Assignment 4

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1 Problem(loney pg 98 Q7)

Find the value of k so that following equation may represent pairs of straight lines,

$$12x^2 - 10xy + 2y^2 + 11x - 5y + k = 0 \tag{1.0.1}$$

2 Solution

The general equation of second degree is given by,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

In vector from the equation (2.0.1) can be expressed as,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{2.0.4}$$

Now, comparing (2.0.1) to (1.0.1) we get, a =12, b=-5, c = 2, d = $\frac{11}{2}$,e = -$\frac{5}{2}$, f = k. Hence, substituting these values in (2.0.3) and (2.0.4) we get,

$$\mathbf{V} = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \tag{2.0.5}$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{2.0.6}$$

(1.0.1) represents pair of straight lines if,

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{2.0.7}$$

$$\begin{vmatrix} 12 & -5 & \frac{11}{2} \\ -5 & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & k \end{vmatrix} = 0 \tag{2.0.8}$$

$$\implies k = 2 \tag{2.0.9}$$

Substituting (2.0.9) in (1.0.1) we get the following pairs of straight lines,

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \tag{2.0.10}$$

Let the pair of straight lines is given by,

$$\mathbf{n_1}^T\mathbf{x} = c1 \tag{2.0.11}$$

$$\mathbf{n_2}^T\mathbf{x} = c2 \tag{2.0.12}$$

Now equating the product of (2.0.11) and (2.0.12) with (2.0.2) we get,

$$(\mathbf{n_1}^T\mathbf{x} - c1)(\mathbf{n_2}^T\mathbf{x} - c2) = \tag{2.0.13}$$

$$\mathbf{x}^T\begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} \frac{11}{2} & -\frac{5}{2} \end{pmatrix}\mathbf{x} + 2 \tag{2.0.14}$$

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \tag{2.0.15}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\mathbf{u} = \begin{pmatrix} -11 \\ 5 \end{pmatrix} \tag{2.0.16}$$

$$c_1c_2 = 2. \tag{2.0.17}$$

Slopes of line is given by roots of polynomial,

$$cm^2 + 2bm + a = 0 \implies 2m^2 - 10m + 12 = 0 \tag{2.0.18}$$

$$m_i = \frac{-b \pm \sqrt{-|V|}}{c} \tag{2.0.19}$$

$$|V| = 1 \tag{2.0.20}$$

$$m_1 = 3 \tag{2.0.21}$$

$$m_2 = 2 \tag{2.0.22}$$

The normal vector to the two lines is given by,

$$\mathbf{n_i} = k_i\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{2.0.23}$$

$$\mathbf{n_1} = k_1\begin{pmatrix}-3\\1\end{pmatrix} \tag{2.0.24}$$

$$\mathbf{n_2} = k_2\begin{pmatrix}-2\\1\end{pmatrix} \tag{2.0.25}$$

Also,

$$k_1k_2 = 2 \tag{2.0.26}$$

Let $k_1 = 2$ and $k_2 = 1$

$$\mathbf{n_1} = \begin{pmatrix}-6\\2\end{pmatrix} \tag{2.0.27}$$

$$\mathbf{n_2} = \begin{pmatrix}-2\\1\end{pmatrix} \tag{2.0.28}$$

We verify obtained $\mathbf{n_1}$ and $\mathbf{n_2}$ using Toeplitz matrix,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix}-6 & 0\\2 & -6\\0 & 2\end{pmatrix}\begin{pmatrix}-2\\1\end{pmatrix} \tag{2.0.29}$$

$$= -2\begin{pmatrix}-6\\2\\0\end{pmatrix} + \begin{pmatrix}0\\-6\\2\end{pmatrix} \tag{2.0.30}$$

$$= \begin{pmatrix}12\\-10\\2\end{pmatrix} \tag{2.0.31}$$

Hence (2.0.15) and (2.0.31) are same. Hence verified. Now substituting it in (2.0.16) we get,

$$c_2\begin{pmatrix}-6\\2\end{pmatrix} + c_1\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}-11\\5\end{pmatrix} \tag{2.0.32}$$

Solve using Row reduction Technique we get,

$$\implies \begin{pmatrix}-2 & -6 & -11\\1 & 2 & 5\end{pmatrix} \tag{2.0.33}$$

$$\xleftrightarrow{R_1 \leftarrow -\frac{R_1}{2}} \begin{pmatrix}1 & 3 & \frac{11}{2}\\1 & 2 & 5\end{pmatrix} \tag{2.0.34}$$

$$\xleftrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix}1 & 3 & \frac{11}{2}\\0 & 1 & \frac{1}{2}\end{pmatrix} \tag{2.0.35}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix}1 & 0 & 4\\0 & 1 & \frac{1}{2}\end{pmatrix} \tag{2.0.36}$$

$$\implies c_1 = 4 \tag{2.0.37}$$

$$c_2 = \frac{1}{2} \tag{2.0.38}$$

substituting the values of c_1, c_2 and (2.0.27) and (2.0.28) to (2.0.11) and (2.0.12) we get equation of two straight lines.

$$\implies \begin{pmatrix}-6 & 2\end{pmatrix}\mathbf{x} = 4 \tag{2.0.39}$$

$$\begin{pmatrix}-2 & 1\end{pmatrix}\mathbf{x} = \frac{1}{2} \tag{2.0.40}$$

Hence the equation of pair of straight lines are,

$$\left(\begin{pmatrix}-6 & 2\end{pmatrix}\mathbf{x} - 4\right)\left(\begin{pmatrix}-2 & 1\end{pmatrix}\mathbf{x} - \frac{1}{2}\right) = 0 \tag{2.0.41}$$

Hence, Plot of the equation (2.0.41) is shown below

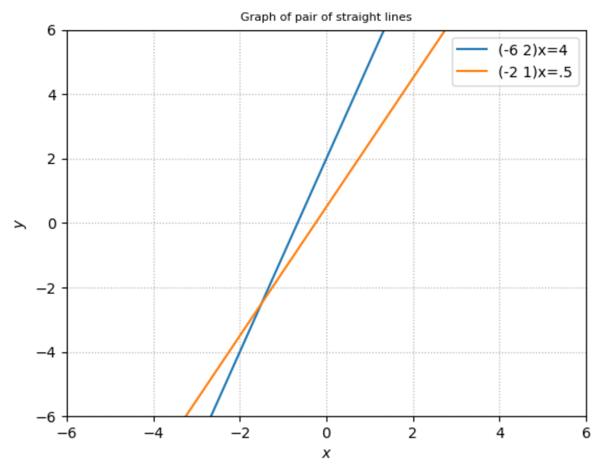

Fig. 0: Pair of lines