Assignment 3

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I. Question 1.36 Geolin.pdf

BE and CF are two equal altitudes of a triangle ABC. Using RHS congruence rule, prove that the triangle ABC is isosceles.

II. Solution

BE and CF are two equal altitudes of a triangle ABC.

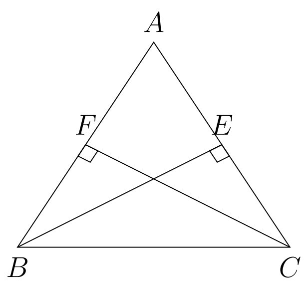

Fig. 1: Triangle with equal altitudes on two sides

Given:-

1) Altitudes are Equal means their magnitude are same

$$\|\mathbf{E}-\mathbf{B}\| = \|\mathbf{F}-\mathbf{C}\| \tag{1}$$

2) Altitude makes right angle at the base therefore $\cos 90 = 0$ therefore FC $\perp$ BF and EB $\perp$ CE where **m** is the directional vectors.

$$\mathbf{m}_{FC}\mathbf{m}_{BF} = 0 \tag{2}$$

$$\mathbf{m}_{EB}\mathbf{m}_{CE} = 0 \tag{3}$$

From equation 2

$$\left(B-F\right)\left(F-C\right)^T = 0 \quad \left(F-C\right)\left(B-F\right)^T = 0 \tag{4}$$

From equation 2 and using equation 4

$$\left(B-C\right)\left(B-C\right)^T \tag{5}$$

$$= \left(B-F+F-C\right)\left(B-F+F-C\right)^T \tag{6}$$

$$= \left(B-F\right)\left(B-F\right)^T + \left(F-C\right)\left(F-C\right)^T \tag{7}$$

$$\|B-C\|^2 = \|B-F\|^2 + \|F-C\|^2 \tag{8}$$

Similarly
From Equation 3

$$\left(E-B\right)\left(E-C\right)^T = 0 \quad \left(E-C\right)\left(B-E\right)^T = 0 \tag{9}$$

From equation 3 and using equation 9

$$\left(B-C\right)\left(B-C\right)^T \tag{10}$$

$$= \left(B-E+E-C\right)\left(B-E+E-C\right)^T \tag{11}$$

$$= \left(B-E\right)\left(B-E\right)^T + \left(E-C\right)\left(E-C\right)^T \tag{12}$$

$$\|B-C\|^2 = \|B-E\|^2 + \|E-C\|^2 \tag{13}$$

Equating Equation 8 and equation 13 and using equation 1

$$\|B-F\|^2 + \|F-C\|^2 = \|B-E\|^2 + \|E-C\|^2 \tag{14}$$

$$\|B-F\|^2 = \|E-C\|^2 \tag{15}$$

$$= \|B-F\| = \|E-C\| \tag{16}$$

Let $\angle FBC = \theta_1$ and $\angle EBC = \theta_2$

$$\left(B-F\right)\left(B-C\right)^T = \|B-F\|\,\|B-C\|\cos\theta_1 \tag{17}$$

$$\cos\theta_1 = \frac{\left(B-F\right)\left(B-C\right)^T}{\|B-F\|\,\|B-C\|} \tag{18}$$

$$\cos\theta_1 = \frac{\left(B-F\right)\left(B-F+F-C\right)^T}{\|B-F\|\,\|B-C\|} \tag{19}$$

$$\cos\theta_1 = \frac{\left(B-F\right)\left(B-F\right)^T + \left(B-F\right)\left(F-C\right)^T}{\|B-F\|\,\|B-C\|} \tag{20}$$

From Equation 4

$$\cos\theta_1 = \frac{\left(B-F\right)\left(B-F\right)^T}{\|B-F\|\,\|B-C\|} \tag{21}$$

$$\cos\theta_1 = \frac{\|B-F\|^2}{\|B-F\|\,\|B-C\|} \tag{22}$$

$$\cos\theta_1 = \frac{\|B-F\|}{\|B-C\|} \tag{23}$$

Similarly for $\angle EBC = \theta_2$

$$\left(C-E\right)\left(B-C\right)^T = \|C-E\|\,\|B-C\|\cos\theta_2 \tag{24}$$

$$\cos\theta_2 = \frac{\left(C-E\right)\left(B-C\right)^T}{\|C-E\|\,\|B-C\|} \tag{25}$$

$$\cos\theta_2 = \frac{\left(C-E\right)\left(B-E+E-C\right)^T}{\|C-E\|\,\|B-C\|} \tag{26}$$

$$\cos\theta_2 = \frac{\left(C-E\right)\left(B-E\right)^T + \left(C-E\right)\left(E-C\right)^T}{\|C-E\|\,\|B-C\|} \tag{27}$$

From Equation 9

$$\cos\theta_2 = \frac{\left(C-E\right)\left(C-E\right)^T}{\|C-E\|\,\|B-C\|} \tag{28}$$

$$\cos\theta_2 = \frac{\|C-E\|^2}{\|C-E\|\,\|B-C\|} \tag{29}$$

$$\cos\theta_2 = \frac{\|C-E\|}{\|B-C\|} \tag{30}$$

From equation 16 we know $\|B-F\| = \|E-C\|$ we conclude

$$\cos\theta_1 = \cos\theta_2 \implies \theta_1 = \theta_2 \tag{31}$$

So the sides opposite to equal angles are equal. Hence AB=AC hence the given Triangle is isosceles.