VECTORS AND VECTOR SPACES

FIELD :→ Suppose there is a set F of objects $x, y, z, \ldots$ and two operations on the elements of F as follows. The first operation, called addition, associates with each pair of elements x, y in F, an element $(x+y)$, the second operation called multiplication, associates with each pair x, y, an element xy, and these two operations satisfy the following properties :

i) PROPERTIES OF ADDITION :→

(a) Closure : $x \in F,\ y \in F \Rightarrow x+y \in F$.

(b) Commutative : $x+y = y+x \Rightarrow x, y \in F$.

(c) Associative : $x+(y+z) = (x+y)+z,\ \forall\ x, y, z \in F$.

(d) Nutral element : There is an unique element zero (0) in F $\ni$
$$x+0 = x\ \forall\ x \in F.$$

(e) Inverse : To each $x \in F$, there corresponds an unique element $(-x)$ in F $\ni$ $x+(-x) = 0$.

ii) PROPERTIES OF MULTIPLICATION :→

(a) Closure : $x \in F,\ y \in F, \Rightarrow xy \in F$.

(b) Commutative : $xy = yx \Rightarrow x, y \in F$.

(c) Associative : $x(yz) = (xy)z \Rightarrow x, y, z \in F$

(d) Nutral element : There is an unique non-zero element 1 in F
$$\ni\ x \times 1 = x\ \forall\ x \in F.$$

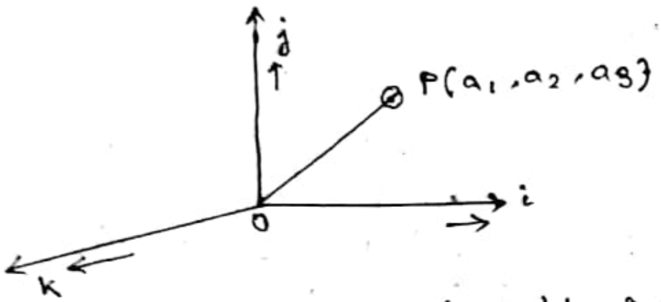
(e) Inverse : To each non-zero $x \in F$, there corresponds an unique element x^{-1} (or, $\frac{1}{x}$) in F $\ni$ $xx^{-1} = 1$.

iii) PROPERTIES OF ADDITION & MULTIPLICATION :→
(DISTRIBUTIVITY)

Multiplication distributes over addition, i.e.,
$$x \times (y+z) = xy + xz\ \forall\ x, y, z \in F$$

The set F together with these two operations is called a field.

Ex:→ $F = \{0, 1\}$.



A vector in elementary physics, is a physical quantity having both magnitude and direction.

$$\overline{OP} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} .$$

Instead of characterising a vector by magnitude and direction, an equally satisfactory description could be achieved by the terminal point of vector originated from the origin.

Hence, we write $\rightarrow$ $\underset{\sim}{a} = (a_1, a_2, a_3)$, where a_i's the ith component.

An ordered array of numbers :→ An ~~ordered~~ array of numbers $(a_1, a_2, a_3, \dots, a_n)$ is said to be an ordered array of numbers if $(a_{i_1}, a_{i_2}, a_{i_3}, \dots, a_{i_n})$ is not the same or equivalent to $(a_{j_1}, a_{j_2}, a_{j_3}, \dots, a_{j_n})$; where $(i_1, i_2, \dots, i_n)$ and $(j_1, j_2, \dots, j_n)$ are two different permutation of $(1, 2, \dots, n)$.

An ordered array of n-numbers $(a_1, a_2, \dots, a_n)$ will be called an ordered n-tuple.

Definition of Vector :→

1) An n-component vector $\underset{\sim}{a}$ is an ordered n-tuple written as a row $(\underset{\sim}{a}_1, \underset{\sim}{a}_2, \dots, \underset{\sim}{a}_n)$ or written as a column

$$\begin{pmatrix} a_1 \\ \underset{\sim}{a}_2 \\ \vdots \\ \underset{\sim}{a}_n \end{pmatrix} .$$

2) ~~An ordered set of elements~~

** An ordered n-tuple of real numbers specifies a point in an n-dimensional space is called an n-component vector.

3) An ordered set of elements of a field is called a vector; the elements are called components. A vector of n components is called an 'n-component vector' ~~and~~ or simply an 'n-vector'. An n-vector can be expressed in a horizontal or vertical line and in accordance, a row or column vector will appear.

TYPES :→

(a) Null Vector :

$$\underset{\sim}{0} = (0, 0, \ldots 0)'$$

all of those elements are zero.

(b) Unit Vector :

$$\underset{\sim}{e}_1 = (1, 0, 0, \ldots\ldots, 0)$$
$$\underset{\sim}{e}_2 = (0, 1, 0, \ldots\ldots, 0)$$
$$\vdots$$
(ith element)
$$\underset{\sim}{e}_i = (0, 0, 0, \ldots, 1, 0, \ldots, 0) \qquad [\bullet\ a_i' \underset{\sim}{e}_i = a_i \quad \forall\ i = 1(1)n.]$$
$$\vdots$$
$$e_n = (0, 0, 0, \ldots\ldots, 0, 1)$$

—are called the unit vectors.

(c) Sum Vector :

$$\underset{\sim}{1} = (1, 1, \ldots\ldots 1)'$$

all of ~~th~~ whose components are unity. $[\bullet\ \underset{\sim}{1}.\underset{\sim}{x} = \sum_{i=1}^{n} x_i]$

VECTOR OPERATIONS :→

Let $\underset{\sim}{a} = (a_1, a_2, \ldots\ldots, a_n)'$ and $\underset{\sim}{b} = (b_1, b_2, \ldots, b_n)'$ be two n-component vectors.

(a) Equality : Then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are said to be equal iff.

$\underset{\sim}{a}_i = \underset{\sim}{b}_i \quad \forall \quad i = 1(1)n$. Then we can say, $\underset{\sim}{a} = \underset{\sim}{b}$.

NOTE : The vectors $(1, 2)$ and $(1, 2, 0)$ are not equal. Two vectors can't be equal unless they have the same number of components.

(b) Addition : The sum of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is defined as

$$\underset{\sim}{a} + \underset{\sim}{b} = (a_1 + b_1, a_2 + b_2, \ldots\ldots, a_n + b_n)'$$

NOTE : This definition is appied only to the vectors which have equal number of components.

(c) Scalar Multiplication : The product of a scalar λ (constant real no.) and a vector $\underset{\sim}{a}$ is defined as

$$\lambda \underset{\sim}{a} = (\lambda \underset{\sim}{a}_1, \lambda \underset{\sim}{a}_2, \ldots\ldots, \lambda \underset{\sim}{a}_n)'$$

(d) Subtraction :

$$\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{a} + (-1)\underset{\sim}{b}$$
$$= (a_1, a_2, \ldots\ldots, a_n)' + (-b_1, -b_2, \ldots, -b_n)'$$
$$= (a_1 - b_1, a_2 - b_2, \ldots\ldots, a_n - b_n)'$$

Some Geometrical Concepts :

(a) Scalar Product : The scalar product of two vectors

$\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is defined to be scalar if

$$\underset{\sim}{a}'\underset{\sim}{b} = \sum_{i=1}^{n} a_i b_i .$$

PROPERTIES :

1) $\underset{\sim}{a}'\underset{\sim}{b} = \underset{\sim}{b}'\underset{\sim}{a}$

2) $(\lambda\underset{\sim}{a})\cdot\underset{\sim}{b} = \lambda(\underset{\sim}{a}\cdot\underset{\sim}{b})$

3) $\underset{\sim}{a}(\underset{\sim}{b}+\underset{\sim}{c}) = \underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{c}$

(b) Distance : The distance of a vector (or a point) $\underset{\sim}{a}$ to the vector (or the point) $\underset{\sim}{b}$ is defined as the scalar,

$$|\underset{\sim}{a}-\underset{\sim}{b}| = \sqrt{(a-b)'(a-b)}$$
$$= \sqrt{\sum_{i=1}^{n}(a_i-b_i)^2}$$

PROPERTIES :

1) $|\underset{\sim}{a}-\underset{\sim}{b}| = |\underset{\sim}{b}-\underset{\sim}{a}|$

2) $|\underset{\sim}{a}-\underset{\sim}{b}| \geq 0$

3) $|\underset{\sim}{a}-\underset{\sim}{b}|+|\underset{\sim}{b}-\underset{\sim}{c}| \geq |\underset{\sim}{c}-\underset{\sim}{a}|$

RESULT :→ Prove that for any two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$,
$(\underset{\sim}{a}\cdot\underset{\sim}{b})^2 \leq |\underset{\sim}{a}|^2 \cdot |\underset{\sim}{b}|^2$. [Cauchy - Schwartz Inequality]

Proof :→ For any scalar λ,

$$|\lambda\underset{\sim}{a}+\underset{\sim}{b}| \geq 0$$

$$\Leftrightarrow |\lambda\underset{\sim}{a}+\underset{\sim}{b}|^2 \geq 0 \qquad \left[\because \underset{\sim}{a}'\underset{\sim}{a} = \sum_{i=1}^{n} a_i^2 = |\underset{\sim}{a}|^2\right]$$

$$\Leftrightarrow (\lambda\underset{\sim}{a}+\underset{\sim}{b})(\lambda\underset{\sim}{a}+\underset{\sim}{b}) \geq 0$$

$$\Leftrightarrow \lambda^2|\underset{\sim}{a}|^2 + 2\lambda(\underset{\sim}{a}\cdot\underset{\sim}{b}) + |\underset{\sim}{b}|^2 \geq 0$$

$$\Leftrightarrow |\underset{\sim}{a}|^2\left\{\lambda^2 + 2\lambda\cdot\frac{(\underset{\sim}{a}\cdot\underset{\sim}{b})}{|\underset{\sim}{a}|^2}\right\} + |\underset{\sim}{b}|^2 \geq 0$$

$$\Leftrightarrow |\underset{\sim}{a}|^2\left\{\lambda + \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}|^2}\right\}^2 + |\underset{\sim}{b}|^2 - \frac{(\underset{\sim}{a}\cdot\underset{\sim}{b})^2}{|\underset{\sim}{a}|^2} \geq 0$$

$$\Leftrightarrow |\underset{\sim}{a}|^2\left\{\lambda + \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}|^2}\right\}^2 + \frac{|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2 - (\underset{\sim}{a}\cdot\underset{\sim}{b})^2}{|\underset{\sim}{a}|^2} \geq 0$$

For, $\lambda = -\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}|^2}$, then $\frac{|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2 - (\underset{\sim}{a}\cdot\underset{\sim}{b})^2}{|\underset{\sim}{a}|^2} \geq 0$

$$\Rightarrow \boxed{(\underset{\sim}{a}\cdot\underset{\sim}{b})^2 \leq |\underset{\sim}{a}|^2\cdot|\underset{\sim}{b}|^2}$$, provided $\underset{\sim}{a}$ & $\underset{\sim}{b}$ have finite length.

'=' holds iff $|\lambda \underset{\sim}{a} + \underset{\sim}{b}| = 0$ for some λ.

iff $\lambda \underset{\sim}{a} + \underset{\sim}{b} = 0$.

iff $\underset{\sim}{b} = -\lambda \underset{\sim}{a}$, for some λ.

iff $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are collinear.

Remark → C-S inequality :-

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i\right)^2 \left(\sum_{i=1}^{n} b_i\right)^2$$

'=' holds iff $b_i = -\lambda a_i \;\forall\; i = 1(1)n$.

iff $b_i \propto a_i \;\forall\; i = 1(1)n$.

RESULT :→ For any three vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, $|\underset{\sim}{a} - \underset{\sim}{b}| + |\underset{\sim}{b} - \underset{\sim}{c}| \geq |\underset{\sim}{c} - \underset{\sim}{a}|$

[Triangle Inequality]

Proof :→ $|\underset{\sim}{c} - \underset{\sim}{a}|^2$

$= |(\underset{\sim}{a} - \underset{\sim}{b}) + (\underset{\sim}{b} - \underset{\sim}{c})|^2$

$= (\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{b} - \underset{\sim}{c}) \cdot (\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{b} - \underset{\sim}{c})$

$= (\underset{\sim}{a} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) + 2(\underset{\sim}{a} - \underset{\sim}{b}) \cdot (\underset{\sim}{b} - \underset{\sim}{c}) + (\underset{\sim}{b} - \underset{\sim}{c})(\underset{\sim}{b} - \underset{\sim}{c})$

$= |\underset{\sim}{a} - \underset{\sim}{b}|^2 + 2 \cdot (\underset{\sim}{a} - \underset{\sim}{b})(\underset{\sim}{b} - \underset{\sim}{c}) + |\underset{\sim}{b} - \underset{\sim}{c}|^2$

$\le |\underset{\sim}{a} - \underset{\sim}{b}|^2 + 2\,|\underset{\sim}{a} - \underset{\sim}{b}|\,|\underset{\sim}{b} - \underset{\sim}{c}| + |\underset{\sim}{b} - \underset{\sim}{c}|^2$ [Applying C-S inequality]

[$\because \underset{\sim}{a} \cdot \underset{\sim}{b} \le |\underset{\sim}{a} \cdot \underset{\sim}{b}| \le |\underset{\sim}{a}| \cdot |\underset{\sim}{b}|$]

$\therefore |\underset{\sim}{c} - \underset{\sim}{a}|^2 \le \{|\underset{\sim}{a} - \underset{\sim}{b}| + |\underset{\sim}{b} - \underset{\sim}{c}|\}^2$

$\Rightarrow |\underset{\sim}{a} - \underset{\sim}{b}| + |\underset{\sim}{b} - \underset{\sim}{c}| \geq |\underset{\sim}{c} - \underset{\sim}{a}|$

'=' holds iff $\underset{\sim}{b} - \underset{\sim}{c} = \lambda(\underset{\sim}{a} - \underset{\sim}{b})$ for some λ.

iff $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ are collinear.

(c) Length (Norm) :- The length of a vector $\underset{\sim}{a}$ is the distance between $\underset{\sim}{a}$ and the origin $\underset{\sim}{0}$.

$\therefore |\underset{\sim}{a}| = |\underset{\sim}{a} - \underset{\sim}{0}|$

$= \sqrt{\underset{\sim}{a}'\underset{\sim}{a}}$

$= \sqrt{\sum_i a_i^2}$

$\therefore |\underset{\sim}{a}|^2 = \underset{\sim}{a}' \cdot \underset{\sim}{a}$.

[C.U.]

(d) **Angle :** The angle (θ) between two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, where $\underset{\sim}{a}, \underset{\sim}{b} \neq 0$, is given by

$$\cos\theta = \frac{\underset{\sim}{a}' \underset{\sim}{b}}{|\underset{\sim}{a}| \cdot |\underset{\sim}{b}|} = \frac{\sum_{i=1}^{n} a_i b_i}{\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}}$$

REMARK: The angle (θ) between two non-null vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is given by $\cos\theta = \frac{\underset{\sim}{a}' \underset{\sim}{b}}{|\underset{\sim}{a}| |\underset{\sim}{b}|}$.

Here for any θ, $|\cos\theta| \leq 1$

$$\Rightarrow \cos^2\theta \leq 1$$

$$\Rightarrow \left(\frac{\underset{\sim}{a}' \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right)^2 \leq 1$$

$$\Rightarrow (\underset{\sim}{a}' \underset{\sim}{b})^2 \leq |\underset{\sim}{a}|^2 |\underset{\sim}{b}|^2$$

Provided $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have finite length.

It is C-S inequality, here '=' holds if $\underset{\sim}{a} = \lambda \underset{\sim}{b}$ for some scalar λ.

Some interpretations by vector operations :

Let $x_1, x_2, \ldots, x_n$ be n-values of a variable x. Then,

$$\bar{x} = \sum_{i=1}^{n} x_i / n.$$

$$= \frac{1}{n} \cdot \underset{\sim}{1}' \underset{\sim}{x} \quad [\text{where } \underset{\sim}{x} = (x_1, x_2, \ldots, x_n)]$$

The deviations $\Rightarrow x_1 - \bar{x}, x_2 - \bar{x}, x_3 - \bar{x}, \ldots\ldots, x_n - \bar{x}$.

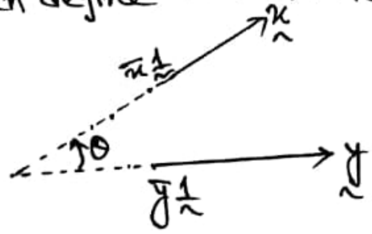
The deviation can be represented as

$$\underset{\sim}{x} - \bar{x} \cdot \underset{\sim}{1} = (x_1, \ldots, x_n) - (\bar{x}, \bar{x}, \ldots, \bar{x})$$
$$= (x_1 - \bar{x}, \ldots\ldots, x_n - \bar{x})$$

As a measure of dispersion, we take the distance between $\underset{\sim}{x}$ and $\bar{x}\underset{\sim}{1}$, i.e. the length of deviations vector

$$|\underset{\sim}{x} - \bar{x}\underset{\sim}{1}| = \sqrt{\sum_i (x_i - \bar{x})^2}$$

Let, $(x_1, y_1), \ldots\ldots, (x_n, y_n)$ be n pairs of values on x & y. Then define the deviation vectors for $\underset{\sim}{x}$ and $\underset{\sim}{y}$.



$$\underset{\sim}{x} - \bar{x} \cdot \underset{\sim}{1} = \underset{\sim}{d}_1 \text{ and}$$

$$\underset{\sim}{y} - \bar{y}\underset{\sim}{1} = \underset{\sim}{d}_2,$$

If θ, the angle between $\underset{\sim}{d_1}$ and $\underset{\sim}{d_2}$, is 0 then x and y are on a line. The smaller the angle, the more the vectors $\underset{\sim}{d_1}$ and $\underset{\sim}{d_2}$ closer to a line.

As a measure of linear relationship, we define $\cos\theta$ as

$$\cos\theta = \frac{\underset{\sim}{d_1} \cdot \underset{\sim}{d_2}}{|\underset{\sim}{d_1}||\underset{\sim}{d_2}|}$$

$$= \frac{(\underset{\sim}{x} - \bar{x}\underset{\sim}{1})(\underset{\sim}{y} - \bar{y}\underset{\sim}{1})}{|\underset{\sim}{x} - \bar{x}\underset{\sim}{1}||\underset{\sim}{y} - \bar{y}\underset{\sim}{1}|}$$

$$= \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2}\sqrt{\sum_i (y_i - \bar{y})^2}}$$

$= r_{xy}$, which is the correlation coefficient between x and y.

PROPERTY: $-1 \leq r_{xy} \leq 1$.

Interpret the cases:
1) $r_{xy} = 0$ where $\theta = 90^\circ$
2) $r_{xy} = 1$ where $\theta = 0$
3) $r_{xy} = -1$ where $\theta = 180^\circ$.

Linear Combination :→

An n-vector $\underset{\sim}{a}$ is said to be a linear combination of the n-vectors $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n$ if for some scalars $\lambda_i, (i = 1, 2, \ldots, n)$, $\underset{\sim}{a}$ can be written as

$$\underset{\sim}{a} = \lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \cdots + \lambda_n \underset{\sim}{a}_n.$$

Euclidean Space :→

An n-dimensional Euclidean space is the collection of all n-component vectors such that:

i) for any $\underset{\sim}{a}, \underset{\sim}{b} \in E^n$; $\underset{\sim}{a} + \underset{\sim}{b} \in E^n$. → Closure property w.r.t. Addition.

ii) for any $\underset{\sim}{a} \in E^n$ and for any scalar λ, $\lambda \underset{\sim}{a} \in E^n$. → Closure property w.r.t. scalar multiplication.

iii) There is a non-negative quantity associated with any two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ called distance between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

In E^n, a vector $\underset{\sim}{a} \in E^n$ is a point in n-dimension Geometry.

C.U.

LINEAR DEPENDENCE :

A set of vectors $\{a_1, a_2, \ldots, a_n\}$ from E^n is said to be linearly dependent if $\exists$ scalars $\lambda_i, (i = 1(1)n)$, not all zero, such that

$$\lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \cdots + \lambda_n \underset{\sim}{a}_n = 0$$

C.U.

LINEAR INDEPENDENCE :

A set of vectors $\{b_1, b_2, \ldots, b_n\}$ is said to be linearly independent if $\lambda_i = 0, (i = 1(1)n)$, is the only solution of

$$\lambda_1 \underset{\sim}{b}_1 + \lambda_2 \underset{\sim}{b}_2 + \cdots + \lambda_n \underset{\sim}{b}_n = \underset{\sim}{0}$$

Examples :

1) Show that $(0,0,1)$, $(0,1,1)$ and $(1,1,1)$ from E^3 are linearly independent.

Soln. →

Let, $\lambda_1(0,0,1) + \lambda_2(0,1,1) + \lambda_3(1,1,1) = \underset{\sim}{0}$ ——— $(*)$

$\Rightarrow (0, 0, \lambda_1) + (0, \lambda_2, \lambda_2) + (\lambda_3, \lambda_3, \lambda_3) = (0, 0, 0)$

$\Rightarrow (\lambda_3, \lambda_2 + \lambda_3, \lambda_1 + \lambda_2 + \lambda_3) = (0, 0, 0)$

$$\Rightarrow \begin{cases} \lambda_3 = 0 \\ \lambda_2 + \lambda_3 = 0 \\ \lambda_1 + \lambda_2 + \lambda_3 = 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0.$$

Hence, $\lambda_1 = \lambda_2 = \lambda_3 = 0$ is the only solution of $(*)$.
Therefore, the vectors are linearly independent.

2) Show that the unit vectors from E^n are linearly independent.

Soln. → Let $\{e_1, e_2, \dots, e_n\}$ be a set of unit vectors from E^n. If $\exists$ scalars λ_i, $i = 1(1)n$, then we can write

$$\lambda_1 \underset{\sim}{e_1} + \lambda_2 \underset{\sim}{e_2} + \dots + \lambda_n \underset{\sim}{e_n} = \underset{\sim}{0} \quad \text{———} (*)$$

$\Rightarrow \lambda_1(1, 0, 0, \dots 0) + \lambda_2(0, 1, 0, \dots, 0) + \dots + \lambda_n(0, \dots, 0, 1) = \underset{\sim}{0}$

$\Rightarrow (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n) = (0, 0, 0, \dots, 0)$

$\therefore \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \dots, \lambda_n = 0.$

$\Rightarrow \lambda_i = 0 \ \forall \ i = 1(1)n$, is the only solution of $(*)$.

$\therefore \{e_1, e_2, \dots, e_n\}$ is linearly independent.

3) Show that $(1,-1,0,0)$, $(1,0,-1,0)$, $(1,0,0,-1)$, $(3,-1,-1,-1)$ are linearly dependent.

Soln. →

Let, $\lambda_1(1,-1,0,0) + \lambda_2(1,0,-1,0) + \lambda_3(1,0,0,-1) + \lambda_4(3,-1,-1,-1) = \underset{\sim}{0}$

$\Rightarrow (\lambda_1 + \lambda_2 + \lambda_3 + 3\lambda_4, -\lambda_1 - \lambda_4, -\lambda_2 - \lambda_4, -\lambda_3 - \lambda_4) = (0,0,0,0)$

$\Rightarrow \begin{cases} \lambda_1 = \lambda_2 = \lambda_3 = -\lambda_4 \\ \text{In particular, } \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, \lambda_4 = -1. \end{cases}$

$\boxed{\text{C.U}}$ Hence the vectors are linearly dependent.

