Computer Science & DA

Probability and Statistics

Continuous Random variable

Lecture No. 03

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Recap of previous lecture

Topic Uniform and Exponential Distribution

p.d.f $\boxed{f(t) = \begin{cases} \mu e^{-\mu t} & , t \geq 0 \\ 0 & , \text{otw} \end{cases}}$ ie $t \sim E\{\mu\}$ & $E(t) = \frac{1}{\mu}$

Note → C.D.F for E. Dist →

$$F(t) = \int_{-\infty}^{t} f(t)\,dt = \int_{-\infty}^{0} (0)\,dt + \int_{0}^{t} f(t)\,dt = \int_{0}^{t} \mu e^{-\mu t}\,dt$$

$$= \mu\left[\frac{e^{-\mu t}}{-\mu}\right]_0^t = -\left[e^{-\mu t} - 1\right] = \boxed{1 - e^{-\mu t}}$$ learn.

Topics to be Covered

Topic Normal Distribution 1

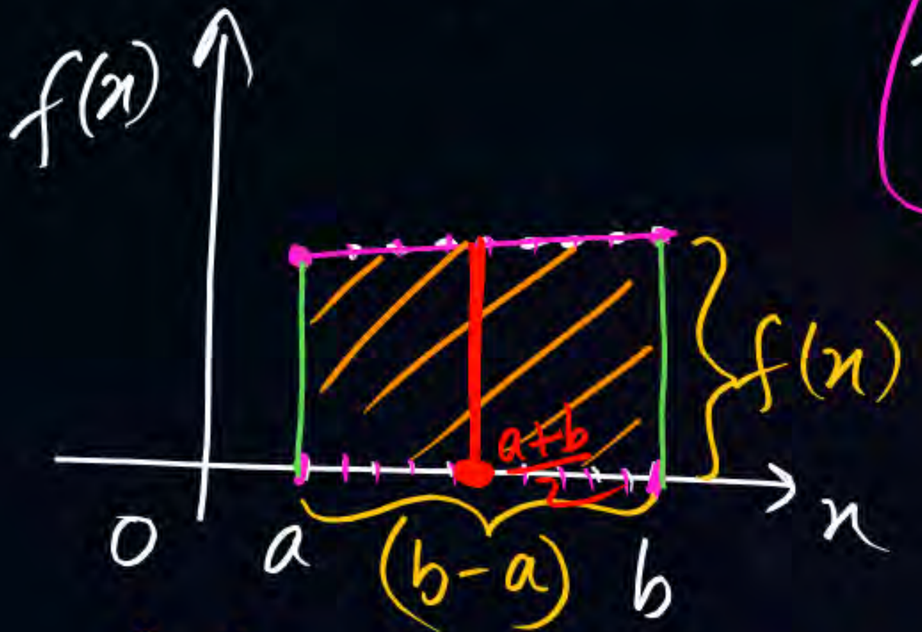
UNIFORM DIST →

Let x is C.R.V defined in $[a, b]$

s.t it's p.d.f is defined as

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otw} \end{cases}$$

then x is Called U.R.V over the interval $[a, b]$



Proof: w.k.that Total Area under $f(x) = 1$

length × Height = 1

$(b-a) \times f(x) = 1 \Rightarrow f(x) = \frac{1}{b-a}$

① $\text{Mean}(x) = E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

$$= \int_a^b x\left(\frac{1}{b-a}\right)dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right] = \boxed{\frac{a+b}{2}}$$

② $\text{Var}(x) = E(x^2) - (E(x))^2 = \cdots\cdots = \frac{(b-a)^2}{12}$

③ $\text{S.D}(\sigma) = +\frac{(b-a)}{\sqrt{12}}$

④ $P(\alpha < x < \beta) = \int_\alpha^\beta f(x)\,dx = ?$

eg: if x is Uniformly Distributed R. Variable b/n 0 & 1 then find $Var(x) = ?$ & also evaluate 2nd Moment

