Theorem: Let rank$(A) = r \,(< n)$, then the system $A\underset{\sim}{x} = \underset{\sim}{0}$ has exactly $(n-r)$ independent solutions.

Proof: Let S denotes the solution space of $A\underset{\sim}{x} = \underset{\sim}{0}$ and R denotes the row-space of A.

Obviously, S will be an orthocomplement of R.

Let, $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_r$ be an orthogonal basis of R. this can be extended to an orthogonal basis of E^n, say, $\{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_r, \underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2, \ldots, \underset{\sim}{\beta}_{n-r}\}$. Here in contradiction we prove that $\underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2, \ldots, \underset{\sim}{\beta}_{n-r}$ is an orthogonal basis of S. $\therefore \dim(S) = n-r$.

Consider a vector $\underset{\sim}{\beta} \in S$, which can't be written as a linear combination of $\underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2, \ldots, \underset{\sim}{\beta}_{n-r}$. Then the set $\underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2, \ldots, \underset{\sim}{\beta}_{n-r}, \underset{\sim}{\beta}$ will be linearly independent and generate the subspace $S_2 \subseteq S$.

Now the set $\{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n, \underset{\sim}{\beta}_1, \ldots, \underset{\sim}{\beta}_{n-r}, \underset{\sim}{\beta}\}$ is a set of non-zero orthogonal vectors and this is a linearly independent set of $(n+1)$ vectors in E^n, which is impossible.

Therefore $\{\underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2, \ldots, \underset{\sim}{\beta}_{n-r}\}$ forms a basis of S. So, $\dim(S) = n-r$, i.e. the system has exactly $(n-r)$ independent solutions.

Solution Space: The vector space generated by the soln. of $A\underset{\sim}{x} = \underset{\sim}{0}$ is the solution space of the system of equation.

i.e. $\{x : A\underset{\sim}{x} = \underset{\sim}{0}\}$ which is the null space of A $\boxed{N(A)}$ → Notation

We know, $\dim(N(A)) = (n-r)$, where $r = R(A)$.

Clearly $\dim(N(A)) = 0$ if A is of full column rank, in that case we have the trivial soln as the only soln.

If $\dim(N(A)) > 0$ we may search for a basis of the soln. space.

[basis is the set of maximum no. of LIN soln.]

Suppose $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_{n-r}\}$ be a basis of the soln. space. Then any soln. of $A\underset{\sim}{x} = \underset{\sim}{0}$ can be written as a linear combination of the basis vectors, i.e., the general soln. to the system of eqn. will be of the form $\sum_{i=1}^{n-r} \lambda_i \underset{\sim}{x}_i$.

NOTE: — If there exists a non-null solution then it must be orthogonal to the non-null rows of A.

Rewrite A as

$$A = (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n)$$

and $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$

Then $A\underset{\sim}{x} = \underset{\sim}{0}$ reduces to

$$\sum_{i=1}^{n} x_i \underset{\sim}{\alpha}_i = \underset{\sim}{0} \quad \text{——} (***)$$

$\therefore$ to search for a non-trivial solution is equivalent to check whether (***) holds good for at least one $x_i \neq 0$.

In other words, the system of equation defined by $A\underset{\sim}{x} = \underset{\sim}{0}$ possesses a non-trivial solution if column of A are linearly dependent.

Ex. 1. Obtain the general soln. to the system of equations:

$$x_1 + x_2 + x_3 + x_4 = 0$$
$$x_1 - 2x_2 + 2x_4 = 0$$
$$x_1 + 7x_2 + 3x_3 - x_4 = 0$$

Soln. → $A\underset{\sim}{x} = \underset{\sim}{0}$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 2 \\ 1 & 7 & 3 & -1 \end{pmatrix} = (\underset{\sim}{\alpha}_1 \ \underset{\sim}{\alpha}_2 \ \underset{\sim}{\alpha}_3 \ \underset{\sim}{\alpha}_4)$$

$\underset{\sim}{\alpha}_1$ and $\underset{\sim}{\alpha}_2$ are linearly independent.

If possible, let $\underset{\sim}{\alpha}_3 = \lambda_1 \underset{\sim}{\alpha}_1 + \lambda_2 \underset{\sim}{\alpha}_2$

$$\Rightarrow \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 - 2\lambda_2 \\ \lambda_1 + 7\lambda_2 \end{pmatrix}$$

$$\Rightarrow \lambda_1 = \frac{2}{3}, \ \lambda_2 = \frac{1}{3}$$

i.e. $\frac{2}{3}\underset{\sim}{\alpha}_1 + \frac{1}{3}\underset{\sim}{\alpha}_2 - \underset{\sim}{\alpha}_3 = 0$

$$\Rightarrow (\underset{\sim}{\alpha}_1 \ \underset{\sim}{\alpha}_2 \ \underset{\sim}{\alpha}_3 \ \underset{\sim}{\alpha}_4) \begin{pmatrix} 2/3 \\ 1/3 \\ -1 \\ 0 \end{pmatrix} = \underset{\sim}{0}$$

i.e. $\begin{pmatrix} 2/3 \\ 1/3 \\ -1 \\ 0 \end{pmatrix}$ is a non trivial solution.

If possible, let $\Rightarrow \underset{\sim}{\alpha}_4 = \lambda_1 \underset{\sim}{\alpha}_1 + \lambda_2 \underset{\sim}{\alpha}_2$

$$\Rightarrow \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 - 2\lambda_2 \\ \lambda_1 + 7\lambda_2 \end{pmatrix}$$

$$\Rightarrow \lambda_2 = -1/3,\ \lambda_1 = 4/3$$

$\Rightarrow$ another non-trivial independent solution is $\begin{pmatrix} 4/3 \\ -1/3 \\ 0 \\ -1 \end{pmatrix}$.

Here, $R(A) = 2 \Rightarrow \dim(N(A)) = 2$.

A basis of the solution space is $\left\{ \begin{pmatrix} 2/3 \\ 1/3 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4/3 \\ -1/3 \\ 0 \\ -1 \end{pmatrix} \right\}$

A basis of the soln. space $= \begin{pmatrix} 2\alpha + 4\beta \\ \alpha - \beta \\ -3\alpha \\ -3\beta \end{pmatrix}$

$\boxed{\text{Or}}$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 2 \\ 1 & 7 & 3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 1 & -1 \\ 0 & -6 & -2 & 2 \end{pmatrix} \begin{matrix} ① R_2' = R_1 - R_2 \\ ② R_3' = R_1 - R_3 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} ① R_2' = R_2/3 \\ ② R_3' = 2R_2 + R_3 \end{matrix}$$

$= H$, an echelon matrix.

Here, rank$(A) = 2$.

$$H\underset{\sim}{x} = 0 \Rightarrow \left. \begin{matrix} x_1 + x_2 + x_3 + x_4 = 0 \\ \text{and, } x_2 + \frac{x_3}{3} - \frac{x_4}{3} = 0 \end{matrix} \right\} \Rightarrow$$

x_1	x_2	x_3	x_4
-2	0	1	1
$2/3$	$-2/3$	1	-1

$\Rightarrow$ General soln. is $\begin{pmatrix} -2\gamma + \frac{2}{3}\delta \\ -\frac{2}{3}\delta \\ \gamma + \delta \\ \gamma - \delta \end{pmatrix}$

NOTE:- If m exceeds n, then at least m−n equation become redem, thus we consider $m \leq n$.

Ex. 2. Obtain the general soln. to the system of equations :—

$$x_1 + 3x_2 + 2x_3 + 3x_4 = 0$$
$$x_2 + 2x_3 + x_4 - x_1 = 0$$
$$3x_1 + 4x_2 + x_3 + 4x_4 = 0$$

Soln.→ $A\underset{\sim}{x} = \underset{\sim}{0}$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 2 & 3 \\ -1 & 1 & 2 & 1 \\ 3 & 4 & 1 & 4 \end{bmatrix} \underset{\sim}{x} = \underset{\sim}{0}$$

or, $(\underset{\sim}{\alpha_1}\ \underset{\sim}{\alpha_2}\ \underset{\sim}{\alpha_3}\ \underset{\sim}{\alpha_4})\underset{\sim}{x} = \underset{\sim}{0}$

Here, $R(A) = 2$.

$\underset{\sim}{\alpha_1}$ and $\underset{\sim}{\alpha_2}$ are linearly independent,

Note that — $\underset{\sim}{\alpha_3} = \underset{\sim}{\alpha_2} - \underset{\sim}{\alpha_1}$

$$\Rightarrow (\underset{\sim}{\alpha_1}\ \underset{\sim}{\alpha_2}\ \underset{\sim}{\alpha_3}\ \underset{\sim}{\alpha_4}) \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \underset{\sim}{0}$$

i.e. $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ is a solution.

Again, $\alpha_2 - \alpha_4 = 0$

$$\Rightarrow (\underset{\sim}{\alpha_1}\ \underset{\sim}{\alpha_2}\ \underset{\sim}{\alpha_3}\ \underset{\sim}{\alpha_4}) \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \underset{\sim}{0}$$

$\Rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ is another linearly independent solution.

A basis of the solution space is $\longrightarrow \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

A basis of the solution space $\longrightarrow \begin{pmatrix} \alpha \\ \beta - \alpha \\ \alpha \\ -\beta \end{pmatrix}$

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Ex. 3. Consider the system of equations:

$$x_1+x_2+x_3+x_4=0$$
$$x_1+2x_2+3x_3+4x_4=0$$
$$2x_1+x_3-x_4=0$$

Soln. $A=\begin{pmatrix}1&1&1&1\\1&2&3&4\\2&1&0&-1\end{pmatrix}$, $\quad x=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$

The sweep-out method method gives —

$$A \sim \begin{pmatrix}1&1&1&1\\0&1&2&3\\0&-1&-2&-3\end{pmatrix} \sim \begin{pmatrix}1&1&1&1\\0&1&2&3\\0&0&0&0\end{pmatrix} \sim \begin{pmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\end{pmatrix}$$

$= R$, a row-reduced echelon matrix.

So, Rank$(A)=2$.

$R\underset{\sim}{x}=\underset{\sim}{0} \Rightarrow x_1-x_3-2x_4=0$ & $x_2+2x_3+3x_4=0$

x_1	x_2	x_3	x_4
1	-2	1	0
3	-5	1	1

$\Rightarrow$ General form is $\rightarrow \begin{pmatrix}\alpha+3\beta\\-2\alpha-5\beta\\\alpha+\beta\\\beta\end{pmatrix}$

Ex. 4. The system of equations

$x-y+z=0$, $\quad x+2y-z=0$, $\quad 2x+y+3z=0$ have

(A) infinite solutions (B) Trivial solution (C) No-solution.

Sol. (B)

$$A=\begin{pmatrix}1&-1&1\\1&2&-1\\2&1&3\end{pmatrix} \sim \begin{pmatrix}1&-1&1\\1&2&-1\\0&0&3\end{pmatrix} \sim \begin{pmatrix}1&-1&1\\0&3&-2\\0&0&3\end{pmatrix}$$

$$\begin{pmatrix}1&-1&1\\0&3&-2\\0&0&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{cases}x-y+z=0\\3y-2z=0\\3z=0\end{cases}$$

$\therefore$ $z=0$, $y=0$, $x=0$ is the only solution of the given system of equations.

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Case of non-homogeneous system: If $\underset{\sim}{b} \neq \underset{\sim}{0}$ for the system of equations $A\underset{\sim}{x} = \underset{\sim}{b}$ then $\underset{\sim}{x} = 0$ can't be a solution. It means that if there is at all a solution then it must be a non-trivial solution.
We will now establish a necessary and sufficient condition for a non-homogeneous system to be consistent.

$$A\underset{\sim}{x} = \underset{\sim}{b} \quad , \underset{\sim}{b} \neq 0.$$

This can be written as — $\sum_{i=1}^{n} x_i \hat{\alpha}_i = \underset{\sim}{b}$.

Augmented Matrix: By the augmented matrix for the system we mean the matrix $(A : b)$ of order $m \times \overline{n+1}$ obtained by adjoining one more column, viz. $\underset{\sim}{b}$. thus

$$(A : b) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} & b_m \end{pmatrix}_{m \times \overline{n+1}}$$

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Theorem: — The system of equation $A\hat{x} = \underset{\sim}{b}$ is compartible (consistent) or has at least one soln. if and only if rank $(A : \hat{b})$ = rank (A).
[Due to Kronecker - Capelli]

Proof:

If Part: Suppose rank $(A : \hat{b})$ = rank(A)

$\therefore$ $\underset{\sim}{b}$ can be written as a linear combination of the columns of A,

$\Rightarrow \underset{\sim}{b} \in V_c(A)$

Suppose $A = (\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_n)$

Therefore there exists scalars $\lambda_1, \lambda_2, \dots, \lambda_n \ni \hat{b} = \sum_{i=1}^{n} \lambda_i \underset{\sim}{\alpha}_i$

$$\Rightarrow (\hat{\alpha}_1, \alpha_2, \dots, \hat{\alpha}_n) \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \underset{\sim}{b}.$$

Hence $\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$ is a solution to the system of equation $A\hat{x} = \underset{\sim}{b}$.

Therefore, the system is consistent (compartable)

Only if part :— Suppose the system $A\underset{\sim}{x} = \underset{\sim}{b}\ (\neq 0)$ is consistent and let $\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$ is a solution of the system.

$$A \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \underset{\sim}{b}$$

$$\Rightarrow \sum_{i=1}^{n} \lambda_i \hat{\alpha}_i = \underset{\sim}{b}$$

i.e. $\underset{\sim}{b}$ is linearly dependent on the columns of A.

$$\Rightarrow \underset{\sim}{b} \in V_c(A)$$

$\therefore$ column rank of A = column rank of $A : b$

i.e. $\operatorname{rank}(A) = \operatorname{rank}(A : b)$.

Theorem :— A non-homogeneous system of n-linear equations in n unknowns $\underset{n\times n}{A}\ \underset{n\times 1}{\underset{\sim}{x}} = \underset{n\times 1}{\underset{\sim}{b}}$ has the unique solution $\underset{\sim}{x} = A^{-1}b$ if and only if $\operatorname{rank}(A) = n$ or A is non-singular.

Proof :→

If part :— Let $\operatorname{rank}(A) = n$.

Then A^{-1} exists. Putting $x = A^{-1}b$ in the L.H.S of the system —

$$AX = A(A^{-1}b) = (AA^{-1}).b = Ib = b = R.H.S$$

$\therefore x = A^{-1}b$ is a solution of the system.

To show that this is the unique solution, let $\underset{\sim}{x} = \hat{\alpha}$ be another solution, we can write $\underset{\sim}{\alpha} = A^{-1}b + \hat{\beta}$, where $\hat{\beta} = \underset{\sim}{\alpha} - A^{-1}b$.

As α is a solution of $A\underset{\sim}{\alpha} = \underset{\sim}{b} \Rightarrow A(A^{-1}b + \underset{\sim}{\beta}) = \underset{\sim}{b}$

$$\Rightarrow (AA^{-1}b) + A\hat{\beta} = \underset{\sim}{b}$$

$$\Rightarrow Ib + A\hat{\beta} = \underset{\sim}{b}$$

$$\therefore A\hat{\beta} = \underset{\sim}{0}$$

If $\hat{\beta} = \underset{\sim}{0}$ then this is obviously satisfied, on the other hand if $\hat{\beta} \neq 0$, then this means that the columns of A will form a linearly dependent set. This means $\operatorname{rank}(A) < n$ which is a contradiction. So, $x = A^{-1}\hat{b}$ is the unique solution.

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Only if part :- Suppose $\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$ is a unique solution of $A\underset{\sim}{x} = \underset{\sim}{b}$.
$\underset{\sim}{x}=\underset{\sim}{0}$ is the unique solution of $A\underset{\sim}{x} = \underset{\sim}{0}$.
so, $\underset{\sim}{x} = \underset{\sim}{0}$ is the only solution $x_i = 0$,
$\Rightarrow \sum_{i=1}^{n} \underset{\sim}{\alpha_i} x_i = 0$ has the only solution $x_i = 0$,
so, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent, so, rank$(A) = n$,
where $\underset{\sim}{\alpha_i}$: ith column vector of A.

Note :-
$$A\underset{\sim}{x} = \underset{\sim}{b} \quad (\underset{\sim}{b} \neq 0)$$
$$\Leftrightarrow \sum_i x_i \underset{\sim}{\alpha_i} = \underset{\sim}{b}$$
If α_i's are linearly independent then the collection constitutes the basis of the column space, in that case the representation of $\underset{\sim}{b}$ interms of columns of A will be unique, i.e. the choice of $(x_1, x_2, \ldots, x_n)'$ is unique. Hence the system possesses a unique solution.
On the other hand, if the columns of A are linearly independent then representation of $\underset{\sim}{b}$ will not be unique.

General solution to a system of non-homogeneous equation :-

Let $A\underset{\sim}{x} = \underset{\sim}{b}\ (\neq 0)$ be a system of non-homogeneous equation where y is a specific solution to the system of eqn. and $\underset{\sim}{z}$ is the general solution to $A\underset{\sim}{x} = \underset{\sim}{0}$, then $(\underset{\sim}{y} + \underset{\sim}{z})$ will be the general solution to $A\underset{\sim}{x} = \underset{\sim}{b}$.

Ex.1. Obtain the general soln. to the following system of eqn. :
$$x_1 + x_2 + x_3 + x_4 = 2$$
$$x_1 - 2x_2 + 2x_4 = -1$$
$$x_1 + 7x_2 + 3x_3 - x_4 = 8$$

Soln. →
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 2 \\ 1 & 7 & 3 & -1 \end{pmatrix}, \quad \underset{\sim}{b} = \begin{pmatrix} 2 \\ -1 \\ 8 \end{pmatrix}$$

$$A\underset{\sim}{x} = \underset{\sim}{b}$$
$$\Leftrightarrow PA\underset{\sim}{x} = P\underset{\sim}{b}$$, where P is an non-singular matrix ∋ $PA = H$, an echelon matrix.
i.e. $H\underset{\sim}{x} = P\underset{\sim}{b}$

Reducing, $(A \mid \underset{\sim}{b}) \underset{\text{trans}}{\overset{\text{r.s.}}{\sim}} (H \mid P\underset{\sim}{b})$.

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & 7 & 3 & -1 & 8 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 3 & 1 & -1 & 3 \\ 0 & -6 & -2 & 2 & -6 \end{array}\right) \quad \begin{array}{l} ① \; R_2' = R_1 - R_2 \\ ② \; R_3' = R_1 - R_3 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1/3 & -1/3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} ① \; R_3' = R_3 + 2R_2 \\ ② \; R_2' = R_2/3 \end{array}$$

$\Rightarrow R(A \vdots b) = R(A) = 2$

Hence it is consistent.

$\boxed{\text{Or}}$.

Note that, $\underset{\sim}{b} = \underset{\sim}{\alpha}_1 + \underset{\sim}{\alpha}_2$,

$$A = (\underset{\sim}{\alpha}_1 \; \underset{\sim}{\alpha}_2 \; \underset{\sim}{\alpha}_3 \; \underset{\sim}{\alpha}_4)$$

so, here $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

The general soln. of $A\underset{\sim}{x} = \underset{\sim}{0}$ is $\begin{pmatrix} 2\alpha + 4\beta \\ \alpha - \beta \\ -3\alpha \\ -3\beta \end{pmatrix}$ [Obtained before]

Thus the general solution of $A\underset{\sim}{x} = \underset{\sim}{b}$ is $\begin{pmatrix} 2\alpha + 4\beta + 1 \\ \alpha - \beta + 1 \\ -3\alpha \\ -3\beta \end{pmatrix}$

NOTE:→ In particular if A is non-singular then the unique solution is $A^{-1}\underset{\sim}{b}$.

Ex. 2. Obtain the general solution to the following system of equation:

$$x_1 + 3x_2 + 2x_3 + 3x_4 = -1$$
$$x_2 + 2x_3 + x_4 - x_1 = -3$$
$$3x_1 + 4x_2 + x_3 + 4x_4 = 2$$

Soln. → Let it be $A\underset{\sim}{x} = \underset{\sim}{b}$

First of all we are to show that the system is consistent i.e. $R(A:b) = R(A)$.

Here, $\begin{pmatrix} 1 & 3 & 2 & 3 \\ -1 & 1 & 2 & 1 \\ 3 & 4 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$

It is to be noted that — $\underset{\sim}{b} = \underset{\sim}{\alpha}_1 - \underset{\sim}{\alpha}_3$.

Hence a specific form is $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$.

Again the general solution of $A\underset{\sim}{x} = \underset{\sim}{0}$ is $\begin{pmatrix} \alpha \\ \beta-\alpha \\ \alpha \\ -\beta \end{pmatrix}$ [Obtained earlier]

Thus the general solution to this system will be $\begin{pmatrix} \alpha+1 \\ \beta-\alpha \\ \alpha-1 \\ -\beta \end{pmatrix}$.

Ex. 3. Show whether the system is consistent or not:

$$x_1 + x_2 + x_3 = 4$$
$$2x_1 + 5x_2 + 2x_3 = 3$$
$$x_1 + 7x_2 - 7x_3 = 5$$

Soln. → $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{pmatrix}$ and the method of sweep-out gives → $A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{pmatrix}$

As such, $R(A) = 2$.

