Assignment 14

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1 Problem Hoffman pg213 Q1

Let $\mathbb{V}$ be a finite-dimensional vector space and let $\mathbb{W}_1$ be any subspace of $\mathbb{V}$.Prove that there is a subspace $\mathbb{W}_2$ of $\mathbb{V}$ such that $\mathbb{V}= \mathbb{W}_1 \oplus \mathbb{W}_2$

2 Solution

Assumption and Claim	Let $\beta = \{\mathbf{u}_1, ..., \mathbf{u}_n\}$ be a basis for $\mathbb{W}_1$.Since $\mathbb{W}_1$ is the subspace of $\mathbb{V}$. therefore let us take $\alpha = \{\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{u}_{n+1}, ..., \mathbf{u}_m\}$ the basis of $\mathbb{V}$ So $\mathbb{W}_2$=span($\{\mathbf{u}_{n+1}, ..., \mathbf{u}_m\}$) Claim that $\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2$.
Proof of $\mathbb{V} = \mathbb{W}_1 + \mathbb{W}_2$	if $\mathbf{v} \in \mathbb{V}$, then $\mathbf{v} = \sum_{i=1}^{m} a_i\mathbf{u}_i = \sum_{i=1}^{n} a_i\mathbf{u}_i + \sum_{i=n+1}^{m} a_i\mathbf{u}_i \in \mathbb{W}_1 + \mathbb{W}_2$ for scalar a_i , $i = 1, ..., m$ This implies that $\mathbb{V} \subseteq \mathbb{W}_1 + \mathbb{W}_2$ But by the defination of $\mathbb{W}_1 + \mathbb{W}_2$ we know that $\mathbb{W}_1 + \mathbb{W}_2 \subseteq \mathbb{V}$. Hence $\mathbb{V} = \mathbb{W}_1 + \mathbb{W}_2$
Proof of $\mathbb{W}_1 \cap \mathbb{W}_2 = \{0\}$	Let $\mathbf{u} \in \mathbb{W}_1 \cap \mathbb{W}_2$ Then $\mathbf{u} = \sum_{i=1}^{n} b_i\mathbf{u}_i = \sum_{i=n+1}^{m} c_i\mathbf{u}_i$ for some scalar $b_1, ...b_n, c_{n+1}, ..., c_m$ $\implies \sum_{i=1}^{n} b_i\mathbf{u}_i + \sum_{i=n+1}^{m} (-c_i)\,\mathbf{u}_i = 0$ But α is linearly independent ,since α is a basis. Hence $b_1 = ... = b_n = c_{n+1}..... = c_m = 0$.This implies $\mathbf{u} = 0$. Thus $\mathbb{W}_1 \cap \mathbb{W}_2 = \{0\}$
Combining Both the proof	$\mathbb{V} = \mathbb{W}_1 + \mathbb{W}_2$ $\mathbb{W}_1 \cap \mathbb{W}_2 = \{0\}$

	From the above two condition we can say that $\mathbb{V}$ is the direct sum of subspaces $\mathbb{W}_1$ and $\mathbb{W}_2$. Hence it is represented as $\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2$ Hence Proved.

TABLE 2: Solution Table