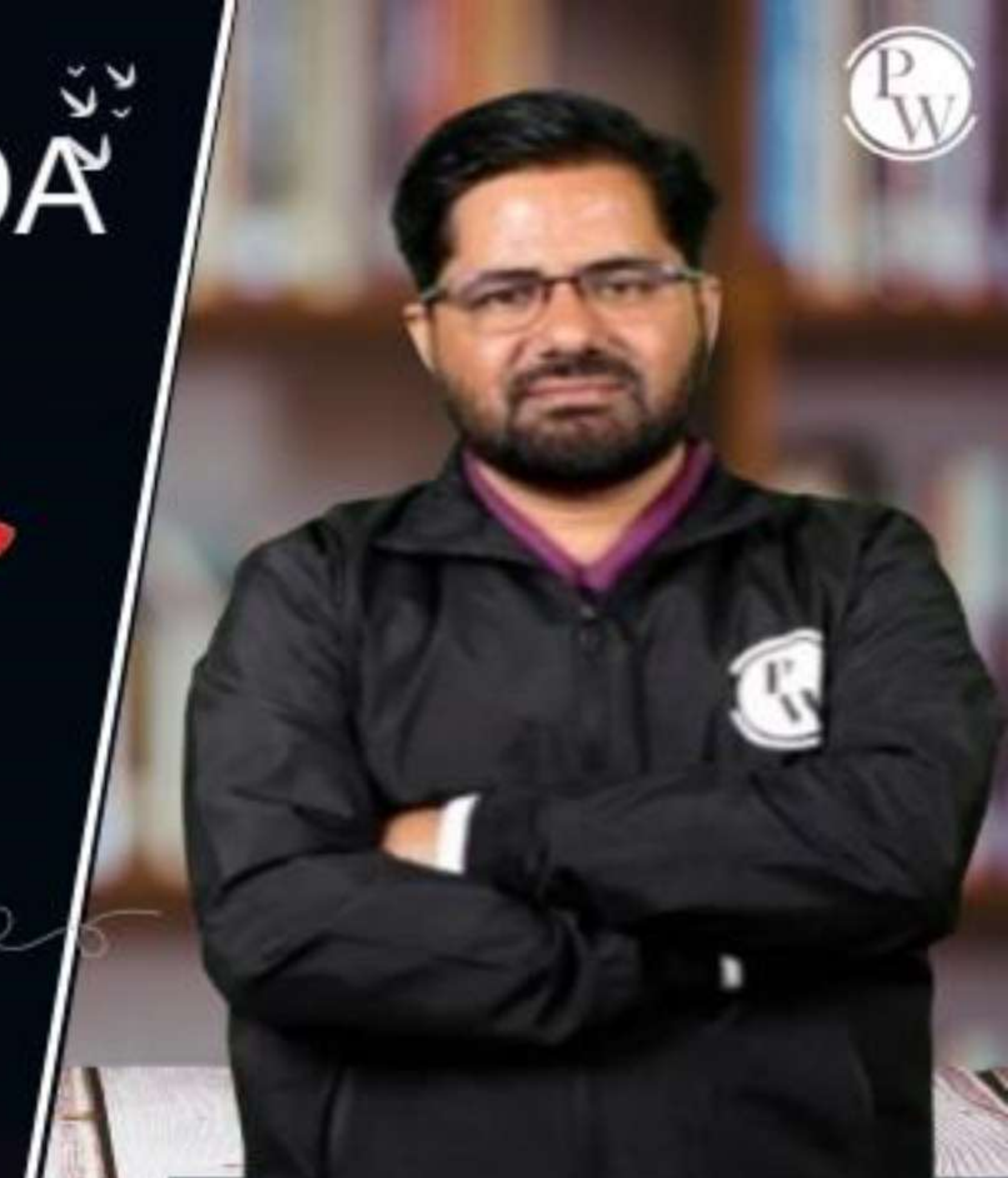
Computer Science & DA

Probability and Statistics

Continuous Random variable

Lecture No. 02



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Recap of previous lecture

Topic Basics of Continuous R Variable.

