Computer Science & DA

Calculus and Optimization

Functions and Limit, Continuity & Differentiability

DPP Discussion Notes

By- Dr. Puneet Sharma Sir

[MCQ]

#Q. The domain of the function $y = f(x) = \sin^{-1}\left(\frac{x^2 - 3x + 2}{x^2 + 2x + 7}\right)$ is-

A. $[1, \infty]$

B. $[-1, 2]$

C. $[-1, \infty]$

D. $[-\infty, 2]$

$\sin y = \frac{x^2-3x+2}{x^2+2x+7} \Rightarrow -1 \le \frac{x^2-3x+2}{x^2+2x+7} \le 1$

$0 \le \frac{x^2-3x+2}{x^2+2x+7} + 1 \le 2 \Rightarrow 0 \le \frac{2x^2-x+9}{x^2+2x+7} \le 2$

$\frac{2x^2-x+9}{x^2+2x+7} \geq 0$ & $\frac{2x^2-x+9}{x^2+2x+7} \le 2$

$2x^2-x+9 \geq 0$

$(\quad)^2 + (\quad) \geq 0$

$\boxed{x \in R}$ — 1

$2x^2-x+9-2x^2-4x-14 \le 0$

$-5x-5 \le 0 \Rightarrow x+1 \geq 0 \Rightarrow \boxed{x \geq -1}$ — 2

Ans $\Rightarrow$ $x \geq -1$ or $[-1, \infty)$

[MCQ]

$-\sqrt{a^2+b^2} \le (a\sin x + b\cos x) \le +\sqrt{a^2+b^2}$

#Q. What is the range of $f(x) = \cos 2x - \sin 2x$?

$y =$

$-\sqrt{2} \le y \le \sqrt{2}$

(A) $[2, 4]$

(B) $[-1, 1]$

(C) $[-\sqrt{2}, \sqrt{2}]$

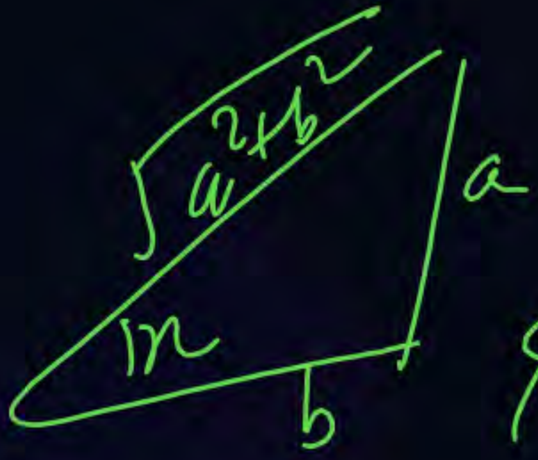
(D) $(-\sqrt{2}, \sqrt{2})$

$$y = a\sin x + b\cos x$$

$$y' = 0$$

$$a\cos x - b\sin x = 0$$

$$\tan x = \frac{a}{b}$$



$$\sin x = \frac{a}{\sqrt{a^2+b^2}}$$

$$\cos x = \frac{b}{\sqrt{a^2+b^2}}$$

$y'' = -ve \Rightarrow f(x)$ will be max for these values of $\sin x, \cos x, \tan x$

$$\text{Max } f(x) = a\left(\frac{a}{\sqrt{a^2+b^2}}\right) + b\left(\frac{b}{\sqrt{a^2+b^2}}\right)$$

$$= \sqrt{a^2+b^2}$$

[NAT]

#Q. A function f (x) is linear and has a value of 29 at x = -2 and 39 at x = 3. Find its value at x = 5.

$f(x) = ax + b$

$f(-2) = 29 \Rightarrow -2a + b = 29$

$f(3) = 39 \Rightarrow 3a + b = 39$

$-5a = -10 \Rightarrow a = 2$

$b = 33$

$= 2x + 33$

$\boxed{f(5) = 43}$

Ans

[MSQ]

#Q. Which of the following function is odd ?

A $x^2 - 2x + 3$

B $\sin x$

C $\sin x + \tan x$

D $\cos x$

$f(-x) = -f(x)$

[MSQ]

#Q. Which of the following functions is periodic ?

