Assignment 13

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1 Problem hoffman pg208/1A

Find an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{P}^{-1}\mathbf{BP}$ are both diagonal where $\mathbf{A}$ and $\mathbf{B}$ are real matrices.

1) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix}$

2 Theorem's

Theorem	According to theorem , if a 2×2 matrix has two characteristics values then the $\mathbf{P}$ that diagonalize $\mathbf{A}$ will necessarily also diagonalize any $\mathbf{B}$ that commutes with $\mathbf{A}$.
Common Basis	Let there exist a $\mathbf{P}$ in basis $\beta = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ of $\mathbb{V}$ consisting of eigen vector which are common to both $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}\mathbf{b}_i = \lambda_i \mathbf{b}_i$ $\mathbf{B}\mathbf{b}_i = \mu_i \mathbf{b}_i$ $\Lambda_A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $\Lambda_B = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ $\Lambda_A = \mathbf{P}^{-1}\mathbf{AP}$ $\Lambda_B = \mathbf{P}^{-1}\mathbf{BP}$

TABLE 1: Theorems

3 Solution

Operations	Matrix A	Matrix B
Characteristic Polynomial	$p(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert$ $= \begin{vmatrix} x-1 & -2 \\ 0 & x-2 \end{vmatrix}$ $= (x-1)(x-2)$	$p(x) = \left\lvert x\mathbf{I} - \mathbf{B} \right\rvert$ $= \begin{vmatrix} x-3 & 8 \\ 0 & x+1 \end{vmatrix}$ $= (x-3)(x+1)$
Characteristic values	$p(x) = 0$ $(x-1)(x-2) = 0$ $\lambda_1 = 1$,$\lambda_2 = 2$	$p(x) = 0$ $(x-3)(x+1) = 0$ $\mu_1 = 3$, $\mu_2 = -1$
Basis for Characteristics Values	For $\lambda_1 = 1$ $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{b_1} = 0$ $\left(\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{b_1} = 0$ $\left(\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}\right)\mathbf{b_1} = 0$ $\mathbf{b_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ For $\lambda_2 = 2$ $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{b_2} = 0$ $\left(\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{b_2} = 0$ $\left(\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}\right)\mathbf{b_2} = 0$ $\mathbf{b_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	For $\mu_1 = 3$ $(\mathbf{B} - \mu_1\mathbf{I})\mathbf{b_1} = 0$ $\left(\begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{b_1} = 0$ $\left(\begin{pmatrix} 0 & -8 \\ 0 & -4 \end{pmatrix}\right)\mathbf{b_1} = 0$ $\mathbf{b_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ For $\mu_2 = -1$ $(\mathbf{B} - \mu_2\mathbf{I})\mathbf{b_2} = 0$ $\left(\begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{b_2} = 0$ $\left(\begin{pmatrix} 4 & -8 \\ 0 & 0 \end{pmatrix}\right)\mathbf{b_2} = 0$ $\mathbf{b_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Verification	$\Lambda_A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\Lambda_A = \mathbf{P^{-1}AP}$ $\implies \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \Lambda_A$	$\Lambda_B = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ $\Lambda_B = \mathbf{P^{-1}BP}$ $\implies \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \Lambda_B$
Answer.The Invertible Matrix **P**	$\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

TABLE 2: Solution Table