

Assignment 15

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1 PROBLEM HOFFMAN PG 230 Q2

Let T be a linear operator on \mathbb{R}^3 which is represented in standard ordered basis by matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (1.0.1)$$

Prove that T has no cyclic vector. What is the T -cyclic subspace generated by the vector $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$?

2 THEOREMS

Theorem 1	<p>T be a linear operator on vector space \mathbb{V} of n dimensional.</p> <p>There exist a cyclic vector for T if and only if minimal polynomial and characteristic polynomial are same.</p> <p>Characteristics Polynomial:-</p> $f(x) = (\mathbf{x} - \lambda_1)^{d_1} \dots (\mathbf{x} - \lambda_k)^{d_k}$ <p>Minimal Polynomial:-</p> $p_a(x) = (\mathbf{x} - \lambda_1) \dots (\mathbf{x} - \lambda_k) \text{ for the given eigen values } \lambda_1 \dots \lambda_k$
Theorem 2	<p>$\mathbb{Z}(\mathbf{a}; T)$ is the subspace spanned by vectors $T^k \mathbf{a}$, $k \geq 0$, and \mathbf{a} is a cyclic vector for T if and only if these vector span \mathbb{V}.</p>
Theorem 3 Cyclic Base	<p>Let \mathbf{a} be any non-zero vector in \mathbb{V} and let $p_a(\text{minimal polynomial})$ be the T-annihilator of \mathbf{a}</p> <p>If the degree of p_a is k, then vectors $\mathbf{a}, T\mathbf{a}, T^2\mathbf{a}, \dots, T^{k-1}\mathbf{a}$ form of a basis for $\mathbb{Z}(\mathbf{a}; T)$</p> <p>if $g(T)\mathbf{a} = 0$.</p>

TABLE 1: Illustration of theorem.

3 SOLUTION

proof of T doesn't have cyclic vector	<p>Characteristics polynomial of the matrix is $\det(x\mathbf{I} - \mathbf{A})$</p> $\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} (x-2) & 0 & 0 \\ 0 & (x-2) & 0 \\ 0 & 0 & (x+1) \end{vmatrix}$ <p>Characteristic Polynomial = $(x-2)^2(x+1)$</p> <p>Minimal Polynomial = $p_a(x) = (x-2)(x+1)$ degree = 2</p> <p>Minimal Polynomial \neq Characteristic Polynomial</p> <p>Thus from Theorem 1 T doesn't have cyclic vector.</p>
Verify that $p_a(x)$ is minimal polynomial	<p>From the above calculation of Characteristics polynomial</p> $f(x) = (x-2)^2(x+1)$ <p>Since the root of characteristics polynomial are repeated so we take minimum order monic polynomial</p> $p_a(x) = (x-2)(x+1)$ <p>By Cayley-Hamilton theorem states that every square matrix satisfies its characteristics equation</p> $p_a(x) = 0 \implies p_a(A) = 0$ $p_a(x) = (x-2)(x+1) = 0 \implies x^2 - x - 2 = 0$ $p_a(\mathbf{A}) = \mathbf{A}^2 - \mathbf{A} - 2\mathbf{I} = 0$ $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$ <p>Thus $p_a(x)$ is the monic polynomial of lowest degree such that $p_a(\mathbf{A}) = 0$</p> <p>So $p_a(x) = (x-2)(x+1)$ is a minimal polynomial.</p>

Cyclic subspace

For the given matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$T(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (2\mathbf{x}, 2\mathbf{y}, -\mathbf{z})$$

$$T \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$$

Since we know $T^2 \mathbf{a} = T(T\mathbf{a})$

$$T^2 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = T \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix}$$

Degree of minimal polynomial is 2 therefore $k=2$

From Theorem 2

$$\mathbb{Z}(\mathbf{a}; T) \text{ spans } \{\mathbf{a}, T\mathbf{a}, T^2\mathbf{a}\} = \begin{pmatrix} 1 & 2 & 4 \\ -1 & -2 & -4 \\ 3 & -3 & 3 \end{pmatrix}$$

which is linearly dependent matrix. $g(T) = \det(\mathbb{Z}(\mathbf{a}; T)) = 0$

$$\begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix} \text{ is a linear combination of } \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}.$$

Therefore from Theorem 3

Cyclic subspace of $\mathbb{Z}(\mathbf{a}; T)$ spans $\{\mathbf{a}, T\mathbf{a}\}$

$$\text{Hence } T\text{-cycle subspace generated by } \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

$$= \text{span} \left(\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} \right)$$

TABLE 2: Solution Table