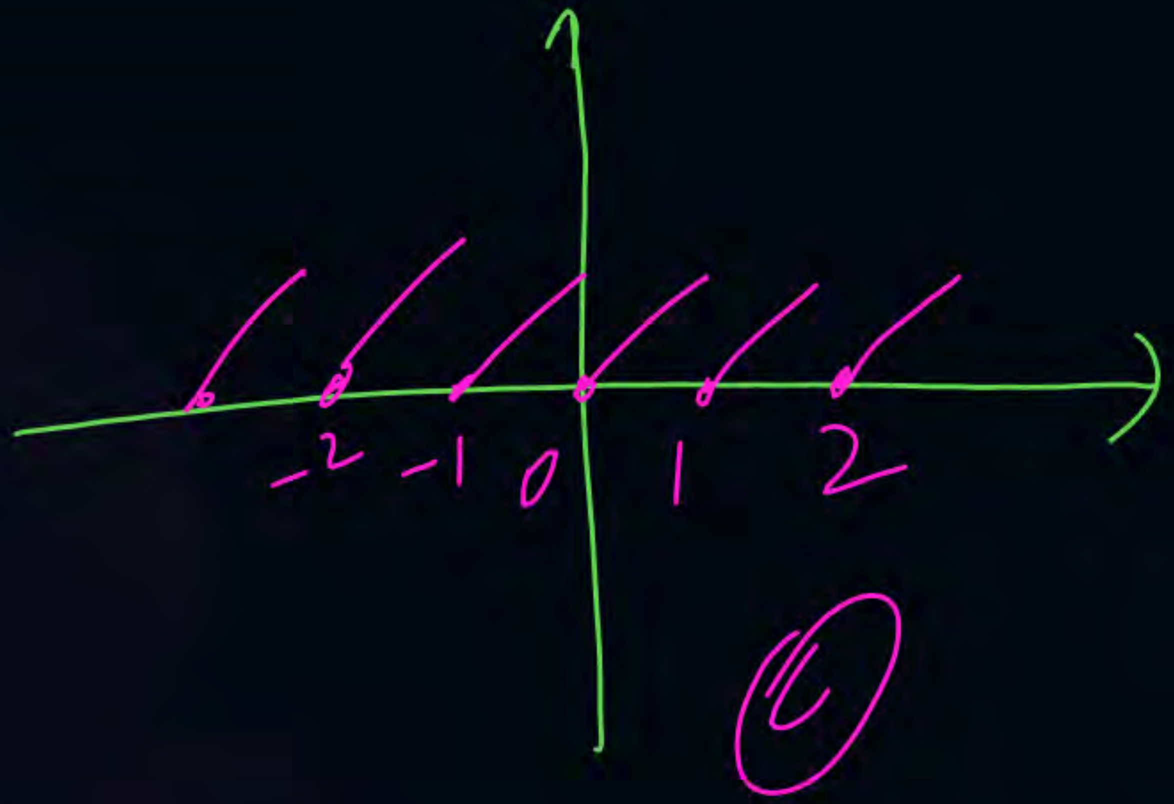
$f(x+T) = f(x)$

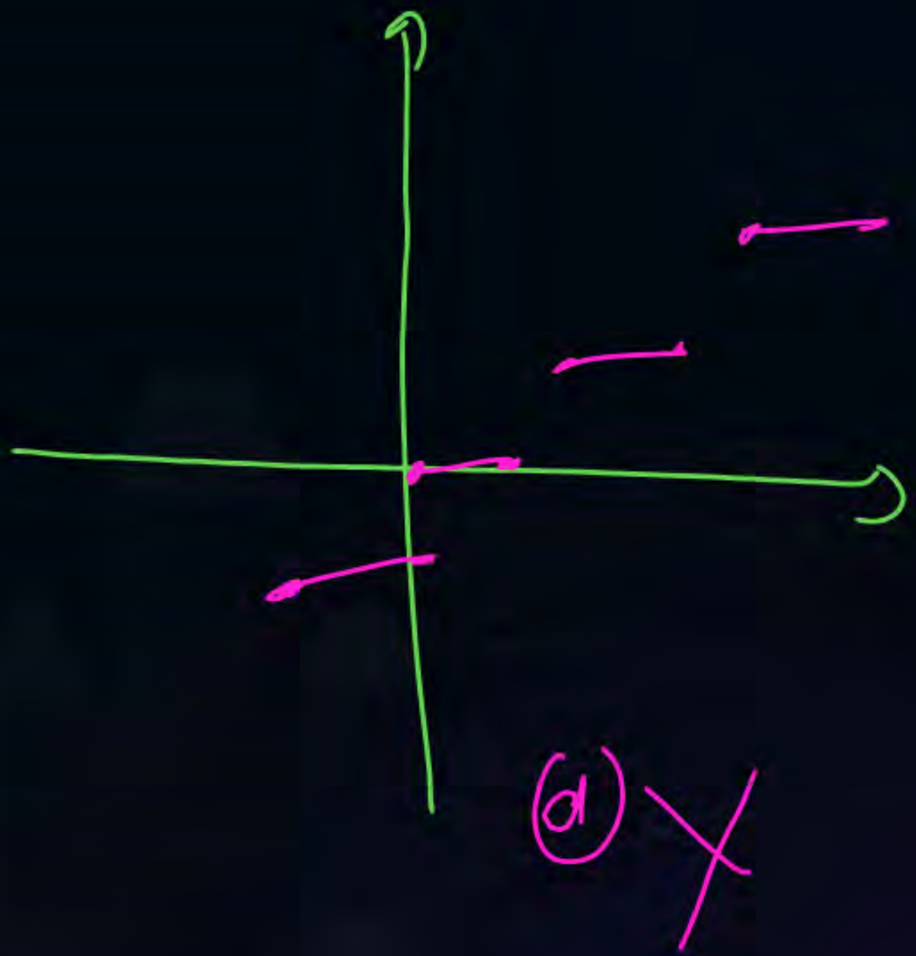
A $\sin x + \cos x$ ✓

B $e^x + \log x$ ✗

C $\{n\}$ ✓

D $[n]$ ✗





[NAT]

#Q. Evaluate $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$ $= \frac{0}{0} = \lim_{x\to 0} \left(\frac{\frac{1}{2\sqrt{4+x}} - 0}{1} \right) = \frac{1}{2\sqrt{4+0}} = \frac{1}{4}$

$= 0.25$

[NAT]

#Q. Evaluate $\lim_{x\to -1}\dfrac{(x+2)(3x-1)}{x^2+3x-2} \approx \dfrac{(-1+2)(-3-1)}{1-3-2} = \dfrac{1\times(-4)}{-4} = 1$

[MCQ]

#Q. At $x = 1$, the function $f(x) = \begin{cases} x^3 - 1, & 1 < x < \infty \\ x - 1, & -\infty < x \le 1 \end{cases}$

A. Continuous and differentiable

B. Continuous and non-differentiable

C. Discontinuous and differentiable

D. Discontinuous and non-differentiable

$LHL = 1 - 1 = 0$

$RHL = 1 - 1 = 0$

$f(1) = 0$

ie it is cont at $x = 1$

$f'(x) = \begin{cases} 3x^2, & 1 < x < \infty \\ 1, & -\infty < x < 1 \end{cases}$

$LHD = 1$ & $RHD = 3$

$\times$ Not Diff

[MCQ]

#Q. If $f(x) = x\left(\sqrt{x} - \sqrt{x+1}\right)$, then -

$$f'(0) = \lim_{x\to 0}\left[\frac{f(x) - f(0)}{x - 0}\right]$$

$$= \lim_{x\to 0}\left[\frac{x\left[\sqrt{x} - \sqrt{x+1}\right] - 0}{x - 0}\right]$$

