Assignment 18

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1 Problem UGC JUNE 2015 Q71

Let S be the set of 3x3 real matrices $\mathbf{A}$ with

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.0.1}$$

Then the set contains:-

1) a Nilpotent Matrix
2) a matrix of rank one
3) a matrix of rank two
4) a non-zero skew symmetric matrix.

2 Essential Framework Required to solve problem

Proof 1 $Rank(\mathbf{A}) = Rank(\mathbf{A}^T\mathbf{A})$	Let $\mathbf{A}x$=0 and $\mathbb{N}(\mathbf{A})$ is the null space of $\mathbf{A}$ Then $\mathbf{A}^T\mathbf{A}$x=0 which means $\mathbb{N}(\mathbf{A}) \subset \mathbb{N}(\mathbf{A}^{\mathbb{T}}\mathbf{A})$ Thus if $\mathbf{A}^T\mathbf{A}$x=0 ,then $x^T\mathbf{A}^T\mathbf{A}x = 0 \implies \|\mathbf{A}x\| = 0$ Which means $\mathbf{A}x = 0$ thus $\mathbb{N}(\mathbf{A}^{\mathbb{T}}\mathbf{A}) \subset \mathbb{N}(\mathbf{A})$ From the Above two condition we can say that $N(\mathbf{A}^T\mathbf{A}) = \mathbb{N}(\mathbf{A})$ $rank(\mathbf{A}) = n - \mathbb{N}(\mathbf{A})$ $rank(\mathbf{A}) = rank(\mathbf{A}^T\mathbf{A})$ Hence Proved.
Proof 2 Rows($\mathbf{A}^T\mathbf{A}$)=Rows($\mathbf{A}$)	Suppose $\mathbf{A} = \begin{pmatrix} \mathbf{a_1} & \dots & \mathbf{a_n} \end{pmatrix}$ where $\mathbf{a_i}$ is the column vector of $\mathbf{A}$ $\mathbf{A}^T\mathbf{A} = \mathbf{A}^T\begin{pmatrix} \mathbf{a_1} & \dots & \mathbf{a_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A}^T\mathbf{a_1} & \dots \mathbf{A}^T\mathbf{a_n} \end{pmatrix}$ For each column of $\mathbf{A}^T\mathbf{A}$

	$\mathbf{A}^T\mathbf{a_i} = \begin{pmatrix}\mathbf{b_1} & \dots\mathbf{b_n}\end{pmatrix}\mathbf{a_i}$where $\mathbf{b_i}$ is the column vector of $\mathbf{A}^T$ and Row of $\mathbf{A}$ $= \begin{pmatrix}\mathbf{b_1} & \dots\mathbf{b_n}\end{pmatrix}\begin{pmatrix}a_{i1}\\ \vdots \\ a_{in}\end{pmatrix} = \sum_{j=1}^{n} a_{ij}b_j$ So column of $\mathbf{A}^T\mathbf{A}$ is the linear combination of rows of $\mathbf{A}$. Since rank($\mathbf{A}^T$)=rank($\mathbf{A}$) so, Row($\mathbf{A}^T\mathbf{A}$) =Column($\mathbf{A}^T\mathbf{A}$)=Row($\mathbf{A}$) Hence Proved.

TABLE 1: Proofs

3 Solution

Option 1 Nilpotent Matrix check	From Proof 2,Set S contained a set of matrix whose First Column is Non-zero. $S \in \text{Set}\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$ Given $\mathbf{A}^T\mathbf{A} = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$ So the only matrix $\mathbf{A}$ which satisfy $\mathbf{A}^T\mathbf{A} = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$, $\mathbf{A}^2 = 0$ such that $\mathbf{A} \in S$ $\mathbf{A} = \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} \in S$ $\mathbf{A}^T\mathbf{A} = \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$ $\mathbf{A}^2 = \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$ which is a nilpotent matrix Option 1 is correct.
Option 2	In Proof 1 we already prove that $Rank(\mathbf{A}) = Rank(\mathbf{A}^T\mathbf{A})$

matrix of rank one check	Since the $Rank(\mathbf{A}^T\mathbf{A}) = 1$ so the $Rank(\mathbf{A}) = 1$ There fore Set S always contains only Rank 1 matrices. Hence Option 2 is correct.
Option 3 matrix of rank two check	Since set S contain only rank 1 matrices and none of rank 2 matrices as already proved above therefore Option 3 is incorrect.
Option 4 non-zero skew . symmetric matrix check	Proved by contradiction Assume Rank of $\mathbf{A}$ is 1 so $\mathbf{A}$ can be written as $\mathbf{A} = \mathbf{u}\mathbf{v}^T$ for any non-zero Columns vectors $\mathbf{u}$, $\mathbf{v}$ with n entries. If A is skew symmetric,we have:- $\mathbf{A}^T = -\mathbf{A}$ $(\mathbf{u}\mathbf{v})^T = -\mathbf{u}\mathbf{v}^T \ \mathbf{v}\mathbf{u}^T = -\mathbf{u}\mathbf{v}^T$ The Column space of these matrices is same.The column space of $\mathbf{v}\mathbf{u}^T$ is span of $\mathbf{v}$,where as the column space of $\mathbf{u}\mathbf{v}^T$ is the span of $\mathbf{u}$, So we must have $\mathbf{v} = k\mathbf{u}$ for some $k \in \mathbb{R}$.So the equation becomes $k\mathbf{u}\mathbf{u}^T = -k\mathbf{u}\mathbf{u}^T$ and since $\mathbf{u} \neq 0$;We can conclude that k=0,which means $\mathbf{v} = 0$ therefore $\mathbf{A} = 0$. This Contradicts our assumption that $\mathbf{A}$has rank 1. Thus real skew symmentric matrix can never have rank=1. Hence option 4 is incorrect.
Answers	Option 1 and Option 2 are correct.

TABLE 2: Solution Table