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WEEKLY TEST-02

DS AND AI

CALCULUS AND OPTIMIZATION

Q1

The least value of the funtion f(x) = 2cos x + x in the closed interval is $\left[0, \frac{\pi}{2}\right]$

(A) 2
(B) $\frac{\pi}{6} + \sqrt{3}$
(C) $\frac{\pi}{2}$
(D) None of these

Q2 Find the interval in which of the following function is decreasing-

f(x) = 10 − 6x − 2x^2

(A) (0,1)
(B) $\left(-\frac{3}{2}, \infty\right)$
(C) $(1, \infty)$
(D) $\left(-\frac{3}{2}, \frac{3}{2}\right)$

Q3 Let $f(x) = \int\limits_{\sin x}^{\cos x} e^{-t^2} dt, \text{Then } f'(\pi/4) \text{ equals}$

(A) $\sqrt{\frac{1}{e}}$
(B) $-\sqrt{\frac{2}{e}}$
(C) $\sqrt{\frac{2}{e}}$
(D) $-\sqrt{\frac{1}{e}}$

Q4 Let a be non-zero real number. Then $\lim\limits_{x \to a} \frac{1}{x^2 - a^2} \int_a^x \sin\left(t^2\right) dt$ equals

(A) $\frac{1}{2a} \sin\left(a^2\right)$
(B) $\frac{1}{2a} \cos\left(a^2\right)$
(C) $-\frac{1}{2a} \sin\left(a^2\right)$
(D) $-\frac{1}{2a} \cos\left(a^2\right)$

Q5 $If \;\; I = \int\limits_{-\pi}^{\pi} \frac{\cos^2 x}{1 + a^x} dx \text{ then}$

(A) $\frac{\pi}{4}$
(B) $\frac{\pi}{8}$
(C) $\frac{\pi}{3}$
(D) $\frac{\pi}{2}$

Q6 Definite integration of $\int\limits_2^3 \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} dx$ is-

(A) $\frac{1}{3}$
(B) $\frac{1}{4}$
(C) $\frac{1}{2}$
(D) 0

Q7 The minimum value of $u = xy + \frac{a^3}{x} + \frac{a^3}{y} \text{is ka}^2$, where k is__________.

(A) 3
(B) 1
(C) 2
(D) 5

Q8 Four small square of side x are cut out of a square of side 12 cm to make a tray by folding the edges. What is the value of x so that the tray has the maximum volume?

(A) 1
(B) 2
(C) 3
(D) 4

Q9 Find $\frac{\partial z}{\partial x}$ for the following function.

$x^2 \sin\left(y^3\right) + xe^{3z} - \cos\left(z^2\right) = 3y - 6z + 8$

(A) $\frac{2x \sin\left(y^3\right) + e^{3z}}{-6 - 3xe^{3z} - 2z \sin\left(z^2\right)}$
(B) $\frac{\sin\left(y^3\right) + e^{3x}}{-6 - 3xe^{3z} - 2z \sin\left(z^2\right)}$
(C) $\frac{e^{3x}}{-6 - 3xe^{3z} - 2z \sin\left(z^2\right)}$
(D) none of them

Q10 If $u = \tan^{-1}(x + y), \text{then } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} =$

(A) $\sin 2u$
(B) $\frac{1}{3} \sin 2u$
(C) $\frac{1}{2} \sin 2u$
(D) none of them

Q11 If $u(x, y) = \frac{x^2 + y^2}{\sqrt{x + y}}$, then value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

(A) 3u/4
(B) 3u/2
(C) 3u/8
(D) 3u/9

Q12 Find the 1st order partial derivatives of the following function wrt to s.

$g(s, t, v) = t^2 \ln(s + 2t)$
$- \ln(3v)\left(s^3 + t^2 - 4v\right)$

(A) $\frac{\partial g}{\partial s} = \frac{t^2}{s + 2t} - 3s^2 \ln(3v)$
(B) $\frac{\partial g}{\partial s} = \frac{t^2}{s + 2t} - 3s \ln(3v)$

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(C) $\frac{\partial g}{\partial s}=\frac{t^2}{s+2t}-3s^2$

(D) none of them

Q13 Find the length of the curve-

$y=\frac{x^5}{6}+\frac{1}{10x^3}\text{between } 1 \le x \le 2?$

(A) 1264/240 (B) 1263/240

(C) 1262/240 (D) 1261/240

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Answer Key

Q1	(C)
Q2	(B)
Q3	(B)
Q4	(A)
Q5	(D)
Q6	(C)
Q7	(A)
Q8	(B)
Q9	(A)
Q10	(C)
Q11	(B)
Q12	(A)
Q13	(D)

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Hints & Solutions

Q1 Text Solution:

f(x) = 2cosx + x

f'(x) = −2 sinx + 1

f'(x) = 0

−2sin x + 1 =0

sin x = $\frac{1}{2}$

x = $\frac{\pi}{6}$

Now f''(x) = −2 cosx

f''$\left(\frac{\pi}{6}\right)$ = −2 × $\frac{\sqrt{3}}{2} < 0$

Thus at $\frac{\pi}{6}$ it's a maxima.

