MA108 Tutorial Solutions

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Note : The questions till Tutorial 2, Question 9 are from the old tutorial sheets, while the ones that follow are from the new ones.

1 Tutorial Sheet 1

1. Classify the following equations (order, linear or non-linear):

   (i) $\frac{d^3y}{dx^3} + 4(\frac{dy}{dx})^2 = y$

   Order : 3, non-linear

   (ii) $\frac{dy}{dx} + 2y = \sin x$

   Order : 1, linear

   (iii) $y\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$

   Order : 2, non-linear

   (iv) $\frac{d^4y}{dx^4} + (\sin x)\frac{dy}{dx} + x^2y = 0$

   Order : 4, linear

   (v) $(1+y^2)\frac{d^2y}{dt^2} + t\frac{d^6y}{dt^6} + y = e^t$

   Order : 6, non-linear

2. Formulate the differential equtions represented by the following functions by eliminating the arbitrary constants a, b and c:

   (i) $y = ax^2$

   $\frac{y}{x^2} = a$

   Differentiating both sides wrt x,

   $\frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = 0$

   (ii) $y - a^2 = a(x-b)^2$

   Differentiating both sides wrt x,

   $\frac{dy}{dx} = 2a(x-b)$

   $\frac{1}{(x-b)}\frac{dy}{dx} = 2a$

Differentiating both sides wrt x,

$\frac{1}{(x-b)}\frac{d^2y}{dx^2} - \frac{1}{(x-b)^2}\frac{dy}{dx} = 0$

$\frac{d^2y}{dx^2} = \frac{dy}{dx}\frac{1}{(x-b)}$

$x - b = \frac{\frac{dy}{dx}}{\frac{d^2y}{dx^2}}$

$a = \frac{1}{2(x-b)}\frac{dy}{dx} = \frac{1}{2}\frac{d^2y}{dx^2}$

Substituting a and $x-b$ in original equation,

$y - \frac{1}{4}(\frac{d^2y}{dx^2})^2 = \frac{1}{2}\frac{d^2y}{dx^2}(\frac{\frac{d^2y}{dx^2}}{\frac{dy}{dx}})^2$

(iii) $x^2 + y^2 = a^2$

Differentiating both sides wrt x,

$2x\frac{dy}{dx} + 2y = 0$

(iv) $(x-a)^2 + (y-b)^2 = a^2$ ... (1)

Differentiating both sides wrt x,

$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$ ... (2)

$\frac{(x-a)}{(y-b)} = -\frac{dy}{dx}$ ... (3)

Differentiating (2) wrt x,

$1 + (y-b)\frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 = 0$

$y - b = -\frac{1+(\frac{dy}{dx})^2}{\frac{d^2y}{dx^2}}$

Substituting $(y-b)$ in (2),

$a = x + (y-b)\frac{dy}{dx}$

$a = x - \frac{1+(\frac{dy}{dx})^2}{\frac{d^2y}{dx^2}}\frac{dy}{dx}$

Dividing (1) on both sides by $(y-b)^2$,

$\frac{(x-a)^2}{(y-b)^2} + 1 = \frac{a^2}{(y-b)^2}$

$(\frac{dy}{dx})^2 + 1 = \frac{(x - \frac{1+(\frac{dy}{dx})^2}{\frac{d^2y}{dx^2}}\frac{dy}{dx})^2}{(\frac{1+(\frac{dy}{dx})^2}{\frac{d^2y}{dx^2}})^2}$

(v) $y = a\sin x + b\cos x + a$ ... (1)

Differentiating both sides wrt x,

$\frac{dy}{dx} = a\cos x - b\sin x$ ... (2)

Multiply (1) by $\sin x$ and (2) by $\cos x$ and add,

$y\sin x + \frac{dy}{dx}\cos x = a\sin^2 x + a\cos^2 x + a\sin x$

$y\sin x + \frac{dy}{dx}\cos x = a(1+\sin x)$

$a = \frac{y\sin x + \frac{dy}{dx}\cos x}{1+\sin x}$ ...(3)

Differentiating both sides of (2) wrt x

$\frac{d^2y}{dx^2} = -(a\sin x + b\cos x)$ ... (4)

Adding (1) and (4),

$y + \frac{d^2y}{dx^2} = a$

From equation (4),

$y + \frac{d^2y}{dx^2} = \frac{y\sin x + \frac{dy}{dx}\cos x}{1+\sin x}$

(vi) $y = a(1-x^2) + bx + cx^3$

Differentiating both sides wrt x

$\frac{dy}{dx} = -2ax + b + 3cx^2$ ...(1)

Differentiating both sides wrt x

$\frac{d^2y}{dx^2} = -2a + 6cx$ ... (2)

Differentiating both sides wrt x

$\frac{d^3y}{dx^3} = 6c$ ...(3)

From (2) and (3),

$\frac{d^2y}{dx^2} = -2a + x\frac{d^3y}{dx^3}$ ...(4)

From (3), (4) and (1),

$\frac{dy}{dx} = x(\frac{d^2y}{dx^2} - x\frac{d^3y}{dx^3}) + b + \frac{x^2}{2}\frac{d^3y}{dx^3}$

From all equations,

$y = -\frac{1}{2}(\frac{d^2y}{dx^2} - x\frac{d^3y}{dx^3})(1-x^2) + (\frac{dy}{dx} - x(\frac{d^2y}{dx^2} - x\frac{d^3y}{dx^3}) - \frac{x^2}{2}\frac{d^3y}{dx^3})x + (\frac{1}{6}\frac{d^3y}{dx^3})x^3$

(vii) $y = cx + f(c)$

Differentiating both sides wrt x

$\frac{dy}{dx} = c$

$y = \frac{dy}{dx} + f(\frac{dy}{dx})$

3. Solve the equation $x^3(\sin y)y' = 2$. Find the particular solution such that $y(x) \to \frac{\pi}{2}$ as $x \to +\infty$.

$x^3(\sin y)\frac{dy}{dx} = 2$

$(\sin y)dy = \frac{2}{x^3}dx$

$\int(\sin y)dy = \int\frac{2}{x^2}dx$

$-\cos y + c = -\frac{2}{x}$

$\frac{2}{x} - \cos y + c = 0$

This is an implicit solution for the given differential equation.

As $x \to \infty$, $\frac{2}{x} \to 0$ and as $y(x) \to \frac{\pi}{2}$, $\cos y \to 0$

$\implies c = 0$

$\frac{2}{x} - \cos y = 0$ is the required particular solution.

4. Prove that a curve with the property that all its normals pass through a point is a circle.

Let (x_0, y_0) be a point on the curve

Equation of normal at (x_0, y_0) is $-\frac{x_0 - x}{y_0 - y} = \frac{dy}{dx}$

Let the point (a, b) always satisfy this equation, i.e. $(x_0, y_0) \equiv (x, y)$

$\implies -\frac{x-a}{y-b} = \frac{dy}{dx}$

$\implies -(x-a)dx = (y-b)dy$

$\implies \int -(x-a)dx = \int (y-b)dy$

$\implies -(x-a)^2 + c = (y-b)^2$

$\implies (x-a)^2 + (y-b)^2 = c$

This is a real curve iff $c = r^2$, where r is a real number.

Therefore, it is the equation of a circle with center at (a, b).

5. Find the values of m for which

   (a) $y = e^{mx}$ is a solution of

      i. $y'' + y' - 6y = 0$

         Substituting $y = e^{mx}$ in the given equation,

         $m^2e^{mx} + me^{mx} - 6e^{mx} = 0$

         $m^2 + m - 6 = 0$

         $(m+3)(m-2) = 0$

         $\implies m = -3$ or $m = 2$

      ii. $y''' - 3y'' + 2y' = 0$

         Substituting $y = e^{mx}$ in the given equation,

         $m^3e^{mx} - 3m^2e^{mx} + 2me^{mx} = 0$

         $m^3 - 3m^2 + 2m = 0$

         $m(m-2)(m-1) = 0$

         $\implies m = 0$ or $m = 2$ or $m = 1$

   (b) $y = x^m$ for $x > 0$ is a solution of

      i. $x^2y'' - 4xy' + 4y = 0$

         Substituting $y = x^m$ in the given equation,

         $x^2m(m-1)x^{m-2} - 4xmx^{m-1} + 4x^m = 0$

         Dividing by x^m,

         $m(m-1) - 4m + 4 = 0$

         $\implies m = 4$ or $m = 1$

      ii. $x^2y''' - xy'' + y' = 0$

         Substituting $y = x^m$ in the given equation,

         $x^2m(m-1)(m-2)x^{m-3} - xm(m-1)x^{m-2} + mx^{m-1} = 0$

         Dividing by x^{m-1},

         $m(m-1)(m-2) - m(m-1) + m = 0$

         $m(m^2 - 3m + 2 - m^2 + m + m) = 0$

$m(2-m)=0$

$\implies m=0$ or $m=2$

6. For each of the following linear differential equations verify that the function given in brackets is a solution of the differential equation.

   Warning : Incomplete.

   (i) $y''+4y=5e^x+3\sin x$ $(y=a\sin 2x+b\cos 2x+e^x+\sin x)$

   (ii) $y''-5y'+6y=0$ $(y_1=e^{3x}, y_2=e^{2x}, c_1y_1+c_2y_2)$

   (iii) $y'''+6y''+11y'+6y=e^{-2x}$ $(y=ae^{-x}+be^{-2x}+ce^{-3x}-xe^{-2x})$

   (iv) $y'''+8y=9e^x=65\cos x$ $(y=ae^{-2x}+e^x(b\cos\sqrt{3}x+c\sin\sqrt{3}x)+8\cos x-\sin x+e^x)$

7. Let ϕ_i be a solution of $y'+ay=b_i(x)$ for i=1,2. Show that $\phi_1+\phi_2$ satisfies $y'+ay=b_1(x)+b_2(x)$. Use this result to find the solutions of $y'+y=\sin x+3\cos 2x$ passing through the origin.

