Assignment 6

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1 Question

The following equation represent parabola.

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \quad (1.0.1)$$

Find the Vertex.

2 Solution

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \quad (2.0.1)$$

$$\mathbf{u} = \begin{pmatrix} -9 \\ -\frac{101}{2} \end{pmatrix} \quad (2.0.2)$$

$$f = 4 \quad (2.0.3)$$

The eigenvalues of $\mathbf{V}$ is given by

$$\left|\lambda \mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 9 & 12 \\ 12 & \lambda - 16 \end{vmatrix} = 0 \quad (2.0.4)$$

$$\lambda^2 - 25\lambda = 0 \quad (2.0.5)$$

$$\lambda_1 = 0, \lambda_2 = 25 \quad (2.0.6)$$

For $\lambda_1 = 0$, the eigen vector $\mathbf{p}$ is given by

$$\mathbf{Vp} = 0 \quad (2.0.7)$$

Row reducing $\mathbf{V}$ yields

$$\implies \begin{pmatrix} -9 & 12 \\ 12 & -16 \end{pmatrix} \xleftrightarrow[R_2 = R_2 + 4R_1]{R_1 = -\frac{R_1}{3}} \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix} \quad (2.0.8)$$

$$\implies \mathbf{p}_1 = \frac{1}{5}\begin{pmatrix} -4 \\ -3 \end{pmatrix} \quad (2.0.9)$$

Similarly, Eigen vector corresponding to $\lambda_2 = 25$

$$\mathbf{p}_2 = \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix} \quad (2.0.10)$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -4 & -3 \\ -3 & 4 \end{pmatrix} \quad (2.0.11)$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \quad (2.0.12)$$

The focal length of the parabola is given by

$$\frac{\left|2\mathbf{u}^T \mathbf{p_1}\right|}{\lambda_2} = \frac{75}{25} = 3 \quad (2.0.13)$$

and its equation is

$$\mathbf{y^T D y} = -2\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad (2.0.14)$$

where

$$\eta = \mathbf{u}^T \mathbf{p_1} = \frac{75}{2} \quad (2.0.15)$$

$$\begin{pmatrix} \mathbf{u^T} + \eta \mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \quad (2.0.16)$$

using (2.0.1),(2.0.3) and (2.0.9)

$$\begin{pmatrix} -39 & -73 \\ 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -19 \\ -21 \\ 28 \end{pmatrix} \quad (2.0.17)$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix} -39 & -73 & -19 \\ 9 & -12 & -21 \\ -12 & 16 & 28 \end{pmatrix} \quad (2.0.18)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + (4/3)R_2} \begin{pmatrix} -39 & -73 & -19 \\ 9 & -12 & -21 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.19)$$

$$\xleftrightarrow{R_1 \leftarrow R_1/(-39)} \begin{pmatrix} 1 & 73/39 & 19/39 \\ 9 & -12 & -21 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.20)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 9R_1} \begin{pmatrix} 1 & 73/39 & 19/39 \\ 0 & -1125/39 & -990/39 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.21)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \times (-39/1125)} \begin{pmatrix} 1 & 73/39 & 19/39 \\ 0 & 1 & 22/25 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.22)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - (73/39)R_2} \begin{pmatrix} 1 & 0 & -29/25 \\ 0 & 1 & 22/25 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.23)$$

Thus the vertex **c** is:

$$\mathbf{c} = \begin{pmatrix} -29/25 \\ 22/25 \end{pmatrix} = \begin{pmatrix} -1.16 \\ 0.88 \end{pmatrix} \tag{2.0.24}$$