$$= \lim_{x\to 0}\left(\sqrt{x} - \sqrt{x+1}\right) = 0 - \sqrt{1} = -1$$

A f(x) is continuous but not differentiable at x = 0

B f(x) is differentiable at x = 0

C f(x) is not differentiable at x = 0

D None of these

[MCQ]

#Q. If $\lim_{x\to\infty}\left(\sqrt{x^2-x+1}-ax\right)=b$, then the ordered pair (a,b) is:

it is in $(\infty-\infty)$ form only when $a>0$

A $\left(-1,\frac{1}{2}\right)$

B $\left(-1,-\frac{1}{2}\right)$

C $\left(1,-\frac{1}{2}\right)$

D $\left(1,\frac{1}{2}\right)$

$$\lim_{x\to\infty}\left(\sqrt{x^2-x+1}-ax\right)\left(\frac{\sqrt{x^2-x+1}+ax}{\sqrt{x^2-x+1}+ax}\right)=b$$

$$\lim_{x\to\infty}\left[\frac{(x^2-x+1)-a^2x^2}{\sqrt{x^2-x+1}+ax}\right]=b$$

$$\lim_{x\to\infty}\left[\frac{(1-a^2)x^2-x+1}{\sqrt{x^2-x+1}+ax}\right]=b \qquad (1-a^2)\Rightarrow 0\Rightarrow a=\pm1\Rightarrow a=1$$

$$\lim_{x\to\infty}\left(\frac{-x+1}{\sqrt{x^2-x+1}+ax}\right)=b$$

$$\lim_{x\to\infty}\left[\frac{-1+\frac{1}{x}}{\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+1}\right]=b$$

$$\frac{-1}{1+1}=b\Rightarrow b=-\frac{1}{2}$$

[MCQ]

#Q. The value of the function $f(x) = \lim_{x \to 0} \frac{x^3 + x^2}{2x^2 - 7x^2}$ is......

