Assignment 7

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1 Problem

1)Find QR decomposition of matrix

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \tag{1.0.1}$$

2)Find the vertex $\mathbf{c}$ of the parabola using SVD for

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \tag{1.0.2}$$

by changing η to $\eta/2$ also verify the result using least squares.

2 Solution

2.1 Part 1:QR Decomposition of V

Let $\mathbf{x}$ and $\mathbf{y}$ be the column vectors of the given matrix.

$$\mathbf{x} = \begin{pmatrix} 9 \\ -12 \end{pmatrix} \tag{2.1.1}$$

$$\mathbf{y} = \begin{pmatrix} -12 \\ 16 \end{pmatrix} \tag{2.1.2}$$

The column vectors can be expressed as follows,

$$\mathbf{x} = k_1 \mathbf{u}_1 \tag{2.1.3}$$

$$\mathbf{y} = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \tag{2.1.4}$$

$$k_1 = \|\mathbf{x}\| \tag{2.1.5}$$

$$\mathbf{u}_1 = \frac{\mathbf{x}}{k_1} \tag{2.1.6}$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{y}}{\|\mathbf{u}_1\|^2} \tag{2.1.7}$$

$$\mathbf{u}_2 = \frac{\mathbf{y} - r_1 \mathbf{u}_1}{\|\mathbf{y} - r_1 \mathbf{u}_1\|} \tag{2.1.8}$$

$$k_2 = \mathbf{u}_2^T \mathbf{y} \tag{2.1.9}$$

The (2.1.3) and (2.1.4) can be written as,

$$\begin{pmatrix} \mathbf{x} & \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.1.10}$$

$$\begin{pmatrix} \mathbf{x} & \mathbf{y} \end{pmatrix} = \mathbf{QR} \tag{2.1.11}$$

Now, $\mathbf{R}$ is an upper triangular matrix and also,

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \tag{2.1.12}$$

Now using equations (2.1.5) to (2.1.9) we get,

$$k_1 = \sqrt{9^2 + 12^2} = 15 \tag{2.1.13}$$

$$\mathbf{u}_1 = \begin{pmatrix} \frac{3}{5} \\ \frac{-4}{5} \end{pmatrix} \tag{2.1.14}$$

$$r_1 = \begin{pmatrix} \frac{3}{5} & \frac{-4}{5} \end{pmatrix} \begin{pmatrix} -12 \\ 16 \end{pmatrix} = -20 \tag{2.1.15}$$

$$\mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.16}$$

$$k_2 = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} -12 \\ 16 \end{pmatrix} = 0 \tag{2.1.17}$$

Thus putting the values from (2.1.13) to (2.1.17) in (2.1.10) we obtain QR decomposition,

$$\begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & 0 \\ \frac{-4}{5} & 0 \end{pmatrix} \begin{pmatrix} 15 & -20 \\ 0 & 0 \end{pmatrix} \tag{2.1.18}$$

2.2 Part 2:Finding Vertex using SVD

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \tag{2.2.1}$$

$$\mathbf{u} = \begin{pmatrix} -9 \\ -\frac{101}{2} \end{pmatrix} \tag{2.2.2}$$

$$f = 19 \tag{2.2.3}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -4 & -3 \\ -3 & 4 \end{pmatrix} \tag{2.2.4}$$

$$\eta = \mathbf{u}^T \mathbf{p_1} = \frac{75}{2} \tag{2.2.5}$$

So the equation of perpendicular line passing through focus and intersecting parabola at vertex c is given as

$$\begin{pmatrix} \mathbf{u^T} + \frac{\eta}{2}\mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.2.6}$$

using (2.2.1),(2.2.2) ,(2.2.3) and (2.2.4)

$$\begin{pmatrix} -24 & \frac{-247}{4} \\ 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -19 \\ -6 \\ \frac{157}{4} \end{pmatrix} \tag{2.2.7}$$

$$\mathbf{Mc} = \mathbf{b} \tag{2.2.8}$$

where

$$\mathbf{M} = \begin{pmatrix} -24 & \frac{-247}{4} \\ 9 & -12 \\ -12 & 16 \end{pmatrix}, b = \begin{pmatrix} -19 \\ -6 \\ \frac{157}{4} \end{pmatrix} \tag{2.2.9}$$

To solve (2.2.8), we perform singular value decomposition on $\mathbf{M}$ given as

$$\mathbf{M} = \mathbf{USV^T} \tag{2.2.10}$$

Substituting the value of $\mathbf{M}$ from (2.2.10) in (2.2.8), we get

$$\mathbf{USV^Tc} = \mathbf{b} \tag{2.2.11}$$

$$\implies \mathbf{c} = \mathbf{VS_+U^Tb} \tag{2.2.12}$$

where, $\mathbf{S_+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigen-vectors of $\mathbf{MM^T}$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M^TM}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M^TM}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 801 & 1182 \\ 1182 & \frac{67409}{16} \end{pmatrix} \tag{2.2.13}$$

$$\mathbf{MM}^T = \begin{pmatrix} \frac{70225}{16} & 525 & -700 \\ 525 & 225 & -300 \\ -700 & -300 & 400 \end{pmatrix} \tag{2.2.14}$$

Eigen values of $\mathbf{M^TM}$ can be found out as

$$\left|\mathbf{M^TM} - \lambda\mathbf{I}\right| = 0 \tag{2.2.15}$$

$$\begin{pmatrix} 801 - \lambda & 1182 \\ 1182 & \frac{67409}{16} - \lambda \end{pmatrix} = 0 \tag{2.2.16}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = 431.539 \tag{2.2.17}$$

$$\lambda_2 = 4582.523 \tag{2.2.18}$$

$$\tag{2.2.19}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} -3.2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} .312 \\ 1 \end{pmatrix} \tag{2.2.20}$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} -.9544 \\ .2982 \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} .2978 \\ .9546 \end{pmatrix} \tag{2.2.21}$$