   The proof follows immediately from the fact that the equation is linear and hence,

   $(\phi_1+\phi_2)'+a(\phi_1+\phi_2)=(\phi_1)'+(\phi_2)'+a\phi_1+a\phi_2$

   $=(\phi_1'+a\phi_1)+(\phi_2'+a\phi_2)$

   $=b_1(x)+b_2(x)$

   Let $b_1=\sin x$ and $b_2=3\cos 2x$

   For ϕ_1, we have,

   $\phi_1'+a\phi_1=\sin x$

   Multiplying both sides by e^{ax},

   $e^{ax}\phi_1'+ae^{ax}\phi_1=e^{ax}\sin x$

   $(\phi_1e^{ax})'=e^{ax}\sin x$

   $\phi_1e^{ax}=\int e^{ax}\sin x\ dx=e^{ax}\frac{(a\sin x-\cos x)}{a^2+1}+c$

   $\phi_1=\frac{a\sin x-\cos x}{a^2+1}+c$

   Similarly,

   $\phi_2=\frac{3(a\cos 2x+2\sin 2x)}{a^2+4}$

   The required solutions are $\phi_1+\phi_2$

8. Obtain the solution of the following differential equations:

   (i) $(x^2+1)dy+(y^2+4)dx=0$; $y(1)=0$

   $\frac{dy}{y^2+4}=-\frac{dx}{x^2+1}$

   $\int\frac{dy}{y^2+4}=-\int\frac{dx}{x^2+1}$

   $\frac{1}{2}\tan^{-1}\frac{y}{2}=-\tan^{-1}x+c$

   $\frac{1}{2}\tan^{-1}\frac{y}{2}+\tan^{-1}x=c$

Putting $x = 1$, $y = 0$,

$c = \frac{\pi}{4}$ is the constant for the particular solution.

(ii) $y' = y \cot x$; $y(\frac{\pi}{2}) = 1$

$\frac{dy}{dx} = \frac{y}{\tan x}$

$\frac{dy}{y} = \frac{\cos x}{\sin x} dx$

$\int \frac{dy}{y} = \int \frac{\cos x}{\sin x} dx$

$\ln y = \ln \sin x + c$

$y = c \sin x$

Putting $x = \frac{\pi}{2}$, $y = 1$,

$c = 1$ is the constant for the particular solution.

(iii) $y' = y(y^2 - 1)$, with $y(0) = 2$ or $y(0) = 1$ or $y(0) = 0$

$\frac{dy}{dx} = y(y^2 - 1)$

$\int \frac{dy}{y(y-1)(y+1)} = \int dx$

Using partial fractions,

$\int (\frac{A}{y} + \frac{B}{y-1} + \frac{C}{y+1}) dy = \int dx$

$A = -1, B = C = \frac{1}{2}$

$\implies -\ln y + \frac{1}{2} \ln |y - 1| + \frac{1}{2} \ln |y + 1| = x + c$

On putting values for (x, y) as $(0, 2), (0, 1)$ and $(0, 0)$, we get the respective particular solutions.

(iv) $(x + 2)y' - xy = 0$; $y(0) = 1$

$(x + 2)\frac{dy}{dx} = xy$

$\frac{dy}{y} = \frac{x}{x+2} dx$

$\int \frac{dy}{y} = \int (\frac{x}{x+2}) dx$

$\int \frac{dy}{y} = \int (1 - \frac{2}{x+2}) dx$

$\ln y = x - 2 \ln |x + 2| + c$

Putting $x = 0$, $y = 1$,

$0 = -2 \ln 2 + c$

$c = \ln 4$ is the constant for the particular solution.

(v) $y' + \frac{y-x}{y+x} = 0$; $y(1) = 1$

Substituting $y = vx$, $y' = v'x + v$,

$v'x + v + \frac{vx-x}{vx+x} = 0$

$v'x + v + \frac{v-1}{v+1} = 0$

$\frac{dv}{dx} x = \frac{1-v}{1+v} - v$

$\frac{dv}{dx} x = \frac{1-2v-v^2}{1+v}$

$\frac{1+v}{1-2v-v^2} dv = \frac{1}{x} dx$

$\int \frac{1+v}{1-2v-v^2} dv = \int \frac{1}{x} dx$

$-\frac{1}{2} \ln |1-2v-v^2| = \ln x + c$

$\frac{1}{\sqrt{|1-2v-v^2|}} = cx$

$\frac{x}{\sqrt{|x^2-2xy-y^2|}} = cx$

$\frac{}{\sqrt{|x^2-2xy-y^2|}} = c$

Putting $x = 1$, $y = 1$,

$c = \frac{1}{\sqrt{2}}$ is the constant for the particular solution.

(vi) $y' = (y-x)^2$; $y(0) = 2$

Substituting $v = y - x$, $v' = y' - 1$,

$v' + 1 = v^2$

$\int \frac{dv}{v^2-1} = \int dx$

$\frac{1}{2} \ln(\frac{v-1}{v+1}) = x + c$

$\ln(c_1 \frac{v-1}{v+1}) = 2x$

$\frac{v-1}{v+1} = c_2 e^{2x}$

$\frac{y-x-1}{y-x+1} = c_2 e^{2x}$

Putting $x = 0$, $y = 2$,

$\frac{1}{3} = c_2$ is the constant for the particular solution.

(vii) $2(y \sin 2x + \cos 2x)dx = \cos 2x \ dy$; $y(\pi) = 0$

Substituting $v = y \cos 2x$, $dv = dy \cos 2x - 2y \sin 2x \ dx$,

$2(y \sin 2x + \cos 2x)dx = dv + 2y \sin 2x \ dx$

$\int 2 \cos 2x \ dx = \int dv$

$\sin 2x = v + c$

$\sin 2x = y \cos 2x + c$

Putting $x = \pi$, $y = 0$,

$c = 0$ is the constant for the particular solution.

(viii) $y' = \frac{1}{(x+1)(x^2+1)}$

$\frac{dy}{dx} = \frac{1}{(x+1)(x^2+1)}$

$\int dy = \int \frac{dx}{(x+1)(x^2+1)}$

Putting $x = \tan\theta$,

$y = \int \frac{\sec^2\theta}{(\tan\theta+1)(\tan^2\theta+1)} d\theta$

$y = \int \frac{\cos\theta}{\sin\theta+\cos\theta} d\theta$

$y = \int \frac{1}{2}(\frac{\cos\theta+\sin\theta}{\cos\theta+\sin\theta} + \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}) d\theta$

$y = \int \frac{1}{2}(1 + \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}) d\theta$

$y = \frac{1}{2}(\theta + \ln|\cos\theta + \sin\theta|) + c$

$y = \frac{1}{2}(\tan^{-1} x + \ln|\frac{1+x}{\sqrt{1+x^2}}|) + c$

This is the general solution for the differential equation.

9. For each of the following differential equations, find the general solution (by substituting $y = vx$)

   (i) $y' = \frac{y^2 - xy}{x^2 + xy}$

   Substituting $y = vx$, $y' = v'x + v$,

   $v'x + v = v\frac{v-1}{v+1}$

   $x\frac{dv}{dx} = v(\frac{v-1}{v+1} - 1)$

   $x\frac{dv}{dx} = -\frac{2}{v+1}$

   $\int (v+1)dv = \int \frac{-2}{x}dx$

   $\frac{v^2}{2} + v = -2\ln|x| + c$

   $\frac{v^2}{2} + v = \ln c_1 x^{-2}$

   $\frac{1}{2}(\frac{y}{x})^2 + \frac{y}{x} = \ln c_1 x^{-2}$ is the general solution for the given differential equation.

   (ii) $x^2 y' = y^2 + xy + x^2$

   Substituting $y = vx$, $y' = v'x + v$,

   $x^2(v'x + v) = v^2x^2 + vx^2 + x^2$

   $v'x + v = v^2 + v + 1$

   $v'x = v^2 + 1$

   $\frac{dv}{v^2+1} = dx$

   $\int \frac{dv}{v^2+1} = \int dx$

   $\tan^{-1} v = x + c$

   $\tan^{-1} \frac{y}{x} = x + c$ is the general solution for the given differential equation.

   (iii) $xy' = y + x\cos^2(\frac{y}{x})$

   Substituting $y = vx$, $y' = v'x + v$,

   $x(v'x + v) = vx + x\cos^2 v$

   $v'x = \cos^2 v$

   $\int \frac{dv}{\cos^2 v} = \int \frac{dx}{x}$

   $\tan v = \ln|x| + c$

   $e^{\tan\frac{y}{x}} = c_1|x|$ is the general solution for the given differential equation.

   (iv) $xy' = y(\ln y - \ln x)$

   Substituting $y = vx$, $y' = v'x + v$,

   $x(v'x + v) = vx\ln v$

   $v'x = v(\ln v - 1)$

   $\int \frac{dv}{v(\ln v - 1)} = \int \frac{dx}{x}$

   $\ln|\ln v - 1| = \ln|x| + c$

$\ln v - 1 = c_1 x$ is the general solution for the given differential equation.

10. Show that the differential equation $\frac{dy}{dx} = \frac{ax+by+m}{cx+dy+n}$ where a, b, c, d, m and n are constants can be reduced to $\frac{dy}{dt} = \frac{ax+by}{cx+dy}$ if $ad - bc \neq 0$. Then find the general solution of the given equations.

    Warning : Incomplete.

    (i) $(1 + x - 2y) + y'(4x - 3y - 6) = 0$

    (ii) $y' = \frac{y-x+1}{y-x+5}$

    (iii) $(x + 2y + 3) + (2x + 4y - 1)y' = 0$

11. Solve the differential equation $\sqrt{1-y^2}\,dx + \sqrt{1-x^2}\,dy = 0$ with the conditions $y(0) = \frac{\pm 1}{2}\sqrt{3}$. Sketch the graphs of the solutions and show that they are each arcs of the same ellipse. Also show that after these arcs are removed, the remaining part of the ellipse does not satisfy the differential equation.