Again, $(A:b) = \begin{pmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -4/3 & -5/3 \\ 0 & 0 & 0 & 11 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -4/3 & -5/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Hence, $R(A:b) = 3$.

Since, $R(A:b) > R(A)$, the system of equations is inconsistent, i.e. does not have a solution.

Ex. 4.

Check the consistency of this equations :

$$x_1 + x_2 + x_3 = 4$$
$$2x_1 + 5x_2 - 2x_3 = 3.$$

Soln.→ We have $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -4/3 \end{pmatrix}$$

Hence, $R(A) = 2$.

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -4/3 & -5/3 \end{array}\right)$$

$R(A \vdots b) = 2$

Since $r(A \vdots b) \geq 2$, $(A \vdots b)$ having one more column than A, and also, $r(A \vdots b) \leq 2$, for $(A \vdots b)$ has just two ~~row~~ rows, we have also $r(A \vdots b) = 2$

As such, the pair of equations reduces to

$$x_1 + x_2 + x_3 = 4$$
$$x_2 - 4/3\, x_3 = -5/3$$

$x_1 + x_3 = 4 - \alpha$

$-\frac{4}{3}x_3 = -\frac{5}{3} - \alpha$

$\Rightarrow x_3 = \frac{5}{4} + \frac{3}{4}\alpha = \frac{5+3\alpha}{4}$

$x_1 = (4-\alpha) - \left(\frac{5+3\alpha}{4}\right) = \frac{11-7\alpha}{4}$

x_1	x_2	x_3
$10/3$	$-1/3$	1
$17/3$	$-5/3$	0

Thus the general solution of the system will be —

$$= \begin{pmatrix} \frac{11-7\alpha}{4} \\ \alpha \\ \frac{5+3\alpha}{4} \end{pmatrix}$$

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Questions from C.U. Question Papers

1) Examine the consistency or otherwise of the system $A\underset{\sim}{x} = \underset{\sim}{b}$ by citing examples of both. (4)

2) Examine the consistency of the following system of linear equations:

(i) $x_1 = 1$ $\quad x_2 = -1$

(ii) $x_3 = 1$ $\quad x_4 = 0$

$$2x_1 + 5x_2 + 3x_3 + 6x_4 = 7$$
$$3x_1 - x_2 + 2x_3 + 8x_4 = 9$$
$$5x_1 + 2x_2 - 9x_3 - x_4 = 17$$

3) State a necessary and sufficient condition for the existence of a solution $A^{n\times n}\underset{\sim}{x}^{n\times 1} = b^{n\times 1}$. Show that if a system of linear equations has two distinct solutions then there exists an infinite number of solutions. (5)

4) Find the value of c for which the following equations admit a solution:

$$4x_1 + 6x_3 = 1$$
$$4x_3 - 2x_2 = 7 + c$$
$$2x_1 - x_2 + 5x_3 = 4$$
(4)

5) Discuss when the system of non-homogeneous linear equations $A\underset{\sim}{x} = \underset{\sim}{b}$ has (i) a unique solution, (ii) no solution. (4)

Linear Equations – Regular case; Cramer's solution:

Consider the n-linear equation in n unknowns $x_1, x_2, \ldots\ldots, x_n$,

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right\}$$

Let $|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$ = Determinant of the coefficients.

If $|A| \neq 0$, the set of equations is said to be regular. then have a unique solution given by

$$\frac{x_1}{|A_1|} = \frac{x_2}{|A_2|} = \cdots\cdots = \frac{x_n}{|A_n|} = \frac{1}{|A|};$$

This is known as Cramer's solution.

3) State a necessary and sufficient condition for the existance of a solution to $\underset{n\times n}{A}\,\underset{\sim}{x}_{n\times 1} = \underset{\sim}{b}_{n\times 1}$. Show that if a system of linear equations has two distinct solutions then $\exists$ an infinite number of solutions. 6

Ans:–

$A = ((a_{ij}))_{m\times n}$

$$(A:b) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} & b_m \end{pmatrix}_{m\times \overline{n+1}}, \text{ where } \underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}_{n\times 1}$$

A necessary and sufficient condition for the existance of a solution to $\underset{n\times n}{A}\,\underset{n\times 1}{x} = \underset{n\times 1}{b}$ is rank$(A \vdots b) =$ rank(A).

Let $\underset{m\times n}{A}\,\underset{n\times 1}{x} = \underset{m\times 1}{b}$ be a system of linear equations having two distinct solution $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$.

$\Rightarrow A\underset{\sim}{x}_1 = \underset{\sim}{b}$ and $A\underset{\sim}{x}_2 = \underset{\sim}{b}$.

$\Rightarrow A(\underset{\sim}{x}_1 - \underset{\sim}{x}_2) = 0$

$\Rightarrow (\underset{\sim}{x} - \underset{\sim}{x}_2)\ (\neq 0)$ is a solution of the corresponding homogeneous system $A\underset{\sim}{x} = \underset{\sim}{0}$.

$\Rightarrow \lambda(x_1 - x_2)$ is a solution of $A\underset{\sim}{x} = \underset{\sim}{0}$ for any scalar λ.

$\Rightarrow x_1 + \lambda(x_1 - x_2)$ is a solution of $A\underset{\sim}{x} = \underset{\sim}{b}$ because

$$\begin{aligned} & A\{x_1 + \lambda(x_1 - x_2)\} \\ &= Ax_1 + A\lambda(x_1 - x_2) \\ &= \underset{\sim}{b} + 0 \\ &= \underset{\sim}{b} \end{aligned}$$

$\Rightarrow$ infinitely many solutions:– $x_1 + \lambda(x_1 - x_2)$ for varying λ – of $A\underset{\sim}{x} = \underset{\sim}{b}$ exists.

4) Find the value of c for which the following equations admit a solution: (4)

$$4x_1 + 6x_3 = 1$$
$$4x_3 - 2x_2 = 7 + c$$
$$2x_1 - x_2 + 5x_3 = 4$$

Ans:~ Given that the system admits a solution ~~for~~ for a value of c.

$\Rightarrow$ rank $(A \vdots b)$ = rank (A)

Here $A = \begin{pmatrix} 4 & 0 & 6 \\ 0 & -2 & 4 \\ 2 & -1 & 5 \end{pmatrix}$ and the method of sweep-out gives $\rightarrow$ $A \sim \begin{pmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

As such, rank$(A) = 2$

Again, $(A \vdots b) = \left(\begin{array}{ccc|c} 4 & 0 & 6 & 1 \\ 0 & -2 & 4 & 7+c \\ 2 & -1 & 5 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 3/2 & 1/4 \\ 2 & -1 & 5 & 4 \\ 0 & -2 & 4 & 7+c \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3/2 & 1/4 \\ 0 & -1 & 2 & 7/2 \\ 0 & -1 & 2 & \frac{7+c}{2} \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 3/2 & 1/4 \\ 0 & 1 & -2 & -7/2 \\ 0 & 0 & 0 & c/2 \end{array}\right)$

So, we know from the given assumption that rank $(A \vdots b) = r(A) = 2$

$\Rightarrow c/2 = 0$

i.e. $c = 0$.

So, for $c = 0$ the equations admit a solution.

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6) Investigate for what values of λ and μ, the system of equations

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

has i) no solution.

ii) a unique solution.

iii) an infinite no. of solution.

Soln →

$$(A \vdots \underset{\sim}{b}) = \left(\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right)$$

$$\sim \left(\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right) \quad R_3' = R_3 - R_2$$

i) No solution $\Rightarrow r(A) < r(A \vdots \underset{\sim}{b})$

$\Rightarrow \lambda = 3$ but $\mu \neq 10$.

$[r(A) = 2 < 3 = r(A \vdots b)]$

ii) Unique solution $\Rightarrow r(A) = 3$

$\Rightarrow \lambda \neq 3$.

iii) An infinite no. of solution

$\Rightarrow r(A) = r(A \vdots b) < 3$

$\Rightarrow \lambda = 3, \mu = 10$.

Theorem:- If A is a square matrix of order n, then A is row-equivalent to the identity matrix I_n if and only if the system of ~~linear~~ equations $A\underset{\sim}{x}=\underset{\sim}{0}$ has the only solution $x_1=0,\ x_2=0, \ldots,\ x_n=0$.

$$I_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A_{n\times n} \sim A_1 \sim \cdots\cdots \sim \text{Identity mtx.}$$

$$\vdots$$

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Proof:-

Only If Part:- Let $A_{n\times n}$ be row-equivalent to I_n. Then $A\underset{\sim}{x}=0$ and $I_n\underset{\sim}{x}=0$ have exactly the same solution(s).

As $I_n\underset{\sim}{x}=0$ has the only solution $x_1=0,\ x_2=0, \ldots, x_n=0$, then the system $A\underset{\sim}{x}=0$ has the only solution $x_1=0,\ x_2=0, \ldots, x_n=0$.

If Part:- Let $A\underset{\sim}{x}=\underset{\sim}{0}$ has the only solution $x_1=0, \ldots, x_n=0$.

Further, let R be an $n\times n$ row-reduced echelon matrix which is row equivalent to A and r be the number of non-zero rows of R. Now $R\underset{\sim}{x}=0$ has no non-zero solution implies $r \geq n$ [because any homogeneous system of m equation in n variables has a non-zero solution if $n>m$]. But since R has n rows, $r=n$ this means that R actually has a leading non-zero entry of 1 in each of the n rows, and as these 1's occur each in a different one of the n columns, R must be the identity matrix I_n of order $n\times n$.

- **What is elementary Congruent Operation?**

A pair of elementary operations — one row operation and the other column operation — is said to constitute an elementary congruent operation if the corresponding elementary matrices are such that one is transpose of the other.

It may be noted that each matrix obtained from a given matrix A by subjecting to a series of congruent elementary operation is congruent to A.

Problems:-

1. Show that the three equations
$$-2x+y+z=a$$
$$x-2y+z=b$$
$$x+y-2z=c$$

Have no solution, unless $a+b+c=0$.

Solution:-

$$[A:b] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 0 & 0 & 0 & a+b+c \end{array}\right]$$

If $a+b+c=0$, then $r(A) = r(A:b) = 2$.
Then the system of equation has infinitely many solutions.

If $a+b+c \neq 0$, then $r(A:b) = 3 \neq r(A) = 2$.
Then these 3 equations has no solution.

2. Use matrices to find the solution set of
$$2x+y+z=1$$
$$x-2y-3z=1$$
$$3x+2y+4z=5$$

Solution:-

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

$$\text{i.e. } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{9}\begin{pmatrix} -2 & -2 & -1 \\ -13 & 5 & 7 \\ 8 & -1 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$ $\therefore$ Solution set is $\{(1,-3,2)\}$.

3. For which value of μ, the following system of equations is inconsistent;
$3x+2y+z=10$, $2x+3y+2z=10$ $x+2y+\mu z=10$.

Solution:- The given system of equations is inconsistent if

$$\begin{vmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & \mu \end{vmatrix} = 0 \Rightarrow 5\mu - 7 = 0$$
$$\Rightarrow \mu = 7/5.$$

END

Quadratic Forms

Linear Form :— An expression of the type $\sum_{i=1}^{n} a_i x_i$, where a_i are constant coefficients and x_i are variables, is called a linear form w.r.t. the variables. It may be denoted by $L(x)$, being looked upon as a function of the vector variable $\underset{\sim}{x} = [x_1, x_2, \dots, x_n]'$. Putting $\underset{\sim}{a} = [a_1, a_2, \dots, a_n]'$. We may write $L(x) = \underset{\sim}{a}'\underset{\sim}{x}$.

Quadratic Form :— An expression of the type $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$, where a_{ij} are constant coefficients while x_i and x_j are variables, is called a quadratic form w.r.t. the variables. It is denoted by $Q(x)$, being looked upon as a function of the vector variable $\underset{\sim}{x}$. Putting $A = (a_{ij})$, we may write $Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x}$.

Def$^{\underline{n}}$.→ A quadratic form in $x_1, x_2, \dots, x_n$ is a second degree homogeneous function in n variables $x_1, x_2, \dots, x_n$, i.e.

$$Q(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{i \neq j} a_{ij} x_i x_j$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$$

Define, $\underset{\sim}{x} = (x_1, x_2, \dots, x_n)'$, and

$$A = ((a_{ij}))_{n \times n}$$

$$\therefore Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j = a_{11} x_1 x_1 + a_{12} x_1 x_2 + \dots + a_{1n} x_1 x_n$$
$$+ a_{21} x_2 x_1 + \dots + a_{2n} x_2 x_n + \dots\dots$$
$$+ a_{n1} x_n x_1 + \dots + a_{nn} x_n x_n$$

Ex. 1. $n = 2$,

$$Q(x, y) = ax^2 + bxy + cxy + dy^2$$
$$= (x \;\; y) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

For a quadratic form $\hat{x}' A \hat{x}$, A is said to be a matrix of the quadratic form. If the matrix of a quadratic form be not symmetric then it can reduced to a symmetric matrix.

$n = 2$,

$$Q(x, y) = ax^2 + bxy + cxy + dy^2$$
$$= ax^2 + \left(\frac{b+c}{2}\right) xy + \left(\frac{c+b}{2}\right) xy + dy^2$$
$$= (x \;\; y) \begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Note :→ The square matrix A in a quadratic form [redacted] $Q(x)$ may, without any loss of generality, be supposed to be a symmetric matrix. For, in case A is not symmetric, we may take another matrix $B = (b_{ij})$ such that

$$b_{ij} = \frac{a_{ij} + a_{ji}}{2} \text{ for all } i, j.$$

which implies that $b_{ii} = a_{ii}$ for all i.

then,

$$b_{ij} = b_{ji} \text{ for all } ij, \text{ so that}$$

$$B = B', \text{ i.e. } B \text{ is a symmetric matrix.}$$

Now,

$$x'Bx = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} x_i x_j$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{a_{ij}+a_{ji}}{2}\right) x_i x_j$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji} x_i x_j\right]$$

$$= \frac{1}{2}\left[x'Ax + x'A'x\right]$$

$$= x'Ax.$$

$(x'Ax)' = x'A'x$
But $x'Ax$ is a scalar,
so → $(x'Ax)' = x'Ax$
$\therefore$ $x'Ax = x'A'x$.

In our discussion, then, we shall always assume A [redacted] is symmetric. We shall also assume that each element of $\underset{\sim}{x}$ can take only real values.

The determinant $|A|$ is said to be the <u>discriminant</u> of the quadratic form $\hat{x}'A\hat{x}$.

Ex.1. $x^2 - 2x_2^2 + 3x_3^2 + 4x_1x_2 - 6x_1x_3 + 5x_2x_3$ is a q.f. with

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & -2 & 5/2 \\ -3 & 5/2 & 3 \end{pmatrix}$$

Ex.2. [redacted] $x_1^2 + 4x_2^2 + 9x_3^2 + 12x_1x_2 - 16x_2x_3 + 8x_1x_3$ is a q.f. with

$$A = \begin{pmatrix} 1 & 6 & 4 \\ 6 & 4 & -8 \\ 4 & -8 & 9 \end{pmatrix}$$

Classification of Quadratic forms / Classification of the matrix of the Quadratic forms :—

Every real quadratic form $Q(\underset{\sim}{x})$ can be put into one of the following broad categories, depending on the range of values that it may assume, i.e. depending on the nature of A:

i) Non-negative definite Quadratic form :— If $Q(\underset{\sim}{x}) \geq 0 \ \forall \ x \in E^n$ then it is said to be n.n.d. quadratic form.

i.e. $Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x}$ is said to be an n.n.d. quadratic form if

$$\underset{\sim}{x}' A \underset{\sim}{x} \geq 0 \ \forall \ \underset{\sim}{x}$$

A is then said to be Non-negative definite matrix.

An n.n.d. matrix will be either positive definite or positive-semi definite matrix.

An n.n.d. matrix is also called a matrix which is atleast p.s.d.q.f.

ii) **Positive definite Quadratic form :— A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x}$ is said to be a p.d. quadratic form if

$$\underset{\sim}{x}' A \underset{\sim}{x} > 0 \ \forall \ \underset{\sim}{x} \neq \underset{\sim}{0}$$
$$= 0 \text{ iff } \underset{\sim}{x} = \underset{\sim}{0}$$

$Q(\underset{\sim}{x})$ is a p.d. quadratic form means A is a p.d. matrix.

iii) **Positive semi-definite Quadratic form :— A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x}$ is said to be a p.s.d. quadratic form if

$$\underset{\sim}{x}' A \underset{\sim}{x} \geq 0 \ \forall \ \underset{\sim}{x}$$
$$= 0 \text{ for atleast one } \underset{\sim}{x} \neq \underset{\sim}{0}.$$

$Q(\underset{\sim}{x})$ is a p.s.d. quadratic form means A is a p.s.d. matrix.

Moreover, An n.n.d. quadratic form which is not p.d. is said to be a positive semi-definite (or p.s.d) quadratic form.

iv) Non-positive Definite Quadratic form :— A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x}$ is said to be non-positive definite quadratic form if

$$\underset{\sim}{x}' A \underset{\sim}{x} \leq 0 \ \forall \ \underset{\sim}{x}.$$

i.e. if $Q(\underset{\sim}{x}) \leq 0 \ \forall \ x \in E^n$, then it is said to be a n.p.d. q.f.

- A n.p.d. $\Longleftrightarrow$ $-A$ n.n.d.

A non-positive definite quadratic form (at least negative semi definite) is either negative definite or negative semi definite quadratic form.

v) ** **Negative definite Quadratic form :** A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x}$ is said to be a negative definite (or n.d.) quadratic form if

$$\underset{\sim}{x}'A\underset{\sim}{x} < 0 \quad \forall \underset{\sim}{x} \neq \underset{\sim}{0}$$
$$= 0 \text{ iff } \underset{\sim}{x} = \underset{\sim}{0}$$

- A : nd $\Leftrightarrow$ $-A$: pd matrix.

vi) **Negative Semi-definite Quadratic Form :** A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x}$ is said to be an n.s.d. quadratic form if

$$\underset{\sim}{x}'A\underset{\sim}{x} \leq 0 \quad \forall \underset{\sim}{x}$$
$$= 0 \text{ for at least one } \underset{\sim}{x} \neq 0.$$

- A : nsd $\Leftrightarrow$ A : psd

An n.p.d. quadratic form which is not n.d. is said to be negative semi-definite (or n.s.d.).

vii) **Indefinite Quadratic form :** A quadratic form $Q(\underset{\sim}{x}) \underset{\sim}{x}'A\underset{\sim}{x}$ is said to be indefinite if

$$Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x} > 0 \text{ for some } \underset{\sim}{x}$$
$$< 0 \text{ for some } \underset{\sim}{x}$$

A q.f. which is neither p.s.d. nor n.s.d. is called indefinite quadratic form.

Ex. 1.

$$Q(x) = 5x_1^2 - 5x_1x_2 + 4x_2^2$$
$$= 5\left(x_1^2 - x_1x_2 + \frac{x_2^2}{4}\right) + \frac{11}{4}x_2^2$$
$$= 5\left(x_1 - \frac{x_2}{2}\right)^2 + \frac{11}{4}x_2^2$$

$\therefore Q(x) > 0 \quad \forall\ x_1, x_2 \neq 0$

$Q(x) = 0 \Rightarrow \left(x_1 - \frac{x_2}{2}\right) = 0$ and $x_2 = 0$

i.e. iff $x_1 = x_2 = 0$.

$\therefore Q(x)$ is a positive definite quadratic form.

Ex. 2. $Q(x) = 3x_1^2 - 6x_1x_2 + 3x_2^2$

$= 3(x_1 - x_2)^2$

$\therefore Q(x) \geq 0 \quad \forall x_1, x_2$

Here, $Q(x) = 0 \Rightarrow x_1 = x_2$

$Q(x) = 0$ for at least one $\underset{\sim}{x} \neq 0$.

$\therefore$ It is positive semidefinite quadratic form.

Ex. 3. $Q(x) = 4x_1^2 - 7x_1x_2 - 2x_2^2$

Here $Q(x)$ may assume values that may be positive, negative or zero. Thus, for instance, in case $x_1 = 1, x_2 = 0$,

$Q(x) = 4$;

when $x_1 = 0, x_2 = 1$,

$Q(x) = -2$;

and when $x_1 = 2, x_2 = 1$ or $x_1 = x_2 = 0$,

$Q(x) = 0$.

As such, we now have an indefinite quadratic form.

Theorem :→ If $x'Ax$ is positive definite (p.d.), then $x'(-A)x$ is negative definite (n.d.). Conversely, if $x'Ax$ is negative definite (n.d.) then $x'(-A)x$ is positive definite (p.d.).

Proof:

We have, $x'(-A)x = \sum_i \sum_j (-a_{ij}) x_i x_j$

$= -\sum_i \sum_j a_{ij} x_i x_j$

$= -x'Ax$. ——— (1)

Case-I → Now, let $x'Ax$ is p.d. q.f.,

then for $x \neq 0$, we have $x'Ax > 0$,

$\Rightarrow -x'Ax < 0$ [Applying (1)]

$\Rightarrow x'(-A)x < 0$.

if $x = 0$, we have $x'Ax = 0$

$\Rightarrow x'(-A)x = 0$

Hence, $x'(-A)x$ must be n.d. q.f.

Case-II → Again, let $x'Ax$ is n.d. q.f.,

then for $x \neq 0$, we have $x'Ax < 0$.

