16-782
Planning & Decision-making in Robotics

Search Algorithms:
A*, Weighted A*, Backward A*

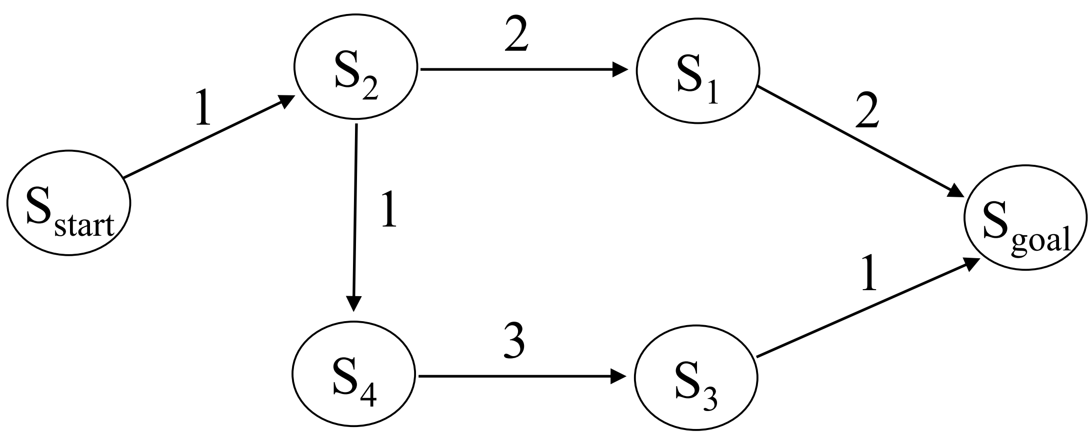
Maxim Likhachev

Robotics Institute

Carnegie Mellon University

Searching Graphs for a Least-cost Path

- Once a graph is constructed (from skeletonization or uniform cell decomposition or adaptive cell decomposition or lattice or whatever else), we need to search it for a least-cost path

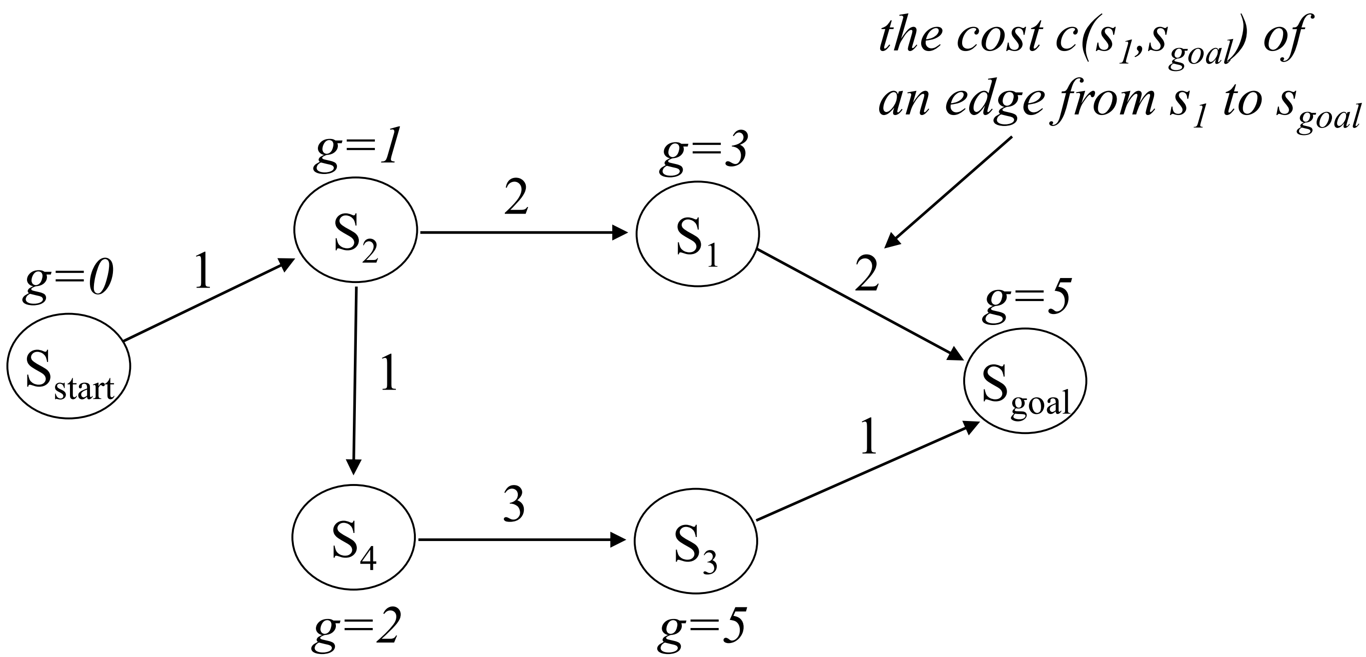


Searching Graphs for a Least-cost Path

- Many searches work by computing optimal g-values for relevant states

  – $g(s)$ – an estimate of the cost of a least-cost path from s_{start} to s

  – optimal values satisfy: $g(s) = \min_{s'' \in pred(s)} g(s'') + c(s'',s)$



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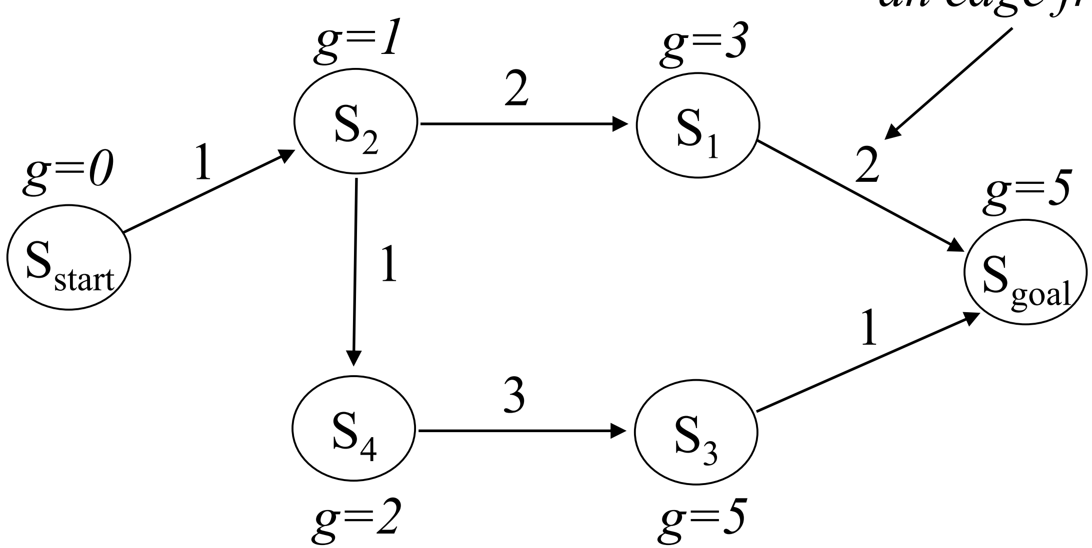
the cost $c(s_1,s_{go}$
an edge from s_1 *to* s_{goal}

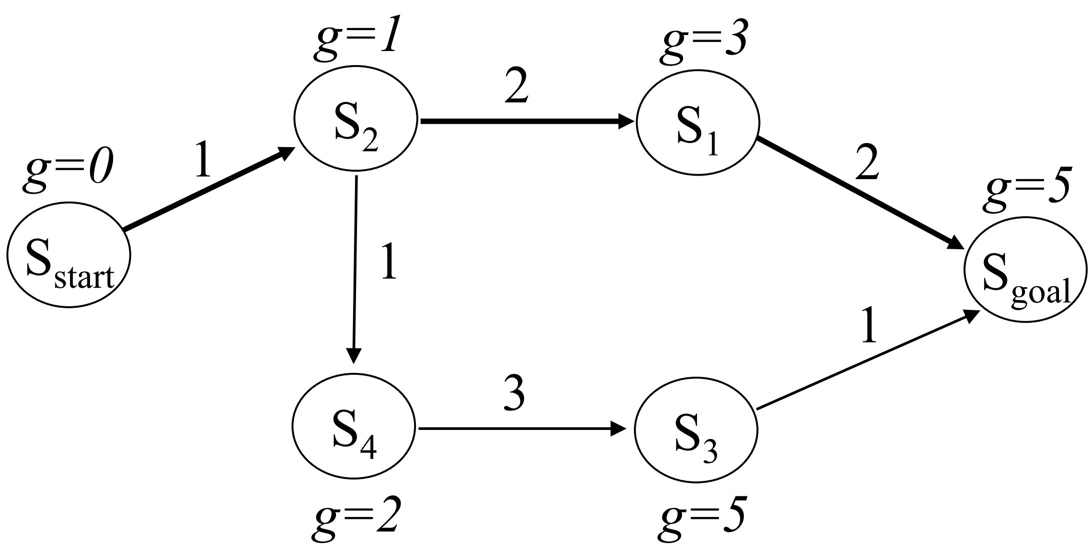
why?