    $\sqrt{1-y^2}\,dx + \sqrt{1-x^2}\,dy = 0$

    Dividing by $\sqrt{1-y^2}\sqrt{1-x^2}$

    $\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{q-y^2}} = 0$

    $\int \frac{dx}{\sqrt{1-x^2}} + \int \frac{dy}{\sqrt{1-y^2}} = 0$

    $\sin^{-1} x + \sin^{-1} y = c$

    Putting $x = 0$ and $y = \frac{\sqrt{3}}{2}$,

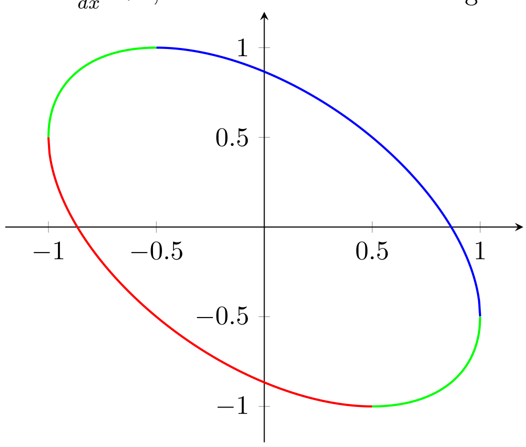
    $c_1 = \frac{\pi}{3}$

    Putting $x = 0$ and $y = -\frac{\sqrt{3}}{2}$,

    $c_2 = -\frac{\pi}{3}$

    (Where c_1 and c_2 are the constants for the two particular solutions)

    Since $\frac{dy}{dx} < 0$, we need to exclude the regions of the solution where $\frac{dy}{dx} > 0$



The blue part of the graph corresponds to $c = \frac{\pi}{3}$ and the red part corresponds to $c = -\frac{\pi}{3}$. The green part of the ellipse is not a solution to the differential equation.

12. The differential equation $y = xy' + f(y')$ is called a Clairaut equation (or Clairaut's equation). Show that the general solution of this equation is the family of straight lines $y = cx + f(c)$. In addition to these, show that it has a special solution given by $f'(p) = -x$ where $p = y'$. This special solution which does not (in general) represent one of the straight lines $y = cx + f(c)$, is called a singular solution. Hint: Differentiate the differential equation.

    $y = xy' + f(y')$

    Differentiation both sides wrt x,

    $y' = y' + xy'' + f(y')y''$

    $y''(x + f(y')) = 0$

    $\implies y'' = 0$ or $x + f(y') = 0$

    $\implies y' = c$ or $f'(p) = -x$, where $p = y'$

    Substituting $y' = c$ in Clairaut's equation,

    $y = cx + f(c)$ is the general solution

    $f'(p) = -x$, where $p = y'$ is the singular solution

13. Determine the general solutions as well as the singular solutions of the following Clairaut equations. *In each of the two examples, sketch the graphs of these solutions.*

    Warning : Incomplete.

    (i) $y = xy' + \frac{1}{y'}$

    Since the equation is in the form of a Clairaut equation with $f(x) = \frac{1}{x}$, we can substitute $y' = c$ to get the general solution.

    $y = cx + \frac{1}{c}$ is the general solution of the differential equation.

    Let $p = y'$ and $f'(p) = -x$,

    $-\frac{1}{p^2} = -x$

    $p = \frac{1}{\sqrt{x}}$

    Substitute $y' = p = \frac{1}{\sqrt{x}}$ in the equation,

    $y = \frac{x}{\sqrt{x}} + \sqrt{x}$

    $\implies y = 2\sqrt{x}$ is the singular solution of the equation.

    (ii) $y = xy' - \frac{y'}{\sqrt{1+y'^2}}$

    Since the equation is in the form of a Clairaut equation with $f(x) = -\frac{x}{\sqrt{1+x^2}}$, we can substitute $y' = c$ to get the general solution.

    $y = cx - \frac{c}{\sqrt{1+c^2}}$ is the general solution of the differential equation.

    Let $p = y'$ and $f'(p) = -x$,

    $\frac{d}{dp}(\frac{p}{\sqrt{1+p^2}}) = -x$

    $\frac{1}{(1+p^2)^{3/2}} = -x$

$p = \sqrt{\frac{1}{x^{2/3}} - 1}$

Substitute $y' = p = \sqrt{\frac{1}{x^{2/3}} - 1}$ in the equation,

$y = x\sqrt{\frac{1}{x^{2/3}} - 1} - \sqrt{1 - x^{2/3}}$

$y = -(1 - x^{2/3})^{3/2}$ is the singular solution of the equation.

14. For the parabola $y = x^2$ find the equation of its tangent at (c, c^2) and find the ordinary differential equation for this one parameter family of tangents. Identify this as a Clairaut equation. More generally, take your favorite curve and determine the ODE for the one parameter family of its tangents and verify that it is a Clairaut's equation. N.B: Exercise 13 shows that the converse is true.

$\frac{dy}{dx} = 2x$

Equation of tangent at (c, c^2) is,

$\frac{y-c^2}{x-c} = 2c$

$y = 2cx - c^2$

Put $c_1 = 2c$

$y = c_1 x - (\frac{c_1}{2})^2$

This is a solution for the Clairaut equation:

$y = xy' - (\frac{y'}{2})^2$

Similarly for other curves, the one parameter family of tangents forms a Clairaut equation.

15. In the preceding exercises, show that in each case, the envelope of the family of straight lines is also a solution of the Clairaut equation.

For the envelope of the family of lines, we have,

$F(x, y, c) = y - 2cx + c^2 = 0$

$\frac{\partial F(x,y,c)}{\partial c} = -2x + 2c = 0$

$\implies x = c, y = c^2 \implies y = x^2$, i.e. the equation of the parabola itself.

Substituting $y = x^2$ in the equation

$y = xy' - (\frac{y'}{2})^2$

$x^2 = 2x^2 - (\frac{2x}{2})^2 \implies 1 = 1$

Thus, the envelope of the family of lines satisfies the Clairaut equation.

Similar for other curves.

16. Show that the differential equation $y' - y^3 = 2x^{-3/2}$ has three distinct solutions of the form $\frac{A}{\sqrt{x}}$ but that only one of these is real valued.

Putting $y = \frac{A}{\sqrt{x}}$ in the given equation,

$-\frac{1}{2}\frac{A}{x^{3/2}} - \frac{A^3}{x^{3/2}} = \frac{2}{x^{3/2}}$

$\frac{2}{x^{3/2}} + \frac{1}{2}\frac{A}{x^{3/2}} + \frac{A^3}{x^{3/2}} = 0$

$\implies Q = 2A^3 + A + 4 = 0$

$\implies A$ can have only one real value, since $Q \to \infty$ as $A \to \infty$; $Q \to -\infty$ as $A \to -\infty$ and $\frac{dQ}{dA} > 0$, i.e. Q is always increasing.

2 Tutorial Sheet 2

1. State the conditions under which the following equations are exact.

   Warning: This solution is suspicious!

   (i) $[f(x) + g(y)]dx + [h(x) + k(y)]dy = 0$

   For the equation to be exact on $\mathbb{R}^2$, it must be closed,

   $\implies \frac{\partial f(x)+g(y)}{\partial y} = \frac{\partial h(x)+k(y)}{\partial x}$

   $\implies g'(y) = h'(x)$

   For this to be true always, both must be equal to zero.

   $\implies g'(y) = h'(x) = c$

   $\implies g(y) = cy, h(x) = cx$

   (ii) $(x^3 + xy^2)dx + (ax^2y + bxy^2)dy = 0$

   For the equation to be exact on $\mathbb{R}^2$, it must be closed,

   $\implies \frac{\partial(x^3+xy^2)}{\partial y} = \frac{\partial(ax^2y+bxy^2)}{\partial x}$

   $\implies 2xy = 2axy + by^2$

   $\implies 2x = 2ax + by$

   $\implies a = 1, b = 0$

   (iii) $(ax^2 + 2bxy + cy^2)dx + (bx^2 + 2cxy + gy^2)dy = 0$

   For the equation to be exact on $\mathbb{R}^2$, it must be closed,

   $\implies \frac{\partial(ax^2+2bxy+cy^2)}{\partial y} = \frac{\partial(bx^2+2cxy+gy^2)}{\partial x}$

   $\implies 2bx + 2cy = 2cy + 2bx$

   $\implies$ The equation is always exact.

2. Solve the following exact equations

   (i) $3x(xy - 2) + (x^3 + 2y)dy = 0$

   Let $M(x, y) = 3x(xy - 2)$ and $N(x, y) = (x^3 + 2y)$

   Let $u(x, y)$ be a function such that $\frac{\partial u(x,y)}{\partial x} = M(x, y)$ and $\frac{\partial u(x,y)}{\partial y} = N(x, y)$

   $\implies u = \int 3x(xy - 2)\ dx + c(y)$

   $\implies u = \int 3x^2y - 6x\ dx + c(y)$

   $\implies u = x^3y - 3x^2 + c(y)$

   Partially differentiating $u(x, y)$ wrt y,

$\frac{\partial u(x,y)}{\partial y} = N(x,y)$

$\implies \frac{\partial(x^3y-3x^2+c(y))}{\partial y} = x^3 + 2y$

$\implies x^3 + c'(y) = x^3 + 2y$

$\implies c'(y) = 2y$

$\implies c(y) = y^2 + c_1$

$\implies$ The solution to the differential equation is $x^3y - 3x^2 + y^2 = c_2$

(ii) $(\cos x \cos y - \cot x)dx - \sin x \sin y \; dy = 0$

Let $M(x,y) = \cos x \cos y - \cot x$ and $N(x,y) = \sin x \sin y$

Let $u(x,y)$ be a function such that $\frac{\partial u(x,y)}{\partial x} = M(x,y)$ and $\frac{\partial u(x,y)}{\partial y} = N(x,y)$

$\implies u = \int \cos x \cos y - \cot x \; dx + c(y)$

$\implies u = \sin x \cos y - \ln \sin x + c(y)$

Partially differentiating $u(x,y)$ wrt y,

$\frac{\partial u(x,y)}{\partial y} = N(x,y)$

$\implies \frac{\partial(\sin x \cos y - \ln \sin x + c(y))}{\partial y} = \sin x \sin y$

$\implies -\sin x \sin y + c'(y) = \sin x \sin y$

$\implies c'(y) = 2 \sin x \sin y$

$\implies c(y) = -2 \sin x \cos y + c_1$

$\implies$ The solution to the differential equation is $\sin x \cos y - \ln \sin x - 2 \sin x \cos y = c_2$

(iii) $e^x y(x+y)dx + e^x(x+2y-1)dy = 0$

Let $M(x,y) = e^x y(x+y) - \cot x$ and $N(x,y) = e^x(x+2y-1)$

Let $u(x,y)$ be a function such that $\frac{\partial u(x,y)}{\partial x} = M(x,y)$ and $\frac{\partial u(x,y)}{\partial y} = N(x,y)$

$\implies u = \int e^x y(x+y) \; dx + c(y)$

$\implies u = \int e^x xy + e^x y^2 \; dx + c(y)$

$\implies u = y \int xe^x \; dx + e^x y^2 + c(y)$

$\implies u = e^x y(x-1) + e^x y^2 + c(y)$

$\implies u = ye^x(x+y-1) + c(y)$

Partially differentiating $u(x,y)$ wrt y,

$\frac{\partial u(x,y)}{\partial y} = N(x,y)$

$\implies \frac{\partial(ye^x(x+y-1)+c(y))}{\partial y} = e^x(x+2y-1)$

$\implies \frac{\partial(ye^x(x-1)+y^2e^x+c(y))}{\partial y} = e^x(x+2y-1)$

$\implies e^x(x-1) + 2ye^x + c'(y) = e^x(x+2y-1)$

$\implies c'(y) = 0$

$\implies c(y) = c_1$

$\implies$ The solution to the differential equation is $ye^x(x+y-1) = c_2$

3. Determine (by inspection suitable) Integrating Factors (IF's) so that the following equations

are exact.