Sol: $x \in (0,1) \Rightarrow f(x) = \begin{cases} \frac{1}{1-0}, & 0<x<1 \\ 0, & otw \end{cases}$

$$Var(x) = \frac{(b-a)^2}{12} = \frac{(1-0)^2}{12} = \frac{1}{12}$$

M-II $Mean(x) = E(x) = \int_0^1 x f(x)\,dx = \int_0^1 x(1)\,dx = \frac{1}{2}$

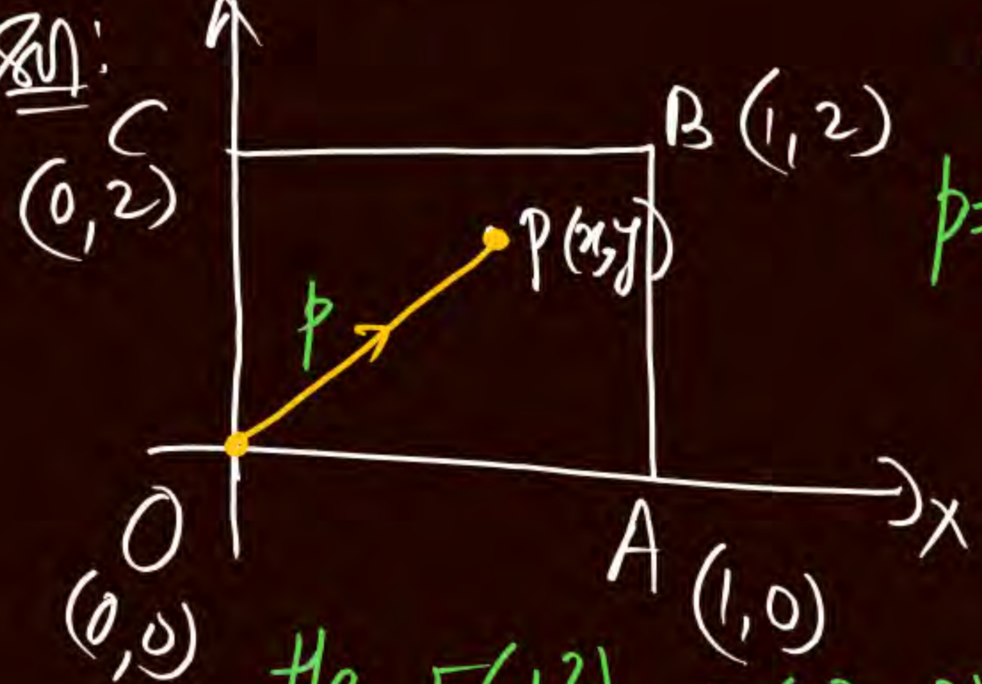
2nd Moment $= E(x^2) = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2(1)\,dx = \frac{1}{3}$

So $Var(x) = E(x^2) - E^2(x)$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12} \text{ Ans}$$

Qe: If p is the length of the position vector of any Random point (x, y) with in the Rectangle formed by $(0,0)$, $(1,0)$, $(1,2)$, $(0,2)$ then find $E(p^2) = ?$

Sol:



$$\overrightarrow{OP} = x\hat{i} + y\hat{j}$$

$$p = |\overrightarrow{OP}| = \sqrt{x^2+y^2}$$

$$p^2 = x^2+y^2$$

then $E(p^2) = E(x^2+y^2) = E(x^2) + E(y^2)$

i.e. $E(p^2) = E(x^2) + E(y^2)$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx + \int_{-\infty}^{\infty} y^2 f(y)\,dy \quad \text{--- (1)}$$

$\because x \in (0,1)$ & x is U.R.V so p.d.f is $f(x) = \begin{cases} \frac{1}{1-0}, & 0<x<1 \\ 0, & otw \end{cases}$

