Assignment 17

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1 Problem UGCDEC2015 Q76

Let **A** be an m x n real matrix and $\mathbf{b} \in \mathbb{R}^m$ with $b \neq 0$.

1) The set of all real solutions of $\mathbf{A}x = \mathbf{b}$ is a vector space.
2) If u nd v are two solutions of $\mathbf{A}x = \mathbf{b}$ then $\lambda u + (1-\lambda)v$ is also a solution of $\mathbf{A}x = \mathbf{b}$
3) For any two solutions u and v of $\mathbf{A}x = \mathbf{b}$, the linear combination $\lambda u + (1-\lambda)v$ is also a solution of $\mathbf{A}x = \mathbf{b}$ only when $0 \leq \lambda \leq 1$.
4) If rank of **A** is n ,then $\mathbf{A}x = \mathbf{b}$ has at most one solution.

2 Solutions

Option 1	Suppose $\mathbb{V}$ is the vector space defined as $\mathbb{V} = \{\mathbf{x} : \mathbf{Ax} = \mathbf{b}\ ,\ \mathbb{R}^n \to \mathbb{R}^m\}$ **v** and **u** are the solution to the equation $\mathbf{Ax} = \mathbf{b}$ such that **u** and $\mathbf{v} \in \mathbb{V}$ $\mathbf{Au} = \mathbf{b} \quad \mathbf{Av} = \mathbf{b}$ Checking Closure under vector addition $\mathbf{A}(\mathbf{u}+\mathbf{v}) = \mathbf{Au} + \mathbf{Av} = \mathbf{b} + \mathbf{b} = 2\mathbf{b} \neq \mathbf{b}$ Which is enclosed under vector addition if and only if $\mathbf{b} = \mathbf{0}$.But here given $\mathbf{b} \neq 0$ means $\mathbf{0} \notin \mathbb{V}$ Hence does not satisfy requirements of vector space. Hence option 1 is incorrect.
Option 2	**Proof 1:** If **u** and **v** are the two solution of $\mathbf{A}x = \mathbf{b}$ $\mathbf{Au} = \mathbf{b} \quad \mathbf{Av} = \mathbf{b}$ For $\lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ to be a solution of $\mathbf{A}x = \mathbf{b}$,it must satisfy this equation. $\mathbf{A}(\lambda\mathbf{u} + (1-\lambda)\mathbf{v}) = \mathbf{b} \implies \mathbf{A}\lambda\mathbf{u} + \mathbf{A}(1-\lambda)\mathbf{v} = \mathbf{b} \implies \mathbf{A}\lambda\mathbf{u} + \mathbf{Av} - \mathbf{A}\lambda\mathbf{v} = \mathbf{b}$ $\mathbf{b}\lambda + \mathbf{Av} - \mathbf{b}\lambda = \mathbf{b} \implies \mathbf{Av} = \mathbf{b}$

	Which satisfies the equation therefore $\lambda\mathbf{u} + (1-\lambda)\,\mathbf{v}$ is the solution of $\mathbf{A}x = \mathbf{b}$ for any λ Since the λ term cancels out therefore vaild for $\lambda \in \mathbb{R}$. **Proof 2 (Through affine Subspace with an Example)**:- Let us suppose the two solution $\mathbf{u}$ and $\mathbf{v}$ be the points on the line given by the equation $\mathbf{A}x = \mathbf{b}$ Let the Line joining these two points is given as $\mathbf{l} = \mathbf{u} - \mathbf{v}$ is line parallel to the given line $\mathbf{A}x = \mathbf{b}$ Therefore $\mathbf{v}$ belongs to solution set and is independent to other linearly independent vectors of $\mathbf{l}$ $\mathbf{x} = \mathbf{v} + \lambda\mathbf{l}$ for $\lambda \in \mathbb{R}$ on substuting $\mathbf{l}$ $\mathbf{x} = \mathbf{v} + \lambda\,(\mathbf{u} - \mathbf{v}) = \mathbf{v} + \lambda\mathbf{u} - \lambda\mathbf{v} = \mathbf{v}\,(1-\lambda) + \lambda\mathbf{u}$ Hence $\mathbf{v}\,(1-\lambda) + \lambda\mathbf{u}$ is also the solution of the equation $\mathbf{Ax} = \mathbf{b}$ for $\lambda \in \mathbb{R}$. Hence Option 2 is correct.
Option 3	Since in Option 2 we have proved that $\mathbf{v}\,(1-\lambda) + \lambda\mathbf{u}$ is a solution for $\mathbf{Ax} = \mathbf{b}$ for any $\lambda \in \mathbb{R}$ therefore λ can be any real value but in option 3 there is restriction on λ which is incorrect. Hence option 3 is incorrect
Option 4	$\mathbf{A}_{mxn}\mathbf{x}_{nx1} = \mathbf{b}_{mx1}$ If $\mathbf{A}$ has Full column rank($\mathbf{A}$) = n then there exist one pivot in each columns and there exists no free variables thus $\mathbf{N}\,(\mathbf{A}) = \mathbf{0}$ so the only solution to $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. So the solution to $\mathbf{Ax} = \mathbf{b}$ $\mathbf{x} = \mathbf{x_p}$ unique solution exists if it exist.It can be either 0 or 1. Hence at most 1 solution is possible . **Proof with example** Let $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{pmatrix}_{4x2} \xleftrightarrow{RREF} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ Hence $n = 2$ pivot columns at both column position

	$\begin{pmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix}$ Hence no solution possible as no combination of $\mathbf{x}$ can gives the solution except $\mathbf{x}=\begin{pmatrix}0\\0\end{pmatrix}$ only if $\mathbf{b}=\mathbf{0} \implies \begin{pmatrix}1 & 3\\2 & 1\\6 & 1\\5 & 1\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ **OR** $\mathbf{x}=\begin{pmatrix}1\\1\end{pmatrix}$ only if $\mathbf{b}$ is addition of columns of $\mathbf{A} \implies \begin{pmatrix}1 & 3\\2 & 1\\6 & 1\\5 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}4\\3\\7\\6\end{pmatrix}$ Hence either no solution possible or one solution possile. Therefore we say at most one solution possible. Option 4 is correct.
Answers	Option 2 and Option 4 are correct

TABLE 1: Solution