① $P(a<x<b)=\int_a^b f(x)\,dx$ = Area under $f(x)$ b/n a & b

② $P(-\infty<x<\infty)=\int_{-\infty}^{\infty} f(x)\,dx$ = Total area under $f(x)=1$

③ $$\boxed{E\{g(x)\}=\int_{-\infty}^{\infty} g(x)\,f(x)\,dx}$$

Topics to be Covered

Topic Uniform and Exponential Distribution

HW $y = f(x) = \begin{cases} 1+x, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \end{cases} = 1-|x|; \; -1 \le x \le 1$

$\text{Mean}(x) = E(x) = 0$

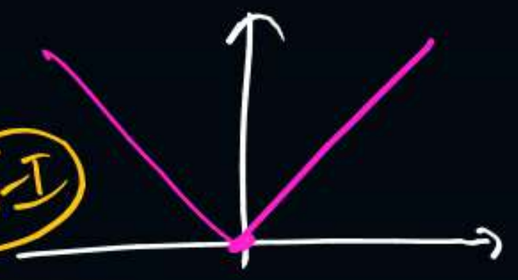
$E(x^2) = \frac{1}{6}$

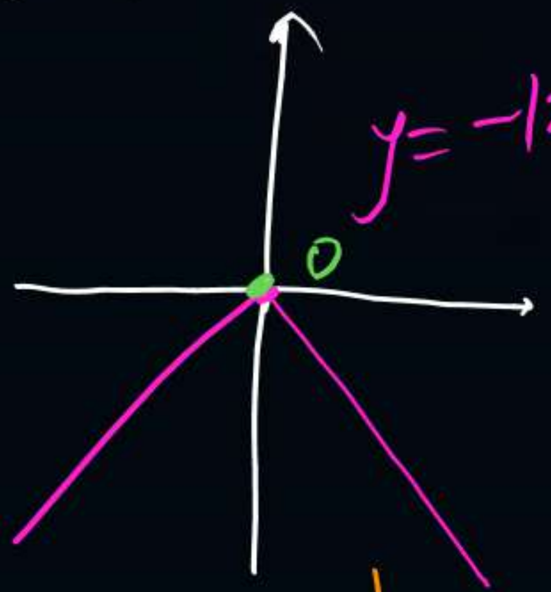
$\text{Var}(x) = \frac{1}{6}$

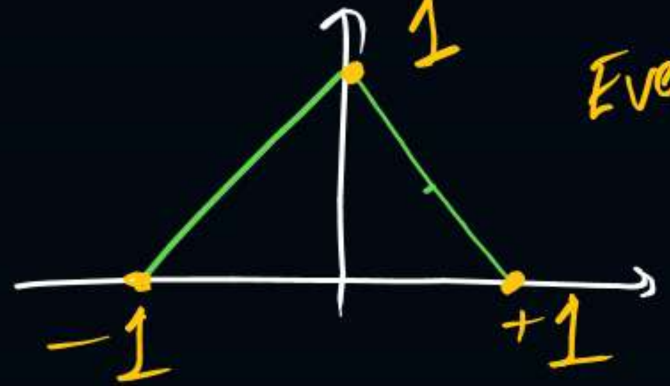
$SD(\sigma) = \frac{+1}{\sqrt{6}}$

$f(x) = \text{Even func}^n$

M-I







Even func^n

$y = (-|x|) + 1$

M-II

Case I: $-1 \le x \le 0$

$y = 1 + x$

$-x + y = 1$

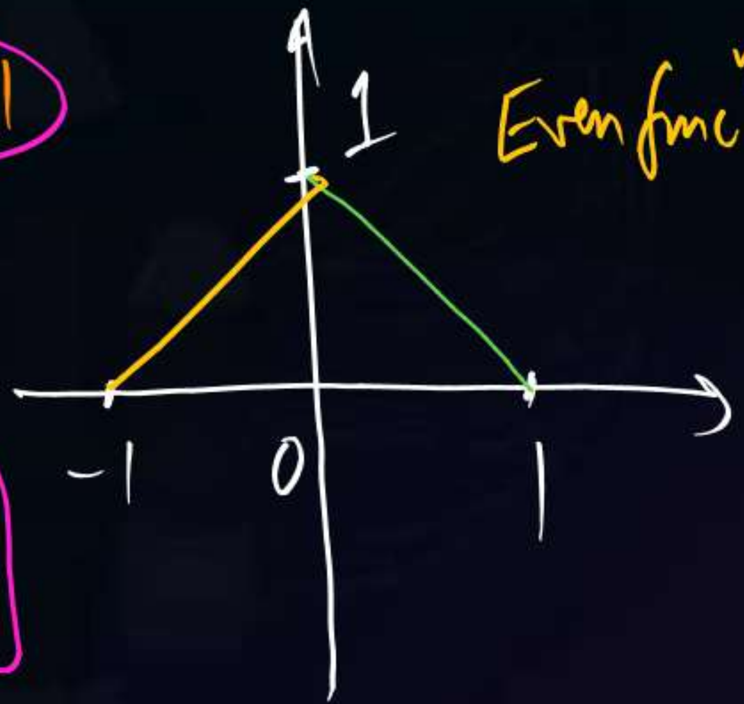
$\boxed{\frac{x}{-1} + \frac{y}{1} = 1}$