4) Examine whether the following vectors are linearly dependent or not ?

i) $\underset{\sim}{\alpha_1} = (4,3,5)$; $\underset{\sim}{\alpha_2} = (1,0,4)$; $\underset{\sim}{\alpha_3} = (3,6,1)$

ii) $\underset{\sim}{\alpha_4} = (2,0,5)$; $\underset{\sim}{\alpha_5} = (1,3,2)$; $\underset{\sim}{\alpha_6} = (2,-12,7)$

Soln. →

Let, $\lambda_1(4,3,5) + \lambda_2(1,0,4) + \lambda_3(3,6,1) + \lambda_4(2,0,5) + \lambda_5(1,3,2) + \lambda_6(2,-12,7) = \underset{\sim}{0}$

$\Rightarrow (4\lambda_1 + \lambda_2 + 3\lambda_3 + 2\lambda_4 + \lambda_5 + 2\lambda_6,\ 3\lambda_1 + 6\lambda_3 + 3\lambda_5 - 12\lambda_6,\ 5\lambda_1 + 4\lambda_2 + \lambda_3 + 5\lambda_4 + 2\lambda_5 + 7\lambda_6) = (0,0,0)$

$$\Rightarrow \lambda_1\begin{pmatrix}4\\3\\5\end{pmatrix} + \lambda_2\begin{pmatrix}1\\0\\4\end{pmatrix} + \lambda_3\begin{pmatrix}3\\6\\1\end{pmatrix} + \lambda_4\begin{pmatrix}2\\0\\5\end{pmatrix} + \lambda_5\begin{pmatrix}1\\3\\2\end{pmatrix} + \lambda_6\begin{pmatrix}2\\-12\\7\end{pmatrix} = 0$$

$$\text{or, } \begin{pmatrix}4 & 1 & 3 & 2 & 1 & 2\\ 3 & 0 & 6 & 0 & 3 & -12\\ 5 & 4 & 1 & 5 & 2 & 7\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\\ \lambda_5\\ \lambda_6\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 & 3 & 2 & 1 & 2 \\ 3 & 0 & 6 & 0 & 3 & -12 \\ 5 & 4 & 1 & 5 & 2 & 7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -3 & 2 & -2 & 14 \\ 1 & 0 & 2 & 0 & 1 & -4 \\ 0 & 4 & -9 & 5 & -3 & 27 \end{pmatrix}$$

① $R_2' = R_2/3$
② $R_1' = R_1 - R_2$
③ $R_3' = R_3 - 5R_2'$

$$\sim \begin{pmatrix} 1 & 1 & -3 & 2 & -2 & 14 \\ 0 & 1 & -5 & 2 & -3 & 18 \\ 0 & 4 & -9 & 5 & -3 & 27 \end{pmatrix}$$

$R_2' = R_1 - R_2$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & -4 \\ 0 & 1 & -5 & 2 & -3 & 18 \\ 0 & 0 & 11 & -3 & 9 & -45 \end{pmatrix}$$

$R_1' = R_1 - R_2$
$R_3' = R_3 - 4R_2$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & -4 \\ 0 & 1 & -4/3 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3/11 & 9/11 & -45/11 \end{pmatrix}$$

$R_2' = R_2 + \frac{1}{3} R_3$
$R_3' = R_3/11$

$= H$, an echelon matrix

$H\underset{\sim}{x} = \underset{\sim}{0}$ gives $\rightarrow$

$$\begin{cases} \lambda_1 + 2\lambda_3 + \lambda_5 - 4\lambda_6 = 0 \\ \lambda_2 - 4/3\, \lambda_3 + \lambda_4 + 3\lambda_6 = 0 \\ \lambda_3 - \frac{3}{11}\lambda_4 + 9/11\, \lambda_5 - \frac{45}{11}\lambda_6 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_1 + 2\lambda_3 + \lambda_5 - 4\lambda_6 = 0 \\ 3\lambda_2 - 4\lambda_3 + 3\lambda_4 + 9\lambda_6 = 0 \\ 11\lambda_3 - 3\lambda_4 + 9\lambda_5 - 45\lambda_6 = 0 \end{cases}$$

---

9) $\lambda_1(4, 3, 5) + \lambda_2(1, 0, 4) + \lambda_3(3, 6, 1) = 0$ - - - (*)

$\Rightarrow (4\lambda_1 + \lambda_2 + 3\lambda_3,\ 3\lambda_1 + 6\lambda_3,\ 5\lambda_1 + 4\lambda_2 + \lambda_3) = (0, 0, 0)$

$\Rightarrow 4\lambda_1 + \lambda_2 + 3\lambda_3 = 0 \Rightarrow \lambda_2 - 5\lambda_3 = 0 \Rightarrow 4\lambda_2 - 20\lambda_3 = 0$

$\lambda_1 + 2\lambda_3 = 0 \Rightarrow \lambda_1 = -2\lambda_3$

$5\lambda_1 + 4\lambda_2 + \lambda_3 = 0 \Rightarrow 4\lambda_2 - 9\lambda_3 = 0$

$\Rightarrow \lambda_3 = 0,\ \lambda_1 = 0,\ \lambda_2 = 0$ is the only soln. of (*).

$\therefore$ the vectors are LIN.

Theorem 1. A set of vectors $\{a_1, a_2, \cdots, a_k\}$ from E^n is linearly dependent if and only if one of these vectors can be written as a linear combination of the others.

Proof:

- **Only if part:** Let $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \cdots, \underset{\sim}{a}_k$ are linearly dependent vectors from E^n. Then $\exists$ scalars λ_i's, not all zero, such that

$$\lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \cdots + \lambda_k \underset{\sim}{a}_k = \underset{\sim}{0} \quad \text{———} (*)$$

Let, non-zero scalar be λ_i, so $(*)$ implies —

$$-\lambda_i \underset{\sim}{a}_i = \lambda_1 \underset{\sim}{a}_1 + \cdots + \lambda_{i-1} \underset{\sim}{a}_{i-1} + \lambda_{i+1} \underset{\sim}{a}_{i+1} + \cdots + \lambda_k a_k$$

$$\Rightarrow \underset{\sim}{a}_i = \left(-\frac{\lambda_1}{\lambda_i}\right) \underset{\sim}{a}_1 + \left(-\frac{\lambda_2}{\lambda_i}\right) \underset{\sim}{a}_2 + \cdots + \left(-\frac{\lambda_{i-1}}{\lambda_i}\right) \underset{\sim}{a}_{i-1} + \left(-\frac{\lambda_{i+1}}{\lambda_i}\right) \underset{\sim}{a}_{i+1} + \cdots + \left(-\frac{\lambda_k}{\lambda_i}\right) a_k$$

$$= \sum_{j(\neq i)=1}^{k} \left(-\frac{\lambda_j}{\lambda_i}\right) \underset{\sim}{a}_j,$$ a linear combination of the vectors $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \cdots, \underset{\sim}{a}_{i-1}, \underset{\sim}{a}_{i+1}, \cdots, a_k$.

Hence, one vector $\underset{\sim}{a}_i$ has been written as a linear combination of the others.

- **If part:** Let $\underset{\sim}{a}_i$ can be written as a linear combination of the other vectors, giving —

$$\underset{\sim}{a}_i = \lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \cdots + \lambda_{i-1} \underset{\sim}{a}_{i-1} + \lambda_{i+1} \underset{\sim}{a}_{i+1} + \cdots + \lambda_k \underset{\sim}{a}_k$$

for some scalars λ_i's.

$$\Rightarrow \lambda_1 \underset{\sim}{a}_1 + \cdots + \lambda_{i-1} \underset{\sim}{a}_{i-1} + (-1) \underset{\sim}{a}_i + \lambda_{i+1} \underset{\sim}{a}_{i+1} + \cdots + \lambda_k \underset{\sim}{a}_k = \underset{\sim}{0}$$

$\Rightarrow \lambda_1 \underset{\sim}{a}_1 + \cdots + \lambda_k \underset{\sim}{a}_k = \underset{\sim}{0}$ is satisfied for $\lambda_i = -1 (\neq 0)$, a non-zero scalar.

So, $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \cdots, \underset{\sim}{a}_k$ are linearly dependent.

Hence, the set of vectors is linearly dependent.

C.U.

Theorem 2. If $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_m\}$ is linearly independent and $\{\hat{a}, \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_m\}$ is linearly dependent, then $\hat{a}$ is a linear combination of $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_m$.

Proof: Since $\{\hat{a}, \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_m\}$ is linearly dependent. Then $\exists$ scalars $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_m$, not all zero, such that

$$\lambda \hat{a} + \lambda_1 \hat{a}_1 + \lambda_2 \hat{a}_2 + \cdots + \lambda_m \hat{a}_m = \underset{\sim}{0} \quad \text{——— (1)}$$

If $\lambda = 0$, then (1) implies that —

$$\lambda_1 \hat{a}_1 + \lambda_2 \hat{a}_2 + \cdots + \lambda_m \hat{a}_m = \underset{\sim}{0} \text{ for at least one } \lambda_i \neq 0.$$

which shows that $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_m\}$ is linearly dependent. But this contradicts our giving condition, so λ is always non zero.

As $\lambda \neq 0$, then (1) gives $\underset{\sim}{a} = -\frac{\lambda_1}{\lambda}\hat{a}_1 - \frac{\lambda_2}{\lambda}\hat{a}_2 - \cdots - \frac{\lambda_m}{\lambda}\hat{a}_m$

i.e. $\underset{\sim}{a}$ is a linear combination of $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_m$.

Hence, the proof.

Example: 1) Show that the set of vectors:

$$\hat{x}_1 = (1, -1, 0, 0, \ldots, 0, 0)$$
$$\hat{x}_2 = (1, 0, -1, 0, \ldots, 0, 0)$$
$$\hat{x}_3 = (1, 0, 0, -1, \ldots, 0, 0)$$
$$\vdots$$
$$\hat{x}_{n-1} = (1, 0, 0, 0, \ldots, 0, -1)$$
$$\hat{x}_n = (n-1, -1, -1, \ldots, -1, -1)$$

is linearly dependent. Also find a linearly independent set of vectors and determine the maximum number of linearly independent vectors in the set.

Soln. Note that — $\hat{x}_n = \hat{x}_1 + \hat{x}_2 + \cdots + \hat{x}_{n-1}$

$\Rightarrow \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n\}$ are linearly dependent.

Now, we consider the equation $\sum_{i=1}^{n-1} \lambda_i \underset{\sim}{x}_i = \underset{\sim}{0}$

$$\Rightarrow \left(\sum_{i=1}^{n-1} \lambda_i, -\lambda_1, -\lambda_2, \ldots, -\lambda_{n-1}\right) = (0, 0, \ldots, 0)$$

$$\Rightarrow \begin{cases} \sum_{i=1}^{n-1} \lambda_i = 0 \\ -\lambda_1 = 0 \\ \vdots \\ -\lambda_{n-1} = 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$$

Hence, $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_{n-1}\}$ is linearly independent.
Here, the collection of n vectors $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_{n-1}, \underset{\sim}{x}_n\}$ is linearly dependent but the collection of $(n-1)$ vectors $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_{n-1}\}$ is linearly independent.

2) If $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n$ are linearly independent and $\sum_{i=1}^{n} \alpha_i \underset{\sim}{x}_i = \sum_{i=1}^{n} \beta_i \underset{\sim}{x}_i$, then show that $\alpha_i = \beta_i \;\; \forall\, i = 1(1)n$.

Soln. →

$$\sum_{i=1}^{n} \alpha_i \underset{\sim}{x}_i = \sum_{i=1}^{n} \beta_i \underset{\sim}{x}_i$$

$$\Rightarrow \sum_{i=1}^{n} (\alpha_i - \beta_i)\underset{\sim}{x}_i = 0$$

$$\Rightarrow \alpha_i - \beta_i = 0 \;\; \forall\, i = 1(1)n.$$

since x_i's are linearly independent.

$$\Rightarrow \alpha_i = \beta_i \;\; \forall\, i = 1(1)n.$$

3) If $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are linearly independent. Then show that $\underset{\sim}{x} + \alpha \underset{\sim}{y}$ and $\underset{\sim}{x} + \beta \underset{\sim}{y}$ are also linearly independent if $\alpha \neq \beta$.

Soln. → Let, $L_1(\underset{\sim}{x} + \alpha \underset{\sim}{y}) + L_2(\underset{\sim}{x} + \beta \underset{\sim}{y}) = \underset{\sim}{0}$

$$\Rightarrow (L_1 + L_2)\underset{\sim}{x} + (L_2\beta + L_1\alpha)\underset{\sim}{y} = \underset{\sim}{0}$$

$\Rightarrow L_1 + L_2 = 0$ & $L_1\alpha + L_2\beta = 0$, since $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are linearly independent.

$$\Rightarrow L_1 = -L_2 \;\; \& \;\; L_1(\alpha - \beta) = 0.$$

$$\Rightarrow L_1 = 0,\; L_2 = 0 \;\; \text{since } \alpha \neq \beta.$$

Hence $(\underset{\sim}{x} + \alpha \underset{\sim}{y})$ and $(\underset{\sim}{x} + \beta \underset{\sim}{y})$ are linearly independent if $\alpha \neq \beta$.

C.U

Theorem 3. Show that a set of vectors containing a null vector can't be linearly independent.

Proof: Let $\{\underset{\sim}{0}, \underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ be a set containing a null vector.

For, $\lambda = 5$, $\lambda_1 = 0$, $\lambda_2 = 0$, ..., $\lambda_n = 0$, the equation

$$5\underset{\sim}{0} + \lambda_1 \underset{\sim}{a_1} + \lambda_2 \underset{\sim}{a_2} + \cdots + \lambda_n \underset{\sim}{a_n} = \underset{\sim}{0}$$ is satisfied.

Hence, $\exists$ a non zero λ, so the set of vectors is linearly dependent.

$\therefore$ A set of vectors containing the null vector can't be linearly independent.

Theorem 4. Any n-vector must be linearly dependent on the unit vectors $\underset{\sim}{e_1}, \underset{\sim}{e_2}, \ldots, \underset{\sim}{e_n}$, which by themselves form an idependent set.

Proof: Let $\underset{\sim}{a}$ be an n-component vector, given by

$$\underset{\sim}{a} = (a_1, a_2, \ldots, a_n)$$

Since $\underset{\sim}{e_1} = (1, 0, \ldots 0)$, $\underset{\sim}{e_2} = (0\ 1\ 0 \ldots 0)$, ...

$$\ldots \underset{\sim}{e_n} = (0, 0, \ldots, 0, 1).$$

We can write $\underset{\sim}{a} = a_1\underset{\sim}{e_1} + a_2\underset{\sim}{e_2} + \cdots + a_n\underset{\sim}{e_n}$.

This shows that $\underset{\sim}{a}$ is, indeed, linearly dependent on e_i $(i = 1(1)n)$.
Since $\underset{\sim}{a}$ is arbitrarily chosen, the first part of the theorem is established.

~~Also suppose~~ second part has been proved earlier, i.e. Unit vectors are linearly independent.

C.U.

Theorem 5. A null vector is linearly dependent on any other set of non-null vectors.

Proof:

C.U. Theorem 6.

Any subset of a linearly independent set of vectors is also linearly independent.

<u>Proof</u>:→ Let $\{a_1, a_2, \ldots, a_n\}$ be linearly independent set of vectors. and $\{a_{i_1}, a_{i_2}, a_{i_3}, \ldots, a_{i_k}\}$ $(k < n)$ is a subset of it.

If possible, let this subset be linearly dependent, so $\exists$ some scalars $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k}$, not all zero, such that

$$\lambda_{i_1} \underset{\sim}{a}_{i_1} + \lambda_{i_2} \underset{\sim}{a}_{i_2} + \cdots + \lambda_{i_k} \underset{\sim}{a}_{i_k} = \underset{\sim}{0}$$

$$\Rightarrow \lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \cdots + \lambda_n \underset{\sim}{a}_n = \underset{\sim}{0} \text{ for at least one } \lambda_{i_j}\ (j = 1(1)k)$$

non-zero and other λ_i's are zero.

Therefore, $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n$ are linearly dependent which contradicts our original assumption.

Hence, any subset of a linearly independent set of vectors is linearly independent.

<u>Theorem 7.</u> Every superset of a linearly dependent set of vectors is also linearly dependent.

<u>Proof</u>:→ Let $\{a_1, a_2, \ldots, a_k\}$ be a linearly dependent set of vectors from $V_n(F)$ and $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_n\}$ be a superset of $\{a_1, a_2, \ldots, a_k\}$. As $\{a_1, a_2, \ldots, a_k\}$ is linearly dependent, $\exists$ scalar λ_i, not all zero, satisfying —

$$\lambda_1 \underset{\sim}{a}_1 + \cdots + \lambda_k \underset{\sim}{a}_k = \underset{\sim}{0}$$

$$\Rightarrow \lambda_1 \underset{\sim}{a}_1 + \cdots + \lambda_k \underset{\sim}{a}_k + 0 \cdot \underset{\sim}{b}_1 + \cdots + 0 \cdot \underset{\sim}{b}_n = \underset{\sim}{0} \text{ for at least one } \lambda_i \neq 0.$$

Hence, $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_n\}$ is linearly dependent.

So, every superset of linearly dependent set of vectors is linearly dependent.

<u>Result</u>:→ Suppose $k (< m)$ is the maximum number of linearly independent vectors in a set of m vectors. Then given any linearly independent subset of k vectors from this set, every other vector in the set can be written as a linear combination of these k vectors.

C.U.

VECTOR SPACE :

Definition :→ A vector space V_n is a collection of n-component vectors which is closed under the operations of addition and scalar multiplication.

i.e. i) If $\underset{\sim}{a}, \underset{\sim}{b} \in V_n$ then $\underset{\sim}{a} + \underset{\sim}{b} \in V_n$ [Closed under Addition]

ii) If $\underset{\sim}{a} \in V_n$, for any arbitrary scalar, $\lambda \underset{\sim}{a} \in V_n$ [Closed under Scalar multiplication]

Ex. → $V_2(R) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| x_1, x_2 \in R \right\}$ is a vector space.

REMARK :→ The concept of distance, scalar product and angle are not defined in vector spaces. Clearly, E^n satisfies the properties of a vector space, so E^n is a vector space. If we define length in V_n as in E^n, then V_n is identical with E^n. Although, E^n is a vector space, it does not follow that every V_n is E_n.

Example :→ Consider the collection of vectors of the form $(x_1, x_2, 0)$ from E^3 or V_3. Show that → the collection is a vector space.

Soln. Define $S_3 = \{(x_1, x_2, 0) : x_1, x_2 \in R\}$

Let $x, y \in S_3$.

Then $\underset{\sim}{x} = (x_1, x_2, 0)$ and $\underset{\sim}{y} = (y_1, y_2, 0)$

Now, $\underset{\sim}{x} + \underset{\sim}{y} = (x_1 + y_1, x_2 + y_2, 0) \in S_3$

and $\alpha \underset{\sim}{x} = (\alpha x_1, \alpha x_2, 0) \in S_3$

Hence, S_3 is closed under addition and scalar multiplication.

NOTE :– It is important to note that there are several subsets of E^n or V^n, which are itself a vector space.

VECTOR SUBSPACE :

C.U. **Definition :→**

1) A vector subspace S_n of an n-dimensional vector space V_n is a subset of V_n which is itself a vector space.

2) A non empty subset S_n of vectors from n-dimensional vector space V_n is called a subspace of V_n if S_n is closed under —

i) Addition, i.e. $\underset{\sim}{a} \in S_n$, $\underset{\sim}{b} \in S_n \Rightarrow \underset{\sim}{a}+\underset{\sim}{b} \in S_n$, and

ii) Scalar multiplication, i.e. $\lambda \in R$, $\underset{\sim}{a} \in S_n \Rightarrow \lambda \underset{\sim}{a} \in S_n$.

Ex :→ $S_2 = \{(x_1, x_2, x_3) : x_1, x_2 \in R\}$ is a vector subspace.

REMARK :→

(a) For a vector space V_n if $\underset{\sim}{x} \in V_n$, then $\alpha \underset{\sim}{x} \in V_n$ for any scalar $\lambda \in R$. For $\alpha = 0$, we get $\underset{\sim}{0} \in V_n$. Hence, the null vector is a member of a vector space.

(b) Any subspace of E^3 is either E^3 itself, a plane through the origin or just the origin itself.

C.U. Example :→ Show that S_1 is a vector subspace, where

$$S_1 = \{(x_1, x_2, x_3) : x_1 + x_2 = x_3\}.$$

Soln. → Clearly, $S_1 \in E^3$.

Consider two vectors $\underset{\sim}{x}$ and $y \in S$.

$\underset{\sim}{x} = (x_1, x_2, x_3)$, where $x_1 + x_2 = x_3$

$y = (y_1, y_2, y_3)$, where $y_1 + y_2 = y_3$.

Now, $\underset{\sim}{x} + y = (x_1+y_1, x_2+y_2, x_3+y_3)$ where $x_1+x_2 = x_3$, $y_1+y_2 = y_3$.

$\therefore \underset{\sim}{x} + y = (u_1, u_2, u_3)$, where $u_i = x_i + y_i$ and $u_1 + u_2 = u_3$.

$\Rightarrow \underset{\sim}{x} + y \in S$, and for any scalar λ,

$$\therefore \lambda \underset{\sim}{x} = (\lambda x_1, \lambda x_2, \lambda x_3)$$
$$= (\alpha_1, \alpha_2, \alpha_3), \text{ where } \alpha_i = \lambda x_i \text{ and}$$
$$\alpha_1 + \alpha_2 = \lambda(x_1 + x_2) = \lambda x_3 = \alpha_3.$$

i.e. $\lambda \underset{\sim}{x} \in S_1$.

Hence S_1 is a vector subspace of E^3.

Example: 1) In each of the following find out whether the subsets given, form subspaces of the vector space V_2:

i) $S = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$

ii) $T = \{(x_1, x_2) : x_1 x_2 \geq 0\}$

Soln.→

i) For $\alpha < 0$, $\alpha \underset{\sim}{x} = (\alpha x_1, \alpha x_2)$, where $\alpha x_1 \leq 0, \alpha x_2 \leq 0$.
$\notin S$.

So, it's not a subspace.

ii) Consider the vectors $(3, 2)$ and $(-2, -4)$ from T.
Note that $(3, 2) + (-2, -4) = (1, -2)$ for which $x_1 x_2 < 0 \notin T$.

So, T is not a subspace.

e.g. 2) Consider a subset S of $V_2(R)$:
$S = \{(x, y) \mid 0 < x, y < \infty\} \subset V_2(R)$. Is S a subspace of $V_2(R)$?

Ans:→ Consider any two vectors $\underset{\sim}{a}_1 = (a_{11}, a_{12})$ and $\underset{\sim}{a}_2 = (a_{21}, a_{22})$ from S.

Now, $\underset{\sim}{a}_1 + \underset{\sim}{a}_2 = (a_{11} + a_{21}, a_{12} + a_{22}) \in S$ because $a_{11} + a_{21}$ and $a_{21} + a_{22}$ are both positive, giving S is closed under addition.

But S is not closed under the operations of multiplication by a scalar λ, because for any vector $\underset{\sim}{a}_1 \in S$ and $\lambda \leq 0$, $\lambda \underset{\sim}{a}_1 = (\lambda a_{11}, \lambda a_{12}) \notin S$.

$\therefore$ S is not a subspace of $V_2(R)$.

e.g. 3) The subset $S = \{(x, y) \mid -\infty < x, y < \infty, x \neq 0, y \neq 0\}$ is not a subspace.

