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Some notions of multivariate positive dependence

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Abstract

The authors study new notions of positive dependence that are associated to multivariate stochastic orders of positive dependence introduced recently by Colangelo et al. [J. Multivar. Anal., to appear]. In particular, they discuss the relationship of these new notions to other existing concepts of positive dependence.

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1. Introduction

Given a pair (X_1, X_2) of random variables, a concept of positive (negative) dependence reflects the tendency of the two random variables to assume concordant (discordant) values. Many notions of positive dependence have been introduced in the literature to mathematically describe this intuitive concept and, in most cases, they are based on the comparison, with respect to some stochastic order, of the distribution function of (X_1, X_2) with the distribution function that the pair would have if the variables were independent.

The work by Kimeldorf and Sampson (1987, 1989) represents an attempt to study dependence notions, orders, and association measures in a unified framework in the bivariate case. In particular, after postulating some desirable properties that positive dependence notions and orders should satisfy, they point out that any positive dependence

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order determines a positive dependence notion by comparing a bivariate distribution with the distribution corresponding to the independence hypothesis.

In recent years, some researchers attempted to generalize various positive dependence concepts from the bivariate case to the multivariate setting. In particular, Joe (1997) and Pellerey and Semeraro (2003), respectively, generalized to the multivariate case the postulates for positive dependence orders and notions introduced by Kimeldorf and Sampson.

In this paper, we study multivariate notions of positive dependence that are connected to some positive dependence orders which were introduced and studied in Colangelo et al. (2005). After introducing some conventions in Section 2, we define in Section 3 the concept of positive dependence notion, and we also list some postulates that any reasonable such notion should satisfy; these postulates were introduced in Pellerey and Semeraro (2003). We also analyze there the relationships among some well known positive dependence notions that have appeared in the literature, reserving a particular attention to the concept of affiliation, which was introduced in Milgrom and Weber (1982).

In Section 4, we discuss some multivariate positive dependence notions, introduced by Ahmed et al. (1978), which generalize the well known bivariate concepts of left-tail decreasing (LTD) and right-tail increasing (RTI) random vectors.

In Section 5, some notions of positive dependence which were first discussed by Harris (1970) are introduced. We determine, in that section, their relationships to the other notions presented in the previous sections. In particular, we show that they imply the notions of Ahmed et al. (1978), and that they are implied by the notion of affiliation.

In Section 6, we study the multivariate notions of positive dependence that are connected to the positive dependence orders $\leq_{\mathrm{uoir}}$ and $\leq_{\mathrm{lodr}}$ introduced in Colangelo et al. (2005). The relationships of these concepts to the other multivariate notions are also discussed in detail. In particular, we show that notions generalizing the bivariate LTD and RTI concepts imply the new concepts. In addition, we show that a conditionally increasing in sequence (CIS) random vector need not be positive dependent according to the new notions, but, on the other hand, this is the case whenever it is conditionally increasing (CI).

Finally, in Section 7 we prove that the notions connected to another pair of strong orders, discussed in Colangelo et al. (2005), are equivalent to the notions associated with $\leq_{\mathrm{uoir}}$ and $\leq_{\mathrm{lodr}}$.

2. Conventions

Some conventions that are used in this paper are the following. By "increasing" and "decreasing," we mean "non-decreasing" and "non-increasing", respectively. A subset $A \subseteq \mathbb{R}^n$ is an increasing set if its indicator function I_A is increasing. For any two n-dimensional vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$, the notation $\boldsymbol{x} \leq \boldsymbol{y}$ means $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$. The minimum and maximum operators are respectively denoted by $\wedge$ and $\vee$ and apply componentwise to vectors. A subset $L \subseteq \mathbb{R}^n$ is called a sublattice if $\boldsymbol{x}, \boldsymbol{y} \in L$ implies that $\boldsymbol{x} \wedge \boldsymbol{y} \in L$ and $\boldsymbol{x} \vee \boldsymbol{y} \in L$. A function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be multivariate totally positive of order two (MTP_2) if it satisfies

$$h(\boldsymbol{x})h(\boldsymbol{y}) \leq h(\boldsymbol{x} \vee \boldsymbol{y})h(\boldsymbol{x} \wedge \boldsymbol{y}), \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n. \tag{1}$$

We denote by Δ_n the class of all n-dimensional distribution functions on $\mathbb{R}^n$ and, for any $F \in \Delta_n$ and $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, F_I represents the k-dimensional marginal distribution function of the underlying random variables with indices $i_1, \ldots, i_k$. We also denote by $\Gamma_n(F_1, \ldots, F_n)$, $n \geq 2$, the Fréchet class with marginal distribution functions $F_1, \ldots, F_n$, i.e., the subclass of Δ_n containing the distribution functions with the univariate marginals $F_1, \ldots, F_n$. The Fréchet upper and lower bounds in each Fréchet class $\Gamma_n(F_1, \ldots, F_n)$ are defined as

$$F^+(x_1, \ldots, x_n) = \min\{F_1(x_1), \ldots, F_n(x_n)\}$$

and

$$F^-(x_1, \dots, x_n) = \max\left\{\sum_{i=1}^{n} F_i(x_i) - n + 1, 0\right\}.$$

These bounds are pointwise sharp and $F^+ \in \Gamma_n(F_1, \dots, F_n)$ for all $n \geq 2$, and $F^- \in \Gamma_2(F_1, F_2)$. For $n > 2$ Dall'Aglio (1972) showed that F^- is a distribution function if, and only if, its support satisfies some suitable conditions.

For every distribution function F of a random variable $\boldsymbol{X}$, denote $\bar{F}(\boldsymbol{x}) = \mathrm{P}(\boldsymbol{X} > \boldsymbol{x})$. For any $F, G \in \Gamma_n(F_1, \dots, F_n)$, F is said to be smaller than G in the positive upper orthant dependence order and we write

$$F \leq_{\text{puod}} G \Leftrightarrow \bar{F}(\boldsymbol{x}) \leq \bar{G}(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^n,$$

while F is said to be smaller than G in the positive lower orthant dependence order and we write

$$F \leq_{\text{plod}} G \Leftrightarrow F(\boldsymbol{x}) \leq G(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^n.$$

Whenever it holds that $F \leq_{\text{puod}} G$ and $F \leq_{\text{plod}} G$, we say that F is smaller than G in the multivariate positive orthant dependence order; this is denoted by $F \leq_{\text{mpod}} G$.

A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be supermodular if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ it satisfies

$$\phi(\boldsymbol{x}) + \phi(\boldsymbol{y}) \leq \phi(\boldsymbol{x} \wedge \boldsymbol{y}) + \phi(\boldsymbol{x} \vee \boldsymbol{y}).$$

For any $F, G \in \Gamma_n(F_1, \dots, F_n)$, F is said to be smaller than G in the supermodular order, denoted by $F \leq_{\text{sm}} G$, if $\mathrm{E}\{\phi(\boldsymbol{X})\} \leq \mathrm{E}\{\phi(\boldsymbol{Y})\}$ for all functions $\phi : \mathbb{R}^n \to \mathbb{R}$ that are supermodular, provided the expectations exist, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are any two random vectors with distribution functions F and G, respectively.