$= \lim_{x \to 0} \frac{x^2(x+1)}{-5x^2} = \lim_{x \to 0} \frac{x+1}{-5}$

$= -\frac{1}{5}$

A. 0

B. $\frac{-1}{7}$

C. $\frac{1}{7}$

D. $-1/5$

[NAT]

#Q. $\lim_{x\to 0} \dfrac{x-\sin x}{1-\cos x}$ is $= \dfrac{0}{0} = \lim_{x\to 0}\left(\dfrac{1-\cos x}{\sin x}\right) \simeq \dfrac{0}{0} = \lim_{x\to 0}\left(\dfrac{\sin x}{\cos x}\right) = \dfrac{0}{1}$

$= 0$

$\boxed{\text{Ans} = 0}$

[NAT]

#Q. $\underset{x\to 0}{\mathrm{Lt}}\left(\dfrac{e^{2x}-1}{\sin(4x)}\right)$ is equal to

M-II

$$\lim_{2x\to 0}\left(\frac{e^{2x}-1}{2x}\right)\times\lim_{4x\to 0}\left(\frac{4x}{\sin 4x}\right)\times\lim_{x\to 0}\left(\frac{2x}{4x}\right)$$

$$= 1 \times 1 \times \frac{1}{2} = 0.5$$

[MSQ]

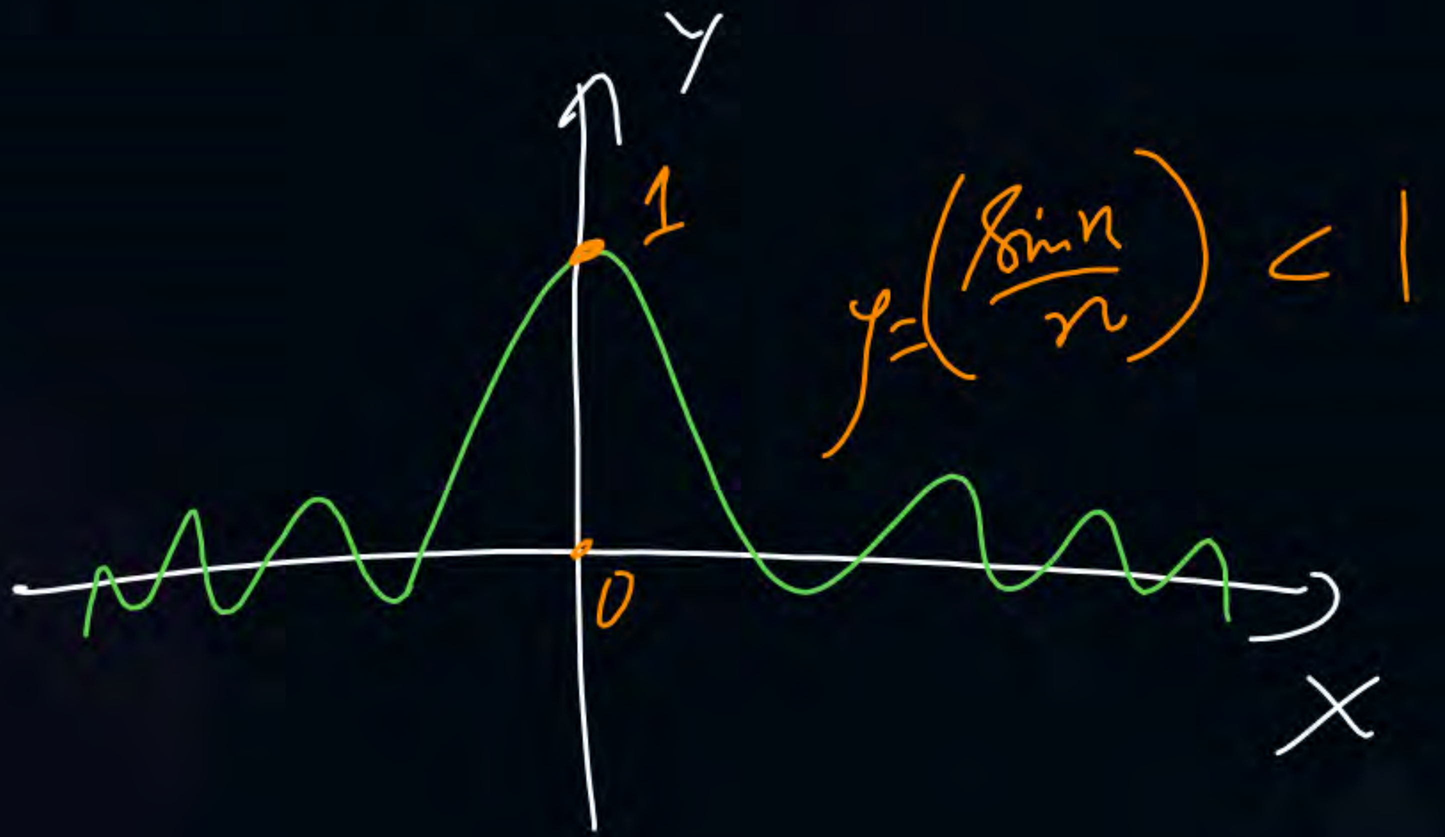
#Q. Which of the following values are correct