Warning : Incomplete.

(i) $ydx + xdy = 0$

The equation is exact, since $M_y = N_x$, hence an integrating factor is 1.

(ii) $d(e^x \sin y) = 0$

The equation is exact, with solution $e^x \sin y = c$

(iii) $dx + (\frac{y}{x})^2 dy = 0$

By inspection, multiplying both sides by the integrating factor x^2,

$x^2 dx + y^2 dy = 0$

The equation is exact, since $M_y = N_x$

(iv) $ye^{x/y}dx + (y - xe^{x/y}dy = 0$

Warning : Incomplete.

(v) $(2x + e^y)dx + xe^y dy = 0$

The equation is exact, since $M_y = N_x$, hence an integrating factor is 1.

(vi) $(x^2 + y^2)dx + xydy = 0$

Trying the integrating factor $\frac{1}{x(x^2+2y^2)}$,

$(\frac{x^2+y^2}{x(x^2+2y^2)})dx + \frac{xy}{x(x^2+2y^2)}dy = 0$

$(\frac{x^2+2y^2}{x(x^2+2y^2)} - \frac{y^2}{x(x^2+2y^2)})dx + \frac{y}{x^2+2y^2}dy = 0$

$(\frac{1}{x} - \frac{y^2}{x(x^2+2y^2)})dx + \frac{y}{x^2+2y^2}dy = 0$

$M_y = 0 - \frac{1}{x}\frac{2y(x^2+2y^2)-y^2(4y)}{(x^2+2y^2)^2}$

$M_y = -\frac{2xy}{(x^2+2y^2)^2}$

$N_x = -\frac{2xy}{(x^2+2y^2)^2}$

$\implies M_y = N_x$

Hence, $\frac{1}{x(x^2+2y^2)}$ is an integrating factor for the equation.

4. Verify that the equation $Mdx + Ndy = 0$ ... (1) can be expressed in the form

$$\frac{1}{2}(Mx + Ny)d(\ln xy) + \frac{1}{2}(Mx - Ny)d\ln(\frac{x}{y}) = 0$$

Expanding the differentials in the second equation,

$\frac{1}{2}(Mx + Ny)d(\ln xy) + \frac{1}{2}(Mx - Ny)d\ln(\frac{x}{y}) = 0$

$\frac{1}{2}(Mx + Ny)d(\ln x + \ln y) + \frac{1}{2}(Mx - Ny)d(\ln x - \ln y) = 0$

$\frac{1}{2}(Mx + Ny)(\frac{dx}{x} + \frac{dy}{y}) + \frac{1}{2}(Mx - Ny)(\frac{dx}{x} - \frac{dy}{y}) = 0$

$Mdx + Ndy = 0$

Hence, the equation can be expressed in the given form.

Hence, show that,

(i) if $Mx + Ny = 0$, then $\frac{1}{Mx-Ny}$ is an IF of (1)

Multiplying the second equation by $\frac{1}{Mx-Ny}$, putting $Mx + Ny = 0$,

$\frac{1}{2}(\frac{dx}{x} - \frac{dy}{y}) = 0$

This equation is exact, since $M_y = N_x$, hence $\frac{1}{Mx-Ny}$ is an integrating factor.

(ii) if $Mx - Ny = 0$, then $\frac{1}{Mx+Ny}$ is an IF of (1)

Multiplying the second equation by $\frac{1}{Mx+Ny}$, putting $Mx - Ny = 0$,

$\frac{1}{2}(\frac{dx}{x} + \frac{dy}{y}) = 0$

This equation is exact, since $M_y = N_x$, hence $\frac{1}{Mx+Ny}$ is an integrating factor.

(iii) if M and N are homogeneous of the same degree then $\frac{1}{Mx+Ny}$ is an IF of (1).

Multiplying the second equation by $\frac{1}{Mx+Ny}$,

$d(\ln xy) + \frac{(Mx-Ny)}{(Mx+Ny)} d\ln(\frac{x}{y}) = 0$

$d(\ln xy) + \frac{(1-\frac{Ny}{Mx})}{(1+\frac{Ny}{Mx})} d\ln(\frac{x}{y}) = 0$

Since Ny and Mx are homogeneous of the same degree, we can express $\frac{Ny}{Mx}$ purely as a function of v, where $v = \frac{y}{x}$

$d(\ln xy) + f(v)d\ln(\frac{x}{y}) = 0$

$d(\ln xy) + f(v)d\ln(\frac{1}{v}) = 0$

$d(\ln vx^2) - \frac{f(v)}{v}dv = 0$

$d(\ln v + 2\ln x) - \frac{f(v)}{v}dv = 0$

$\frac{1}{v}dv + \frac{2}{x}dx - \frac{f(v)}{v}dv = 0$

$\frac{2}{x}dx + g(v)dv = 0$, where $g(v)$ is another function of v.

This equation is exact, since $M_v = N_x$, hence $\frac{1}{Mx+Ny}$ is an integrating factor.

5. If $\mu(x, y)$ is an IF of $Mdx + Ndy = 0$ then prove that

$$M_y - N_x = N\frac{\partial}{\partial x}ln|\mu| - M\frac{\partial}{\partial y}ln|\mu|$$

$M_y - N_x = N\frac{\partial}{\partial x}ln|\mu| - M\frac{\partial}{\partial y}ln|\mu|$

$\iff M_y - N_x = \frac{N}{\mu}\frac{\partial\mu}{\partial x} - \frac{M}{\mu}\frac{\partial\mu}{\partial y}$ ... (assume $\mu \neq 0$)

$\iff M_y\mu + M\frac{\partial\mu}{\partial y}\mu = N_x\mu + N\frac{\partial\mu}{\partial x}$

$\iff (M\mu)_y = (N\mu)_x$

Which is true, since the equation $\mu Mdx + \mu Ndy = 0$ is exact since μ is an IF.

Use the relation to prove that if $\frac{1}{N}(M_y - N_x) = f(x)$ then there exists an IF $\mu(x)$ given by $\exp(\int_a^x f(t)dt)$ and if $\frac{1}{M}(M_y - N_x) = g(y)$, then there exists an IF $\mu(y)$ given by

$\exp(-\int_a^y g(t)dt)$. Further if $M_y - N_x = f(x)N - g(y)M$ then $\mu(x,y)$ $=\exp(\int_a^x f(x')dx' + \int_a^y g(y')dy')$ is an IF, where a is any constant.

Can be proved trivially by substituting μ as appropriate and showing L.H.S. = R.H.S.

Determine an IF for the following differential equations:

(i) $y(8x-9y)dx + 2x(x-3y)dy = 0$

$\frac{1}{N}(M_y - N_x) = \frac{1}{2x(x-3y)}8x - 18y - (4x - 6y)$

$\frac{1}{N}(M_y - N_x) = \frac{1}{2x(x-3y)}4(x-3y)$

$\frac{1}{N}(M_y - N_x) = \frac{2}{x} = f(x)$

$\implies \mu = \exp(\int_a^x f(t)dt)$ is an IF for the equation.

$\mu = \exp(\int_a^x \frac{2}{t}dt)$

$\mu = \exp(2\ln(\frac{x}{a}))$

$\mu = \frac{x^2}{a^2}$

(ii) $3(x^2+y^2)dx + (x^3 + 3xy^2 + 6xy)dy = 0$

$\frac{1}{N}(M_y - N_x) = \frac{1}{3(x^2+y^2)}(6y - (3x^2 + 3y^2 + 6y))$

$\frac{1}{N}(M_y - N_x) = -1 = f(x)$

$\implies \mu = \exp(\int_a^x f(t)dt)$ is an IF for the equation.

$\mu = \exp(\int_a^x -1dt)$

$\mu = \exp(a - x)$

(iii) $(4xy + 3y^2 - x)dx + x(x+2y)dy = 0$

$\frac{1}{N}(M_y - N_x) = \frac{1}{x(x+2y)}(4x + 6y - (2x + 2y))$

$\frac{1}{N}(M_y - N_x) = \frac{1}{x(x+2y)}2(x+2y)$

$\frac{1}{N}(M_y - N_x) = \frac{2}{x} = f(x)$

$\implies \mu = \exp(\int_a^x f(t)dt)$ is an IF for the equation.

$\mu = \exp(\int_a^x \frac{2}{t}dt)$

$\mu = \exp(2\ln(\frac{x}{a}))$

$\mu = \frac{x^2}{a^2}$

6. Find the general solution of the following differential equations.

Incomplete, only hints given

(i) $(y - xy') + a(y^2 + y') = 0$

Separable, use partial fractions.

