Assignment 9

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1 Question Hoffman PG26 Q1

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{pmatrix} \tag{1.0.1}$$

Find a row-reduced echelon matrix **R** which is row-equivalent to **A** and an invertible 3x3 matrix **P** such that $\mathbf{R} = \mathbf{P}\ \mathbf{A}$.

2 Solution

Given

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{pmatrix} \tag{2.0.1}$$

Row reduce **A** by applying the elementary row operations and equivalently at each operations find the elementary matrix **E**

$$\mathbf{A}|\mathbf{I} = \left(\begin{array}{cccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 3 & 5 & 0 & 1 & 0 \\ 1 & -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.0.2}$$

$$\xleftrightarrow{R_2=R_2+R_1} \left(\begin{array}{cccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 1 & -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\implies \mathbf{e_1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.0.3}$$

$$\xleftrightarrow{R_3=R_3-R_1} \left(\begin{array}{cccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 0 & -4 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\implies \mathbf{e_2} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{2.0.4}$$

$$\xleftrightarrow{R_1=R_1-R_2} \left(\begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 0 & -4 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\implies \mathbf{e_3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{2.0.5}$$

$$\xleftrightarrow{R_3=R_3+2R_2} \left(\begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 0 & 0 & 8 & 11 & 1 & 2 & 1 \end{array}\right)$$

$$\implies \mathbf{e_4} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \tag{2.0.6}$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}} \left(\begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 1 & 2 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 8 & 11 & 1 & 2 & 1 \end{array}\right)$$

$$\implies \mathbf{e_5} = \begin{pmatrix} 0 & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 2 & 1 \end{pmatrix} \tag{2.0.7}$$

$$\xleftrightarrow{R_3=\frac{R_3}{8}} \left(\begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 1 & 2 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{11}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{array}\right)$$

$$\implies \mathbf{e_6} = \begin{pmatrix} 0 & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{pmatrix} \tag{2.0.8}$$

$$\xleftrightarrow{R_1=R_1+3R_3} \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & -\frac{7}{8} & \frac{3}{8} & -\frac{1}{4} & \frac{3}{8} \\ 0 & 1 & 2 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{11}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{array}\right)$$

$$\implies \mathbf{e_7} = \begin{pmatrix} \frac{3}{8} & -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{pmatrix} \tag{2.0.9}$$

$$\xleftrightarrow{R_2=R_2-2R_3}\left(\begin{array}{cccc|ccc}1 & 0 & 0 & -\frac{7}{8} & \frac{3}{8} & -\frac{1}{4} & \frac{3}{8}\\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{4}\\ 0 & 0 & 1 & \frac{11}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8}\end{array}\right)$$

$$\implies \mathbf{e_8} = \begin{pmatrix}\frac{3}{8} & -\frac{1}{4} & \frac{3}{8}\\ \frac{1}{4} & 0 & -\frac{1}{4}\\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8}\end{pmatrix} \quad (2.0.10)$$

Hence,row reduced echelon matrix that is row equivalent to $\mathbf{A}$ is

$$\mathbf{R} = \begin{pmatrix}1 & 0 & 0 & -\frac{7}{8}\\ 0 & 1 & 0 & -\frac{1}{4}\\ 0 & 0 & 1 & \frac{11}{8}\end{pmatrix} \quad (2.0.11)$$

where,

$$\mathbf{E} = \mathbf{e_1 e_2 e_3 e_4 e_5 e_6 e_7 e_8} \quad (2.0.12)$$

are the elementary matrices that transform $\mathbf{A}$ to $\mathbf{R}$

$$\mathbf{e_1 e_2 e_3 e_4 e_5 e_6 e_7 e_8 A} = \mathbf{R} \implies \mathbf{P} = \mathbf{e_1 e_2 e_3 e_4 e_5 e_6 e_7 e_8} \quad (2.0.13)$$

Since elementary matrices are invertible and the product of invetible matrices is invertible ,thus

$$\mathbf{P} = \mathbf{e_1 e_2 e_3 e_4 e_5 e_6 e_7 e_8} \quad (2.0.14)$$

is invertible.
From (2.0.10)

$$\mathbf{P} = \begin{pmatrix}\frac{3}{8} & -\frac{1}{4} & \frac{3}{8}\\ \frac{1}{4} & 0 & -\frac{1}{4}\\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8}\end{pmatrix} \quad (2.0.15)$$

$$\mathbf{R} = \begin{pmatrix}1 & 0 & 0 & -\frac{7}{8}\\ 0 & 1 & 0 & -\frac{1}{4}\\ 0 & 0 & 1 & \frac{11}{8}\end{pmatrix} \quad (2.0.16)$$

such that $\mathbf{R} = \mathbf{PA}$.