A $y = \frac{\sin x}{x} < 1$

B $\lim_{x \to 0} \frac{\sin x}{x} = 1$

C $\lim_{x \to 0} \frac{\sin x}{x} = 0$

D $\lim_{x \to 0} \frac{\sin x}{x} = -1$



[MSQ]

#Q. For the given function $f(x)=\begin{cases} \dfrac{x^2}{2} & ; \quad 0 \le x < 1 \\ 2x^2-3x+\dfrac{3}{2} & ; \quad 1 \le x \le 2 \end{cases}$

$\Rightarrow LHL = \frac{1}{2}$

$\Rightarrow RHL = \frac{1}{2}$

$\& f(1) = \frac{1}{2}$

A $f(x)$ is continuous $\forall x \in [0,2]$

B $f'(x)$ is continuous $\forall x \in [0,2]$

C $f''(x)$ is discontinuous at $x=1$

D $f''(x)$ is discontinuous $\forall x \in [0,2]$

$g(x) = f'(x) = \begin{cases} x, & 0 \le x < 1 \\ 4x-3, & 1 \le x \le 2 \end{cases}$

$LHL = 1, \; RHL = 1$

$h(x) = f''(x) = \begin{cases} 1, & 0 \le x < 1 \\ 4, & 1 \le x \le 2 \end{cases}$

$LHL = 1$

$RHL = 4$

[NAT]

#Q. Let $\alpha, \beta, \in \mathbb{R}$ be such that $\lim\limits_{x \to 0} \boxed{\dfrac{x^2 \sin(\beta x)}{\alpha x - \sin x}} = 1$. Then $6(\alpha + \beta)$ equals.

$$= 6\left(1 - \frac{1}{6}\right) = 5 \text{ Ans}$$

$$\lim_{x \to 0} \left[\frac{2x \sin \beta x + x^2 \cos \beta x (\beta)}{\alpha - \cos x}\right] = 1$$

if we take $\boxed{\alpha = 1}$ then LHS is in $\frac{0}{0}$ form.

$$\lim_{x \to 0} \left[\frac{2x \sin \beta x + \beta x^2 \cos \beta x}{1 - \cos x}\right] = 1$$

using L'Hosp Rule twice ..... $\Rightarrow$ $2\beta + 2\beta + 2\beta = -1 \Rightarrow \beta = -\frac{1}{6}$

[MCQ]

#Q. A function $f(x) = 1 - x^2 + x^3$ is defined in the closed interval [–1, 1]. The value of x, in the open interval (1, 1) for which the mean value theorem is satisfied, is

$\Rightarrow f'(x) = 3x^2 - 2x$

$f(-1) = 1 - 1 + (-1) = -1$

$f(1) = 1 - 1 + 1 = 1$

A) $-\frac{1}{2}$

B) $-\frac{1}{3}$

C) $\frac{1}{3}$

D) $\frac{1}{2}$

$$\frac{f(1) - f(-1)}{1 - (-1)} = f'(c)$$

$$\frac{1 - (-1)}{1 - (-1)} = 3c^2 - 2c$$

$$\boxed{3c^2 - 2c - 1 = 0}$$

$$3c^2 - 3c + c - 1 = 0$$

$$(3c + 1)(c - 1) = 0$$

$$c = \cancel{1}, -\frac{1}{3}$$

[MCQ]