A random vector $\boldsymbol{X} = (X_1, \dots, X_n)$ is said to be conditionally increasing in sequence (CIS) if

$$\mathrm{P}(X_i > x_i \,|\, X_1 = x_1, \dots, X_{i-1} = x_{i-1})$$

is increasing in $x_1, \dots, x_{i-1}$ for all x_i, $i \in \{2, \dots, n\}$; see Barlow and Proschan (1975). A random vector $\boldsymbol{X} = (X_1, \dots, X_n)$ is said to be conditionally increasing (CI) if $(X_{i_1}, \dots, X_{i_n})$ is CIS for every permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$; see, e.g., Müller and Scarsini (2001).

3. Some preliminaries

A positive dependence notion is any criterion which can mathematically describe the tendency of the components of an n-variate random vector to assume concordant values. It is evident that any positive dependence notion is uniquely characterized by the subclass $\mathcal{P}_n^+$ of Δ_n containing all the n-variate distribution functions which exhibit positive dependence with respect to that criterion.

Let $\mathcal{F}_n^+$ and $\mathcal{I}_n$ denote the subclasses of Δ_n consisting, respectively, of the collection of the Fréchet upper bounds, and of the distribution functions corresponding to the independence hypothesis. Let $\mathcal{P}_{2,\text{PQD}}^+$ be the class of all the bivariate distribution functions that are positive quadrant dependent in the sense of Lehmann (1966), i.e., satisfying $F(x_1, x_2) \geq F_1(x_1)F_2(x_2)$ for all $(x_1, x_2) \in \mathbb{R}^2$, where F_1 and F_2 are the marginal distribution functions of F.

Pellerey and Semeraro (2003), generalizing the postulates proposed by Kimeldorf and Sampson (1989) in the bivariate case, suggested a list of desirable properties that any multivariate positive dependence notion, corresponding to the set $\mathcal{P}_n^+$ of Δ_n, should fulfill:

B1. If $F \in \mathcal{P}_n^+$ then $F_{i,j} \in \mathcal{P}_{2,\mathrm{PQD}}^+$ for all $i, j \in \{1, \dots, n\}$ with $i < j$.
B2. $\mathcal{F}_n^+ \subseteq \mathcal{P}_n^+$.
B3. $\mathcal{I}_n \subseteq \mathcal{P}_n^+$.
B4. If $(X_1, \dots, X_n) \sim F \in \mathcal{P}_n^+$, then $(\phi_1(X_1), \dots, \phi_n(X_n)) \sim G \in \mathcal{P}_n^+$ for all increasing functions $\phi_1, \dots, \phi_n : \mathbb{R} \to \mathbb{R}$.
B5. If $(X_1, \dots, X_n) \sim F \in \mathcal{P}_n^+$, then $(X_{i_1}, \dots, X_{i_n}) \sim G \in \mathcal{P}_n^+$ for all permutations $(i_1, \dots, i_n)$ of $(1, \dots, n)$.
B6. If $\{F_n : n \geq 1\} \subseteq \mathcal{P}_n^+$ is such that $F_n \xrightarrow{\mathrm{d}} F$ ($\xrightarrow{\mathrm{d}}$ denotes weak convergence), then $F \in \mathcal{P}_n^+$.
B7. If $(X_1, \dots, X_n) \sim F \in \mathcal{P}_n^+$, then $\boldsymbol{X}_I \sim G \in \mathcal{P}_{\mathrm{card}(I)}^+$ for all $I \subseteq \{1, \dots, n\}$.

We note that any multivariate positive dependence order $\preceq$ defines a positive dependence notion through the class

$$\mathcal{P}_{n,\preceq}^+ = \{F \in \Delta_n : F^\perp \preceq F\}, \tag{2}$$

which contains all the distribution functions that exhibit more positive dependence than the distribution function that corresponds to the independence hypothesis in the same Fréchet class. It is easy to verify that whenever $\preceq$ satisfies the postulates proposed by Joe (1997) for a positive dependence order, the corresponding notion fulfills all the properties B1–B7. For example, the orders $\leq_{\mathrm{puod}}$, $\leq_{\mathrm{plod}}$, $\leq_{\mathrm{mpod}}$, and $\leq_{\mathrm{sm}}$ satisfy Joe's postulates; see Joe (1997) and Müller and Scarsini (2000). Therefore the corresponding notions, which in the sequel will be denoted as $\mathcal{P}_{n,\mathrm{PUOD}}^+$, $\mathcal{P}_{n,\mathrm{PLOD}}^+$, $\mathcal{P}_{n,\mathrm{MPOD}}^+$, and $\mathcal{P}_{n,\mathrm{PSMD}}^+$, satisfy properties B1–B7.

Not all the positive dependence notions that have appeared in the literature can be derived via a corresponding multivariate stochastic order by condition (2). This is the case for the notion of association, introduced by Esary et al. (1967), and, at least not in the bivariate case, for the notions of CIS and CI; see the definitions of the latter two in Section 2. These will be denoted in the sequel by $\mathcal{P}_{n,\mathrm{A}}^+$, $\mathcal{P}_{n,\mathrm{CIS}}^+$, and $\mathcal{P}_{n,\mathrm{CI}}^+$, respectively. The definition of association will not be given here, but it should be pointed out that $\mathcal{P}_{n,\mathrm{A}}^+$ satisfies B1–B7, whereas this is not the case for $\mathcal{P}_{n,\mathrm{CIS}}^+$ and for $\mathcal{P}_{n,\mathrm{CI}}^+$ in view of their conditional nature.

It is well known that

$$\mathcal{P}_{n,\mathrm{CI}}^+ \subset \mathcal{P}_{n,\mathrm{CIS}}^+ \subset \mathcal{P}_{n,\mathrm{A}}^+ \subset \mathcal{P}_{n,\mathrm{PSMD}}^+ \subset \mathcal{P}_{n,\mathrm{MPOD}}^+ \subset \mathcal{P}_{n,\mathrm{PUOD}}^+ \cap \mathcal{P}_{n,\mathrm{PLOD}}^+;$$

see Müller and Stoyan (2002), Christofides and Vaggelatou (2004).

An interesting positive dependence notion was studied in Karlin and Rinott (1980). Assume that the random vector $\boldsymbol{X} = (X_1, \dots, X_n)$ has a probability measure which is absolutely continuous with respect to some σ-finite product measure $\lambda = \lambda_1 \times \cdots \times \lambda_n$. Then $\boldsymbol{X}$ is called MTP_2 if its density is an MTP_2 function, i.e., if it satisfies (1) for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, except perhaps on a set of null λ-measure. Milgrom and Weber (1982) extended this definition to arbitrary probability measures and called the concept affiliation. Their extension allows us to consider distributions that are not dominated by product measures, like, for instance, the Marshall–Olkin distribution or the Fréchet upper bound.