Case II: $0 \le x \le 1$

$y = 1 - x$

$x + y = 1$

$\boxed{\frac{x}{1} + \frac{y}{1} = 1}$



Even func^n

Q: if $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is p.d.f for x

and $g(x) = e^{3x/4}$ then find $E\{g(x)\} = ?$

Sol: $E\{g(x)\} = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx$

$$= \int_{-\infty}^{0} g(x)(0)\, dx + \int_{0}^{\infty} g(x)(e^{-x})\, dx$$

$$= 0 + \int_{0}^{\infty} e^{\frac{3x}{4}} \cdot e^{-x}\, dx = \int_{0}^{\infty} e^{-\frac{x}{4}}\, dx$$

$$= \left(\frac{e^{-\frac{x}{4}}}{-\frac{1}{4}} \right)_0^{\infty} = -4\left[e^{-\infty} - e^{0}\right] = -4(0-1) = 4$$

Q: if $f(x) = \begin{cases} kx+1; & 0 \leq x \leq 4 \\ 0, & otw \end{cases}$ is density func^n for C.R.V x then $k = ?$

p.d.f

(a) $-\frac{3}{8}$ (b) $-\frac{8}{3}$, (c) $\frac{8}{3}$, (d) $f(x)$ can not be p.d.f for any k.