#Q. The value of c in the Lagrange's mean value theorem of the function $f(x) = x^3 - 4x^2 + 8x + 11$ when $x \in [0,1]$ is :

$f(0) = 11$

$f(1) = 16$

$f'(x) = 3x^2 - 8x + 8$

A $\frac{4-\sqrt{5}}{3}$

B $\frac{\sqrt{7}-2}{3}$

C $\frac{2}{3}$

D $\frac{4-\sqrt{7}}{3}$

$\frac{16-11}{1-0} = 3c^2 - 8c + 8$

$3c^2 - 8c + 3 = 0$

$ax^2 + bx + c = 0 \rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\boxed{c = \frac{4-\sqrt{7}}{3}}$

[MCQ]

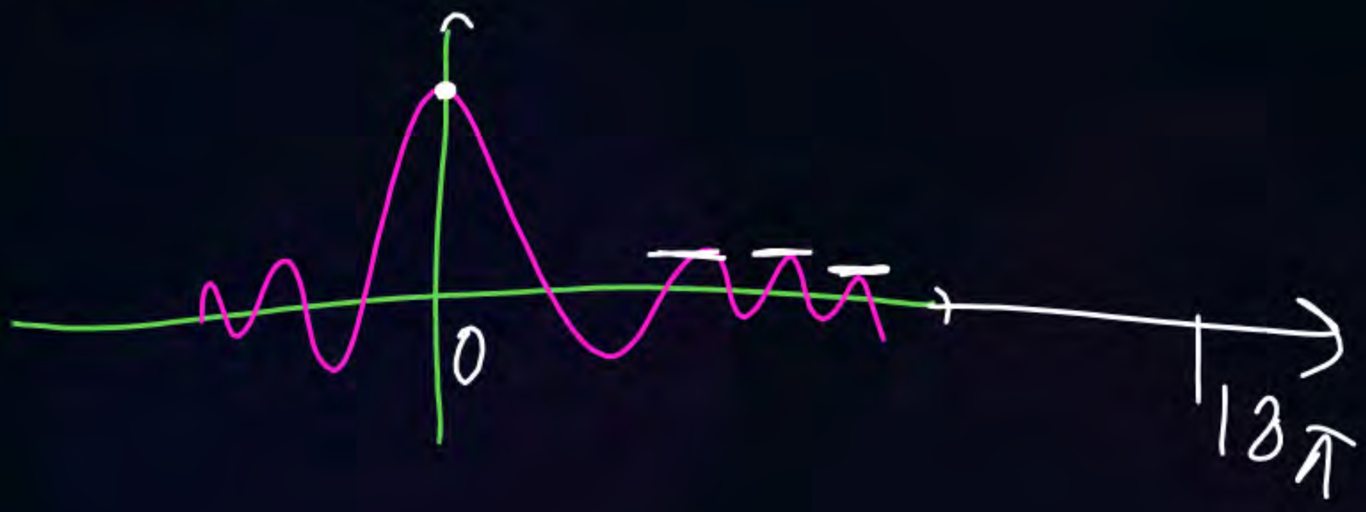
#Q. $f(x)=\frac{\sin(x)}{x}$, How many points exist such that $\boxed{f'(c)=0}$ in the interval $[0,18\pi]$

A 18

B 17

C 8

D 9



$f(x) = \frac{\sin x}{x}$

$(0, \pi) \Rightarrow$ R.Th Not app $\because f(0) = 1$ & $f(\pi) = \frac{0}{\pi} = 0$

$(\pi, 2\pi) \Rightarrow$ $f(\pi) = 0$, $f(2\pi) = 0$ i.e R.Th is app. and there will exist one point c_1

$(2\pi, 3\pi)$ $f(2\pi) = 0$, $f(3\pi) = 0$ . . . . c_2

$(3\pi, 4\pi) \to c_3$

$(4\pi, 5\pi) \to c_4$ - - - - - - - - $(17\pi, 18\pi) \to c_{17}$

[NAT]

#Q. Find a point on the parabola $y = (x+2)^2$, where the tangent is parallel to the chord joining $(-2, 0)$ and $(0, 4)$.