Definition 1. Let $\boldsymbol{X}$ be a random vector on $\mathbb{R}^n$ with probability measure μ. Then $\boldsymbol{X}$ or μ are said to be affiliated if

$$\mu(A \cap B \big| L) \geq \mu(A \big| L)\mu(B \big| L) \tag{3}$$

for all increasing sets A and B and all sublattices $L \subset \mathbb{R}^n$.

Whenever μ is absolutely continuous with respect to a σ-finite product measure, affiliation is equivalent to the requirement that the density f of μ is MTP_2.

It is easy to show that affiliation satisfies postulates B1, B4, and B5. Affiliation also satisfies postulate B3 because independent random variables have an MTP_2 density with respect to their probability measure which is a σ-finite product measure. Furthermore, affiliation satisfies postulate B7; just consider upper sets A and B, and a sublattice L in $\mathbb{R}^{\mathrm{card}(I)}$ as upper sets and a sublattice in $\mathbb{R}^n$, in order to verify (3) for $\boldsymbol{X}_I$, $I \subseteq \{1, \ldots, n\}$. Müller and Stoyan (2002) proved that also B2 holds. In addition, it is not difficult to prove, by using (3) and postulate B4, that the notion of affiliation satisfies the following closure property.

Proposition 1. *If the random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ is affiliated, then $(\phi_1(X_1), \ldots, \phi_n(X_n))$ is affiliated for all decreasing functions $\phi_1, \ldots, \phi_n : \mathbb{R} \to \mathbb{R}$. Conversely, if $(\phi_1(X_1), \ldots, \phi_n(X_n))$ is affiliated for some strictly decreasing functions $\phi_1, \ldots, \phi_n : \mathbb{R} \to \mathbb{R}$, then $\boldsymbol{X}$ is affiliated.*

It is well known that MTP_2 implies the CI notion. Müller and Stoyan (2002) proved that affiliation implies CI too. They also provide a counterexample showing that none of the other positive dependence concepts discussed in this section imply affiliation.

4. Multivariate notions generalizing LTD and RTI

Two well known bivariate positive dependence notions are the left-tail decreasing (LTD) and the right-tail increasing (RTI) concepts introduced by Esary and Proschan (1972). Given a bivariate random vector $\boldsymbol{X} = (X_1, X_2)$, they defined X_2 to be left-tail decreasing in X_1 if $\mathrm{P}(X_2 \le x_2 \mid X_1 \le x_1)$ is decreasing in x_1, and X_2 to be right-tail increasing in X_1 if $\mathrm{P}(X_2 > x_2 \mid X_1 > x_1)$ is increasing in x_1, whenever the conditional probabilities are well defined. Ahmed et al. (1978) generalized the RTI notion to the multivariate case, while Block and Ting (1981) studied its relationships to other multivariate positive dependence concepts. Ebrahimi and Ghosh (1981) generalized the LTD property to the multivariate case and, in particular, studied in detail the negative dependence counterpart of it.

Definition 2. Let $\boldsymbol{X}$ be a random vector with values in $\mathbb{R}^n$. Then

(a) $\boldsymbol{X}$ is said to be left tail decreasing in sequence (LTDS) if

$$\mathrm{P}(X_i \le x_i \mid X_1 \le x_1, \ldots, X_{i-1} \le x_{i-1})$$

is decreasing in $x_1, \ldots, x_{i-1}$ for all x_i, $i \in \{2, \ldots, n\}$;

(b) $\boldsymbol{X}$ is said to be right tail increasing in sequence (RTIS) if

$$\mathrm{P}(X_i > x_i \mid X_1 > x_1, \ldots, X_{i-1} > x_{i-1})$$

is increasing in $x_1, \ldots, x_{i-1}$ for all x_i, $i \in \{2, \ldots, n\}$.

The following result gives one of the main properties of the positive dependence notions defined above. The proof is based on standard arguments and is therefore omitted.

Proposition 2. *Let $\boldsymbol{X}$ be a random vector with values in $\mathbb{R}^n$.*

(a) *If $\boldsymbol{X}$ is LTDS (RTIS) then $(\phi_1(X_1), \ldots, \phi_n(X_n))$ is LTDS* (RTIS) *for all increasing functions $\phi_1, \ldots, \phi_n : \mathbb{R} \to \mathbb{R}$. Conversely, if $(\phi_1(X_1), \ldots, \phi_n(X_n))$ is LTDS (RTIS) for some strictly increasing functions $\phi_1, \ldots, \phi_n : \mathbb{R} \to \mathbb{R}$, then $\boldsymbol{X}$ is LTDS (RTIS).*

(b) *If $\boldsymbol{X}$ is LTDS (RTIS) then $(\phi_1(X_1), \ldots, \phi_n(X_n))$ is RTIS (LTDS) for all decreasing functions $\phi_1, \ldots, \phi_n : \mathbb{R} \to \mathbb{R}$. Conversely, if $(\phi_1(X_1), \ldots, \phi_n(X_n))$ is RTIS (LTDS) for some strictly decreasing functions $\phi_1, \ldots, \phi_n : \mathbb{R} \to \mathbb{R}$, then $\boldsymbol{X}$ is LTDS (RTIS).*

Remark 1. It is well known that the notions PUOD and PLOD enjoy the same properties that are stated above for the RTIS and the LTDS concepts of positive dependence. In particular, $\boldsymbol{X}$ is PUOD if, and only if, $-\boldsymbol{X}$ is PLOD.

Using Proposition 2, and some simple additional arguments, it is easy to verify that the LTDS and RTIS notions satisfy all the postulates introduced in Section 3, with the obvious exception of B5.

It is interesting to consider the relationships between the notions defined above and the multivariate positive dependence concepts described in Section 3. It is easy to verify that

$$\mathcal{P}^+_{n,\mathrm{LTDS}} \subset \mathcal{P}^+_{n,\mathrm{PLOD}} \quad \text{and} \quad \mathcal{P}^+_{n,\mathrm{RTIS}} \subset \mathcal{P}^+_{n,\mathrm{PUOD}},$$

while Example 1 below, adapted from Block and Ting (1982), shows that $\mathcal{P}^+_{n,\mathrm{RTIS}}$ is not contained in $\mathcal{P}^+_{n,\mathrm{PLOD}}$ and therefore neither in $\mathcal{P}^+_{n,\mathrm{A}}$, $\mathcal{P}^+_{n,\mathrm{PSMD}}$, and $\mathcal{P}^+_{n,\mathrm{MPOD}}$. By Proposition 2 and Remark 1, it follows that also $\mathcal{P}^+_{n,\mathrm{LTDS}}$ is not contained in $\mathcal{P}^+_{n,\mathrm{A}}$, $\mathcal{P}^+_{n,\mathrm{PSMD}}$, and $\mathcal{P}^+_{n,\mathrm{MPOD}}$ since

$$\mathcal{P}^+_{n,\mathrm{LTDS}} \not\subseteq \mathcal{P}^+_{n,\mathrm{PUOD}}.$$

This is in contrast to what happens in the bivariate case, where the notions of LTD and RTI are known to imply association. In addition, notice that from the bivariate case we know that neither RTIS nor LTDS imply CIS or CI.