Sol: let $f(x)$ is p.d.f so

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_{0}^{4} f(x)\, dx = 1$$

$$\int_{0}^{4} (kx+1)\, dx = 1$$

$$\left(k \cdot \frac{x^2}{2} + x\right)_0^4 = 1$$

$$(k \cdot 8 + 4 - 0) = 1$$

$$k = -\frac{3}{8}$$

Hence,

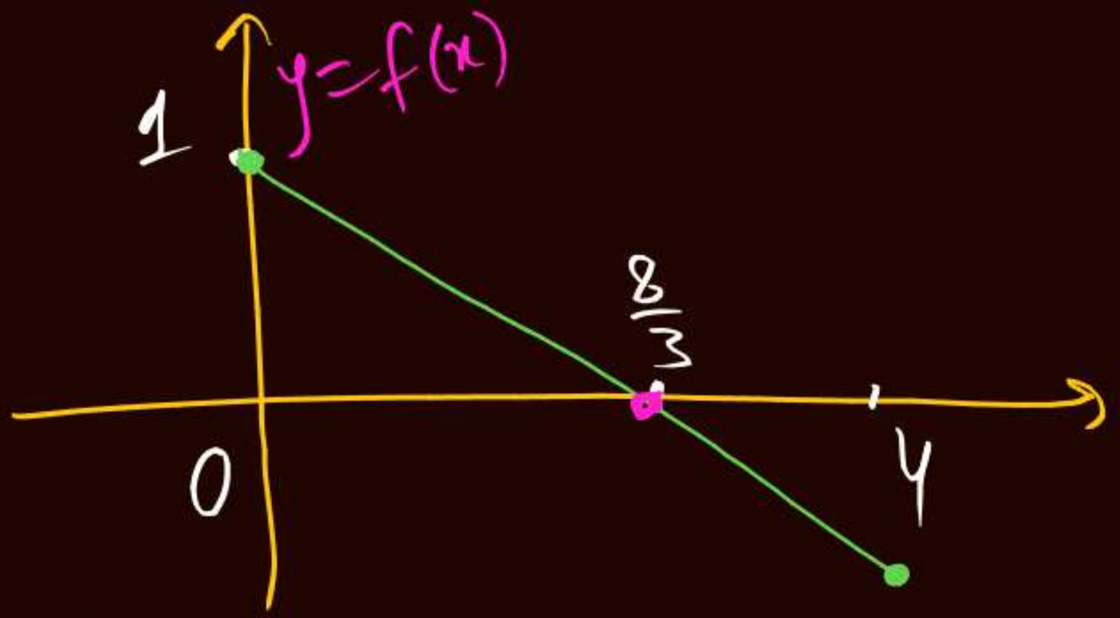
$$f(x) = \begin{cases} -\frac{3x}{8} + 1; & 0 \leq x \leq 4 \\ 0, & otw \end{cases}$$

CROSS Check:

M-I $y = -\frac{3x}{8} + 1 \quad ; \; 0 \le x \le 4$

$3x + 8y = 8$

$\frac{x}{8/3} + \frac{y}{1} = 1$



$\because$ Graph of f(x) lies below X axis so it can not be p.d.f

M-II

$$P\left(0 \le x < \frac{8}{3}\right) = \int_0^{8/3} f(x)\,dx = \int_0^{8/3} \left(-\frac{3}{8}x + 1\right) dx = \cdots\cdots = \frac{4}{3}$$

> 1 (Not possible)

$$P\left(\frac{8}{3} < x \le 4\right) = \int_{8/3}^{4} f(x)\,dx = \int_{8/3}^{4} \left(-\frac{3}{8}x + 1\right) dx = \cdots\cdots = -\frac{1}{3}$$

< 0 (Not possible)

While Total Prob $= \frac{4}{3} + \left(\frac{-1}{3}\right) = 1$

(x) Cummulative Distribution Function →

(C.D.F)

Let x is C.R.V and $f(x)$ is it's p.d.f then it's C.D.f is denoted by $F(x)$ and it is defined as;

$$\boxed{F(x) = \int_{-\infty}^{x} f(x)\,dx}$$

$F(-\infty) = 0$

$F(+\infty) = 1$

Note ① Graph of p.d.f can not lies below X axis. (T)

② " of C.D.F lies in b/w 0 & 1 on Y axis (T)

③ C.D.F = Int of p.d.f & p.d.f = Diff of C.D.F

Q. if $f(x) = a e^{-b|x|}$ is p.d.f then $F(5) = ?$

Sol: $F(5) = \int_{-\infty}^{5} f(x)\,dx = \int_{-\infty}^{0} a e^{-b|x|}\,dx + \int_{0}^{5} a e^{-b|x|}\,dx$

$$= \int_{-\infty}^{0} a e^{-b(-x)}\,dx + \int_{0}^{5} a e^{-b(+x)}\,dx$$

$$= a\left(\frac{e^{bx}}{b}\right)_{-\infty}^{0} + a\left(\frac{e^{-bx}}{-b}\right)_{0}^{5}$$