$\Rightarrow x'(-A)x < 0$.

for $x = 0$, we have $x'Ax = 0$

$\Rightarrow x'(-A)x = 0$

also hence, $x'(-A)x$ must be p.d. q.f.

Congruence of matrix :→ A square matrix B of order n over a field F is said to be congruent to another square matrix A of the same order over the same field F if there exists a non-singular matrix P over F such that $B = P'AP$.

Result :→ The relation of 'congruence of matrices' is an equivalence relation in the set of all $n \times n$ matrices over a field F.

* **Result :→** Every matrix congruent to a symmetric matrix is also a symmetric matrix.

Proof :→ Let A be a symmetric matrix, so $A = A^T$.

Suppose B is congruent to A, then $B = P'AP$.

$$\therefore B' = (P'AP)'$$
$$= P'A'P$$
$$= P'AP \quad [\because A \text{ is symmetric}]$$
$$= B$$

$\therefore B'$ is also congruent and B is a symmetric matrix.

Congruence of Quadratic form or Equivalance of Quadratic Forms :-

Defn :- Two quadratic forms $x'Ax$ and $y'By$ over a field F are said to be congruent or equivalent over F if these respective matrices A and B are congruent over F. Thus $x'Ax$ is congruent to $y'By$ if $\exists$ a matrix $P \ni B = P'AP$. Since congruence of matrices is an equivalence relation, therefore, congruence of quadratic form is also an equivalence relation.

The linear transformation of a quadratic form :- Consider a quadratic form $x'Ax$ and a non-singular linear tramformation $x = Py$ so that P is a non-singular matrix.

Putting $x = Py$, we get →
$$x'Ax = (Py)'A(Py)$$
$$= y'P'APy$$
$$= y'(P'AP)y = y'By \quad \left[\begin{array}{l}\text{where} \\ P'AP = B\end{array}\right]$$

Since B is congruent to a symmetric matrix A, So, B is also symmetric. Thus, $y'By$ is a quadratic form. It is called a linear tramformation of the form $x'Ax$ by the non-singular matrix P. The matrix of the quadratic form $y'By$ is $B = P'AP$. Thus, the quadratic form $y'By$ is congruent to $x'Ax$.

Result :→ The ranges of values of two congruent quadratic form are the same.

Congruent reduction of a symmetric matrix :-

Theorem :- If A be any n rowed non-zero symmetric matrix of rank r over a field F then ∃ an n-rowed non-singular matrix P over F such that $P'AP = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$ where A_1 is a non-singular diagonal matrix of order r over F and O is the null matrix of suitable order. [For Practical]

Proof :→ We shall proof the theorem by induction on n, the order of the given matrix. If n = 1, the theorem is obviously true. Let us suppose that the theorem is true for all symmetric matrices of order (n−1). Then we have to show that it is also true for an n×n symmetric matrix A.

Let $A = (a_{ij})_{n\times n}$ be a symmetric matrix of rank r over a field F. First we shall show that ∃ a matrix $B = (b_{ij})_{n\times n}$ over F congruent to A such that $b_{11} \neq 0$.

Case I → If $a_{11} \neq 0$, then we take B = A.

Case II → If $a_{11} = 0$, but some diagonal element of A, say, $a_{ii} \neq 0$, then applying the congruent operation $R_i \leftrightarrow R_1$, $C_i \leftrightarrow C_1$ to A, we obtain a matrix B congruent to A such that $b_{11} = a_{ii} \neq 0$.

Case III → Suppose that each diagonal element of A is 0. Since A is a non-zero matrix, let a_{ij} be a non-zero element of A. Then $a_{ij} = a_{ji} \neq 0$. Applying the congruent operation $R_i \leftrightarrow R_i + R_j$, $C_i \leftrightarrow C_i + C_j$ to A, we obtain a matrix $D = [d_{ij}]$ congruent to A ∋ $d_{ii} = a_{ij} + a_{ji} = 2a_{ij} \neq 0$.

Now applying the congruent operation $R_i \leftrightarrow R_1$, $C_i \leftrightarrow C_1$ to D, We obtain a matrix $B = (b_{ij})_{n\times n}$ congruent to D and therefore, also congruent to A ∋ $b_{11} = d_{ii} \neq 0$. Thus there always exists a matrix $B = [b_{ij}]_{n\times n}$ congruent to a symmetric matrix ∋ the leading element of B is not zero. Since B is congruent to a symmetric matrix, therefore B itself is a symmetric matrix. Since $b_{11} \neq 0$, therefore all elements in the first row and first column of B, except the leading element, can be made zero by suitable congruent operations. We thus have a matrix $C = \begin{bmatrix} a_{11} & 0 \cdots 0 \\ 0 & \\ \vdots & B_1 \\ 0 & \end{bmatrix}$ congruent to B and therefore, also congruent to A $\ni$ B_1 is a square matrix of order $(n-1)$, since C is congruent to a symmetric matrix A, therefore C is also a symmetric matrix and consequently B_1 is also a symmetric matrix of order $(n-1)$. Therefore by our induction hypothesis, it can be reduced to a diagonal matrix by congruent operations. If the congruent operations applied to B_1 for this purpose applied to C, they will not affect row one and column one of C. So C can be reduced to a diagonal matrix by congruent operations. Thus A is congruent to a diagonal matrix, say, $\text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_k, 0, 0, \ldots, 0]$. Thus $\exists$ a non-singular matrix $P \ni P'AP = \text{diag}[\lambda_1, \ldots, \lambda_k, 0, \ldots, 0]$.

Since rank$(A) = r$, and the rank of a matrix does not change on multiplication by a non-singular matrix, Therefore, the rank of $\text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots, 0]$ is also r. So, precisely r elements of $\text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots, 0]$ are non-zero. $\therefore k = r$ and thus $P'AP = \text{diag}[\lambda_1, \ldots, \lambda_r, 0, \ldots, 0]$

Thus A can be reduced to diagonal form by Congruent operations.

Theorem: → If A be any n-rowed non-zero symmetric matrix of rank r over the field of real numbers, then $\exists$ a real non-singular matrix $P \ni P'AP = \{1, \ldots, 1, -1, \ldots, -1, 0, \ldots, 0\}$ where 1 appears $s\ (= 0, 1, \ldots, r)$ times and -1 appears $(r-s)$ times.

Proof :→ As A is a non-zero symmetric matrix of rank r over the field of real numbers. There exists a non-singular matrix Q over the real field $\ni Q'AQ$ is a diagonal matrix with first r diagonal entries nonzero and remaining $(n-r)$ entries zero, i.e. $Q'AQ = \text{diag}\{\lambda_1, \dots, \lambda_r, 0, \dots, 0\}$, now let s of the λ_j's are positive and remaining $(r-s)$ λ_j's are negative. Performing suitable elementary congruent operations on $Q'AQ$ of the type ; interchanging ith and jth row and ith and j-th column we get a diagonal matrix whose 1st s diagonal entries are positive and remaining $(r-s)$ entries are negative. Without loss of generality let $\lambda_1, \dots, \lambda_s$ are positive and $\lambda_{s+1}, \lambda_{s+2}, \dots, \lambda_r$ are negative then $\exists$ real numbers $\beta_1, \beta_2, \dots, \beta_r \ni$ $\lambda_1 = \beta_1^2, \lambda_2 = \beta_2^2, \dots, \lambda_s = \beta_s^2, \beta_{s+1}^2 = -\lambda_{s+1}, \dots, \beta_r^2 = -\lambda_r$

Let $R = \text{diag}\{1/\beta_1, 1/\beta_2, \dots, 1/\beta_r, 1, \dots, 1\}$ and

$$P = QR$$

$$\Rightarrow P'AP = R'Q'AQR$$

$$= R'(Q'AQ)R$$

$$= \text{diag}\{1/\beta_1, \dots 1/\beta_r, 1, \dots, 1\}\,\text{diag}\{\lambda_1, \dots, \lambda_r, 0, \dots, 0\}\,\text{diag}\{1/\beta_1, \dots, 1/\beta_r, 1, \dots, 1\}$$

$$= \text{diag}\{\underbrace{1, 1, \dots, 1}_{s\text{ times}}, \underbrace{-1, -1, \dots, -1}_{(r-s)\text{ times}}, \underbrace{0, 0, \dots, 0}_{(n-r)\text{ times}}\}$$

$$R'(Q'AQ)R = \begin{pmatrix} 1/\beta_1 & & & & 0 \\ & \ddots & & & \\ & & 1/\beta_r & & \\ & & & 1 & \\ 0 & & & & \ddots \ 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_r & & \\ & & & 0 & \\ 0 & & & & \ddots \ 0 \end{pmatrix} \begin{pmatrix} 1/\beta_1 & & & & 0 \\ & \ddots & & & \\ & & 1/\beta_r & & \\ & & & 1 & \\ 0 & & & & \ddots \ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1/\beta_1^2 & & & & 0 \\ & \ddots & & & \\ & & \lambda_r/\beta_r^2 & & \\ & & & 0 & \\ 0 & & & & \ddots \ 0 \end{pmatrix}$$

$[\beta_1^2 = \lambda_1, \dots, \beta_s^2 = \lambda_s,$
$\beta_{s+1}^2 = -\lambda_{s+1}, \dots,$
$\beta_r^2 = -\lambda_r]$

$$= \begin{pmatrix} \underbrace{1 \ 1 \ \dots \ 1}_{s\text{ times}} & & 0 \\ & \underbrace{-1 \ \dots \ -1}_{r-s\text{ times}} & \\ 0 & & \underbrace{0 \ \dots \ 0}_{n-r\text{ times}} \end{pmatrix} = \text{diag}\{1, \dots, 1, -1, \dots, -1, 0, \dots, 0\}$$

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SOME ELEMENTARY RESULTS :→

- **Result 1.** A positive definite (p.d.) quadratic form remains positive definite under a non-singular linear transformation.

Proof:→ Let $Q(x) = x'Ax$ is a p.d. quadratic form and $y = Bx$ be a ~~[illegible]~~ non-singular linear transformation, under this non-singular linear transformation $x = B^{-1}y$. Therefore,

$$
\begin{aligned}
Q(x) &= (B^{-1}y)'A(B^{-1}y) \\
&= y'[(B^{-1})'AB^{-1}]y \\
&= y'Cy \quad [\text{where } C = (B^{-1})'AB^{-1}] \\
&= Q^*(y), \text{ a quadratic form in } y_i\text{'s } (i = 1(1)n) \qquad \text{--- (1)}
\end{aligned}
$$

From (1) we see that — the range value of $Q^*(y)$ is the same as the range value $Q(x)$.

$$\therefore\ Q^*(y) \geq 0 \quad \forall\ y \quad [\text{since } Q(x) \text{ is p.d.}]$$

Now $y \neq 0 \Rightarrow x = B^{-1}y \neq 0$

Let $B^{-1} = (c_1, c_2, \dots, c_n)$, since B^{-1} is non-singular. So, $c_1, c_2, \dots, c_n$ are linearly independent.

So, $B^{-1}y = \underset{\sim}{c}_1 y_1 + \underset{\sim}{c}_2 y_2 + \dots + \underset{\sim}{c}_n y_n = 0$ iff $y = \underset{\sim}{0}$.

$\therefore\ Q^*(y) = 0$ if $\underset{\sim}{y} = \underset{\sim}{0}$ and $Q^*(y) > 0$ if $\underset{\sim}{y} \neq 0$.

$\therefore\ Q^*(y)$ is positive definite (p.d.).

- **Result 2.** If $x'Ax$ is a real quadratic form of rank r, i.e. the rank of the ~~matrix~~ symmetric matrix A is r, then $\exists$ a non-singular linear transformation $x = Py$ which transforms $x'Ax$ to $y'(P'AP)y = y_1^2 + y_2^2 + \cdots + y_s^2 - y_{s+1}^2 - y_{s+2}^2 - \cdots - y_r^2$.

Proof: $\rightarrow$

We know that for an n-rowed non zero matrix A of rank r $\exists$ a non-singular matrix P $\ni$

$$P'AP = \operatorname{diag}\{\underbrace{1, 1, \ldots, 1}_{s\text{ times}}, \underbrace{-1, -1, \ldots, -1}_{r-s\text{ times}}, \underbrace{0, 0, \ldots, 0}_{n-r\text{ times}}\}$$

Let us consider now the non-singular transformation

$$x = Py$$

$$\therefore x'Ax = (Py)'A(Py)$$

$$= y'P'AP\,y$$

$$= y'(P'AP)y$$

$$= y'\left[\operatorname{diag}\{\underbrace{1, 1, \ldots, 1}_{s\text{ times}}, \underbrace{-1, \ldots, -1}_{r-s\text{ times}}, \underbrace{0, \ldots, 0}_{n-r\text{ times}}\}\right] y$$

$$= y_1^2 + y_2^2 + \cdots + y_s^2 - y_{s+1}^2 - y_{s+2}^2 - \cdots - y_r^2.$$

[C.U.]

Result 3. If $x'Ax$ is a real p.d. quadratic form in n variables $x_1, x_2, \dots, x_n$ then it can be reduced by a non-singular linear transformation to the form

$$Q^*(y) = y'y = \sum_{i=1}^{n} y_i^2$$

Proof:→ Since $x'Ax$ is a real p.d. q.f. of rank r, then $\exists$ a non-singular matrix P which transform $x'Ax$ to the form

$$Q^*(y) = y_1^2 + \dots + y_s^2 - y_{s+1}^2 - \dots - y_r^2$$

Now, since $Q(x) > 0$ if $x \neq 0$
$= 0$ if $x = 0$,

then $Q^*(y) \geq 0 \ \forall\, y$

Now in order that $Q^*(y)$ to be p.d., there must be no-zero coefficient in the expression of $Q^*(y)$ since when $y_1 = 0, \dots, y_r = 0, y_{r+1} \neq 0, \dots, y_n \neq 0$.

i.e. $y \neq 0$ then also $Q^*(y) = 0$, i.e. the condition for a p.d. q.f. is violated. Hence $r = n$.

$$\therefore Q^*(y) = y_1^2 + \dots + y_s^2 - y_{s+1}^2 - \dots - y_n^2$$

Now if $(y_1^2 + \dots + y_s)^2 < (y_{s+1}^2 + \dots + y_n^2)$

Then $Q^*(y) < 0$, again the condition of a p.d.q.f. is violated. So, to satisfy the condition no negative co-efficient will be there. Hence $s = n$.

$$\therefore Q^*(y) = y'y = \sum_{i=1}^{n} y_i^2.$$

[C.U.]

Result 4. The determinant of a p.d. q.f. is positive.

Proof:→ Let $x'Ax$ be a real p.d.q.f.. Since $x'Ax$ is p.d., then $\exists$ a non-singular transformation $y = Px \ni$

$$P'AP = I$$

$\therefore$ Taking determinant, $|P'||A||P| = 1$

or, $|A||P|^2 = 1 \quad [\because |P| = |P'|]$

or, $|A| = \dfrac{1}{|P|^2} > 0$

$\therefore$ The determinant of a real p.d.q.f. is positive.

[C.U.]

Result 4.* Suppose A is a p.d. matrix then show that
(a) all diagonal elements of A are positive.
(b) all principal submatrices are p.d.
(c) A^{-1} is a positive-definite matrix.

Proof:→

(a) A : p.d. , $\Rightarrow \underset{\sim}{x}' A \underset{\sim}{x} > 0 \quad \forall \underset{\sim}{x} \neq \underset{\sim}{0}$

$A = ((a_{ij}))_{n\times n}$ and $\underset{\sim}{x} = (x_1, x_2, \dots, x_n)'$.

Choose $\underset{\sim}{x} = \underset{\sim}{e}_i = (0, 0, \dots, 0, \underset{i\text{th component}}{1}, 0, \dots, 0, 0)$

$$\therefore \underset{\sim}{x}' A \underset{\sim}{x} \Big|_{\underset{\sim}{x} = \underset{\sim}{e}_i} = a_{ii} > 0$$

(b) A : p.d.

$\Rightarrow \underset{\sim}{x}' A \underset{\sim}{x} > 0 \ \forall \underset{\sim}{x} \neq \underset{\sim}{0}$.

Let $A = ((a_{ij}))_{n\times n}$

Partition $\underset{\sim}{x}$ and A as $\underset{\sim}{x} = \begin{pmatrix} \underset{\sim}{x}_{(1)m\times 1} \\ \underset{\sim}{x}_{(2)} \end{pmatrix}_{n\times 1}$

$$A = \begin{pmatrix} A_{11\, m\times m} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}_{n\times n}$$

Note that, $\underset{\sim}{x}' A \underset{\sim}{x}$

$$= \begin{pmatrix} \underset{\sim}{x}'_{(1)} & \underset{\sim}{x}'_{(2)} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \underset{\sim}{x}_{(1)} \\ \underset{\sim}{x}_{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} \underset{\sim}{x}'_{(1)} A_{11} + \underset{\sim}{x}'_{(2)} A_{21} & \underset{\sim}{x}'_{(1)} A_{12} + \underset{\sim}{x}'_{(2)} A_{22} \end{pmatrix} \begin{pmatrix} \underset{\sim}{x}_{(1)} \\ \underset{\sim}{x}_{(2)} \end{pmatrix}$$

$$= \underset{\sim}{x}'_{(1)} A_{11} \underset{\sim}{x}_{(1)} + \underset{\sim}{x}'_{(2)} A_{21} \underset{\sim}{x}_{(1)} + \underset{\sim}{x}'_{(1)} A_{12} \underset{\sim}{x}_{(2)} + \underset{\sim}{x}'_{(2)} A_{22} \underset{\sim}{x}_{(2)}$$

Choose, $\underset{\sim}{x} \neq 0$, $\underset{\sim}{x}_{(1)} \neq \underset{\sim}{0}$ and $\underset{\sim}{x}_{(2)} = 0$.

$\underset{\sim}{x}' A \underset{\sim}{x} > 0 \quad \forall \underset{\sim}{x} \neq \underset{\sim}{0}$.

$\Rightarrow \underset{\sim}{x}' A \underset{\sim}{x} \Big|_{\underset{\sim}{x}_{(2)} = \underset{\sim}{0}} = \underset{\sim}{x}'_{(1)} A_{11} \underset{\sim}{x}_{(1)} > 0 \quad \forall \underset{\sim}{x}_{(1)} \neq \underset{\sim}{0}$.

$\Rightarrow A_{11}$ is p.d. of order $m\times m$, where A_{11} is principal submatrix of A. Hence the fact holds good for any 'm'.

(c) A be a p.d. matrix.

Then $\exists$ a non-singular matrix P show that
$$P'AP = I_n$$
$$\Rightarrow \quad (P'AP)^{-1} = I_n$$
or, $P^{-1}A^{-1}(P')^{-1} = I_n$

i.e. $P^{-1}A^{-1}(P^{-1})' = I_n$.

$\therefore \quad Q'A^{-1}Q = I_n$

[Taking $(P^{-1})' = Q$, i.e. $P^{-1} = Q'$, since P is non-singular, Q is also non-singular]

So, A^{-1} is also congruent to I_n,

$\Rightarrow A^{-1}$ is p.d., i.e. $x'A^{-1}x$ is p.d.

● <u>Result 5.</u> Suppose A be a p.d. matrix and P be a non-singular matrix, then show that $P'AP$ or PAP' will also be p.d. matrix.

<u>Proof :-</u> A be a p.d. matrix $\Rightarrow \underset{\sim}{x}'A\underset{\sim}{x} > 0 \;\forall\; \underset{\sim}{x} \neq \underset{\sim}{0}$.

Then $\exists$ a non-singular transformation $\underset{\sim}{x} = P\underset{\sim}{y}$,

As P is non-singular matrix
$$\underset{\sim}{x}'A\underset{\sim}{x} > 0$$
$$\Rightarrow \underset{\sim}{y}'(P'AP)\underset{\sim}{y} > 0$$

In order to show $P'AP$ is p.d. matrix it is enough to verify $\underset{\sim}{x} \neq \underset{\sim}{0}$
$$\Rightarrow \underset{\sim}{y} \neq \underset{\sim}{0}$$

Rewrite P and $\underset{\sim}{y}$ as —

$P = (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n)$ and $\underset{\sim}{y} = (y_1, y_2, \ldots, y_n)$

Then $P\underset{\sim}{y} = \sum_{i=1} y_i \underset{\sim}{\alpha}_i$

$\therefore \sum y_i \underset{\sim}{\alpha}_i = \underset{\sim}{x}$

As P is non-singular matrix, then columns of P must be linearly independent i.e. α_i's are LIN vectors. Hence in order to get a non-null $\underset{\sim}{x}$ at least one y_i must be non-zero. i.e. $\underset{\sim}{x} \neq 0$

$\Rightarrow \underset{\sim}{y} \neq \underset{\sim}{0}$ $\therefore \underset{\sim}{y}'P'AP\underset{\sim}{y} > 0 \;\forall\; \underset{\sim}{y} \neq 0$

$\Rightarrow P'AP$ is p.d.

• **Result 6.** Suppose P be a non singular matrix, show that $P'P$ or PP' will be a p.d. matrix.

Proof: → Let $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)' \neq \underset{\sim}{0}$

i.e. $x_i \neq 0$ for at least one i.

$\therefore \underset{\sim}{x}'\underset{\sim}{x} = \Sigma x_i^2 > 0$ for at least one $x_i \neq 0$.