Example 1 ($\mathcal{P}^+_{n,\mathrm{RTIS}} \not\subseteq \mathcal{P}^+_{n,\mathrm{PLOD}}$). Let (X_1, X_2, X_3) be a random vector where the X_i are identically distributed Bernoulli random variables with parameter 0.7, and their joint distribution is determined by the probabilities

$$\mathrm{P}(X_1 = 0, X_2 = 0, X_3 = 1) = 0.2, \qquad \mathrm{P}(X_1 = 1, X_2 = 1, X_3 = 1) = 0.5,$$

$$\mathrm{P}(X_1 = 0, X_2 = 1, X_3 = 0) = \mathrm{P}(X_1 = 1, X_2 = 0, X_3 = 0) = \mathrm{P}(X_1 = 1, X_2 = 1, X_3 = 0) = 0.1.$$

An easy calculation shows that (X_1, X_2, X_3) is RTIS. On the other hand,

$$\mathrm{P}(X_1 \le 0, X_2 \le 0, X_3 \le 0) = 0 < 0.3^2 = \prod_{i=1}^{3} \mathrm{P}(X_i = 0),$$

so (X_1, X_2, X_3) is not PLOD.

Ahmed et al. (1978) and Block and Ting (1982) claimed that CIS does not imply RTIS. The following example shows, more generally, that $\mathcal{P}^+_{n,\mathrm{CI}} \not\subseteq \mathcal{P}^+_{n,\mathrm{RTIS}}$ and that $\mathcal{P}^+_{n,\mathrm{CI}} \not\subseteq \mathcal{P}^+_{n,\mathrm{LTDS}}$. This also shows that none of the concepts, considered in Section 3 above (with the exception of affiliation), imply the notions introduced in Definition 2.

Example 2 ($\mathcal{P}^+_{n,\mathrm{CI}} \not\subseteq \mathcal{P}^+_{n,\mathrm{RTIS}}$ and $\mathcal{P}^+_{n,\mathrm{CI}} \not\subseteq \mathcal{P}^+_{n,\mathrm{LTDS}}$). Let (X_1, X_2, X_3) be a random vector with probability masses 1/6 at the points $(-1, -1, 0)$, $(0, -1, 0)$, $(-1, 0, 1)$, $(1, 0, 1)$, $(0, 2, 2)$, and $(1, 2, 2)$. It is not hard to verify that $\boldsymbol{X}$ is CI. Now notice that

$$\mathrm{P}(X_3 > 1 \,|\, X_1 > -1, X_2 > -1) = 2/3 \not\le 1/2 = \mathrm{P}(X_3 > 1 \,|\, X_1 > 0, X_2 > -1),$$

so $\boldsymbol{X}$ is not RTIS. Also,

$$\mathrm{P}(X_3 \le 0 \big| X_1 \le -1, X_2 \le 0) = 1/2 \not\ge 2/3 = \mathrm{P}(X_3 \le 0 \big| X_1 \le 0, X_2 \le 0),$$

so $\boldsymbol{X}$ is not LTDS.

In the next section, we will prove that affiliation implies both LTDS and RTIS. It is trivial to verify that this implication is strict.

Two other notions of positive dependence may be defined by requiring the vectors under consideration to be RTIS or LTDS for all permutations of the variables.

Definition 3. Let $\boldsymbol{X}$ be a random vector with values in $\mathbb{R}^n$. Then

(a) $\boldsymbol{X}$ is said to be multivariate left tail decreasing (MLTD) if $(X_{i_1}, \ldots, X_{i_n})$ is LTDS for all permutations $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$;
(b) $\boldsymbol{X}$ is said to be multivariate right tail increasing (MRTI) if $(X_{i_1}, \ldots, X_{i_n})$ is RTIS for all permutations $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$.

An obvious version of Proposition 2 holds for the MLTD and the MRTI concepts, and they satisfy all the postulates B1–B7. In addition,

$$\mathcal{P}^+_{n,\mathrm{MRTI}} \subset \mathcal{P}^+_{n,\mathrm{RTIS}} \quad \text{and} \quad \mathcal{P}^+_{n,\mathrm{MLTD}} \subset \mathcal{P}^+_{n,\mathrm{LTDS}},$$

while they are not implied by any of the other positive dependence notions so far discussed (with the exception of affiliation). A straightforward calculation also shows that the vector in Example 1 is MRTI, so that

$$\mathcal{P}^+_{n,\mathrm{MRTI}} \not\subseteq \mathcal{P}^+_{n,\mathrm{MPOD}} \quad \text{and} \quad \mathcal{P}^+_{n,\mathrm{MLTD}} \not\subseteq \mathcal{P}^+_{n,\mathrm{MPOD}}.$$

Thus MRTI and MLTD do not imply any other notion of positive dependence among the ones previously introduced. In Section 5, we will prove that affiliation implies both MLTD and MRTI.

5. The multivariate LCSD and RCSI notions

In this section we study some notions of multivariate dependence which were introduced in Harris (1970).

Definition 4. Let $\boldsymbol{X}$ be a random vector with values in $\mathbb{R}^n$. Then

(a) $\boldsymbol{X}$ is said to be left corner set decreasing (LCSD) if $\mathrm{P}(\boldsymbol{X} \le \boldsymbol{x} \big| \boldsymbol{X} \le \boldsymbol{x}')$ is decreasing in $\boldsymbol{x}'$ for all $\boldsymbol{x} \in \mathbb{R}^n$;
(b) $\boldsymbol{X}$ is said to be right corner set increasing (RCSI) if $\mathrm{P}(\boldsymbol{X} > \boldsymbol{x} \big| \boldsymbol{X} > \boldsymbol{x}')$ is increasing in $\boldsymbol{x}'$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

Lee (1985) noted that if $\mathrm{P}(\boldsymbol{X} > \boldsymbol{x})$ is positive on a product space then $\boldsymbol{X}$ is RCSI if, and only, $\mathrm{P}(\boldsymbol{X} > \boldsymbol{x})$ is an MTP_2 function. Similarly, if $\mathrm{P}(\boldsymbol{X} \le \boldsymbol{x})$ is positive on a product space then $\boldsymbol{X}$ is LCSD if, and only, $\mathrm{P}(\boldsymbol{X} \le \boldsymbol{x})$ is an MTP_2 function.