$\because y \in (0,2)$ & y is also U.R.V so it's p.d.f is

$$f(y) = \begin{cases} \frac{1}{2-0}, & 0<y<2 \\ 0, & otw \end{cases}$$

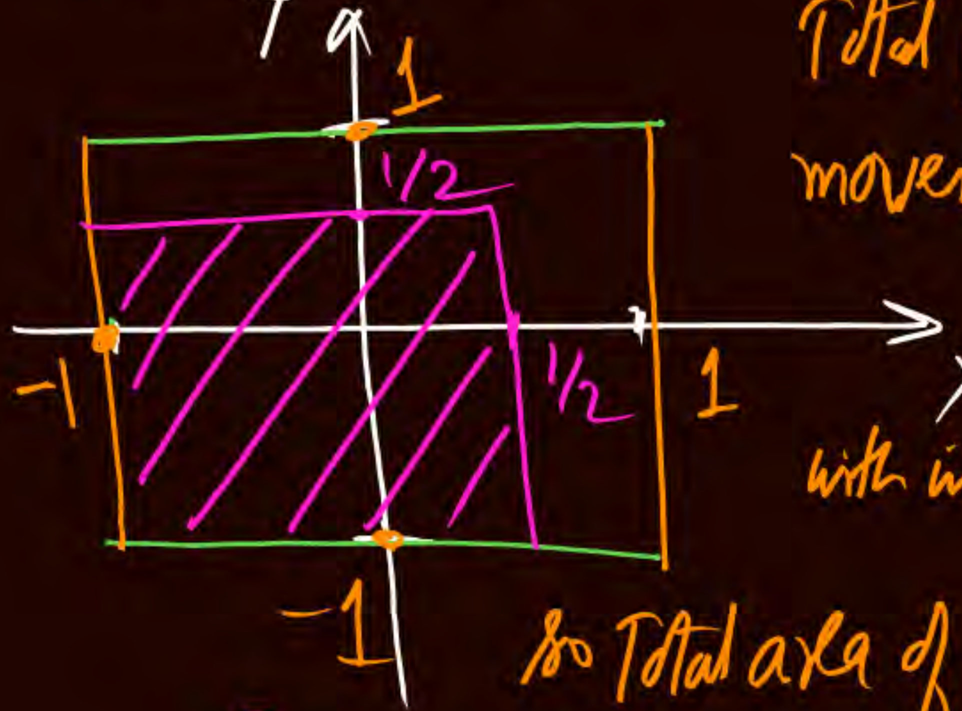
$$E(p^2) = \int_0^1 x^2 (1)\,dx + \int_0^2 y^2 \left(\frac{1}{2}\right) dy$$

$$= \left(\frac{x^3}{3}\right)_0^1 + \left(\frac{y^3}{6}\right)_0^2 = \frac{1}{3} + \frac{4}{3} = \frac{5}{3}$$

GATE / AIEEE

If x and y are two Ind R. Variables defined in $[-1,1]$ then $P\left[\max(x,y) < \frac{1}{2}\right] = ?$

Sol: M-I (using Class 12^{th}) $\rightarrow$



Total possibility for the movement of (x,y) is with in the Rectangle

so Total area of Rectangle $= (2)(2)$

fav area for $(x,y) = \left(\frac{3}{2}\right)\left(\frac{3}{2}\right) = \frac{9}{4}$

Req Prob $= \frac{\text{fav area}}{\text{Total area}} = \frac{9/4}{4} = \frac{9}{16}$

M-II

$\because x \in [-1,1]$ & x is U.R.V so $f(x) = \begin{cases} \frac{1}{1-(-1)}, & -1 \le x \le 1 \\ 0, & \text{otw} \end{cases}$

$\because y \in [-1,1]$ & y is also U.R.V so $f(y) = \begin{cases} \frac{1}{1-(-1)}, & -1 \le y \le 1 \\ 0, & \text{otw} \end{cases}$

$P\left[\max(x,y) < \frac{1}{2}\right] = P\left[-1 \le (x,y) < \frac{1}{2}\right]$

$= P\left[-1 \le x < \frac{1}{2}\right] \cdot P\left[-1 \le y < \frac{1}{2}\right]$ ($\because x$ & y are Ind)

$= \int_{-1}^{1/2} f(x)\,dx \cdot \int_{-1}^{1/2} f(y)\,dx = \int_{-1}^{1/2} \left(\frac{1}{2}\right)dx \int_{-1}^{1/2} \left(\frac{1}{2}\right)dy$

$= \frac{1}{4}\left(\frac{3}{2}\right)\left(\frac{3}{2}\right) = \frac{9}{16}$ An

Q:- At a certain Traffic junction, the cycle of Traffic light is 2 mins of green (vehicle does not stop) and 3 mins of Red (vehicle stops). If arrival time of the vehicle at the junction is uniformly distributed over 5 mins cycle then find Expected waiting time of the vehicle at the junction.