Consider the non-singular transformation $\underset{\sim}{x} = P\underset{\sim}{y}$, where P is a non-singular matrix.

$$\therefore \underset{\sim}{x}'\underset{\sim}{x} > 0$$

$$\Rightarrow \underset{\sim}{y}'P'P\underset{\sim}{y} > 0$$

In order to show $P'P$ is a p.d. matrix it is enough to verify that

$$\underset{\sim}{x} \neq \underset{\sim}{0} \Rightarrow \underset{\sim}{y} \neq \underset{\sim}{0}.$$

Rewrite P as $\quad P = (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n)$

then $P\underset{\sim}{y} = \sum_{i=1}^{n} y_i \underset{\sim}{\alpha}_i$ [where $\underset{\sim}{y} = (y_1, \ldots, y_n)'$]

$$\therefore \Sigma y_i \underset{\sim}{\alpha}_i = \underset{\sim}{x}$$

As P is a non-singular matrix, then columns of P must be LIN, i.e. $\underset{\sim}{\alpha}_i$'s are LIN vectors. Hence in order to get a non-null $\underset{\sim}{x}$ at least one y_i must be non-zero.

i.e. $\underset{\sim}{x} \neq 0$,

$\Rightarrow \underset{\sim}{y} \neq 0$.

$\therefore \underset{\sim}{y}'P'P\underset{\sim}{y} > 0 \;\forall\; \underset{\sim}{y} \neq 0$.

$\Rightarrow P'P$ is p.d.

- **Result 7.** (Lagrange's method / Method of Completing the square)

If $A_{n\times n}$ be a p.d. matrix then there exists an n.s. matrix $P \ni P'AP$ or $PAP' = \Lambda$, Λ is a diagonal with positive elements.

Proof :→ (Practical)

$$A = ((a_{ij}))_{n\times n}$$

Partition A as $\begin{bmatrix} a_{11} & \underset{\sim}{a}_{(1)}' \\ \underset{\sim}{a}_{(1)} & A_{22} \end{bmatrix}$

Now A is a p.d. matrix $\Rightarrow a_{11} > 0$, hence through row and column operation A can be reduced to

$$P_1AP_1' = \begin{bmatrix} a_{11} & \underset{\sim}{0}' \\ \underset{\sim}{0} & A_{22}^{(1)} \end{bmatrix}, \text{ where } A_{22}^{(1)} = ((a_{ij}^{(1)})), i,j = 2(1)n.$$

thus there exists a non-singular matrix $P_1 \ni$

$$P_1AP_1' = \begin{bmatrix} a_{11} & \underset{\sim}{0}' \\ \underset{\sim}{0} & A_{22}^{(1)} \end{bmatrix}$$

Here
$$P_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\frac{a_{21}}{a_{11}} & 0 & 0 & \cdots & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -\frac{a_{n1}}{a_{11}} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Partition P_1AP_1' as
$$= \begin{pmatrix} a_{11} & 0 & \underset{\sim}{0}' \\ 0 & a_{22}^{(1)} & \underset{\sim}{a}_{(2)}^{(1)\prime} \\ \underset{\sim}{0} & \underset{\sim}{a}_{(2)}^{(1)} & A_{33}^{(1)} \end{pmatrix}$$

as A is a p.d. matrix and P_1 be an n.s. matrix, so P_1AP_1' must be p.d. matrix.

Hence $a_{22}^{(1)} > 0$

therefore there exists an n.s. matrix P_2 such that

$$P_2P_1AP_1'P_2' = \begin{pmatrix} a_{11} & 0 & \underset{\sim}{0}' \\ 0 & a_{22}^{(1)} & \underset{\sim}{0}' \\ \underset{\sim}{0} & \underset{\sim}{0} & A_{33}^{(2)} \end{pmatrix}$$

~~as P_1AP_1' is a p.d. matrix~~

$$A_{33}^{(2)} = ((a_{ij}^{(2)})) \quad , i,j = 3(1)n$$

where, $P_2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -\frac{a_{n2}^{(1)}}{a_{22}^{(1)}} & 0 & \cdots & 1 \end{bmatrix}$

as P_1AP_1' is a p.d. matrix and P_2 is an n.s. matrix, so $P_2P_1AP_1'P_2'$ must be a p.d. matrix, hence $a_{33}^{(2)} > 0$.

Proceeding in this way it can be shown that

$$P_{n-1}P_{n-2}\cdots P_2P_1AP_1'P_2'\cdots P_{n-1}'$$
$$= \text{diag}\left(a_{11}, a_{22}^{(1)}, a_{33}^{(2)}, \ldots, a_{nn}^{(n-1)}\right)$$

, a matrix with +ve elements.

where P_i's are n.s. matrices.

Define $P = P_{n-1}P_{n-2}\cdots P_2P_1$.

Clearly P is an n.s. matrix as a product of n.s. matrices.

$$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

where, $\lambda_i = a_{ii}^{(i-1)}$, $i = 1, 2, \ldots, n$.

$\lambda_i > 0 \ \forall \ i$.

therefore $PAP' = \Lambda$, a diagonal matrix with positive elements.

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- **Result 8.** If $A_{n\times n}$ be a p.s.d. matrix with rank$(A) = r < n$ then $\exists$ an n.s. matrix $P \ni PAP' = \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0)$ when $\lambda_i > 0$, $i = 1, 2, \dots, r$.

Proof:→
Since rank$(A_{n\times n}) = r < n$, $\exists$ an n.s. matrix $P \ni$

$$P'AP = \begin{bmatrix} \Lambda_{r\times r} & 0_{r\times n-r} \\ 0 & 0 \end{bmatrix}_{n\times n}$$

, where Λ is a diagonal matrix with non-null elements.

now, A is p.s.d. matrix.

$\Rightarrow P'AP$ is also p.s.d. matrix as P is n.s.

Let $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_r)$

Note that, $\underset{\sim}{y}' P'AP \underset{\sim}{y}$

$$= \begin{pmatrix} \underset{\sim}{y}'_{(1)} & \underset{\sim}{y}'_{(2)} \end{pmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \underset{\sim}{y}_{(1)} \\ \underset{\sim}{y}_{(2)} \end{pmatrix}$$

, where $\underset{\sim}{y} = \begin{pmatrix} \underset{\sim}{y}_{(1)\, r\times 1} \\ \underset{\sim}{y}_{(2)} \end{pmatrix}_{n\times 1}$

$$= \underset{\sim}{y}'_{(1)} \Lambda \underset{\sim}{y}_{(1)}$$

$$= \sum_{i=1}^{r} \lambda_i y_i^2$$

, where $\underset{\sim}{y}_{(1)} = (y_1, y_2, \dots, y_r)'$

since $\lambda_i \neq 0$, $i = 1(1)r$.

and $\sum_{i=1}^{r} \lambda_i y_i^2 \geq 0$ as $P'AP$ is p.s.d., we have $\lambda_i > 0$, $i = 1(1)r$.

C.U

- **Result 9.** If A be a p.d. matrix then it can be written as $A = P'P$ or PP' where P is non-singular matrix.

Proof:→ As A be a p.d. matrix, $\exists$ an n.s. matrix $Q \ni$

$$QAQ' = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n),\ \lambda_i > 0\ \forall i.$$

$$= \Lambda^{1/2} \Lambda^{1/2}$$

[assume A is of order $n\times n$]

, when $\Lambda^{1/2} = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$

$$\Rightarrow A = Q^{-1} \Lambda^{1/2} \Lambda^{1/2} (Q')^{-1}$$

[Q^{-1} exists and is n.s. as Q is n.s.]

$$= Q^{-1} \Lambda^{1/2} \Lambda^{1/2} (Q^{-1})'$$

[Q is n.s.]

$$= PP'$$

[P is n.s. because it's a product of two non-singular matrices]

C.U.

Result 10. If A be a p.d. matrix, then A can be written as $B'B$ (or BB') when B is also p.d. matrix.

Proof:→ If A be a p.d. matrix then $\exists$ an orthogonal matrix $Q \ni QAQ' = \Lambda$

$$= \text{diag}(\lambda_1, \dots, \lambda_n), \quad \lambda_i > 0 \ \forall i$$

$$A = Q^{-1}\Lambda^{1/2}\Lambda^{1/2}(Q')^{-1}$$ [Q^{-1} exists as it is equals to Q' as Q is an orthogonal matrix as,

$$= Q^{-1}\Lambda^{1/2} Q Q' \Lambda^{1/2}(Q^{-1})'$$

$$= (Q'\Lambda^{1/2}Q)(Q'\Lambda^{1/2}Q) \quad [\because Q^{-1} = Q'] \quad \Lambda^{1/2} = \text{diag}(\sqrt{\lambda_1}, \dots \sqrt{\lambda_n})]$$

$$= BB'$$

Q is an orthogonal matrix,

$\Rightarrow Q$ is n.s., Λ is a p.d. matrix,

$\Rightarrow Q'\Lambda^{1/2}Q$ is also a p.d. matrix,

i.e. B is a p.d. matrix.

Result 11. If A be a p.s.d. matrix then A can be written as PP' or $P'P$ where P is a singular matrix.

Proof:→ Let A be a matrix of order $n \times n$ with $R(A) = r < n$.

then $\exists$ an n.s. matrix $Q \ni$

$$QAQ' = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_r, 0, \dots, 0)$$

$$= \Lambda, \text{ say, where } \lambda_i > 0, \ i = 1(1)r.$$

$$\Rightarrow A = Q^{-1}\Lambda^{1/2}\Lambda^{1/2}(Q')^{-1},$$ [Q^{-1} exists and is n.s. as Q is n.s.]

$$\Lambda^{1/2} = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_r}, 0, \dots, 0)$$

Note that, $|P| = |Q^{-1}||\Lambda^{1/2}| = 0$ [$\because P$ is a singular matrix]

$$A = Q^{-1}\Lambda^{1/2}\Lambda^{1/2}(Q^{-1})'$$

$$= PP' \text{ where } P \text{ is a singular matrix.}$$

[redacted] C.U.

Result 12. If A be a real p.s.d. matrix with $\sum_i \sum_j a_{ij} = 0$, then show that $\sum_j a_{ij} = 0 \;\forall\, i$.

Proof :→ $\square$ $A = ((a_{ij}))_{n\times n}$

Given $\sum_i \sum_j a_{ij} = 0 = \underset{\sim}{1}_n' A \underset{\sim}{1}_n$

since A is a p.s.d. matrix, $\exists$ a singular matrix $P \ni$

$$A = P'P$$

$\therefore \underset{\sim}{1}_n' A \underset{\sim}{1}_n = \underset{\sim}{1}_n' P'P \underset{\sim}{1}_n = \underset{\sim}{x}'\underset{\sim}{x}$, where $\underset{\sim}{x} = P\underset{\sim}{1}_n = (x_1, x_2, \ldots, x_n)'$

$$= \sum_{i=1}^{n} x_i^2$$

Now, $\sum_{i=1}^{n} x_i^2 = 0 \Rightarrow x_i = 0 \;\forall\, i$.

$\therefore \underset{\sim}{x} = \underset{\sim}{0}$

or, $P\underset{\sim}{1}_n = \underset{\sim}{0}$

or, $P'P\underset{\sim}{1}_n = \underset{\sim}{0}$

or, $A\underset{\sim}{1}_n = \underset{\sim}{0}$

or, $\sum_j a_{ij} = 0 \;\forall\, i$ ///

OR

$\sum_i \sum_j a_{ij} = 0 \Rightarrow \underset{\sim}{1}' A \underset{\sim}{1} = 0$ ——(1)

We know, $P'AP = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{n\times n}$

$$\therefore A = (P^{-1})' \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$$

$$= (P^{-1})' \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$$

$= B'B$, say, where $B = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$.

$\therefore A = B'B$.

From(1) → $\underset{\sim}{1}' B'B\underset{\sim}{1} = 0$

$\Rightarrow (B\underset{\sim}{1})' B\underset{\sim}{1} = 0$ $[\because \underset{\sim}{Y}'\underset{\sim}{Y} = \underset{\sim}{0}$ only when $\underset{\sim}{Y} = \underset{\sim}{0}]$

$\therefore B\underset{\sim}{1} = \underset{\sim}{0}$

$\therefore B'(B\underset{\sim}{1}) = \underset{\sim}{0}$ (Premultiplying by B')

$\therefore A\underset{\sim}{1} = \underset{\sim}{0}$, i.e. $\sum_j a_{ij} = 0 \;\forall\, i$

[C.V.]

Result 13. If $A = ((a_{ij}))_{n\times n}$ be a symmetric p.d. matrix then $|A| \le \prod_{i=1}^{n} a_{ii}$.
When does the equality hold?

Proof: → Let $A = ((a_{ij}))_{n\times n}$ be a p.d. matrix;

Partition A as: $A = \begin{bmatrix} A_{n-1} & \underset{\sim}{a}(n) \\ \underset{\sim}{a}'(n) & a_{nn} \end{bmatrix}$

$$|A| = |A_{n-1}|\,|\underbrace{a_{nn} - \underset{\sim}{a}'(n) A^{-1}_{n-1} \underset{\sim}{a}(n)}_{\text{scalar}}|$$

Now A_{n-1} is the principal submatrix of order $\overline{n-1}$ obtained from A. As A is p.d. matrix. ⇒ A_{n-1} will be a p.d. matrix.

⇒ A^{-1}_{n-1} is p.d. matrix.

⇒ $\underset{\sim}{a}_n' A^{-1}_{n-1} \underset{\sim}{a}_n \geq 0$

$\therefore |A| = |A_{n-1}|\,|a_{nn} - \underset{\sim}{a}'(n) A^{-1}_{n-1} \underset{\sim}{a}(n)|$

$\le |A_{n-1}|\, a_{nn}$

Again partition A_{n-1} as

$$A_{n-1} = \begin{bmatrix} A_{n-2} & \underset{\sim}{a}_{n-1} \\ \underset{\sim}{a}'_{n-1} & a_{\overline{n-1}\times\overline{n-1}} \end{bmatrix}$$

Similar arguments lead to

$\therefore |A_{n-1}| \le |A_{n-2}|\, a_{\overline{n-1}\times\overline{n-1}}$

$\therefore |A| \le |A_{n-2}|\, a_{\overline{n-1}\times\overline{n-1}}\, a_{nn}$

proceeding this way. ⋮

$$\le \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} a_{33} \cdots a_{nn}$$

$\le (a_{11} a_{22} - a_{12}^2)\, a_{33} \cdots a_{nn}$

$\le a_{11} a_{22}\, a_{33} \cdots a_{nn}$

$\le \prod_{i=1}^{n} a_{ii}$

'=' holds when A be a p.d. diagonal matrix.

C.U

- **Result 14.** If $A = ((a_{ij}))_{n\times n}$ be a non-singular matrix then

$$|AA'| \le \prod_i \left(\sum_j a_{ij}^2\right)$$

Proof:→ Since A is an n.s. matrix $\Rightarrow AA'$ must be p.d. matrix.

Let, $B = AA' = ((b_{ij}))$ Clearly, $b_{ij} = \sum_k a_{ik}a_{jk}$

by previous result:→

$$|B| \le \prod_i b_{ii}$$

$$= \prod_i \sum_k a_{ik}^2$$

Hence the result.

~~Note that, $AA' = B$ is n.n.d.~~

~~By the~~

Problem:— Reduce the quadratic form $5x^2+y^2+10z^2-4yz-10zx$ to the normal form and show that it is positive definite.

Solution:— The associated symmetric mtx is

$$A = \begin{pmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ -5 & -2 & 10 \end{pmatrix}$$

Let us apply congruence operations on A to reduce it to the normal form

$$A \xrightarrow{R_3+R_1} \begin{pmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & -2 & 5 \end{pmatrix} \xrightarrow{C_3+C_1} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 5 \end{pmatrix}$$

$$\xrightarrow{R_3+2R_2} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_3+2C_2} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{\sqrt{5}}R_1} \begin{pmatrix} \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\frac{1}{\sqrt{5}}C_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The normal form is $x^2+y^2+z^2$

The rank of the quadratic form is 3.

∴ the quadratic form is positive definite.

Canonical reduction of Quadratic Forms :→

Let A be a p.d. matrix then $\underset{\sim}{x}' A \underset{\sim}{x} = Q(\underset{\sim}{x}) > 0$ $\forall\ \underset{\sim}{x} \neq \underset{\sim}{0}$.

$$Q(\underset{\sim}{x}) \longrightarrow Q(\underset{\sim}{y}) = \underset{\sim}{y}' \Lambda \underset{\sim}{y} = \sum_{i=1}^{n} \lambda_i y_i^2 \text{, where } \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n),\ \lambda_i > 0\ \forall i$$

and $\underset{\sim}{y} = (y_1, \dots, y_n)'$

$$Q(\underset{\sim}{y}) \longrightarrow Q(\underset{\sim}{z}) = \sum_{i=1}^{n} z_i^2$$

$\exists$ an n.s. matrix $P \ni P'AP = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n),\ \lambda_i > 0\ \forall i$

Choose $\underset{\sim}{y} \ni \underset{\sim}{x} = P\underset{\sim}{y}$, then we get —

$$Q(\underset{\sim}{x}) \Big|_{\underset{\sim}{x} = P\underset{\sim}{y}} = x'Ax = \underset{\sim}{y}'P'AP\underset{\sim}{y} = \underset{\sim}{y}' \Lambda \underset{\sim}{y} = \sum_{i=1}^{n} \lambda_i y_i^2$$

Choose $\underset{\sim}{z} \ni \underset{\sim}{y} = \Lambda^{-1/2} \underset{\sim}{z}$, where $\Lambda^{-1/2} = \text{diag}\left(\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}, \dots, \frac{1}{\sqrt{\lambda_n}}\right)$

$$Q(\underset{\sim}{y}) \Big|_{\underset{\sim}{y} = \Lambda^{-1/2}\underset{\sim}{z}} = y' \Lambda y = \underset{\sim}{z}' \underbrace{\Lambda^{-1/2} \Lambda^{1/2}}_{I} \underbrace{\Lambda^{1/2} \Lambda^{-1/2}}_{I} \underset{\sim}{z}$$

$$= \underset{\sim}{z}'\underset{\sim}{z}$$

$$= \sum_{i=1}^{n} z_i^2 \text{, where } \Lambda^{1/2} = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$$

Defn. of canonical form :→ If by any real non-singular linear transformation a real quadratic form be expressed as a sum and difference of the squares of the new variables then this later expression is called the canonical ~~~~ form of the given form.

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C.U.

Result * If A be a p.s.d. matrix of order $n \times n$ with $\text{rank}(A) = r < n$, then $Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x} \longrightarrow Q(\underset{\sim}{y}) = \sum_{i=1}^{r} \lambda_i y_i^2$ (or)

* Every quadratic form $\underset{\sim}{x}' A \underset{\sim}{x}$ can be reduced to a canonical form $\sum_{i=1}^{n} \lambda_i y_i^2 = \underset{\sim}{y}' \Lambda \underset{\sim}{y}$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ by an n.s. transformation of variables.

Proof :→ Let, $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values and $\underset{\sim}{u}_1, \underset{\sim}{u}_2, \ldots, \underset{\sim}{u}_n$ be the corresponding eigen vectors.

We can take these vectors $\underset{\sim}{u}_i$'s as orthogonal.

Take $Q = (\underset{\sim}{u}_1, \underset{\sim}{u}_2, \ldots, \underset{\sim}{u}_n)$, obviously Q is non-singular.

Then $\underset{\sim}{x} = Q\underset{\sim}{y}$ is a non-singular transformation.

It reduces the q.f. to,
$$\underset{\sim}{x}' A \underset{\sim}{x} = (Q\underset{\sim}{y})' A (Q\underset{\sim}{y})$$
$$= \underset{\sim}{y}' (Q' A Q) \underset{\sim}{y}$$
$$= \underset{\sim}{y}' \Lambda \underset{\sim}{y} \quad (\#)$$
$$= \sum_{i=1}^{n} \lambda_i y_i^2$$

Thus the theorem is established.

[(#) → As A be a square matrix of order n and $\lambda_1, \ldots, \lambda_n$ be the eigen values, then an real n.s. matrix Q exists, ∋
$$Q' A Q = \text{diag}(\lambda_1, \ldots, \lambda_n) = \Lambda$$ is true. Using this result there.]

Corollary: We have a connection between the nature of the eigen values and the nature of the matrix (or, q.f.).

If $\lambda_i > 0$ for each i, then $\underset{\sim}{x}' A \underset{\sim}{x}$ is positive definite (p.d.)

$\lambda_i < 0$ for each i, then $\underset{\sim}{x}' A \underset{\sim}{x}$ is negative definite (n.d.)

$\lambda_i \geq 0$ $\forall$ i and $\lambda_i = 0$ for some i, then $\underset{\sim}{x}' A \underset{\sim}{x}$ is p.s.d.

$\lambda_i \leq 0$ $\forall$ i and $\lambda_i = 0$ for some i, then $\underset{\sim}{x}' A \underset{\sim}{x}$ is n.s.d.

$\lambda_i > 0$ for some i & $\lambda_i < 0$ for some i, then $\underset{\sim}{x}' A \underset{\sim}{x}$ is indefinite.

[C.V.]

● **Result** The necessary and sufficient condition for a quadratic form $x'Ax$ to be positive definite is the leading principal minors of its matrix A are all positive. i.e.

