Assignment 16

Pulkit Saxena

1 Problem Hoffman Pg 242 Q7A

Find the minimal polynomials and the rational forms of the following real matrices

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{1.0.1}$$

2 Theorems

Theorem 1	A Rational canonical form is a matrix $\mathbf{R}$ that is Direct sum of companion matrix. $\mathbf{R} = \mathbf{C(p_1)} \oplus \cdots \oplus \mathbf{C(p_r)}$ $$\mathbf{R} = \begin{pmatrix} \mathbf{C(p_1)} & 0 & 0 & \dots & 0 \\ 0 & \mathbf{C(p_2)} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{C(p_r)} \end{pmatrix} \tag{2.0.1}$$ where $\mathbf{C(p_i)}$ is the k_i x k_i companion matrix of p_i where polynomial $p_1, p_2 \dots p_r$ are called invariant factors for Given Matrix .Where k_i denotes the degree of annihilator of p_i. This representation is called rational form.
Theorem 2	If $p_i(x) = x + a_0$ then its companion matrix $\mathbf{C(p)}$ is 1 x 1 matrix as $\left(-a_0\right)$. If $k_i \geq 2$ then $p(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ then its companion matrix is $$\mathbf{C(p_i)} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ 0 & 0 & 1 & \dots & 0 & -a_3 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -a_{k-1} \end{pmatrix} \tag{2.0.2}$$

TABLE 1: Illustration of theorem.

3 Solution

Given	$\mathbf{A} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$
Characteristics and Minimal Polynomial	Characteristics polynomial of the matrix is $det(x\mathbf{I} - \mathbf{A})$ $\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} (x) & 1 & 1 \\ -1 & (x) & 0 \\ 1 & 0 & (x) \end{vmatrix} = x\left(x^2\right) - 1\,(-x) - x = x^3$ Characteristic Polynomial $= x^3$ Minimal Polynomial can be x, x^2 or x^3 of lowest degree satisfying $p(\mathbf{A}) = 0$ Let take $p\,(x) = x \implies p(\mathbf{A}) = \mathbf{A} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \neq 0$ Let take $p\,(x) = x^2 \implies p(\mathbf{A}) = \mathbf{A}^2 = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \neq 0$ Take $p\,(x) = x^3 \implies p\,(\mathbf{A}) = \mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$ Thus minimal polynomial $p\,(x) = x^3$.
Companion Matrix and Rational Form	Since Characteristics polynomial=Minimal polynomial=Invariant factors $p\,(x) = x^3 + 0x^2 + 0x + 0$ So Companion Matrix is of dimention 3x3 and from theorem 2 $\mathbf{C}(\mathbf{p}) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ Since there is only one minimal polynomial of degree 3 which is equal to characteristics equation therefore Rational matix=companion matrix

	$\mathbf{R} = \mathbf{C}(\mathbf{p}) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ which is in rational form.

TABLE 2: Solution Table