Searching Graphs for a Least-cost Path

- Least-cost path is a greedy path computed by backtracking:
  - start with s_{goal} and from any state s move to the predecessor state s' such that

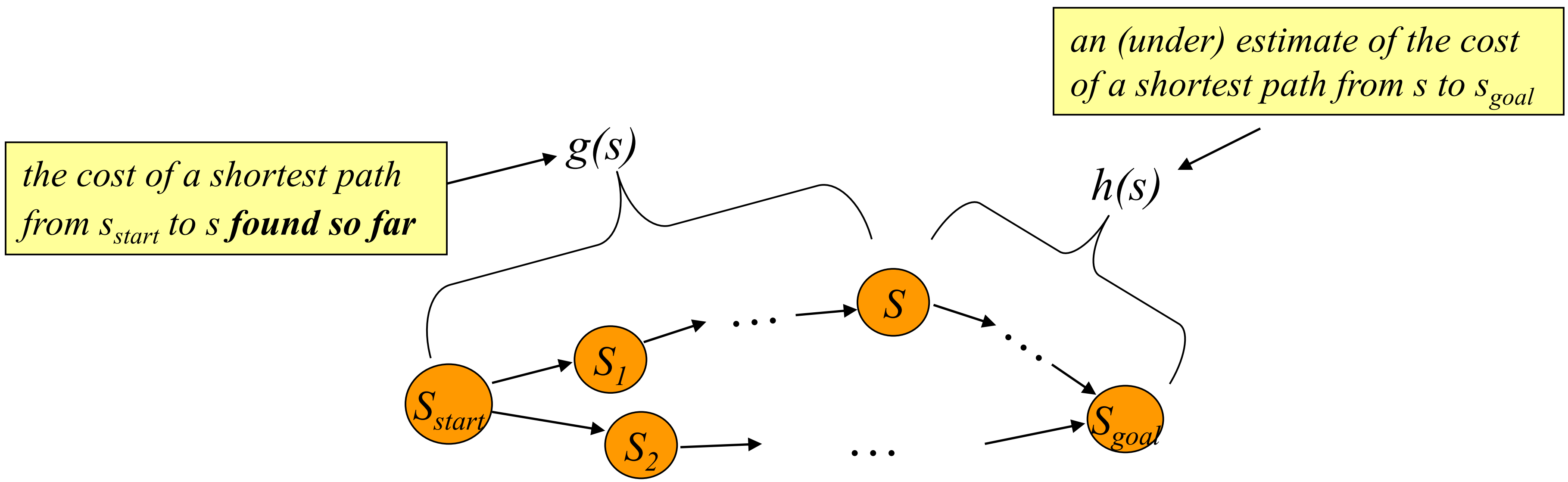
$$s' = \arg\min_{s'' \in pred(s)} (g(s'') + c(s'', s))$$



A* Search [Hart, Nillson, Raphael, '68]

- Computes optimal g-values for relevant states

at any point of time:



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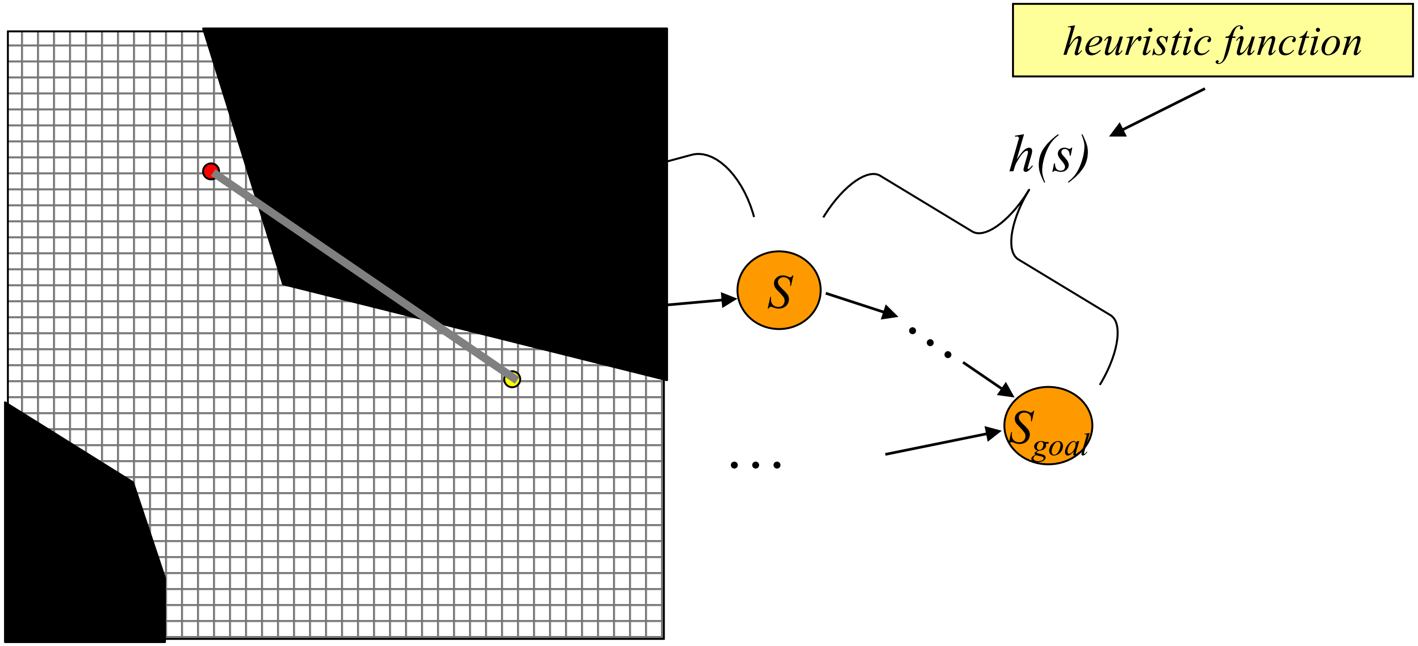


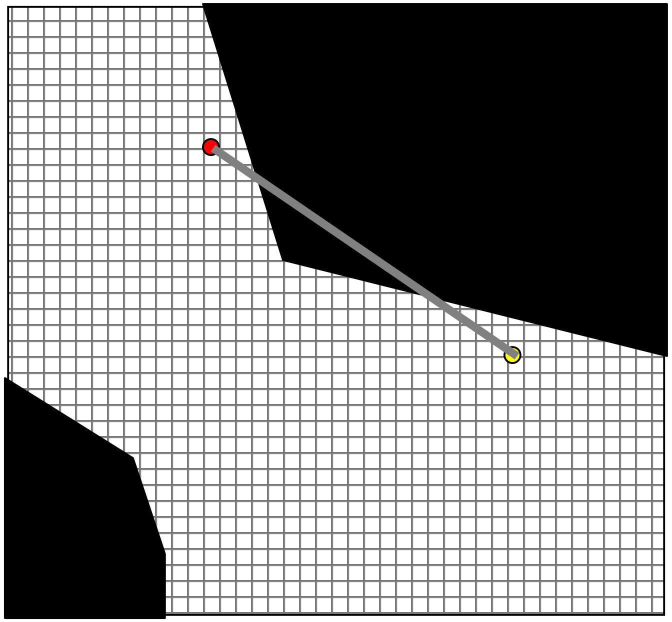
one popular heuristic function – Euclidean distance

A* Search

minimal cost from s to s_{goal}

- Heuristic function must be:
  - admissible: for every state s, $h(s) \leq c^*(s,s_{goal})$
  - consistent (satisfy triangle inequality):

    $h(s_{goal},s_{goal}) = 0$ *and for every* $s \neq s_{goal}$, $h(s) \leq c(s,succ(s)) + h(succ(s))$
  - admissibility <u>provably</u> follows from consistency and often (<u>not always</u>) consistency follows from admissibility



A* Search

- Computes optimal g-values for relevant states

Main function

$g(s_{start}) = 0$; all other g-values are infinite; $OPEN = \{s_{start}\}$;

ComputePath();

publish solution;

set of candidates for expansion

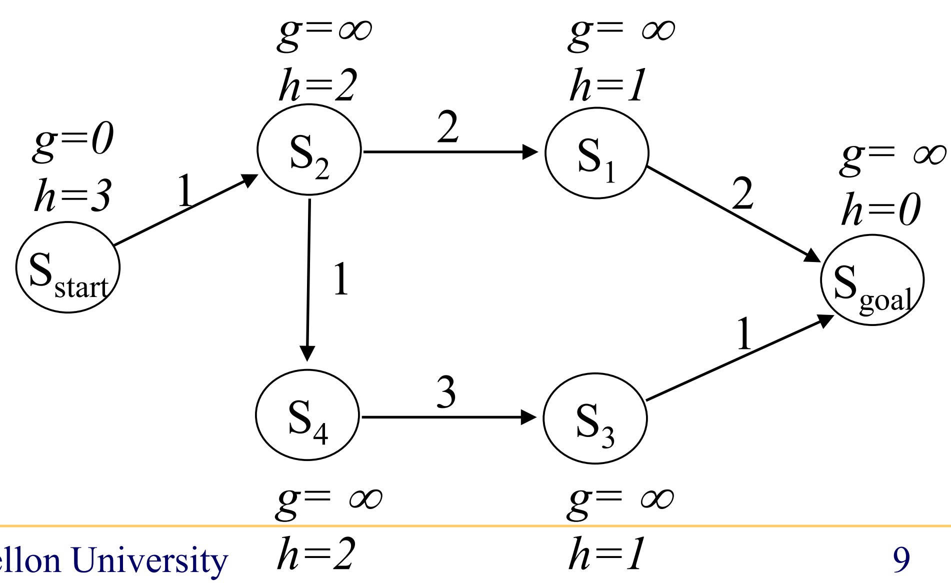
ComputePath function

while(s_{goal} is not expanded and $OPEN \neq 0$)

  remove s with the smallest $[f(s) = g(s)+h(s)]$ from $OPEN$;

  expand s;

for every expanded state
g(s) is optimal
(if heuristics are consistent)



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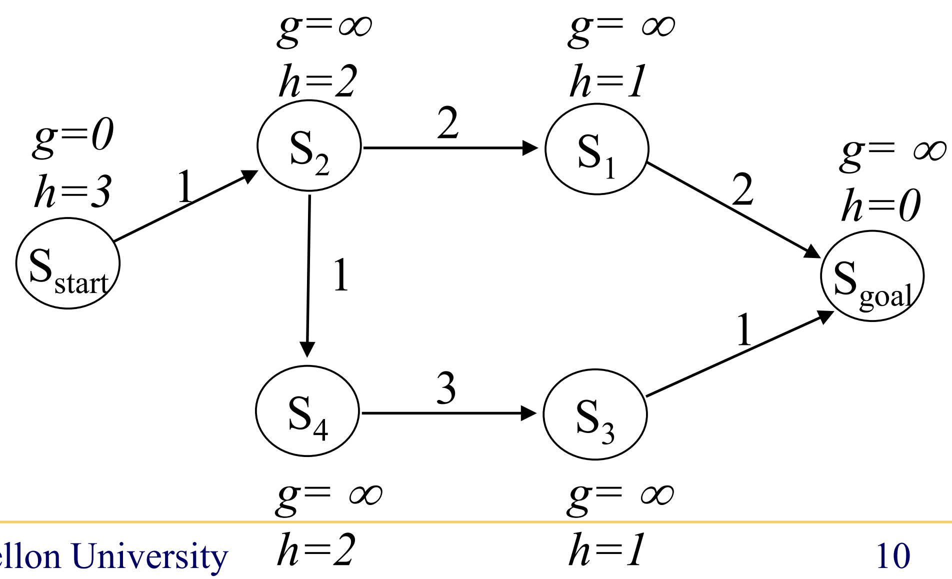