Now let's check for extremities

f(0) = 2 and f $\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3}$

f $\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

Thus, least value at $\left(\frac{\pi}{2}\right)$.

Q2 Text Solution:

f(x) = $f(x) = \frac{d(10-6x-2x^2)}{dx}$

= − 6 − 4 x

f'(x) < 0

−6 −4x < 0

4x +6 > 0

$x > \frac{-3}{2}$

Q3 Text Solution:

Applying leibnitz rule

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$$

$$f(x) = \int_{\sin x}^{\cos x} e^{-t^2}dt$$

$$f'(x) = e^{-\cos^2 x}(-\sin x) - e^{-\sin^2 x}(\cos x)$$

$$f'\left(\frac{\pi}{4}\right) = e^{-1/2}\cdot\left(-\frac{1}{\sqrt{2}}\right) - e^{-1/2}\left(\frac{1}{\sqrt{2}}\right)$$

$$= -\sqrt{\frac{2}{e}}$$

Q4 Text Solution:

$$\lim_{x\to a}\frac{1}{x^2-a^2}\int_a^x \sin\left(t^2\right)dt \qquad \because \left(\frac{0}{0} form\right)$$

$$\lim_{x\to a}\frac{\frac{d}{dx}\int_a^x \sin t^2 dt}{\frac{d}{dx}(x^2-a^2)}$$

$$= \lim_{x\to a}\frac{\sin x^2}{2x}$$

$$= \frac{1}{2a}\sin a^2$$

Q5 Text Solution:

$$I = \int_{-\pi}^{\pi}\frac{\cos^2(-x)}{1+a^{-x}}dx \ : \ I = \int_{-\pi}^{\pi}\frac{a^x\cos^2 x}{1+a^x}dx$$

adding

$$2I = \int_{-\pi}^{\pi}\cos^2 x dx = 2$$

$$\int_0^{\pi}\cos^2 x dx \left[\because f(x) = \cos^2 x = f(-x)\right]$$

$$= 2\int_0^{\pi}\cos^2\cdot x dx \Rightarrow \left[x + \frac{\sin 2x}{2}\right]_0^{\pi} = \pi$$

2I = π

$I = \frac{\pi}{2}$

Q6 Text Solution:

Using property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_2^3\frac{\sqrt{2+3-x}}{\sqrt{5-(2+3-x)}+\sqrt{5-x}}dx = \int_2^3\frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx$$

$$I = \int_2^3\frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$$

Adding both we get

$$2I = \int_2^3\left(\frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}} + \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\right)dx = \int_2^3 1\cdot dx$$

$$= 3 - 2 = 1$$

Q7 Text Solution:

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$u = xy + \frac{a^3}{x} + \frac{a^3}{y}$ is ka^2

Evaluating the partial derivatives – and equating to 0

$u_x = 0$

$y - \frac{a^3}{x^2} = 0$

$x^2 y = a^3$

$u_y = 0$

$x - \frac{a^3}{y^2} = 0$

$y^2 x = a^3$

Equating $u_x = u_y$

we get $x = y$

Now putting $x = y$ *in* u_x

$x - \frac{a^3}{x^2} = 0$

$x = a = y$

$u_{xx} = \frac{2a^3}{x^3}, u_{yy} = \frac{2a^3}{y^3}$

$u_{xy} = 1$

Now checking for –

$u_{xx} . u_{yy} - (u_{xy})^2 = 4 - 1 = 3 > 0$

Now chceking for, $u_{xx}(a, a) = 2 > 0$

Thus at (a, a) *the function* $u(x, y)$ *has minima.*

$u(x, y) = a^2 + a^2 + a^2 = ka^2$

Thus $k = 3$

Q8 Text Solution:

Given side of the square = 12 cm.

Four small square of side x are cut out of a square.