$$= \frac{a}{b}\left[1-0\right] - \frac{a}{b}\left[e^{-5b} - 1\right]$$

$$= \frac{a}{b}\left[2 - e^{-5b}\right]$$

Q. if $f(x) = a\,e^{-b|x|}$ is p.d.f then Evaluate C.D.F at x where $x > 0$

Sol: $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_{-\infty}^{0} a\,e^{-b|x|}\,dx + \int_{0}^{x} a\,e^{-b|x|}\,dx$$

$$= \int_{-\infty}^{0} a\,e^{-b(-x)}\,dx + \int_{0}^{x} a\,e^{-b(+x)}\,dx$$

$$= \text{---------} = \frac{a}{b}\left[2 - e^{-bx}\right]$$

(ii) find C.D.F at x where $x < 0$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} a\,e^{-b|x|}\,dx$$

$$= \int_{-\infty}^{x} a\,e^{-b(-x)}\,dx = a\left(\frac{e^{bx}}{b}\right)_{-\infty}^{x}$$

$$= \frac{a}{b}\left[e^{bx} - e^{-\infty}\right] = \frac{a}{b}e^{bx}$$ Ans

(iii) Also Evaluate $P(x > 0) = ?$

$$P(x>0) = \int_{0}^{\infty} f(x)\,dx = \int_{0}^{\infty} a\,e^{-b|x|}\,dx = \text{.....} = ? = \frac{a}{b}$$

Q. if $f(x) = \begin{cases} \frac{1}{8}, & 0 \le x \le 2 \\ \frac{3}{8}, & 2 < x \le 4 \\ 0, & otw \end{cases}$ is p.d.f for x then find it's Distribution func^n

Sol.

if $x < 0$ then $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} (0)\,dx = 0$

if $0 \le x \le 2$ then $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{0} (0) + \int_{0}^{x} f(x)\,dx$

$= 0 + \int_{0}^{x} \frac{1}{8}\,dx = \frac{x}{8}$

if $2 < x \le 4$ then $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$

$= 0 + \int_{0}^{2} \left(\frac{1}{8}\right) dx + \int_{2}^{x} \left(\frac{3}{8}\right) dx$

$= \frac{1}{8}(2-0) + \frac{3}{8}(x-2) = \frac{3x}{8} - \frac{1}{2}$

if $x > 4$, $F(x) = \int_{-\infty}^{x} f(x)\,dx = 1$

Doubt

(1) $F(5) = \int_{-\infty}^{0} (0)\,dx + \int_{0}^{2} \left(\frac{1}{8}\right) dx + \int_{2}^{4} \left(\frac{3}{8}\right) dx + \int_{4}^{5} (0)\,dx = 0 + ? + ? + 0 = 1$

(2) $2 < x \le 4$; let $x = 3.5$

$$F(3.5) = \int_{-\infty}^{3.5} f(x)\,dx = \int_{-\infty}^{0} + \int_{0}^{3.5}$$

$$= \cdots = ??$$

(3) $0 \le x \le 2$; let $x = 1.7$

$$F(1.7) = \int_{-\infty}^{1.7} f(x)\,dx = \int_{-\infty}^{0} + \int_{0}^{1.7}$$

$$= \cdots = ?$$

Final Conclusion:-

p.d.f is $f(x) = \begin{cases} \frac{1}{8}, & 0 \le x \le 2 \\ \frac{3}{8}, & 2 < x \le 4 \\ 0, & \text{otw} \end{cases}$