Sol:- $x = \{$Arrival time of vehicle at junction$\}$

& ATQ, $x \in (0, 5)$ & x is U.R.V so

it's p.d.f $f(x) = \begin{cases} \frac{1}{5-0}, & 0 < x < 5 \\ 0, & \text{otw} \end{cases}$

So waiting time $= g(x) = \begin{cases} 0, & 0 < x < 2 \\ 5-x, & 2 < x < 5 \end{cases}$

$$E\{g(x)\} = \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx = \int_0^5 g(x)\left(\frac{1}{5}\right)dx$$

$$= \frac{1}{5}\left[\int_0^2 (0)\,dx + \int_2^5 (5-x)\,dx\right]$$

$$= \frac{1}{5}\left[0 + \left(5x - \frac{x^2}{2}\right)_2^5\right] = 0.9 \text{ mins}$$

$$= 0.9 \times 60 \text{ sec} = 54 \text{ sec.}$$

(M-II) By common sense $\rightarrow$

$$\bar{X} = \frac{\Sigma X}{N} \begin{cases} \bar{X}_1 = \dfrac{\Sigma X_1}{N_1} \\ \bar{X}_2 = \dfrac{\Sigma X_2}{N_2} \end{cases}$$

& $\bar{X} = \dfrac{\Sigma X_1 + \Sigma X_2}{N_1 + N_2} = \dfrac{N_1\bar{X}_1 + N_2\bar{X}_2}{N_1 + N_2}$

Av. wt time $= 0$ Av. wt time $= \frac{3}{2} = 1.5$

$$\underbrace{0 \longrightarrow 2}_{\text{Green}} \underbrace{\longrightarrow 5}_{\text{Red light}}$$

So Av. wt time $= \dfrac{2 \times 0 + 3 \times \frac{3}{2}}{2+3} = \dfrac{9}{10}$