Let A be a matrix of order $n\times n$, $A=((a_{ij}))_{n\times n}$, then

$$a_{11}>0,\ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}>0,\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}>0,\ \ldots\ldots,\ |A|>0.$$

Proof :→ Necessity part (Only if part) :→

Let $Q(\underset{\sim}{x})=\underset{\sim}{x}'A\underset{\sim}{x}$ be a p.d.q.f. in n variable $x_1, x_2, \ldots, x_n$ and let $m(\leq n)$ be a natural number, putting $x_{m+1}=x_{m+2}=\ldots\ldots=x_n=0$ in the p.d.q.f. $\underset{\sim}{x}'A\underset{\sim}{x}$, we arrive to another p.d.q.f. in m variables $x_1, x_2, \ldots, x_m$, The determinant of Whose matrix is the leading principal minor of the matrix A.

Since, the determinant of every p.d.q.f. is positive.

∴ every leading principal minor of the matrix A is positive.

Sufficiency part (If part) :→ Method of induction will be used to proof this part. For a single variable $\underset{\sim}{x}$, the q.f.

$Q(\underset{\sim}{x}) = ax_1^2>0$ for $x_1\neq 0$ if $a>0$.

Now, suppose that the theorem is true for m variables.

Consider any q.f. in $(m+1)$ variables with the corresponding symmetric matrix $A \ni$ the leading pricipal minors of A is positive.

Let us partition of A as follows:

$$A_{m+1\times m+1}=\begin{pmatrix} \underset{m\times m}{B} & \underset{m\times 1}{b_1} \\ \underset{1\times m}{b_1'} & K \end{pmatrix}$$

the leading principal minors of A and B are all positive as the theorem is supposed to be true for m variables ∃ a n.s. matrix P over the real field $\ni$ $P'BP=I_m$

Let us now determine $C\ni$

$\begin{pmatrix} P' & 0 \\ c' & 1 \end{pmatrix}\begin{pmatrix} B & b_1 \\ b_1' & k \end{pmatrix}\begin{pmatrix} P & c \\ 0 & 1 \end{pmatrix}$ has its R.H. top corner element zero.

the product matrix equals to $\begin{pmatrix} P'B & P'b_1 \\ c'B+b_1' & c'b_1+K \end{pmatrix}\begin{pmatrix} P & c \\ 0 & 1 \end{pmatrix}$

$$=\begin{pmatrix} P'BP & P'BC+P'b_1 \\ C'BP+b_1'P & C'BC+b_1'C+c_1'b_1+k \end{pmatrix}$$

$\therefore$ We have $P'BC + P'\underset{\sim}{b}_1 = 0$

$$\Rightarrow P'BC = -P'\underset{\sim}{b}_1$$

$$\Rightarrow BC = -\underset{\sim}{b}_1$$

$$\therefore C = -(B^{-1}\underset{\sim}{b}_1)$$

Under this choice of C, we also have

$$C'BP + b_1'P = 0 \quad \text{and}$$

$$C'BC + C'\underset{\sim}{b}_1 = \underset{\sim}{b}_1'(B^{-1})'BB^{-1}\underset{\sim}{b} - \underset{\sim}{b}_1'(B^{-1})'\underset{\sim}{b}_1$$

$$= \underset{\sim}{b}_1'(B^{-1})'\underset{\sim}{b}_1 - \underset{\sim}{b}_1'(B^{-1})'\underset{\sim}{b}_1$$

$$= 0$$

$\therefore$ Product matrix $= \begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{b}_1'C + k \end{pmatrix}$

Taking determinant in both sides,

$$|P'|\,|A|\,|P| = |I_m|\,(\underset{\sim}{b}_1'C + k)$$

$$\text{or, } |A|\{|P|\}^2 = (\underset{\sim}{b}_1'C + k)$$

Now by assumption, $|A| > 0$ and $|P| \neq 0$.

So, $(\underset{\sim}{b}_1'C + k) > 0$

Let $\underset{\sim}{b}_1'C + k = \beta^2$

$$\therefore \begin{pmatrix} P' & 0 \\ C' & 1 \end{pmatrix} A \begin{pmatrix} P & C \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \beta^2 \end{pmatrix}$$

Premultiplying and postmultiplying both sides by

$$\begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \beta^{-1} \end{pmatrix}$$

$$\begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \beta^{-1} \end{pmatrix} \begin{pmatrix} P' & 0 \\ C' & 1 \end{pmatrix} A \begin{pmatrix} P & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \beta^{-1} \end{pmatrix} = I_{m+1}$$

or, $Q'AQ = I_{m+1}$, where $Q = \begin{pmatrix} P & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \beta^{-1} \end{pmatrix}$

Since A is congruent to I_{m+1}, therefore A is p.d., i.e., the corresponding q.f. is p.d. Hence, by induction result follows.

C.U.

- Corollary : The necessary and sufficient condition for a q.f. $Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x}$ to be negative definite is that the leading principal minors of A starting from the first are alternatively negative and positive, i.e.

$$a_{11} < 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0, \ldots\ldots, (-1)^n |A| \gtrless 0$$ according as n is even or odd.

Proof :→ $\underset{\sim}{x}'A\underset{\sim}{x}$ is negative definite if and only if $\underset{\sim}{x}'(-A)\underset{\sim}{x}$ is positive definite. But by virtue of the above theorem, $\underset{\sim}{x}'(-A)\underset{\sim}{x}$ is positive definite if and only if

$$-a_{11} > 0, \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \begin{vmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{vmatrix} > 0, \ldots\ldots$$

Thus the necessary and sufficient conditions for $x'(-A)\underset{\sim}{x}$ to be p.d. or, equivalently, $\underset{\sim}{x}'A\underset{\sim}{x}$ to be n.d. is the following :

$$-a_{11} > 0, \text{ i.e. } a_{11} < 0,$$

$$(-1)^2 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \text{ i.e. } \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0,$$

$$(-1)^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \text{ i.e. } \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0, \ldots\ldots$$

$(-1)^n |A| \gtrless 0$ according as n is even or odd.

Hence the proof.

Ex.1. Consider the quadratic form: $4x_1^2 - 10x_1x_2 + 7x_2^2$.

Here, $A = \begin{pmatrix} 4 & -5 \\ -5 & 7 \end{pmatrix}$

As such, $a_{11} = 4 > 0$, $|A| = 3 > 0$.

Consequently, the quadratic form is positive definite (p.d.)

Ex.2. Consider now the quadratic form: $4x_1^2 - 10x_1x_2 + 3x_2^2$.

Here, $A = \begin{pmatrix} 4 & -5 \\ -5 & 3 \end{pmatrix}$

As such, $a_{11} = 4 > 0$, $|A| = -13 < 0$,

so, that this quadratic form is neither positive definite (p.d.) nor negative definite (n.d.).

Orthogonal Matrix :-

A matrix A is said to be an orthogonal matrix if

$$A^TA = AA^T = I$$

i.e. $A^T = A^{-1}$.

Rewrite $\underset{n\times n}{A} = (\underset{\sim}{\alpha_1}, \underset{\sim}{\alpha_2}, \ldots, \underset{\sim}{\alpha_n})$,

$\therefore A^TA = I$

then, $\Rightarrow \begin{pmatrix} \underset{\sim}{\alpha_1}' \\ \underset{\sim}{\alpha_2}' \\ \vdots \\ \underset{\sim}{\alpha_n}' \end{pmatrix} (\underset{\sim}{\alpha_1}, \underset{\sim}{\alpha_2}, \ldots, \underset{\sim}{\alpha_n}) = I$

i.e. $((\alpha_i' \alpha_j)) = I$

$\Leftrightarrow \underset{\sim}{\alpha_i}' \underset{\sim}{\alpha_j} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

i.e. $(\underset{\sim}{\alpha_1}\ \underset{\sim}{\alpha_2}, \ldots, \underset{\sim}{\alpha_n})'$ are mutually orthogonal vectors.

Again if $\underset{n\times n}{A} = \begin{pmatrix} \underset{\sim}{\beta_1}' \\ \underset{\sim}{\beta_2}' \\ \vdots \\ \underset{\sim}{\beta_n}' \end{pmatrix}$ then $A^T = (\underset{\sim}{\beta_1}, \underset{\sim}{\beta_2}, \ldots, \underset{\sim}{\beta_n})$

$$AA^T = \begin{pmatrix} \underset{\sim}{\beta_1}' \\ \vdots \\ \underset{\sim}{\beta_n}' \end{pmatrix} (\underset{\sim}{\beta_1}, \underset{\sim}{\beta_2}, \ldots \underset{\sim}{\beta_n})$$

$$= I$$

i.e. $((\underset{\sim}{\beta_i}' \underset{\sim}{\beta_j})) = I$

$\Rightarrow \underset{\sim}{\beta_i}' \underset{\sim}{\beta_j} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

i.e. $(\underset{\sim}{\beta_1}, \ldots, \underset{\sim}{\beta_n})$ are mutually orthogonal vectors.

Ex:→

I, $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$, $\begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & & \frac{-(n-1)}{\sqrt{n(n-1)}} \end{bmatrix}_{n\times n}$, etc.

C.U.

Result: → Show that the following matrix is orthogonal $(I-S)(I+S)^{-1}$ when S is a skew-symmetric matrix, and $(I-S)$ is non-singular.

We have to show –
i) $A'A = I$ or,
ii) $AA' = I$

Proof: →
$$\{(I-S)(I+S)^{-1}\}^T \{(I-S)(I+S)^{-1}\}$$
$$= \{(I+S)^{-1}\}^T (I-S)^T (I-S)(I+S)^{-1}$$
$$= \{(I+S)^T\}^{-1} (I-S')(I-S)(I+S)^{-1}$$
$$= (I+S')^{-1}(I+S)(I-S)(I+S)^{-1}$$
$$= (I-S)^{-1}(I+S)(I-S)(I+S)^{-1}$$

[Note that, $(I+S)(I-S) = I - S + S - S^2 = I - S^2 = (I-S)(I+S)$] — (1)

$$= \underbrace{(I-S)^{-1}(I-S)}\,\underbrace{(I+S)(I+S)^{-1}}$$
$$= I \cdot I$$
$$= I$$

OR

$$(I-S)(I+S)^{-1}\{(I-S)(I+S)^{-1}\}'$$
$$= (I-S)(I+S)^{-1}\{(I+S)^{-1}\}'(I-S')$$
$$= (I-S)(I+S)^{-1}(I+S')^{-1}(I-S')$$
$$= (I-S)(I+S)^{-1}(I-S)^{-1}(I+S)$$
$$= (I-S)\left[(I+S)(I-S)\right]^{-1}(I+S)$$
[Using (1)]
$$= (I-S)(I-S)^{-1}(I+S)^{-1}(I+S)$$
$$= I I$$
$$= I.$$

1) If $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

Show that the matrix $((\delta_{ij} + x_i x_j))$ is p.d., $i, j = 1(1)p$. (ISI)

<u>**Soln**</u> →

Let $A = ((\delta_{ij} + x_i x_j))$

$= I_n + \underset{\sim}{x}\underset{\sim}{x}^T$, where, $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Consider an associated Q.F.

$$\underset{\sim}{u}' A \underset{\sim}{u} = \underset{\sim}{u}' (I_n + xx^T) \underset{\sim}{u}$$

$$= \underset{\sim}{u}' I_n \underset{\sim}{u} + \underset{\sim}{u}' \underset{\sim}{x} \underset{\sim}{x}' \underset{\sim}{u}$$

$$= \underset{\sim}{u}'\underset{\sim}{u} + (\underset{\sim}{x}'\underset{\sim}{u})'(\underset{\sim}{x}'\underset{\sim}{u})$$

For $\underset{\sim}{u} \neq \underset{\sim}{0}$, $\quad \underset{\sim}{u}'\underset{\sim}{u} = \sum_{i=1}^{n} u_i^2 > 0$

$\Rightarrow \underset{\sim}{u}' A \underset{\sim}{u} > 0$

Hence, $\underset{\sim}{u}' A \underset{\sim}{u}$ is positive definite,

i.e. $A = ((\delta_{ij} + x_i x_j))$ is p.d.

Quadratic Form

- N.n.d.
  - p.d.
  - p.s.d.
- N.p.d.
  - n.d.
  - N.s.d.
- Indefinite

<u>Remark</u>: →

1) If $\underset{\sim}{x}' A \underset{\sim}{x}$ is n.n.d./p.d./p.s.d. then $\underset{\sim}{x}'(-A)\underset{\sim}{x}$ is n.p.d./n.d./n.s.d.

2) If $\underset{\sim}{x}' A \underset{\sim}{x}$ is n.n.d. or n.p.d. or Indefinite, then A is said to be n.n.d. or n.p.d. or indefinite. If a quadratic form has some definiteness, then the associated matrix has the same definiteness.

EXAMPLE:
1) $x_1^2 + 2x_2^2$ is p.d.
2) $x_1^2 + (x_1 - x_2)^2$ is p.d.
3) $Q(x_1, x_2) = (x_1 - x_2)^2$ is p.s.d.
4) $Q(x_1, x_2) = x_1^2 + x_1 x_2$ is indefinite.
5) $Q(x_1, x_2) = x_1^2 - x_2^2$ is indefinite.

Theorem:— A non-singular transformation of the variables changes a quadratic form into another q.f. with the same definiteness.

Proof:→

Consider a quadratic form $\underset{\sim}{x}' A \underset{\sim}{x}$ and an n.s. transformation

$\underset{\sim}{y} = P\underset{\sim}{x} \Rightarrow \underset{\sim}{x} = P^{-1}\underset{\sim}{y}$, P is n.s.

Note that $\underset{\sim}{x}' A \underset{\sim}{x} = \underset{\sim}{y}' \{(P^{-1})' A (P^{-1})\} \underset{\sim}{y} = \underset{\sim}{y}' B \underset{\sim}{y}$

Clearly, the range sets of $\underset{\sim}{x}' A \underset{\sim}{x}$ and $\underset{\sim}{y}' B \underset{\sim}{y}$ are identical.

Note that $\underset{\sim}{x} = \underset{\sim}{0}$

$\Leftrightarrow \underset{\sim}{y} = 0$, since P is n.s.

$[\because \underset{\sim}{y} = P\underset{\sim}{x} \Rightarrow \underset{\sim}{x} = P^{-1}\underset{\sim}{y}]$

Now, $\underset{\sim}{x}' A \underset{\sim}{x} = 0$

or, $\underset{\sim}{x} \neq 0 \Rightarrow \underset{\sim}{y} \neq 0$ as P is n.s.

[That is, $\underset{\sim}{x}' A \underset{\sim}{x} = 0$ iff $\underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{y}' B \underset{\sim}{x} = 0$ iff $\underset{\sim}{y} = \underset{\sim}{0}$

or, $\underset{\sim}{x}' A \underset{\sim}{x} > 0$ for all $\underset{\sim}{x} \neq 0$

$\Rightarrow \underset{\sim}{y}' B \underset{\sim}{x} > 0 \quad \forall \quad \underset{\sim}{y} \neq 0$]

Problem:- Find the values of p for which $(1-p)I_n + p\underset{\sim}{1}\underset{\sim}{1}'$ is p.d.?

Soln.

$(1-p)I_n + p\underset{\sim}{1}\underset{\sim}{1}'$

$$= \begin{bmatrix} 1 & p & p & \cdots & p \\ p & 1 & p & \cdots & p \\ \vdots & \vdots & \vdots & & \vdots \\ p & p & p & \cdots & 1 \end{bmatrix}$$

The k^{th} order principal minor is

$$\Delta_k = \begin{bmatrix} 1 & p & p & \cdots & p \\ p & 1 & p & \cdots & p \\ \vdots & \vdots & \vdots & & \vdots \\ p & p & p & \cdots & 1 \end{bmatrix}_{k\times k} = (1+\overline{k-1}\,p)(1-p)^{k-1}$$

$(1-p)I_n + p\underset{\sim}{1}\underset{\sim}{1}'$ is p.d.

iff $\Delta_k > 0, \ \forall \ k = 1(1)n.$

iff $\Delta_1 = 1 > 0, \ \Delta_2 = (1+p)(1-p) > 0$

$\Delta_k = (1+\overline{k-1}\,p)(1-p)^{k-1} > 0 \ \forall \ k = 3(1)n.$

iff $-1 < p < 1, \ (1+\overline{k-1}\,p) > 0 \ \forall \ k = 3(1)n$

iff $-\frac{1}{k-1} < p < 1, \ k = 2(1)n.$

iff $-\frac{1}{n-1} < p < 1.$

Theorem: $\Rightarrow$ A is n.n.d., iff $A = B'B$, for some B.

Proof: $\rightarrow$

If part: — Let, $A = B'B$

Then $\underset{\sim}{x}' A \underset{\sim}{x} = \underset{\sim}{x}'(B'B)\underset{\sim}{x}$

$\Rightarrow (B\underset{\sim}{x})'(B\underset{\sim}{x}) = y'y \geq 0$, where $y = B\underset{\sim}{x}$

$\therefore \underset{\sim}{x}' A \underset{\sim}{x} \geq 0 \ \forall x$

$\Rightarrow A$ is n.n.d..

Only if part: — Let A be n.n.d. then $\exists$ a n.s. matrix $C \ni C'AC = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, $r = \text{rank}(A)$.

$$\Rightarrow A = (C')^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} (C)^{-1}$$

$$= \left\{ \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} C^{-1} \right\}^T \left\{ \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} C^{-1} \right\}$$

$$= B^T B, \text{ where } B = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} C^{-1},$$

with $\text{rank}(B) = r$.

Corollary: — A is p.d. iff $A = B'B$, for some non-singular B.

Proof: $\rightarrow$ **If part:** — Let $A = B'B$, B is n.s.

Then, $\underset{\sim}{x}' A \underset{\sim}{x} = (B\underset{\sim}{x})^T (B\underset{\sim}{x}) = y^T y \geq 0 \ \forall \underset{\sim}{y}$

Now, $\underset{\sim}{x}' A \underset{\sim}{x} = 0$

$\Leftrightarrow \hat{y}'\hat{y} = 0$

$\Leftrightarrow \underset{\sim}{y} = 0 \Rightarrow B\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \underset{\sim}{x} = B^{-1}\underset{\sim}{0} = \underset{\sim}{0}$

Hence, A is p.d. $\quad$ [A is p.d. $\Rightarrow |A| > 0$, $\Rightarrow r(A_{n \times n}) = n$]

Only if part: — Let A is p.d.

Then $\exists$ a n.s. matrix $C \ni C'AC = I_n$

$\Rightarrow A = (C^{-1})' C^{-1} = B'B$, where $B = C^{-1}$ is n.s.

— X —

Example:—

1) If $\underset{\sim}{x}' A \underset{\sim}{x}$ is p.s.d. then S.T. $|A| = 0$.

Soln. → A is p.s.d.

$\therefore \exists$ an n.s. mtx $C \ni C^T A C = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where

$r = \text{rank}(A) < n$.

$$A = \left\{ \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} C^{-1} \right\}^T \left\{ \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} C^{-1} \right\}$$

$\therefore A = B^T B$, where $B = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} C^{-1}$

with rank$(B) = r < n$

$\therefore |A| = |B^T||B| = |B|^2 = 0$ $\left[\because r(\underset{n\times n}{B}) < n \Rightarrow |B| = 0 \right]$

2) If ~~[illegible]~~ A is p.d., then S.T. A^{-1} is also p.d.

Soln. → A is p.d.

$\Leftrightarrow \underset{\sim}{x}' A \underset{\sim}{x} > 0 \quad \forall \underset{\sim}{x} \neq \underset{\sim}{0}$.

Also, $|A| > 0$, i.e. A is n.s.

Consider the non-singular transformation:

$\underset{\sim}{y} = A \underset{\sim}{x}$

Now, $\underset{\sim}{x}' A \underset{\sim}{x} > 0 \quad \forall \underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow (A^{-1}\underset{\sim}{y})' A (A^{-1}\underset{\sim}{y}) > 0 \quad \forall A^{-1}\underset{\sim}{y} \neq 0$

$\Rightarrow \underset{\sim}{y}'(A')^{-1} A A^{-1} \underset{\sim}{y} > 0 \quad \forall \underset{\sim}{y} \neq 0$ as A is n.s.

$\Rightarrow \underset{\sim}{y}' A^{-1} \underset{\sim}{y} > 0, \quad \forall \underset{\sim}{y} \neq \underset{\sim}{0}$ as A is symmetric (WL

$\therefore A^T = A$.

$\Rightarrow A^{-1}$ is p.d.

Problem:—1. Reduce the equation $3x^2+5y^2+3z^2+2xy+2yz+2zx=1$ into canonical form.

Solution:-

$$A=\begin{pmatrix}3&1&1\\1&5&1\\1&1&3\end{pmatrix},\quad \underset{\sim}{X}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$\therefore \underset{\sim}{X}^T A\underset{\sim}{X}=1.$

The ch. equation of A is $\begin{vmatrix}3-x&1&1\\1&5-x&1\\1&1&3-x\end{vmatrix}=0$

or, $(x-2)(x-3)(x-6)=0.$

$\therefore$ The eigen values of A are 2, 3, 6.

