No (rectangular parallelepiped) body is perfect

"All in all I'm just another brick in the wall."

This is yet another try of mine to explain why I believe that no perfect cuboid exists. A cuboid is a rectangular parallelepiped where all edge and all face diagonal lengths are integer numbers. See Fig.1 on the next page.

1. What is a perfect cuboid?

A perfect cuboid is a rectangular parallelepiped where not only all edge and all face diagonal lengths, but also the space diagonal's length are integer numbers.

Many years of (computerized) searching for a solution did not yield a result and there is neither a proof nor a disproof for the existence of a perfect cuboid until today.

2. What are Pythagorean Triples?

Pythagoras gave proof of the well known $a^2+b^2=c^2$.

The triple of numbers a , b , and c is a **Pythagorean Triple** (*PT*). The numbers in the triple are the lengths of the **two cathetus** and the **hypotenuse** of a rectangular triangle.

In case the three numbers have no common divisor this triple is a **Primitive Pythaogrean Triple** (*PPT*). Its numbers are mutually prime.

3. Two Axioms for Pythagorean Triples

A1 In every *PPT* the one cathetus is odd, the other cathetus is even, and the hypotenuse is odd.

A2 Every *PT* is in exactly one way the multiple of a *PPT* and an integer factor.

4. The face triangles of the cuboid

The three *PT* on the faces of the cuboid are a,b,d , a,c,e , and b,c,f . They are multiples of the three *PPT* a_1,b_1,d_1 , a_2,c_1,e_1 , and b_2,c_2,f_1 and a factor $F\in\mathbb{N}$ in this way

$$a=F_1a_1 \quad b=F_1b_1 \quad d=F_1d_1$$

$$a=F_2a_2 \quad c=F_2c_1 \quad e=F_2e_1$$

$$b=F_3b_2 \quad c=F_3c_2 \quad f=F_3f_1$$

See Fig. 2 and Fig. 3 on the following page.

Fig. 1

Cuboid in 3D

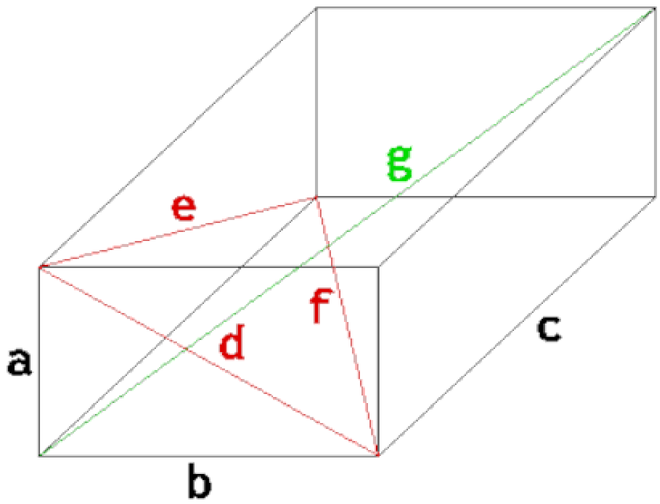


Fig. 2

Cuboid faces flattened

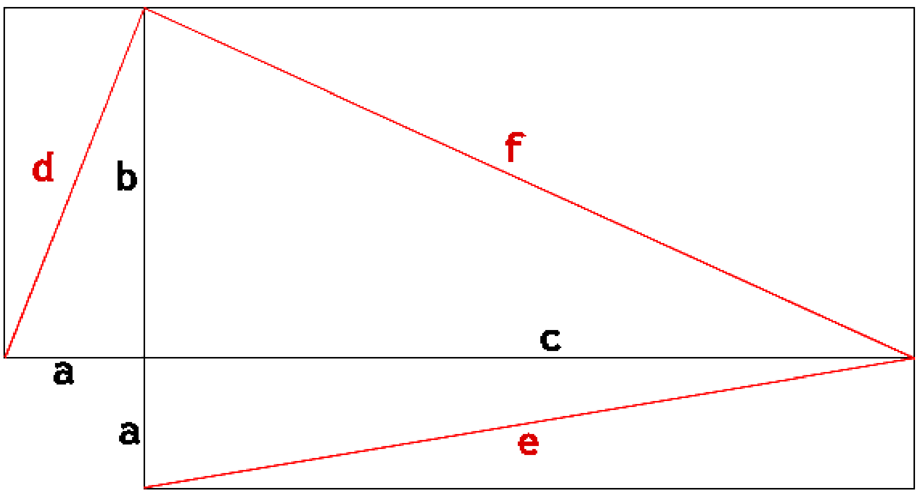
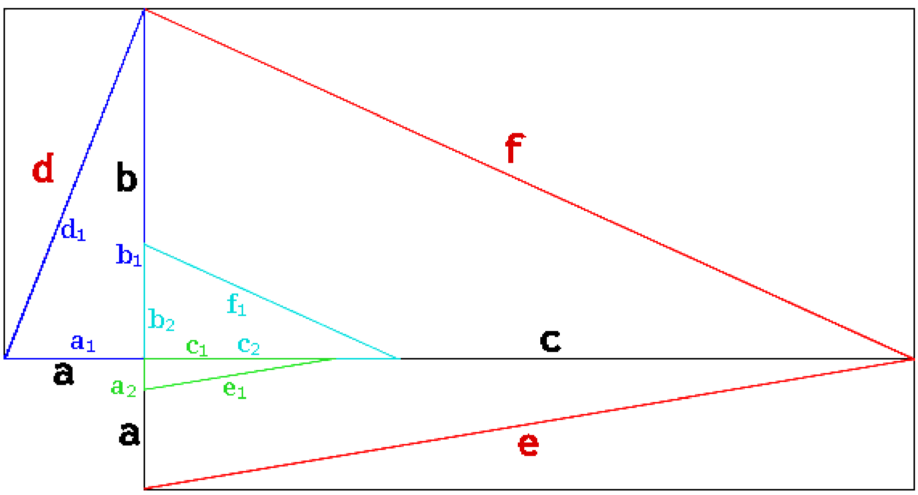


Fig. 3

Cuboid faces with their PPT



5. Lemma

The edges of every cuboid are each the multiples of two cathetus of two distinct of the three PPT on the faces. This are the three tuples $a_1 a_2$, $b_1 b_2$, $c_1 c_2$. In every cuboid one of the tuples has two odd numbers, one has two even numbers, and the third one has one odd and one even number.