& it's C.D.F is

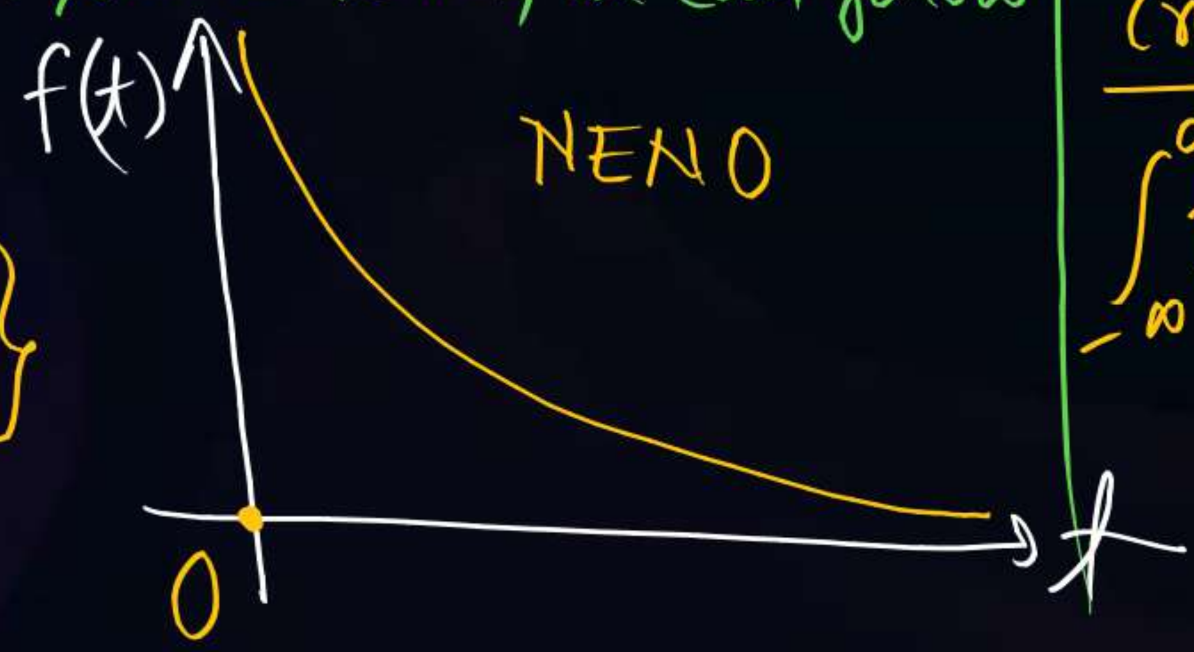
$$F(x) = \begin{cases} 0, & x < 0 \\ x/8, & 0 \le x \le 2 \\ \frac{3x}{8} - \frac{1}{2}, & 2 < x \le 4 \\ 1, & x > 4 \end{cases}$$

Exponential Distribution

(*) Arrivals of Customers in a Queue is governed by <u>POISSON</u> Distribution while their waiting time in a Queue is governed by <u>Exponential</u> Distribution.

(*) When in a Question we have feeling of waiting time or service time, we can follow E. Dist.

(*) $t = \{\text{waiting time}\}$



Def^n: Let t is C.R.V. s.t. it's p.d.f is defined as

$$\boxed{f(t) = \begin{cases} \mu e^{-\mu t}, & t \geq 0 \\ 0, & t < 0 \end{cases}}$$

then t is called E.R.V with parameter μ & it is denoted by $t \sim E\{\mu\}$

Cross check:→

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_0^{\infty} \mu e^{-\mu t}\,dt = \mu\left[\frac{e^{-\mu t}}{-\mu}\right]_0^{\infty}$$

$$= -1\left[e^{-\infty} - e^{0}\right] = 1$$

① $\mu \longrightarrow$ parameter of E-Dist.

or $\mu \longrightarrow$ Service Rate of service provider

② $\text{Mean}(t) = E(t) = \int_{-\infty}^{\infty} t \cdot f(t)\,dt = \int_{0}^{\infty} t \cdot f(t)\,dt$

$$= \int_{0}^{\infty} t \cdot \mu e^{-\mu t} \cdot dt = \cdots\cdots = \frac{1}{\mu} \approx \text{Av. waiting time}$$

③ $\text{Var}(t) = E(t^2) - E^2(t) = \cdots\cdots\cdots = \frac{1}{\mu^2}$

④ $SD(\sigma) = +\sqrt{\text{Var}(t)} = \frac{1}{\mu}$

eg In E-Dist, $\boxed{\text{Mean} = SD}$

⑤ $P(t_1 < t < t_2) = \int_{t_1}^{t_2} f(t)\,dt$

⑥ Inter Arrival time b/n two successive arrivals follow Exponential Dist.