It is easy to verify that the RCSI and the LCSD notions satisfy postulates B1–B7. Also, an obvious version of Proposition 2 holds for the RCSI and the LCSD concepts. The following result is due to Ahmed et al. (1978).

Proposition 3. *Let $\boldsymbol{X}$ be a random vector with values in $\mathbb{R}^n$.*

$x_1 \backslash x_2$	0	1	2
2	0	1/6	1/6
1	1/6	0	1/6
0	1/6	1/6	0

(a) *If $\boldsymbol{X}$ is RCSI, then $\boldsymbol{X}$ is MRTI.*
(b) *If $\boldsymbol{X}$ is LCSD, then $\boldsymbol{X}$ is MLTD.*

It is easy to see that these implications are strict, even in the bivariate case, as the following simple example shows.

Example 3 ($\mathcal{P}^+_{n,\mathrm{MRTI}} \not\subseteq \mathcal{P}^+_{n,\mathrm{RCSI}}$ and $\mathcal{P}^+_{n,\mathrm{MLTD}} \not\subseteq \mathcal{P}^+_{n,\mathrm{LCSD}}$). Let (X_1, X_2) be a random vector with probability mass function

An easy calculation shows that $\boldsymbol{X}$ is MRTI. Let now $\boldsymbol{x} = (0, 1)$, $\boldsymbol{x}' = (0, 0)$ and $\boldsymbol{x}'' = (1, 0)$. Then

$$\mathrm{P}(\boldsymbol{X} > \boldsymbol{x} \big| \boldsymbol{X} > \boldsymbol{x}') = 2/3 \not\leq 1/2 = \mathrm{P}(\boldsymbol{X} > \boldsymbol{x} \big| \boldsymbol{X} > \boldsymbol{x}''),$$

so $\boldsymbol{X}$ is not RCSI. Clearly $-\boldsymbol{X}$ is MLTD, but it is not LCSD.

It thus follows that also the relations

$$\mathcal{P}^+_{n,\mathrm{RCSI}} \subset \mathcal{P}^+_{n,\mathrm{RTIS}} \quad \text{and} \quad \mathcal{P}^+_{n,\mathrm{LCSD}} \subset \mathcal{P}^+_{n,\mathrm{LTDS}}$$

are strict. A straightforward calculation shows that the random vector in Example 1 is RCSI so that

$$\mathcal{P}^+_{n,\mathrm{RCSI}} \not\subseteq \mathcal{P}^+_{n,\mathrm{PLOD}}$$

and, in view of the relationship between RCSI and LCSD, we see that

$$\mathcal{P}^+_{n,\mathrm{LCSD}} \not\subseteq \mathcal{P}^+_{n,\mathrm{PUOD}}.$$

Therefore none of concepts introduced in Definition 4 imply any other notion of positive dependence which we have discussed so far.

Finally, in Example 2 we considered a random vector which is CI but which is neither RTIS nor LTDS, so that by Proposition 3 it follows that none of the notions CI, CIS, association, PSMD, MPOD, PUOD, and PLOD imply either RCSI or LCSD.

Block and Ting (1982) proved that the MTP_2 concept implies the RCSI and the LCSD notions. We now show that this result continues to hold for the notion of affiliation. It is easy to see that this implication is strict.

Theorem 1. *Affiliation implies RCSI and LCSD.*

Proof. We only prove that affiliation implies RCSI as the other implication follows by Proposition 1 and the relationship between the RCSI and LCSD notions. Assume that $\boldsymbol{X} = (X_1, \ldots, X_n)$ is affiliated and let μ be its probability measure. Let $\boldsymbol{x}', \boldsymbol{x}'' \in \mathbb{R}^n$ with $\boldsymbol{x}' \leq \boldsymbol{x}''$, and for all $\boldsymbol{y} \in \mathbb{R}^n$ define the sets

$$A = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} > \boldsymbol{x}''\}, \quad B_{\boldsymbol{y}} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} > \boldsymbol{x}' \vee \boldsymbol{y}\},$$

$$C = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} > \boldsymbol{x}'\}, \quad D_{\boldsymbol{y}} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} > \boldsymbol{x}'' \vee \boldsymbol{y}\}.$$

It is easy to verify that C is a sublattice, and that the sets A and $B_{\boldsymbol{y}}$ are increasing. Furthermore, $A \cap C = A$, $B_{\boldsymbol{y}} \cap C = B_{\boldsymbol{y}}$, and $A \cap B_{\boldsymbol{y}} \cap C = A \cap B_{\boldsymbol{y}} = D_{\boldsymbol{y}}$. Therefore, by (3),

$$\mathrm{P}(\boldsymbol{X} > \boldsymbol{x}'' \vee \boldsymbol{y})\mathrm{P}(\boldsymbol{X} > \boldsymbol{x}') = \mu(C)\mu(D_{\boldsymbol{y}}) \geq \mu(A)\mu(B_{\boldsymbol{y}}) = \mathrm{P}(\boldsymbol{X} > \boldsymbol{x}'')\mathrm{P}(\boldsymbol{X} > \boldsymbol{x}' \vee \boldsymbol{y})$$

for all $\boldsymbol{y} \in \mathbb{R}^n$. It is easy to see that the latter inequality is equivalent to RCSI. □

6. The orthant ratio notions of positive dependence

Colangelo et al. (2005) introduced the following positive dependence orders. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ be two random vectors with respective distribution functions F and G in the Fréchet class $\Gamma_n(F_1, \ldots, F_n)$. We say that $\boldsymbol{X}$ is smaller than $\boldsymbol{Y}$ in the lower orthant decreasing ratio order (denoted by $\boldsymbol{X} \leq_{\mathrm{lodr}} \boldsymbol{Y}$ or $F \leq_{\mathrm{lodr}} G$) if

$$\frac{\mathrm{P}(\boldsymbol{Y} \leq \boldsymbol{x})}{\mathrm{P}(\boldsymbol{X} \leq \boldsymbol{x})} \quad \text{is decreasing in } \boldsymbol{x} \in \{\boldsymbol{x} : \mathrm{P}(\boldsymbol{Y} \leq \boldsymbol{x}) > 0\}, \tag{4}$$

where in (4) we use the convention $a/0 \equiv \infty$ whenever $a > 0$. We say that $\boldsymbol{X}$ is smaller than $\boldsymbol{Y}$ in the upper orthant increasing ratio order (denoted by $\boldsymbol{X} \leq_{\mathrm{uoir}} \boldsymbol{Y}$ or $F \leq_{\mathrm{uoir}} G$) if

$$\frac{\mathrm{P}(\boldsymbol{Y} > \boldsymbol{x})}{\mathrm{P}(\boldsymbol{X} > \boldsymbol{x})} \quad \text{is increasing in } \boldsymbol{x} \in \{\boldsymbol{x} : \mathrm{P}(\boldsymbol{Y} > \boldsymbol{x}) > 0\},$$

where here, again, we use the convention $a/0 \equiv \infty$ whenever $a > 0$. These orders were shown to satisfy all the postulates proposed by Joe (1997) for a stochastic order of positive dependence but one; in particular, it is not true that any Fréchet upper bound is maximal in its Fréchet class with respect to these orders. The following result from Colangelo et al. (2005) shows the relationship between the orders $\leq_{\mathrm{uoir}}$ and $\leq_{\mathrm{lodr}}$.