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  for every successor s' of s such that s' not in $CLOSED$

    if $g(s') > g(s) + c(s,s')$

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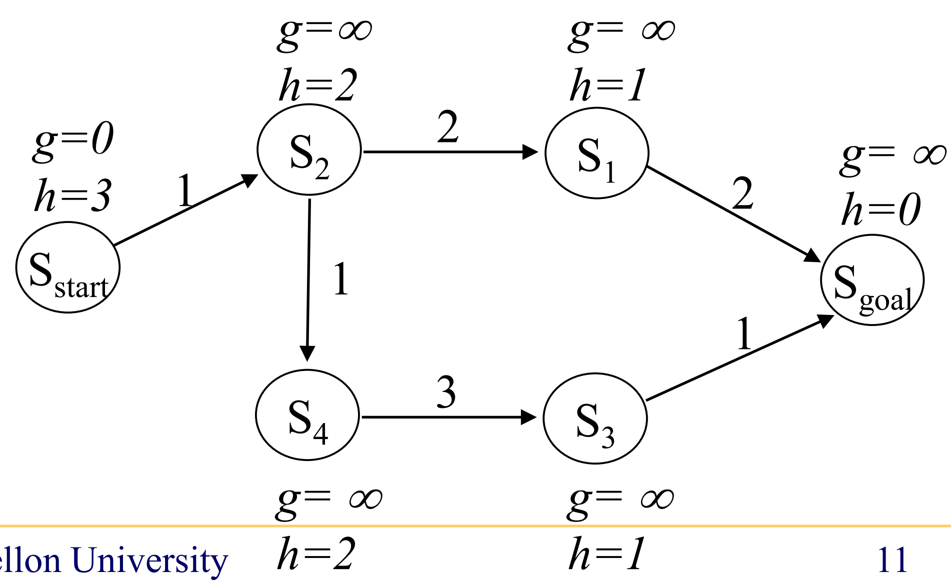
set of states that have already been expanded

tries to decrease $g(s')$ using the found path from s_{start} to s



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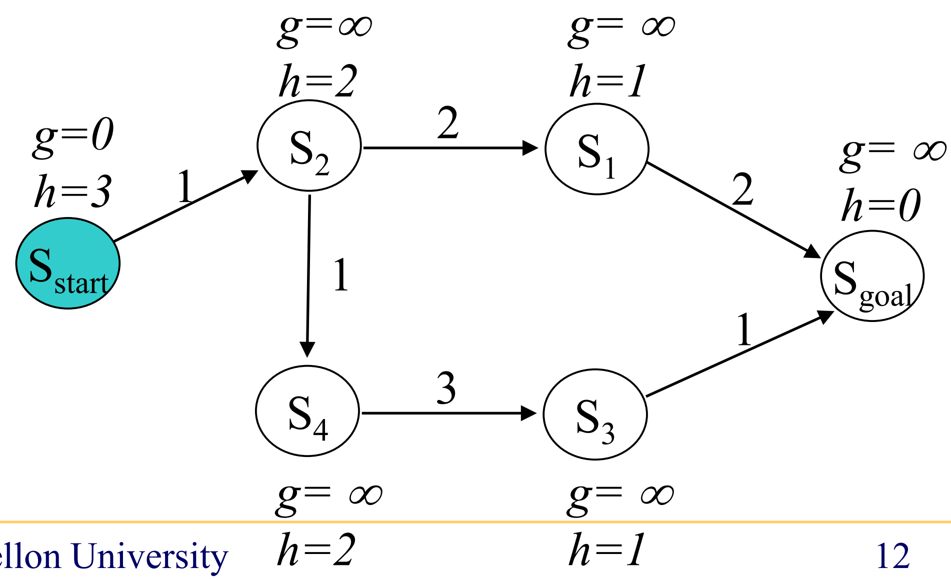
next state to expand: s_{start}



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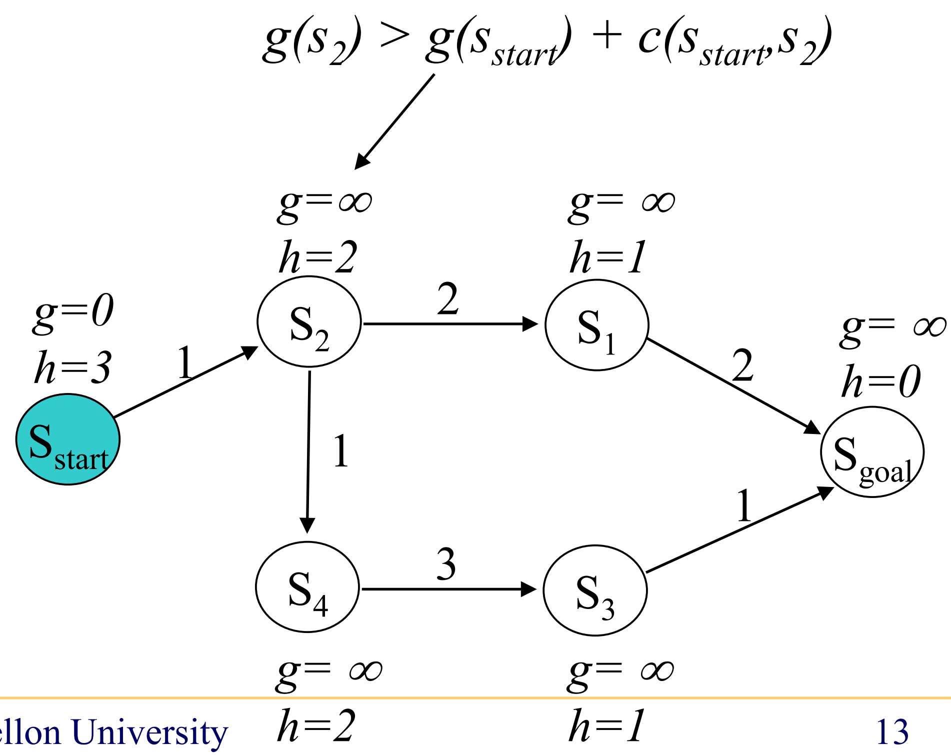

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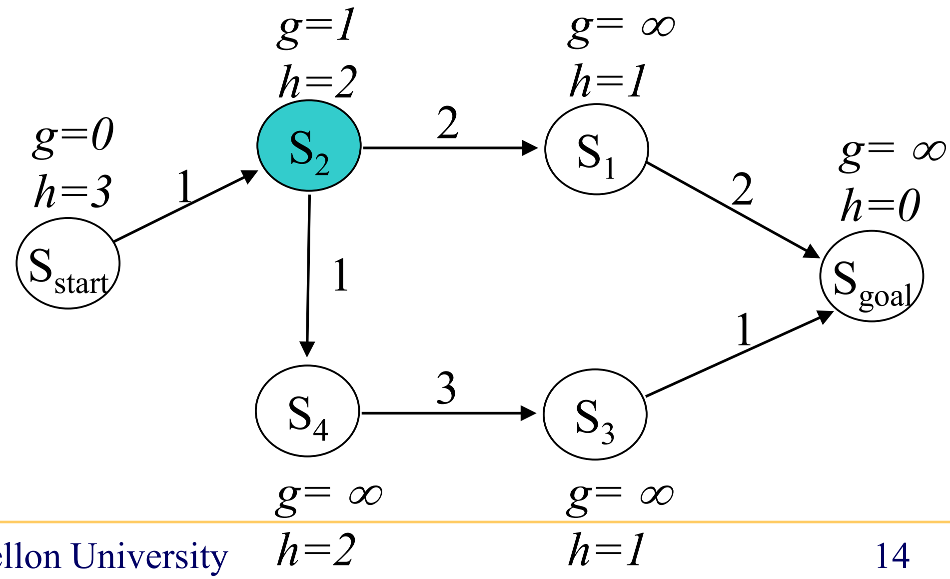

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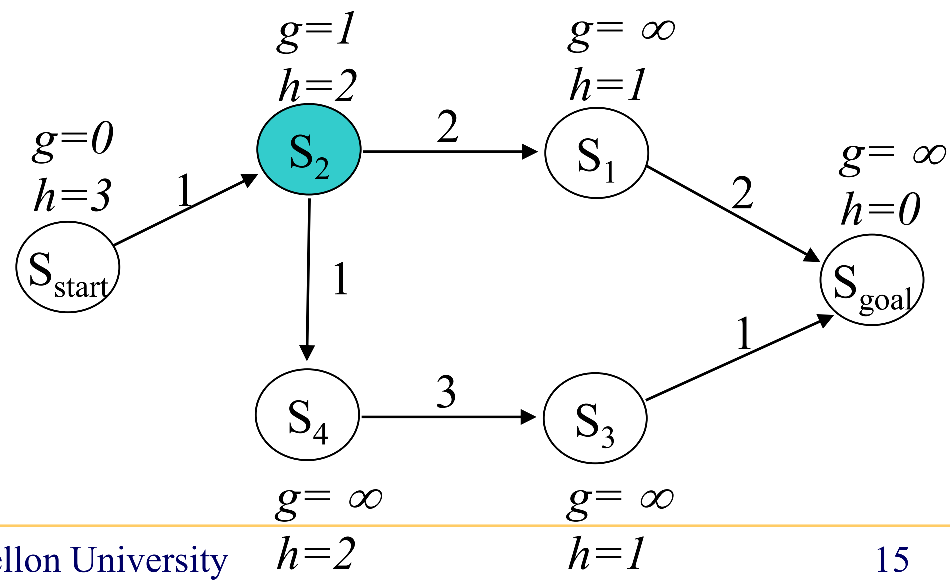
$CLOSED = \{s_{start}\}$

$OPEN = \{s_2\}$

next state to expand: s_2



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insert s' into $OPEN$;