Ans:→ For any vector, $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in S$ and $\lambda = 0$, $\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin S$,

So, S is not closed under multiplication by a scalar, giving that S is not a subspace.

e.g. 4) The subset $S = \{(x,y) \mid y = a + bx,\ -\infty < x, y < \infty;\ a \neq 0,\ b \neq 0\}$ is not a subspace.

ANS :→ Consider any two vectors $\underset{\sim}{a}_1 = (a_{11}, a_{12})$ and $\underset{\sim}{a}_2 = (a_{21}, a_{22})$ from S.

Now, $\underset{\sim}{a}_1 + \underset{\sim}{a}_2 = (a_{11} + a_{21}, a_{12} + a_{22})$

$= (a_{11} + a_{21}, \underbrace{2a + b(a_{11} + a_{21})})$ → [this does not satisfy $y = a + bx$]

$y = a + bx$
$a_{12} = a + ba_{11}$
$a_{22} = a + ba_{21}$

$\Rightarrow \underset{\sim}{a}_1 + \underset{\sim}{a}_2 \notin S.$

So, giving S is not closed under addition. {Here, $x = (a_{11} + a_{21})$}

So, S is not a subspace.

Construction of a vector Subspace :

RESULT :→ Consider a set $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \dots, \underset{\sim}{a}_n\}$ of vectors from V_n. Then show that the collection of all possible linear combinations of $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \dots, \underset{\sim}{a}_n\}$ is a vector subspace of V_n.

Proof :→ The collection of all possible linear combinations of $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \dots, \underset{\sim}{a}_n$ is

$$S = \left\{ \sum_{i=1}^{n} l_i a_i : l_i \in \mathbb{R},\ i = 1(1)n \right\}$$

Let, $\underset{\sim}{x}, \underset{\sim}{y} \in S$.

$\therefore \underset{\sim}{x} = \sum_{i=1}^{n} l_i \underset{\sim}{a}_i$ and $\underset{\sim}{y} = \sum_{i=1}^{n} \lambda_i \underset{\sim}{a}_i$, for some l_i's and λ_i's.

$\therefore \underset{\sim}{x} + \underset{\sim}{y} = \sum_{i=1}^{n} (l_i + \lambda_i) \underset{\sim}{a}_i$

$= \sum_{i=1}^{n} l_i \underset{\sim}{a}_i \in S,$

and $\alpha \underset{\sim}{x} = \alpha \left(\sum_{i=1}^{n} l_i \underset{\sim}{a}_i \right)$

$= \sum_{i=1}^{n} (\alpha l_i) \underset{\sim}{a}_i$

$= \sum_{i=1}^{n} l_i \underset{\sim}{a}_i \in S.$

Clearly, $S \subseteq V_n$.

Hence, S is a vector subspace of V_n.

Theorem 8. The intersection $S \cap T$ of any two subspaces of V is itself a subspace of V, although $S \cup T$ may not be the same.

Proof:→ Since S and T are both subspaces, each contains the zero vector. Hence the zero vector is in $S \cap T$, so that $S \cap T$ is non-empty. Again, let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be vectors in $S \cap T$ and let k be a scalar (in F).

By defn. of $S \cap T$, both $\underset{\sim}{a}$ and $\underset{\sim}{b}$ belongs to S as well as T, and because S as well as T is a subspace, $k\underset{\sim}{a} + \underset{\sim}{b} \in S$ and $k\underset{\sim}{a} + \underset{\sim}{b} \in T$. Hence, $k\underset{\sim}{a} + \underset{\sim}{b} \in S \cap T$. —①

Lemma: A non-empty subset S of a vector space V is a subspace of V iff for each pair of vectors $\underset{\sim}{a}, \underset{\sim}{b} \in S$ and each scalar $k \in F$, the vector $k\underset{\sim}{a} + \underset{\sim}{b} \in S$.

Proof:→ Suppose the condition is true. Then since S is non-empty, $\exists$ a vector $\underset{\sim}{c} \in S$ and hence $(-1)\underset{\sim}{c} + \underset{\sim}{c} = \underset{\sim}{0} \in S$. Then if $\underset{\sim}{a} \in S$ and the scalar $k \in F$, then the vector $k\underset{\sim}{a} = k\underset{\sim}{a} + \underset{\sim}{0} \in S$. In particular $(-1)\underset{\sim}{a} = -\underset{\sim}{a} \in S$. Finally, if $\underset{\sim}{a} \in S$, $b \in S$, then $\underset{\sim}{a} + \underset{\sim}{b} \in S$. Thus S is seen to be a subspace of V.
Conversely, if S is a subspace of V, $\underset{\sim}{a} \in S$ and $\underset{\sim}{b} \in S$ and the scalar $k \in F$, then $k\underset{\sim}{a} \in S$ and so, $k\underset{\sim}{a} + \underset{\sim}{b} \in S$. □

By this lemma ① gives $S \cap T$ is a subspace.

It should be also be noted that the union of two subspaces S and T, denoted by $S \cup T$, and defined as the set of all vectors that belong to either S or T, may not be a subspace.

Spanning Set :-

Definition :- A set $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_r\}$ of vectors from a vector space V_n is said to be span or generate V_n if every vector in V_n can be written as a linear combination of $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_r$.

Then we say that $\{a_1, a_2, \ldots, a_r\}$ is a spanning set or generating set of the vector space V_n.

REMARK :→ The collection of all possible linear combinations of $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_r\}$ is $S = \left\{\sum_{i=1}^{r} l_i \underset{\sim}{a}_i : l_i \in R\right\}$ which is a vector space.

The spanning set of S is $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_r\}$ or the span of $\{a_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_r\}$ is $S = \left\{\sum_{i=1}^{r} l_i \underset{\sim}{a}_i : l_i \in R\right\}$.

Example :- For V_2, the unit vectors $\underset{\sim}{e}_1, \underset{\sim}{e}_2$ form a spanning set. Show that any two linearly independent vectors in V_2 form a spanning set of V_2. Also, show that — $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, (1,1)\}$ forms a spanning set of V_2.

Soln.→ Let $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2\}$ be two linearly independent vectors from V_2.

If possible, let $\underset{\sim}{x} = l_1 \underset{\sim}{a}_1 + l_2 \underset{\sim}{a}_2$, where $\underset{\sim}{x} \in V_2$.

$$\underset{\sim}{x} = l_1(a_{11}, a_{12}) + l_2(a_{21}, a_{22})$$

$$\therefore (x_1, x_2) = (l_1 a_{11} + l_2 a_{21}, \ l_1 a_{12} + l_2 a_{22})$$

$$\Rightarrow \begin{cases} x_1 = l_1 a_{11} + l_2 a_{21} \\ x_2 = l_1 a_{12} + l_2 a_{22} \end{cases}$$

$$\Rightarrow \begin{cases} l_1 = \dfrac{x_1 a_{12} - x_2 a_{21}}{a_{11} a_{22} - a_{12} a_{21}} \\[2ex] l_2 = \dfrac{x_1 a_{12} - x_2 a_{11}}{a_{12} a_{21} - a_{11} a_{22}} \end{cases}$$

as $\underset{\sim}{a}_1 = (a_{11}, a_{12})$ and $\underset{\sim}{a}_2 = (a_{21}, a_{22})$ are linearly independent.

$$\Rightarrow \underset{\sim}{a}_2 \neq \lambda \underset{\sim}{a}_1$$

$$\Rightarrow \frac{a_{11}}{a_{21}} \neq \frac{a_{12}}{a_{22}}$$

$$\Rightarrow a_{11} a_{22} - a_{12} a_{21} \neq 0.$$

Hence, $\exists$ scalars l_1 and $l_2 \ni \underset{\sim}{x} = l_1 \underset{\sim}{a}_1 + l_2 \underset{\sim}{a}_2$

Hence, $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2\}$ spans V_2.

Again, for $\underset{\sim}{x} \in V_2$, $\underset{\sim}{x} = x_1 \underset{\sim}{e}_1 + x_2 \underset{\sim}{e}_2 + 0(1,1)$

$$\Rightarrow \{e_1, e_2, (1,1)\} \text{ spans } V_2.$$

Theorem 9. Any set of vectors which span V, a vector space and containing the smallest possible number of vectors must be linearly independent.

Proof:→ Let $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ be a smallest spanning set of V.
If possible let $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ is linearly dependent.
Then, someone of them can be written as a linear combination of the others.
Let this one vector is $\underset{\sim}{a_n}$.
Then, $\underset{\sim}{a_n} = \sum_{i=1}^{n-1} \lambda_i \underset{\sim}{a_i}$ for some scalar λ_i.
Then, for any $\underset{\sim}{x} \in V$,

$$\underset{\sim}{x} = \sum_{i=1}^{n} l_i \underset{\sim}{a_i}$$

$$= \sum_{i=1}^{n-1} l_i \underset{\sim}{a_i} + l_n \underset{\sim}{a_n}$$

$$= \sum_{i=1}^{n-1} l_i \underset{\sim}{a_i} + l_n \left(\sum_{i=1}^{n-1} \lambda_i \underset{\sim}{a_i}\right)$$

$$= \sum_{i=1}^{n-1} (l_i + l_n \lambda_i) \underset{\sim}{a_i}$$

This implies $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ is a spanning set, which is a contradiction to the fact that $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ is the smallest spanning set of V.
Hence, our assumption is not correct, i.e. $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ must be linearly independent.

Remark:→ Converse of the theorem, a spanning set of vectors of V which is linearly independent, must be the minimal spanning.

Proof:→ Let $\{\underset{\sim}{a_1}, \ldots, \underset{\sim}{a_n}\}$ be a spanning and linearly independent set of vectors of V.
If possible, let the spanning set is not minimal.
Then, it is possible to have a set of vectors, say, $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_{n-1}}\}$ which is a spanning set of V.
Then $\underset{\sim}{a_n}$ can be written as a linear combination of $\{\underset{\sim}{a_1}, \ldots, \underset{\sim}{a_{n-1}}\}$.
$\Rightarrow \{\underset{\sim}{a_1}, \ldots, \underset{\sim}{a_{n-1}}, \underset{\sim}{a_n}\}$ is linearly dependent which is a contradiction, since $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ is linearly independent.
Hence, a linearly independent spanning set of vectors is minimal spanning.

BASIS :

C.U Definition :

1) A linearly independent set of vectors that generates a subspace, is called a basis of the subspace.

2) A set of vectors $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ is said to constitute a basis of a subspace S if —

i) $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ spans S ;

ii) $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ is a set of linearly independent vectors.

Ex:→ • The unit vectors $\underset{\sim}{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \underset{\sim}{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \underset{\sim}{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ form a basis of $V_3(R)$ because these vectors are linearly independent and spans $V_3(R)$.

PROBLEMS:→

1) Do the vectors $\underset{\sim}{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \underset{\sim}{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \underset{\sim}{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ form a basis of

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| z = 0, -\infty < x, y < \infty \right\} ?$$

Ans:→ No, because in $\underset{\sim}{e_3}$, $z \neq 0$, so, that defies the definition of spanning set.

2) Does $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| z = 0, -\infty < x, y < \infty \right\}$ form a subspace of $V_3(R)$?

U.P

Ans:→ Considering any two vectors $\underset{\sim}{a_1}$ and $\underset{\sim}{a_2}$ from S.

$\underset{\sim}{a_1} = (a_{11}, a_{12}, 0)$

$\underset{\sim}{a_2} = (a_{21}, a_{22}, 0)$

Now, $\underset{\sim}{a_1} + \underset{\sim}{a_2} = (a_{11} + a_{21}, a_{12} + a_{22}, 0) \in S$.

because $a_{11} + a_{21} > -\infty$ and $a_{12} + a_{22} < \infty$.

So, S is closed under addition.

Now, for any scalar λ,

$$\lambda \underset{\sim}{a_1} = (\lambda a_{11}, \lambda a_{12}, 0) \in S.$$

So, S is closed under multiplication by a scalar.

$\therefore$ S is a subspace of $V_3(R)$.

3) Show that $S = \{(x, y, z) : x+y+z=0 ; x, y, z \in \mathbb{R}\}$ is a subspace.

Soln → Consider a vector $(x, y, z) \in S$.

Here, $x+y+z=0$

$\Rightarrow \lambda(x+y+z) = 0$

$\Rightarrow (\lambda x + \lambda y + \lambda z) = 0$

$\therefore (\lambda x, \lambda y, \lambda z) \in S$.

i.e. S is closed under scalar multiplication.

Consider

$(x, y, z) \in S$ & $(u, v, w) \in S$

$\Rightarrow x+y+z=0$, $\Rightarrow u+v+w = 0$

$\Rightarrow (x+y+z) = \underset{\sim}{0}$ & $\Rightarrow (u+v+w) = \underset{\sim}{0}$

i.e. $(x+u) + (y+v) + (z+w) = 0$

$\Rightarrow (x+u, y+v, z+w) \in S$

$\therefore$ S is closed under vector addition.

Therefore S is a vector space. (Ans)

C.U

4) Show that all the vectors $(\underset{\sim}{x_1}, \underset{\sim}{x_2}, \underset{\sim}{x_3})$ in a vector space V_3 which obey $x_1 - x_2 = 0$ form a subspace V and find a basis of this subspace.

(OR)

Show that, the set of vectors $S = \{(x_1, x_2, x_3) \mid x_1 - x_2 = 0, x_1, x_2, x_3 \in \mathbb{R}\}$ forms a subspace and find it's basis.

Soln.

i) Considering any two vectors $\underset{\sim}{a_1}$ and $\underset{\sim}{a_2}$ from S.

$\underset{\sim}{a_1} = (a_{11}, a_{12}, a_{13})$ &

$\underset{\sim}{a_2} = (a_{21}, a_{22}, a_{23})$

As $x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$.

Here, $a_{11} = a_{12}$

$a_{21} = a_{22}$

Now, $\underset{\sim}{a_1} + \underset{\sim}{a_2} = (a_{11} + a_{21}, a_{12} + a_{22}, a_{13} + a_{23}) \in S$

$$= (a_{11} + a_{21}, a_{11} + a_{21}, a_{13} + a_{23}) \in S.$$

because $a_{11} + a_{21} \in R$ and $a_{13} + a_{23} \in R$.

So, S is closed under addition.

Now, for any scalar λ,

$$\lambda \underset{\sim}{a_1} = (\lambda a_{11}, \lambda a_{12}, \lambda a_{13}) \in S.$$

So, S is closed under multiplication by a scalar.

$\therefore$ S forms a subspace.

ii)

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| x_1 - x_2 = 0, x_1, x_2, x_3 \in R \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_1 \\ x_3 \end{pmatrix} \middle| x_1, x_3 \in R \right\}$$

$$= \left\{ x_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| x_1, x_3 \in R \right\}$$

$\Rightarrow$ $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ span S.

So, more over these two vectors are linearly independent.

So, they are the basis of S.

[NOTE:→ Here $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are not spanning set because e_1, e_2, e_3 are not in the subspace.]

5) The vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ form a basis of $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| x = y ; x, y, z \in R \right\}$.

Does another basis of S exists?

Ans:→ $\left\{ \begin{pmatrix} a_{11} \\ a_{11} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ b_{31} \end{pmatrix} ; a_{11} \text{ and } b_{31} \in R \right\} \longrightarrow$ a basis.

If we put any another set then for this particular subspace the new set would be a linearly dependent set, which contradicts our definition of basis.

C.U

Theorem 10. The set of unit vectors $\{e_1, e_2, \ldots, e_n\}$ is a basis of $V_n(F)$.

Proof:⇒ The vectors $e_1, e_2, \ldots, e_n$ are in $V_n(F)$ and any vector $a = (a_1, a_2, \ldots, a_n)$ can be written as
$$a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$$, a linear combination of n-unit vectors $\{e_1, e_2, \ldots, e_n\}$. So, $e_1, e_2, \ldots, e_n$ span or generate $V_n(F)$.

Further, these unit vectors $e_1, \ldots, e_n$ are linearly independent because for some scalars λ_i,
$$\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n = 0$$
$$\Rightarrow (\lambda_1, \lambda_2, \ldots, \lambda_n) = (0, \ldots, 0)$$
$$\Rightarrow \lambda_i = 0 \text{ for all } i = 1(1)n.$$
$\Rightarrow e_1, e_2, \ldots, e_n$ are linearly independent.

So, $\{e_1, e_2, \ldots, e_n\}$ being linearly independent generating set of $V_n(F)$, form a basis of $V_n(F)$.

C.U.

Theorem 11. A vector subspace has more than one basis.

Proof:⇒ Let $a_1, a_2, \ldots, a_k$ are basis vectors of a subspace S.

These implies for any scalar $\gamma \neq 0$,

To show, $\gamma a_1, a_2, \ldots, a_k$ is also a basis of S, consider
$$\lambda_1 \gamma a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k = 0$$ where λ_i's are scalar, $i = 1(1)k$.
$$\Rightarrow \lambda_1 \gamma = 0, \quad \lambda_i = 0 \text{ for all } i = 2(1)k.$$
As $a_1, a_2, \ldots, a_k$ are linearly independent,
$$\Rightarrow \lambda_1 = 0, \quad \lambda_i = 0 \ \forall\ i = 2(1)k.$$
$\Rightarrow \gamma a_1, a_2, \ldots, a_k$ are linearly independent. —— (1)

Further, let a be any vector from S and hence, ∃ scalars α_i, such that
$$a = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_k a_k$$, because $\{a_1, a_2, \ldots, a_k\}$ is a basis of S.
$$\Rightarrow a = \frac{\alpha_1}{\gamma}(\gamma a_1) + \alpha_2 a_2 + \cdots + \alpha_k a_k,$$
$\Rightarrow a$ is also a linear combination of $\{\gamma a_1, a_2, \ldots, a_k\}$ —— (2)

Hence, From (1) & (2), we get —

$\{\gamma a_1, a_2, \ldots, a_k\}$ is also a basis of S.

(C.U)

Theorem 12. Representation of a vector ~~[illegible]~~ belonging to a subspace S, in terms of basis vectors of S, is unique.

Proof:→ Let $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \dots, \underset{\sim}{a}_k\}$ be a given basis of a subspace S and $\underset{\sim}{a}$ be a vector belonging to S. Suppose $\underset{\sim}{a}$ can be written in two ways as as a linear combination of $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \dots, \underset{\sim}{a}_k\}$ as follows :—

$$\underset{\sim}{a} = \lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \dots + \lambda_k \underset{\sim}{a}_k \quad \text{——— (1)}$$

and $$\underset{\sim}{a} = \mu_1 \underset{\sim}{a}_1 + \mu_2 \underset{\sim}{a}_2 + \dots + \mu_k \underset{\sim}{a}_k \quad \text{——— (2)}$$

$$(1)-(2) \Rightarrow (\lambda_1 - \mu_1)\underset{\sim}{a}_1 + (\lambda_2 - \mu_2)\underset{\sim}{a}_2 + \dots + (\lambda_k - \mu_k)\underset{\sim}{a}_k = \underset{\sim}{0}$$

$\Rightarrow \lambda_i - \mu_i = 0 \quad \forall\, i = 1(1)k$, because $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \dots, \underset{\sim}{a}_k$ being basis vectors are linearly independent.

$$\Rightarrow \lambda_i = \mu_i \quad \forall\, i = 1(1)k.$$

Hence, the representation of any vector in terms of basis is unique.

Theorem 13. The representation of any vector in terms of an arbitrary set of spanning vectors is not ~~[illegible]~~ unique.

Proof:→ Consider a spanning set $\{\underset{\sim}{a}_1, \dots, \underset{\sim}{a}_n\}$ of vectors of a vector space V_n. If it is linearly independent, then it will form a basis of V_n. Then the representation will be unique.

But if it is linearly dependent, the someone of them can be written as a linear combination of the others. Let, this one vector be $\underset{\sim}{a}_n$.

Then, $\underset{\sim}{a}_n = \sum_{i=1}^{n-1} \lambda_i \underset{\sim}{a}_i$, for some λ_i.

Again, for any $\underset{\sim}{x} \in V$,

$$\underset{\sim}{x} = \sum_{i=1}^{n} l_i \underset{\sim}{a}_i \quad \text{, for some scalars } l_i. \quad \text{——— } (*)$$

and $$\underset{\sim}{x} = \sum_{i=1}^{n-1} l_i \underset{\sim}{a}_i + l_n \underset{\sim}{a}_n = \sum_{i=1}^{n-1} l_i \underset{\sim}{a}_i + l_n \left(\sum_{i=1}^{n-1} \lambda_i \underset{\sim}{a}_i \right)$$

$$\therefore \underset{\sim}{x} = \sum_{i=1}^{n-1} (l_i + l_n \lambda_i)\underset{\sim}{a}_i + 0.\underset{\sim}{a}_n \quad \text{——— } (**)$$

$(*)$ & $(**)$ shows ⇒ the representation is not unique.

NOTE:→ How ~~[illegible]~~ many choices are there of forming basis?

ANS. Infinite number of choices for the representation of any vector in terms of an arbitrary set of spanning vectors.

Theorem 14. (Change of Basis Technique)

If $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_p\}$ be a basis of V_n and $\underset{\sim}{y} \in V_n$ be $\ni$ $\underset{\sim}{y} = \sum_{i=1}^{p} \lambda_i \underset{\sim}{x}_i$, $\lambda_1 \neq 0$. Show that $\{\underset{\sim}{y}, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_p\}$ will also be a basis of V_n.

Proof: Consider a vector $\underset{\sim}{z} \in V_n$. As $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_p\}$ is a basis of V_n, $\underset{\sim}{z}$ can be written as a linear combination of the basis vectors.

$\therefore \exists$ scalars θ_i, $i = 1(1)p$ such that

$$\underset{\sim}{z} = \sum_{i=1}^{p} \theta_i \underset{\sim}{x}_i \quad \cdots\cdots (1)$$

Given that, $\underset{\sim}{y} = \sum_{i=1}^{p} \lambda_i \underset{\sim}{x}_i$; $\lambda_1 \neq 0$

$$\Rightarrow \underset{\sim}{x}_1 = \frac{1}{\lambda_1}\underset{\sim}{y} - \sum_{i=2}^{p} \frac{\lambda_i}{\lambda_1} x_i \quad [\lambda_1 \neq 0] \quad \cdots\cdots (2)$$

Combining (1) & (2) $\quad \underset{\sim}{z} = \frac{\theta_1}{\lambda_1}\underset{\sim}{y} + \sum_{i=2}^{p}\left(\theta_i - \frac{\theta_1}{\lambda_1}\lambda_i\right)\underset{\sim}{x}_i$

$\therefore \{\underset{\sim}{y}, \underset{\sim}{x}_2, \underset{\sim}{x}_3, \ldots, \underset{\sim}{x}_p\}$ spans V_n $\quad \cdots\cdots (3)$

For the scalars $\alpha_1, \alpha_2, \ldots, \alpha_p$ consider

$$\alpha_1 \underset{\sim}{y} + \sum_{i=2}^{p} \alpha_i \underset{\sim}{x}_i = \underset{\sim}{0}$$

$$\text{i.e. } \alpha_1 \sum_{i=1}^{p} \lambda_i \underset{\sim}{x}_i + \sum_{i=2}^{p} \alpha_i \underset{\sim}{x}_i = \underset{\sim}{0}$$

$$\Rightarrow \alpha_1 \lambda_1 x_1 + \sum_{i=2}^{p} (\alpha_i + \alpha_1 \lambda_i)\underset{\sim}{x}_i = 0$$

$$\Rightarrow \alpha_1 \lambda_1 = 0, \quad \alpha_i + \alpha_1 \lambda_i = 0 \quad \forall\ i = 2(1)p.$$

as $x_1, x_2, \ldots, x_p$ are LIN vectors.