The eigen vector corresponding to the eigen value 2 are $c\begin{pmatrix}1\\0\\-1\end{pmatrix}; c\neq 0.$

The " " " " " 3 " $c\begin{pmatrix}1\\-1\\1\end{pmatrix}, c\neq 0.$

The " " " " " " 6 " $c\begin{pmatrix}1\\2\\1\end{pmatrix}, c\neq 0.$

Let $\underset{\sim}{\alpha}=\begin{pmatrix}1\\0\\-1\end{pmatrix}, \underset{\sim}{\beta}=\begin{pmatrix}1\\-1\\1\end{pmatrix}, \underset{\sim}{\gamma}=\begin{pmatrix}1\\2\\1\end{pmatrix}$. Then the set $\{\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}\}$ is an orthogonal set. The orthonormal set of eigen vectors is

$$\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix}\right\}.$$

Let $P=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\\0&-\frac{1}{\sqrt{3}}&\frac{2}{\sqrt{6}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\end{pmatrix}$. Then P is an orthogonal mtx.

Let us apply the orthogonal transformation $\underset{\sim}{X}=P\underset{\sim}{X}'$, where $\underset{\sim}{X}'=\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$.

Then the equation transforms to $(X')^T(P^TAP)X'=1.$

$P^TAP=(P^{-1}AP)$ is a diagonal matrix D which has the same eigen value as those of A.

$$AP=\begin{pmatrix}3&1&1\\1&5&1\\1&1&3\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\\0&-\frac{1}{\sqrt{3}}&\frac{2}{\sqrt{6}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\end{pmatrix}=\begin{pmatrix}2\cdot\frac{1}{\sqrt{2}}&3\cdot\frac{1}{\sqrt{3}}&6\cdot\frac{1}{\sqrt{6}}\\2\cdot 0&3\cdot-\frac{1}{\sqrt{3}}&6\cdot\frac{2}{\sqrt{6}}\\2\cdot-\frac{1}{\sqrt{2}}&3\cdot\frac{1}{\sqrt{3}}&6\cdot\frac{1}{\sqrt{6}}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\\0&-\frac{1}{\sqrt{3}}&\frac{2}{\sqrt{6}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}2&0&0\\0&3&0\\0&0&6\end{pmatrix}$$

$$=PD, \text{ where } D=\begin{pmatrix}2&0&0\\0&3&0\\0&0&6\end{pmatrix}.$$

So, $P^{-1}AP=D=\begin{pmatrix}2&0&0\\0&3&0\\0&0&6\end{pmatrix}$

The equation transforms to $(X')^TDX'=1$

i.e. to $2x'^2+3y'^2+6z'^2=1.$

2. Reduce the equation $7x^2 - 2xy + 7y^2 - 16x + 16y - 8 = 0$ into canonical form and determine the nature of the conic.

Solution:— Let $A = \begin{pmatrix} 7 & -1 \\ -1 & 7 \end{pmatrix}$, $B = \begin{pmatrix} -16 & 16 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

Then the equation takes the form $X^T A X + BX - 8I_1 = 0$

The eigen values of A are $8, 6$.

The eigen vectors corresponding the eigen values 8 and 6 are

$c\begin{pmatrix} 1 \\ -1 \end{pmatrix}, c \neq 0;\ d\begin{pmatrix} 1 \\ 1 \end{pmatrix}, d \neq 0$, respectively.

The orthonormal set of eigen vectors is $\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

Let $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Then P is an orthogonal matrix.

$P^T A P = (P^{-1}AP)$ is a diagonal matrix which has the same eigenvalues as those of A.

So, $P^T A P = \begin{pmatrix} 8 & 0 \\ 0 & 6 \end{pmatrix}$. $BP = \begin{pmatrix} -16\sqrt{2} & 0 \end{pmatrix}$.

By the orthogonal transformation $X = PX'$, where $X' = \begin{pmatrix} x' \\ y' \end{pmatrix}$, the equation transforms to $8x'^2 + 6y'^2 - 16\sqrt{2}x' - 8 = 0$

or, $8(x' - \sqrt{2})^2 + 6y'^2 = 24$.

Let us apply the translation $x'' = x' - \sqrt{2}$,

$y'' = y'$

∴ The equation transforms to $8x''^2 + 6y''^2 = 24$.

The canonical form is $8x^2 + 6y^2 = 24$.

The equation represents an ellipse.

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EIGEN VALUES & EIGEN VECTORS

■ Differentiation with respect to a vector :→

Let $f \equiv f(x_1, x_2, \ldots, x_n)$ be a function of the n real variables $x_1, x_2, \ldots, x_n$.

Suppose further that the partial derivatives $\frac{\partial f}{\partial x_i}$ $(i = 1(1)n)$ exists for $x_1 = a_1, x_2 = a_2, \ldots, x_n = a_n$.

Then by the vector of partial derivatives, denoted by

$$\left[\frac{\partial f}{\partial \underset{\sim}{x}}\right]_{\underset{\sim}{x}=\underset{\sim}{a}} = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right]'_{\substack{x_1=a_1\\ x_2=a_2\\ \vdots\\ x_n=a_n}}$$

E.g. Suppose $f(\underset{\sim}{x}) = Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x}$, where A is a symmetric matrix, $A = ((a_{ij}))_{n\times n}$

$$\begin{aligned} f(\underset{\sim}{x}) = & \left(a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2\right) \\ & + 2x_1(a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n) \\ & + 2x_2(a_{23}x_3 + a_{24}x_4 + \cdots + a_{2n}x_n) \\ & + 2x_3(a_{34}x_4 + a_{35}x_5 + \cdots + a_{3n}x_n) \\ & + \\ & + 2x_{n-1}(a_{\overline{n-1}n}x_n) \end{aligned}$$

We have, $$\frac{\partial f}{\partial x_1} = 2a_{11}x_1 + 2a_{12}x_2 + \cdots + 2a_{1n}x_n$$
$$= 2\sum_{j=1}^{n} a_{1j}x_j = 2\underset{\sim}{a}_1'\underset{\sim}{x}$$

In the same way, $$\frac{\partial f}{\partial x_2} = 2\sum_{j=1}^{n} a_{2j}x_j = 2\underset{\sim}{a}_2'\underset{\sim}{x}$$

$$\vdots$$

$$\frac{\partial f}{\partial x_n} = 2\sum_{j=1}^{n} a_{nj}x_j = 2\underset{\sim}{a}_n'\underset{\sim}{x}$$

$$\therefore \left[\frac{\partial f}{\partial \underset{\sim}{x}}\right]_{\underset{\sim}{x}=\underset{\sim}{a}} = 2\left(\underset{\sim}{a}_1'\underset{\sim}{x} + \underset{\sim}{a}_2'\underset{\sim}{x} \cdots\cdots \underset{\sim}{a}_n'\underset{\sim}{x}\right)'$$
$$= 2\begin{pmatrix} \underset{\sim}{a}_1' \\ \underset{\sim}{a}_2' \\ \vdots \\ \underset{\sim}{a}_n' \end{pmatrix}\underset{\sim}{x}$$

Maxima and minima of a real function : →

Suppose the function f has partial derivatives of both first and second orders in the neighbourhood of the point $\underset{\sim}{x} = \underset{\sim}{a}$. Let us write

$$g_i = \frac{\partial f}{\partial x_i}\bigg|_{\underset{\sim}{x}=\underset{\sim}{a}}, \quad g_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}\bigg|_{\underset{\sim}{x}=\underset{\sim}{a}}$$

Define,

$$g = [g_1, g_2, \ldots, g_n] = \frac{\partial f}{\partial \underset{\sim}{x}}\bigg|_{\underset{\sim}{x}=\underset{\sim}{a}}$$

$$G = ((g_{ij}))$$

so that G is a symmetric matrix.

By Taylor's series for f at $x = a+h$, where $h = [h_1, h_2, \ldots h_n]$ we may, for sufficiently small h_i, write

$$f(a+h) = f(a) + h'g + \frac{1}{2!} h'Gh + R,$$

where R is a reminder term $\ni R \to 0$ as $h \to 0$.

A necessary condition for f to have a maximum or a minimum at $\underset{\sim}{a}$ is that $g = 0$, since this ensures that f is stationary at a. Whether this is a maximum or a minimum or neither a maximum nor a minimum depends on the quadratic form $h'Gh$. One may then assert :

i) If G is a positive definite matrix, then f is a minimum at $\underset{\sim}{a}$.

ii) If G is a negative definite, then f is a maximum at $\underset{\sim}{a}$.

iii) If G is indefinite, then f is neither a maximum nor a minimum at $\underset{\sim}{a}$.

Method of Lagrangian Multiplier →

$f(x)$

$g(\underset{\sim}{x}) = c$

$$Z = f(x_1, x_2, \ldots, x_n) + \lambda\left(g(x_1, x_2, \ldots, x_n) - c\right)$$

$$\frac{\partial Z}{\partial x_1} = 0, \ \ldots\ldots, \ \frac{\partial Z}{\partial x_n} = 0, \ \frac{\partial Z}{\partial \lambda} = 0$$

$\underset{\sim}{x}'A\underset{\sim}{x}$

$\underset{\sim}{x}'\underset{\sim}{x} = 1$.

$$g = \underset{\sim}{x}'A\underset{\sim}{x} - \lambda \underset{\sim}{x}'\underset{\sim}{x}$$

$$\therefore g = x'(A - \lambda I)\underset{\sim}{x}.$$

And, $$\frac{\partial g}{\partial \underset{\sim}{x}} = 2(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$$

if $\underset{\sim}{x} \neq \underset{\sim}{0}$, then this implies rank $(A - \lambda I) < n$

$$\therefore |A - \lambda I| = 0.$$

An optimization Problem :→ Suppose $\underset{\sim}{x}'A\underset{\sim}{x}$ is a quadratic form in the n variables $x_1, x_2, \ldots, x_n$ which is to be maximized or minimized subject to the condition $\underset{\sim}{x}'\underset{\sim}{x} = 1$.

Introducing a Lagrange multiplier λ, we are then led to consider the quadratic form

$$\underset{\sim}{x}'A\underset{\sim}{x} - \lambda\underset{\sim}{x}'\underset{\sim}{x} = \underset{\sim}{x}'(A - \lambda I)\underset{\sim}{x} \quad \text{——— (1)}$$

To get a maximum or a minimum, we equate.

$$\frac{\partial}{\partial x_1}\left(\underset{\sim}{x}'(A-\lambda I)\underset{\sim}{x}\right) = 0, \; \ldots\ldots, \; \frac{\partial}{\partial x_n}\left(\underset{\sim}{x}'(A-\lambda I)\underset{\sim}{x}\right) = 0$$

This yields $(A-\lambda I)\underset{\sim}{x} = 0$ ——— (2)

One solution of the system of linear equations (2) is, of course, $\underset{\sim}{x} = 0$. But we are not interested in this trivial solution since it does not meet the condition $\underset{\sim}{x}'\underset{\sim}{x} = 1$. Here, a non-trivial solution exists iff rank $(A - \lambda I) < n$.

i.e. $|A - \lambda I| = 0$ ——— (3)

***Definitions :→** The values of the scalar parameter λ satisfying $|A - \lambda I_n| = 0$ are called the eigen values or latent roots or eigen roots or characteristic roots of the square matrix $(A_{n \times n})$.

Any non-null vector satisfying $(A - \lambda I)\underset{\sim}{x} = 0$ corresponding to an eigen value is called an eigen vector or latent vector or characteristic vector of A.

In other words, let λ_0 be an eigen value of A and x_0 be a non-null vector satisfying $(A - \lambda_0 I_n)\underset{\sim}{x} = 0$. We call this x_0 as an eigen vector of A corresponding to λ_0.

□ Let λ be a characteristic root of $A_{n \times n}$. If r be the rank of $(A - \lambda I)$, then there are $(n - r)$ linearly independent characteristic vectors of A corresponding to this λ.

* ## Characteristic Polynomial and Equations and Space :-

The determinant $|A - \lambda I|$, which has λ in each diagonal element, is a polynomial in λ, called the Characteristic polynomial for A. The heighest power of λ, of course, comes from the product of the n diagonal elements, thus the term with the heighest power of λ is $(-\lambda)^n$, so that the characteristic polynomial of a square matrix of order n is a polynomial of degree n. Putting $|A - \lambda I| = f(\lambda)$, we may, therefore write

$$f(\lambda) = b_0 + b_1(-\lambda) + b_2(-\lambda)^2 + \cdots + b_{n-1}(-\lambda)^{n-1} + (-\lambda)^n \quad (*)$$

From a fundamental theorem of algebra, we know that the equation $(*) = 0$,

i.e. $|A - \lambda I| = 0$

or, $f(\lambda) = 0$,

has n roots, say $\lambda_1, \lambda_2, \ldots, \lambda_n$. These roots, i.e. eigen values, need not be all distinct, but if each distinct root is counted a number of times equal to its multiplicity, then we have in total n roots. Further, each distinct root may be either real or complex and the complex roots must occur in pairs. The equation is called the characteristic equation for A.

For any value of λ different from the roots of the characteristic equation, the only solution of $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$ is $\underset{\sim}{x} = \underset{\sim}{0}$. On the other hand, if λ is put equal to any of the eigen values, say to λ_i, then $|A - \lambda_i I| = 0$ and so there is at least one $\underset{\sim}{x} \neq \underset{\sim}{0}$ which satisfies $(A - \lambda I)\underset{\sim}{x} = 0$, i.e. there exists at least one eigen vector. The maximum number of independent vectors satisfying

$$(A - \lambda_i I)\underset{\sim}{x} = 0$$

is the nullity of $(A - \lambda_i I)$. In other words, the highest number of independent eigenvectors corresponding to the eigen value λ_i is the same as the nullity of $(A - \lambda_i I)$, i.e. $n - r$, where r is the rank of the matrix A.

Every non-zero linear combination of this (n-r) independent characteristic vectors is also a characteristic vector of A corresponding to this λ. The set of all these linear combinations together with the null vector is a subspace. The subspace is called the characteristic space of A related to the characteristic root λ. It may be noted that the characteristic space of A corresponding to the characteristic root λ is actually the null space of $(A-\lambda I)$.

*What do you mean by a characteristic value problem?

A problem which arises frequently in application of linear algebra, is that of finding value of a scalar λ for which $\exists$ non-null vectors $\underset{\sim}{x}$ satisfying

$A\underset{\sim}{x}=\lambda \underset{\sim}{x}$, A be a square matrix, i.e., we verify wheather for a non-null vector $\underset{\sim}{x}$ the matrix A operating on it, produces a scalar multiple of $\underset{\sim}{x}$. Such a problem is known as a characteristic value problem. A well known example of an eigen value problem emarges from a problem of constrained optimization. Suppose we are to maximize the quadratic form

$Q(\underset{\sim}{x})=\underset{\sim}{x}'A\underset{\sim}{x}$, with respect to the variables $x_1, x_2, \ldots, x_n$

Subject to the restriction that the length of the vector remain constant $\sum_i x_i^2 = c$

Then by the Lagranges method of optimisation, we maximize,

$$f(x)=\underset{\sim}{x}'A\underset{\sim}{x}+\lambda(c-\underset{\sim}{x}'\underset{\sim}{x})$$

w.r.t. $\underset{\sim}{x}$, when λ is a Lagranges multiplier.

Clearly $\frac{\partial}{\partial x} f(\underset{\sim}{x}) = 2A\underset{\sim}{x} - 2\lambda\underset{\sim}{x} = 0$

i.e. $A\underset{\sim}{x}=\lambda\underset{\sim}{x}$

Again for a given λ we search for a non-null vector $\underset{\sim}{x}$ satisfying $A\underset{\sim}{x}=\lambda\underset{\sim}{x}$

is equivalent to obtain a non-null vector from the null space of $(A-\lambda I)$.

We have a non-trivial solution to

$$(A-\lambda I)\underset{\sim}{x}=0$$

if $R(A-\lambda I)<n$ [assuming A is of order $n\times n$]

i.e. there will be a non-null $\underset{\sim}{x}$ satisfying $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

if $|A-\lambda I| = 0$

Let, $A = ((a_{ij}))_{n\times n}$.

Then $|A - \lambda I_n| = 0$

$$\equiv \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad \text{——— (1)}$$

Note that LHS of (1) is a polynomial in λ of degree n since the highest order term in λ comes out to be $(-\lambda)^n$. Thus solution to the equation $|A-\lambda I|=0$ has n-roots on λ, namely $\lambda_1, \lambda_2, \ldots, \lambda_n$, these roots are known as characteristic ~~values~~ roots or, eigen values or, latent roots of matrix A. and a non-null vector satisfying $A\underset{\sim}{x} = \lambda_i \underset{\sim}{x}$ is ~~the~~ called the eigen vector, clearly the maximum number of LIN vectors $\underset{\sim}{x}$ satisfying $A\underset{\sim}{x} = \lambda_i \underset{\sim}{x}$ is the nullity of $(A-\lambda_i I)$, these roots need not to be distinct but if a root is counted as many time as its multiplicity, we have n-roots in all.

Again the eigen values may real or complexed.

As $|A-\lambda I|$ is a polynomial in $(-\lambda)$ of degree n, it can be written as

$$|A-\lambda I| = b_0 + b_1(-\lambda) + \cdots + b_{n-1}(-\lambda)^{n-1} + (-\lambda)^n$$

$$= (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots\cdots(\lambda_n-\lambda)$$

$$\Rightarrow b_0 = \prod_i \lambda_i$$

$$b_1 = \sum_{i_1 > i_2 > \cdots > i_{n-1}} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n-1}}$$

$$\vdots$$

$$b_{n-2} = \sum_{i>j} \lambda_i \lambda_j$$

$$b_{n-1} = \sum_i \lambda_i$$

Example :→ Consider the matrix

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

or, $(4-\lambda)(1-\lambda) - 4 = 0$

or, $\lambda(\lambda - 5) = 0$.

As such, the eigen values are $\lambda_1 = 0, \lambda_2 = 5$.

To determine the eigenvectors $\underset{\sim}{x}$ corresponding to λ_1, we have to solve the linear homogeneous equations

$$(A - \lambda_1 I)\underset{\sim}{x} = \underset{\sim}{0},$$

i.e. the equations

$$4x_1 + 2x_2 = 0,$$

$$\Rightarrow \quad 2x_1 + x_2 = 0.$$

these gives $\quad x_2 = -2x_1$

Take $x_1 = \alpha \, (\neq 0)$, so, $x_2 = -2\alpha$

Any vector of the form $\begin{pmatrix} \alpha \\ -2\alpha \end{pmatrix}$, $\alpha \neq 0$ is an eigen ~~value~~ vector corresponding to $\lambda_1 = 0$.

To determine the eigen vector corresponding to $\lambda_2 \, (=5)$, we have to solve the following linear homogeneous eqn.

$$(A - \lambda_2 I)\underset{\sim}{x} = \underset{\sim}{0}.$$

$$\Rightarrow \left\{ \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \right\} \underset{\sim}{x} = \underset{\sim}{0}$$

$$\Rightarrow -x_1 + 2x_2 = 0$$

i.e, $2x_1 - 4x_2 = 0$

or, $x_1 = 2x_2$

Take $x_1 = \beta$, $x_2 = \beta/2$.

Any vector of the form $\begin{pmatrix} \beta \\ \beta/2 \end{pmatrix}$, $\beta \neq 0$ is an eigen vector corresponding to an eigen value $\lambda_2 = 5$.

*① **Problem:** → Obtain the eigen values and hence the eigen vectors of the matrix $A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$.

Soln. → $\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Rightarrow \begin{bmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underset{\sim}{0}$$

$$\therefore \begin{vmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(1-\lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda = 0$$

$$\therefore \lambda(\lambda - 3) = 0$$

i.e. $\lambda = 0, 3$.

when $\lambda = 0$, $\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \sqrt{2}x_1 + x_2 = 0$ —— ① • vectors of unit length

& $x_1^2 + x_2^2 = 1$ [as eigen ~~values~~ are ~~orthogonal~~] —— ②

Let $x_2 = a$,

$\therefore x_1 = -\frac{a}{\sqrt{2}}$ [from ①]

Putting these values in the eqn. ②,

$$\left(\frac{-a}{\sqrt{2}}\right)^2 + a^2 = 1$$

$$\Rightarrow a = \sqrt{\frac{2}{3}}$$

$$\therefore x_2 = \sqrt{\frac{2}{3}}, \quad x_1 = -\frac{1}{\sqrt{3}}$$

when $\lambda = 3$, $\begin{bmatrix} 2-3 & \sqrt{2} \\ \sqrt{2} & 1-3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow -x_1 + \sqrt{2}x_2 = 0$ —— ③

& $x_1^2 + x_2^2 = 1$ —— ④

Let $x_2 = b$, $\therefore x_1 = \sqrt{2}\cdot b$ [From (3)]

Putting these values into eqn. (4),

$$(\sqrt{2}b)^2 + b^2 = 1$$

$$\Rightarrow b = \sqrt{1/3}$$

$$\therefore x_2 = \sqrt{1/3}, \quad x_1 = \sqrt{2/3}$$

Hence, the eigen values are 0 and 3 and eigen vectors are

$$\begin{pmatrix} -1/\sqrt{3} \\ \sqrt{2/3} \end{pmatrix}, \begin{pmatrix} \sqrt{2/3} \\ \sqrt{1/3} \end{pmatrix}.$$

* Similar Matrices :-

Defn. → 1) Two square matrices A and B of the same order, say n are said to be similar if there exists a non-singular matrix P of order n, such that $B = PAP^{-1}$.