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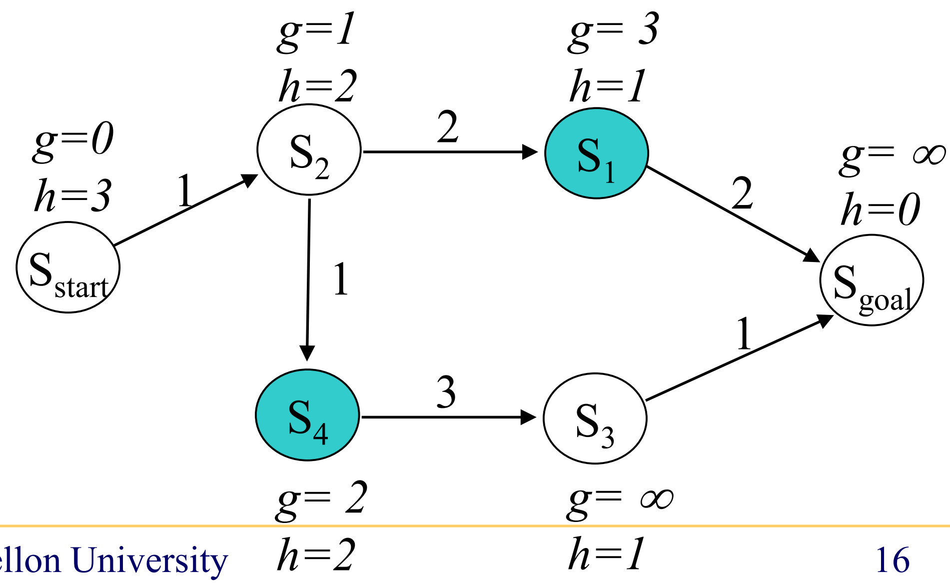
$OPEN = \{s_1, s_4\}$

next state to expand: s_1



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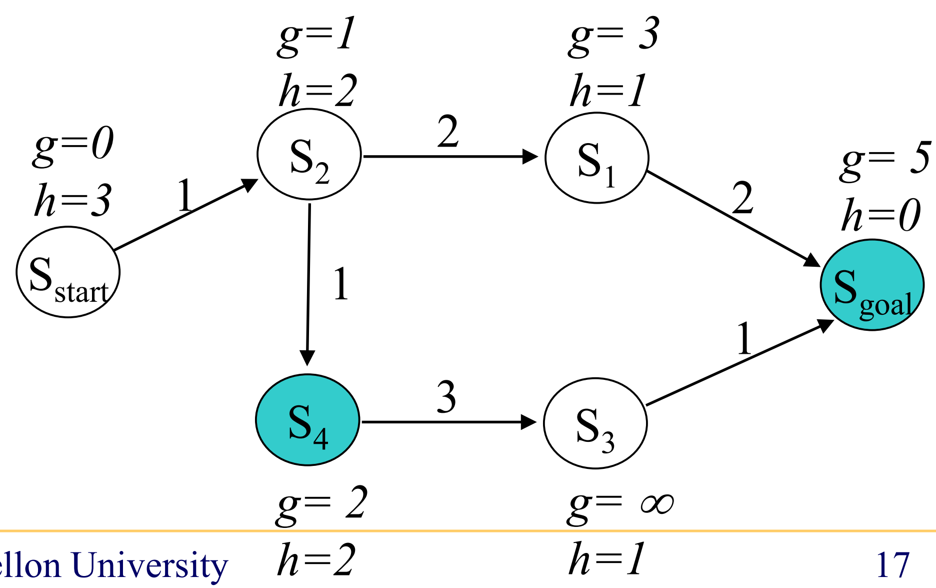
$OPEN = \{s_4, s_{goal}\}$

next state to expand: s_4



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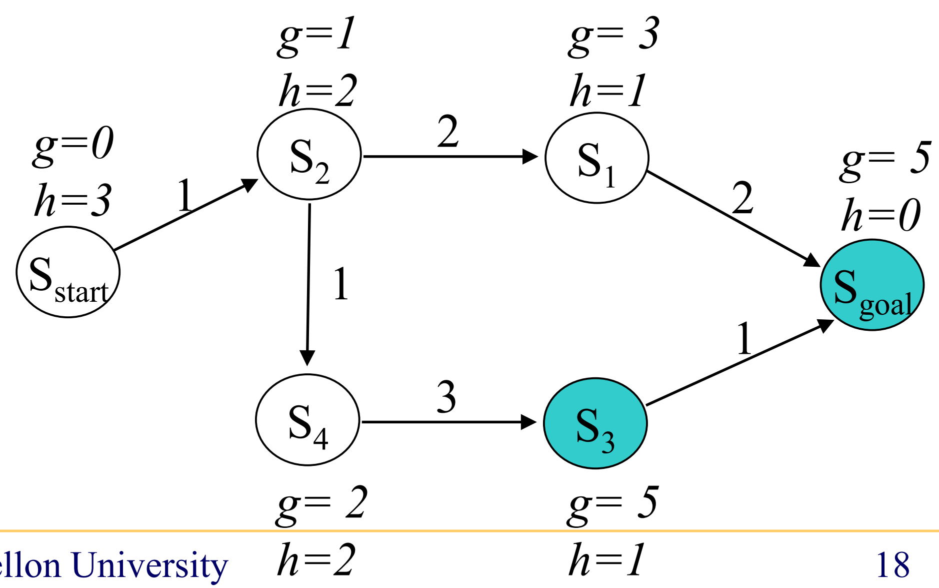
$OPEN = \{s_3, s_{goal}\}$

next state to expand: s_{goal}



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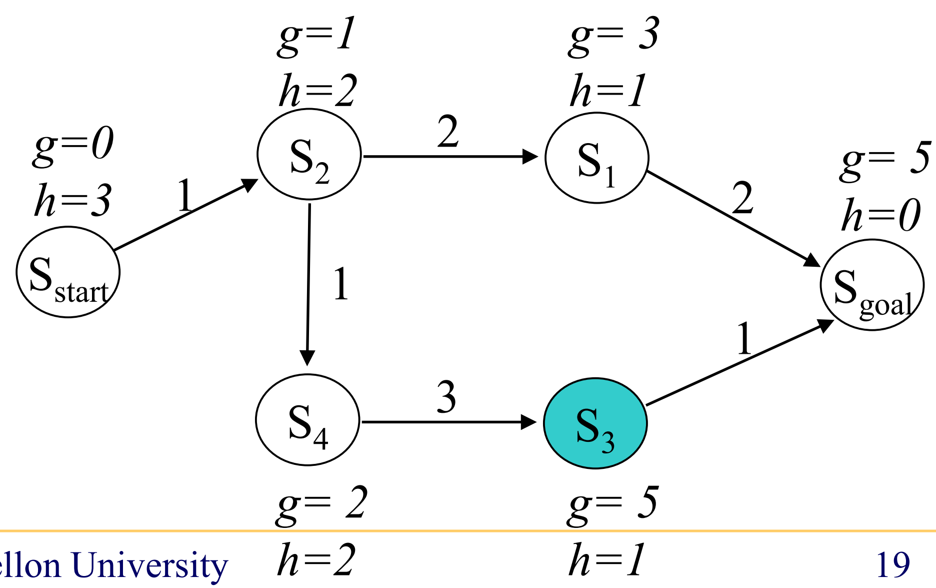
$OPEN = \{s_3\}$

done



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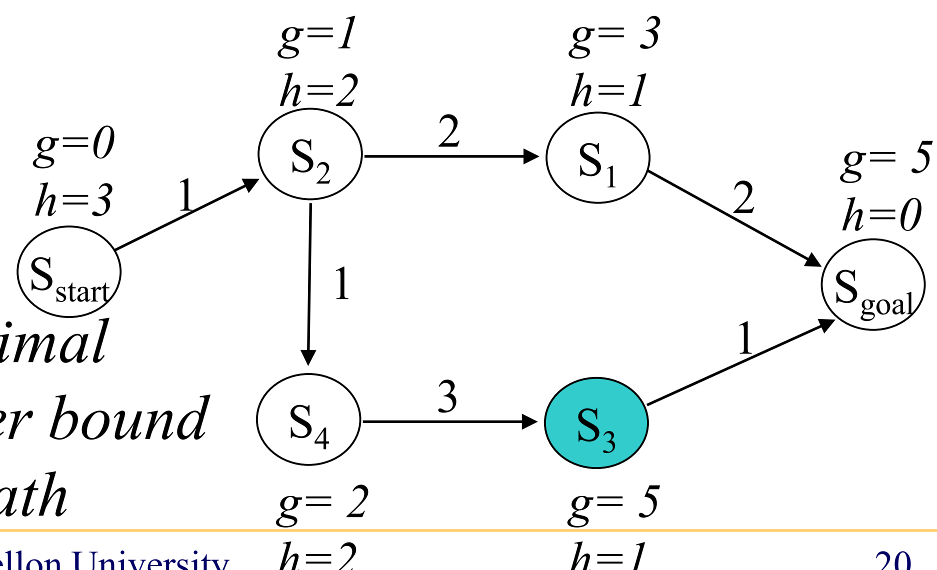

for every expanded state g(s) is optimal
for every other state g(s) is an upper bound
we can now compute a least-cost path

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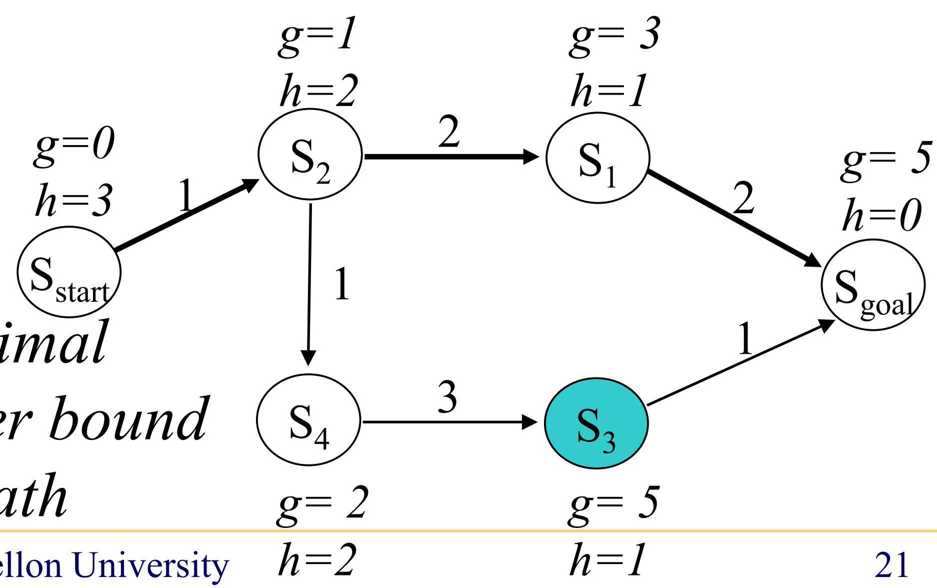