(ii) $[y + xf(x^2+y^2)]dx + [yf(x^2+y^2) - x]dy = 0$

Incomplete

(iii) $(x^3 + y^2\sqrt{x^2+y^2})dx - xy\sqrt{x^2+y^2}dy = 0$

$\frac{1}{N}(M_y - N_x) = f(x)$, hence μ =exp($\int_a^x f(t)dt$) is an IF.

(iv) $(x+y)^2y' = 1$

Substitute $v = (x+y)$

(v) $y' - x^{-1}y = x^{-1}y^2$

Homogenous, substitute $y = vx$.

(vi) $x^2y' + 2xy = \sinh 3x$

Express both sides as derivatives of functions

(vii) $y' + y\tan x = \cos^2 x$

Use IF $\sec x$

(viii) $(3y - 7x + 7)dx + (7y - 3x + 3)dy = 0$

Shift origin to intersection of the lines and solve as homogeneous.

7. Solve the following homogeneous equations

Substitute $y = vx$ in all

(i) $(x^3 + y^3)dx - 3x^2dy = 0$

(ii) $(x^2 + 6y^2)dx + 4xydy = 0$

(iii) $xy' = y(\ln y - \ln x)$

(iv) $xy' = y + x\cos^2 \frac{y}{x}$

8. Solve the following first order linear equations.

Incomplete, only hints given

(i) $xy' - 2y = x^4$

Use IF $\frac{1}{x^3}$

(ii) $y' + 2y = e^{-2x}$

Use IF e^{2x}

(iii) $y' = 1 + 3y\tan x$

Use IF $\cos^3 x$

(iv) $y' = \text{cosec}x + y\cot x$

Use IF $\text{cosec}x$

(v) $y' = \text{cosec}x - y\cot x$

Use IF $\sin x$

(vi) $y' - my = c_1e^{mx}$

Use IF e^{-mx}

9. A differential equation of the form $y' + f(x)y = g(x)y^\alpha$ is called a Bernoulli equation. Note that if $\alpha = 0$ or 1 it is linear and for other values it is nonlinear. Show that the transformation $y^{1-\alpha} = u$ converts it into a linear equation. Use this to solve the following equations.

   Divinding the equation by y, $\frac{y'}{y} + f(x) = g(x)y^{\alpha-1}$

   Substituting $y^{1-\alpha} = u$ and $y' = \frac{y^\alpha}{1-\alpha}u'$ in the equation,

   $\frac{y^\alpha}{y(1-\alpha)}u' + f(x) = g(x)u^{-1}$

   Multiplying by u,

   $\frac{u'}{1-\alpha} + f(x)u = g(x)$

   This equation is linear, hence shown.

   Incomplete: Equations left out

IMPORTANT : The solutions till this point were for the older tutorial sheets, the ones that follow are for the newer ones

10. This is blank because it doesn't exist in the newer tutorial sheet

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11. A differential equation of the form $y' = P(x)+Q(x)y+R(x)y^2$ is called Ricatti's equation. In general, the equation cannot be solved by elementary methods. But if a particular solution $y = y_1$ is known, then the general solution is given by $y(x) = y_1(x) + u(x)$ where u satisfies the Bernoulli equation

$$\frac{du}{dx} - (Q + 2Ry_1)u = Ru^2$$

   (i) Use the method to solve $y' + x^3y - x^2y^2 = 1$, given $y_1 = x$

   $P(x) = 1; Q(x) = -x^3; R(X) = x^2$

   Thus, the Bernoulli equation we need to solve is

   $\frac{du}{dx} - (-x^3 + 2x^2 \times x)u = x^2u^2$

   $\frac{du}{dx} - ux^3 = x^2u^2$

   Substituting $v = \frac{1}{u}$,

   $-\frac{1}{v^2}\frac{dv}{dx} - \frac{x^3}{v} = \frac{x^2}{v^2}$

   $\frac{dv}{dx} + vx^3 = -x^2$

   Using IF $e^{x^4/4}$

   $ve^{x^4/4} = -\int x^2e^{x^4/4}dx$

   Thus, the solution to the equation $y' + x^3y - x^2y^2 = 1$ is

   $y(x) = x - \frac{e^{x^4/4}}{\int x^2e^{x^4/4}dx}$

   (ii) Use the method to solve $y' = x^3(y - x)^2 + x^{-1}y$

   Solve similarly as above

12. Determine by Picard's method, successive approximations to the solutions of the following initial value problems. Compare your results with the exact solutions.

 (i) $y' = 2\sqrt{y}; y(1) = 0$

 $\phi_0 = 0$

 $\phi_1(t) = \int f(s, \phi_0(s))ds$

 We get all $\phi(s)$ as 0, which is the only solution we can find using Picard's method (unless we guess $\phi_0(x) = x^2$).

 Might be wrong

 (ii) $y' - xy = 1; y(0) = 1$

 $\phi_0 = 1$

 $\phi_1(x) = 1 + \int_0^x (1+s)ds = 1 + x + \frac{x^2}{2}$

 $\phi_2(x) = 1 + \int_0^x (1 + s + s^2 + \frac{s^3}{2})ds = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$

 $\phi_3(x) = 1 + \int_0^x (1 + s + s^2 + \frac{s^3}{2} + \frac{s^4}{3} + \frac{s^5}{8})ds = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$

 ϕ_n follows this pattern.

 (iii) $y' = x - y^2; y(0) = 1$

 $\phi_0 = 1$

 $\phi_1 = 1 + \int_0^x (s-1)ds = 1 - s + \frac{s^2}{2}$

 $\phi_2 = 1 + \int_0^x s - (1 - s + \frac{s^2}{2})^2$

 I'm too bored to continue ... this doesn't even seem right ...

3 Tutorial Sheet 3

1. Find the values of m for which

 (i) $y = e^{mx}$ is a solution of $y'' + y' - 6y = 0$

 Plugging in $y = e^{mx}$ in the equation,

 $e^{mx}(m^2 + m - 6) = 0$

 $\implies m = -3$ or $m = 2$ ... since $e^{mx} > 0$

 (ii) $y = x^m$ for $x > 0$ is a solution of $x^2y'' - 4xy' + 4y = 0$

 Plugging in $y = x^m$ in the equation,

 $m(m-1)x^m - 4mx^m + 4x^m = 0$

 $m^2 - m - 4m + 4 = 0$ ... since $x > 0$

 $\implies m = 4$ or $m = 1$

2. Find the curve $y(x)$ through the origin for which $y'' = y'$ and the tangent at the origin is $y = x$.

Consider $u = y'$, we have $u' = u \implies \frac{du}{dx} = u \implies u = ce^x$

$\implies \frac{dy}{dx} = u = ce^x \implies y = ce^x + d$

As the curve has a tangent at origin, it satisfies $(0, 0)$

$\implies d = -1$

Also, we have $\frac{dy}{dx} = 1$ at $(0, 0)$

$\implies c = 1$

i.e. $y = e^x - 1$

3. Find the general solutions of the following differential equations.

   (i) $y'' - y' - 2y = 0$

   Try $y = e^{mx}$

   $e^{mx}(m^2 - m - 2) = 0$

   $\implies m = 2$ or $m = -1$

   i.e. $y = e^{2x}$ and $y = e^{-x}$ are two solutions. Since these are linearly independent, they form the basis of the solution space of the equation. Thus, the general solution is, $y = ae^{2x} + be^{-x}$

   (ii) $y'' - 2y' + 5y = 0$

   Try $y = e^{mx}$

   $e^{mx}(m^2 - 2m + 5) = 0$

   $m = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2}$

   $m = 1 \pm 2i$

   $\implies y = e^x e^{2xi}$ or $y = e^x e^{-2xi}$

   $\implies y = e^x(\cos 2x + i \sin 2x)$ or $y = e^x(\cos 2x - i \sin 2x)$

   $\implies y = e^x \cos 2x$ and $y = e^x \sin 2x$ are also solutions to the equation (linear combinations).

   Since these are linearly independent, they form the basis of the solution space of the equation. Thus, the general solution is,

   $y = e^x(a \sin 2x + b \cos 2x)$

4. Find the differential equation of the form $y'' + ay' + by = 0$ where a and b are constants for which the following functions are solutions:

   (i) $e^{-2x}, 1$

   Plugging $y = 1$ in the equation, we get $b = 0$

   Plugging $y = e^{-2x}$,

   $4e^{-2x} - 2ae^{-2x} = 0$

   $\implies a = 2$

(ii) $e^{-(\alpha+i\beta)x}, e^{-(\alpha-i\beta)x}$

Since the ODE is homogeneous, the linear combinations are also solutions

$\implies y = e^{-\alpha x}\cos\beta x$ and $y = e^{-\alpha x}\sin\beta x$ are solutions.

Incomplete : Do the math that follows

5. Are the following statements true or false. If the statement is true, prove it, if it is false, give a counter example showing it is false. Here Ly denotes $y'' + P(x)y' + Q(x)y$

   (i) If $y_1(x)$ and $y_2(x)$ are linearly independent on an interval I; then they are linearly independent on any interval containing I.

   True, easy to show.

   (ii) If $y_1(x)$ and $y_2(x)$ are linearly dependent on an interval I; then they are linearly dependent on any subinterval of I.

   True, if $c_1y_1 + c_2y_2 = 0$ on $I \implies c_1y_1 + c_2y_2 = 0$ on all subintervals.

   (iii) If $y_1(x)$ and $y_2(x)$ are linearly independent solution of $L(y) = 0$ on an interval I, they are linearly independent solution of $L(y) = 0$ on any interval J contained in I.

   True, use Wronskian. Because the equation is homogeneous second order, $W(x) = W(a)e^{Ax}$ and $W(x) = 0$ for some x means $W(a) = 0$, i.e. $W(x) = 0$ for I (and J). **Improve.**

   (iv) If $y_1(x)$ and $y_2(x)$ are linearly dependent solutions of $L(y) = 0$ on an interval I, they are linearly dependent on any interval J contained in I.

   True, easy to show.

6. Are the following pairs of functions linearly independent on the given interval?

   (i) $\sin 2x, \cos(2x + \frac{\pi}{2}); x > 0$

   (ii) $x^3, x^2|x|; -1 < x < 1$

   (iii) $x|x|, x^2; 0 \le x \le 1$

   (iv) $\log x, \log x^2; x > 0$

   (v) $x, x^2, \sin x; x \in \mathbb{R}$

   (ii) and (v) are linearly independent, others dependent.