Now, $\lambda_1 \neq 0$, $\lambda_1 \alpha_1 = 0 \Rightarrow \alpha_1 = 0$

$\alpha_i + \alpha_1 \lambda_i = 0 \ \forall\ i = 2(1)p$

$\Rightarrow \alpha_2 = \alpha_3 = \cdots = \alpha_p = 0$

$\therefore \alpha_1 y + \sum_{i=2}^{p} \alpha_i x_i = \underset{\sim}{0} \Rightarrow \alpha_i = 0 \ \forall\ i$

Hence, $\{\underset{\sim}{y}, \underset{\sim}{x}_2, \underset{\sim}{x}_3, \ldots, x_p\}$ is a set of LIN vectors. $\quad \cdots\cdots (4)$

(3) & (4) $\Rightarrow$ hence we get the result.

Remark: It is important to realize that "the choice of basis for a vector space is not unique".

Theorem 15. A set of n linearly independent n component vectors form a basis of E^n.

Proof:→ Let $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n\}$ be a set of LIN n component vectors.

Note that $\underset{\sim}{x}_1 \neq \underset{\sim}{0} \in E^n$ [as it belongs to the set of LIN vectors]

Further assume that

$$\underset{\sim}{x}_1 = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{pmatrix}, \quad x_{1i} \neq 0 \text{ for at least one } i, \ i = 1(1)n.$$

Clearly, $\underset{\sim}{x}_1 = x_{11}\underset{\sim}{e}_1 + x_{12}\underset{\sim}{e}_2 + \cdots + x_{1n}\underset{\sim}{e}_n$

WLG, let $x_{11} \neq 0$, then $\{\underset{\sim}{x}_1, \underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_n\}$ will be a basis of E^n, as $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_n\}$ is a basis of E^n.

Again, $\underset{\sim}{x}_2 \neq \underset{\sim}{0} \in E^n$, $\{\underset{\sim}{x}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3, \ldots, \underset{\sim}{e}_n\}$ is a basis of E^n then $\exists$ scalars $\theta_1, \theta_2, \ldots, \theta_n$, not all zero, such that

$$\underset{\sim}{x}_2 = \theta_1 \underset{\sim}{x}_1 + \sum_{i=2}^{n} \theta_i \underset{\sim}{e}_i .$$

if possible let, $\theta_1 \neq 0$ and $\theta_2 = \theta_3 = \cdots = \theta_n = 0$

then $\underset{\sim}{x}_2 = \theta_1 \underset{\sim}{x}_1$ which is not possible as both of the vectors belongs to the set of LIN vectors.

So, there must be some non-zero θ_i's among $\theta_2, \theta_3, \ldots, \theta_n$.

WLG; let $\theta_2 \neq 0$.

Hence, $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4, \ldots, \underset{\sim}{e}_n\}$ will be a basis of E^n.

Proceeding in this way it can be shown that $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n\}$ is a basis of E^n.

Corollary:—

If $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n\}$ be a set of linearly independent n-component vectors then for any n-component vector, $\underset{\sim}{y} \neq \underset{\sim}{0}$, $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n, \underset{\sim}{y}\}$ must be a set of LD vectors.

C.U

Theorem 17. Number of vectors in a basis of a vector space is unique in the sense of that any two bases have the same number of vectors.

Proof:→ Consider a vector space V_n, if possible let $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_p\}$ and $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_q\}$ be two bases of V_n. We are to show $p=q$.

Now, let us assume that $p \neq q$.

Let $p<q$.

Consider $\underset{\sim}{y}_1 (\neq 0) \in V_n$ as $\underset{\sim}{y}_1$ is a basis vector, must be non-null.

Now $\exists$ scalars $\theta_1, \theta_2, \ldots, \theta_p$ with at least one $\theta_i \neq 0 \ni$ $(i=1(1)p)$. $\underset{\sim}{y}_1 = \sum_{i=1}^{p} \theta_i \underset{\sim}{x}_i$ [$\because \{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_p\}$ is a basis of V_n]

WLG let $\theta_1 \neq 0$, $\{\underset{\sim}{y}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3, \ldots, \underset{\sim}{x}_p\}$ will be a basis of V_n.

Consider $\underset{\sim}{y}_2 (\neq 0) \in V_n$.

Thus $\exists$ scalars $\lambda_1, \lambda_2, \ldots, \lambda_p$ with at least one $\lambda_i \neq 0 \ \forall \ i=1(1)p$ such that

$$\underset{\sim}{y}_2 = \lambda_1 \underset{\sim}{y}_1 + \sum_{i=2}^{p} \lambda_i \underset{\sim}{x}_i \quad [\because \{\underset{\sim}{y}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3, \ldots, \underset{\sim}{x}_p\} \text{ is a basis of } V_n]$$

If possible, let, $\lambda_1 \neq 0$, $\lambda_2 = \lambda_3 = \cdots = \lambda_p = 0$, then

$\underset{\sim}{y}_2 = \lambda_1 \underset{\sim}{y}_1$, which is not true as both the vectors belong to the same basis.

There must be some non-zero λ_i among $\lambda_2, \lambda_3, \ldots, \lambda_p$.

WLG, let $\lambda_2 \neq 0$, then $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \underset{\sim}{x}_3, \ldots, \underset{\sim}{x}_p\}$ will be a basis of V_n.

Proceeding in this way it can be shown that $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_p\}$ will be a basis of V_n.

Now $\exists$ $\underset{\sim}{y}_{p+1} \in V_n$.

Hence it can be written as a linear combination of the vectors in the basis $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_p\}$, i.e. $\underset{\sim}{y}_{p+1}$ is linearly dependent on $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_p\}$ which is not possible as $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_p, \underset{\sim}{y}_{p+1}\}$ belongs to the basis $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_q\}$.

$$\therefore p \nless q.$$

Similar arguments lead to $p \ngtr q$, so,

$$p = q.$$

- **Corollary:**

(1) Every basis of $V_n(F)$ contains exactly n vectors.

Proof:→ We know that $-\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$ is a basis for V_n or E^n. As any two bases for a vector space have the same number of vectors, hence ~~a ba~~ every basis of $V_n(F)$ contains exactly n vectors.

(2) Any set of $(n+1)$ vectors from V_n or E^n is linearly dependent.

Proof:– Let $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{n+1}\}$ be a set of $n+1$ vectors from E^n.
If possible, suppose $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{n+1}\}$ is linearly independent.
then any subset of $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{n+1}\}$, namely $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n\}$ containing n vectors is linearly independent.
Now, $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n\}$ forms a basis for E^n.
Then $\hat{a}_{n+1}$ can be expressed as a linear combination of $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n\}$ which contradicts with the fact that $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{n+1}\}$ is linearly independent.
By contradiction, hence the proof.

(3) S.T. it is not possible to have $(n+1)$ mutually orthogonal vectors (non-zero) with n components.

Proof:→ Let $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{n+1}\}$ is a set of $(n+1)$ vectors.
If possible, suppose that the vectors are mutually orthogonal.
But as we know mutually orthogonal vectors are necessarily LIN, so any subset of it containing n vectors will be linearly independent.

OR

* Any set of mutually orthogonal non-null vectors from E^n is LIN.
If possible, let the $(n+1)$ vectors are mutually orthogonal, and they will be LIN, which contradicts the fact that "any set containing the $(n+1)$ vectors from E^n is necessarily LD".
Hence the conclusion. *

Dimension :

Definition : 1) The number of vectors in a basis of a subspace is called the dimension of the subspace.

2) Dimension of a vector space V_n, denoted by $d(V_n)$ is defined as ~~the~~ number of linearly independent vectors required to span V_n, i.e. number of vectors in any basis of V_n. clearly —

i) $d(E^n) = n$.

ii) $d(V_n) \leq n$.

iii) $d(V_n) \geq p$ if we have an arbitrary collection of p linearly independent vectors in V_n.

Example:

1) $\dim(V_2(R)) = 2$
because $\underset{\sim}{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ & $\underset{\sim}{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$ form a basis of $V_2(R)$.

2) Can you find the $\dim(S)$ where $S = \left\{\begin{pmatrix}x\\1\end{pmatrix} \middle| x \in R\right\}$

ANS: $\begin{pmatrix}a_{11}\\1\end{pmatrix}, \begin{pmatrix}a_{12}\\1\end{pmatrix}$ As, $\begin{pmatrix}a_{11}\\1\end{pmatrix} + \begin{pmatrix}a_{12}\\1\end{pmatrix} = \begin{pmatrix}a_{11}+a_{12}\\2\end{pmatrix} \notin S$

Here S is not closed under addition and thus S is not a subspace, so $\dim(S)$ is not defined in this case.

3) $S = \left\{\begin{pmatrix}x\\x\end{pmatrix} \middle| x \in R\right\}$. Calculate $\dim(S)$?

Soln. → $\underset{\sim}{a} = \begin{pmatrix}a_1\\a_2\end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix}b_1\\b_2\end{pmatrix} \in S$; $\underset{\sim}{a} + \underset{\sim}{b} = \begin{pmatrix}a_1+b_1\\a_2+b_2\end{pmatrix} \in S$,

$\lambda \underset{\sim}{a} = \begin{pmatrix}\lambda a_1\\\lambda a_2\end{pmatrix} \in S \;\forall\, \lambda \in R.$

So, S is a subspace.

$$S = \left\{\begin{pmatrix}x\\x\end{pmatrix} \middle| x \in R\right\} = \left\{x\begin{pmatrix}1\\1\end{pmatrix} \middle| x \in R\right\}.$$

So, $\begin{pmatrix}1\\1\end{pmatrix}$ forms a basis of S, giving $\dim(S) = 1$.

4) $S_1 = \left\{\begin{pmatrix}x\\0\end{pmatrix} \middle| x \in R\right\}$ and $S_2 = \left\{\begin{pmatrix}0\\y\end{pmatrix} \middle| y \in R\right\}$ are two subspace of $V_2(R)$. Find $\dim(S_1+S_2)$?

Soln. →
$$S_1 + S_2 = \left\{\begin{pmatrix}x\\0\end{pmatrix} + \begin{pmatrix}0\\y\end{pmatrix} \middle| x, y \in R\right\}$$
$$= \left\{\begin{pmatrix}x\\y\end{pmatrix} \middle| x, y \in R\right\} = V_2(R)$$

Subspace		Dimension
S_1	=	1
S_2	=	1
S_1+S_2	=	2

5) $S_1 = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} \middle| x \in R \right\}$ and $S_2 = \left\{ \begin{pmatrix} y \\ -y \end{pmatrix} \middle| y \in R \right\}$ are two subspaces of $V_2(R)$. Find $\dim(S_1+S_2)$?

Soln →

$$S_1+S_2 = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} + \begin{pmatrix} y \\ -y \end{pmatrix} \middle| x, y \in R \right\}$$

$$= \left\{ \begin{pmatrix} x+y \\ x-y \end{pmatrix} \middle| x, y \in R \right\}$$

$$= \left\{ (x+y)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (x-y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \middle| x, y \in R \right\}$$

$\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ form a basis of S_1+S_2.

$\boxed{\text{C.U.}}$ $\therefore \dim(S_1+S_2) = 2$.

$\underset{\wedge}{a} = \begin{pmatrix} a_1+a_2 \\ a_1-a_2 \end{pmatrix}, \underset{\wedge}{b} = \begin{pmatrix} b_1+b_2 \\ b_1-b_2 \end{pmatrix}$

$$\underset{\wedge}{a} + \underset{\wedge}{b} = \begin{pmatrix} a_1+b_1 + b_1+b_2 \\ a_1+b_1-(a_2+b_2) \end{pmatrix}$$

$\in (S_1+S_2)$

$\therefore \lambda \underset{\wedge}{a} = \lambda \begin{pmatrix} a_1+a_2 \\ a_1-a_2 \end{pmatrix} \in S_1$

$\therefore S_1 + S_1$ is also a subspace.

6) Consider the vectors $\underset{\wedge}{x_1} = (1, 3, 2)$ and $x_2 = (-2, 4, 3)$ in V_3 on F^3. S.T. span of $\{\underset{\wedge}{x_1}, \underset{\wedge}{x_2}\}$ is given by $S = \{(\xi_1, \xi_2, \xi_3) : \xi_1 - 7\xi_2 + 10\xi_3 = 0\}$

Soln → span of $\{\underset{\wedge}{x_1}, \underset{\sim}{x_2}\}$ is

$$S = \{ l_1 x_1, l_2 x_2 : l_1, l_2 \in R \}$$

$$= \{ l_1(1,3,2), l_2(-2,4,3) : l_1, l_2 \in R \}$$

$$= \{ (l_1 - 2l_2, 3l_1 + 4l_2, 2l_1 + 3l_2) : l_1, l_2 \in R \}$$

Let, $\xi \in S$, then $S = \{ (\xi_1, \xi_2, \xi_3) : \xi_1, \xi_2, \xi_3 \in R \}$

$\xi_1 = l_1 - 2l_2$, $\xi_2 = 3l_1 + 4l_2$, $\xi_3 = 2l_1 + 3l_2$

$$\xi_1 - 7\xi_2 + 10\xi_3 = l_1 - 2l_2 - 21l_1 - 28l_2 + 20l_1 + 30l_2$$

$$\therefore \xi_1 - 7\xi_2 + 10\xi_3 = 0.$$

$$S = \{ (\xi_1, \xi_2, \xi_3) : \xi_1 - 7\xi_2 + 10\xi_3 = 0 \}$$

Theorem 18. If S_n and V_n be two disjoint vector space, i.e. $S_n \cap V_n = \{\underset{\sim}{0}\}$, then $\dim(S_n + V_n) = \dim(S_n) + \dim(V_n)$.

Proof: Let $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N\}$ be a basis of S_n and $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_s\}$ be a basis of V_n. In this case it is enough to show that $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N, \underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_s\}$ is a basis of $S_n + V_n$, in that case we have

$$\dim(S_n + V_n) = \dim(S_n) + \dim(V_n) = N + S.$$

Consider a vector $\underset{\sim}{z} \in (S_n + V_n)$.

$$\underset{\sim}{z} = \underset{\sim}{z}_1 + \underset{\sim}{z}_2$$

where $\underset{\sim}{z}_1 \in S_n$, $\underset{\sim}{z}_2 \in V_n$

Clearly, $\underset{\sim}{z}_1$ & $\underset{\sim}{z}_2$ be respectively written as,

$\underset{\sim}{z}_1 = \sum_{i=1}^{N} \theta_i \underset{\sim}{x}_i$ since $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N\}$ spans S_n,

&

$\underset{\sim}{z}_2 = \sum_{i=1}^{S} \lambda_i \underset{\sim}{y}_i$ $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_s\}$ spans V_n.

Thus, $$\underset{\sim}{z} = \sum_{i=1}^{N} \theta_i \underset{\sim}{x}_i + \sum_{i=1}^{S} \lambda_i \underset{\sim}{y}_i$$

Hence, $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N, \underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_s\}$ spans $(S_n + V_n)$. —— ①

Now for scalars α_i's and β_i's consider

$$\sum_{i=1}^{N} \alpha_i \underset{\sim}{x}_i + \sum_{i=1}^{S} \beta_i \underset{\sim}{y}_i = \underset{\sim}{0}$$

$$\Rightarrow \sum_{i=1}^{N} \alpha_i \underset{\sim}{x}_i = -\sum_{i=1}^{S} \beta_i \underset{\sim}{y}_i \quad \text{—— ②}$$

Note that $\rightarrow \sum_{i=1}^{S} \beta_i \underset{\sim}{y}_i \in V_n$.

$\Rightarrow$ R.H.S of ② belongs to V_n

$\Rightarrow$ L.H.S. of ② belongs to V_n

Again L.H.S of ② belongs to S_n

So, L.H.S. of ② belongs to $V_n \cap S_n$.

Now, we have, $V_n \cap S_n = \{\underset{\sim}{0}\}$

$$\therefore \sum_{i=1}^{N} \alpha_i \underset{\sim}{x}_i = \underset{\sim}{0} \text{ ------ ③}$$

$\Rightarrow \alpha_i = 0 \ \forall \ i = 1(1)N$ as x_i's are LIN.

② & ③ implies $\Rightarrow \sum_{i=1}^{S} \beta_i \underset{\sim}{y}_i = \underset{\sim}{0} \Rightarrow \beta_i = 0 \ \forall \ i = 1(1)s$ as y_i's are LIN.

$$\therefore \sum_{i=1}^{N} \alpha_i \underset{\sim}{x}_i + \sum_{i=1}^{S} \beta_i \underset{\sim}{y}_i = 0 \Rightarrow \alpha_i = 0 \ \& \ \beta_i = 0 \ \forall \ i$$

$\therefore \{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N, \underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_s\}$ is a set of LIN vectors. —— ④

① & ④ $\Rightarrow \dim(S_n + V_n) = \dim(S_n) + \dim(V_n)$.

Result: For two subspaces W_1 & W_2 of V_n, S.T.

i) $W_1 \cap W_2$ is also a subspace of V_n,

ii) $\dim(W_1 \cap W_2) \le \min\{\dim(W_1), \dim(W_2)\}$.

Proof:

i) Let, $\underset{\sim}{a}, \underset{\sim}{b} \in W_1 \cap W_2$

$\Rightarrow \underset{\sim}{a}, \underset{\sim}{b} \in W_i$ for $i = 1, 2$.

$\Rightarrow \underset{\sim}{a}, \underset{\sim}{b} \in W_1$.

So, $\underset{\sim}{a} + \underset{\sim}{b} \in W_1$ and $\underset{\sim}{a} + \underset{\sim}{b} \in W_2$ as W_1 & W_2 being two subspaces closed under addition. — ①

$\Rightarrow \underset{\sim}{a} + \underset{\sim}{b} \in W_1 \cap W_2$.

For any scalar $\lambda \in F$,

$\lambda \underset{\sim}{a} \in W_1$ and $\lambda \underset{\sim}{b} \in W_2$.

$\therefore \lambda \underset{\sim}{a} \in W_1 \cap W_2$.

$\therefore W_1 \cap W_2$ is also closed under the property of multiplication. — ②

① & ② $\Rightarrow$ So, $W_1 \cap W_2$ is also a subspace of V_n.

Example:

$$W_1 = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \middle| x, y \in \mathbb{R} \right\},$$

$$W_2 = \left\{ \begin{pmatrix} x \\ x \\ z \end{pmatrix} \middle| x, z \in \mathbb{R} \right\}$$

$$\therefore W_1 \cap W_2 = \left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \middle| x \in \mathbb{R} \right\}$$

is also a subspace.

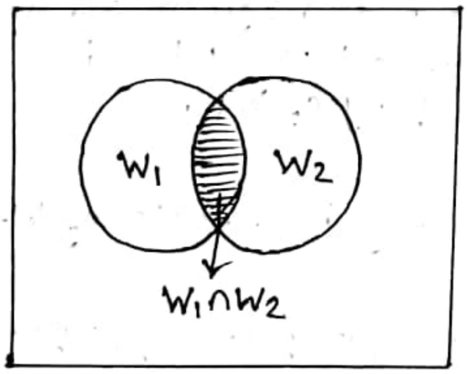
$\therefore \dim(W_1) = 2$,

$\therefore \dim(W_2) = 2$,

$\dim(W_1 \cap W_2) = 1$

$\therefore \dim(W_1 \cap W_2) \le \min\{\dim(W_1), \dim(W_2)\}$

ii)



$V_n(F)$

Clearly, the maximum no. of LIN vectors in $(W_1 \cap W_2)$ $\le$ the maximum no. of LIN vectors in W_1 $[\because W_1 \cap W_2 \subseteq W_1]$

$\Rightarrow \dim(W_1 \cap W_2) \le \dim(W_1)$

Similarly, $W_1 \cap W_2 \subseteq W_2$

$\Rightarrow \dim(W_1 \cap W_2) \le \dim(W_2)$

$\therefore \dim(W_1 \cap W_2) \le \min\{\dim(W_1), \dim(W_2)\}$

Remark: 1. $\dim(W_1 \cap W_2) \le \sqrt{\dim(W_1)\dim(W_2)}$.

2. Let V be a vector space over a field F and W be a subspace of V. Then $\dim(V/W) = \dim V - \dim W$.

Result: Prove that $S_n' \cap S_n''$ is also a vector subspace of V_n.
Let $S_2' = \{\lambda \underset{\sim}{e_1}\}$ and $S_2'' = V_2$. What is $S_2' \cap S_2''$?

Proof: Let $\hat{x}, \hat{y} \in S_n' \cap S_n''$

Then $\underset{\sim}{x} \in S_n'$ and $\hat{x} \in S_n''$; $\hat{y} \in S_n'$ and $y \in S_n''$

$\therefore \hat{x} + \hat{y} \in S_n'$ as S_n' is closed under vector addition.

$\therefore \hat{x} + y \in S_n' \cap S_n''$

For any λ, $\lambda \in R$, $\lambda \hat{x} \in S_n'$ and $\lambda y \in S_n''$ as S_n' and S_n'' are closed under scalar multiplication.

$\therefore \lambda \hat{x} \in S_n' \cap S_n'' \quad \forall \lambda$.

Hence, $S_n' \cap S_n''$ is a vector subspace of V_n.

$$V_2 = \{ \alpha \hat{e}_1 + \lambda \underset{\sim}{e_2} : \alpha, \lambda \in R\} = S_2''$$

$$S_2' = \{ \alpha \hat{e}_1 \}$$

$$\therefore S_2' \cap S_2'' = S_2' \cap V_2 = \{ \alpha \hat{e}_1 \} = S_2'.$$

Theorem 19. For two subspaces W_1 and W_2 of $V_n(F)$.

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Proof: Let $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_p\}$ be a basis of $W_1 \cap W_2$.
As this set is LIN & contains vectors from W_1 and W_2,
It can be extended so as to constitute a basis
$S_1 = \{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_p, \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_q\}$ of W_1.
For similar reason
$S_2 = \{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_p, \hat{c}_1, \hat{c}_2, \ldots, \hat{c}_r\}$ is a basis of W_2, contains vectors a_i's $\forall i = 1(1)p$.

We now prove that $S = \{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_p, \hat{b}_1, \ldots, \hat{b}_q, \hat{c}_1, \ldots, \hat{c}_n\}$ is a basis of $W_1 + W_2$.

Let $\underset{\sim}{d} \in W_1 + W_2$

$\Rightarrow \hat{d} = \hat{d}_1 + \hat{d}_2$ where $\hat{d}_1 \in W_1$ & $\underset{\sim}{d_2} \in W_2$

$$\Rightarrow \underset{\sim}{d} = \left[\sum_{i=1}^{p} \lambda_{1i} \hat{a}_i + \sum_{i=1}^{q} \lambda_{2i} \hat{b}_i \right] + \left[\sum_{i=1}^{p} \gamma_{1i} \hat{a}_i + \sum_{i=1}^{n} \gamma_{2i} c_i \right]$$

(for some scalar λ and γ)

$$= \sum_{i=1}^{p} (\lambda_{1i} + \gamma_{1i}) \hat{a}_i + \sum_{i=1}^{q} \lambda_{2i} \hat{b}_i + \sum_{i=1}^{n} \gamma_{2i} \hat{c}_i$$

$\Rightarrow$ S is a spanning set of $(W_1 + W_2)$.