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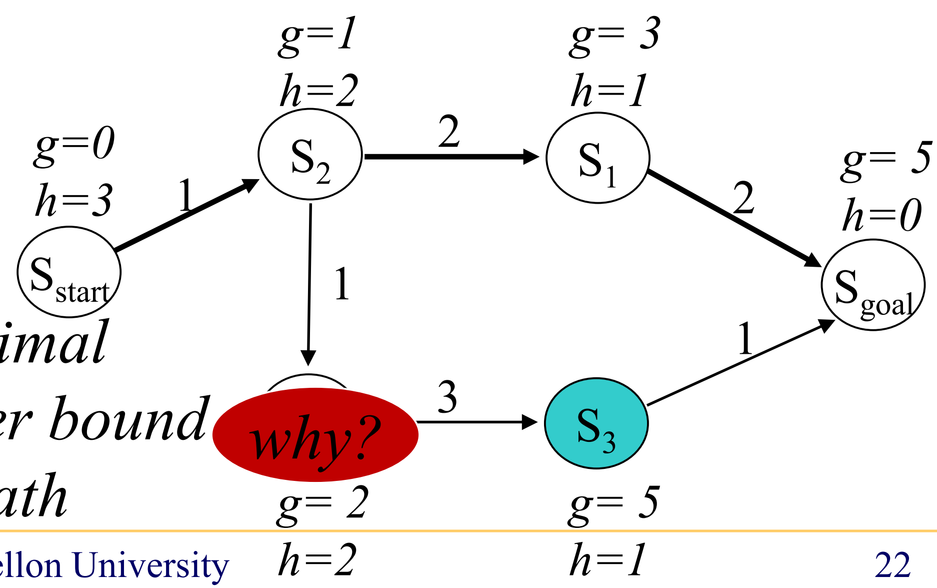

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A* Search

- Is guaranteed to return an optimal path (in fact, for every expanded state) – optimal in terms of the solution

- Performs <u>provably minimal number of state expansions</u> required to guarantee optimality – optimal in terms of the computations

A* Search

- Is guaranteed to return an optimal path (in fact, for every expanded state) – optimal in terms of the solution

Sketch of proof by induction for $h = 0$:

assume all previously expanded states have optimal g-values

next state to expand is s: $f(s) = g(s)$ – min among states in OPEN

OPEN separates expanded states from never seen states

thus, path to s via a state in OPEN or an unseen state will be worse than $g(s)$ (assuming positive costs)

Effect of the Heuristic Function

- A* Search: expands states in the order of $f = g+h$ values

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Sketch of proof of optimality by induction for consistent h:

1. *assume all previously expanded states have optimal g-values*
2. *next state to expand is s: f(s) = g(s)+h(s) – min among states in OPEN*
3. *assume g(s) is suboptimal (i.e., proof by contradiction)*
4. *then there must be at least one state s' on an optimal path from start to s such that it is still in OPEN*
5. $g(s') + h(s') \geq g(s)+h(s)$
6. *but* $g(s') + c^*(s',s) < g(s) =>$

   $$g(s') + c^*(s',s) + h(s) < g(s) + h(s) =>$$

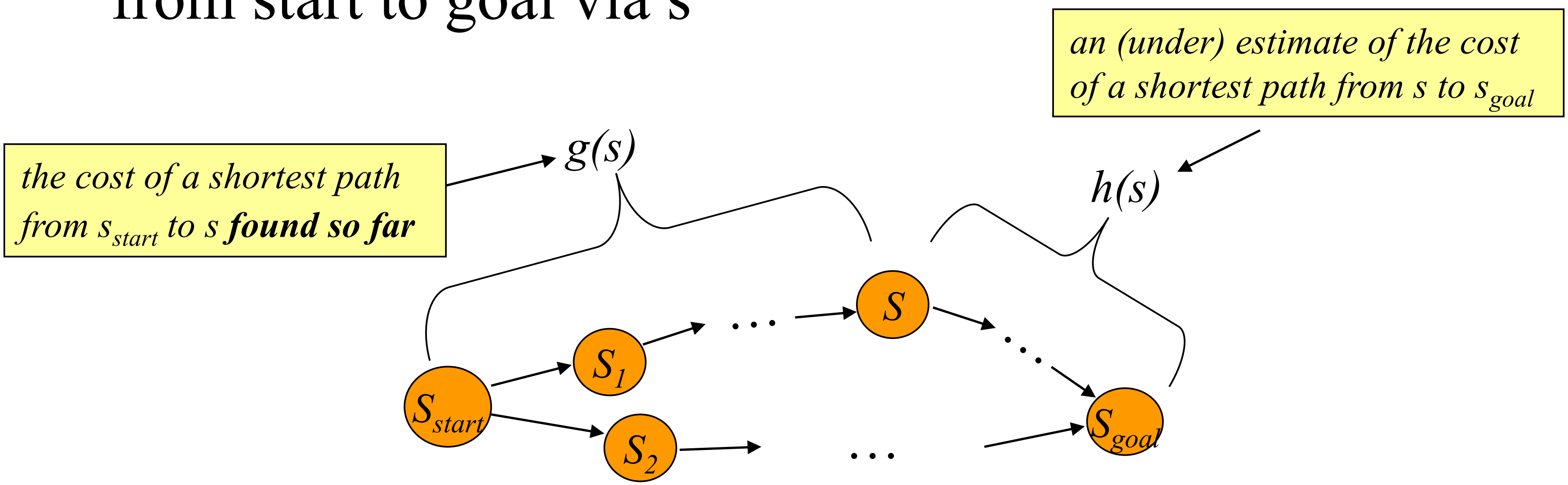
   $$g(s') + h(s') < g(s) + h(s) \text{ (= contradiction)}$$

7. *thus it must be the case that g(s) is optimal*

Effect of the Heuristic Function

- A* Search: expands states in the order of $f = g+h$ values
- Dijkstra's: expands states in the order of $f = g$ values (pretty much)

- Intuitively: f(s) – estimate of the cost of a least cost path from start to goal via s



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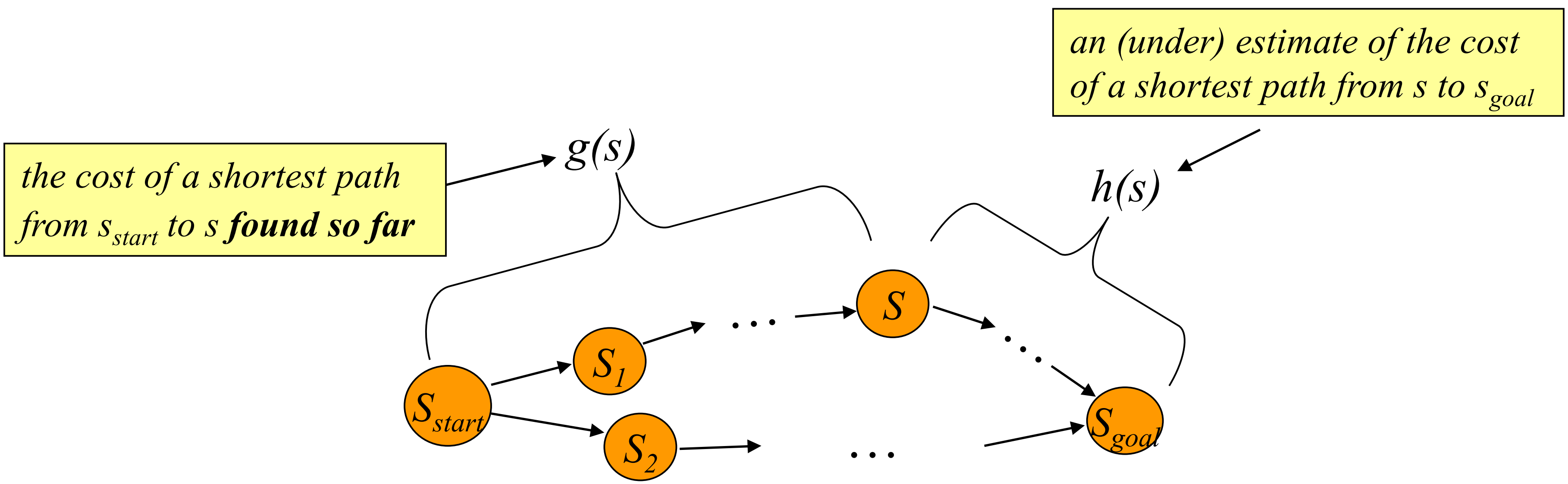
- **Weighted A*:** expands states in the order of $f = g+\varepsilon h$ values, $\varepsilon > 1$ = bias towards states that are closer to goal



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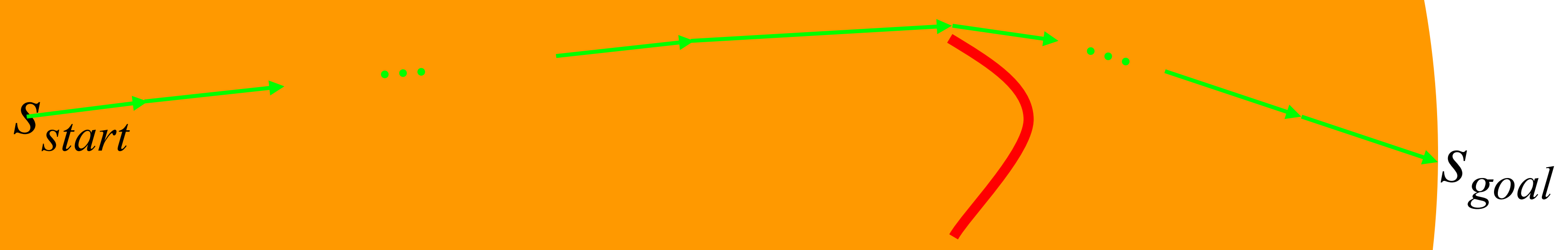
What are the states expanded?