7. Solve the following:

   $y'' - 4y' + 3y = 0, y(0) = 1, y'(0) = -5$

   Putting $y = e^{mx}$,

   $m^2 - 4m + 3 = 0$

   $\implies m = 1$ or $m = 3$

Since the equation is homogeneous,

$y = c_1 e^x + c_2 e^{3x}$

$y(0) = 1 \implies c_1 + c_2 = 1$

$y'(0) = -5 \implies c_1 + 3c_2 = -5$

$\implies c_2 = -3$ and $c_1 = 4$

$\implies y = 4e^x - 3e^{3x}$

8. Solve the Cauchy-Euler equations:

   (i) $x^2 y'' - 2y = 0$

   Putting $y = x^m$,

   $m(m-1) - 2 = 0$

   $\implies m^2 - m - 2 = 0$

   $\implies m = -1$ or $m = 2$

   Since the equation is homogeneous,

   $y = c_1 x^2 + c_2 \frac{1}{x}$

   (ii) $x^2 y'' + 2xy' - 6y = 0$

   Putting $y = x^m$,

   $m(m-1) + 2m - 6 = 0$

   $m^2 + m - 6 = (m-2)(m+3) = 0$

   $\implies m = -3$ or $m = 2$

   Since the equation is homogeneous,

   $y = c_1 x^2 + c_2 \frac{1}{x^3}$

9. Find the solution of $x^2 y'' - xy' - 3y = 0$ satisfying $y(1) = 1$ and $y(x) \to 0$ as $x \to \infty$

   Putting $y = x^m$,

   $m(m-1) - m - 3 = 0$

   $m^2 - 2m - 3 = 0$

   $\implies m = 3$ or $m = -1$

   Since the equation is homogeneous,

   $y = c_1 x^3 + c_2 \frac{1}{x}$

   $y(1) = 1 \implies c_1 + c_2 = 1$

   $\lim_{x \to \infty} y(x) = 0 \implies c_1 = 0$

   $\implies c_2 = 1$

   $\implies y = \frac{1}{x}$

10. Show that every solution of the constant coefficient equation $y'' + \alpha y + \beta y = 0$ tends to zero as $x \to \infty$ if and only if the real parts of the roots of the characteristic polynomial are

negative.

Let m_1 and m_2 be the solutions of the characteristic polynomial

$\implies y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

If any of m_1 or m_2 are positive, then that term would go to infinity as $x \to \infty$. Also, if m_1 and m_2 are negative, then the limit is zero.

11. Let $y_1(x)$ and $y_2(x)$ be two solutions of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$, $a < x < b$, and let $W(x)$ be the Wronskian of these two solutions. Prove that $W'(x) = -p(x)W(x)$. If $W(x_0) = 0$ for some x_0 with $a < x_0 < b$, the prove that $W(x) = 0$ for each x with $a < x < b$.

$W'(x) = -p(x)W(x)$

$\iff W'(x) + p(x)W(x) = 0$

$$\iff \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}' + p(x) \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = 0$$

$$\iff \begin{vmatrix} y_1'(x) & y_2'(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} + \begin{vmatrix} y_1(x) & y_2(x) \\ y_1''(x) & y_2''(x) \end{vmatrix} + p(x) \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = 0$$

$$\iff \begin{vmatrix} y_1(x) & y_2(x) \\ y_1''(x) & y_2''(x) \end{vmatrix} + \begin{vmatrix} y_1(x) & y_2(x) \\ p(x)y_1'(x) & p(x)y_2'(x) \end{vmatrix} = 0$$

$$\iff \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'' + p(x)y_1'(x) & y_2'' + p(x)y_2'(x) \end{vmatrix} = 0$$

$$\iff \begin{vmatrix} y_1(x) & y_2(x) \\ -q(x)y_1(x) & -q(x)y_2(x) \end{vmatrix} = -q(x) \begin{vmatrix} y_1(x) & y_2(x) \\ y_1(x) & y_2(x) \end{vmatrix} = 0$$

Which is true!

We have $W'(x) = \frac{dW(x)}{dx} = -p(x)W(x)$

$\implies \frac{dW(x)}{W(x)} = -p(x)dx$

$\implies \int \frac{dW(x)}{W(x)} = \int -p(x)dx$

$\implies \ln |CW(x)| = \int -p(x)dx$

$\implies W(x) = Ce^{\int -p(x)dx}$

$\implies W(x) = Ce^{P(x)}$

If $W(x_0) = 0$, then we have $C = 0$, since $e^{P(x)} > 0$

Thus, we have $W(x_0) = 0 \implies W(x) = 0$ for $a < x < b$

12. Let $y = y_1(x)$ be a solution of $y'' + p(x)y' + q(x)y = 0$. Let I be an interval where $y_1(x)$ does

not vanish, and $a \in I$ be any element. Prove that the general solution is given by

$$y = y_1(x)[c_2 + c_1\psi(x)] \text{ where } \psi(x) = \int_a^x \frac{\exp(-\int_a^t p(u)du)}{y_1^2(t)} dt$$

13. For each of the following ODEs, you are given one solution. Find the second solution.

4 Tutorial Sheet 4

1. Solve the following initial value problems.

   (i) $(D^2 + 5D + 6)y = 0, y(0) = 2, y'(0) = -3$

   The characteristic equation for the constant coefficient ODE is

   $m^2 + 5m + 6 = 0$

   $\implies m = -2$ or $m = -3$

   $\implies y = c_1e^{-2x} + c_2e^{-3x}$

   $y(0) = 2 \implies c_1 + c_2 = 2$

   $y'(0) = -3 \implies -2c_1 - 3c_2 = -3 \implies 2c_1 + 3c_2 = 3$

   $\implies c_1 = 3, c_2 = -1$

   $\implies y = 3e^{-2x} - e^{-3x}$

   (ii) $(D + 1)^2y = 0, y(0) = 1, y'(0) = 2$

   The characteristic equation for the constant coefficient ODE is

   $(m + 1)^2 = 0$

   $\implies m = -1$

   $\implies y = c_1e^{-x} + c_2xe^{-x}$ ... since $m_1 = m_2$

   $y(0) = 1 \implies c_1 = 1$

   $\implies y'(0) = 1 + c_2(0 + 1) = 2 \implies c_2 = 1$

   $\implies y = e^{-x} + xe^{-x}$

   (iii) $(D^2 + 2D + 2)y = 0, y(0) = 1, y'(0) = -1$

   The characteristic equation for the constant coefficient ODE is

   $m^2 + 2m + 2 = 0$

   $\implies m = \frac{-2\pm\sqrt{4-8}}{2}$

   $\implies m = -1 \pm i$

   $\implies y = c_1e^{-x}e^{ix} + c_2e^{-x}e^{-ix}$

   $\implies y = e^{-x}(c_1e^{ix} + c_2e^{-ix})$

   Taking appropriate linear combination,

   $y = e^{-x}(c_1 \sin x + c_2 \cos x)$

$y(0) = 1 \implies c_2 = 1$

$y'(0) = -1 \implies c_1 - c_2 = -1$

$\implies c_1 = 0$

$\implies e^{-x}\cos x$

2. Solve the following initial value problems

   (i) $(x^2D^2 - 4xD + 4)y = 0, y(1) = 4, y'(1) = 1$

   (ii) $(4x^2D^2 + 4xD - 1)y = 0, y(4) = 2, y'(4) = -1/4$

   (iii) $(x^2D^2 - 5xD + 8)y = 0, y(1) = 5, y'(1) = 18$

   Easy, solve as earlier.

3. Using the Method of Undetermined Coefficients, determine a particular solution of the following equations. Also find the general solutions of these equations.

   (i) $y'' + 2y' + 3y = 27x$

   Try $y = c_1x + c_0$

   $2c_1 + 3(c_1x + c_0) = 27x$

   $2c_1 + 3c_0 + 3c_1x = 27x$

   $\implies c_1 = 9$ and $18 + 3c_0 = 0 \implies c_0 = -6$

   $\implies y = 9x - 6$ is a particular solution.

   The general solution of the corresponding homogeneous differential equation is

   $y = e^{-x}(a\cos\sqrt{2}x + b\sin\sqrt{2}x)$

   $\implies$ The general solution of the non-homogeneous ODE is

   $y = 9x - 6 + e^{-x}(a\cos\sqrt{2}x + b\sin\sqrt{2}x)$

   (ii) $y'' + y' - 2y = 3e^x$

   Try $y = ce^x \implies 0 = 3e^x$

   Try $y = cxe^x$

   $c(x+2)e^x + c(x+1)e^x - 2cxe^x = 3e^x$

   $\implies c = 1$

   Thus, a particular solution of the ODE is $y = xe^x$

   The general solution of the corresponding homogeneous ODE is $y = ae^x + be^{-2x}$

   $\implies$ the general solution of the non-homogeneous ODE is $y = ae^x + be^{-2x} + xe^x$

   (iii) $y'' + 4y' + 4y = 18\cos x$

   Try $y = a\cos x + b\sin x$

   $\implies -a + 4b + 4a = 18$ and $-b - 4a + 4b = 0$

   $\implies 4b + 3a = 18$ and $3b - 4a = 0$

$\implies 16b + 12a = 72$ and $9b - 12a = 0$

$\implies 25b = 72 \implies b = \frac{72}{25}$

$\implies a = \frac{9\times 72}{12}$

$\implies a = 54$

P(this is wrong) $\approx$ 1

(iv) $y'' + 4y' + 3y = \sin x + 2\cos x$

Try $y = a\cos x + b\sin x$

$-a + 4b + 3a = 2$ and $-b - 4a + 3b = 1$

$\implies a + 2b = 1$ and $-4a + 2b = 1$

$\implies a = 0$ and $b = 1/2$

$\implies y = \frac{1}{2}\sin x$ is a particular solution.