Theorem 2. *Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ be two random vectors in the same Fréchet class. If $\boldsymbol{X} \leq_{\mathrm{lodr}} (\leq_{\mathrm{uoir}}) \boldsymbol{Y}$, then*

$$(\phi_1(X_1), \ldots, \phi_n(X_n)) \leq_{\mathrm{uoir}} (\leq_{\mathrm{lodr}}) (\phi_1(Y_1), \ldots, \phi_n(Y_n)) \tag{5}$$

for any decreasing functions $\phi_1, \ldots, \phi_n : \mathbb{R} \to \mathbb{R}$. Conversely, if condition (5) holds for some strictly decreasing functions $\phi_1, \ldots, \phi_n : \mathbb{R} \to \mathbb{R}$, then $\boldsymbol{X} \leq_{\mathrm{lodr}} (\leq_{\mathrm{uoir}})\boldsymbol{Y}$.

Although F^+ need not dominate, with respect to $\leq_{\mathrm{uoir}}$ and $\leq_{\mathrm{lodr}}$, every distribution function in the class $\Gamma_n(F_1, \ldots, F_n)$ to which it belongs, the next result, which is essentially from Colangelo et al. (2005), shows that F^+ is larger, with respect to these orders, than the distribution function $F^\perp$ that corresponds to the independence hypothesis.

Proposition 4. *Let $F^\perp$ and F^+ denote, respectively, the distribution function under the independence hypothesis, and the Fréchet upper bound in the Fréchet class $\Gamma_n(F_1, \ldots, F_n)$. Then*

$$F^\perp \leq_{\mathrm{lodr}} F^+ \quad \text{and} \quad F^\perp \leq_{\mathrm{uoir}} F^+.$$

Define the classes $\mathcal{P}^+_{n,\mathrm{PLODRD}}$ and $\mathcal{P}^+_{n,\mathrm{PUOIRD}}$ by condition (2), replacing $\preceq$ there by $\leq_{\mathrm{lodr}}$ and $\leq_{\mathrm{uoir}}$, as follows.

Definition 5. Let $\boldsymbol{X}$ be an n-dimensional random vector with distribution function F. Then

(a) $\boldsymbol{X}$ and F are said to be positively lower orthant decreasing ratio dependent (PLODRD) if

$$\frac{\mathrm{P}(X_1 \le x_1, \dots, X_n \le x_n)}{\mathrm{P}(X_1 \le x_1) \cdots \mathrm{P}(X_n \le x_n)} \quad \text{is decreasing in } \boldsymbol{x} \in \{\boldsymbol{x} : \mathrm{P}(\boldsymbol{X} \le \boldsymbol{x}) > 0\};$$

(b) $\boldsymbol{X}$ and F are said to be positively upper orthant increasing ratio dependent (PUOIRD) if

$$\frac{\mathrm{P}(X_1 > x_1, \dots, X_n > x_n)}{\mathrm{P}(X_1 > x_1) \cdots \mathrm{P}(X_n > x_n)} \quad \text{is increasing in } \boldsymbol{x} \in \{\boldsymbol{x} : \mathrm{P}(\boldsymbol{X} > \boldsymbol{x}) > 0\}.$$

In view of the properties of the orders $\le_{\text{lodr}}$ and $\le_{\text{uoir}}$ and Proposition 4, it is trivial to see that the positive dependence notions PLODRD and PUOIRD satisfy all the postulates B1–B7. In addition, Theorem 2 implies that an obvious version of Proposition 2 holds for these notions.

We now turn our attention to study the relationship of the notions PLODRD and PUOIRD to the notions that were introduced earlier. A simple calculation shows that

$$F \le_{\text{lodr}} G \Longrightarrow F \le_{\text{plod}} G \quad \text{and} \quad F \le_{\text{uoir}} G \Longrightarrow F \le_{\text{puod}} G,$$

and therefore it is easy to see that

$$\mathcal{P}^+_{n,\text{PUOIRD}} \subset \mathcal{P}^+_{n,\text{PUOD}} \quad \text{and} \quad \mathcal{P}^+_{n,\text{PLODRD}} \subset \mathcal{P}^+_{n,\text{PLOD}}.$$

The following result shows that

$$\mathcal{P}^+_{n,\text{MRTI}} \subset \mathcal{P}^+_{n,\text{PUOIRD}} \quad \text{and} \quad \mathcal{P}^+_{n,\text{MLTD}} \subset \mathcal{P}^+_{n,\text{PLODRD}}.$$

Theorem 3. *Let $\boldsymbol{X}$ be a random vector with distribution function $F \in \Delta_n$.*

(a) *If F is MLTD, then F is PLODRD.*
(b) *If F is MRTI, then F is PUOIRD.*

Proof. We just prove part (a) as part (b) directly follows from it by applying Theorem 2 and Proposition 2. Let the marginal distribution functions of F be $F_1, \dots, F_n$. Note that

$$\frac{F(\boldsymbol{x})}{\prod_{i=1}^n F(x_i)} = \frac{\prod_{i=2}^n \mathrm{P}(X_i \le x_i \,|\, X_1 \le x_1, \dots, X_{i-1} \le x_{i-1})}{\prod_{i=2}^n F_i(x_i)}$$

for all $\boldsymbol{x} \in \{\boldsymbol{x} \in \mathbb{R}^n : F(\boldsymbol{x}) > 0\}$. If F is MLTD then each term of the product at the numerator is decreasing in x_1. Therefore the ratio is decreasing in x_1. Since the MLTD notion is closed under permutations, the ratio decreases in each x_i, yielding that F is PLODRD. □

That the implications stated in Theorem 3 are strict is a consequence of Example 2 and Theorem 4 below. It is easy to verify that for bivariate random vectors the following stronger result holds.

Corollary 1. *Let $\boldsymbol{X} = (X_1, X_2)$ be a random vector with distribution function F. Then*

$x_1 \backslash x_2$	0	1	2
2	0	0	$1/5$
1	$1/5$	0	$1/5$
0	$1/5$	$1/5$	0

(a) *$F \in \mathcal{P}^+_{2,\mathrm{PLODRD}}$ if, and only if, (X_1, X_2) is MLTD;*
(b) *$F \in \mathcal{P}^+_{2,\mathrm{PUOIRD}}$ if, and only if, (X_1, X_2) is MRTI.*

Corollary 1 thus shows that, at least in the bivariate case, the orders $\leq_{\mathrm{lodr}}$ and $\leq_{\mathrm{uoir}}$ can be viewed as comparing, respectively, the degree of LTD-ness and RTI-ness of different random vectors. Colangelo et al. (2005) compared these orders to other stochastic orders, studied by Avérous and Dortet-Bernadet (2000) and by Hollander et al. (1990), which generalize the same positive dependence notions, and showed that no relationship exists between them.

