Assignment 3

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1 Question 1.36 Geolin.pdf

BE and CF are two equal altitudes of a triangle ABC. Using RHS congruence rule, prove that the triangle ABC is isosceles.

2 Solution

BE and CF are two equal altitudes of a triangle ABC.

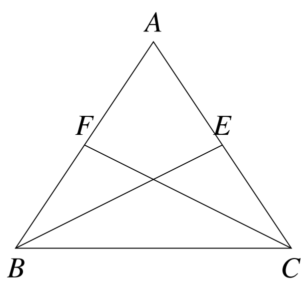

Fig. 0: Triangle with equal altitudes on two sides

Given:-
1) Altitudes are Equal means their magnitude are same

$$\|\mathbf{E}-\mathbf{B}\| = \|\mathbf{F}-\mathbf{C}\| \tag{2.0.1}$$

2) Altitude makes right angle at the base therefore $\cos 90 = 0$ therefore FC $\perp$ BF and EB $\perp$ CE where $\mathbf{m}$ is the directional vectors.

$$\mathbf{m}_{FC}\mathbf{m}_{BF} = 0 \tag{2.0.2}$$

$$\mathbf{m}_{EB}\mathbf{m}_{CE} = 0 \tag{2.0.3}$$

From (2.0.2)

$$(\mathbf{B}-\mathbf{F})(\mathbf{F}-\mathbf{C})^T = \mathbf{0} \quad (\mathbf{F}-\mathbf{C})(\mathbf{B}-\mathbf{F})^T = \mathbf{0} \tag{2.0.4}$$

From (2.0.2) and using (2.0.4)

$$(\mathbf{B}-\mathbf{C})(\mathbf{B}-\mathbf{C})^T \tag{2.0.5}$$

$$= (\mathbf{B}-\mathbf{F}+\mathbf{F}-\mathbf{C})(\mathbf{B}-\mathbf{F}+\mathbf{F}-\mathbf{C})^T) \tag{2.0.6}$$

$$= (\mathbf{B}-\mathbf{F})(\mathbf{B}-\mathbf{F})^T + (\mathbf{F}-\mathbf{C})(\mathbf{F}-\mathbf{C})^T \tag{2.0.7}$$

$$\|\mathbf{B}-\mathbf{C}\|^2 = \|\mathbf{B}-\mathbf{F}\|^2 + \|\mathbf{F}-\mathbf{C}\|^2 \tag{2.0.8}$$

Similarly
From (2.0.3)

$$(\mathbf{E}-\mathbf{B})(\mathbf{E}-\mathbf{C})^T = \mathbf{0} \quad (\mathbf{E}-\mathbf{C})(\mathbf{B}-\mathbf{E})^T = \mathbf{0} \tag{2.0.9}$$

From (2.0.3) and using (2.0.9)

$$(\mathbf{B}-\mathbf{C})(\mathbf{B}-\mathbf{C})^T \tag{2.0.10}$$

$$= (\mathbf{B}-\mathbf{E}+\mathbf{E}-\mathbf{C})(\mathbf{B}-\mathbf{E}+\mathbf{E}-\mathbf{C})^T \tag{2.0.11}$$

$$= (\mathbf{B}-\mathbf{E})(\mathbf{B}-\mathbf{E})^T + (\mathbf{E}-\mathbf{C})(\mathbf{E}-\mathbf{C})^T \tag{2.0.12}$$

$$\|\mathbf{B}-\mathbf{C}\|^2 = \|\mathbf{B}-\mathbf{E}\|^2 + \|\mathbf{E}-\mathbf{C}\|^2 \tag{2.0.13}$$

Equating (2.0.8) and (2.0.13) and using (2.0.1)

$$\|\mathbf{B}-\mathbf{F}\|^2 + \|\mathbf{F}-\mathbf{C}\|^2 = \|\mathbf{B}-\mathbf{E}\|^2 + \|\mathbf{E}-\mathbf{C}\|^2 \tag{2.0.14}$$

$$\|\mathbf{B}-\mathbf{F}\|^2 = \|\mathbf{E}-\mathbf{C}\|^2 \tag{2.0.15}$$

$$= \|\mathbf{B}-\mathbf{F}\| = \|\mathbf{E}-\mathbf{C}\| \tag{2.0.16}$$

Let $\angle FBC = \theta_1$ and $\angle EBC = \theta_2$

$$(\mathbf{B}-\mathbf{F})(\mathbf{B}-\mathbf{C})^T = \|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|\cos\theta_1 \tag{2.0.17}$$

$$\cos\theta_1 = \frac{(\mathbf{B}-\mathbf{F})(\mathbf{B}-\mathbf{C})^T}{\|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{2.0.18}$$

$$\cos\theta_1 = \frac{(\mathbf{B}-\mathbf{F})(\mathbf{B}-\mathbf{F}+\mathbf{F}-\mathbf{C})^T}{\|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{2.0.19}$$

$$\cos\theta_1 = \frac{(\mathbf{B}-\mathbf{F})(\mathbf{B}-\mathbf{F})^T + (\mathbf{B}-\mathbf{F})(\mathbf{F}-\mathbf{C})^T}{\|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{2.0.20}$$

From (2.0.4)

$$\cos\theta_1 = \frac{(\mathbf{B}-\mathbf{F})(\mathbf{B}-\mathbf{F})^T}{\|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{2.0.21}$$

$$\cos\theta_1 = \frac{\|\mathbf{B}-\mathbf{F}\|^2}{\|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{2.0.22}$$

$$\cos\theta_1 = \frac{\|\mathbf{B}-\mathbf{F}\|}{\|\mathbf{B}-\mathbf{C}\|} \tag{2.0.23}$$

Similarly for $\angle EBC = \theta_2$

$$(\mathbf{C}-\mathbf{E})(\mathbf{B}-\mathbf{C})^T = \|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|\cos\theta_2 \tag{2.0.24}$$

$$\cos\theta_2 = \frac{(\mathbf{C}-\mathbf{E})(\mathbf{B}-\mathbf{C})^T}{\|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{2.0.25}$$

$$\cos\theta_2 = \frac{(\mathbf{C}-\mathbf{E})(\mathbf{B}-\mathbf{E}+\mathbf{E}-\mathbf{C})^T}{\|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{2.0.26}$$

$$\cos\theta_2 = \frac{(\mathbf{C}-\mathbf{E})(\mathbf{B}-\mathbf{E})^T + (\mathbf{C}-\mathbf{E})(\mathbf{E}-\mathbf{C})^T}{\|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{2.0.27}$$

From (2.0.9)

$$\cos\theta_2 = \frac{(\mathbf{C}-\mathbf{E})(\mathbf{C}-\mathbf{E})^T}{\|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{2.0.28}$$

$$\cos\theta_2 = \frac{\|\mathbf{C}-\mathbf{E}\|^2}{\|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{2.0.29}$$

$$\cos\theta_2 = \frac{\|\mathbf{C}-\mathbf{E}\|}{\|\mathbf{B}-\mathbf{C}\|} \tag{2.0.30}$$

From (2.0.16) we know $\|\mathbf{B}-\mathbf{F}\| = \|\mathbf{E}-\mathbf{C}\|$ we conclude

$$\cos\theta_1 = \cos\theta_2 \implies \theta_1 = \theta_2 \tag{2.0.31}$$

So the sides opposite to equal angles are equal. Hence AB=AC hence the given Triangle is isosceles.