6. Proof

Assume there exists a smallest solution for a cuboid where the three tuples $a_1 a_2$, $b_1 b_2$, $c_1 c_2$ each have one odd an one even number. As a result each of the cuboid's edges a , b , and c as well as the face diagonals d , e , and f lengths would share a common factor of 2. This means the smallest solution for a cuboid would be evenly divisble by 2 and the result be a yet smaller cuboid. This is absurd and so the assumption must be wrong.

7. Consequence for the existence of a perfect cuboid

As a direct consequence of this lemma and proof, a perfect cuboid can not exist. One of its edges is the multiple of an odd number – in two ways. The opposite face diagonal is always a multiple of an odd number because it is the hypotenuse of a *PT*. According to axiom A1 there is no *PPT* with two odd cathetus and thus there is, according to axiom A2, no *PT* where the space diagonal could be the integer hypotenuse.

This consequence can be also stated as:

The diophantine equation $A^2+B^2+C^2=D^2$ does not have a solution.

No sum of three perfect squares is itself a perfect square.

9. The smallest non perfect cuboid

The smallest non-perfect cuboid has the edge lengths $a=44 \quad b=117 \quad c=240$

The face diagonal between edges a and b is

$$d=\sqrt{a^2+b^2}=\sqrt{44^2+117^2}=\sqrt{1936+13689}=\sqrt{15625}=125$$

The face diagonal between edges a and c is

$$e=\sqrt{a^2+c^2}=\sqrt{44^2+240^2}=\sqrt{1936+57600}=\sqrt{59536}=244$$

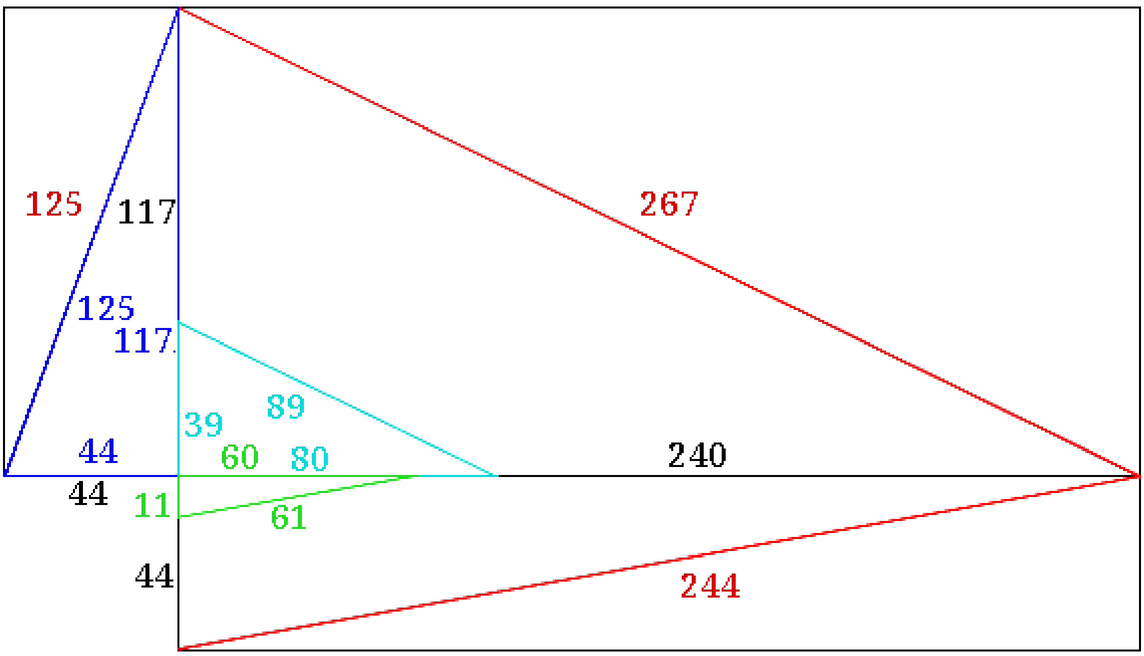
The face diagonal between edges b and c is

$$f=\sqrt{b^2+c^2}=\sqrt{117^2+240^2}=\sqrt{13689+57600}=\sqrt{71289}=267$$

The length of the space diagonal is

$$g=\sqrt{a^2+b^2+c^2}=\sqrt{44^2+117^2+240^2}=\sqrt{1936+13689+57600}=\sqrt{73225}=267.601182\ldots$$

The reason that the space diagonal is not an integer number is because of the arrangement of the *PPT* which are the base for the three *PT*. Always one edge of the cuboid is the multiple of two odd cathetus of two *PPT* and can thus not be the even cathetus of the *PPT* where the other cathetus is the face diagonal on the opposite *PT*, which is always the multiple of an odd number anyway.



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