Theorem 3 also shows that

$$\text{affiliation} \Rightarrow \text{RCSI} \Rightarrow \text{PUOIRD}$$

and that

$$\text{affiliation} \Rightarrow \text{LCSD} \Rightarrow \text{PLODRD},$$

and it is simple to verify that the implications are strict. In addition, from the characterization of the properties PLODRD and PUOIRD in the bivariate case, it is easy to see that they do not imply each other, and that

$$\mathcal{P}^+_{n,\mathrm{RTIS}} \nsubseteq \mathcal{P}^+_{n,\mathrm{PUOIRD}} \quad \text{and} \quad \mathcal{P}^+_{n,\mathrm{LTDS}} \nsubseteq \mathcal{P}^+_{n,\mathrm{PLODRD}};$$

the reverse implications also do not hold in view of Example 2 and Theorem 4 below. In Example 1, we considered a random vector that is MRTI but not PLOD; this implies that

$$\mathcal{P}^+_{n,\mathrm{PUOIRD}} \nsubseteq \mathcal{P}^+_{n,\mathrm{PLOD}},$$

and, by Remark 1 and Theorem 3, that

$$\mathcal{P}^+_{n,\mathrm{PLODRD}} \nsubseteq \mathcal{P}^+_{n,\mathrm{PUOD}}.$$

This also implies that both PLODRD and PUOIRD do not necessarily imply the positive dependence notions CI, CIS, association, PSMD, and MPOD.

The following examples present CIS random vectors that are neither PLODRD nor PUOIRD. It follows that none of the notions CIS, association, PSMD, MPOD, PUOD, and PLOD necessarily imply the PLODRD and the PUOIRD notions.

Example 4 ($\mathcal{P}^+_{n,\mathrm{CIS}} \nsubseteq \mathcal{P}^+_{n,\mathrm{PLODRD}}$). Let (X_1, X_2) be a random vector with probability mass function and let $(X_3 | X_1 = x_1, X_2 = x_2)$ assume each of the values $x_1 + x_2 - 1$ and $x_1 + x_2 + 1$ with probability $1/2$. It is easy to see that (X_1, X_2, X_3) is CIS. Denoting the distribution function of (X_1, X_2, X_3) by F, we see, for $\boldsymbol{x} = (0, 0, 2) \leq (0, 1, 2) = \boldsymbol{y}$, that $F(\boldsymbol{x})/F^{\perp}(\boldsymbol{x}) = 50/28 \ngeq 50/21 = F(\boldsymbol{y})/F^{\perp}(\boldsymbol{y})$. So (X_1, X_2, X_3) is not PLODRD.

Example 5 ($\mathcal{P}^+_{n,\mathrm{CIS}} \nsubseteq \mathcal{P}^+_{n,\mathrm{PUOIRD}}$). Let (X_1, X_2) be a random vector with probability mass function

$x_1 \backslash x_2$	0	1	2
2	0	1/5	1/5
1	1/5	0	1/5
0	1/5	0	0

and let $(X_3 \big| X_1 = x_1, X_2 = x_2)$ be as in Example 4. Again, (X_1, X_2, X_3) is CIS. Denoting the distribution function of (X_1, X_2, X_3) by F, we see, for $\boldsymbol{x} = (1, 0, 1) \le (1, 1, 1) = \boldsymbol{y}$, that $\overline{F}(\boldsymbol{x})/\overline{F^{\perp}}(\boldsymbol{x}) = 50/21 \not\le 50/28 = \overline{F}(\boldsymbol{y})/\overline{F^{\perp}}(\boldsymbol{y})$. So (X_1, X_2, X_3) is not PUOIRD.

We now proceed to proving that, contrary to the CIS notion, the stronger CI condition implies both the PLODRD and the PUOIRD concepts. We need the following notation and lemma. For any random vector $\boldsymbol{X}$, and a random variable Y, let $\boldsymbol{X} \uparrow_{\rm st} Y$ denote the fact that $\boldsymbol{X}$ is stochastically increasing in Y; i.e., that $\mathrm{E}\{h(\boldsymbol{X}) \big| Y = y\}$ is increasing in y for every increasing function h for which the above conditional expectations exist.

Lemma 1. *Let $\boldsymbol{X} = (X_1, \dots, X_n)$ be a random vector with values in $\mathbb{R}^n$. If $\boldsymbol{X}$ is CIS, then $(X_2, \dots, X_k) \uparrow_{\rm st} X_1$ for $k \in \{2, \dots, n\}$.*

This lemma is a minor variation of Theorem 5.3 of Block et al. (1985), and its proof is similar to the proof of that result. We omit the details.

We now can obtain the main result of this section.

Theorem 4. *Let $\boldsymbol{X}$ be a random vector with distribution function $F \in \Delta_n$. If $F \in \mathcal{P}^+_{n,\mathrm{CI}}$ then $F \in \mathcal{P}^+_{n,\mathrm{PLODRD}} \cap \mathcal{P}^+_{n,\mathrm{PUOIRD}}$.*

Proof. Let $F \in \mathcal{P}^+_{n,\mathrm{CI}}$, then, by Lemma 1, $\boldsymbol{X}_{J_i} \uparrow_{\rm st} X_i$ where $J_i = \{1, \dots, n\} \setminus \{i\}$ for $i \in \{1, \dots, n\}$. In order to prove that $F^{\perp} \le_{\rm lodr} F$, it suffices to verify that $F(\boldsymbol{x})/F_i(x_i)$ is decreasing in x_i whenever $F(\boldsymbol{x}) > 0$ for $i \in \{1, \dots, n\}$, where $F_1, \dots, F_n$ are the marginals of F. We will show that this is true for $i = 1$, and the stated result will then follow from the fact that the CI notion is closed under permutations. Note that $F(\boldsymbol{x})/F_1(x_1)$ is decreasing in x_1 if, and only if,

$$\int_{-\infty}^{x_1} \frac{F_{2,\dots,n|1}(x_2, \dots, x_n \big| t)}{\int_{-\infty}^{x_1} \mathrm{d}F_1(t)} \,\mathrm{d}F_1(t) \ge \int_{-\infty}^{x_1+h} \frac{F_{2,\dots,n|1}(x_2, \dots, x_n \big| t)}{\int_{-\infty}^{x_1+h} \mathrm{d}F_1(t)} \,\mathrm{d}F_1(t) \tag{6}$$

for all $h \ge 0$, where $F_{2,\dots,n|1}(\cdot, \dots, \cdot \big| t)$ denotes the conditional distribution function of $(X_2, \dots, X_n)$ given that $X_1 = t$. Some straightforward algebra shows that the inequality (6) is equivalent to

$$\int_{t=-\infty}^{x_1} \int_{s=x_1}^{x_1+h} F_{2,\dots,n|1}(x_2, \dots, x_n \big| t) \,\mathrm{d}F_1(s) \,\mathrm{d}F_1(t) \ge \int_{t=-\infty}^{x_1} \int_{s=x_1}^{x_1+h} F_{2,\dots,n|1}(x_2, \dots, x_n \big| s) \,\mathrm{d}F_1(s) \,\mathrm{d}F_1(t),$$

which is true since $F_{2,\dots,n|1}(x_2, \dots, x_n \big| t)$ is decreasing in t. Hence $F \in \mathcal{P}^+_{n,\mathrm{PLODRD}}$. The proof that $F \in \mathcal{P}^+_{n,\mathrm{PUOIRD}}$ is similar. □