To show that S is LIN, we consider the equation $\rightarrow$

$$\sum_{i=1}^{p} k_{1i}\underset{\sim}{a_i} + \sum_{i=1}^{q} k_{2i}\underset{\sim}{b_i} + \sum_{i=1}^{r} k_{3i}\underset{\sim}{c_i} = 0 \qquad (\text{k's are scalars})$$

$$\Rightarrow \sum_{i=1}^{p} k_{1i}\underset{\sim}{a_i} + \sum_{i=1}^{q} k_{2i}\underset{\sim}{b_i} = -\sum_{i=1}^{r} k_{3i}\underset{\sim}{c_i} \quad \text{——} (1)$$

(1) shows that $-\sum_{i=1}^{r} k_{3i}\underset{\sim}{c_i} \in W_1$,

as it is a Linear combination of the basis vector of W_1, more over being a linear combination of the basis vectors of W_2,

$$\sum_{i=1}^{r} k_{3i}\underset{\sim}{c_i} \in W_2 .$$

$\therefore \sum_{i=1}^{r} k_{3i}\underset{\sim}{c_i} \in (W_1 \cap W_2)$, and hence

$-\sum_{i=1}^{r} k_{3i}\underset{\sim}{c_i} = \sum_{i=1}^{p} \delta_i \underset{\sim}{a_i}$ for some scalars δ_i, as $\{a_1, a_2, \ldots, a_p\}$ is a basis of W_1.

$$\Rightarrow \sum_{i=1}^{p} \delta_i \underset{\sim}{a_i} + \sum_{i=1}^{r} k_{3i}\underset{\sim}{c_i} = 0$$

$\Rightarrow \delta_i = 0$ and $k_{3i} = 0 \ \forall i$, because the set $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_p}, \underset{\sim}{c_1}, \ldots, \underset{\sim}{c_r}\}$ is LIN.

Thus, $\sum_{i=1}^{p} k_{1i}\underset{\sim}{a_i} + \sum_{i=1}^{q} k_{2i}\underset{\sim}{b_i} = 0$

$\Rightarrow \begin{cases} k_{1i} = 0 \ \forall\ i = 1(1)p, \\ k_{2i} = 0 \ \forall\ i = 1(1)q, \end{cases}$ because $\{\underset{\sim}{a_1}, a_2, \ldots, a_p, \underset{\sim}{b_1}, \ldots, \underset{\sim}{b_q}\}$ is also LIN set of vectors.

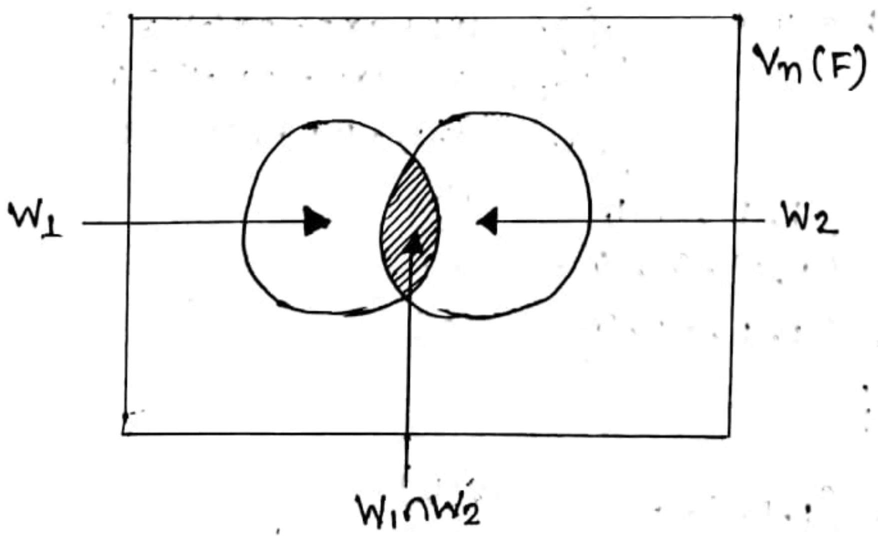
So, the set S containing all the $(p+q+r)$ vectors is LIN.

$\therefore$ S is a basis of $W_1 + W_2$.

Now, $\dim(W_1 + W_2) = p + q + r$

$$= (p+q) + (p+r) - p$$

$$= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$



Orthogonal Vectors :

Definition :

1) Two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are said to be orthogonal if $\underset{\sim}{a}'\underset{\sim}{b} = 0$. We generally use the notation $a \perp b$ to mean that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are orthogonal. Clearly, the null vector $\underset{\sim}{0}$ is orthogonal to every vector.

2) A set of vectors $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n\}$ from E^n is said to be mutually orthogonal if $\underset{\sim}{a}_i{}'\underset{\sim}{a}_j = 0 \ \forall \ i \neq j$.

NOTE: It is noted that the unit vectors are orthogonal since $\underset{\sim}{e}_i{}'\underset{\sim}{e}_j = 0 \ \forall \ i \neq j$.

Let θ be the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then

$$\cos\theta = \frac{\underset{\sim}{a}'\underset{\sim}{b}}{\|\underset{\sim}{a}\| \|\underset{\sim}{b}\|}, \text{ where, } \|\underset{\sim}{a}\| = \sqrt{\sum_{i=1}^{n} a_i^2},$$

$$\|\underset{\sim}{b}\| = \sqrt{\sum_{i=1}^{n} b_i^2}$$

Now, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are said to be orthogonal then, $\underset{\sim}{a}'\underset{\sim}{b} = 0$

i.e. $\cos\theta = 0$

i.e. $\theta = \frac{\pi}{2}$

i.e. vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

Example:

(a) $\underset{\sim}{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ are orthogonal vectors.

(b) The set containing $\underset{\sim}{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \underset{\sim}{a}_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ and $\underset{\sim}{a}_3 = \begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix}$ is orthogonal because

$$\underset{\sim}{a}_1{}'\underset{\sim}{a}_2 = 1\times 0 + 1\times 2 + 2\times(-1) = 0$$
$$\underset{\sim}{a}_2{}'\underset{\sim}{a}_3 = 0 + 1\times 2 + 2\times(-1) = 0$$
$$\underset{\sim}{a}_1{}'\underset{\sim}{a}_3 = 1\times(-5) + 1\times 1 + 2\times 2 = 0$$

(c)

$$\underset{\sim}{a}_1 = (1, 1, 1, 1, \ldots, 1)$$
$$\underset{\sim}{a}_2 = (1, -1, 0, 0, \ldots, 0)$$
$$\underset{\sim}{a}_3 = (1, 1, -2, 0, \ldots, 0)$$
$$\vdots$$
$$\underset{\sim}{a}_n = (1, 1, 1, \ldots \ldots -(n-1))$$

is a set of orthogonal vectors.

(d) Let W_1 be the subspace generated by the vectors $\begin{pmatrix}1\\1\\2\end{pmatrix}$ and $\begin{pmatrix}0\\2\\-1\end{pmatrix}$ and W_2, the subspace generated by $\begin{pmatrix}-5\\1\\2\end{pmatrix}$ are orthogonal.

$$W_1 = \left\{\lambda_1\begin{pmatrix}1\\1\\2\end{pmatrix} + \lambda_2\begin{pmatrix}0\\2\\-1\end{pmatrix} : \lambda_1, \lambda_2 \in R\right\}$$

$$W_2 = \left\{\lambda\begin{pmatrix}-5\\1\\2\end{pmatrix} : \lambda \in R\right\}$$

For, $\underset{\sim}{a} = \lambda_1\begin{pmatrix}1\\1\\2\end{pmatrix} + \lambda_2\begin{pmatrix}0\\2\\-1\end{pmatrix}$ and $\underset{\sim}{b} = \lambda\begin{pmatrix}-5\\1\\2\end{pmatrix}$,

$$\underset{\sim}{a}'\underset{\sim}{b} = 0.$$

Orthonormal Vectors :

Definition :– $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n\}$ is said to be a set of orthonormal vectors if

$$\underset{\sim}{a}_i'\underset{\sim}{a}_j = 0 \quad \forall \quad i \neq j$$
$$\underset{\sim}{a}_i'\underset{\sim}{a}_j = 1 \quad \forall \quad i = j \rightarrow \text{(with unit length)}$$

If we define a set of vectors $\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \ldots, \underset{\sim}{b}_n\}$ and $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n\}$, form a set of mutually orthogonal vectors, such that

$$\underset{\sim}{b}_i = \frac{\underset{\sim}{a}_i}{\|\underset{\sim}{a}_i\|} \quad \forall \ i$$

then $\{\underset{\sim}{b}_i\}$ will be a set of orthonormal vectors.

Orthogonal Basis :

- Definition :– A basis of a subspace of E^n is called orthogonal if the basis vectors are orthogonal. A basis is called orthonormal if the basis vectors are of unit length in addition to orthogonal.

Example :– The vectors $\begin{pmatrix}1\\1\end{pmatrix}$ & $\begin{pmatrix}2\\-2\end{pmatrix}$ form an orthogonal basis of E^2 because $\begin{pmatrix}1\\1\end{pmatrix}'\begin{pmatrix}2\\-2\end{pmatrix} = 0$.

The vectors $\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$ & $\begin{pmatrix}2/2\sqrt{2}\\-2/2\sqrt{2}\end{pmatrix}$ being of unit length form an orthonormal basis of E^2.

In particular, $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_n\}$ forms an orthonormal basis for E^n.

Theorem 20. Any set of orthogonal vectors not containing the null vector is linearly independent.

Proof: Let $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n\}$ be a set of mutually orthogonal non-null vectors from E^n. To show that these vectors are LIN, we consider the equation —

$$\lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \cdots + \lambda_n \underset{\sim}{a}_n = \underset{\sim}{0} \quad \text{——— (1)}$$

By considering the scalar products of $\underset{\sim}{a}_i$ and the equation (1), $i = 1(1)n$, we get —

$$\lambda_1 \underset{\sim}{a}_1' \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_1' \underset{\sim}{a}_2 + \cdots + \lambda_n \underset{\sim}{a}_1' \underset{\sim}{a}_n = 0$$

$$\lambda_1 \underset{\sim}{a}_2' \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2' \underset{\sim}{a}_2 + \cdots + \lambda_n \underset{\sim}{a}_2' \underset{\sim}{a}_n = 0$$

$$\vdots$$

$$\lambda_1 \underset{\sim}{a}_n' \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_n' \underset{\sim}{a}_2 + \cdots + \lambda_n \underset{\sim}{a}_n' \underset{\sim}{a}_n = 0$$

$$\lambda_1 \underset{\sim}{a}_1' \underset{\sim}{a}_1 + \lambda_2 \times 0 + \cdots + \lambda_n \times 0 = 0$$

$\Rightarrow$
$$\lambda_1 \times 0 + \lambda_2 \underset{\sim}{a}_2' \underset{\sim}{a}_2 + \cdots + \lambda_n \times 0 = 0$$

$$\vdots$$

$$\lambda_1 \times 0 + \lambda_2 \times 0 + \cdots + \lambda_n \underset{\sim}{a}_n' \underset{\sim}{a}_n = 0$$

$\Rightarrow$ $\lambda_1 = 0, \lambda_2 = 0, \ldots, \lambda_n = 0$ as a_i's are non-null vectors, gives $\underset{\sim}{a}_i' \underset{\sim}{a}_i \neq 0$, $i = 1(1)n$.

$\Rightarrow$ The set of vectors $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n\}$ is LIN.

Theorem 21. Any set of n mutually orthogonal, non-null vectors from E^n forms a basis for E^n.

Proof: Let $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3, \ldots, \underset{\sim}{v}_n\}$ be a set of non-null orthogonal vectors from E^n.

Consider the equation

$$\lambda_1 \underset{\sim}{v}_1 + \lambda_2 \underset{\sim}{v}_2 + \cdots + \lambda_n \underset{\sim}{v}_n = \underset{\sim}{0}$$

Note that

$$\underset{\sim}{v}_i (\lambda_1 \underset{\sim}{v}_1 + \lambda_2 \underset{\sim}{v}_2 + \cdots + \lambda_n \underset{\sim}{v}_n) = \underset{\sim}{v}_i \cdot \underset{\sim}{0}$$

$$\Rightarrow \lambda_1 (\underset{\sim}{v}_i \cdot \underset{\sim}{v}_1) + \cdots + \lambda_i (\underset{\sim}{v}_i' \underset{\sim}{v}_i) + \cdots + \lambda_n (\underset{\sim}{v}_i' \underset{\sim}{v}_n) = \underset{\sim}{0}$$

$$\Rightarrow \lambda_i |\underset{\sim}{v}_i|^2 = 0 \quad \forall\ i = 1(1)n$$

$\Rightarrow \lambda_i = 0$, since $|v_i| \neq 0\ \forall\ i = 1(1)n$, i.e. v_i's are non-null.

Hence the set $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_n\}$ is LIN.

We know any set of n LIN set of vectors forms a basis for E^n. Hence, $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_n\}$ forms a basis of E^n.

Result: — If $\{x_1, x_2, \ldots, x_{n-1}\}$ be a set of mutually orthogonal n-component vectors & if $\exists$ two non-null vectors y and z such that each of y and z is orthogonal to any vector belonging to the given set. Then y must be a scalar multiple of z.

Soln → Note that $\{x_1, x_2, \ldots, x_{n-1}, z\}$ is an orthogonal basis of E^n. Now, $y(\neq 0) \in E^n$

$\because \exists$ scalars $\theta_1, \theta_2, \ldots, \theta_n$, at least one $\theta_i \neq 0 \ni$

$$y = \sum_{i=1}^{n-1} \theta_i x_i + \theta_n z$$

$$\Rightarrow y' x_1 = \sum_{i=1}^{n-1} \theta_i x_i' x_1 + \theta_n z' x_1$$

$\because \theta_1 x_1' x_1 = 0$

$\therefore \theta_1 = 0$ as $x_1' x_1 \neq 0$

Similarly it can be shown that $\theta_2 = \theta_3 = \cdots = \theta_{n-1} = 0$.

Thus $y = \theta_n z$ $[\theta_n \neq 0]$.

Orthogonal Basis: —

Definition: — 1) If we consider a set of n mutually orthogonal non-null vectors from E^n then it forms a basis for E^n. Hence, a set of n mutually orthogonal non-null vectors from E^n is known as an Orthogonal Basis for E^n.

Orthonormal Basis: — If a_i is a non-null vector, then $v_i = \frac{a_i}{|a_i|}$ is a vector of unit length.

For a set $\{a_1, a_2, \ldots, a_n\}$ of mutually orthogonal non-null vectors from E^n, we obtain a set $u_i = \frac{a_i}{|a_i|}$, $i = 1(1)n$.

Then, $$u_i \cdot v_j = \frac{a_i \cdot a_j}{|a_i||a_j|} = \begin{cases} 0 & \forall\, i \neq j \\ 1 & \forall\, i = j \end{cases}$$

The set $\{a_1, \ldots, a_n\}$ of mutually orthogonal vectors of unit length and forms a basis for E^n. This type of set of vectors from E^n is known as an orthonormal basis for E^n.

GRAM-SCHMIDT ORTHOGONALISATION PROCESS :

~~By this process a set of independent vectors from E^n can be converted to an~~

Any set of n given linearly independent vectors from E^n can be converted into an orthonormal basis by a procedure known as the Schmidt orthogonalization process. Let us suppose that $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \cdots, \underset{\sim}{a}_n$ are n LIN vectors from E^n. We select any vectors from this set, for example, $\underset{\sim}{a}_1$. This vectors are non-null vectors i.e. $\underset{\sim}{a}_i \neq 0$, $i = 1(1)n$.

Let us define the vector of unit length $\underset{\sim}{u}_1$ as

$$\underset{\sim}{u}_1 = \frac{\underset{\sim}{a}_1}{|\underset{\sim}{a}_1|}.$$

To obtain a vector $\underset{\sim}{v}_2$ orthogonal to $\underset{\sim}{u}_1$, we subtract from $\underset{\sim}{a}_2$ a scalar multiple of $\underset{\sim}{u}_1$; i.e. $\underset{\sim}{v}_2$ is expressed as

$$\underset{\sim}{v}_2 = \underset{\sim}{a}_2 - \lambda_1 \underset{\sim}{u}_1,$$

where λ_1 is determined so that $\underset{\sim}{u}_1' \underset{\sim}{v}_2 = 0$.

$$\text{or, } \underset{\sim}{u}_1'(\underset{\sim}{a}_2 - \lambda_1 \underset{\sim}{u}_1) = 0$$

$$\text{or, } \underset{\sim}{u}_1' \underset{\sim}{a}_2 - \lambda_1 = 0$$

$$\text{or, } \lambda_1 = \underset{\sim}{u}_1' \underset{\sim}{a}_2$$

Therefore, $v_2 = a_2 - (\underset{\sim}{u}_1' \underset{\sim}{a}_2)\underset{\sim}{u}_1$.

A second unit length vector orthogonal to $\underset{\sim}{u}_1$ is defined by

$$\underset{\sim}{u}_2 = \frac{\underset{\sim}{v}_2}{|\underset{\sim}{v}_2|};$$

this can be done since $|v_2| \neq 0$ because

$v_2 = 0$ when, $\underset{\sim}{a}_2 = (\underset{\sim}{u}_1' \underset{\sim}{a}_2)\underset{\sim}{u}_1$

$$= (\underset{\sim}{u}_1' \underset{\sim}{a}_2) \cdot \frac{\underset{\sim}{a}_1}{|\underset{\sim}{a}_1|} = \left(\frac{\underset{\sim}{u}_1' \underset{\sim}{a}_2}{|\underset{\sim}{a}_1|}\right)\underset{\sim}{a}_1$$

It is possible only when $\underset{\sim}{a}_1 = 0$, but $\underset{\sim}{a}_1 \neq 0$ as $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \cdots, \underset{\sim}{a}_n$ are non-zero LIN vectors. so, $|v_2| \neq 0$

To obtain a vector $\underset{\sim}{v}_3$ orthogonal to both $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$, we write –

$$\underset{\sim}{v}_3 = \underset{\sim}{a}_3 - (\underset{\sim}{u}_1' \underset{\sim}{a}_3)u_1 - (\underset{\sim}{u}_2' \underset{\sim}{a}_3)\underset{\sim}{u}_2$$

The third unit vector which is orthogonal to $\underset{\sim}{u}_1, \underset{\sim}{u}_2$ is

$$\underset{\sim}{u}_3 = \frac{\underset{\sim}{v}_3}{|\underset{\sim}{v}_3|}.$$

This is also valid as $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_3$ are LIN; giving $|v_3| \neq 0$.

This procedure is continued until an orthonormal basis is obtained. In general,

$$\underset{\sim}{v}_n = \underset{\sim}{a}_n - \sum_{i=1}^{n-1} (\underset{\sim}{u}_i' \underset{\sim}{a}_n) \underset{\sim}{u}_i ,$$

$$\underset{\sim}{u}_n = \frac{\underset{\sim}{v}_n}{|\underset{\sim}{v}_n|}.$$

Orthonormal vectors are $u_1, u_2, \ldots$
Orthogonal vectors are $v_1, v_2, \ldots$

Example :— Using the Schmidt process, construct an orthonormal basis from $\underset{\sim}{a}_1 = [2, 3, 0]$, $\underset{\sim}{a}_2 = [6, 1, 0]$, $\underset{\sim}{a}_3 = [0, 2, 4]$.

Soln →

Let scalars are $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \lambda_3 \underset{\sim}{a}_3 = (0, 0, 0)$$

$$\Rightarrow \lambda_1 (2, 3, 0) + \lambda_2 (6, 1, 0) + \lambda_3 (0, 2, 4) = (0, 0, 0)$$

$$\therefore 2\lambda_1 + 6\lambda_2 = 0 ; \quad 3\lambda_1 + \lambda_2 + 2\lambda_3 = 0 ; \quad 4\lambda_3 = 0$$

$$\therefore \lambda_3 = 0, \therefore 3\lambda_1 + \lambda_2 = 0, \quad \lambda_1 + 3\lambda_2 = 0$$

$$\therefore \lambda_2 = -3\lambda_1 ; \therefore \lambda_1 + 3(-3\lambda_1) = 0 \Rightarrow \lambda_1 - 9\lambda_1 = 0 \Rightarrow \lambda_1 = 0$$

$$\therefore \lambda_2 = 0$$

$\therefore$ The vectors are LIN.

$$\underset{\sim}{u}_1 = \frac{\underset{\sim}{a}_1}{|\underset{\sim}{a}_1|} = \frac{[2, 3, 0]}{\sqrt{13}} = [0.554, 0.831, 0] ;$$

$$\underset{\sim}{v}_2 = \underset{\sim}{a}_2 - (\underset{\sim}{u}_1' \underset{\sim}{a}_2) \underset{\sim}{u}_1 ;$$

$$(\underset{\sim}{u}_1' \underset{\sim}{a}_2) = (0.554, 0.831, 0)' (6, 1, 0)$$
$$= 4.16$$

$$(\underset{\sim}{u}_1' \underset{\sim}{a}_2) \underset{\sim}{u}_1 = [2.30, 3.45, 0] ;$$

$$\underset{\sim}{v}_2 = [3.70, -2.45, 0] ;$$

$$\underset{\sim}{u}_2 = \frac{\underset{\sim}{v}_2}{|\underset{\sim}{v}_2|} = [0.831, -0.554, 0] ;$$

$$\underset{\sim}{v}_3 = \underset{\sim}{a}_3 - (\underset{\sim}{u}_1' \underset{\sim}{a}_3) \underset{\sim}{u}_1 - (\underset{\sim}{u}_2' \underset{\sim}{a}_3) \underset{\sim}{u}_2 ;$$

$$\underset{\sim}{u}_1' \underset{\sim}{a}_3 = 1.664, \quad \underset{\sim}{u}_2' \underset{\sim}{a}_3 = -1.106 ;$$

$$(\underset{\sim}{u}_1' \underset{\sim}{a}_3) \underset{\sim}{u}_1 = [0.921, 1.386, 0], \quad (\underset{\sim}{u}_2' \underset{\sim}{a}_3) \underset{\sim}{u}_2 = [-0.921, 0.614, 0] ;$$

$$\underset{\sim}{v}_3 = [0, 0, 4] ;$$

$$\underset{\sim}{u}_3 = [0, 0, 1] .$$

Example 2) Construct an orthogonal basis for E^3 with $(1,1,1)$ as the starting vector.

Soln →

$$\underset{\sim}{a}_1 = (1,1,1).$$

$$\underset{\sim}{u}_1 = \frac{(1,1,1)}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}(1,1,1).$$

Note that $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ forms a basis for E^3.

Now, $(1,1,1) = 1.\underset{\sim}{e}_1 + 1.\underset{\sim}{e}_2 + 1.\underset{\sim}{e}_3$

$\therefore \{(1,1,1), \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ forms a basis for E^3, by replacement theorem.