$a = -2$, $b = 0$

$$\boxed{f(x) = (x+2)^2}$$

$f(-2) = 0$

$f(0) = 4$

$f'(x) = 2(x+2)$

By LMVT, $\dfrac{4-0}{0-(-2)} = 2(c+2)$

$2c + 4 = 2$

$c = -1$

& $f(c) = 1$

So Req Point $= (c, f(c))$

$= (-1, 1)$ Ans

[NAT]

#Q. Consider the function $f(x) = (x-2)\log x$ for $x \in [1, 2]$ show that the equation $x \log x + x = 2$ has at least one solution lying between 1 and 2.

$f(x) = (x-2)\log x \,;\ [1, 2]$

$f'(x) = (x-2)\left(\frac{1}{x}\right) + \log x\,(1)$

$= 1 - \frac{2}{x} + \log x$

$$\boxed{f'(x) = (x-2) + x\log x}$$

$x\log x + x - 2 = 0$

for $f(x)$: $f(1) = 0$, $f(2) = 0$ ie $\boxed{f(1) = f(2)} = 0$

ie $x = 1$ & 2 are the roots of $f(x)$

By R.T. $\exists\, c \in (1, 2)$ s.t. $f'(c) = 0$

ie $x = c$ is the root of $f'(x)$

"Between any two roots of $f(x)$ there will be at least one root of $f'(x)$"

R.Th: $f(x)$ is cont & Diff & $\boxed{f(a)=f(b)}$ then $\exists\ c \in (a,b)$ s.t. $f'(c)=0$

$= 0$

[MCQ]

#Q. If $f(x) = e^x - e^{-x}$ and $g(x) = |\cos x - \sin x|$, then on the interval $\left[0, \frac{\pi}{2}\right]$ Cauchy's mean value theorem is -

$$g(x) = \begin{cases} \cos x - \sin x & ; \ 0 < x < \frac{\pi}{4} \\ -(\cos x - \sin x) & ; \ \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases}$$

A Applicable

B not applicable as $g(0) = g\left(\frac{\pi}{2}\right)$

C not applicable as $g'\left(\frac{\pi}{4}\right) = 0$

D not applicable as g(x) contains || (i.e., mod) function

$$g'(x) = \begin{cases} (-\sin x - \cos x), & 0 < x < \frac{\pi}{4} \\ (\sin x + \cos x), & \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases}$$

$g'\left(\frac{\pi}{4}^-\right) = -ve$

$g'\left(\frac{\pi}{4}^+\right) = +ve$

[SUB]

#Q. Verify Cauchy's mean value theorem for the functions $f(x)=\sqrt{x}$ and $g(x)=\frac{1}{\sqrt{x}}$ in the interval [a, b], where $\boxed{a > 0.}$

$f'(x)=\frac{1}{2\sqrt{x}}$ & $g'(x)=\frac{d}{dx}(x)^{-1/2}$

$=-\frac{1}{2}x^{-3/2}$

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}=\frac{\frac{1}{2}c^{-1/2}}{-\frac{1}{2}c^{-3/2}}$$

$$\frac{(\sqrt{b}-\sqrt{a})\sqrt{ab}}{(+\sqrt{a}-\sqrt{b})}=-c$$

$$-\sqrt{ab}=-c$$

$$\boxed{c=\sqrt{ab}}$$

//

[MCQ]

#Q. If $f(x) = e^x$ and $g(x) = e^{-x}$, then the value of c by Cauchy mean value theorem in [a, b] is given by

$$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}$$

A. a + b

B. $\frac{1}{2}(a+b)$

C. a·b

D. None of these

$$\frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = -e^{2c}$$

$$\frac{(e^b - e^a)}{(e^a - e^b)} \times (e^a \cdot e^b) = -e^{2c}$$

$$-e^a e^b = -e^{2c}$$

$$e^{a+b} = e^{2c} \Rightarrow c = \frac{a+b}{2}$$

[MCQ]

#Q. Cauchy's mean value theorem is applicable only

A. For only one function

B. For two functions

C. For one or two functions both

D. None of these

[NAT]

BOLZANO Th.

#Q. Use the intermediate value theorem to prove that the equation $\boxed{e^x = 4 - x^3}$ is solvable on the interval [–2, –1].

Let $\boxed{f(x) = e^x + 4 + x^3}$

$f(-2) = e^{-2} + 4 + (-2)^3 = -ve$

$f(-1) = e^{-1} + 4 + (-1)^3 = +ve$

Hence verified.