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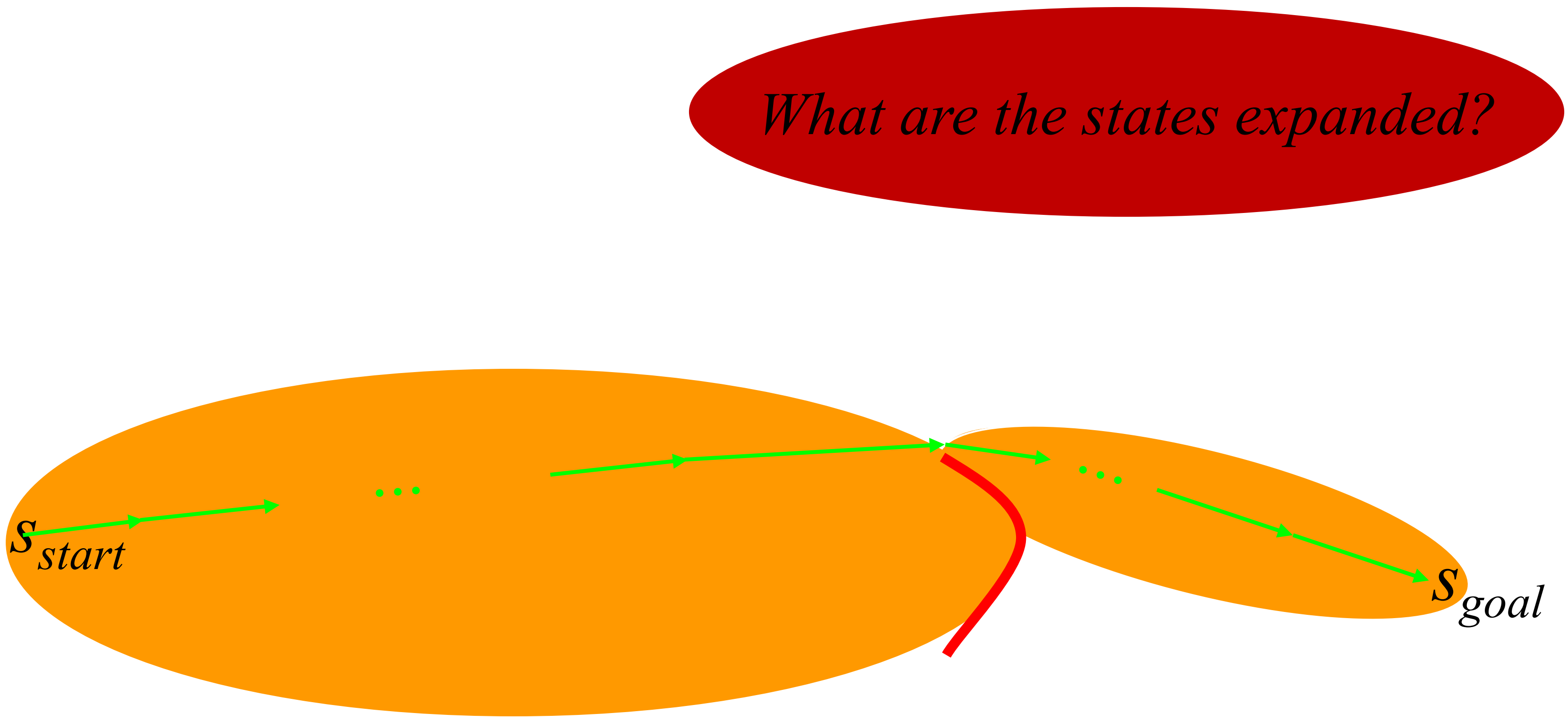
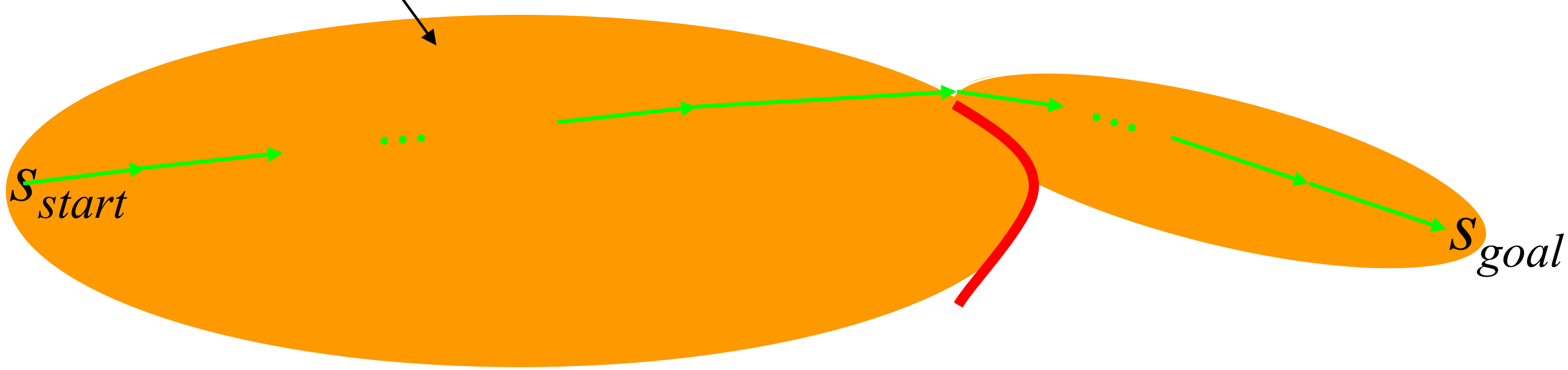


Effect of the Heuristic Function

- A* Search: expands states in the order of $f = g + h$ values

for large problems this results in A quickly running out of memory (memory: O(n))*



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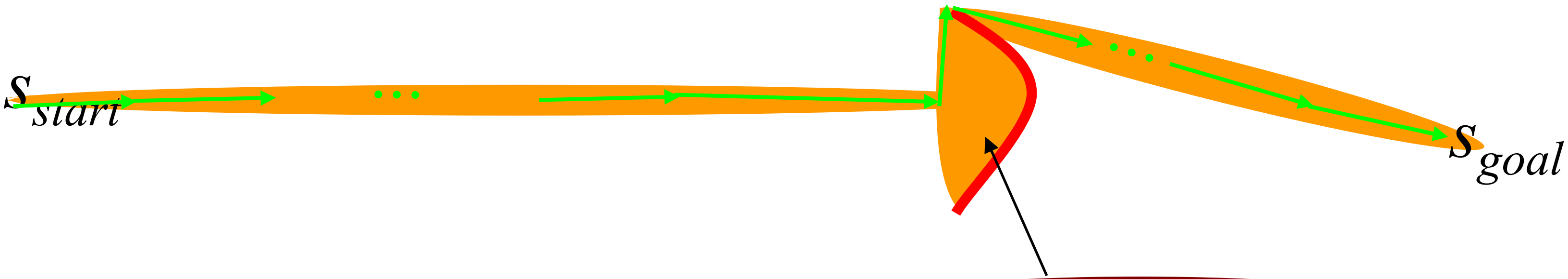

key to finding solution fast:
shallow minima for h(s)-h(s) function*

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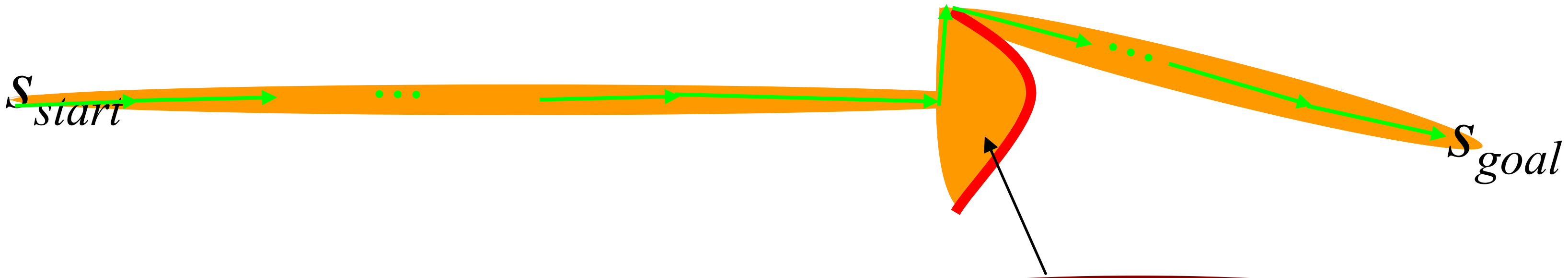
what states are expanded?

No one knows. Topic for research.



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Effect of the Heuristic Function

- Weighted A* Search:
  - trades off optimality for speed
  - ε-suboptimal:

    $$cost(solution) \leq \varepsilon \cdot cost(optimal\ solution)$$

  - in many domains, it has been shown to be orders of magnitude faster than A*
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- Weighted A* Search
  - with re-expansions (no Closed List) [Pohl, '70]
  - without re-expansions (with Closed List) [Likhachev et al., '04]
    - same sub-optimality guarantees <u>but no more than 1 expansion per state</u>

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Is it guaranteed to expand no more states than A?*

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Backward A* Search

- Searches from goal towards states
- g-values are cost-to-goals

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What needs to be changed?

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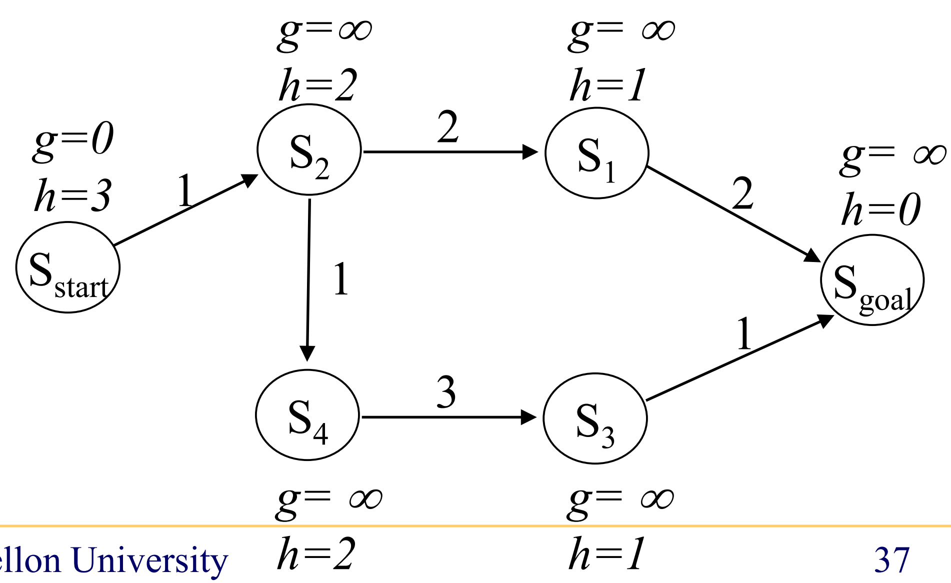

