(a) Let $z \in \mathbb{R}^n$ be an n-vector. Show that $A = zz^T$ is positive semidefinite.

(b) Let $z \in \mathbb{R}^n$ be a *non-zero* n-vector. Let $A = zz^T$. What is the null-space of A? What is the rank of A?

(c) Let $A \in \mathbb{R}^{n\times n}$ be positive semidefinite and $B \in \mathbb{R}^{m\times n}$ be arbitrary, where $m, n \in \mathbb{N}$. Is BAB^T PSD? If so, prove it. If not, give a counterexample with explicit A, B.

3. **[0 points] Eigenvectors, eigenvalues, and the spectral theorem**

The eigenvalues of an $n \times n$ matrix $A \in \mathbb{R}^{n\times n}$ are the roots of the characteristic polynomial $p_A(\lambda) = \det(\lambda I - A)$, which may (in general) be complex. They are also defined as the the values $\lambda \in \mathbb{C}$ for which there exists a vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$. We call such a pair (x, λ) an *eigenvector, eigenvalue* pair. In this question, we use the notation $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ to denote the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$, that is,

$$\operatorname{diag}(\lambda_1, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

(a) Suppose that the matrix $A \in \mathbb{R}^{n\times n}$ is diagonalizable, that is, $A = T\Lambda T^{-1}$ for an invertible matrix $T \in \mathbb{R}^{n\times n}$, where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is diagonal. Use the notation $t^{(i)}$ for the columns of T, so that $T = [t^{(1)} \ \cdots \ t^{(n)}]$, where $t^{(i)} \in \mathbb{R}^n$. Show that $At^{(i)} = \lambda_i t^{(i)}$, so that the eigenvalues/eigenvector pairs of A are $(t^{(i)}, \lambda_i)$.

Can show, justifying every step w/ a property from the lin.alg section. I think I can do it

A matrix $U \in \mathbb{R}^{n\times n}$ is orthogonal if $U^T U = I$. The spectral theorem, perhaps one of the most important theorems in linear algebra, states that if $A \in \mathbb{R}^{n\times n}$ is symetric, that is, $A = A^T$, then A is *diagonalizable by a real orthogonal matrix*. That is, there are a diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$ and orthogonal matrix $U \in \mathbb{R}^{n\times n}$ such that $U^T AU = \Lambda$, or, equivalently,

$$A = U\Lambda U^T.$$

Let $\lambda_i = \lambda_i(A)$ denote the ith eigenvalue of A.

(b) Let A be symmetric. Show that if $U = [u^{(1)} \ \cdots \ u^{(n)}]$ is orthogonal, where $u^{(i)} \in \mathbb{R}^n$ and $A = U\Lambda U^T$, then $u^{(i)}$ is an eigenvector of A and $Au^{(i)} = \lambda_i u^{(i)}$, where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Can show, justifying every step w/ a property from the lin.alg section. I think I can do it

(c) Show that if A is PSD, then $\lambda_i(A) \geq 0$ for each i.