2*g/M 0; 0 0 0 1 0 0; 0 0 -((M*g)+(m1*g))/(M*L1) 0
-g*m2/(M*L1) 0; 0 0 0 0 0 1; 0 0 -m1*g/(M*L2) 0 -((M*g)+(m2*g))/(M*L2) 0];

B = [0; 1/M; 0; 1/(L1*M); 0; 1/(L2*M)];

K = 1;

Y = C*X;

dX = (A+B*K)*X + L*(Y-C*X);

end
```

Non-Linear Observer

```
function dX = nonlinearObserver(t,X,C,F,L)

m1 = 100;

m2 = 100;

M = 1000;

l1 = 20;

l2 = 10;

g = 9.81;

x = X(1);

dx = X(2);

th1 = X(3);

dth1 = X(4);

th2 = X(5);

dth2 = X(6);

dX = zeros(6,1);

Y = C*[x;0;th1;0;th2;0];

corr = L*(Y-C*X);


dX(1) = dx + corr(1);

dX(2) = (F-m1*(g*sin(th1)*cos(th1)+l1*sin(th1)*(dth1)^2)-
m2*(g*sin(th2)*cos(th2)+l2*sin(th2)*(dth2)^2))/(M+m1*(sin(th1))^2+m2*(sin(th2))^2) +
corr(2);
```

```
dX(3) = dth1 + corr(3);

dX(4) = ((cos(th1))/l1)*(F-m1*(g*sin(th1)*cos(th1)+l1*sin(th1)*(dth1)^2)-
m2*sin(th2)*cos(th2)+l2*sin(th2)*(dth2)^2)/(M+m1*(sin(th1))^2+m2*(sin(th2))^2)-(g*sin(th1))/
l1 + corr(4);

dX(5) = dth2 + corr(5);

dX(6) = ((cos(th2))/l2)*(F-m1*(g*sin(th1)*cos(th1)+l1*sin(th1)*(dth1)^2)-
m2*sin(th2)*cos(th2)+l2*sin(th2)*(dth2)^2)/(M+m1*(sin(th1))^2+m2*(sin(th2))^2)-(g*sin(th2))/
l2 + corr(6);

end
```