Finally, notice that it is trivial to verify that the PLODRD notion does not imply the RCSI, MRTI, and the RTIS notions, while the PUOIRD notion does not imply the LCSD, MLTD, and the LTDS notions. That the reverse implications do not hold follows easily from Example 1 in view of the discussion at the end of Section 4.

In Fig. 1, we summarize the implications that were derived in this paper; a variation of that figure, with affiliation replacing MTP_2, is obviously also valid.

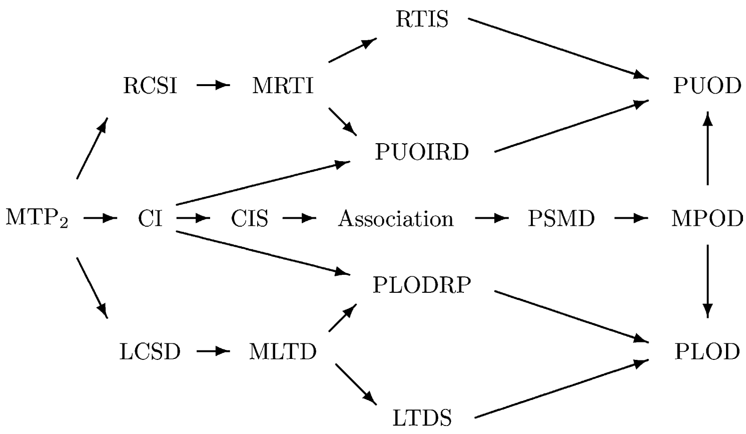

Fig. 1. Chart of implications.

7. Some further remarks

Colangelo et al. (2005) studied a pair of positive dependence orders that are stronger than $\leq_{\text{lodr}}$ and $\leq_{\text{uoir}}$. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ be two random vectors with respective distribution functions F and G in the Fréchet class $\Gamma_n(F_1, \ldots, F_n)$. Then $\boldsymbol{X}$ is said to be less dependent than $\boldsymbol{Y}$ in the strong lower orthant decreasing ratio sense, denoted by $F \leq_{\text{slodr}} G$ or $\boldsymbol{X} \leq_{\text{slodr}} \boldsymbol{Y}$, if

$$F(\boldsymbol{x})G(\boldsymbol{y}) \leq F(\boldsymbol{x} \vee \boldsymbol{y})G(\boldsymbol{x} \wedge \boldsymbol{y}), \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n,$$

whereas $\boldsymbol{X}$ is said to be less dependent than $\boldsymbol{Y}$ in the strong upper orthant increasing ratio sense, denoted by $F \leq_{\text{suoir}} G$ or $\boldsymbol{X} \leq_{\text{suoir}} \boldsymbol{Y}$, if

$$\bar{F}(\boldsymbol{x})\bar{G}(\boldsymbol{y}) \leq \bar{F}(\boldsymbol{x} \wedge \boldsymbol{y})\bar{G}(\boldsymbol{x} \vee \boldsymbol{y}), \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

Colangelo et al. (2005) verified that these orders fulfill most of the Joe (1997) postulates for a dependence stochastic order, and they proved an obvious analog of Theorem 2 for these orders. In addition, it is easy to verify that the orders $\leq_{\text{slodr}}$ and $\leq_{\text{suoir}}$ are indeed respectively stronger than $\leq_{\text{lodr}}$ and $\leq_{\text{uoir}}$; i.e.,

$$\boldsymbol{X} \leq_{\text{slodr}} \boldsymbol{Y} \Longrightarrow \boldsymbol{X} \leq_{\text{lodr}} \boldsymbol{Y} \quad \text{and} \quad \boldsymbol{X} \leq_{\text{suoir}} \boldsymbol{Y} \Longrightarrow \boldsymbol{X} \leq_{\text{uoir}} \boldsymbol{Y}. \tag{7}$$

Under additional assumptions, the converse is also true, as the following result from Colangelo et al. (2005) shows.

Lemma 2. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two random vectors with respective distribution functions F and G.*

(a) *If either F or G is MTP$_2$, then $\boldsymbol{X} \leq_{\text{lodr}} \boldsymbol{Y} \Longrightarrow \boldsymbol{X} \leq_{\text{slodr}} \boldsymbol{Y}$.*
(b) *If either $\bar{F}$ or $\bar{G}$ is MTP$_2$, then $\boldsymbol{X} \leq_{\text{uoir}} \boldsymbol{Y} \Longrightarrow \boldsymbol{X} \leq_{\text{suoir}} \boldsymbol{Y}$.*

In the earlier notation, let us now define $\mathcal{P}^+_{n,\text{PSLODRD}}$ and $\mathcal{P}^+_{n,\text{PSUOIRD}}$ to be the positive dependence notions associated with the orders $\leq_{\text{slodr}}$ and $\leq_{\text{suoir}}$. We have the following result.

Theorem 5. *For any $n \geq 2$, the following relations hold*:

$$\mathcal{P}^+_{n,\text{PSLODRD}} = \mathcal{P}^+_{n,\text{PLODRD}} \quad \textit{and} \quad \mathcal{P}^+_{n,\text{PSUOIRD}} = \mathcal{P}^+_{n,\text{PUOIRD}}.$$

Proof. Let $F \in \mathcal{P}^+_{n,\mathrm{PSLODRD}}$, then $F \geq_{\mathrm{slodr}} F^\perp$ and, in view of (7), $F \geq_{\mathrm{lodr}} F^\perp$, so that $F \in \mathcal{P}^+_{n,\mathrm{PLODRD}}$. Conversely, suppose that $F \in \mathcal{P}^+_{n,\mathrm{PLODRD}}$. Then $F \geq_{\mathrm{lodr}} F^\perp$ and, since $F^\perp$ is an MTP_2 function, we can apply Lemma 2 to show that $F \geq_{\mathrm{slodr}} F^\perp$, so that $F \in \mathcal{P}^+_{n,\mathrm{PSLODRD}}$. The second equivalence follows by a similar argument, upon noticing that $\overline{F^\perp}$ is also an MTP_2 function. $\square$

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