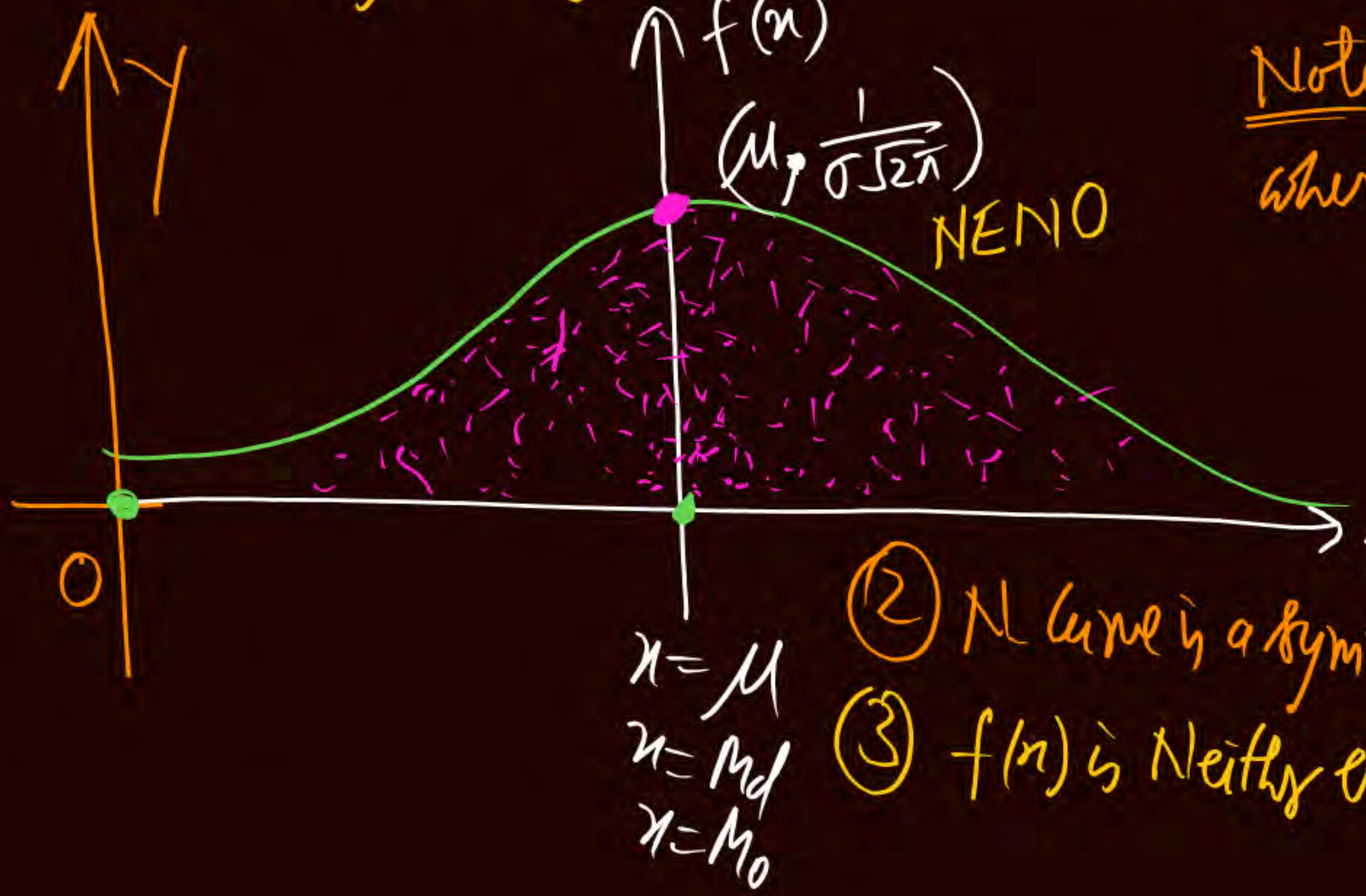
$= 0.9$ mins ≈ 54 sec

$x \in (a, b)$ & x is U.R.V then Av of $(x) = \dfrac{a+b}{2}$

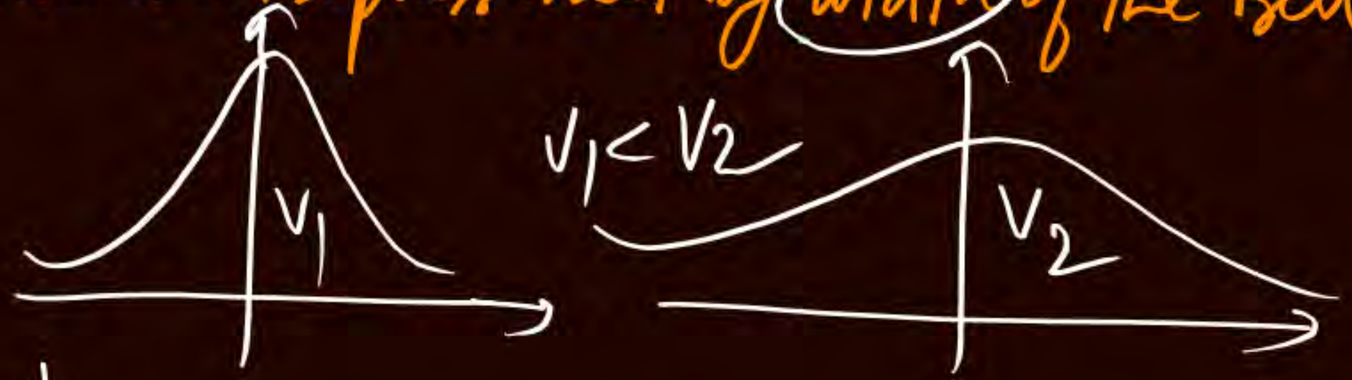
NORMAL DIST / Gaussian Dist / Bell Curve

eg (Height, wt, Marks Dist, Intelligence etc)

Whenever Random Variable has a tendency to accumulate about it's average value, then such types of variables are called N.R.V.



Note ① N. Curve is a Bell shaped curve where Variance is represented by width of the Bell.



② N Curve is a symmetrical curve with symmetry about $x=\mu$

③ f(x) is Neither even Nor odd.