Dimension of the tray is :-

Side length of bar of the tray = 12 – 2x cm

Height of the tray is x cm

Volume of the tray is V = $(12 - 2x)^2$ x cm^3

V = $(144 + 4x^2 - 48x)$ x

V = $4x^3 - 48x^2 + 144x$

$\frac{dv}{dx} = 12x^2 - 96x + 144$

for max volume put $\frac{dv}{dx} = 0$

$12x^2 - 96x + 144 = 0$

$x^2 - 8x + 12 = 0$

(x – 2) (x – 6) = 0

x = 2, x = 6

$\frac{d^2v}{dx^2} = 24x - 96$

at x = 2 → $\frac{d^2v}{dx^2} = -48$

Thus, x = 2 is maxima.

Volume is maximum when length of x is 2 cm.

Q9 Text Solution:

Okay, we are basically being asked to do implicit differentiation here and recall that we are assuming that z is in fact z (x, y) when we do our derivative work.

Let's get $\frac{\partial z}{\partial x}$ first and that requires us to differentiate with respect to x.

Differentiating the equation with respect to x gives,

$2x \sin(y^3) + e^{3z} + 3\frac{\partial z}{\partial x} xe^{3z} + 2z\frac{\partial z}{\partial x}\sin(z^2)$

$= -6\frac{\partial z}{\partial x}$

Solving for $\frac{\partial z}{\partial x}$ gives

$2x \sin(y^3) + e^{3x}$ →

$= (-6 - 3xe^{3z} - 2z\sin(z^2))\frac{\partial z}{\partial x}$

$$\boxed{\frac{\partial z}{\partial x} = \frac{2x \sin(y^3) + e^{3z}}{-6 - 3xe^{3z} - 2z\sin(z^2)}}$$

Q10 Text Solution:

Given that

tan u = x + y is a homogeneous of degree = 1

$\frac{\partial u}{\partial x} = \cos^2 u, \frac{\partial u}{\partial y} = \cos^2 u$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}(x + y)$

$\cos^2 u.(x + y)$

$\cos^2 u. \tan u = \frac{1}{2}\sin 2u$

$$\Rightarrow \boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u}$$

Q11 Text Solution:

Given u (x, y) = $\frac{x^2 + y^2}{\sqrt{x + y}}$

We can say that

$\Rightarrow u(\lambda x, \lambda y) = \frac{\lambda^2 x^2 + \lambda^2 y^2}{\sqrt{\lambda x + \lambda y}}$

$\Rightarrow u(\lambda x, \lambda y) = \frac{\lambda^2 (x^2 + y^2)}{\lambda^{1/2}\sqrt{x + y}}$

$\Rightarrow u(\lambda x, \lambda y) = \frac{\lambda^{3/2}(x^2+y^2)}{\sqrt{x+y}} u$

is a homogeneous function of degree $\frac{3}{2}$.

By Euler's Theorem,

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$

Q12 Text Solution:

For this problem It looks like we'll have three 1st order partial derivatives to compute.

Here are the three 1st order partial denvatives for this problem.

$$\boxed{\begin{aligned} &\frac{\partial g}{\partial s} = gs = \frac{t^2}{s+2t} - 3s^2 \ln(3u) \\ &\frac{\partial g}{\partial s} = gt = 2t\ln(s+2t) + \frac{2t^2}{s+2t} - 2s^2 \ln(3u) \\ &\frac{\partial g}{\partial s} = gv = 4\ln(3u) - \frac{s^3+t^2-4u}{u} \end{aligned}}$$

so wrt to s it will be

$\frac{\partial g}{\partial s} = \frac{t^2}{s+2t} - 3s^2 \ln(3v)$

Q13 Text Solution:

We can find the arc length to be $\frac{1261}{240}$ by the integral.

$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

Let us look at some details.

By taking the derivative,

$\frac{dy}{dx} = \frac{5x^4}{6} - \frac{3}{10x^4}$

So, the integrand looks like:

$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\left(\frac{5x^4}{6}\right) + \frac{1}{2} + \left(\frac{3}{10x^4}\right)^2}$

by completing the square

$= \sqrt{\left(\frac{5x^4}{6} + \frac{3}{10x^4}\right)^2} = \frac{5x^4}{6} + \frac{3}{10x^4}$

Now, we can evaluate the integral.

$L = \int_1^2 \left(\frac{5x^4}{6} + \frac{3}{10x^4}\right) dx = \left[\frac{x^5}{6} - \frac{1}{10x^3}\right]_1^2$

$= \frac{1261}{240}$

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