Assignment 16

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1 Problem Hoffman Pg 242 Q7

Find the minimal polynomials and the rational forms of the following real matrices

1) $\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$
2) $\begin{pmatrix} c & 0 & -1 \\ 0 & c & 1 \\ 1 & 1 & c \end{pmatrix}$
3) $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

2 Theorems

Theorem 1	A Rational canonical form is a matrix $\mathbf{R}$ that is Direct sum of companion matrix. $\mathbf{R} = \mathbf{C}(\mathbf{p_1}) \oplus \cdots \oplus \mathbf{C}(\mathbf{p_r})$ $$\mathbf{R} = \begin{pmatrix} \mathbf{C}(\mathbf{p_1}) & 0 & 0 & \dots & 0 \\ 0 & \mathbf{C}(\mathbf{p_2}) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{C}(\mathbf{p_r}) \end{pmatrix} \quad (2.0.1)$$ where $\mathbf{C}(\mathbf{p_i})$ is the k_i x k_i companion matrix of p_i where polynomial $p_1, p_2 \dots p_r$ are called invariant factors for Given Matrix .Where k_i denotes the degree of annihilator of p_i. This representation is called rational form.
Theorem 2	If $p_i(x) = x + a_0$ then its companion matrix $\mathbf{C}(\mathbf{p})$ is 1 x 1 matrix as $\left(-a_0\right)$. If $k_i \geq 2$ then $p(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ then its companion matrix is $$\mathbf{C}(\mathbf{p_i}) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ 0 & 0 & 1 & \dots & 0 & -a_3 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -a_{k-1} \end{pmatrix} \quad (2.0.2)$$

TABLE 1: Illustration of theorem.

3 Solution

Given part 1	$\mathbf{A} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$
Characteristics and Minimal Polynomial	Characteristics polynomial of the matrix is $det(x\mathbf{I} - \mathbf{A})$ $\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} (x) & 1 & 1 \\ -1 & (x) & 0 \\ 1 & 0 & (x) \end{vmatrix} = x\left(x^2\right) - 1\,(-x) - x = x^3$ Characteristic Polynomial $= x^3$ Minimal Polynomial can be x, x^2 or x^3 of lowest degree satisfying $p(\mathbf{A}) = 0$ Let take $p\,(x) = x \implies p(\mathbf{A}) = \mathbf{A} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \neq 0$ Let take $p\,(x) = x^2 \implies p(\mathbf{A}) = \mathbf{A}^2 = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \neq 0$ Take $p\,(x) = x^3 \implies p\,(\mathbf{A}) = \mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$ Thus minimal polynomial $p\,(x) = x^3$.
Companion Matrix and Rational Form	Since Characteristics polynomial=Minimal polynomial=Invariant factors $p\,(x) = x^3 + 0x^2 + 0x + 0$ So Companion Matrix is of dimention 3x3 and from theorem 2 $\mathbf{C}(\mathbf{p}) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ Since there is only one minimal polynomial of degree 3 which is equal to characteristics equation therefore Rational matix=companion matrix

	$\mathbf{R} = \mathbf{C}(\mathbf{p}) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ which is in rational form.
Given Part 2	$\mathbf{A} = \begin{pmatrix} c & 0 & -1 \\ 0 & c & 1 \\ 1 & 1 & c \end{pmatrix}$
Characteristics and Minimal Polynomial	Characteristics polynomial of the matrix is $det(x\mathbf{I} - \mathbf{A})$ $\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} (x-c) & 0 & 1 \\ 0 & (x-c) & -1 \\ -1 & -1 & (x-c) \end{vmatrix}$ $= (x-c)\left((x-c)^2 + 1\right) - (x-c) = (x-c)^3$ Characteristic Polynomial $= (x-c)^3$ Minimal Polynomial can be $(x-c), (x-c)^2$ or $(x-c)^3$ of lowest degree satisfying $p(\mathbf{A}) = 0$ Let take $p(x) = (x-c) \implies p(\mathbf{A}) = \mathbf{A} - c\mathbf{I} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \neq 0$ Let take $p(x) = (x-c)^2 \implies p(\mathbf{A}) = (\mathbf{A} - c\mathbf{I})^2 = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$ Take $p(x) = (x-c)^3 \implies p(\mathbf{A}) = (\mathbf{A} - c\mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$ Thus minimal polynomial $p(x) = (x-c)^3$.
Companion Matrix and Rational Form	Since Characteristics polynomial=Minimal polynomial=Invariant factors $p(x) = x^3 - 3cx^2 + 3c^2x - c^3$ So Companion Matrix is of dimention 3x3 and from theorem 2

	$\mathbf{C(p)} = \begin{pmatrix} 0 & 0 & c^3 \\ 1 & 0 & -3c^2 \\ 0 & 1 & 3c \end{pmatrix}$ Since there is only one minimal polynomial of degree 3 which is equal to characteristics equation therefore Rational matix=companion matrix $\mathbf{R} = \mathbf{C(p)} = \begin{pmatrix} 0 & 0 & c^3 \\ 1 & 0 & -3c^2 \\ 0 & 1 & 3c \end{pmatrix}$ which is in rational form.

Given part 3	$\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$
Characteristics and Minimal Polynomial	Characteristics polynomial of the matrix is $det(x\mathbf{I} - \mathbf{A})$ det$(x\mathbf{I} - \mathbf{A})$= $\begin{vmatrix} (x - \cos\theta) & -\sin\theta \\ \sin\theta & (x - \cos\theta) \end{vmatrix}$ $=x^2 - 2\cos\theta x + 1$ Characteristic Polynomial =Minimal Polynomial=$x^2 - 2\cos\theta x + 1$
Companion Matrix and Rational Form	Since Characteristics polynomial=Minimal polynomial=Invariant factors $p(x) = x^2 - 2\cos\theta x + 1$ So Companion Matrix is of dimention 2x2 and from theorem 2 $\mathbf{C(p)} = \begin{pmatrix} 0 & -1 \\ 1 & 2\cos\theta \end{pmatrix}$ Since there is only one minimal polynomial of degree 3 which is equal to characteristics equation therefore Rational matix=companion matrix $\mathbf{R} = \mathbf{C(p)} = \mathbf{C(p)} = \begin{pmatrix} 0 & -1 \\ 1 & 2\cos\theta \end{pmatrix}$ which is in rational form.

TABLE 2: Solution Table