(4) Normal curve is a $\underbrace{\text{UNIMODAL}}_{Me = Md = Mo}$ curve i.e. Mean = Md = Mode

Def^n: Let x is C.R.V s.t it's p.d.f is defined as

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$$

then x is called N.R.V with parameters μ & σ^2 & it is denoted as $x \sim N\{\mu, \sigma^2\}$

Note:- (1) Cross check :-

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \text{------------} = 1$$

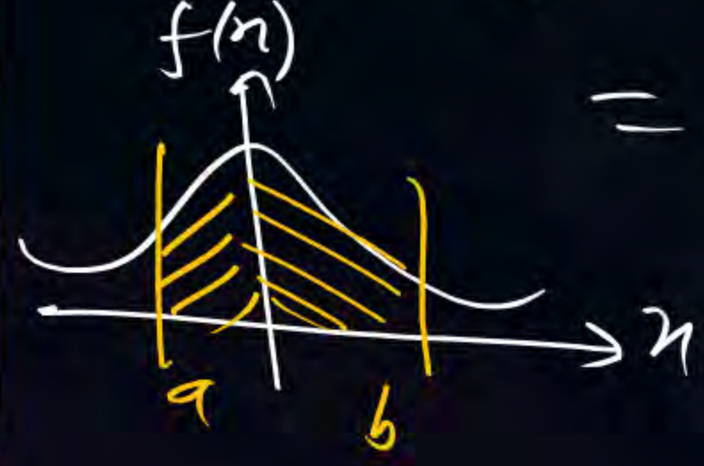
using β-γ func^n

(2) Highest value of N. Curve is $\frac{1}{\sigma\sqrt{2\pi}}$ & it occurs at $x = \mu$

(3) $P(a < x < b) = ? = \int_a^b f(x)\,dx = \ldots\ldots$ Not easy

M-II $P(a < x < b) =$ area under $f(x)$ b/w a & b

$= \text{-----} =$ Not easy



Note (1) if $Z = \frac{x-\mu}{\sigma}$ then find $E(Z)$ & $Var(Z)$

Sol: $E(Z) = E\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\left[E(x-\mu)\right]$

$= \frac{1}{\sigma}\left[E(x) - E(\mu)\right]$

$\mu_Z = \frac{1}{\sigma}\left[E(x) - \mu\right] = \frac{1}{\sigma}\left[\mu - \mu\right] = 0$

Now $Var(Z) = Var\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma^2} Var(x-\mu)$

$= \frac{1}{\sigma^2}\left[Var(x) - Var(\mu)\right]$

$= \frac{1}{\sigma^2}\left[Var(x) - 0\right] = \frac{\sigma^2}{\sigma^2} = 1$

$\Rightarrow \sigma_Z = 1$

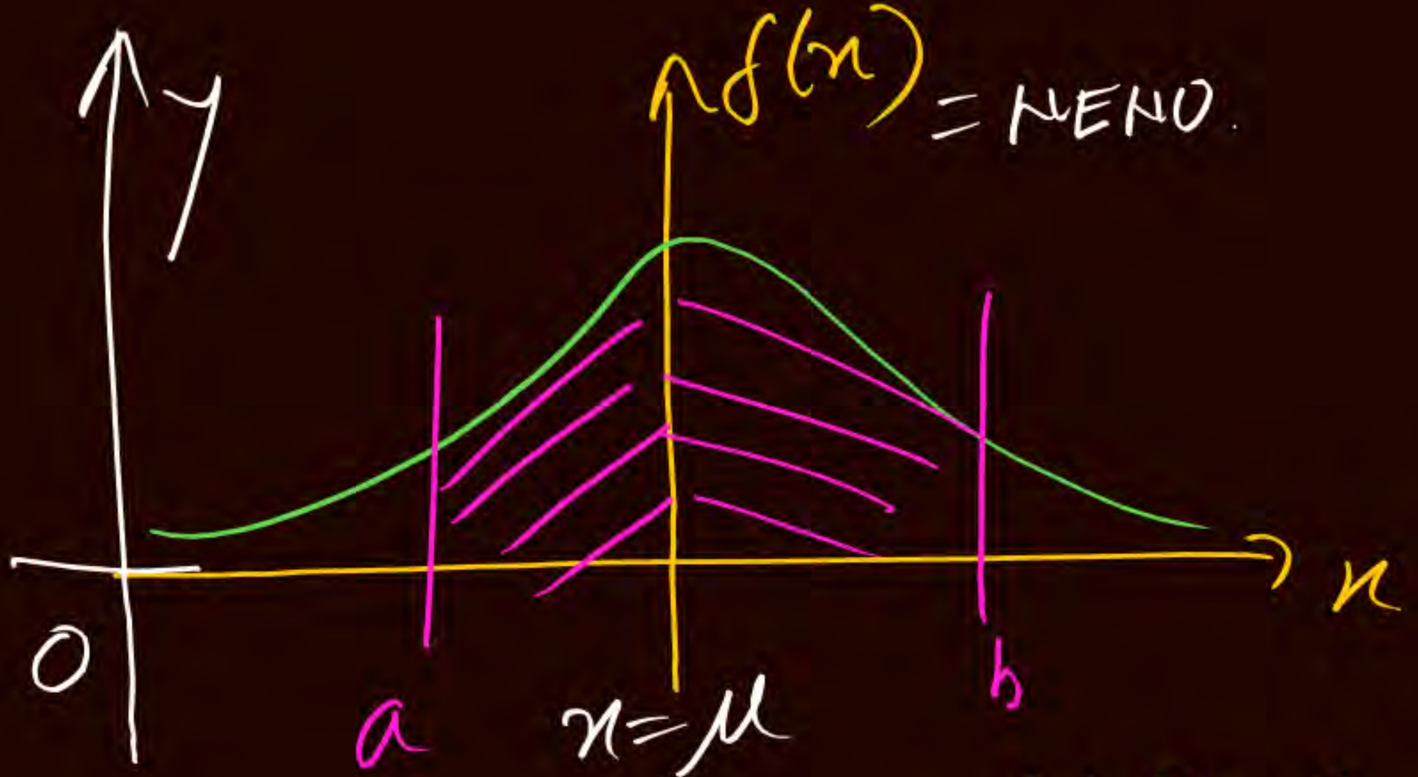
Standard Normal Variable (Z) $\rightarrow$