Now, $\underset{\sim}{v}_1 = (1,1,1)$ and $\underset{\sim}{u}_1 = \frac{\underset{\sim}{v}_1}{|\underset{\sim}{v}_1|} = \frac{1}{\sqrt{3}}(1,1,1)$

$$\therefore \underset{\sim}{v}_2 = \underset{\sim}{a}_2 - (\underset{\sim}{a}_2' \underset{\sim}{u}_1).\underset{\sim}{u}_1$$

$$= \underset{\sim}{e}_2 - (\underset{\sim}{e}_2' \underset{\sim}{u}_1).\underset{\sim}{u}_1$$

$$= (0,1,0) - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}(1,1,1)$$

$$= \left(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right).$$

$$\underset{\sim}{u}_2 = \frac{\left(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)}{\sqrt{2/3}}.$$

$$= \frac{-1}{\sqrt{6}}(1,-2,1)$$

$$\underset{\sim}{v}_3 = \underset{\sim}{a}_3 - (\underset{\sim}{a}_3' \underset{\sim}{u}_1).\underset{\sim}{u}_1 - (\underset{\sim}{a}_3' \underset{\sim}{u}_2).\underset{\sim}{u}_2$$

$$= \underset{\sim}{e}_3 - (\underset{\sim}{e}_3' \underset{\sim}{u}_1).\underset{\sim}{u}_1 - (\underset{\sim}{e}_3' \underset{\sim}{u}_2).\underset{\sim}{u}_2$$

$$= (0,0,1) - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}(1,1,1) + \frac{2}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}}(1,-2,1)$$

$$= (0,-1,1).$$

$$\underset{\sim}{u}_3 = \frac{\underset{\sim}{v}_3}{|\underset{\sim}{v}_3|} = \frac{1}{\sqrt{2}}(0,-1,1).$$

Hence, $\left\{(1,1,1), -\frac{1}{3}(1,-2,1), (0,-1,1)\right\}$ is an orthogonal basis for E^3.

Also, $\left\{\frac{1}{\sqrt{3}}(1,1,1), -\frac{1}{\sqrt{6}}(1,-2,1), \frac{1}{\sqrt{2}}(0,-1,1)\right\}$ is an orthonormal basis for E^3.

[C.U.]

Example 3> Construct an orthonormal basis for E^4 with $\frac{1}{2}(1,1,1,1)$ as the starting vector.

Soln. →

$$\underset{\sim}{a}_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$\underset{\sim}{u}_1 = \frac{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\sqrt{1/4+1/4+1/4+1/4}} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

Note that, $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4\}$ forms a basis for E^4.

Now, By replacement theorem $\left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4\right\}$ is also a basis for E^4.

Since $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}\underset{\sim}{e}_1 + \frac{1}{2}\underset{\sim}{e}_2 + \frac{1}{2}\underset{\sim}{e}_3 + \frac{1}{2}\underset{\sim}{e}_4$.

Now, we shall apply orthogonalization process to the set

$$\left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4\right\}, \quad \underset{\sim}{u}_1 = \frac{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\sqrt{1/4+1/4+1/4+1/4}}$$

$$\therefore \underset{\sim}{v}_2 = \underset{\sim}{a}_2 - (\underset{\sim}{a}_2, \underset{\sim}{u}_1)\cdot\underset{\sim}{u}_1$$

$$= \underset{\sim}{e}_2 - (\underset{\sim}{e}_2, \underset{\sim}{u}_1)\cdot\underset{\sim}{u}_1$$

$$= (0,1,0,0) - \frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$= \left(-\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$$

$$\underset{\sim}{u}_2 = \frac{\left(-\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}\right)}{\sqrt{1/16 + 9/16 + 9/16 + 1/16}}$$

$$= -\frac{1}{4}\Big/\sqrt{\frac{12}{16}}\,(1,-3,1,1)$$

$$= -\frac{1}{2\sqrt{3}}(1,-3,1,1)$$

$$\underset{\sim}{v}_3 = \underset{\sim}{e}_3 - (\underset{\sim}{u}_1, \underset{\sim}{e}_3)\,\underset{\sim}{u}_1 - (\underset{\sim}{u}_2, \underset{\sim}{e}_3)\,\underset{\sim}{u}_2$$

$$= (0,0,1,0) - \frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) - \frac{1}{(2\sqrt{3})^2}(1,-3,1,1)$$

$$= \left(-\frac{1}{3}, 0, \frac{2}{3}, -\frac{1}{3}\right)$$

$$\underset{\sim}{u}_3 = \frac{\left(-\frac{1}{3}, 0, \frac{2}{3}, -\frac{1}{3}\right)}{\sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}}} = -\frac{1}{\sqrt{6}}(1,0,-2,1)$$

$$\underset{\sim}{v}_4 = \underset{\sim}{e}_4 - (\underset{\sim}{u}_1'\underset{\sim}{e}_4)\underset{\sim}{u}_1 - (\underset{\sim}{u}_2'\underset{\sim}{e}_4)\underset{\sim}{u}_2 - (\underset{\sim}{u}_3'\underset{\sim}{e}_4)\underset{\sim}{u}_3$$

$$= (0,0,0,1) - \frac{1}{2}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) - \frac{1}{(2\sqrt{3})^2}(1,-3,1,1) - \frac{1}{(\sqrt{6})^2}(1,0,-2,1)$$

$$= \left(-\frac{1}{2},0,0,\frac{1}{2}\right)$$

$$\underset{\sim}{u}_4 = \frac{\left(-\frac{1}{2},0,0,\frac{1}{2}\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}}} = -\frac{1}{2}\cdot\sqrt{2}(1,0,0,-1) = -\frac{1}{\sqrt{2}}(1,0,0,-1)$$

Now, $\left\{\frac{1}{2}(1,1,1,1), -\frac{1}{2\sqrt{3}}(1,-3,1,1), -\frac{1}{\sqrt{6}}(1,0,-2,1), -\frac{1}{\sqrt{2}}(1,0,0,-1)\right\}$ is an orthonormal basis for E^3.

Result: → Let $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$ be two LIN vectors and $\underset{\sim}{b}\,(\neq 0)$ is orthogonal to $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$. Show that $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{b}$ are LIN.

Proof: → Let, $S(\underset{\sim}{a}_1,\underset{\sim}{a}_2)$ and $S(\underset{\sim}{b})$ be two subspaces generated by $\{\underset{\sim}{a}_1,\underset{\sim}{a}_2\}$ and $\{\underset{\sim}{b}\}$ respectively. As $\underset{\sim}{b}(\neq 0)$ is orthogonal to $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$ both, $S(\underset{\sim}{a}_1,\underset{\sim}{a}_2)$ and $S(\underset{\sim}{b})$ are mutually orthogonal subspaces, implying —

$$S(\underset{\sim}{a}_1,\underset{\sim}{a}_2) \cap S(\underset{\sim}{b}) = \{\underset{\sim}{0}\}$$

To show $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{b}$ are linearly independent, consider—

$$\lambda_1\underset{\sim}{a}_1 + \lambda_2\underset{\sim}{a}_2 + \lambda\underset{\sim}{b} = \underset{\sim}{0}$$

$$\Rightarrow \lambda_1\underset{\sim}{a}_1 + \lambda_2\underset{\sim}{a}_2 = -\lambda\underset{\sim}{b}$$

$$\Rightarrow \begin{cases} \lambda_1\underset{\sim}{a}_1 + \lambda_2\underset{\sim}{a}_2 = 0 & \text{as } S(\underset{\sim}{a}_1,\underset{\sim}{a}_2)\cap S(\underset{\sim}{b}) = \{\underset{\sim}{0}\} \\ \lambda\underset{\sim}{b} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_1 = 0,\ \lambda_2 = 0 & \text{as } \underset{\sim}{a}_1, \underset{\sim}{a}_2 \text{ are LIN.} \\ \lambda = 0 & \text{as } b \neq 0. \end{cases}$$

$\Rightarrow \underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{b}$ are LIN.

Problem :- Let S_1 and S_2 be two vector subspaces generated by $\{(2,3,0,-1),(-1,0,2,3)\}$ and $\{(1,3,2,2),(3,3,-2,-4)\}$, respectively. Show that S_1 and S_2 are identical.

Solution : Here the two sets are both LIN.
To show $S_1 = S_2$, we consider the following equations with unknown scalars $\lambda_1, \lambda_2, \mu_1, \mu_2$:

$$\lambda\begin{pmatrix}2\\3\\0\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}-1\\0\\2\\3\end{pmatrix} = \mu_1\begin{pmatrix}1\\3\\2\\2\end{pmatrix} + \mu_2\begin{pmatrix}3\\3\\-2\\-4\end{pmatrix}$$

$$\text{or, } \begin{pmatrix}2 & -1 & -1 & -3\\ 3 & 0 & -3 & -3\\ 0 & 2 & -2 & 2\\ 1 & 3 & -2 & 4\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \mu_1\\ \mu_2\end{pmatrix} = \underset{\sim}{0} \quad \text{——— (1)}$$

We reduce the co-efficient matrix A of (1) via elementary row operations :

$$A = \begin{pmatrix}2 & -1 & -1 & -3\\ 3 & 0 & -3 & -3\\ 0 & 2 & -2 & 2\\ 1 & 3 & -2 & 4\end{pmatrix} \sim \begin{pmatrix}1 & 0 & -1 & -1\\ 0 & 1 & -1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} = A_1 \quad \text{——— (2)}$$

$$\sim \begin{pmatrix}1/2 & 1/2 & 0 & 1\\ -1/2 & 1/2 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} = A_2 \quad \text{——— (3)}$$

(2) $\Rightarrow$

$$\Rightarrow \left.\begin{aligned}\lambda_1 &= \mu_1 + \mu_2\\ \lambda_2 &= \mu_1 - \mu_2\end{aligned}\right\} \to (4) \qquad (3) \Rightarrow \;\Rightarrow \left.\begin{aligned}\mu_1 &= (\lambda_1+\lambda_2)/2\\ \mu_2 &= (\lambda_1-\lambda_2)/2\end{aligned}\right\} \text{——— (5)}$$

For $\underset{\sim}{a} \in S_1$, $\exists\ \lambda_1, \lambda_2 \ni \lambda_1 \underset{\sim}{a_1} + \lambda_2 \underset{\sim}{a_2} = \underset{\sim}{a}$. Putting there λ_1, λ_2 in (5), we get μ_1, μ_2 so that (1) is satisfied, and thus $\underset{\sim}{a}$ being a linear combination of $(1,3,2,2)$ and $(3,3,-2,-4)$ lies in S_2. Hence, $S_1 \subseteq S_2$. For the same reason, using (4), $S_2 \subseteq S_1$, and thus we have $S_1 = S_2$.

C.U.

2) Obtain the maximum collection of linearly independent vectors using the following :

$\underset{\sim}{\alpha}_1 = (1, 0, 1, 1, 1, 0)'$, $\underset{\sim}{\alpha}_2 = (1, 0, 1, 1, 0, 1)'$,
$\underset{\sim}{\alpha}_3 = (1, 0, 1, 0, 1, 1)'$, $\underset{\sim}{\alpha}_4 = (1, 1, 0, 1, 1, 0)'$,
$\underset{\sim}{\alpha}_5 = (1, 1, 0, 1, 0, 1)'$, $\underset{\sim}{\alpha}_6 = (1, 1, 0, 0, 1, 1)'$

Also find a vector which is orthogonal to each of vectors belonging to the maximal collection.

Soln. ⇒ $\exists$ any λ ∋

$$\underset{\sim}{\alpha}_2 = \lambda \underset{\sim}{\alpha}_1$$

$$\Rightarrow \left\{ \begin{array}{l} \lambda_1 = 1 \\ \lambda = 1 \\ \lambda = 1 \\ \lambda = 0 \end{array} \right.$$

which is not possible either all λi's are zero, then $\underset{\sim}{\alpha}_2$ and $\underset{\sim}{\alpha}_1$ are LIN.

If possible, let $\underset{\sim}{\alpha}_3 = \lambda_1 \underset{\sim}{\alpha}_1 + \lambda_2 \underset{\sim}{\alpha}_2$

$$\Rightarrow \left\{ \begin{array}{l} \lambda_1 + \lambda_2 = 1 \\ \lambda_1 + \lambda_2 = 0 \end{array} \right\} \text{ which is not possible.}$$

$\therefore \{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\alpha}_3\}$ is LIN.

If possible, let $\underset{\sim}{\alpha}_4 = \lambda_1 \underset{\sim}{\alpha}_1 + \lambda_2 \underset{\sim}{\alpha}_2 + \lambda_3 \underset{\sim}{\alpha}_3$

$$\Rightarrow \left\{ \begin{array}{l} \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ \lambda_1 + \lambda_2 = 1 \\ \lambda_1 + \lambda_3 = 1 \\ \lambda_2 + \lambda_3 = 0 \end{array} \right\} \text{ which is not possible.}$$

$\therefore \{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\alpha}_3, \underset{\sim}{\alpha}_4\}$ is LIN.

If possible, let $\underset{\sim}{\alpha}_5 = \lambda_1 \underset{\sim}{\alpha}_1 + \lambda_2 \underset{\sim}{\alpha}_2 + \lambda_3 \underset{\sim}{\alpha}_3 + \lambda_4 \underset{\sim}{\alpha}_4$

$$\Rightarrow \left\{ \begin{array}{l} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ \lambda_4 = 1 \\ \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_1 + \lambda_3 + \lambda_4 = 0 \\ \lambda_2 + \lambda_3 = 1 \end{array} \right.$$

$\Rightarrow \lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 0, \lambda_4 = 1.$

i.e. $\underset{\sim}{\alpha}_5$ is LD of $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\alpha}_3, \underset{\sim}{\alpha}_4$.

If possible, let $\underset{\sim}{\alpha}_6 = \lambda_1 \underset{\sim}{\alpha}_1 + \lambda_2 \underset{\sim}{\alpha}_2 + \lambda_3 \underset{\sim}{\alpha}_3 + \lambda_4 \underset{\sim}{\alpha}_4$

$$\Rightarrow \left\{ \begin{array}{l} \lambda_4 = 1 \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_1 + \lambda_2 + \lambda_4 = 0 \\ \lambda_2 + \lambda_3 = 1 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = 0 \\ \lambda_3 = 1 \\ \lambda_4 = 1 \end{array} \right.$$

Hence, $\underset{\sim}{\alpha}_6$ is LD of $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\alpha}_3, \underset{\sim}{\alpha}_4$.

Hence, a maximal collection of linearly independent vectors will be . $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\alpha}_3, \underset{\sim}{\alpha}_4$.

The choice of this 4 LIN vectors is not unique.

Let the vector be $\underset{\sim}{x} = \{x_1, x_2, x_3, \ldots, x_6\}'$

We are to find x_i's $\ni$

$$\underset{\sim}{\alpha}_i' \underset{\sim}{x} = 0 \quad \forall\ i = 1(1)4.$$

$$\Rightarrow \begin{cases} x_1 + x_3 + x_4 + x_5 = 0 \\ x_1 + x_3 + x_4 + x_6 = 0 \\ x_1 + x_3 + x_5 + x_6 = 0 \\ x_1 + x_2 + x_4 + x_5 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = x_3 \\ x_4 = x_5 = x_6 \end{cases}$$

$$\underset{\sim}{x} = \begin{pmatrix} -\beta - 2\alpha \\ \beta \\ \beta \\ \alpha \\ \alpha \\ \alpha \end{pmatrix}$$

choose, $\alpha = 1, \beta = 1$, then $\underset{\sim}{x} = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

w.r.t. this particular problem we have exactly 2 LIN vectors each of which is orthogonal to $\underset{\sim}{\alpha}_i, i = 1(1)4$.

A choice of such two vectors (LIN) are: $\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

Orthogonal spaces: — A vector $\underset{\sim}{a}$ is orthogonal to every vector in S_n, i.e. if $\underset{\sim}{a}'\underset{\sim}{b} = 0 \ \forall\ \underset{\sim}{b} \in S_n$.

So, two subspaces S_n' and S_n'' of E^n are said to be orthogonal if $\underset{\sim}{a}'\underset{\sim}{b} = 0 \ \forall\ \underset{\sim}{a} \in S_n'$ and $\forall\ \underset{\sim}{b} \in S_n''$;

~~i.e. if any vector, $\underset{\sim}{a} \in S_n'$, is orthogonal to S_n'', i.e. if~~

Orthocomplement space of Vector space :-

Defn: 1) Let S_n be a subspace of E^n. The set of vectors orthogonal to every vector in S_n, is called the orthogonal complement of S_n, and is denoted by $O(S_n)$.

2) Let S_n be a subspace of E^n, then orthocomplement of S_n is defined as – $O(S_n) = \{\underset{\sim}{x} : \underset{\sim}{x}'\underset{\sim}{y} = 0 \ \forall \ \underset{\sim}{y} \in S_n\}$.

Clearly S_n and $O(S_n)$ are two distinct subspaces of E^n, i.e. $S_n \cap O(S_n) = \{\underset{\sim}{0}\}$.

NOTE:- E^n is partitioned into S_n and $O(S_n)$, i.e. S_n and $O(S_n)$ are disjoint. Therefore, if we have a basis of S_n and a basis of $O(S_n)$, combining these two, we get a basis of E^n.

Basis Combination :- Consider a subspace S_n of V_n. Let the number of vectors in a basis of S_n be s and that a basis of V_n be r, then a basis of S_n can be extended to a basis of V_n by augmenting $r-s$ LIN vectors, not belonging to S_n.

Ex. Find the basis of the orthocomplement of span of $\{\underset{\sim}{\alpha_1}, \underset{\sim}{\alpha_2}\}$ where $\underset{\sim}{\alpha_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ and $\underset{\sim}{\alpha_2} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$.

Soln.→ It is enough to find two LIN vectors each of which is orthogonal to both $\underset{\sim}{\alpha_1}$ and $\underset{\sim}{\alpha_2}$. Let us denote those vectors by $\underset{\sim}{\beta_1}$ and $\underset{\sim}{\beta_2}$, then $\{\underset{\sim}{\beta_1}, \underset{\sim}{\beta_2}\}$ will be a basis of the required vector space.

Consider a non-null vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ which is orthogonal to $\underset{\sim}{\alpha_i}$, $i=1,2$.

$$x_1 + x_2 - x_3 - x_4 = 0$$
$$x_1 - x_2 + x_3 - x_4 = 0$$

$$x_1 = x_4$$
$$x_2 = x_3$$

$\therefore \underset{\sim}{\beta_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\underset{\sim}{\beta_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Theorem 22. $O(S_n)$ is a vector subspace of E^n.

Proof:→ Let $\{\underset{\sim}{b_1}, \underset{\sim}{b_2}, \ldots, \underset{\sim}{b_n}\}$ be a basis of S_n.

Consider two vectors $\underset{\sim}{x}$ and $\underset{\sim}{y} \in O(S_n)$.

Then $\underset{\sim}{x}$ & $\underset{\sim}{y}$ are orthogonal to S_n.

$\Rightarrow \underset{\sim}{x} \perp \underset{\sim}{b_i}, \; i = 1(1)n$

and $\underset{\sim}{y} \perp \underset{\sim}{b_i}, \; i = 1(1)n$.

$\Rightarrow (\underset{\sim}{x} + \underset{\sim}{y}) \underset{\sim}{b_i} = \underset{\sim}{0} \quad \forall \; i = 1(1)n$

$\Rightarrow (\underset{\sim}{x} + \underset{\sim}{y}) \perp \underset{\sim}{b_i} = \underset{\sim}{0} \quad \forall \; i = 1(1)n$

$\Rightarrow (\underset{\sim}{x} + \underset{\sim}{y})$ is orthogonal to S_n.

$\Rightarrow (\underset{\sim}{x} + \underset{\sim}{y}) \in O(S_n)$.

Again, for any scalar λ,

$(\lambda \cdot \underset{\sim}{x}) \cdot \underset{\sim}{b_i} = \lambda (\underset{\sim}{x} \cdot \underset{\sim}{b_i}) = 0 \quad \forall \; i = 1(1)n$

$\Rightarrow \lambda \underset{\sim}{x} \perp b_i, \; \forall \; i = 1(1)n$.

$\Rightarrow \lambda \underset{\sim}{x}$ is orthogonal to S_n.

$\Rightarrow \lambda \underset{\sim}{x} \in O(S_n)$.

Hence, $O(S_n)$ is a vector space and $O(S_n) \subseteq E^n$.

Hence, $O(S_n)$ is a vector subspace of E^n.

Example:→ i) $S_3 = \{(x_1, 0, x_3) : x_1, x_3 \in \mathbb{R}\}$

ii) $S_3 = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$

Soln.→ i) Let, $\underset{\sim}{x} \in S_3$.

Then, $\underset{\sim}{x} = (x_1, 0, x_3)$

$= x_1 (1, 0, 0) + x_3 (0, 0, 1)$

Clearly, $\{\underset{\sim}{e_1}, \underset{\sim}{e_3}\}$ forms a basis for S_3. Let $\underset{\sim}{y}$ is orthogonal to S_3.

$\therefore \underset{\sim}{y}$ is orthogonal to $\underset{\sim}{e_1}$ and $\underset{\sim}{e_3}$.

$\Rightarrow \underset{\sim}{y} \cdot \underset{\sim}{e_1} = 0 \quad \Rightarrow y_1 = 0 = y_3$

$\Rightarrow \underset{\sim}{y} \cdot \underset{\sim}{e_3} = 0 \quad \therefore \underset{\sim}{y} = (0, y_2, 0) \perp S_3$.

$\therefore O(S_3) = \{(0, y_2, 0) : y_2 \in \mathbb{R}\}$

Remark: $\dim(S_3) = 2$

$\dim(O(S_3)) = 1$

$\therefore \dim(S_3) + \dim O(S_3) = 3 = \dim(E^3)$.

ii) $S_3 = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$

$\therefore$ Let $\underset{\sim}{x} \in O(S_3)$

$\underset{\sim}{x} = (x_1, x_2, -x_1 - x_2)$ $\quad [\because x_1 + x_2 + x_3 = 0 \quad \therefore x_3 = -x_1 - x_2]$

$= x_1(1, 0, -1) + x_2(0, 1, -1)$

Clearly, $\{(1, 0, -1), (0, 1, -1)\}$ forms a basis for S_3.

Let $\underset{\sim}{y}$ is orthogonal to S_3.

$\therefore \underset{\sim}{y} \perp (1, 0, -1)$, $\underset{\sim}{y} \perp (0, 1, -1)$.

Theorem 23. Every vector space V has a basis.

Soln. $\rightarrow$ If $V = \{\underset{\sim}{0}\}$ then the basis of V can be formed by using the vector $\underset{\sim}{0}$ only.

Let us consider the case, where V is a non-trivial vector space. Then a basis of V can be formed by choosing sequencially non-null vectors $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots\}$ from $V \ni$ no $\underset{\sim}{a_i}$ is dependent on its predecessors.

At first we choose a non-null vector $\underset{\sim}{a_1}$ from V & see whether the other vectors in V can be expressed as a scalar multiple of it. If yes then we stop and the basis of V is formed by $\underset{\sim}{a_1}$ along. If no then we choose a second non-null vector $\underset{\sim}{a_2} \in V$, which is not a scalar multiple of $\underset{\sim}{a_1}$ and check whether the remaining vectors can be expressed as a linear combination of $\underset{\sim}{a_1}$ and $\underset{\sim}{a_2}$. If yes then we stop but if no then we continue picking up another vector in V which is LIN of the previous two and so on.

In this process it may so happen that after the k^{th} stage no independent vector is left in V, in which case $\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots, \underset{\sim}{a_k}$ constitute a basis of V, and V is said to be finite (k) dimensional vector space.

Problem: → i) If S be a set of vectors and $O(S)$ is the set of vectors which is orthogonal to S. Then S.T. $S \subset O_{(O(S))}$. Discuss the situation where $S = O_{(O(S))}$. Discuss the decomposition is unique.

Soln. →

Hints: Consider a vector $\underset{\sim}{x} \in S$ then $\underset{\sim}{x}'\underset{\sim}{y} = 0 \; \forall \; \underset{\sim}{y} \in O(S)$.

Here, $S \subset O_{(O(S))}$

When S is a subspace then $x \in O_{(O(S))}$, then $x \in S$.

In that case, $O_{(O(S))} = S$.