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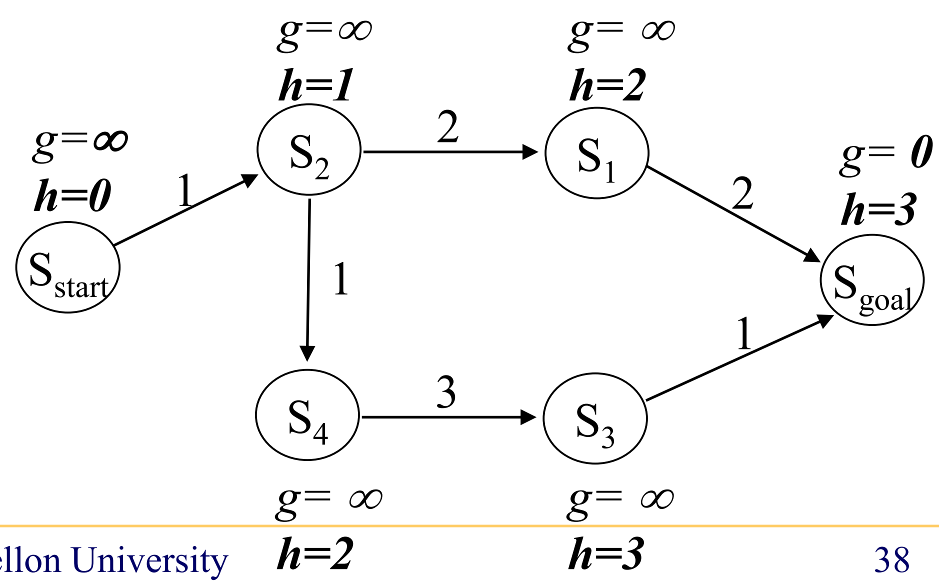

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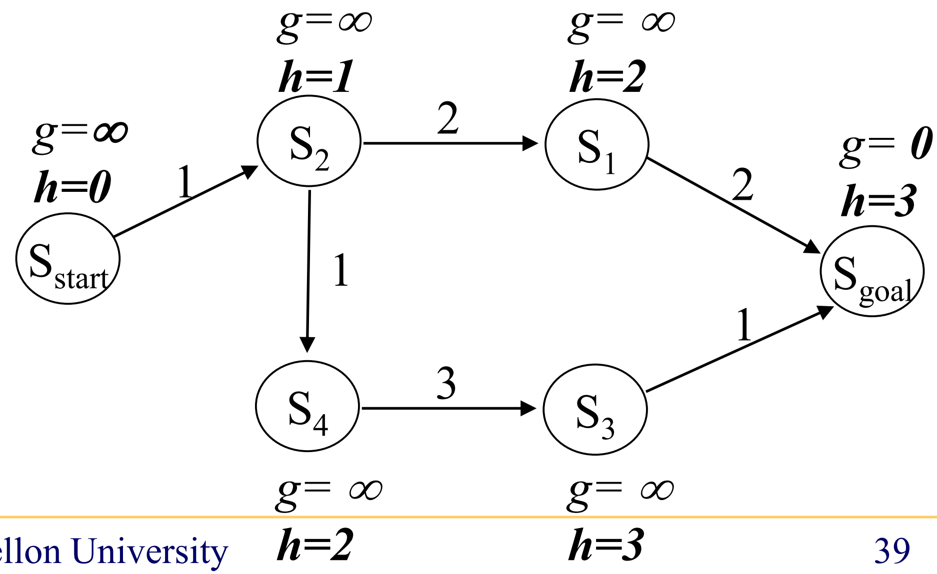

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remove s with the smallest $[f(s) = g(s)+h(s)]$ from $OPEN$;

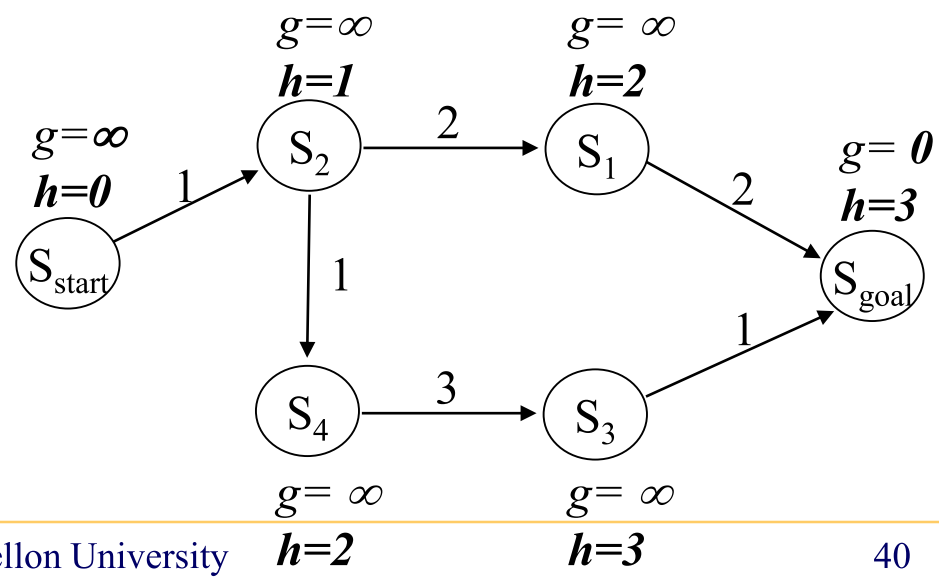
insert s into $CLOSED$;

for every **predecessor** s' of s such that s' not in $CLOSED$

if $g(s') > \boldsymbol{c(s',s)} + \boldsymbol{g(s)}$

$g(s') = \boldsymbol{c(s',s)} + \boldsymbol{g(s)}$;

insert s' into $OPEN$;



Backward A* Search

- Searches from goal towards states
- g-values are cost-to-goals

ComputePath function

while($\boldsymbol{s_{start}}$ is not expanded and $OPEN \neq 0$)

remove s with the smallest $[f(s) = g(s)+h(s)]$ from $OPEN$;

insert s into $CLOSED$;

for every **predecessor** s' of s such that s' not in $CLOSED$

if $g(s') > \boldsymbol{c(s',s) + g(s)}$

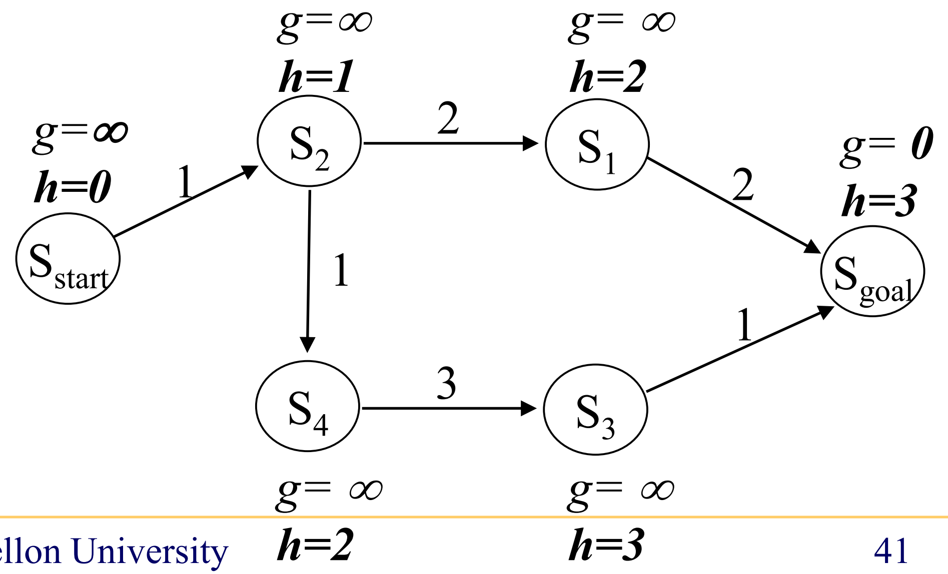
$g(s') = \boldsymbol{c(s',s) + g(s)}$;

insert s' into $OPEN$;

$CLOSED = \{\}$

$OPEN = \{s_{goal}\}$

next state to expand: s_{goal}



Backward A* Search

- Searches from goal towards states
- g-values are cost-to-goals

ComputePath function

while($\boldsymbol{s_{start}}$ is not expanded and $OPEN \neq 0$)

remove s with the smallest $[f(s) = g(s)+h(s)]$ from $OPEN$;

insert s into $CLOSED$;

for every **predecessor** s' of s such that s' not in $CLOSED$

if $g(s') > \boldsymbol{c(s',s) + g(s)}$

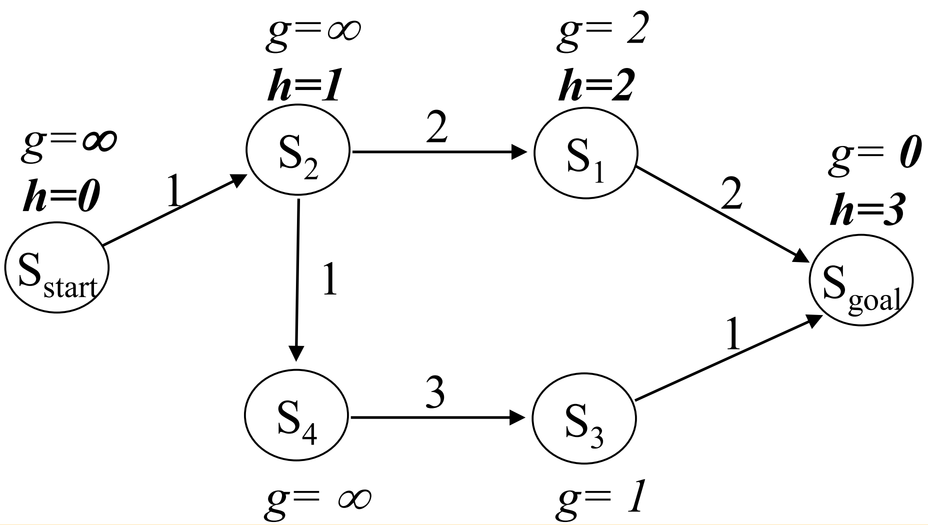
$g(s') = \boldsymbol{c(s',s) + g(s)}$;

insert s' into $OPEN$;