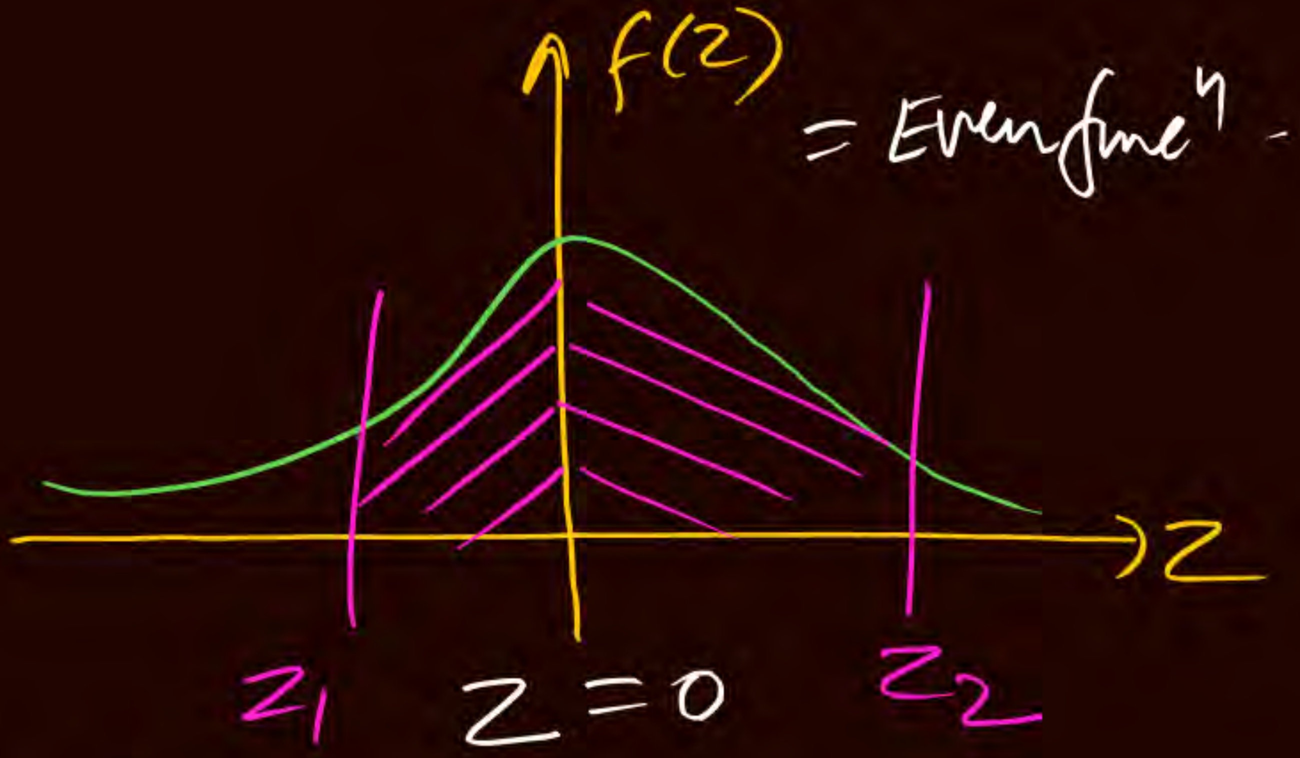
Let $\frac{x-\mu}{\sigma} = Z$ then $\mu_Z = 0$, $\sigma_Z = 1$