☑ If possible suppose $\underset{\sim}{x} = \underset{\sim}{y}_1 + \underset{\sim}{z}_1 \; \forall \; \underset{\sim}{y}_1 \in S_n$ and $\underset{\sim}{z}_1 \in O(S_n)$

Now, $\underset{\sim}{y} \in S_n$, $\underset{\sim}{y}_1 \in S_n \; \therefore \; \underset{\sim}{y} - \underset{\sim}{y}_1 \in S_n$,

$\underset{\sim}{z} \in O(S_n)$, $\underset{\sim}{z}_1 \in O(S_n)$, $\therefore \; \underset{\sim}{z} - \underset{\sim}{z}_1 \in O(S_n)$

Then, $\underset{\sim}{x} - \underset{\sim}{x} = \underset{\sim}{y} - \underset{\sim}{y}_1 + \underset{\sim}{z} - \underset{\sim}{z}_1$

$\Rightarrow \underset{\sim}{y} - \underset{\sim}{y}_1 = -(\underset{\sim}{z} - \underset{\sim}{z}_1)$

So, here, $(\underset{\sim}{y} - \underset{\sim}{y}_1)$ and $(\underset{\sim}{z} - \underset{\sim}{z}_1) \in S_n \cap O(S_n)$.

Since $\underset{\sim}{0}$ is the common vector between $S_n \cap O(S_n)$

We get, $\underset{\sim}{y} - \underset{\sim}{y}_1 = \underset{\sim}{0}$, $\underset{\sim}{z} - \underset{\sim}{z}_1 = \underset{\sim}{0}$

$\therefore$ The above decomposition is unique.

2) If S_r is a r-dimensional subspace of the n-dimensional vector space E^n, then S.T. $\dim[O(S_r)] = n - \dim(S_r)$.

Proof:→ Let $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_r}\}$ be an orthonormal basis for S_r then $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_r}\}$ can be extended to $A = \{\underset{\sim}{a_1}, \ldots, \underset{\sim}{a_{r+1}}, \ldots, \underset{\sim}{a_n}\}$ where $\underset{\sim}{a_{r+1}}, \ldots, \underset{\sim}{a_n} \in A \ni A$ forms an orthonormal basis for E^n. Consider a vector $\underset{\sim}{b} \in E^n \ni \underset{\sim}{b}'\underset{\sim}{a_i} = 0 \;\forall\; i = 1(1)r$.

Hence, $\underset{\sim}{b}$ is orthogonal to every vector in S_r, since any vector in S_r can be expressed as a linear combination of $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_r}\}$

Now, $\underset{\sim}{b} \in E^n$ and $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ forms a basis for V_n so $\exists$ scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \ni \underset{\sim}{b} = \sum_{i=1}^{n} \lambda_i \underset{\sim}{a_i}$.

Now, $\underset{\sim}{a_i}\underset{\sim}{b} = 0$

$\Rightarrow \underset{\sim}{a_i}(\lambda_1 \underset{\sim}{a_1} + \lambda_2 \underset{\sim}{a_2} + \cdots + \lambda_n \underset{\sim}{a_n}) = 0$

$\Rightarrow \lambda_1 \underset{\sim}{a_1}'\underset{\sim}{a_1} = 0$ [since $\underset{\sim}{a_i}'\underset{\sim}{a_j} = 0 \;\forall\; i \neq j$]

$\Rightarrow \lambda_1 = 0$ [$\because \|\underset{\sim}{a_1}\| = 1$]

Similarly, $\underset{\sim}{a_2}'\underset{\sim}{b} = 0 \Rightarrow \lambda_2 = 0$

$\vdots$

$\underset{\sim}{a_r}'\underset{\sim}{b} = 0 \Rightarrow \lambda_r = 0$

Hence, $\underset{\sim}{b} = \sum_{i=r+1}^{n} \lambda_i \underset{\sim}{a_i}$

Hence, every vector in $O(S_r)$ can be expressed as a linear combination of $\{\underset{\sim}{a_{r+1}}, \underset{\sim}{a_{r+2}}, \ldots, \underset{\sim}{a_n}\}$, in other words $\{\underset{\sim}{a_{r+1}}, \underset{\sim}{a_{r+2}}, \ldots, \underset{\sim}{a_n}\}$ spans $O(S_r)$.

Since $\{\underset{\sim}{a_{r+1}}, \underset{\sim}{a_{r+2}}, \ldots, \underset{\sim}{a_n}\}$ is a subset of lineably independent set $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$, $\therefore \{\underset{\sim}{a_{r+1}}, \underset{\sim}{a_{r+2}}, \ldots, \underset{\sim}{a_n}\}$ is LIN and it forms a basis for $O(S_r)$.

$\therefore \dim[O(S_r)] = n - r = n - \dim(S_r)$.

[C.U.] Show that for a matrix $A^{n\times n}$, A is idempotent if and only if $R(A)+R(I_n-A)=n$.

Soln. → If Part: → $R(A)+R(I_n-A)=n$ —— i

$V_c(I_n) \subseteq V_c\{A\} + V_c\{(I_n-A)\}$ dim[

$\Rightarrow \dim[V_c(I_n)] \leq \dim[V_c\{A\} + V_c(I_n-A)\}]$

$V_c(A) \cap V_c(I_n-A) = \underset{\sim}{0}$

Considering another matrix $B \ni B=(I_n-A)A$

so, every column vector of $B \in V_c(I_n-A)$

Again, $B = A(I_n-A)$

So, every column vector of $B \in V_c(A)$

$\therefore B \in V_c(A) \cap V_c(I_n-A)$

$\therefore$ every column vectors of B is $\underset{\sim}{0}$.

$\therefore B = 0$

$\therefore A(I_n-A)=0$

$\therefore A^2 = A$, i.e. A is an idempotent matrix.

Only if part: →

A is an idempotent matrix of order n.

i.e. $A^2=A$

$\Rightarrow A(I_n-A)=0$

Now, the rank of sum of two matrices can't exceed of the sum of their ranks, so we have

$\text{Rank}[A+I_n-A] \leq \text{Rank}(A) + \text{Rank}(I_n-A)$

$\Rightarrow \text{Rank}(A)+\text{Rank}(I_n-A) \geq n$ —— ① [As, $\text{Rank}(I_n)=n$]

From Sylvester's inequality,

$\text{Rank}[A(I_n-A)] \geq \text{Rank}(A)+\text{Rank}(I_n-A) - n$

$\Rightarrow \text{Rank}(A)+\text{Rank}(I_n-A) \leq n$ —— ② [$\because$ the product of two matrices is a zero matrix, then its rank must be zero]

Comparing ① & ②, we get

$\text{Rank}(A)+\text{Rank}(I_n-A) = n.$

C.U.

Show that the choice of basis is not unique by giving suitable example.

ANS:- Consider a vector subspace $V_2(F)$.

As we know that $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}\}$ form a basis of $V_2(F)$.

Now, to show that the choice of basis is not unique we are to show that $\{\lambda \underset{\sim}{e_1}, \underset{\sim}{e_2}\}$ will form a basis of $V_2(F)$ for any $\lambda \neq 0$.

Considering $\underset{\sim}{a} = \begin{pmatrix} \underset{\sim}{a_1} \\ \underset{\sim}{a_2} \end{pmatrix} \in V_2(F)$

$$\underset{\sim}{a} = \frac{a_1}{\lambda}(\lambda \underset{\sim}{e_1}) + a_2 \underset{\sim}{e_2}$$

So, $\underset{\sim}{a}$ can be written as a linear combination of ~~basis~~ $\{\lambda \underset{\sim}{e_1}, \underset{\sim}{e_2}\}$.

To show $\{\lambda \underset{\sim}{e_1}, \underset{\sim}{e_2}\}$ linearly independent, considering the equation with scalars ν_1, ν_2

$$\nu_1 \lambda \underset{\sim}{e_1} + \nu_2 \underset{\sim}{e_2} = \underset{\sim}{0}$$

$\Rightarrow \nu_1 \lambda = 0, \ \nu_2 = 0$, as $\underset{\sim}{e_1}, \underset{\sim}{e_2}$ are LIN.

$\Rightarrow \nu_1 = 0$ as $\lambda \neq 0$, $\nu_2 = 0$

So, $\{\lambda \underset{\sim}{e_1}, \underset{\sim}{e_2}\}$ being LIN vectors span $V_2(F)$. So, $\{\lambda \underset{\sim}{e_1}, \underset{\sim}{e_2}\}$ form a basis of $V_2(F)$.

Hence, the result is proved by e.g.

— x —

WORKED EXAMPLES:-

1. If $U = L\{(1,2,1),(2,1,3)\}$, $W = L\{(1,0,0),(0,0,1)\}$, show that U and W are subspace of $\mathbb{R}^3$. Determine $\dim U$, $\dim W$, $\dim(U\cap W)$, Deduce that $\dim(U+W) = 3$.

Solution:- Let $\alpha = (1,2,1)$, $\beta = (2,1,3)$, $\gamma = (1,0,0)$, $\delta = (0,0,1)$

$\{\alpha, \beta\}$ is linearly independent and therefore U is a subspace of $\mathbb{R}^3$ of dimension 2.

$\{\gamma, \delta\}$ is linearly independent and therefore W is a subspace of $\mathbb{R}^3$ of dimension 2.

Let λ be a vector in $U\cap W$. Then $\lambda = a\alpha + b\beta$ for some real a & b. also $\lambda = c\gamma + d\delta$ for some real c & d.

therefore, $a(1,2,1) + b(2,1,3) = c(1,0,0) + d(0,0,1)$

$\Rightarrow a+2b = c,\ 2a+b = 0,\ a+3b = d.$

so, $a = -b/2,\ c = \frac{3}{2}b,\ d = \frac{5}{2}b,$

$\therefore \lambda = b(3/2, 0, 5/2)\ ;\ b\in\mathbb{R}$

Hence $U\cap W$ be a subspace of dimension 1.

So, $\dim(U+W) = \dim(U) + \dim(W) - \dim(U\cap W)$
$= 2+2-1 = 3.$

2. Find a basis and the dimension of the subspace W of $\mathbb{R}^3$, where $W = \{(x,y,z)\in\mathbb{R}^3 : x+y+z = 0\}$.

Sol. Let $\xi = (a,b,c)\in W$. Then $a,b,c\in\mathbb{R}$ and $a+b+c = 0$.

therefore $\xi = (a, b, -a-b) = a(1,0,-1) + b(0,1,-1)$

Let $\alpha = (1,0,-1)$, $\beta = (0,1,-1)$.

We find α and β are linearly independent in $W = L\{\alpha, \beta\}$.

So, $\{\alpha, \beta\}$ is a basis of W and $\dim(W) = 2$.

3. Find a basis and the dimension of the subspace W of $\mathbb{R}^3$, where $W = \{(x,y,z)\in\mathbb{R}^3 : x+2y+z = 0,\ 2x+y+3z = 0\}$.

Sol.

Let $\xi = (a,b,c)\in W$,

$a+2b+c = 0,\ 2a+b+3c = 0\ ;\ a,b,c\in\mathbb{R}.$

Solving, we have $\frac{a}{5} = \frac{b}{-1} = \frac{c}{-3} = k$

ξ takes the form $k\begin{pmatrix}5\\-1\\-3\end{pmatrix}'$, where k is an arbitrary real no.

Therefore, $W = L\{\alpha\}$, where $\alpha = (5,-1,-3)$.

Since $\{\alpha\}$ is a linearly independent set, $\{\alpha\}$ is a basis of W and $\dim(W) = 1$.

4. Extend the set of vectors $\{(2,3,-1),(1,-2,-4)\}$ to an orthogonal basis of the euclidean space $\mathbb{R}^3$ with standard inner product and then find the associated orthonormal basis.

Solution:- Let $\underset{\sim}{\alpha}_1=(2,3,-1)$, $\underset{\sim}{\alpha}_2=(1,-2,-4)$ $\therefore$ $\underset{\sim}{\alpha}_1' \underset{\sim}{\alpha}_2 = \underset{\sim}{0}$.
$\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2$ are orthogonal vectors. Let $\underset{\sim}{\alpha}_3=(0,0,1)$. Then $\{\alpha_1,\alpha_2,\alpha_3\}$ is linearly independent because $\begin{vmatrix} 2 & 3 & -1 \\ 1 & -2 & -4 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$.

So, $\{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\alpha}_3\}$ is a basis of $\mathbb{R}^3$.
Let $\beta=\underset{\sim}{\alpha}_3 - c_1\underset{\sim}{\alpha}_1 - c_2\underset{\sim}{\alpha}_2$, where $c_1=\dfrac{(\alpha_3,\alpha_1)}{(\alpha_1,\alpha_1)}$, $c_2=\dfrac{(\alpha_3,\alpha_2)}{(\alpha_2,\alpha_2)}$.
Then β is orthogonal to $\underset{\sim}{\alpha}_1$ and $\underset{\sim}{\alpha}_2$ and $L\{\underset{\sim}{\alpha}_1,\underset{\sim}{\alpha}_2,\underset{\sim}{\alpha}_3\}=L\{\underset{\sim}{\alpha}_1,\underset{\sim}{\alpha}_2,\beta\}$
Therefore, $\{\underset{\sim}{\alpha}_1,\underset{\sim}{\alpha}_2,\underset{\sim}{\beta}\}$ is an orthogonal basis of $\mathbb{R}^3$.

$c_1=-1/14$, $c_2=-4/21$ and therefore

$$\beta=(0,0,1)+\frac{1}{14}(2,3,-1)+\frac{4}{21}(1,-2,-4)=\left(\frac{1}{3},-\frac{1}{6},\frac{1}{6}\right).$$

Hence an extended orthogonal basis is $\left\{(2,3,-1),(1,-2,-4),\left(\frac{1}{3},-\frac{1}{6},+\frac{1}{6}\right)\right\}$

and the associated orthonormal basis is

$$\left\{\frac{1}{\sqrt{14}}(2,3,-1),\ \frac{1}{\sqrt{21}}(1,-2,-4),\ \frac{1}{\sqrt{6}}(2,-1,1)\right\}.$$

5. Use Gram-Schmidt process to obtain an orthogonal basis from the basis set $\{(1,0,1),(1,1,1),(1,3,4)\}$ of the Euclidean space $\mathbb{R}^3$ with standard inner product.

Solution:- Let $\underset{\sim}{\alpha}_1=(1,0,1)$, $\underset{\sim}{\alpha}_2=(1,1,1)$, $\underset{\sim}{\alpha}_3=(1,3,4)$.
Let $\beta_1=\alpha_1$, $\beta_2=\alpha_2-c_1\beta_1$, where c_1 is the scalar component of α_2 along β_1.
Then β_2 is orthogonal to β_1 and

$$L\{\beta_1,\beta_2\}=L\{\beta_1,\alpha_2\}=L\{\alpha_1,\alpha_2\}$$

$c_1=\dfrac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}=1$. Therefore, $\beta_2=\alpha_2-\beta_1=(0,1,0)$.

Let $\beta_3=\alpha_3-d_1\beta_1-d_2\beta_2$; where d_1, d_2 are scalar component of α_3 along β_1, β_2, respectively.
Then β_3 is orthogonal to β_1, β_2 and $L\{\beta_1,\beta_2,\beta_3\}=L\{\alpha_1,\alpha_2,\alpha_3\}$

$$d_1=\frac{(\alpha_3,\beta_1)}{(\beta_1,\beta_1)}=\frac{(\alpha_3,\alpha_1)}{(\alpha_1,\alpha_1)}=\frac{5}{2},\quad d_2=\frac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}=\frac{3}{1}=3.$$

Therefore $\beta_3=(1,3,4)-\frac{5}{2}(1,0,1)-3(0,1,0)=\frac{3}{2}(-1,0,1)$.
Therefore, an orthogonal basis is

$$\left\{(1,0,1),(0,1,0),\frac{3}{2}(-1,0,1)\right\}.$$

END

LINEAR TRANSFORMATION

Linear mapping or Linear Transformation:—

Definition:— Let V and W be vector spaces over the same field F. A mapping $T: V \to W$ is said to be a linear mapping if it satisfies the following conditions —

1. $T(\alpha+\beta) = T(\alpha)+T(\beta) \ \forall \ \alpha, \beta \in V$.
2. $T(c\alpha) = cT(\alpha) \ \forall \ c$ in F and $\forall \ \alpha \in V$.

These two conditions can be replaced by the single condition —

$$T(a\alpha+b\beta) = aT(\alpha)+bT(\beta) \ \forall \ a, b \text{ in } F \text{ and all } \alpha, \beta \text{ in } V.$$

Note:— 1. A linear mapping $T: V \to W$ is also a homomorphism of V to W.

2. Generally a linear mapping T is a transformation from one vector V to another vector space W, both over the same field of scalars. But the co-domain space W may be the space V itself. In this case T is said to be a linear mapping on V.

Examples:—

1. The identity mapping: The mapping $T: V \to V$ defined by $T(\alpha) = \alpha$ $\forall \ \alpha$ in V, is a linear mapping. This is called the identity mapping on V and is denoted by I_V.

2. The zero mapping: The mapping $T: V \to W$ defined by $T(\alpha) = 0$ (zero vector of V) $\forall \ \alpha \in V$, is called the zero mapping.

3. Let P the vector space of all real polynomials. The mapping $D: P \to P$ defined by $Dp(x) = \frac{d}{dx} p(x)$, $p(x) \in P$ is a linear mapping.

4. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x_1, x_2, x_3) = (x_1, x_2, 0)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.

→ Let $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3) \in \mathbb{R}^3$.

Then $\alpha+\beta = (x_1+y_1, x_2+y_2, x_3+y_3)$

$$\begin{aligned} T(\alpha+\beta) &= (x_1+y_1, x_2+y_2, 0) \\ &= (x_1, x_2, 0) + (y_1, y_2, 0) \\ &= T(\alpha) + T(\beta). \end{aligned}$$

For $c \in \mathbb{R}$, $c\alpha = (cx_1, cx_2, cx_3)$,

$$\begin{aligned} T(c\alpha) &= (cx_1, cx_2, 0) \\ &= c(x_1, x_2, 0) \\ &= cT(\alpha). \end{aligned}$$

Therefore T is a linear mapping.

Remark:— $\underset{\sim}{Y}^{m \times 1} = A^{m \times n} \underset{\sim}{X}^{n \times 1}$ is a linear mapping from E^n into E^m as $A(\underset{\sim}{x}+\underset{\sim}{y}) = A\underset{\sim}{x}+A\underset{\sim}{y}$ & $A(k\underset{\sim}{x}) = kA(\underset{\sim}{x})$.

5. Let V be the vector space of all real valued functions continuous on the closed interval $[a,b]$ and let $T: V \to \mathbb{R}$ be defined by $T(f) = \int_a^b f(t)\,dt$, $f \in V$. Then T is a linear functional.

6. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x_1, x_2, x_3) = (x_1+1, x_2+1, x_3+1)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.

$\rightarrow$ Let $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3) \in \mathbb{R}^3$

Then $\alpha+\beta = (x_1+y_1, x_2+y_2, x_3+y_3)$

$$T(\alpha+\beta) = (x_1+y_1+1, x_2+y_2+1, x_3+y_3+1)$$
$$= (x_1, x_2, x_3) + (y_1, y_2, y_3) + (1,1,1)$$
$$\neq T(\alpha) + T(\beta).$$

Therefore, T is not a linear mapping.

Properties of a linear Transformation :—

Theorem :— Let V and W be vector spaces over a field F and $T: V \to W$ be a linear mapping. Then

where $\underset{\sim}{0}$ is the null vector in V and W respectively.

(i) $T(\underset{\sim}{0}) = \underset{\sim}{0}$,

(ii) $T(-\alpha) = -T(\alpha)\ \forall\, \alpha \in V$.

(iii) $T(\alpha-\beta) = T(\alpha) - T(\beta)\ \forall\, \alpha, \beta \in V$.

(iv) $T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) = a_1T(\alpha_1) + a_2T(\alpha_2) + \cdots + a_nT(\alpha_n)$;

where $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$ and $a_1, a_2, \ldots, a_n \in F$.

Proof :— (i) Let $\alpha \in V$, Then $T(\alpha) \in W$, We have

$$T(\alpha) + \underset{\sim}{0} = T(\alpha) \quad [\because \underset{\sim}{0} \text{ is a zero vector of } V \text{ and } T(\alpha) \in W]$$
$$= T(\alpha + \underset{\sim}{0})$$

Now, in the vector space W, we have

$$T(\alpha) + \underset{\sim}{0} = T(\alpha) + T(\underset{\sim}{0})$$
$$\Rightarrow T(\underset{\sim}{0}) = \underset{\sim}{0}$$

(ii) $$T[\alpha + (-\alpha)] = T(\alpha) + T(-\alpha)$$
$$\Rightarrow T(0) = T(\alpha) + T(-\alpha)$$
$$\Rightarrow \underset{\sim}{0} = T(\alpha) + T(-\alpha)$$
$$\Rightarrow T(-\alpha) = -T(\alpha)\ \forall\, \alpha \in V.$$

(iii) $$T(\alpha - \beta) = T[\alpha + (-\beta)]$$
$$= T(\alpha) + T(-\beta)$$
$$= T(\alpha) - T(\beta)$$

(iv) By induction the proof can easily be done.

Kernel of T OR Null Space of T :—

Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping. The set of all vectors $\alpha \in V \ni T(\alpha) = \underset{\sim}{0}$, $\underset{\sim}{0}$ being the null vector in W, is said to be the kernel of T or the null space of T, and denoted by KerT or N(T).

$$\ker T \ (N(T)) = \{\alpha \in V : T(\alpha) = \underset{\sim}{0}\}$$

*Theorem:- $T: V \to W$ is a linear mapping, then T is injective or one-to-one mapping if and only if $\ker T = \{\underset{\sim}{0}\}$.

Proof:- Only if part:- Let T is injective, $T(\underset{\sim}{0}) = \underset{\sim}{0}$ is the only pre imag

$\therefore \ker T = \{\underset{\sim}{0}\}$.

If part:- $\ker(T) = \{\underset{\sim}{0}\}$, and α, β be two elements of $V \ni$

$T(\alpha) = T(\beta)$ in W.

$\underset{\sim}{0} = T(\alpha) - T(\beta)$

$\Rightarrow T(\alpha - \beta) = \underset{\sim}{0}$, since T is linear.

$\Rightarrow \alpha - \beta \in \ker T$ and since $\ker(T) = \{0\}$, so $\alpha = \beta$.

Thus $T(\alpha) = T(\beta) \Rightarrow \alpha = \beta$ and therefore T is injective.

Note:- $A\underset{\sim}{x} = \underset{\sim}{0}$ has only trivial solution $\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \text{rank}(A^{m \times n}) = n$.

Theorem:- Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping. then KerT is a subspace of V.

Image of a linear mapping:— Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping. The images of the elements of V under the mapping T form a subset of W. This subset is said to be the image of T and is denoted by ImT or, it is also called Range of T (R(T)).

$$\text{Im}(T) \text{ or } R(T) = \{T(\underset{\sim}{\alpha}) : \underset{\sim}{\alpha} \in V\}.$$

Theorem:— Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping. Then ImT is a subspace of W.

Proof:- Obviously R(T) is a non-empty subset of W.