The general solution of the corresponding homogeneous ODE is $y = ae^{-x} + be^{-3x}$

$\implies$ the general solution of the non-homogeneous ODE is $y = \frac{1}{2}\sin x + ae^{-x} + be^{-3x}$

(v) $y'' - 2y' + 2y = 2e^x\cos x$

Try $y = (a\cos x + b\sin x)e^x$

$(((a+b)\cos x+(b-a)\sin x)e^x)'-2(((a+b)\cos x+(b-a)\sin x)e^x)+2(a\cos x+b\sin x)e^x = 2e^x\cos x$

$((2b\cos x-2a\sin x)e^x)-2(((a+b)\cos x+(b-a)\sin x)e^x)+2(a\cos x+b\sin x)e^x = 2e^x\cos x$

$(b\cos x - a\sin x) - ((a+b)\cos x + (b-a)\sin x) + (a\cos x + b\sin x) = \cos x$

$(b\cos x - a\sin x) - ((a+b)\cos x + (b-a)\sin x) + (a\cos x + b\sin x) = \cos x$

$0 = \cos x$, i.e. the guess did not work

Also, the solution to the homogeneous ODE is $y = (a\cos x + b\sin x)e^x$

Try $y = x(a\cos x + b\sin x)e^x + x^2(c\cos x + d\sin x)e^x$

Incomplete: Do this dirty calculation

Get $a = d = -\frac{1}{2}$ and $b = c = 0$

$\implies$ the general solution of the non-homogeneous ODE is

$y = (\frac{1}{2}x^2\sin x + \frac{1}{2}x\cos x + a\cos x + b\sin x)e^x$

(vi) $y'' + y = x\cos x + \sin x$

Try $y_1 = (xa\cos x + xb\sin x)$

$2(b\cos x - a\sin x) - y_1 + y_1 = x\cos x + \sin x$

$2(b\cos x - a\sin x) = x\cos x + \sin x$

Try $y_2 = (x^2c\cos x + x^2d\sin x)$

$4x(d\cos x - c\sin x) + 2(c\cos x + d\sin x) - y_2 + y_2 = x\cos x + \sin x$

$4x(d\cos x - c\sin x) + 2(c\cos x + d\sin x) = x\cos x + \sin x$

Put $c = 0, d = \frac{1}{4}$

LHS $= \frac{1}{2}\sin x$ and RHS$= \sin x$

Using $b = 0, a = \frac{-1}{4}$,

$y = y_1 + y_2 = \frac{1}{4}(x^2 \sin x - x\cos x)$ is a particular solution of the ODE

$\implies$ the general solution of the non-homogeneous ODE is

$y = a\cos x + b\sin x + \frac{1}{4}x^2 \sin x - \frac{1}{4}x\cos x$

4. Solve the following initial value problems.

(i) $y'' + y' - 2y = 14 + 2x - 2x^2, y(0), y'(0) = 0$

Try $y = ax^2 + bx + c$

$\implies 2a + 2ax + b - 2ax^2 - 2bx - 2c = 14 + 2x - 2x^2 \implies a = 1$

$\implies 2 + 2x + b - 2bx - 2c = 14 + 2x \implies b = 0$

$\implies 2 - 2c = 14 \implies c = -6$

Try $y = x^2 - 6$ is a particular solution of the ODE

The general solution of the corresponding homogeneous ODE is $y = ae^x + be^{-2x}$

$\implies$ the general solution of the non-homogeneous ODE is

$y = x^2 - 6 + ae^x + be^{-2x}$

$y(0) = 0 \implies -6 + a + b = 0$

$y'(0) = 0 \implies a - 2b = 0$

$\implies b = 2$ and $a = 4$

$\implies$ the solution to the IVP is

$y = x^2 - 6 + 4e^x + 2e^{-2x}$

(ii) $y'' + y' - 2y = -6\sin 2x - 18\cos 2x; y(0) = y'(0) = 2$

Try $y = a\cos 2x + b\sin 2x$

$\implies -6(a\cos 2x + b\sin 2x) - 2a\sin 2x + 2b\cos 2x - -6\sin 2x - 18\cos 2x$

$\implies (-6a + 2b)\cos 2x + (-2a - 6b)\sin 2x = -6\sin 2x - 18\cos 2x$

$\implies -6a + 2b = -18$ and $-2a - 6b = -6$

$\implies a = 3, b = 0$

$\implies y = 3\cos 2x$ is a particular solution of the ODE.

The general solution of the corresponding homogeneous ODE is $y = ae^x + be^{-2x}$

$\implies$ the general solution of the non-homogeneous ODE is

$y = ae^x + be^{-2x} + 3\cos 2x$

$y(0) = 2 \implies a + b + 3 = 2$

$y'(0) = 2 \implies a - 2b = 2$

$\implies a = 0, b = -1$

$\implies$ the solution to the IVP is

$y = 3\cos 2x - e^{-2x}$

(iii) $y'' - 4y' + 3y = 4e^{3x}, y(0) = -1, y'(0) = 3$

Try $y = axe^{3x}$

$\implies a(3(3x+1)e^{3x} + 3e^{3x}) - 4a(3x+1)e^{3x} + 3axe^{3x} = 4e^{3x}$

$\implies a(9x+6) - 4a(3x+1) + 3ax = 4$

$\implies 6a - 4a = 2a = 4 \implies a = 2$

$\implies y = 2xe^{3x}$ is a particular solution of the ODE

The general solution of the corresponding homogeneous ODE is $y = ae^{x} + be^{3x}$

$\implies$ the general solution of the non-homogeneous ODE is

$y = ae^{x} + (b + 2x)e^{3x}$

$y(0) = -1 \implies a + b = -1$

$y'(0) = 3 \implies a + 3b + 2 = 3$

$\implies a = -2, b = 1$

$\implies$ the solution to the IVP is

$y = -2e^{x} + (2x+1)e^{3x}$

5. For each of the following equations, write down the form of the particular solution. Do not go further and compute the Undetermined Coefficients.

Warning : Incomplete

(i) $y'' + y = x^3 \sin x$

(ii) $y'' + 2y' + y = 2x^2 e^{-x} + x^3 e^{2x}$

(iii) $y' + 4y = x^3 e^{-4x}$

6. Using the Method of Variation of Parameters, determine a particular solution for each of the following.

(i) $y'' - 5y' + 6y = 2e^{x}$

The general solution of the homogeneous ODE is $y = ae^{2x} + be^{3x}$

$\implies y_1 = e^{2x}, y_2 = e^{3x}$

$r(x) = 2e^{x}$ and $W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x}$

By variation of parameters, $y = -y_1 \int \frac{y_2 r(x)}{W(y_1,y_2)} dx + y_2 \int \frac{y_1 r(x)}{W(y_1,y_2)} dx$

$\implies y = -e^{2x} \int \frac{2e^{x}e^{3x}}{e^{5x}} dx + e^{3x} \int \frac{2e^{x}e^{2x}}{e^{5x}} dx$

$\implies y = -2e^{2x} \int e^{-x} dx + 2e^{3x} \int e^{-2x} dx$

$\implies y = 2e^{2x}e^{-x} - e^{3x}e^{-2x}$

$\implies y = e^{x}$ is the required particular solution

(ii) $y'' + y = \tan x, 0 < x < \frac{\pi}{2}$

The general solution of the homogeneous ODE is $y = a \sin x + b \cos x$

$\implies y_1 = \cos x, y_2 = \sin x$

$r(x) = \tan x$ and $W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

By variation of parameters, $y = -y_1 \int \frac{y_2 r(x)}{W(y_1,y_2)} dx + y_2 \int \frac{y_1 r(x)}{W(y_1,y_2)} dx$

$\implies y = -\cos x \int \frac{\sin x \tan x}{1} dx + \sin x \int \frac{\cos x \tan x}{1} dx$

$\implies y = -\cos x \int \frac{\sin^2 x}{\cos x} dx - \sin x \cos x$ is the required particular solution (this integral can be solved further if necessary).

(iii) $y'' + 4y' + 4y = x^{-2} e^{-2x}, x > 0$

The general solution of the homogeneous ODE is $y = ae^{-2x} + bxe^{-2x}$

$\implies y_1 = e^{-2x}, y_2 = xe^{-2x}$

$r(x) = x^{-2} e^{-2x}$ and $W(y_1, y_2) = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & (-2x+1)e^{-2x} \end{vmatrix} = e^{-4x}$

By variation of parameters, $y = -y_1 \int \frac{y_2 r(x)}{W(y_1,y_2)} dx + y_2 \int \frac{y_1 r(x)}{W(y_1,y_2)} dx$

$\implies y = -e^{-2x} \int \frac{xe^{-2x} x^{-2} e^{-2x}}{e^{-4x}} dx + xe^{-2x} \int \frac{e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} dx$

$\implies y = -e^{-2x} \int x^{-1} dx + xe^{-2x} \int x^{-2} dx$

$\implies y = -e^{-2x} \ln|x| + xe^{-2x}(-\frac{1}{x})$

$\implies y = -e^{-2x}(\ln|x| + 1)$ is the required particular solution.

(iv) $y'' + 4y = 3 \text{ cosec } 2x, 0 < x < \frac{\pi}{2}$

Incomplete

(v) $x^2 y'' - 2xy' + 2y = 5x^3 \cos x$

Incomplete

(vi) $xy'' - y' = (3 + x) x^3 e^x$

Incomplete

7. Prove that the functions $e^{r_1 x}, e^{r_2 x}, ..., e^{r_n x}$, where $r_1, r_2, ..., r_n$ are distinct real numbers, are linearly independent.

   Writing the Wronskian for any two functions,

   $W = \begin{vmatrix} e^{r_p x} & e^{r_q x} \\ r_p e^{r_p x} & r_q e^{r_q x} \end{vmatrix} \neq 0$, so we have all linearly independent.

8. For the following non-homogeneous equations, a solution y_1 of the corresponding homogeneous equation is given. Find a second solution y_2 of the corresponding homogeneous equation and the general solution of the non-homogeneous equation using the Method of Variation of Parameters.

9. A linear DE with constant coefficients has characteristic polynomial $f(x) = 0$. If all the roots of $f(x)$ are negative, prove that every solution of the differential equation approaches 0 as $x \to +\infty$. What can you conclude about the behavior of all solutions in the interval $[0, +1)$, if all the roots of f(x) are non-positive?