2) A and B be two square matrices, are said to be similar if they have same characteristics equation and hence same set of eigen values.

Result : → 1) If A and B are similar matrices, then they must have the same eigenvalues.

Proof : → Since A and B are similar, we have

$$B = PAP^{-1} \quad \text{for some n.s. matrix } P.$$

Now,

$$B - \lambda I = PAP^{-1} - \lambda I$$
$$= P(A - \lambda I)P^{-1}$$

Taking determinants of both sides, we have

$$|B - \lambda I| = |P|\,|A - \lambda I|\,|P^{-1}|$$
$$= |A - \lambda I|\,|P|\,|P^{-1}|$$
$$= |A - \lambda I| \quad \text{since } |P||P^{-1}| = |PP^{-1}| = |I| = 1.$$

Thus, the two matrices A and B have the same characteristic equation and hence the same eigenvalues.

(OR)

$$|A - \lambda I| = 0$$

$$\Rightarrow |PAP^{-1} - P\lambda IP^{-1}| = 0$$

$$\Rightarrow |PAP^{-1} - \lambda I| = 0$$

$$\Rightarrow |B - \lambda I| = 0 \quad [\because B = PAP^{-1}, \text{ as A and B are similar matrices}]$$

Therefore A and PAP^{-1} (or, B) give same characteristic equation.

IMPORTANT THEOREMS ON EIGEN VALUES & EIGEN VECTORS :-

Theorem :→ If λ is an eigenvalue of the square matrix A, then $\lambda - k$ is an eigen value of $(A - kI)$.

Proof :→ Since λ is an eigen value of A, there exists $\underset{\sim}{x} \neq \underset{\sim}{0}$, ∋

$$(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$$

i.e. $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

$$\Rightarrow A\underset{\sim}{x} - kI\underset{\sim}{x} = \lambda \underset{\sim}{x} - kI\underset{\sim}{x}$$

$$\therefore (A - kI)\underset{\sim}{x} = (\lambda - k)\underset{\sim}{x}$$

$\Rightarrow (\lambda - k)$ is an eigen value of $(A - kI)$.

Theorem :→ If λ is an eigenvalue of A, then λ^m is an eigenvalue of A^m, for any positive integer m.

Proof :→ Since λ is an eigen value of A, ∃ $\underset{\sim}{x} \neq \underset{\sim}{0}$ ∋ $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$

$$\Rightarrow A(A\underset{\sim}{x}) = A(\lambda\underset{\sim}{x})$$

$$\Rightarrow A^2\underset{\sim}{x} = \lambda^2\underset{\sim}{x}$$

$\Rightarrow \lambda^2$ is an eigen value of A^2.

We have, $A^2\underset{\sim}{x} = \lambda^2\underset{\sim}{x}$

$$\Rightarrow A(A^2\underset{\sim}{x}) = A(\lambda^2\underset{\sim}{x}) = \lambda^2 A\underset{\sim}{x}$$

$$\Rightarrow A^3\underset{\sim}{x} = \lambda^3\underset{\sim}{x}$$

$\Rightarrow \lambda^3$ is an eigen value of A^3.

In the same way, we can prove that for any $m = 2, 3, 4, \ldots$ λ^m is an eigen value of A^m.

Theorem :→ If λ is a characteristic roots of $A_{n\times n}$, then $k\lambda$ is a ch. roots of kA.

Proof :→ As λ is a ch. root of $A_{n\times n}$, then ∃ some non-zero $\underset{\sim}{x}$ such that $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$

$$\Rightarrow kA\underset{\sim}{x} = k\lambda\underset{\sim}{x}$$

$$\Rightarrow (kA)\underset{\sim}{x} = (k\lambda)\underset{\sim}{x}$$

$\Rightarrow k\lambda$ is an eigen value of kA.

Theorem:→ If λ is an eigen value of an n.s. matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of $Adj(A)$.

Proof:→ Since λ is a ch. roots of A, then $\frac{1}{\lambda}$ is a ch. root A^{-1}. (By the next theorem)
We know that, $A^{-1} = \frac{Adj(A)}{|A|}$ and $k\lambda$ is a ch. root of kA, for any scalar k.
So, λ^{-1} is a characteristic root of $\frac{Adj(A)}{|A|}$, giving
$\frac{|A|}{\lambda}$ is an eigen value of $Adj(A)$.

***

Theorem:→ Suppose A is n.s. and λ is an eigen value of A, then λ^{-1} is an eigen value of A^{-1}. Further A and A^{-1} has the same set of eigen vectors.

Proof:→ Let A be an n.s. matrix and λ is an eigen value of A, then there exists $\underset{\sim}{x} \neq \underset{\sim}{0}$ such that $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$ ——(1)
Premultiplying both sides by A^{-1}, we get

$$A^{-1}(A\underset{\sim}{x}) = A^{-1}\lambda x$$
$$\Rightarrow (A^{-1}A)\underset{\sim}{x} = \lambda(A^{-1}x)$$
$$\text{or, } \underset{\sim}{x} = \lambda(A^{-1}x)$$
$$\text{or, } \frac{1}{\lambda}\underset{\sim}{x} = A^{-1}x$$
$$\text{i.e. } \lambda^{-1}\underset{\sim}{x} = A^{-1}x \quad \text{——(2)}$$

This implies that λ^{-1} is an eigen value of A^{-1}.

From (1) & (2), we see that $\underset{\sim}{x}$ is an eigen vector of A corresponding to λ iff $\underset{\sim}{x}$ is an eigen vector of A^{-1} corresponding to λ^{-1}. Thus A and A^{-1} has the same set of eigen vectors.

Theorem:→ If λ be an eigen value of $A_{n\times n}$ and $\underset{\sim}{x}$ be the corresponding eigen vector then for any n.s. matrix $P_{n\times n}$, λ is also an eigen value of $P^{-1}AP$ and $P^{-1}\underset{\sim}{x}$ is the corresponding eigen vector.

Proof:→ Since λ is an eigen value of $A_{n\times n}$ and $\underset{\sim}{x}$ be a corresponding eigen vector $\Rightarrow$ $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$
Premultiplying both sides by P^{-1}, we get → $P^{-1}A\underset{\sim}{x} = P^{-1}\lambda\underset{\sim}{x}$
Let $x = Py$ or $y = P^{-1}x$, where P is an n.s. matrix, so P^{-1} exists.

$$\text{or, } P^{-1}APy = P^{-1}\lambda Py = \lambda P^{-1}Py$$
$$\text{or, } P^{-1}APy = \lambda y$$

$\therefore$ λ is an eigen value of $P^{-1}AP$ treating as the corresponding eigen vector where $y = P^{-1}x$.

Theorem: → Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distincted eigen values of A and $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_k$ be the corresponding eigen vectors. then show that $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_k$ are Linearly independent.

Proof: → Since $\underset{\sim}{x}_i$ is the characteristic vector corresponding to the eigen value λ_i of A, where $i = 1(1)k$, ∋

$$A\underset{\sim}{x}_i = \lambda_i \underset{\sim}{x}_i$$

To show that the ch. vectors $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_k$ are LIN, let us consider the equation

$$\delta_1 \underset{\sim}{x}_1 + \delta_2 \underset{\sim}{x}_2 + \cdots + \delta_k \underset{\sim}{x}_k = \underset{\sim}{0} \quad \text{——— (1)}$$

Premultiplying both sides by A, we get →

$$A.\delta_1 \underset{\sim}{x}_1 + A\delta_2 \underset{\sim}{x}_2 + \cdots + A\delta_k \underset{\sim}{x}_k = \underset{\sim}{0}$$

$$\Rightarrow \delta_1 (A\underset{\sim}{x}_1) + \delta_2 (A\underset{\sim}{x}_2) + \cdots + \delta_k (A\underset{\sim}{x}_k) = \underset{\sim}{0}$$

$$\text{or, } \delta_1 \lambda_1 \underset{\sim}{x}_1 + \delta_2 \lambda_2 \underset{\sim}{x}_2 + \cdots + \delta_k \lambda_k \underset{\sim}{x}_k = \underset{\sim}{0} \quad \text{——— (2)} \quad [\because A\underset{\sim}{x}_i = \lambda_i \underset{\sim}{x}_i]$$

Premultiplying ~~both side~~ by A, once again

$$\delta_1 \lambda_1^2 \underset{\sim}{x}_1 + \delta_2 \lambda_2^2 \underset{\sim}{x}_2 + \cdots + \delta_k \lambda_k^2 \underset{\sim}{x}_k = \underset{\sim}{0} \quad \text{——— (3)}$$

Proceeding similarly, we will get

$$\delta_1 \lambda_1^m \underset{\sim}{x}_1 + \delta_2 \lambda_2^m \underset{\sim}{x}_2 + \cdots + \delta_k \lambda_k^m \underset{\sim}{x}_k = \underset{\sim}{0} \quad \text{——— (m)}$$

for $m = 3, 4, \ldots, k-1$.

Now we write these k equations in matrix form

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_k^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \lambda_3^{k-1} & \cdots & \lambda_k^{k-1} \end{pmatrix} \begin{pmatrix} \delta_1 \underset{\sim}{x}_1 \\ \delta_2 \underset{\sim}{x}_2 \\ \delta_3 \underset{\sim}{x}_3 \\ \vdots \\ \delta_k \underset{\sim}{x}_k \end{pmatrix} = \underset{\sim}{0}$$

Since, the λ_i's are all distinct, the above matrix coefficients are all singular. Hence the equation has the solution $\delta_1 \underset{\sim}{x}_1 = \underset{\sim}{0}, \delta_2 \underset{\sim}{x}_2 = \underset{\sim}{0}, \ldots, \delta_k \underset{\sim}{x}_k = \underset{\sim}{0}$.

But since $\underset{\sim}{x}_i$'s are eigen vectors $\underset{\sim}{x}_i \neq 0 \ \forall i = 1(1)k$.

So, $\delta_1 = \delta_2 = \cdots = \delta_k = 0$.

Hence, $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_k$ are Linearly Independent.

* **Theorem :→** The eigen vectors corresponding to different eigen values constitute a set of mutually orthogonal vector.

Proof :→ The eigen values of $A_{n\times n}$ are all distinct. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct eigen values of a symmetric matrix $A_{n\times n}$. Let us suppose, λ_i and λ_j are two distinct eigen values of A and further assume that $\underset{\sim}{x}_i$ and $\underset{\sim}{x}_j$ are the eigen vectors corresponding to λ_i and λ_j, respectively.

Then we have, $A\underset{\sim}{x}_i = \lambda_i \underset{\sim}{x}_i$

implying. $\underset{\sim}{x}_j' A\underset{\sim}{x}_i = \lambda_i \underset{\sim}{x}_j'\underset{\sim}{x}_i$ ——— (1)

and $A\underset{\sim}{x}_j = \lambda_j \underset{\sim}{x}_j$

implying $\underset{\sim}{x}_i' A\underset{\sim}{x}_j = \lambda_j \underset{\sim}{x}_i'\underset{\sim}{x}_j$ ——— (2)

Noting that, $\underset{\sim}{x}_i'\underset{\sim}{x}_j = \underset{\sim}{x}_j'\underset{\sim}{x}_i$ and $\underset{\sim}{x}_i' A\underset{\sim}{x}_j = \underset{\sim}{x}_j' A\underset{\sim}{x}_i$ (since A is symmetric), Now, (1) − (2) implies

$$\underset{\sim}{x}_i'\underset{\sim}{x}_j(\lambda_i - \lambda_j) = 0$$

Now, $\lambda_i - \lambda_j \neq 0$ [as λ_i, λ_j are two distinct values]

$$\Rightarrow \underset{\sim}{x}_i'\underset{\sim}{x}_j = 0$$

Hence the eigen vectors are orthogonal.

NOTE :→ If $\underset{\sim}{x}$ be an eigen vector corresponding to the eigen value λ of the matrix $A_{n\times n}$ then a multiple of $\underset{\sim}{x}$ will as well be an eigen vector corresponding to λ. Thus it is usual to assume that the eigen vectors are of length unity.

If A be an matrix of order $n\times n$ having distinct eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the eigen vectors corresponding the eigen values constitute an orthonormal basis of E_n.

[Example of this is Problem No. (1).]

Theorem :→ The value of the quadratic form $x'Ax$ corresponding to an eigen vector x_i for any eigen value λ_i of $\hat{A}$, which satisfies the condition $x_i'x_i = 1$, must be λ_i itself.

Proof :→ Since λ_i is an eigen value of A and $\underset{\sim}{x}_i$ a corresponding eigen vector, we have $A\underset{\sim}{x}_i = \lambda_i \underset{\sim}{x}_i$

$$\Rightarrow \underset{\sim}{x}_i' A\underset{\sim}{x}_i = \lambda_i \underset{\sim}{x}_i'\underset{\sim}{x}_i$$
$$= \lambda_i,$$

since $\underset{\sim}{x}_i'\underset{\sim}{x}_i = 1$.

Theorem :→ If A is a real symmetric matrix, then the eigen values of A must be all real. / The characteristic roots of a real symmetric matrix are all real.

Proof :→

1st Method :– Let λ be the root of a real symmetric matrix A, then there exists a non-zero vector $\underset{\sim}{x}$ such that

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

Premultiplying both sides by $\underset{\sim}{x}'$ we obtain

$$\underset{\sim}{x}'A\underset{\sim}{x} = \lambda \underset{\sim}{x}'\underset{\sim}{x} \quad \text{——— (1)}$$

Transposing (1) we get

$$(\underset{\sim}{x}'A\underset{\sim}{x})' = (\lambda \underset{\sim}{x}'\underset{\sim}{x})'$$

$$\Rightarrow \underset{\sim}{x}'A\underset{\sim}{x} = \underset{\sim}{x}'x\lambda \qquad [\text{since } A' = A, \text{ as A is symmetric}]$$

Thus, $\lambda = \dfrac{\underset{\sim}{x}'A\underset{\sim}{x}}{\underset{\sim}{x}'\underset{\sim}{x}}$ is real since $\underset{\sim}{x}'A\underset{\sim}{x}$ and $\underset{\sim}{x}'\underset{\sim}{x}$ are both real matrices of single elements. ///.

Method 2 :– Suppose λ be a complex ~~eigen vector~~ eigen value of symmetric matrix A, in particular λ may reduce to a real eigenvalues, then ∃ at least one vector $\underset{\sim}{x}$ satisfying

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x} \quad \text{——— (1)}$$

Taking the complex conjugate, we have

$$A\underset{\sim}{x}^* = \lambda^* \underset{\sim}{x}^* \quad \text{——— (2)}$$

where $\underset{\sim}{x}^*$ is conjugate of $\underset{\sim}{x}$ and λ^* is conjugate of λ.

If $\underset{\sim}{x} = (x_1\ x_2 \cdots x_n)'$ and $\underset{\sim}{x}^* = (x_1^*\ x_2^* \cdots x_n^*)'$, assuming A is of order $n \times n$, then $\underset{\sim}{x}_i^*$ is conjugate of $\underset{\sim}{x}_i\ \forall i$.

Premultiplying (1) by $(\underset{\sim}{x}^*)'$ and (2) by $\underset{\sim}{x}'$ respectively, we obtain

$$\underset{\sim}{x}^{*\prime}A\underset{\sim}{x} = \lambda \underset{\sim}{x}^{*\prime}\underset{\sim}{x} \quad \text{——— (3)}$$

$$\underset{\sim}{x}'A\underset{\sim}{x}^* = \lambda^* \underset{\sim}{x}^{*\prime}\underset{\sim}{x} \quad \text{——— (4)}$$

Subtracting (3) & (4), we get

$$\underset{\sim}{x}^{*\prime}A\underset{\sim}{x} - \underset{\sim}{x}'A\underset{\sim}{x}^* = (\lambda - \lambda^*)\underset{\sim}{x}^{*\prime}\underset{\sim}{x}$$

since A is a symmetric matrix,

$$[\underset{\sim}{x}^{*\prime}A\underset{\sim}{x}]' = \underset{\sim}{x}'A'\underset{\sim}{x}^* = \underset{\sim}{x}'A\underset{\sim}{x}^*$$

We get, $(\lambda-\lambda^*)\underset{\sim}{x}^{*\prime}\underset{\sim}{x}=0$,

Let, $\underset{\sim}{x}=\begin{bmatrix} x_{11}+ix_{12} \\ x_{21}+ix_{22} \\ \vdots \\ x_{n1}+ix_{n2}\end{bmatrix}$

$\Rightarrow \underset{\sim}{x}^*=\begin{bmatrix} x_{11}-ix_{12} \\ x_{21}-ix_{22} \\ \vdots \\ x_{n1}-ix_{n2}\end{bmatrix}$

Thus $\underset{\sim}{x}^{*\prime}\underset{\sim}{x}=\sum_{i=1}^{n}(x_{i1}^2+x_{i2}^2)>0$ [since x is non-null]

Therefore $\lambda_i=\lambda_i^*$

Thus if $\lambda=a+ib$, then $\lambda^*=a-ib$, and

$\lambda=\lambda^* \Rightarrow b=0$.

$\therefore$ The roots are real.

☑ Another Method :→ Let A be a real symmetric matrix and let $\underset{\wedge}{x}+i\underset{\sim}{y}$ be the real/complex eigen vector corresponding to the eigen value λ, where $\underset{\wedge}{x}$ and $\underset{\sim}{y}$ are both vectors with real elements, at least one of them being non-null.

$\therefore$ $A(\underset{\wedge}{x}+i\underset{\sim}{y})=\lambda(\underset{\sim}{x}+i\underset{\sim}{y})$

Premultiplying both sides by the conjugate transpose $(\underset{\wedge}{x}-i\underset{\sim}{y})'$, we get

$(\underset{\wedge}{x}-i\underset{\sim}{y})'A(\underset{\sim}{x}+i\underset{\sim}{y})=\lambda(\underset{\wedge}{x}-i\underset{\sim}{y})'(\underset{\wedge}{x}+i\underset{\sim}{y})$

or, $\left\{(\underset{\sim}{x}'A\underset{\sim}{x}+\underset{\wedge}{y}'A\underset{\wedge}{y})+i(\underset{\wedge}{x}'A\underset{\sim}{y}-\underset{\sim}{y}'A\underset{\sim}{x})\right\}=\lambda\left\{(\underset{\wedge}{x}'\underset{\sim}{x}+\underset{\sim}{y}'\underset{\sim}{y})+i(\underset{\wedge}{x}'\underset{\sim}{y}-\underset{\sim}{y}'\underset{\sim}{x})\right\}$

Since A is symmetric, $\underset{\wedge}{x}'A\underset{\sim}{y}-\underset{\sim}{y}'A\underset{\sim}{x}=0$ and $\underset{\wedge}{x}'\underset{\sim}{y}-\underset{\sim}{y}'\underset{\wedge}{x}=0$

$\therefore$ the imaginary part vanishes from both sides.

$\therefore$ $\underset{\sim}{x}'A\underset{\sim}{x}+\underset{\sim}{y}'A\underset{\wedge}{y}=\lambda(\underset{\wedge}{x}'\underset{\wedge}{x}+\underset{\sim}{y}'\underset{\sim}{y})$

It gives $\lambda=\dfrac{\underset{\wedge}{x}'A\underset{\wedge}{x}+\underset{\sim}{y}'A\underset{\sim}{y}}{\underset{\wedge}{x}'\underset{\sim}{x}+\underset{\sim}{y}'\underset{\wedge}{y}}$

Since $\underset{\wedge}{x}\neq 0$ or $\underset{\sim}{y}\neq 0$, then $\underset{\wedge}{x}'\underset{\wedge}{x}+\underset{\wedge}{y}'\underset{\sim}{y}\neq 0$

$\therefore$ λ is real whatever be the nature of the eigen vector.

[*** → Corollary later]

(*) Diagonalisation of a symmetric matrix using orthogonal transformation

Example :- Let us consider the symmetric matrix

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & \sqrt{3} \\ 0 & \sqrt{3} & 6 \end{pmatrix}$$

$$\therefore |A - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 4-\lambda & \sqrt{3} \\ 0 & \sqrt{3} & 6-\lambda \end{vmatrix}$$

$$= (7-\lambda)(3-\lambda)(3-\lambda)$$

Hence the eigen values are $\lambda_1 = 7$, $\lambda_2 = \lambda_3 = 3$.

Corresponding to these, we shall find three eigenvectors that are orthonormal.

(i) An eigenvector corresponding to λ_1 is a solution of

$$A\underset{\sim}{x} = 7\underset{\sim}{x}$$

i.e. of the system

$$3x_1 = 7x_1$$
$$4x_2 + \sqrt{3}x_3 = 7x_2$$
$$\sqrt{3}x_2 + 6x_3 = 7x_3 .$$

The first of the equations, gives

$$x_1 = 0,$$

and from second and three, we get, $x_2 = \dfrac{x_3}{\sqrt{3}}$.

As such, an eigenvector corresponding to λ_1 has the form $\left[0, \frac{1}{\sqrt{3}}x_3, x_3\right]$. In order to normalize it; i.e. to make its length equal to unity, we have to take.

$$\frac{1}{3}x_3^2 + x_3^2 = 1$$

i.e. $x_3 = \pm \dfrac{\sqrt{3}}{2}$.

