Assignment 12

Pulkit Saxena

1 Problem ugcjune2017 Q75

Which of the following 3x3 matrices are diagonizable over $\mathbb{R}$?

1. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$ 2. $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 3. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$ 4. $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

2 Explanation

Test for diagonalizability	Let $\mathbf{W}_i$ be the eigenspace corresponding to eigenvalue λ_i of $\mathbf{A}$ 1)$\mathbf{A}$ is diagonalizable 2) characteristic polynomial of $\mathbf{A}$ is f = $(\mathbf{x} - \lambda_1)^{d_1}....(\mathbf{x} - \lambda_k)^{d_k}$ and $dim(\mathbf{W}_i) = d_i$ 3) $\sum_{i=1}^{k} \mathbf{W_i} = n$
Concept for diagonalization	A linear operator $\mathbf{A}$ on a n-dimensional space $\mathbb{V}$ is diagonalizable , if and only if $\mathbf{A}$ has n distinct characteristic vectors or null spaces corresponding to the characteristic values

TABLE 1: Illustration of theorem.

3 Solution

Option A	Given matrix is $\mathbf{A}=\begin{pmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 0 & 0 & 6\end{pmatrix}$
Finding Characteristics polynomial	Characteristics polynomial of the matrix **A** is $det(xI - A)$ $\det(xI - A)=\begin{vmatrix}(x-1) & -3 & -2\\ 0 & (x-4) & -5\\ 0 & 0 & x-6\end{vmatrix}$ Characteristic Polynomial $= (x-1)(x-4)(x-6)$
Testing diagonalizability over $\mathbb{R}$	1) As the characteristics polynomial is product of linear factors over $\mathbb{R}$. 2) To find characteristic values of the operator $\det(xI - A) = 0$ which gives $\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 6$ Thus over $\mathbb{R}$ matrix **A** has three distinct characteristic values. There will be atleast one characteristics vector i.e., one dimension with each characteristics value . Thus $dimW_i = d_i$ 3) $\sum_i \mathbf{W_i} = n = 3$, which is equal to dim of A.
Conclusion on Option A	Option A satisfy all three condition of Diagonalizability over $\mathbb{R}$.

Option B	Given matrix is $\mathbf{A}=\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$
Finding Characteristics polynomial	Characteristics polynomial of the matrix **A** is $det(xI - A)$ $\det(xI - A)=\begin{vmatrix}x & -1 & 0\\ 1 & x & 0\\ 0 & 0 & x-1\end{vmatrix}$ Characteristic Polynomial $= (x-1)(x+i)(x-i)$

Testing diagonalizability over $\mathbb{R}$	1) As the characteristics polynomial is not the product of linear factors over $\mathbb{R}$. Thus **A** is not diagonalizable over $\mathbb{R}$.
Conclusion on Option B	Option B does not satisfy condition 1.
Option C	Given matrix is $\mathbf{A}=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$
Finding Characteristics polynomial	Characteristics polynomial of the matrix **A** is $det(xI - A)$ $\det(xI - A)= \begin{vmatrix} (x-1) & -2 & -3 \\ -2 & (x-1) & -4 \\ -3 & -4 & x-1 \end{vmatrix}$ Characteristic Polynomial = $(x + 3.19)(x + 0.877)(x - 7.07)$
Testing diagonalizability over $\mathbb{R}$	1) As the characteristics polynomial is product of linear factors over $\mathbb{R}$. 2) To find characteristic values of the operator $\det(xI - A) = 0$ which gives $\lambda_1 = -3.19, \lambda_2 = -0.887, \lambda_3 = 7.07$ Thus over $\mathbb{R}$ matrix **A** has three distinct characteristic values. There will be atleast one characteristics vector i.e., one dimension with each characteristics value . Thus $dimW_i = d_i$ 3) $\sum_i \mathbf{W_i} = n = 3$, which is equal to dim of A.
Conclusion on Option C	Option C satisfy all three condition of Diagonalizability over $\mathbb{R}$.
Option D	Given matrix is $\mathbf{A}=\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
Finding Characteristics	Characteristics polynomial of the matrix **A** is $det(xI - A)$

polynomial	$\det(xI - A) = \begin{vmatrix} x & -1 & -2 \\ 0 & x & -1 \\ 0 & 0 & x \end{vmatrix}$ Characteristic Polynomial = $(x)(x)(x) = x^3$
Testing diagonalizability over $\mathbb{R}$	1) As the characteristics polynomial is product of linear factors over $\mathbb{R}$. 2) To find characteristic values of the operator $\det(xI - A) = 0$ $\lambda_1 = 0$ $d_1 = 3$ $\mathbf{W}_1 = \mathbf{A} - \lambda_1 \mathbf{I} \implies \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - 0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $dimW_1 = 2$ $dimW_i \neq d_i$ Algebric Multiplicity is not equal to Geometric Multiplicity.
Conclusion on Option D	Option D does not satisfy second condition of Diagonalizability.

Answer	Option A and Option C are Diagonalizable over $\mathbb{R}$.