ShreveVladZhukov3

December 14, 2018

```
In [1]: from IPython.display import display, Markdown, Latex
        def printmd(s):
            display(Markdown(s))
```

1 Exercise 3.1

(i) $\forall \omega \in \Omega : Z > 0$, since $\forall \omega \in \Omega \overline{P}(\omega) > 0, P(\omega) > 0$

(ii) $\overline{E}\frac{1}{Z}$

$$\sum_{\omega \in \Omega} \frac{\overline{P}(\omega) P(\omega)}{\overline{P}(\omega)} = 1$$

(iii) $\overline{E}[\frac{1}{Z}Y] = \sum_{\omega \in \Omega} \frac{\overline{P}(\omega)P(\omega)}{\overline{P}(\omega)} Y = \mathbb{E}Y$

2 Exercise 3.2

(i) $\overline{P}(\Omega) = \sum_{\omega \in \Omega} Z(\omega)P(\omega) = \mathbb{E}Z = 1$

(ii) $\overline{\mathbb{E}}Y = \mathbb{E}[ZY]$

$\overline{\mathbb{E}}Y = \sum_{\omega \in \Omega} \overline{P}(\omega)Y = \sum_{\omega \in \Omega} P(\omega)\frac{\overline{P}(\omega)}{P(\omega)}Y = \mathbb{E}[ZY]$

(iii) $P(A) = \sum_{\omega \in A} P(\omega) = 0$ since $P(\omega) \geq 0 => P(\omega) = 0 \; \forall \omega \in \Omega => \overline{P}(A) = \sum P(\omega)Z(\omega) = 0$

(iv) $\overline{P}(A) = \sum_{\omega} \overline{P}(\omega) = \sum ZP(\omega) = 0$ Since $P(Z > 0) = 1$, then $P(\omega) = 0 \; \forall \omega \in A => P(A) = 0$

(v)

$P(A) = 1 <=> P(\overline{A}) = 0$
$P(\overline{A}) = 0 <=> \overline{P}(\overline{A}) = 0 <=> \overline{P}(A) = 1$

3 Exercise 3.3

Calculate M values:

```
In [2]: p = 1/3
        q = 2/3
        def S(a):
            res = 4
            for i in a:
                if i == "H":
                    res *= 2
                else:
                    res /= 2
            return res


        def M(a):
            if len(a) == 3:
                return S(a)
            return q * M(a + "H") + p * M(a + "T")


        m2 = ["HH", "HT", "TH", "TT"]
        m1 = ["H", "T"]
        for w in m2 + m1:
            print("M_{}({})={}".format(len(w), w, M(w)))
```

```
M_2(HH)=24.0
M_2(HT)=6.0
M_2(TH)=6.0
M_2(TT)=1.5
M_1(H)=18.0
M_1(T)=4.5
```

$M_3 = S_3$

3.1 Check M_n is martingale

```
In [3]: assert(q * M("HH") + p * M("HT") == M("H"))
        assert(q * M("TH") + p * M("TT") == M("T"))
        assert(q * M("H") + p * M("T") == M(""))
```

So M_n is martingale

4 Exercise 3.4

I)

$$\zeta_3(HHH) = \frac{Z_3}{(r+1)^3} = \frac{27}{64}\frac{4^3}{5^3} = \frac{27}{125}$$

$$\zeta_3(HHT) = \zeta_3(HTH) = \zeta_3(THH) = \frac{27}{32}\frac{64}{125} = \frac{54}{125}$$
$$\zeta_3(HTT) = \zeta_3(THT) = \zeta_3(TTH) = \frac{27}{16}\frac{64}{125} = \frac{108}{125}$$
$$\zeta_3(TTT) = \frac{216}{125}$$

```
In [4]: zeta = {(3, 0): 27 / 125, (2, 1): 54 / 125, (1, 2): 108 / 125, (0, 3): 216 / 125}
```

II) Number of paths is 8. They match $v_n(s, y)$ values:

```
In [5]: def mean_path(a):
            current_value = 4
            res = current_value
            for i in a:
                if i == "H":
                    current_value *= 2
                    res += current_value
                else:
                    current_value /= 2
                    res += current_value
            return current_value, res

        paths = ["HHH", "HHT", "HTH", "HTT", "THH", "THT", "TTH", "TTT"]
        get_price = lambda sy: max(0, 0.25 * sy[1] - 4)
        for path in paths:
            v_args = list(mean_path(path))
            v_args.append(get_price(v_args))
            printmd("$v_3({}, {}) = {}$".format(*v_args))
```