   Any constant coefficient ODE of degree n has solution as $\sum_{i=1}^{n} b_i e^{a_i x}$ where a_i are roots of the characteristic polynomial. If all roots are negative then $y \to 0$ as $x \to \infty$, since each term goes to zero as x increases.

10. In each case, find a linear differential equation with constant coefficients satisfied by all the given functions.

    (i) $u_1(x) = e^x, u_2(x) = e^{-x}, u_3(x) = e^{2x}, u_4(x) = e^{-2x}$

    From the characteristic polynomial, $(D-1)(D+1)(D-2)(D+2)y = 0$

    Or $(D^2 - 1)(D^2 - 2)y = 0$

    which is a linear ODE of degree 4 with constant coefficients.

    (ii) $u_1(x) = e^{-2x}, u_2(x) = xe^{-2x}, u_3(x) = x^2e^{-2x}$

    From the characteristic polynomial, $(D+2)^3y = 0$

    which is a linear ODE of degree 3 with constant coefficients.

    (iii) $u_1(x) = 1, u_2(x) = x, u_3(x) = e^x, u_4(x) = xe^x$

    From the characteristic polynomial, $(D-1)^2y = 0$ is satisfied by u_3 and u_4

    For $u_1(x) = 1$ and $u_2(x) = x$, $D^2y = 0$ is the corresponding equation

    $\implies$ the required linear ODE is $D^2(D-1)^2y = 0$

5 Tutorial Sheet 5

1. In each case, find a linear differential equation with constant coefficients satisfied by all the given functions.

   (i) $u_1(x) = x^2, u_2(x) = e^x, u_3(x) = xe^x$

   For u_1, we need $D^3y = 0$ and $(D-1)^2y = 0$ for u_2 and u_3

   $\implies D^3(D-1)^2y = 0$ is satisfied by all u.

   (ii) $u_1(x) = e^{-2x}\cos 3x, u_2(x) = e^{-2x}\sin 3x, u_3(x) = e^{-2x}, u_4(x) = xe^{-2x}$

   Taking linear combinations of u_1 and u_2, we get $u = e^{-(2\pm 3i)x}$ also satisfies the equation.

   Forming the equation, for u_1 and u_2, we have $(D+2-3i)(D+2+3i)y = 0$ and for u_3 and u_4, we have $(D+2)^2y = 0$

   $\implies y = (D+2)^2(D^2+4D+13)y = 0$ is satisfied by all u.

2. Let $y_1(x), y_2(x), ..., y_n(x)$ be n linearly independent solutions of the nth order homogeneous linear differential equation $y^{(n)} + p_1 y^{(n-1)} + ... + p_{n-1}(x)y + p_n(x)y = 0$. Prove that $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x) + ... + c_n(x)y_n(x)$ is a solution of the non-homogeneous equation

$$y^{(n)} + p_1 y^{(n-1)} + ... + p_{n-1}(x)y + p_n(x)y = r(x)$$

where $c_1(x), c_2(x), ..., c_n(x)$ are given by $c_i(x) = \int \frac{D_i(x)}{\omega(x)} dx$ where $D_i(x)$ is the determinant of the matrix obtained from the matrix defining the Wronskian $\omega(x)$ by replacing its ith column by $[0\ 0\ 0\ ...\ r(x)]^T$

Incomplete, proof given in lecture slides for Lecture 15

3. Three solutions of a certain second order non-homogeneous linear differential equation are

$$y_1(x) = 1 + e^{x^2} \quad y_2(x) = 1 + xe^{x^2} \quad y_3(x) = (1+x)e^{x^2} - 1$$

Find the general solution of the equation.

Let $u_1 = y_3 - y_2 = e^{x^2} - 2$ and $u_2 = y_2 - y_1 = (x-1)e^{x^2}$, then u_1 and u_2 are solutions of the corresponding homogeneous differential equation.

Thus, the general solution of the non-homogeneous DE is $c_1 u_1 + c_2 u_2 + y_1(x)$.

i.e. $y = a(e^{x^2} - 2) + b(x-1)e^{x^2} + 1 + e^{x^2}$

4. Let $r_1, ..., r_n$ be distinct real numbers and $Q_1, ..., Q_n$ be polynomials with real coefficients, none of which is the zero polynomial. Prove that the functions

$$u_i(x) = Q_i(x)e^{r_i x}, : 1 \leq i \leq n$$

are linearly independent.

5. Let $m_1, ..., m_k$ be positive integers and $r_1, ...r_k$ be distinct real numbers. Set $n = m_1 + ... + m_k$. For each pair of integers satisfying: $1 \leq p \leq k, 1 \leq q \leq m_p$, let

$$u_{p,q} = x^{q-1}e^{r_p x}$$

. For instance, if $p = 1$, then the functions are

$$u_{1,1}(x) = e^{r_1 x}, u_{2,1}(x) = xe^{r_1 x}, ..., u_{m_1,1}(x) = x^{m_1 - 1}e^{r_1 x}$$

Prove that the n functions u_p, q defined above are linearly independent.

Hint: Use the earlier exercise.

For $p_1 = p_2 = k$, we have the functions linearly independent due to linear independence of x^k (this can also be verified by taking the Wronkskian).

For $p_1 \neq p_2$, the functions are linearly independent due to the result in Q.4

Hence all the functions are linearly independent.

6. Solve the following ODE's, given that $y_1 = x$ is a solution. [Hint: Put $y = ux$ in the ODE. Then put $u' = v$ in the new equation, to get an equation of lower order to solve.]

   (i) $x^3y''' - 3x^2y'' + (6-x^2)xy' - (6-x^2)y = 0$

   Putting $y = ux$,

   $x^3(u+xu')'' - 3x^2(u+xu')' + (6-x^2)x(u+xu') - (6-x^2)ux = 0$

   $x^3(2u'+xu'')' - 3x^2(2u'+xu'') + (6-x^2)x^2u' = 0$

   $x^3(3u''+xu''') - 3x^2(2u'+xu'') + (6-x^2)x^2u' = 0$

   $x^3(xu''') - x^4u' = 0$

   $\implies x = 0$ or $u''' = u'$

   Put $u' = v$ in $u''' = u'$,

   $v'' = v$, i.e. $m^2 = 1$ i.e. $v = ae^x + be^{-x}$

   $u = \int vdx = \int ae^x + be^{-x}dx = c_1e^x + c_2e^{-x}$

   (ii) $y''' + (x^2+1)y'' - 2x^2y' + 2xy = 0$

   Putting $y = ux$,

   $(3u''+xu''') + (x^2+1)(2u'+xu'') - 2x^2(u+xu') + 2x(ux) = 0$ $(3u''+xu''') + 2x^2u' + x^3u'' + 2u' + xu'' - 2x^2(xu') = 0$

   $xu''' + (x^3+x+3)u'' - 2(x^3-x^2-1)u' = 0$

   Now what???

7. Solve the following differential equations using the annihilator method.

   (i) $y^{(4)} + 2y^{(2)} + y = \sin x$

   $r(x) = \sin x$ is in the null space of the operator $A = D^2 + 1$

   $\implies AL(y) = (D^2+1)(D^4+2D^2+1) = (D^2+1)^3$

   Roots are $\pm i$

   $\implies$ the general solution of $AL(y) = 0$ is

   $c_1e^{ix} + c_2e^{-ix} + c_3xe^{ix} + c_4xe^{-ix} + c_5x^2e^{ix} + c_6x^2e^{-ix}$

   Since the first four terms are annihilated, we have a solution of the form $c_5x^2e^{ix} + c_6x^2e^{-ix}$ or $ax^2\sin x + bx^2\cos x$

   Plugging this in the ODE, we can get the values of a and b by comparing coefficients on both sides (due to linear independence)

   Incomplete : Do this

   Answer is $a = -\frac{1}{8}, b = 0$

(ii) $y^{(4)} - y^{(3)} - 3y^{(2)} + 5y' - 2y = xe^x + 2e^{-2x}$

$r(x) = xe^x + e^{-2x}$ is in the null space of the operator $A = (D-1)^2(D+2)$

$\implies AL(y) = (D-1)^2(D+2)(D^4 - D^3 - 3D^2 + 5D - 2) = (D-1)^5(D+2)^2$

$\implies$ the general solution of $AL(y) = 0$ is

$c_1e^x + c_2xe^x + c_3x^2e^x + c_4x^3e^x + c_5x^4e^x + c_6e^{-2x} + c_7xe^{-2x}$

Annihilating c_1, c_2, c_3 and c_6, we have a solution of the form $ax^3e^x + bx^4e^x + cxe^{-2x}$

Plugging this in the ODE, we can get the values of a, b and c by comparing coefficients on both sides (due to linear independence)

Incomplete : Do this

Answer is $a = -\frac{1}{54}, b = \frac{1}{72}, c = -\frac{2}{27}$

8. Solve the following differential equations

(i) $x^2y'' + 2xy' + y = x^3$

Finding a complementary solution, we have $m(m-1) + 2m + 1 = 0$

or, $m^2 + m + 1 = 0 \implies m = \frac{-1 \pm i\sqrt{3}}{2} = \omega, \omega^2$

$\implies$ the complementary solution is $y_1 = c_1\frac{\sin(\frac{\sqrt{3}}{2}\ln x)}{\sqrt{x}} + c_2\frac{\cos(\frac{\sqrt{3}}{2}\ln x)}{\sqrt{x}}$ Rewriting the differential equation, we get,

$y'' + \frac{2}{x}y' + \frac{1}{x^2}y = x$

Now apply variation of parameters on the complementary solution to get the particular solution y_p. Then the general solution is $y = y_1 + y_p$

Alternatively, since RHS is x^3, we can guess a particular solution to be cx^3 and get the value of c.

Answer is $y_p = \frac{x^3}{13}$

(ii) Proceed similarly as the above problem.

9. Find a particular solution of the folllowing inhomogeneous Cauchy-Euler equations.

(i) $x^2y'' - 6y = \ln x$

Put $y = c\ln x + d$

$\implies -c - 6c\ln x - 6d = \ln x$

$\implies c = -\frac{1}{6}, d = \frac{1}{36}$

Please send all queries, suggestions and corrections to varunpatil [at] iitb.ac.in

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