Assignment 3

Pulkit Saxena

I. Question 1.36 Geolin.pdf

BE and CF are two equal altitudes of a triangle ABC. Using RHS congruence rule, prove that the triangle ABC is isosceles.

II. Solution

BE and CF are two equal altitudes of a triangle ABC.

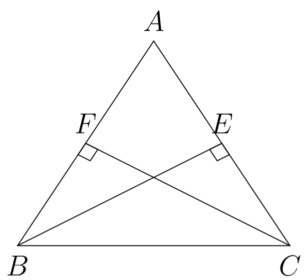

Fig. 1: Triangle with equal altitudes on two sides

Given:-
1) Altitudes are Equal means their magnitude are same

$$\|\mathbf{E}-\mathbf{B}\| = \|\mathbf{F}-\mathbf{C}\| \tag{1}$$

2) Altitude makes right angle at the base therefore $\cos 90 = 0$ therefore FC $\perp$ BF and EB $\perp$ CE where $\mathbf{m}$ is the directional vectors.

$$\mathbf{m}_{FC}\mathbf{m}_{BF} = 0 \tag{2}$$
$$\mathbf{m}_{EB}\mathbf{m}_{CE} = 0 \tag{3}$$

From (2)

$$\left(\mathbf{B}-\mathbf{F}\right)\left(\mathbf{F}-\mathbf{C}\right)^T = \mathbf{0} \quad \left(\mathbf{F}-\mathbf{C}\right)\left(\mathbf{B}-\mathbf{F}\right)^T = \mathbf{0} \tag{4}$$

From (2) and using (4)

$$\left(\mathbf{B}-\mathbf{C}\right)\left(\mathbf{B}-\mathbf{C}\right)^T \tag{5}$$
$$= \left(\mathbf{B}-\mathbf{F}+\mathbf{F}-\mathbf{C}\right)\left(\mathbf{B}-\mathbf{F}+\mathbf{F}-\mathbf{C}\right)^T) \tag{6}$$

$$= \left(\mathbf{B}-\mathbf{F}\right)\left(\mathbf{B}-\mathbf{F}\right)^T + \left(\mathbf{F}-\mathbf{C}\right)\left(\mathbf{F}-\mathbf{C}\right)^T \tag{7}$$

$$\|\mathbf{B}-\mathbf{C}\|^2 = \|\mathbf{B}-\mathbf{F}\|^2 + \|\mathbf{F}-\mathbf{C}\|^2 \tag{8}$$

Similarly
From (3)

$$\left(\mathbf{E}-\mathbf{B}\right)\left(\mathbf{E}-\mathbf{C}\right)^T = 0 \quad \left(\mathbf{E}-\mathbf{C}\right)\left(\mathbf{B}-\mathbf{E}\right)^T = \mathbf{0} \tag{9}$$

From (3) and using (9)

$$\left(\mathbf{B}-\mathbf{C}\right)\left(\mathbf{B}-\mathbf{C}\right)^T \tag{10}$$
$$= \left(\mathbf{B}-\mathbf{E}+\mathbf{E}-\mathbf{C}\right)\left(\mathbf{B}-\mathbf{E}+\mathbf{E}-\mathbf{C}\right)^T \tag{11}$$

$$= \left(\mathbf{B}-\mathbf{E}\right)\left(\mathbf{B}-\mathbf{E}\right)^T + \left(\mathbf{E}-\mathbf{C}\right)\left(\mathbf{E}-\mathbf{C}\right)^T \tag{12}$$

$$\|\mathbf{B}-\mathbf{C}\|^2 = \|\mathbf{B}-\mathbf{E}\|^2 + \|\mathbf{E}-\mathbf{C}\|^2 \tag{13}$$

Equating (8) and (13) and using (1)

$$\|\mathbf{B}-\mathbf{F}\|^2 + \|\mathbf{F}-\mathbf{C}\|^2 = \|\mathbf{B}-\mathbf{E}\|^2 + \|\mathbf{E}-\mathbf{C}\|^2 \tag{14}$$

$$\|\mathbf{B}-\mathbf{F}\|^2 = \|\mathbf{E}-\mathbf{C}\|^2 \tag{15}$$
$$= \|\mathbf{B}-\mathbf{F}\| = \|\mathbf{E}-\mathbf{C}\| \tag{16}$$

Let $\angle FBC = \theta_1$ and $\angle EBC = \theta_2$

$$\left(\mathbf{B}-\mathbf{F}\right)\left(\mathbf{B}-\mathbf{C}\right)^T = \|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|\cos\theta_1 \tag{17}$$

$$\cos\theta_1 = \frac{\left(\mathbf{B}-\mathbf{F}\right)\left(\mathbf{B}-\mathbf{C}\right)^T}{\|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{18}$$

$$\cos\theta_1 = \frac{\left(\mathbf{B}-\mathbf{F}\right)\left(\mathbf{B}-\mathbf{F}+\mathbf{F}-\mathbf{C}\right)^T}{\|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{19}$$

$$\cos\theta_1 = \frac{\left(\mathbf{B}-\mathbf{F}\right)\left(\mathbf{B}-\mathbf{F}\right)^T + \left(\mathbf{B}-\mathbf{F}\right)\left(\mathbf{F}-\mathbf{C}\right)^T}{\|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{20}$$

From (4)

$$\cos\theta_1 = \frac{(\mathbf{B}-\mathbf{F})(\mathbf{B}-\mathbf{F})^T}{\|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{21}$$

$$\cos\theta_1 = \frac{\|\mathbf{B}-\mathbf{F}\|^2}{\|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{22}$$

$$\cos\theta_1 = \frac{\|\mathbf{B}-\mathbf{F}\|}{\|\mathbf{B}-\mathbf{C}\|} \tag{23}$$

Similarly for $\angle EBC = \theta_2$

$$(\mathbf{C}-\mathbf{E})(\mathbf{B}-\mathbf{C})^T = \|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|\cos\theta_2 \tag{24}$$

$$\cos\theta_2 = \frac{(\mathbf{C}-\mathbf{E})(\mathbf{B}-\mathbf{C})^T}{\|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{25}$$

$$\cos\theta_2 = \frac{(\mathbf{C}-\mathbf{E})(\mathbf{B}-\mathbf{E}+\mathbf{E}-\mathbf{C})^T}{\|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{26}$$

$$\cos\theta_2 = \frac{(\mathbf{C}-\mathbf{E})(\mathbf{B}-\mathbf{E})^T + (\mathbf{C}-\mathbf{E})(\mathbf{E}-\mathbf{C})^T}{\|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{27}$$

From (9)

$$\cos\theta_2 = \frac{(\mathbf{C}-\mathbf{E})(\mathbf{C}-\mathbf{E})^T}{\|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{28}$$

$$\cos\theta_2 = \frac{\|\mathbf{C}-\mathbf{E}\|^2}{\|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{29}$$

$$\cos\theta_2 = \frac{\|\mathbf{C}-\mathbf{E}\|}{\|\mathbf{B}-\mathbf{C}\|} \tag{30}$$

From (16) we know $\|\mathbf{B}-\mathbf{F}\| = \|\mathbf{E}-\mathbf{C}\|$ we conclude

$$\cos\theta_1 = \cos\theta_2 \implies \theta_1 = \theta_2 \tag{31}$$

So the sides opposite to equal angles are equal. Hence AB=AC hence the given Triangle is isosceles.