Hence we obtain $\mathbf{V}$ of (2.2.10) as follows,

$$\mathbf{V} = \begin{pmatrix} -.9544 & .2978 \\ .2982 & .9546 \end{pmatrix} \tag{2.2.22}$$

Similarly,eigen values of $\mathbf{MM}^T$ are,

$$\left|\mathbf{MM^T} - \lambda\mathbf{I}\right| = 0 \tag{2.2.23}$$

$$\begin{pmatrix} \frac{70225}{16} - \lambda & 525 & -700 \\ 525 & 225 - \lambda & -300 \\ -700 & -300 & 400 - \lambda \end{pmatrix} = 0 \tag{2.2.24}$$

$$\lambda_3 = 431.539 \tag{2.2.25}$$

$$\lambda_4 = 4582.531 \tag{2.2.26}$$

$$\lambda_5 = 0 \tag{2.2.27}$$

Hence the corresponding eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} .27637 \\ -.75 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -5.6536 \\ -.75 \\ 1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ 1.33 \\ 1 \end{pmatrix} \tag{2.2.28}$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} .2159 \\ -.5859 \\ .78125 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -.9764 \\ -.1295 \\ .1727 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ .8 \\ .6 \end{pmatrix} \tag{2.2.29}$$

Hence we obtain $\mathbf{U}$ of (2.2.10) as follows,

$$\mathbf{U} = \begin{pmatrix} .2159 & -.9764 & 0 \\ -.5859 & -.1295 & .8 \\ .78125 & .1727 & .6 \end{pmatrix} \tag{2.2.30}$$

After computing the singular values from eigen values $\lambda_3, \lambda_4, \lambda_5$ we get $\mathbf{S}$ of (2.2.10) as follows,

$$\mathbf{S} = \begin{pmatrix} 20.7735 & 0 \\ 0 & 67.6943 \\ 0 & 0 \end{pmatrix} \tag{2.2.31}$$

From (2.2.10) we get the Singular Value Decomposition of $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix} .2159 & -.9764 & 0 \\ -.5859 & -.1295 & .8 \\ .78125 & .1727 & .6 \end{pmatrix} \begin{pmatrix} 20.7735 & 0 \\ 0 & 67.6943 \\ 0 & 0 \end{pmatrix} \tag{2.2.32}$$

$$\begin{pmatrix} -.9544 & .2978 \\ .2982 & .9546 \end{pmatrix}^T \tag{2.2.33}$$

$$= \begin{pmatrix} -24 & 61.75 \\ 9 & -12 \\ -12 & 16 \end{pmatrix} \tag{2.2.34}$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} 20.7735 & 0 & 0 \\ 0 & 67.6943 & 0 \end{pmatrix} \tag{2.2.35}$$

From (2.2.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 30.0754 \\ 26.1070 \\ 18.75 \end{pmatrix} \tag{2.2.36}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 1.4477 \\ .38566 \end{pmatrix} \tag{2.2.37}$$

$$\mathbf{c} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -1.266 \\ .8 \end{pmatrix} \tag{2.2.38}$$

$$\implies \mathbf{c} = \begin{pmatrix} -1.266 \\ .8 \end{pmatrix} \tag{2.2.39}$$

Verifying the solution of (2.2.39) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{c} = \mathbf{M}^T\mathbf{b} \tag{2.2.40}$$

Evaluating the R.H.S in (2.2.40) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{c} = \begin{pmatrix} -69 \\ \frac{7493}{4} \end{pmatrix} \tag{2.2.41}$$

$$\implies \begin{pmatrix} 801 & 1182 \\ 1182 & \frac{67409}{16} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -69 \\ \frac{7493}{4} \end{pmatrix} \tag{2.2.42}$$

Solving the augmented matrix of (2.2.42) we get,

$$\begin{pmatrix} 801 & 1182 & -69 \\ 1182 & \frac{67409}{16} & \frac{7493}{4} \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{801}R_1} \begin{pmatrix} 1 & \frac{394}{267} & \frac{-23}{267} \\ 1182 & \frac{67409}{16} & \frac{7493}{4} \end{pmatrix} \tag{2.2.43}$$

$$\xleftrightarrow{R_2 = R_2 - 1182R_1} \begin{pmatrix} 1 & \frac{394}{267} & \frac{-23}{267} \\ 0 & \frac{3515625}{1424} & \frac{703125}{356} \end{pmatrix} \tag{2.2.44}$$

$$\xleftrightarrow{R_2 = \frac{1424}{3515625}R_2} \begin{pmatrix} 1 & \frac{394}{267} & \frac{-23}{267} \\ 0 & 1 & \frac{4}{5} \end{pmatrix} \tag{2.2.45}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{394}{267}R_2} \begin{pmatrix} 1 & 0 & \frac{-19}{15} \\ 0 & 1 & \frac{4}{5} \end{pmatrix} \tag{2.2.46}$$

From equation (2.2.46), solution is given by,

$$\mathbf{c} = \begin{pmatrix} \frac{-19}{15} \\ \frac{4}{5} \end{pmatrix} \tag{2.2.47}$$

$$\mathbf{c} = \begin{pmatrix} -1.266 \\ .8 \end{pmatrix} \tag{2.2.48}$$

Comparing results of $\mathbf{c}$ from (2.2.39) and (2.2.48), we can say that the solution is verified.