Assignment 15

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1 Problem Hoffman Pg 230 Q2

Let T be a linear operator on $\mathbb{R}^3$ which is represented in standard ordered basis by matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1.0.1}$$

Prove that T has no cyclic vector.What is the T-cyclic subspace generated by the vector $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$?

2 Theorems

Theorem 1	T be a linear operator on vector space $\mathbb{V}$ of n dimensional. There exist a cyclic vector for T if and only if minimal polynomial and characteristic polynomial are same. Characteristics Polynomial:- $f(x) = (\mathbf{x} - \lambda_1)^{d_1}............(\mathbf{x} - \lambda_k)^{d_k}$ Minimal Polynomial:- $p_a(x) = (\mathbf{x} - \lambda_1)............(\mathbf{x} - \lambda_k)$ for the given eigen values $\lambda_1.....\lambda_k$
Theorem 2	$\mathbb{Z}(\mathbf{a}; T)$is the subspace spanned by vectors $T^k\mathbf{a}$, $k \geq 0$, and $\mathbf{a}$ is a cyclic vector for T if and only if these vector span $\mathbb{V}$,the α is called cyclic vector of T.
Theorem 3 Cyclic Base	Let $\mathbf{a}$ be any non-zero vector in $\mathbb{V}$ and let $p_a(minimal polynomial)$ be the T-annihilator of $\mathbf{a}$ If the degree of p_a is k , then vectors $\mathbf{a}, T\mathbf{a}, T^2\mathbf{a}, ..., T^{k-1}\mathbf{a}$ form of a basis for $\mathbb{Z}(\mathbf{a}; T)$ if $g(T)\,\mathbf{a} = 0$.

TABLE 1: Illustration of theorem.

3 Solution

proof of T doesn't have cyclic vector	Characteristics polynomial of the matrix is $det(x\mathbf{I} - \mathbf{A})$ $\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} (x-2) & 0 & 0 \\ 0 & (x-2) & 0 \\ 0 & 0 & (x+1) \end{vmatrix}$ Characteristic Polynomial = $(x-2)^2(x+1)$ Minimal Polynomial=$p_a(x)$=$(x-2)(x+1)$ degree =2 Minimal Polynomial $\neq$ Characteristic Polynomial Thus from Theorem 1 T doesn't have cyclic vector.
Verify that $p_a(x)$ is minimal polynomial	From the above calculation of Characteristics polynomial f$(x) = (x-2)^2\,(x+1)$ Since the root of characteristics polynomial are repeated so we take minimum order monic polynomial $p_a\,(x) = (x-2)\,(x+1)$ By Cayley-Hamilton theorem states that every square matrix satisfies its characteristics equation $p_a\,(x) = 0 \implies p_a\,(A) = 0$ $p_a\,(x) = (x-2)\,(x+1)$=0 $\implies x^2 - x - 2 = 0$ $p_a\,(\mathbf{A}) = \mathbf{A}^2 - \mathbf{A} - 2\mathbf{I} = 0$ $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$ Thus $p_a\,(x)$ is the monic polynomial of lowest degree such that $p_a\,(\mathbf{A}) = 0$ So $p_a\,(x) = (x-2)\,(x+1)$ is a minimal polynomial.

Cyclic subspace	For the given matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $T(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (2\mathbf{x}, 2\mathbf{y}, -\mathbf{z})$ $T\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$ Since we know $T^2\mathbf{a} = T(T\mathbf{a})$ $T^2\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = T\begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix}$ Degree of minimal polynomial is 2 therefore k=2 From Theorem 2 $\mathbb{Z}(\mathbf{a}; T)$ spans $\{\mathbf{a}, T\mathbf{a}, T^2\mathbf{a}\} = \begin{pmatrix} 1 & 2 & 4 \\ -1 & -2 & -4 \\ 3 & -3 & 3 \end{pmatrix}$ which is linearly dependent matrix.$g(T) = \det(\mathbb{Z}(\mathbf{a}; T)) = 0$ $\begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$. Therefore from Theorem 3 Cyclic subspace of $\mathbb{Z}(\mathbf{a}; T)$ spans $\{\mathbf{a}, T\mathbf{a}\}$ Hence T-cycle subspace generated by $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ $= \text{span}\left(\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}\right)$

TABLE 2: Solution Table