$CLOSED = \{\}$

$OPEN = \{s_1, s_3\}$

next state to expand: s_1



Backward A* Search

- Searches from goal towards states
- g-values are cost-to-goals

ComputePath function

while($\boldsymbol{s_{start}}$ is not expanded and $OPEN \neq 0$)

remove s with the smallest $[f(s) = g(s)+h(s)]$ from $OPEN$;

insert s into $CLOSED$;

for every **predecessor** s' of s such that s' not in $CLOSED$

if $g(s') > \boldsymbol{c(s',s) + g(s)}$

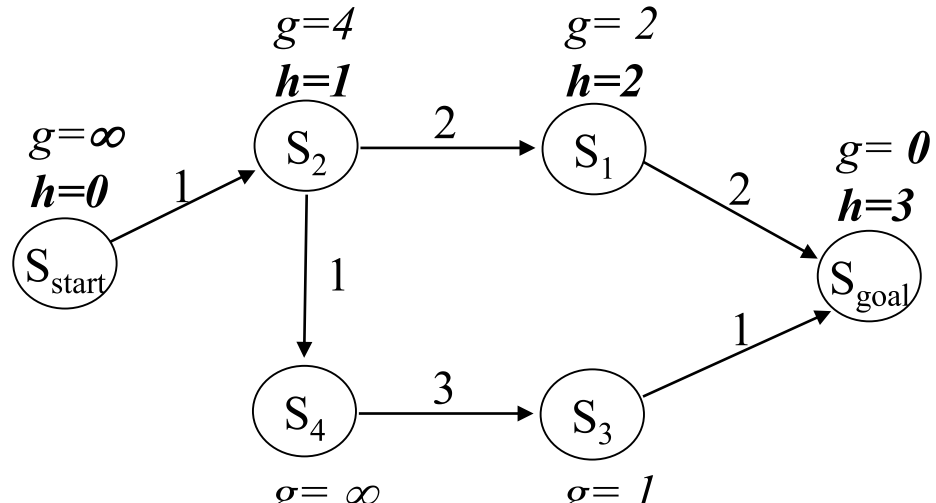
$g(s') = \boldsymbol{c(s',s) + g(s)}$;

insert s' into $OPEN$;

$CLOSED = \{\}$

$OPEN = \{s_2, s_3\}$

next state to expand: s_3



Backward A* Search

- Searches from goal towards states
- g-values are cost-to-goals

ComputePath function

while($\boldsymbol{s_{start}}$ is not expanded and $OPEN \neq 0$)

remove s with the smallest $[f(s) = g(s)+h(s)]$ from $OPEN$;

insert s into $CLOSED$;

for every **predecessor** s' of s such that s' not in $CLOSED$

if $g(s') > \boldsymbol{c(s',s) + g(s)}$

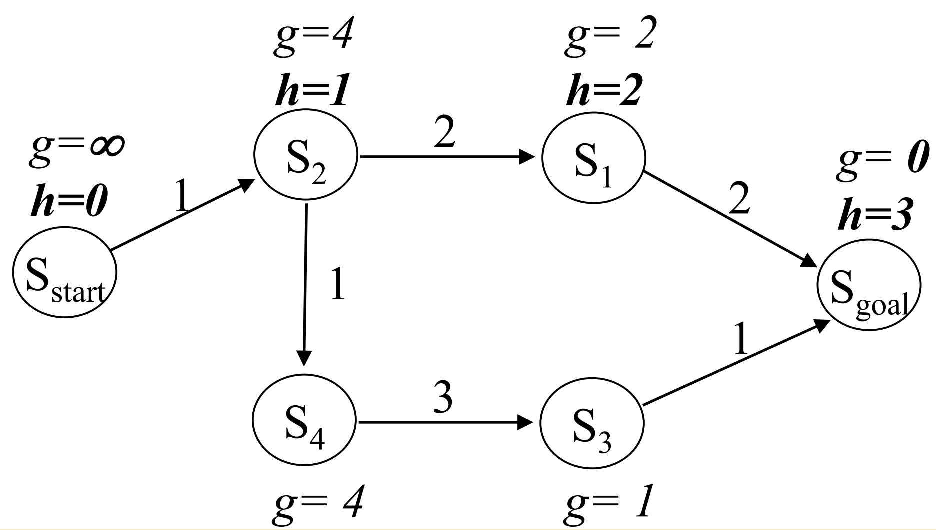
$g(s') = \boldsymbol{c(s',s) + g(s)}$;

insert s' into $OPEN$;

$CLOSED = \{\}$

$OPEN = \{s_2, s_4\}$

next state to expand: s_2



Backward A* Search

- Searches from goal towards states
- g-values are cost-to-goals

ComputePath function

while(s_{start} is not expanded and $OPEN \neq 0$)

remove s with the smallest $[f(s) = g(s)+h(s)]$ from $OPEN$;

insert s into $CLOSED$;

for every **predecessor** s' of s such that s' not in $CLOSED$

if $g(s') > \mathbf{c(s',s) + g(s)}$

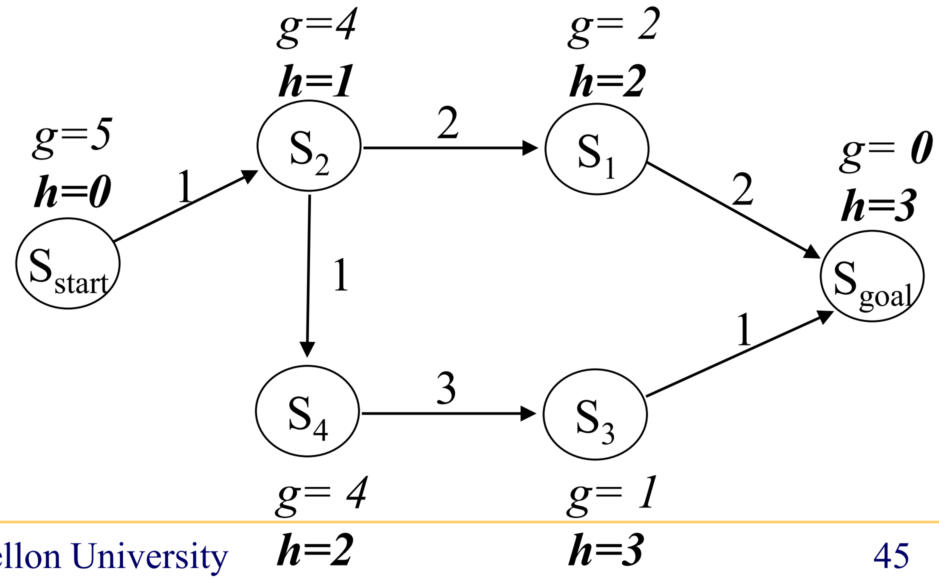
$g(s') = \mathbf{c(s',s) + g(s)}$;

insert s' into $OPEN$;

$CLOSED = \{\}$

$OPEN = \{s_{start}, s_4\}$

next state to expand: s_{start}



Backward A* Search

- Searches from goal towards states
- g-values are cost-to-goals

ComputePath function

while(s_{start} is not expanded and $OPEN \neq 0$)

remove s with the smallest $[f(s) = g(s)+h(s)]$ from $OPEN$;

insert s into $CLOSED$;

for every **predecessor** s' of s such that s' not in $CLOSED$

if $g(s') > \boldsymbol{c(s',s) + g(s)}$

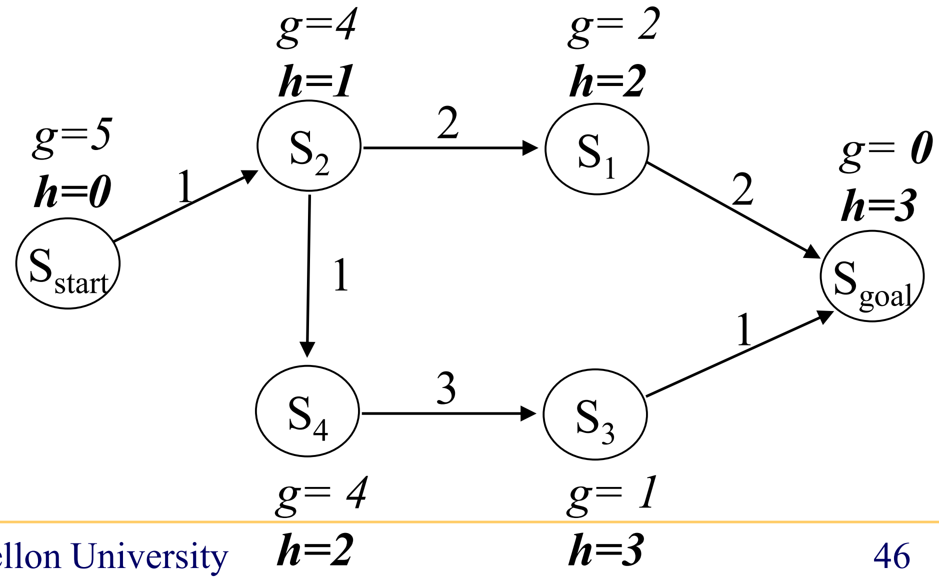
$g(s') = \boldsymbol{c(s',s) + g(s)}$;

insert s' into $OPEN$;

$CLOSED = \{\}$

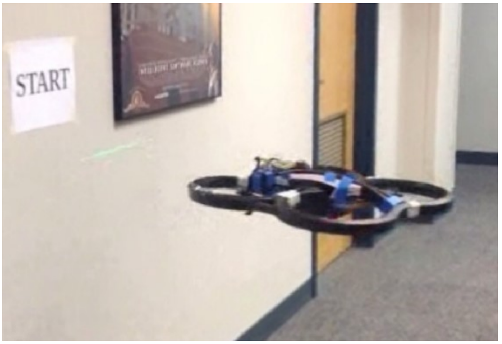
$OPEN = \{s_4\}$

done



Using A* to Compute a Policy

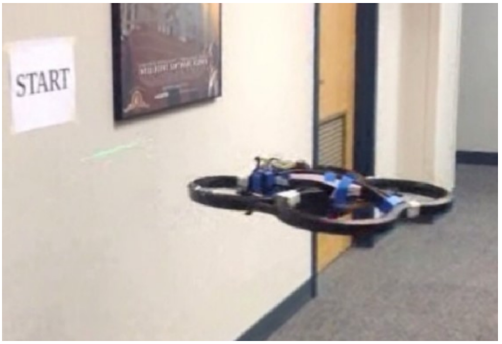
- Imagine planning for the agent that can easily deviate off the path



- Can A* compute least-cost paths from **all** the states of interest?

Using A* to Compute a Policy

- Imagine planning for the agent that can easily deviate off the path



- Can A* compute least-cost paths from **all** the states of interest?
  - Run Backward A* search until all states of interest have been expanded

Using A* to Compute a Policy

- Backward A* search to compute least-cost paths for all states $s \epsilon \Phi$

ComputePath function
while(**at least one state in Φ hasn't been expanded** and $OPEN \neq 0$)
  remove s with the smallest $[f(s) = g(s)+h(s)]$ from $OPEN$;
  insert s into $CLOSED$;
  for every predecessor s' of s such that s' not in $CLOSED$
    if $g(s') > c(s',s) + g(s)$
      $g(s') = c(s',s) + g(s);$
      insert s' into $OPEN$;

- Guaranteed to compute least-cost paths for all $s \epsilon \Phi$ that can reach goal

What You Should Know…

- A*
  - How it works
  - Theoretical properties
  - Proof for its optimality
- Weighted A*
- Backwards A*
- A* can be used to compute a policy and not just a single path