$v_3(32, 60) = 11.0$
$v_3(8.0, 36.0) = 5.0$
$v_3(8.0, 24.0) = 2.0$
$v_3(2.0, 18.0) = 0.5$
$v_3(8.0, 18.0) = 0.5$
$v_3(2.0, 12.0) = 0$
$v_3(2.0, 9.0) = 0$
$v_3(0.5, 7.5) = 0$
by replication in multiperiod binomial model:

$$V_n(\omega_1\omega_2..\omega_n) = \frac{1}{1+r}[\overline{p}V_{n+1}(\omega_1\omega_2..\omega_n H) + \overline{q}V_{n+1}(\omega_1\omega_2..\omega_n T)]$$

$\overline{q} = \frac{1+r-d}{u-d} = \frac{1+1/4-1/2}{2-1/2} = 1/2 = 1 - 1/2 = \overline{p}$

$$v_n(s, y) = \frac{2}{5}(v_{n+1}(2s, y + 2s) + v_{n+1}(s/2, y + s/2))$$

In [6]:
```
paths2 = ["HH", "HT", "TH", "TT"]
values2 = []
for path in paths2:
    v_args = list(mean_path(path))
    v_args.append(2.0 / 5.0 * (get_price(mean_path(path + "H")) + get_price(mean_path(
    printmd("$v_2({}, {}) = {}$".format(*v_args))
    values2.append(v_args[-1])
```

$v_2(16, 28) = 6.4$
$v_2(4.0, 16.0) = 1.0$
$v_2(4.0, 10.0) = 0.2$
$v_2(1.0, 7.0) = 0.0$

In [7]:
```
values2
Hargs = list(mean_path("H")) +[ 2.0/5.0 * (values2[0] + values2[1])]
printmd("$v_1({}, {}) = {}$".format(*Hargs))
Targs = list(mean_path("T")) +[ 2.0/5.0 * (values2[2] + values2[3])]
printmd("$v_1({}, {}) = {}$".format(*Targs))
v1 = [Hargs[-1]] + [Targs[-1]]
printmd("$v_0 = {}$".format(2.0/5.0 * (sum(v1))))
```

$v_1(8, 12) = 2.9600000000000004$
$v_1(2.0, 6.0) = 0.08000000000000002$
$v_0 = 1.2160000000000002$

In [13]:
```
def get_prob_path(a):
    res = 1.0
    for i in a:
        if i == "H":
            res *= q
        else:
            res *= p
    return res
```

back to the problem. By 3.1.10

$$v_0(4, 4) = \mathbb{E}[\zeta V_N] = \sum_{\omega \in \Omega} V_N(\omega)\zeta(\omega)P(\omega)$$

In [26]:
```
res = 0
for path in paths:
    current = get_price(mean_path(path)) * zeta[path.count("H"), path.count("T")] * ge
    res += current
# check computation results are the same
assert(res == 2.0/5.0 * (sum(v1)))
```

By 3.2.6

$$V_n = \frac{1}{\zeta_n}\mathbb{E}_n[\zeta_N V_N]$$

$$\zeta_2(HT) = \frac{Z_2(HT)}{(1+r)^2} = \frac{9}{8}\frac{16}{25} = \frac{18}{25} = \zeta_2(TH)$$

$$V_2(HT) = v_2(4,16) = \frac{25}{18}\cdot(\frac{2}{3}\cdot\frac{54}{125}\cdot 2 + \frac{1}{3}\cdot\frac{108}{125}\cdot 0.5) = 1$$

$$V_2(TH) = v_2(4,10) = \frac{25}{18}(\frac{2}{3}\cdot\frac{54}{125}\cdot 1/2 + \frac{1}{3}\cdot\frac{108}{125}\cdot 0) = 0.144$$

