Fermat‘s last theorem

Pierre de Fermat was a french mathematician and jurist. His most famous and long time unproved theorem says that the diophantine equation $a^n+b^n=c^n$ with $a,b,c\in\mathbb{N}$ does not have solutions for natural numbers $n>2$.

On the margin of his issue of the *Arithmetika* of Diophant Fermat scribbled the note:

> “Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duas ejusdem nominis fas est dividere: cujus rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.”
>
> „It is impossible to divide a cube into two cubes, or a fourth power into two fourth powers, or in general any power greater than 2 into two powers of that same exponent: I have a truly simple proof for this yet the margin here is too small to make it fit.“

How may this proof have looked like? Perhaps Fermat thought about or meant something like the following, geometric approach to the problem…

Cubes are (also) hexagonal pyramids

Every cube with a volume of a natural (integer) number can also be imagined as a hexagonal pyramid. Every layer of elements of such a pyramid, starting with 1 element at the top, is equal to the difference of the previous elements to the next natural number‘s third power.

The pyramid for $2^3=8$ thus consists of two layers with $1+7$ elements, the pyramid for $3^3=27$ has $1+7+19$ elements, the one for $4^3=64$ has $1+7+19+37$ elements, etc.

Obviously in every layer there are multiples of 6 elements arranged around a central element: $1=1+0\times6$, $7=1+1\times6$, $19=1+3\times6$, $37=1+6\times6$, etc.

The sequence of naturl numbers $1,7,19,37,\ldots$ is also known as the *Hex numbers* or *Hexagonal centered numbers*.

Addition of two hexagonal pyramids

The addition of two such pyramids, consisting of imagined elements of unit volume, can be understood as *imposing* the bigger above the smaller (or equally big) pyramid, which shifts up the elements above the pyramid. When adding a bigger pyramid to the smallest cube‘s pyramid 1, the result is a pyramid with a single element on the top of the uppermost layer. The addition of the next greater cube‘s (8) pyramid leads to the bigger pyramid *wearing a cap* of shifted up elements – perhaps the game trap-*the*-*cap* helps imagining that. This c*ap* can never reach down to the bottom layer of the bigger pyramid, because we add the smaller (or equally big) pyramid *below* the bigger one.

Since every cube‘s number of elements is equivalent to a regular, complete hexagonal pyramid, and the sum of two of them can never be a complete hexagonal pyramid again, there is as a consequence no cube which is the sum of two cubes.

Powers larger than three

This geometric approach to the problem can similarly be used for every power larger than 3 as well. Also fourth powers can be seen as pyramids with elements in layers, starting at the top with a single element and the layers below consisting of elements which are equal to the difference to the next natural number‘s fourth power. The start of the sequence of these numbers is: $1, 15, 65, 175, \ldots$ which is known as the sequence of *rhombic dodecahedral numbers*.

You can add this type of pyramids in the same way, by imposing the bigger pyramid above the smaller one, shifting up the elements from the inside. The results are comparable to the ones observed when adding hexagonal pyramids. There’s always a *cap* which makes the result becoming something which is not a regular, complete *rhombic dodecahedral pyramid*.

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