[NAT]

#Q. Check whether there is a solution to the equation $\boxed{x^5 - 2x^3 - 2 = 0}$ between the interval [0,2].

$$\boxed{f(x) = x^5 - 2x^3 - 2}$$

$f(0) = -2 = -ve$

$f(2) = 32 - 16 - 2 = +ve$

by B.Th, $\exists\ \alpha \in (0,2)$ s.t. $f(\alpha) = 0$

$$\boxed{\alpha^5 - 2\alpha^3 - 2 = 0}$$

[MCQ]

#Q. The Value of c in the Lagrange's mean value theorem of the function $f(x) = x^3 - 4x^2 + 8x + 11$, when $x \in [0,1]$ is:

Same as $\boxed{Q18}$

PPP-1

(A) $\frac{4-\sqrt{5}}{3}$

(B) $\frac{\sqrt{7}-2}{3}$

(C) $\frac{2}{3}$

(D) $\frac{4-\sqrt{7}}{3}$

[MCQ]

#Q. The expansion of $f(x) = e^x \cos x$ at $x = 0$.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

A) $1 + x - \frac{2x^3}{3!} + \ldots$

B) $1 + x - \frac{x^3}{3!} + \ldots$

C) $1 + x - \frac{x^2}{2!} + \ldots$

D) $1 + x - \frac{2x^2}{2!} + \ldots$

$$= 1 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \cdots$$

$$f(x) = e^x \cos x \Rightarrow f(0) = 1$$

$$f'(x) = e^x(-\sin x) + (e^x)\cos x \Rightarrow f'(0) = 1$$

$$f''(x) = e^x(-\cos x) + (-\sin x)(e^x) + e^x(-\sin x) + e^x(\cos x) \Rightarrow f''(0) = -1+1 = 0$$

$$f'''(x) = -[\cancel{e^x(-\sin x)} + e^x \cos x] - 2[e^x \cos x + \sin x\, e^x] + [\cancel{e^x(-\sin x)} + e^x \cos x] \Rightarrow f'''(0) = -2$$

[MCQ]

#Q. The third term in the expansion $f(x) = \frac{x-1}{x+1}$ of $\boxed{\text{about}}$ the point $x = 1$ using Taylor's series is :

$\Downarrow$

let $(x-1) = t$

A $\frac{(x-1)^2}{2}$

B $\frac{(x-1)^2}{4}$

C $+\frac{(x-1)^3}{8}$

D $\frac{(x-1)^3}{4}$

$$f(x)=\frac{x-1}{x+1}=\frac{t}{t+2}=\frac{(t+2)-2}{(t+2)}=1-\frac{2}{t+2}=1-\frac{2}{2\left(1+\frac{t}{2}\right)}$$

$x-1=t$

$\Rightarrow x=t+1$

$$=1-\left(1+\frac{t}{2}\right)^{-1}=1-\left[1-\frac{t}{2}+\left(\frac{t}{2}\right)^2-\left(\frac{t}{2}\right)^3\cdots\right]$$

$$\boxed{(1-x)^{-1}=1+x+x^2+x^3+\cdots \quad \infty\ GP}$$

$$(1+x)^{-1}=1-x+x^2-x^3\cdots$$

$$=\frac{t}{2}-\frac{t^2}{4}+\frac{t^3}{8}+\cdots$$

$$f(x)=+\frac{(x-1)}{2}-\frac{(x-1)^2}{4}+\frac{(x-1)^3}{8}+\cdots$$

[MCQ]

#Q. Find the Taylor series expansion of the function $\boxed{\cos h(x)}$ centered at x = 0.

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

A $1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \infty$

B $\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} \ldots \infty$

C $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \infty$

D $1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots \infty$

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \frac{e^x + e^{-x}}{2}$

[MCQ]

#Q. Let Maclaurin series of some f(x) be given recursively, where a_n denotes the coefficient of x^n in the expansion. Also given $a_n = a_{n-1}/n$ and $a_0 = 1$, which of the following functions could be f(x)?

A. e^x

B. e^{2x}

C. $c + e^x$

D. No closed form exists

$$\boxed{a_n = \frac{a_{n-1}}{n}}$$

$$a_0 = 1$$
$$a_1 = \frac{a_0}{1} = 1 = \frac{1}{1!}$$
$$a_2 = \frac{a_1}{2} = \frac{1}{2} = \frac{1}{2!}$$
$$a_3 = \frac{a_2}{3} = \frac{1}{6} = \frac{1}{3!}$$
$$a_4 = \frac{a_3}{4} = \frac{1}{24} = \frac{1}{4!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

THANK - YOU