Taking the positive value of x_3, we may then have as our eigenvector $\underset{\sim}{u_1}$, given by

$$\underset{\sim}{u_1} = \left[0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right]$$

(ii) An eigen vector $\underset{\sim}{x} = [x_1, x_2, x_3]$ corresponding to λ_2 must be orthogonal to $\underset{\sim}{u_1}$ and it may also be of unit lenght, we must have

$$\frac{1}{2}x_2 + \frac{\sqrt{3}}{2}x_3 = 0$$

and $$x_1^2 + x_2^2 + x_3^2 = 1.$$

Taking $x_1 = 1$, then solving this we will get $x_2 = x_3 = 0$.

Hence, an eigenvector of unit ~~leg~~ length corresponding to λ_2 and orthogonal to $\underset{\sim}{u_1}$ is $\underset{\sim}{u_2} = [1, 0, 0]$

(iii) To obtain an eigen vector corresponding to λ_3 which is of unit length and orthogonal to u_1 and u_2, we have first to solve the equations

$$\frac{1}{2}x_2 + \frac{\sqrt{3}}{2}x_3 = 0$$

$$x_1 = 0$$

and $$x_1^2 + x_2^2 + x_3^2 = 1.$$

so, $x_3 = \pm 1/2$, considering the positive value, we get $x_2 = -\frac{\sqrt{3}}{2}$

then $\underset{\sim}{u_3} = [0, -\frac{\sqrt{3}}{2}, \frac{1}{2}]$.

Consider now a square matrix Q whose columns are the three eigenvectors $\underset{\sim}{u_1}, \underset{\sim}{u_2}, \underset{\sim}{u_3}$. thus

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 0 & 1/2 \end{pmatrix}$$

One should note that

$$Q'AQ = \begin{pmatrix} 0 & 1/2 & \sqrt{3}/2 \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & 1/2 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & \sqrt{3} \\ 0 & \sqrt{3} & 6 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Thus $Q'AQ$ is a diagonal matrix with the three eigen values as its diagonal elements.

Theorem:→ If A be a real symmetric matrix of order $n\times n$ then there exists a real non-singular orthogonal matrix Q such that $Q'AQ = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, where the columns of Q are the eigen vectors of length unity corresponding to the eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$ of A, may not be all distinct.

Proof:→ Method 1.

Let $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_n$ be eigenvectors of A corresponding to eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$, respectively, which are so taken as to make them mutually orthogonal and of unit length.

Define, $Q = (\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_n)$.

Clearly, Q is an orthogonal matrix since $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_n\}$ is a set of mutually orthonormal vectors. We thus have—

i) $A\underset{\sim}{x}_i = \lambda_i \underset{\sim}{x}_i \; (i = 1(1)n)$;

ii) $\underset{\sim}{x}_i' \underset{\sim}{x}_j = 0$ if $i \neq j \; (i, j = 1(1)n)$; and

iii) $\underset{\sim}{x}_i' \underset{\sim}{x}_j = 1$ if $i = j = 1(1)n$.

Consequently,

$$Q'AQ = \begin{pmatrix} \underset{\sim}{x}_1' \\ \underset{\sim}{x}_2' \\ \vdots \\ \underset{\sim}{x}_n' \end{pmatrix} A (\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_n)$$

$$= \begin{pmatrix} \underset{\sim}{x}_1' \\ \vdots \\ \underset{\sim}{x}_n' \end{pmatrix} (A\underset{\sim}{x}_1, A\underset{\sim}{x}_2, \dots, A\underset{\sim}{x}_n).$$

$$= ((\underset{\sim}{x}_i' A \underset{\sim}{x}_j))$$

$$= ((\underset{\sim}{x}_i' \lambda_j \underset{\sim}{x}_j))$$

$$= ((\lambda_j \underset{\sim}{x}_i' \underset{\sim}{x}_j))$$

$$= ((\lambda_j \delta_{ij})), \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$= \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n).$$

Method 2. Let A be a real symmetric matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of A, now corresponding to each λ_i, $\exists$ a characteristic vector or eigen vector $\underset{\sim}{u}_i$, now we have already proved that the set of eigen vectors are linearly independent.

Now, every LIN set of vectors can be converted to an orthonormal set by Gram-schmidt orthogonalization process. So, $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n$ can be taken as orthonormal, now we take $Q = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n)$

the ith diagonal element of

$$
\begin{aligned}
Q'AQ &= \hat{u}_i' A \underset{\sim}{u}_i \\
&= \underset{\sim}{u}_i' \lambda_i \underset{\sim}{u}_i \quad [\because A\hat{u}_i = \lambda_i \hat{u}_i \, \forall i = 1(1)n] \\
&= \lambda_i (\hat{u}_i' \hat{u}_i) \\
&= \lambda_i \quad [\because \hat{u}_i' \hat{u}_i = 1, \text{ since } \hat{u}_i\text{'s are orthonormal}]
\end{aligned}
$$

the (i,j)th element of $(i \neq j)$

$$
\begin{aligned}
Q'AQ &= \hat{u}_i' A \hat{u}_j \\
&= \hat{u}_i' \lambda_j \hat{u}_j \quad [\because A\hat{u}_j = \lambda_j \hat{u}_j] \\
&= \lambda_j (\hat{u}_i' \underset{\sim}{u}_j) \\
&= 0 \quad [\because \hat{u}_i' \underset{\sim}{u}_j = 0 \text{ for } i \neq j]
\end{aligned}
$$

$\therefore Q'AQ = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$

NOTE:→ Take $Q'AQ = \Lambda$

If λ_i's are all positive, $\lambda_i^{1/2}$ is defined,

$\therefore \Lambda^{1/2} = (\lambda_1^{1/2}, \lambda_2^{1/2}, \ldots, \lambda_n^{1/2})$

$$
\begin{aligned}
\therefore A &= (Q')^{-1} \Lambda Q^{-1} \\
&= (Q^{-1})' \Lambda^{1/2} \Lambda^{1/2} Q^{-1}
\end{aligned}
$$

We call $Q^{-1} = P$.

$$
\begin{aligned}
&= P' \Lambda^{1/2} \Lambda^{1/2} P \\
&= (\Lambda^{1/2} P)' (\Lambda^{1/2} P) \\
&= B'B, \quad [B = \Lambda^{1/2} P]
\end{aligned}
$$

$\boxed{A = P' \Lambda P}$ → It is called singular value decomposition of a matrix. In terms of eigen values and eigen vectors, we can write a matrix A as

$$A = \lambda_1 \hat{u}_1 \hat{u}_1' + \lambda_2 \hat{u}_2 \hat{u}_2' + \cdots\cdots + \lambda_n \hat{u}_n \hat{u}_n'$$

Cayley Hamilton Theorem :

Every square matrix satisfies it's own characteristic equations

$\Rightarrow$ Let A be a square matrix of order n, I be the identity matrix of order n and O be the zero matrix of order n.. The theorem ~~state~~ states that if

$$|A-\lambda I| = (-1)^n(\lambda^n + P_1\lambda^{n-1} + P_2\lambda^{n-2} + \cdots + P_n) = 0$$

be the characteristic equation then $(A)^n + P_1(A)^{n-1} + \cdots + P_n I = 0$

Problem 1. $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. Use C-H theorem to express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a linear polynomial in A.

Proof : Ch. eqn. $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0$$

or, $(1-\lambda)(3-\lambda)+2 = 0$

or, $\lambda^2 - 4\lambda + 5 = 0$

$\therefore$ the matrix A satisfies $A^2 - 4A + 5I = 0$

$A^2 = 4A - 5I$

$A^3 = 4A^2 - 5A$

$\vdots$

$A^6 = 4A^5 - 5A^4$

$A^4 = 4A^3 - 5A^2$

$A^4 - 4A^3 = -5A^2$

$\therefore 3A^4 - 12A^3 = -15A^2$

$\therefore A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$

$= -5A^4 + 8A^4 - 12A^3 + 14A^2$

$= 3A^4 - 12A^3 + 14A^2$

$= -15A^2 + 14A^2$

$= -A^2$

$= 5I - 4A$

2. $A=\begin{bmatrix}1&0&2\\0&-1&1\\0&1&0\end{bmatrix}$, Find A^9 and A^{-1} by using C-H theorem?

Proof :→ $A=\begin{bmatrix}1&0&2\\0&-1&1\\0&1&0\end{bmatrix}$

Ch. equn. s.

$$|A-\lambda I|=0$$

$$\begin{vmatrix}1-\lambda&0&2\\0&-1-\lambda&1\\0&1&-\lambda\end{vmatrix}=0$$

$$\Rightarrow \lambda^3-2\lambda+1=0$$

The matrix A satisfies $A^3-2A+I=0$.

$$A^3=2A-I.$$

$$\therefore A^9=(A^3)^3=(2A-I)^3$$

$$=8A^3-12A^2-I+6A$$

$$=16A-8I-12A^2-I+6A$$

$$=22A-12A^2-9I$$

$$=\begin{bmatrix}22&0&44\\0&-22&22\\0&22&0\end{bmatrix}-\begin{bmatrix}12&24&24\\0&24&-12\\0&-12&12\end{bmatrix}-\begin{bmatrix}9&0&0\\0&9&0\\0&0&9\end{bmatrix}$$

$$=\begin{bmatrix}1&-24&20\\0&-55&34\\0&34&-21\end{bmatrix}$$

Here, $|A|=-1\neq 0$, $\therefore A^{-1}$ exists.

$$A^3-2A+I=0$$

$$\Rightarrow A^{-1}=2I-A^2$$

$$=\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}-\begin{bmatrix}1&2&2\\0&2&-1\\0&-1&1\end{bmatrix}$$

$$=\begin{bmatrix}1&-2&-2\\0&0&1\\0&1&1\end{bmatrix}$$

[II] Questions from C.U. Previous Papers

1) What is characteristic equation of matrix ? Discuss the nature and the number of roots of such equation. (4)

2) If A and B are two square matrices of the same order then show that AB and BA have the same set of characteristic roots. (6)

3) Discuss the nature of the characteristic roots of a positive semi-definite quadratic form. (3)

4) If $\underset{\sim}{\alpha}$ is a non-null p-component column vector, then find the characteristic roots of $\underset{\sim}{\alpha}\underset{\sim}{\alpha}'$. (3)

5) If A and B are two non-singular matrices of the same order and $C = BAB^{-1}$ and λ is any scalar, prove that —

$$|C+\lambda I| = |A+\lambda I|. \quad (3)$$

(***)

Corollary: → If A is a real skew-symmetric, then all the eigen values are purely imaginary or zero.

Hints: → From (3) & (4) of method 2) of the original theorem, we have,

$$0 = (\lambda+\lambda^*)\underset{\sim}{x}'\underset{\sim}{x}^* \quad \text{as} \quad \underset{\sim}{x}'A\underset{\sim}{y} = (\underset{\sim}{x}'A\underset{\sim}{y})'$$

$$= \underset{\sim}{y}'A'\underset{\sim}{x}$$

$$\Rightarrow \lambda = -\lambda^*$$

$$\Rightarrow \lambda \text{ is purely imaginary.} \qquad = -\underset{\sim}{y}'A\underset{\sim}{x}$$

[I] **Result:** — Show that if λ is a ch. root of the mtx A then $K+\lambda$ is a ch. root of the mtx $A+KI$.

Sol. Let λ be a ch. root of the mtx A and $\underset{\sim}{x}$ be a corresponding vector. Then $\underset{\sim}{x}$ is a non-zero vector $\ni$ $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$.

Now, $(A+KI)\underset{\sim}{x} = A\underset{\sim}{x} + KI\underset{\sim}{x}$

$$= \lambda\underset{\sim}{x} + K\underset{\sim}{x}$$

$$= (\lambda+K)\underset{\sim}{x} \quad \text{——— (1)}$$

Since $\underset{\sim}{x} \neq 0$, therefore from the relation (1), we see that the scalar $\lambda+K$ is a characteristic value of the mtx $(A+KI)$ and $\underset{\sim}{x}$ is a corresponding ch. vector.

Result.

* If A be a symmetric matrix of order $n \times n$ with eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then $\det(A) = \prod_{i=1}^{n} \lambda_i$.

Proof :→

Method 1. Since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the matrix A, the characteristic polynomial of the matrix A can be written in the factor form

$$|A - \lambda I| = f(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) \cdots (\lambda_n - \lambda)$$

Putting $\lambda = 0$ in both sides of the above equation, we get

$$|A| = \prod_{i=1}^{n} \lambda_i \quad ///.$$

Method 2. We know—

For a symmetric matrix $A_{n\times n}$ having distinct eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the orthogonal matrix P diagonalises A, where $P'AP = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ where the columns of P are the eigen vectors of length unity corresponding to the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Here, $P'AP = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ which implies

$$|P'AP| = \det[\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)]$$

$$= \prod_{i=1}^{n} \lambda_i$$

$$\Rightarrow |A||P'P| = \prod_{i=1}^{n} \lambda_i$$

$$\therefore |A| = \prod_{i=1}^{n} \lambda_i \quad \text{, since } P'P = I \text{ as P is orthogonal.}$$

31 Result:— Prove that the characteristic roots of Hermitian mtx are real.

Sol. Let A be a Hermitian matrix, λ be an ch. root of A and $\underset{\sim}{x}$ be the corresponding eigen value. then

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}.$$

Premultiplying both sides by x^{θ}, we get

$$\underset{\sim}{x}^{\theta} A \underset{\sim}{x} = \underset{\sim}{x}^{\theta} \lambda \underset{\sim}{x}.$$

Taking conjugate transpose of the both sides we get

$$(\underset{\sim}{x}^{\theta} A \underset{\sim}{x})^{\theta} = (\lambda \underset{\sim}{x}^{\theta} \underset{\sim}{x})^{\theta}$$

$$\Rightarrow \underset{\sim}{x}^{\theta} A \underset{\sim}{x} = \underset{\sim}{x}^{\theta} \underset{\sim}{x} \lambda$$

$$\Rightarrow \lambda = \frac{\underset{\sim}{x}^{\theta} A \underset{\sim}{x}}{\underset{\sim}{x}^{\theta} \underset{\sim}{x}}$$ is real since $\underset{\sim}{x}^{\theta} A \underset{\sim}{x}$ and $\underset{\sim}{x}^{\theta} \underset{\sim}{x}$ are real.

(C.U.)

Ex. Show that the eigen values of a p.d. / n.n.d. symmetric matrix are all positive / non-negative.

Soln. → By definition,

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}, \quad \underset{\sim}{x} \neq \underset{\sim}{0}.$$

$$\Rightarrow \underset{\sim}{x}' A \underset{\sim}{x} = \lambda \underset{\sim}{x}' \underset{\sim}{x}$$

$$\Rightarrow \lambda = \frac{\underset{\sim}{x}' A \underset{\sim}{x}}{\underset{\sim}{x}' \underset{\sim}{x}}.$$

Note that, $\underset{\sim}{x} \neq 0$,

$$\underset{\sim}{x}' \underset{\sim}{x} > 0$$

If A is p.d. symmetric matrix, then

$$\underset{\sim}{x}' A \underset{\sim}{x} > 0 \quad \forall\ \underset{\sim}{x} \neq \underset{\sim}{0}$$

$$\text{and } \lambda = \frac{\underset{\sim}{x}' A \underset{\sim}{x}}{\underset{\sim}{x}' \underset{\sim}{x}} > 0$$

If A is n.n.d. symmetric matrix, then

$$\underset{\sim}{x}' A \underset{\sim}{x} \geq 0 \quad \forall\ \underset{\sim}{x} \neq \underset{\sim}{0}$$

$$\text{and } \lambda = \frac{\underset{\sim}{x}' A \underset{\sim}{x}}{\underset{\sim}{x}' \underset{\sim}{x}} \geq 0$$

Ex. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Find an orthogonal matrix Q such that $Q'AQ = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, where λ_1, λ_2 are the eigen values of A. Hence, find A^8?

ANS: →

$$|A - \lambda I_2| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = -1, 3.$$

$$\lambda = -1, \quad \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow x_2 = -x_1$$

$$\Rightarrow \underset{\sim}{x} = x_1 (1, -1)$$

$\underset{\sim}{u}_1 = \frac{1}{\sqrt{2}}(1, -1)$ is an orthonormal eigen vector corresponding to $\lambda = -1$.

when $\lambda = 3$, $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\Rightarrow x_1 = x_2$

$\therefore \underset{\sim}{x} = x_2(1,1)$

$\therefore \underset{\sim}{u}_2 = \frac{1}{\sqrt{2}}(1,1)$ is an orthonormal eigen vector corresponding to $\lambda = 3$.

Hence $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is an orthogonal mtx

such that $Q'AQ = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$.

$Q'A^8Q = Q'AQ\,Q'A^7Q$ $[\because QQ' = I_n]$

$= \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} Q'A^7Q$

$= \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}^8 = \begin{pmatrix} (-1)^8 & 0 \\ 0 & 3^8 \end{pmatrix}$

[Alternative way: $\rightarrow$ We know that, if A has an eigen value λ, then A^m has an eigen value λ^m, and A, A^m have the same set of eigen vectors.

$Q'AQ = \text{diag}(\lambda_1, \ldots, \lambda_n)$

and $Q'A^mQ = \text{diag}(\lambda_1^m, \ldots\ldots, \lambda_n^m)$]

$A^8 = (Q')^{-1}\begin{pmatrix} (-1)^8 & 0 \\ 0 & 3^8 \end{pmatrix}$

$= Q\begin{pmatrix} 1 & 0 \\ 0 & 3^8 \end{pmatrix}Q'$

$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^8 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^8 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 & -3^8 \\ 1 & 3^8 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Ex. If A is a p.d. symmetric matrix, then S.T. $\exists$ a p.d. matrix $B \ni A = BB$.

Soln. $\exists$ an orthogonal matrix $Q \ni Q'AQ = \text{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i > 0 \;\forall\, i = 1(1)n$.

$$Q'AQ = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \sqrt{\lambda_n} \end{pmatrix} Q Q' \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \sqrt{\lambda_n} \end{pmatrix}$$

$$\Rightarrow A = Q \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \sqrt{\lambda_n} \end{pmatrix} Q'Q \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \sqrt{\lambda_n} \end{pmatrix} Q^T$$

$$= BB, \text{ where } B = Q \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \sqrt{\lambda_n} \end{pmatrix} Q^T$$

$\Rightarrow Q'BQ$ is p.d.

$\Rightarrow B$ is p.d.

Ex. If A and B are two square matrices, then show that AB and BA have the same characteristic roots.

Soln. $\rightarrow$ If rank$(A^{n\times n}) = r$, then $\exists$ two n.s. matrices P and $Q \ni PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

We have, $PABP^{-1} = PAQQ^{-1}BP^{-1}$

$$= \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

$\left[\text{where, } Q^{-1} B^{n\times n} P^{-1} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \; C_{11} \text{ is of order } r \times r \right]$

$$= \begin{pmatrix} C_{11}^{r\times n} & C_{12} \\ 0 & 0 \end{pmatrix}$$

Now, $Q^{-1}BAQ = Q^{-1}BP^{-1}PAQ$

$$= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_{11} & 0 \\ C_{21} & 0 \end{pmatrix}$$

Note that, the ch. equation of AB is

$$0 = |AB - \lambda I_n|$$

$$= |PABP^{-1} - \lambda I_n|$$

$$= \left| \begin{pmatrix} C_{11} & C_{12} \\ 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} \right|$$

$$\left[\because |AB - \lambda I_n| = |P^{-1}|\,|PABP^{-1} - \lambda P P^{-1}|\,|P| \right.$$

$$= \frac{1}{|P|} |PABP^{-1} - \lambda I_n|\,|P|$$

$$\left. = |PABP^{-1} - \lambda I_n| \right]$$

$$= \begin{vmatrix} C_{11} - \lambda I_r & C_{12} \\ 0 & -\lambda I_{n-r} \end{vmatrix}$$

$$= |C_{11} - \lambda I_r|\,|-\lambda I_{n-r}|$$

$$= (-\lambda)^{n-r} |C_{11} - \lambda I_r|$$

and the ch. equn. of BA is

$$0 = |BA - \lambda I_n| = |Q^{-1} BAQ - \lambda I_n|$$

$$= \begin{vmatrix} C_{11} - \lambda I_r & 0 \\ 0 & -\lambda I_{n-r} \end{vmatrix}$$

$$= (-1)^{n-r} |C_{11} - \lambda I_r|$$

Since AB & BA have the same characteristic equation, they have the same set of eigen values.

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Problem:- 1. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$. If $A = P^{-1}DP$, then the matrix D is equal to

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. (C) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solution:-

A and P are non-singular matrices.

If $A = P^{-1}DP$

then D be a diagonal matrix having eigen values as diagonal entry.

then we find eigen values of A.

characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)\left[(1-\lambda)^2 - 1\right] = 0$$

$$\Rightarrow \lambda(3-\lambda)(\lambda - 2) = 0$$

$\therefore \lambda = 0, 2, 3.$

So, D becomes $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2. A real quadratic form $\underset{\sim}{X}'A\underset{\sim}{X}$ is positive definite if

(A) All eigen values of $A > 0$ (B) All eigen values of $A < 0$
(C) All eigen values of $A = 0$ (D) None.

Solution:- $Q(X) = X'AX$ is positive definite

$\Leftrightarrow Q(\underset{\sim}{X}) \geq 0$; $\underset{\sim}{X} \neq 0$

$\Leftrightarrow$ All eigen values of $A > 0$.

END