Q: If the Average length of the phone Call at public Telephone booth is of 10 mins and when you are about to start your Call, someone arrives immediately ahead of you and starts calling then find the Prob that you will have to wait b/w 10 & 20 mins?

Sol: $t = \{\text{waiting time at P.C.O}\} = \{\text{Length of Phone Call}\}$

Average waiting time $= \frac{1}{\mu} = 10 \text{ mins} \Rightarrow \mu = \frac{1}{10}$ per min & $f(t) = \begin{cases} \mu e^{-\mu t}, & t \geq 0 \\ 0, & \text{otw} \end{cases}$

$P(10 < t < 20) = ? = \int_{10}^{20} f(t)\,dt = \int_{10}^{20} \mu e^{-\mu t} \cdot dt = \mu \left[\frac{e^{-\mu t}}{-\mu}\right]_{10}^{20} = e^{-10\mu} - e^{-20\mu} = \frac{1}{e} - \frac{1}{e^2} = \frac{0.232}{1}$

$= \frac{232}{1000}$

i.e. out of 1000 persons facing same problem,

232 persons will have to wait b/w 10 & 20 mins.

Q: Consider a company that produces ceiling fan with parameter 0.0003 hr^{-1}, then find the % of fans that will provide more than 10000 hrs service?

Sol: $\mu = 0.0003\ (hr)^{-1}$,

Let Company produces N=100 fans

For single fan: $\rightarrow$

$t = \{$ Service hrs of this single fan $\}$

$$f(t) = \begin{cases} \mu e^{-\mu t}, & t \geq 0 \\ 0, & otw \end{cases}$$

$$P(t > 10000) = ? = 1 - P(0 \leq t \leq 10000)$$

$$= 1 - \int_0^{10000} f(t)\,dt = 1 - \int_0^{10000} \mu e^{-\mu t}\,dt = \cdots\cdots = e^{-10000\mu}$$

$$= e^{-3} = \frac{1}{e^3} = 0.0497 = \frac{0.049}{1} = \frac{4.9}{100} \simeq \frac{5}{100}$$

Hence only 5% fans will provide more than 10000 hrs service.

ANALYSIS :

1. Company will not plan such type of warranty (T)

2. Company has produced very $\boxed{\text{Bad}}$ fan (Senseless Quest.)

is not well defined

3. $\mu = 0.0003\,(hr)^{-1} \Rightarrow$ Av service Hrs $= \dfrac{1}{\mu} = \dfrac{1}{0.0003\,(hr)^{-1}} = 3333$ hrs.

Q: Traffic is moving at the Rate of $\boxed{360 \text{ Veh/hr}}$ at certain highway locations and arrivals of vehicles at the junction is governed by Poisson Distribution then find the prob. that GAP b/w two successive vehicles lies b/w 6 & 10 seconds?

Sol: $t = \{\text{GAP b/w two Successive vehicles in seconds}\}$

ATQ, $\lambda = 360 \text{ Veh/hr} = 6 \text{ veh/minute} = \frac{1}{10} \text{ Veh/sec}$.

ie Av. time b/w two successive Vehicles $= 10 \text{ sec} = \frac{1}{\mu} \Rightarrow \mu = \frac{1}{10}$

$$P(6 < t < 10) = \int_6^{10} f(t)\,dt = \int_6^{10} \mu e^{-\mu t}\,dt = \mu\left(\frac{e^{-\mu t}}{-\mu}\right)_6^{10} = e^{-6\mu} - e^{-10\mu} = \frac{1}{e^{0.6}} - \frac{1}{e} = \frac{0.1809}{1} = \frac{18}{100}$$

THANK - YOU