Challenging Problem

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1 Problem

Let W_1 and W_2 be subspaces of a finite-dimensional vector space $\mathbb{V}$. Prove that

1) $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$
2) $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$

2 Solutions

Proof of $W_1 + W_2 = W_1 \cup W_2$	If W_1 and W_2 are subspace of vector space $\mathbb{V}$ over a Field F then 1) $W_1 + W_2$ is a subspace of $\mathbb{V}$ 2)span$(W_1 \cup W_2) = W_1 + W_2$ or $W_1 + W_2 = W_1 \cup W_2$ **Proof of $W_1 + W_2$ is a subspace of $\mathbb{V}$** Let $a_1, a_2 \in W_1$ and $b_1, b_2 \in W_2$ and two scalar α and $\beta \in$ F $\alpha\mathbf{a} + \beta\mathbf{b} = \alpha(a_1 + a_2) + \beta(b_1 + b_2) = \alpha a_1 + \beta b_1 + \alpha a_2 + \beta b_2$ $\implies W_1 + W_2 \in \mathbb{V}$ **Proof of span**$(W_1 \cup W_2) = W_1 + W_2$ **or span**$(W_1 + W_2) = W_1 \cup W_2$ $0 \in W_2$ Let $a_1 \in W_1$ $a_1 = a_1 + 0 \implies a_1 \in W_1 + W_2$ thus $W_1 \subseteq W_1 + W_2$ Similarly $W_2 \subseteq W_1 + W_2$ **We need to show** $W_1 + W_2 \subseteq$ **span**$(W_1 \cup W_2)$ **and span**$(W_1 \cup W_2) \subseteq W_1 + W_2$ Let $a = a_1 + b_1$ be any element of $W_1 + W_2$ Then $a_1 \in W_1$ and $a_2 \in W_2$ therefore $a_1 \in W_1 \cup W_2$ and $b_1 \in W_1 \cup W_2$ We can write $a_1 + b_1 = 1a_1 + 1b_1$ Thus a_1 and b_1 is a linear combination of finite number of elements $a_1, b_1 \in W_1 \cup W_2$ $\implies a_1 + b_1 \in$ span$(W_1 \cup W_2)$ $\implies W_1 + W_2 \in$ span$(W_1 \cup W_2)$ **Now to prove span**$(W_1 \cup W_2) \subseteq W_1 + W_2$ span$(W_1 \cup W_2)$ is the smallest subspace containing $W_1 \cup W_2$ and $W_1 + W_2$is a subspace of containing $W_1 \cup W_2 \implies span(W_1 \cup W_2) \subseteq W_1 + W_2$ Hence we proved $W_1 + W_2 = W_1 \cap W_2$. From this proof the above equation can be modified as De-Morgan's law.

Proof $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$	From the above proof we can Modify as De-Morgan's Law as $(W_1 \cup W_2)' = W_1' \cap W_2'$ Let $x \in (W_1 \cup W_2)'$ $\implies x \notin (W_1 \cup W_2)$ $\implies x \notin W_1$ and $x \notin W_2)$ $\implies x \in W_1'$ and $x \in W_2')$ $\implies x \in W_1' \cap W_2'$ $\implies (W_1 \cup W_2)' = W_1' \cap W_2'$ Therefore $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$. Hence proved.
Proof $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$	From the above proof we can Modify as De-Morgan's Law as $(W_1 \cap W_2)' = W_1' \cup W_2'$ Let $x \in (W_1 \cap W_2)'$ $\implies x \notin (W_1 \cap W_2)$ $\implies x \notin W_1$ OR $x \notin W_2)$ $\implies x \in W_1'$ OR $x \in W_2')$ $\implies x \in W_1' \cup W_2'$ $\implies (W_1 \cup W_2)' = W_1' \cup W_2'$. Therefore $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$Hence proved.

TABLE 1: Solution Table