5 Exercise 3.5

$$\overline{p} = \frac{1+r-d}{u-d}$$

$$\overline{q} = \frac{u-1-r}{u-d}$$

$$P(HH) = \frac{4}{9}, P(HT) = \frac{2}{9}, P(TH) = \frac{2}{9}, P(TT) = \frac{1}{9}$$

$$u_0 = 2, d_0 = 1/2$$

$$u_1(H) = 3/2, d_1(H) = 1$$

$$u_1(T) = 4, d_1(T) = 1$$

Calculate $\overline{P}$ $\overline{P_0} = \frac{1+1/4-1/2}{2-1/2} = 1/2$
$\overline{Q_0} = 1/2$
$\overline{P_1}(H) = \frac{1+1/4-1}{3/2-1} = 1/2$
$\overline{Q_1}(H) = 1/2$
$\overline{P_1}(T) = \frac{1+1/2-1}{4-1} = 1/6$
$\overline{Q_1}(T) = 5/6$
$\overline{P}(HH) = \overline{P}_0\overline{P}_1(H) = 1/2\cdot 1/2 = 1/4$
$\overline{P}(HT) = \overline{P}_0\overline{Q}_1(H) = 1/4$
$\overline{P}(TH) = \overline{Q}_0\overline{P}_1(T) = 1/2\cdot 1/6 = 1/12$
$\overline{P}(TT) = \overline{Q}_0\overline{Q}_1(T) = 1/2\cdot 5/6 = 5/12$

I)

$Z(HH) = \frac{1}{4}\cdot\frac{9}{4} = \frac{9}{16}$
$Z(HT) = \frac{1}{4}\cdot\frac{9}{2} = \frac{9}{8}$
$Z(TH) = \frac{1}{12}\cdot\frac{9}{2} = \frac{3}{8}$
$Z(TT) = \frac{5}{12}\cdot 9 = \frac{15}{4}$

II)

$Z_1(H) = P_1(H|\omega_0 = H)Z_2(HH) + P_1(T|\omega_0 = T)Z_2(HT) = \frac{2}{3}\frac{9}{16} + \frac{1}{3}\cdot\frac{9}{8} = \frac{3}{8} + \frac{3}{8} = \frac{3}{4}$ $Z_1(T) = P_1(H|\omega_0 = T)Z_2(TH) + P_1(T|\omega_0 = T)Z_2(TT) = \frac{2}{3}\frac{3}{8} + \frac{1}{3}\cdot\frac{15}{4} = \frac{1}{4} + \frac{5}{4} = \frac{3}{2}$

iii)

Let H/T denote H or T

$$V_1(H/T) = \frac{1}{Z_1(H/T)(1+r_1(H/T))}\cdot\sum V_2 Z_2 P_1$$

where $V_2 = (S_2 - 7)^+$
$V_2(HH) = 5$
$V_2(HT) = 1$
$V_2(TH) = 1$
$V_2(TT) = 0$
calculate $V_1(H)$

In [47]:
```
v = [5, 1]
p = [2/3, 1/3]
z = [9/16, 9/8]
res = 0
for i, j, k in zip(v, p, z):
    res += i * j * k
1/(3/4 * (1 + 1/4)) * res
```

Out[47]: 2.4

calculate $V_1(T)$

In [55]:
```
2/3 * 3/8 * (1 / ((3/2)*(1 + 1/2)))
```

Out[55]: 0.1111111111111111

calculate V_0

$$V_0 = \mathbb{E}[\frac{Z_2}{(1+r_0)(1+r_1)}V_2] =$$

$$= \frac{1}{(1+1/4)(1+1/4)}\cdot(Z(HH)V_2(HH)P(HH) + Z(HT)V_2(HT)P(HT) + Z(TH)V(TH)P(TH)) =$$

$$= \frac{4}{5}\cdot(\frac{4}{5}\frac{9}{16}\cdot 5\cdot\frac{1}{9} + \frac{2}{3}\frac{9}{8}\frac{2}{9} + \frac{3}{8}\frac{2}{9}) = 1.00(4)$$

In [68]:
```
(4/5) * (4/5 * 9/16 * 4 /9 * 5 + 4/5 * 9/8 * 2/9 + 2/3 * 3/8 * 2/9)
```

Out[68]: 1.0044444444444445

6 Exercise 3.6

$U(x) = ln(x)$ is utility function

By Lagrange multipliers method we get:

$$L(x, \lambda) = \sum_{m=1}^{M} p_m U(x_m) + \lambda(X_0 - \sum_{m=1}^{M} p_m x_m \zeta_m)$$

$$\forall k = 1..M : p_k U'(x_k) - \lambda p_k \zeta_k = 0$$

$$U'(x_k) = \lambda \zeta_k$$

(3.3.25)