in that situation, Z is called S.N. Variable & it's p.d.f is given as;

$$\boxed{f(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(Z)^2}}$$ i.e $Z \sim N\{0,1\}$

Note: S.N.V is free from any parameter





$$Z = \frac{x-\mu}{\sigma}$$

At $x=\mu$, $Z = 0$

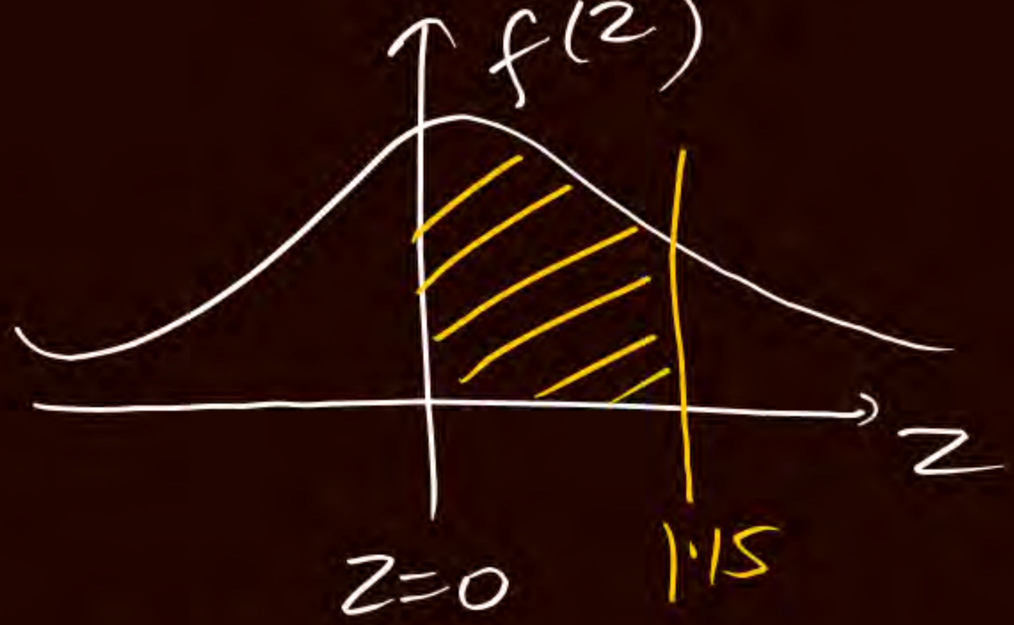
At $x=a$, $z_1 = \dfrac{a-\mu}{\sigma}$

At $x=b$, $z_2 = \dfrac{b-\mu}{\sigma}$

$P(a<x<b) = \,? = P(z_1<z<z_2)$ = Area under S.N. Curve b/w z_1 & z_2

= Use Normal Table = Ans

eg: $P(0 \le z \le 1.15) = ?$ = (Area under N. Curve b/w $z=0$ to 1.15)



M-I
using N. Table

M-II
using Calci

M-III
using information given in the Quest.

M-II $P(0<z<1.15) = \int_0^{1.15} f(z)\,dz = \int_0^{1.15} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{z^2}{2}\right)} \cdot dz = 0.3746$

M-III it is given that area under N. Curve b/w $z=0$ to 1.15 is 0.3746

THANK - YOU