Since $U(x) = \log(x)$

$$x_k = \frac{1}{\lambda \zeta_k}$$

$$X_n = I(\frac{\lambda Z}{(1+r)^N})$$

and by (3.3.26)

$$\mathbb{E}\left[\frac{Z}{(1+r)^N} I(\frac{\lambda Z}{(1+r)^N})\right] = X_0$$

where I in our case is $I = \frac{1}{x}$

$$X_0 = \mathbb{E}\left[\frac{Z}{(1+r)^N} \frac{(1+r)^N}{\lambda Z}\right]$$

we have $\lambda = \frac{1}{X_0}$

Hence $x_k = \frac{X_0}{\zeta_k}$

...check Δ

7 Exercise 3.7

Given: $U(x) = \frac{x^p}{p}, p < 1, p \neq 0$

Show:

$X_n = \frac{x_0(1+r)^N Z^{\frac{1}{p-1}}}{\mathbb{E}\left[Z^{\frac{p}{p-1}}\right]}$

Solution:

Again, by Lagrange method

$$L(x, \lambda) = \sum_{m=1}^{M} p_m U(x_m) + \lambda(X_0 - \sum_{m=1}^{M} p_m x_m \zeta_m)$$

from previous task

$$U'(x_k) = \lambda \zeta_k$$

...check Δ
$U'(x_k) = x_k^{p-1} = \lambda \frac{Z}{(1+r)^N}$
$x_k = \frac{\lambda^{\frac{1}{p-1}} Z^{\frac{1}{p-1}}}{(1+r)^{\frac{N}{p-1}}}$

$$\mathbb{E}\big[\frac{Z}{(1+r)^N} I\big(\frac{\lambda Z}{(1+r)^N}\big)\big] = X_0$$

$I(x) = x^{\frac{1}{p-1}}$ So we have

$$\mathbb{E}\big[\frac{Z}{(1+r)^N}\big(\frac{\lambda Z}{(1+r)^N}\big)^{\frac{1}{p-1}}\big] = X_0$$

$$X_0 = \frac{1}{(1+r)^N} \lambda^{\frac{1}{p-1}} \frac{1}{(1+r)^{\frac{N}{p-1}}} \mathbb{E} Z^{\frac{1}{p-1}+1}$$

$$\lambda = \frac{X_0^{p-1}(1+r)^{N(p-1)+N}}{(\mathbb{E} Z^{\frac{p}{p-1}})^{p-1}}$$

$$x_k = \frac{X_0(1+r)^{N+N/(p-1)}}{\mathbb{E} Z^{\frac{p}{p-1}}} \frac{Z^{\frac{1}{p-1}}}{(1+r)^{N/(p-1)}}$$

$$= X_n = \frac{x_0(1+r)^N Z^{\frac{1}{p-1}}}{\mathbb{E}\big[Z^{\frac{p}{p-1}}\big]}$$

8 Exercise 3.8

$x = I(y)$ is maximum point

$$\forall x : U(x) - yx \leq U(I(y)) - yI(y)$$

I)
$(U(x) - yx)'_x = 0$
y = U′(x)
Then $x = I(y)$ Since function concave and $x = I(y)$ is extreme point it's maximum point.

$$\forall x : U(x) - yx \leq U(I(y)) - yI(y)$$

II)

$$\mathbb{E}\big[U(X_n)\big] - \mathbb{E}\big[\frac{\lambda Z}{(1+r)^N} X_n\big] \leq \mathbb{E}\big[U(I(\frac{\lambda Z}{(1+r)^N}))\big] - \mathbb{E}\big[\frac{\lambda Z}{(1+r)^N} I(\frac{\lambda Z}{(1+r)^N})\big]$$

(3.3.26) is

$$\mathbb{E}\big[\frac{Z}{(1+r)^N} I\big(\frac{\lambda Z}{(1+r)^N}\big)\big] = X_0$$

(3.3.19) is

$$\overline{\mathbb{E}}\frac{X_N}{(1+r)^N} = X_0$$

$\mathbb{E}\big[\frac{\lambda Z}{(1+r)^N}X_n\big] = \lambda X_0$
$\mathbb{E}\big[U(I(\frac{\lambda Z}{(1+r)^N}))\big] = \mathbb{E}\big[U(X_N^*)\big]$
$\mathbb{E}\big[\frac{\lambda Z}{(1+r)^N}I(\frac{\lambda Z}{(1+r)^N})\big] = \overline{\mathbb{E}}\big[\frac{\lambda}{(1+r)^N}X_N^*\big] = \lambda X_0$