Let $\beta_1, \beta_2 \in R(T)$, Then $\exists$ vectors $\underset{\sim}{\alpha_1}, \underset{\sim}{\alpha_2}$ in $V \ni T(\alpha_1) = \beta_1, T(\alpha_2) = \beta_2$

Let a, b be any element of the field F. We have

$$a\beta_1 + b\beta_2 = aT(\alpha_1) + bT(\alpha_2)$$
$$= T(a\alpha_1 + b\alpha_2) \quad [\because T \text{ is a linear mapping}]$$

Now, V is a vector space, therefore $\alpha_1, \alpha_2 \in V$ and

$$a, b \in F \Rightarrow a\alpha_1 + b\alpha_2 \in V.$$

Consequently, $T(a\alpha_1 + b\alpha_2) = a\beta_1 + b\beta_2 \in R(T)$.

Therefore R(T) is a subspace of W.

Theorem:- Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping $\ni$ $\ker T = \{0\}$. Then the image of a LIN set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ in V are LIN in W.

Proof:- To proof the Linear independence of the vectors $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$ in W. Let us consider the relation

$$c_1 T(\alpha_1) + c_2 T(\alpha_2) + \cdots + c_n T(\alpha_n) = \underset{\sim}{0}, \text{ where } c_1, \ldots, c_n \in F.$$

$$\Rightarrow T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n) = \underset{\sim}{0}, \text{ since } T \text{ is linear.}$$

$$\Rightarrow c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = \underset{\sim}{0}, \text{ since } \ker(T) = \{0\}.$$

$$\Rightarrow c_1 = c_2 = \cdots = c_n = 0, \text{ since } \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \text{ is}$$

linearly independent. This proves linear independence of the vectors $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$ in W.

~~**Theorem:-** Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping $\ni$ $\ker T = \{\underset{\sim}{0}\}$. Then the image of a LIN set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ in V are LIN in W.~~

~~**Proof:-**~~

Theorem:- Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of V, then the vectors $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$ generate $\operatorname{Im} T$ or $R(T)$.

Proof:- Let $\xi \in \operatorname{Im} T$. Then $\exists$ an element α in V $\ni$

$$T(\alpha) = \xi.$$

Let $\underset{\sim}{\alpha} = c_1\underset{\sim}{\alpha}_1 + c_2\underset{\sim}{\alpha}_2 + \cdots + c_n\underset{\sim}{\alpha}_n$ for some scalars $c_i \in F$.

Then
$$\xi = T(c_1\underset{\sim}{\alpha}_1 + c_2\underset{\sim}{\alpha}_2 + \cdots + c_n\underset{\sim}{\alpha}_n)$$
$$= c_1 T(\underset{\sim}{\alpha}_1) + c_2 T(\underset{\sim}{\alpha}_2) + \cdots + c_n T(\underset{\sim}{\alpha}_n), \text{ since } T \text{ is linear.}$$

Since each $T(\alpha_i) \in \operatorname{Im} T$, it follows that $\operatorname{Im} T$ is generated by the vectors $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$.

Note:- $T: V \to W$ is a linear transformation. If $\{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n\}$ is a basis for V, then $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ is a spanning set of W. Further, if T is one-to-one, i.e. $\ker(T) = \{\underset{\sim}{0}\}$, then $\{T(\underset{\sim}{\alpha}_1), T(\underset{\sim}{\alpha}_2), \ldots, T(\underset{\sim}{\alpha}_n)\}$ is a basis for W.

Proof:- The above two theorem show it.

WORKED EXAMPLES:—

1. A mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$. Show that T is a linear mapping. Find $\ker T$ and dimension of $\ker T$. Is it one-one? Find $\operatorname{Im} T$ and the dimension of $\operatorname{Im} T$.

Solution:—

1st Part:— Let $\underset{\sim}{\alpha} = (a_1, a_2, a_3) \in \mathbb{R}^3$, $\underset{\sim}{\beta} = (b_1, b_2, b_3) \in \mathbb{R}^3$.

Then $T(\alpha) = (a_1 + a_2 + a_3,\ 2a_1 + a_2 + 2a_3,\ a_1 + 2a_2 + a_3)$,

$T(\beta) = (b_1 + b_2 + b_3,\ 2b_1 + b_2 + 2b_3,\ b_1 + 2b_2 + b_3)$.

$\alpha + \beta = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

$$T(\alpha+\beta) = \big((a_1+b_1) + (a_2+b_2) + (a_3+b_3),\ 2(a_1+b_1) + (a_2+b_2) + 2(a_3+b_3),\ (a_1+b_1) + 2(a_2+b_2) + (a_3+b_3)\big)$$

$$= \big((a_1+a_2+a_3) + (b_1+b_2+b_3),\ (2a_1+a_2+2a_3) + (2b_1+b_2+2b_3),\ (a_1+2a_2+a_2) + (b_1+2b_2+b_3)\big)$$

$$= T(\alpha) + T(\beta)$$

Let $c \in \mathbb{R}$, $c\alpha = (ca_1, ca_2, ca_3)$

$$T(c\underset{\sim}{\alpha}) = (ca_1 + ca_2 + ca_3,\ 2ca_1 + ca_2 + 2ca_3,\ ca_1 + 2ca_2 + ca_3)$$

$$= c(a_1 + a_2 + a_3,\ 2a_1 + a_2 + 2a_3,\ a_1 + 2a_2 + a_3)$$

$$= cT(\underset{\sim}{\alpha})$$

$\therefore T$ is a linear mapping.

2nd Part:— $\ker(T) = \{\underset{\sim}{x} : T(\underset{\sim}{x}) = \underset{\sim}{0}\}$

$T(x_1, x_2, x_3) = (0, 0, 0)$

$\Rightarrow$ $x_1 + x_3 + x_3 = 0$, $2x_1 + x_2 + 2x_3 = 0$, $x_1 + 2x_2 + x_3 = 0$

From 1st two equations,

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1} = k$$

$\therefore x_1 = k, x_2 = 0, x_3 = -k$. the last equation is satisfied.

$\therefore (x_1, x_2, x_3) = k(1, 0, -1)$; $k \in \mathbb{R}$

$\therefore \ker(T) = \{k(1, 0, -1) : k \in \mathbb{R}\}$

$\therefore \dim \ker T = 1$.

As $\ker T \neq \{0\}$, so the linear mapping T is <u>not</u> one-to-one.

3rd Part:-

If $\{\alpha_1, \alpha_2, \alpha_3\}$ be a basis of the domain space $\mathbb{R}^3$, $\operatorname{Im}T$ is the linear span of the vectors $T(\alpha_1), T(\alpha_2), T(\alpha_3)$.

$\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ is a basis of $\mathbb{R}^3$.

$T(e_1) = (1,2,1), T(e_2) = (1,1,2), T(e_3) = (1,2,1)$

$\Rightarrow \{T(e_1), T(e_2), T(e_3)\}$ is a spanning set of $\operatorname{Im}T$.

Since $T(e_1) = T(e_3)$, $\operatorname{Im}T$ is the linear span of the vectors $(1,2,1)$ and $(1,1,2)$.

Hence $\operatorname{Im}T = L\{(1,2,1), (1,1,2)\}$

$\therefore \dim(\operatorname{Im}T) = 2$.

Alter:-

$$T(x) = (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3)$$

$$T(x) = x_1(1,2,1) + x_2(1,1,2) + x_3(1,2,1)$$

$\therefore T(x)$ is a linear combination of the vectors $(1,2,1), (1,1,2)$

Hence $\operatorname{Im}T = \{l_1(1,2,1) + l_2(1,1,2) : l_1, l_2 \in \mathbb{R}\}$

$= L\{(1,2,1), (1,1,2)\}$

$\therefore \dim(\operatorname{Im}T) = 2$.

2. A linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^4$ is defined by $T(x_1, x_2, x_3) = (x_2 + x_3, x_3 + x_1, x_1 + x_2, x_1 + x_2 + x_3)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$. Find $\operatorname{Ker}T$. Verify that the set $\{T(e_1), T(e_2), T(e_3)\}$ is linearly independent in $\mathbb{R}^4$. Find $\operatorname{Im}T$ and the dimension of $\operatorname{Im}T$.

Solution:- 1st Part:-

$$\operatorname{Ker}T = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : T(x_1, x_2, x_3) = (0,0,0,0)\}$$

Let $(x_1, x_2, x_3) \in \operatorname{Ker}T$.

Then $x_2 + x_3 = 0$, $x_3 + x_1 = 0$, $x_1 + x_2 = 0$, $x_1 + x_2 + x_3 = 0$.

The solution is $x_1 = x_2 = x_3 = 0$.

Therefore $\operatorname{Ker}T = \{0\}$.

2nd Part:-

$T(e_1) = (0,1,1,1)$, $T(e_2) = (1,0,1,1)$, $T(e_3) = (1,1,0,1)$.

To examine linear independence of the set $\{T(e_1), T(e_2), T(e_3)\}$,

let $c_1 T(e_1) + c_2 T(e_2) + c_3 T(e_3) = 0$, $c_i \in \mathbb{R}$.

Then $c_1(0,1,1,1) + c_2(1,0,1,1) + c_3(1,1,0,1) = (0,0,0,0)$

$\Rightarrow c_2 + c_3 = 0$, $c_1 + c_2 = 0$, $c_1 + c_3 = 0$, $c_1 + c_2 + c_3 = 0$.

The solution is $c_1 = c_2 = c_3 = 0$.

This proves that $\{T(e_1), T(e_2), T(e_3)\}$ is linearly independent in $\mathbb{R}^4$.

3rd Part:— $\operatorname{Im}T$ is the linear span of the vectors $T(\alpha_1), T(\alpha_2), T(\alpha_3)$ where $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of $\mathbb{R}^3$.

$\{\underset{\sim}{e}_1 = (1,0,0), \underset{\sim}{e}_2 = (0,1,0), \underset{\sim}{e}_3 = (0,0,1)\}$ is a basis of $\mathbb{R}^3$.

$T(\underset{\sim}{e}_1) = (0,1,1,1)$, $T(\underset{\sim}{e}_2) = (1,0,1,1)$, $T(\underset{\sim}{e}_3) = (1,1,0,1)$.

Therefore, $\operatorname{Im}T = L\{(0,1,1,1), (1,0,1,1), (1,1,0,1)\}$.

The set $\{(0,1,1,1), (1,0,1,1), (1,1,0,1)\}$ is linearly independent.

Therefore the dimension of $\operatorname{Im}T$ is 3.

3. Show that the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x+y, y+z, z+x)$ is a linear transformation. Also find the dimension of $\ker(T)$ and S.T. the transformation is one-to-one. Find $\operatorname{Im}T$. Is it onto?

Solution:— 1st Part:— ~~Let~~, $\cancel{\underset{\sim}{\alpha} = (a_1, a_2, a_3) \in \mathbb{R}^3}$

$\cancel{\underset{\sim}{\beta} = (b_1, b_2, b_3) \in \mathbb{R}^3}$

$\cancel{T(a\underset{\sim}{\alpha} + b\underset{\sim}{\beta}) = T(aa_1 + a}$

Let $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^3$, then

$$T(\alpha\underset{\sim}{x} + \beta\underset{\sim}{y}) = T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)$$

$$= \left(\alpha(x_1+x_2) + \beta(y_1+y_2), \alpha(x_2+x_3) + \beta(y_2+y_3), \alpha(x_3+x_1) + \beta(y_3+y_1)\right)$$

$$= \alpha(x_1+x_2, x_3+x_2, x_1+x_3) + \beta(y_1+y_2, y_2+y_3, y_3+y_1)$$

$$= \alpha T(\underset{\sim}{x}) + \beta T(\underset{\sim}{y}).$$

$\therefore T$ is a linear transformation.

2nd Part:— $\ker T = \{\underset{\sim}{x} : T(\underset{\sim}{x}) = \underset{\sim}{0}\}$

$T(\underset{\sim}{x}) = \underset{\sim}{0} \Rightarrow x+y=0,\ y+z=0,\ z+x=0$

$\Rightarrow x=0, y=0, z=0$

$\therefore \ker T = \{0\}$ and hence T is one-to-one.

$\therefore \dim(\ker T) = 0$.

3rd Part:— $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ is a basis for $\mathbb{R}^3$.

$\Rightarrow \{T(\underset{\sim}{e}_1), T(\underset{\sim}{e}_2), T(\underset{\sim}{e}_3)\}$ is a basis for $\operatorname{Im}(T)$.

$\Rightarrow \{(1,0,1), (1,1,0), (0,1,1)\}$ is a basis for $\operatorname{Im}(T)$.

$\therefore \operatorname{Im}T = L\{(1,0,1), (1,1,0), (0,1,1)\}$

$\operatorname{Im}T = \mathbb{R}^3$ and therefore T is an onto mapping.

4. Determine $T : \mathbb{R}^3 \to \mathbb{R}^2$ which maps the basis vectors $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ of $\mathbb{R}^3$ to $(1,1), (2,3), (-1,2)$ of $\mathbb{R}^2$. Find $T(\underset{\sim}{x})$.

Solution:– Let, $\underset{\sim}{x} \in \mathbb{R}^3$

Then, $\underset{\sim}{x} = x_1 \underset{\sim}{e}_1 + x_2 \underset{\sim}{e}_2 + x_3 \underset{\sim}{e}_3$

$$T(\underset{\sim}{x}) = x_1 T(\underset{\sim}{e}_1) + x_2 T(\underset{\sim}{e}_2) + x_3 T(\underset{\sim}{e}_3)$$
$$= x_1(1,1) + x_2(2,3) + x_3(-1,2)$$
$$= (x_1 + 2x_2 - x_3,\ x_1 + 3x_2 + 2x_3).$$

5. Determine the linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ which maps the basis vectors $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ of $\mathbb{R}^3$ to the vectors $(1,1), (2,3), (3,2)$, respectively.

(i) Find $T(1,1,0)$, $T(6,0,-1)$. (ii) Find $\ker T$ and $\operatorname{Im} T$.

(iii) Prove that T is not one-to-one, but onto.

Solution:– Let $\xi = (x,y,z)$ be an arbitrary vector in $\mathbb{R}^3$.

$$\xi = x(1,0,0) + y(0,1,0) + z(0,0,1).$$

Since T is linear,

$$T(\xi) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$$
$$= x(1,1) + y(2,3) + z(3,2)$$
$$= (x+2y+3z),\ (x+3y+2z).$$

So, T is defined by $T(x,y,z) = (x+2y+3z,\ x+3y+2z)$;
$(x,y,z) \in \mathbb{R}^3$.

(i) $T(1,1,0) = (3,4)$; $T(6,0,-1) = (3,4)$.

(ii) Let $(x,y,z) \in \ker T$. Then $T(x,y,z) = (0,0)$

$$\Rightarrow x + 2y + 3z = 0,\ x + 3y + 2z = 0$$

The solution is $\dfrac{x}{-5} = \dfrac{y}{1} = \dfrac{z}{1} = k$

$$\Rightarrow (x,y,z) = k(-5,1,1), \text{ where } k \in \mathbb{R}.$$

Consequently, $\ker T = L\{\underset{\sim}{\alpha}\}$, where $\underset{\sim}{\alpha} = (-5,1,1)$.

$\operatorname{Im} T$ is the linear span of the vectors $T(\alpha_1), T(\alpha_2), T(\alpha_3)$ where $\{\alpha_1, \alpha_2, \alpha_3\}$ is any basis of $\mathbb{R}^3$.

Since $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis of $\mathbb{R}^3$,

$$\operatorname{Im} T = L\{(1,1), (2,3), (3,2)\}.$$

(iii) Since $\ker T \neq \{\underset{\sim}{0}\}$, T is not one-to-one.

$\operatorname{Im} T = L\{(1,1), (2,3), (3,2)\}$.

These vectors are linearly dependent in $\mathbb{R}^2$.

The subset $\{(1,1), (2,3)\}$ is LIN in $\mathbb{R}^2$.

Therefore, $\dim \operatorname{Im} T = 2$.

Since $\operatorname{Im} T$ is a subspace of $\mathbb{R}^2$ and $\dim \operatorname{Im} T = 2$,

$\operatorname{Im} T = \mathbb{R}^2$. Therefore T is onto.

Nullity and Rank of a linear mapping:—

Let V and W be vector spaces over a field F and $T: V \to W$ be a linear mapping.

Then $\ker T$ is a subspace of V. The dimension of $\ker T$ is called the nullity of T. $\operatorname{Im} T$ is a subspace of W. The dimension of $\operatorname{Im} T$ is called the rank of T.

$$\text{Nullity}(T) = \dim(\ker T) = \dim N(T)$$
$$\text{Rank}(T) = \dim(\operatorname{Im} T) = \dim R(T)$$

Note:- If V be a finite dimensional vector space then $\ker T$, being a subspace of V, is finite dimensional. The number of vectors in a basis of V is finite. As $\operatorname{Im} T$ is generated by the images of the vectors in a basis of V, $\operatorname{Im} T$ is also finite dimensional.

Theorem:- Let V and W be vector spaces over a field F and V is finite dimensional. If $T: V \to W$ be a linear mapping then

$$\dim \ker T + \dim \operatorname{Im} T = \dim V.$$

OR

$$\text{Nullity of } T + \text{Rank of } T = \dim V.$$

Proof:-

Case I:- Let $\ker T = V$, then $\operatorname{Im} T$ consists of $\underset{\sim}{0}$, where $\underset{\sim}{0}$ is the null element in W. $\therefore \dim \operatorname{Im} T = 0$.

$\therefore \dim \ker T + \dim \operatorname{Im} T = \dim V + 0 = \dim V$.

Case-II:- Let $\ker T = \{\underset{\sim}{0}\}$, and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of V.

Then $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ is a basis of $\operatorname{Im} T$.

$\therefore \dim \ker T = 0$, $\dim V = n$, $\dim \operatorname{Im} T = n$.

$\therefore$ The theorem holds good.

Case III:- ~~Let $\ker T = \{\underset{\sim}{0}\}$ and let $\{\alpha$~~

Let $\ker T$ be a non-trivial proper subspace of V. Let $\{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_k\}$ be a basis of $\ker T$. This basis of $\ker T$ can be extended to a basis of V and let the extended basis of V be $\{\underset{\sim}{\alpha}_1, \ldots, \underset{\sim}{\alpha}_k, \underset{\sim}{\alpha}_{k+1}, \ldots, \underset{\sim}{\alpha}_n\}$.

$\operatorname{Im} T$ is generated by the vectors $T(\underset{\sim}{\alpha}_1), T(\underset{\sim}{\alpha}_2), \ldots, T(\underset{\sim}{\alpha}_n)$,

As, $T(\alpha_1) = T(\alpha_2) = \cdots = T(\alpha_k) = \underset{\sim}{0}$, $\operatorname{Im} T$ is generated by the vectors $T(\alpha_{k+1}), T(\alpha_{k+2}), T(\alpha_{k+3}), \ldots, T(\alpha_n)$.

Considering the equation with c_{k+i} scalars with $i = 1(1)(n-k)$.
$= 1(1)\overline{n-k}$.

$$c_{k+1}T(\alpha_{k+1}) + c_{k+2}T(\alpha_{k+2}) + \cdots + c_n T(\alpha_n) = \underset{\sim}{0}, \text{ where } c_j \in F.$$

$$\Rightarrow T(c_{k+1}\alpha_{k+1} + c_{k+2}\alpha_{k+2} + \cdots + c_n\alpha_n) = \underset{\sim}{0}$$

$\Rightarrow c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n$ is in $\ker T$.

since $\{\alpha_1, \ldots, \alpha_k\}$ is a basis for $\ker T$.

$\therefore$ The linear independence of $\{\alpha_1, \ldots, \alpha_n\}$ requires $c_{k+1} = c_{k+2} = \cdots = c_n = 0$.

$\therefore \{T(\alpha_{k+1}), \ldots, T(\alpha_n)\}$ is LIN and is a basis for $\operatorname{Im} T$.

So, $\dim(\operatorname{Im} T) = n - k$ and $\dim \ker T + \dim \operatorname{Im} T = k + (n-k) = n$
$= \dim V$.

Theorem:- Let V and W be finite dimensional vector spaces of the same dimensional over a field F and $T : V \to W$ be a linear mapping. Then T is one-to-one $\Leftrightarrow$ T is onto.

Proof:- Let T be one-to-one.

$$\ker(T) = \{\underset{\sim}{0}\}$$

$$\dim \ker T = 0.$$

$\because$ $\dim \ker T + \dim \operatorname{Im} T = \dim V$

$\Rightarrow \dim \operatorname{Im} T = \dim V$

$\Rightarrow \operatorname{Im} T = \dim W.$

But $\operatorname{Im} T$ is a subspace of W and so $\operatorname{Im} T = W$.

Hence T is onto.

Conversely, let T is onto. Then $\operatorname{Im} T = W$.

$\because$ $\dim \ker T + \dim \operatorname{Im} T = \dim V$

$\Rightarrow \dim \ker T + \dim W = \dim V$

As, $\dim W = \dim V$, $\therefore \dim \ker T = 0$

$\Rightarrow \ker T = \{\underset{\sim}{0}\}$

$\Rightarrow T$ is one-to-one.

Ex.1:- Let V be a vector space and T is a linear transformation from V into V. Prove that the following two statements about T are equivalent:

(i) $R(T) \cap N(T) = \{0\}$

(ii) $T[T(\underset{\sim}{\alpha})] = \underset{\sim}{0} \Rightarrow T(\underset{\sim}{\alpha}) = \underset{\sim}{0}$.

Solution:- To show (i) $\Rightarrow$ (ii)

$$R(T) \cap N(T) = \{\underset{\sim}{0}\}$$

$\Rightarrow T(\underset{\sim}{\alpha}) = \underset{\sim}{0}$

$\Rightarrow T(\underset{\sim}{\alpha}) \in R(T) \cap N(T)$

$\Rightarrow T(\underset{\sim}{\alpha}) \in N(T)$

$\Rightarrow T(T(\underset{\sim}{\alpha})) = \underset{\sim}{0}$.

To show (ii) $\Rightarrow$ (i)

$\alpha \in R(T) \cap N(T)$

$\therefore \alpha \in R(T)$, $\alpha \in N(T)$, $\therefore T(\underset{\sim}{\alpha}) = \underset{\sim}{0}$

Now let $T(\beta) = \alpha$

$\Rightarrow T(T(\beta)) = T(\alpha)$

$= \underset{\sim}{0}$ but $T(\beta) = \alpha \neq \underset{\sim}{0}$

This contradicts the hypothesis,

$\therefore \alpha \notin R(T) \cap N(T)$; $\therefore R(T) \cap N(T) = \{\underset{\sim}{0}\}$.

WORKED EXAMPLES:— (Continued)

1. Let T be a linear operator on V and let $\text{Rank}\,T^2 = \text{Rank}\,T$, then show that $\text{Range}\,T \cap \ker T = \{\underset{\sim}{0}\}$.

Solution:— $T: V \to V$, $T^2: V \to V$

$\text{Rank}(T) = \dim V - \dim \ker T$

$\text{Rank}(T^2) = \dim V - \dim \ker T^2$

$\Rightarrow \dim \ker T = \dim \ker T^2$.

we claim $\ker T = \ker T^2$

$\underset{\sim}{x} \in \ker T \Rightarrow T(\underset{\sim}{x}) = \underset{\sim}{0} \Rightarrow T^2(\underset{\sim}{x}) = T(\underset{\sim}{0}) = \underset{\sim}{0}$

$\underset{\sim}{x} \in \ker T^2 \Rightarrow \ker(T) \subseteq \ker(T^2)$

Now, $\underset{\sim}{x} \in \text{Range}\,T \cap \ker T$

$\Rightarrow \underset{\sim}{x} \in \text{Range}\,T$ and $\underset{\sim}{x} \in \ker T$

$\Rightarrow T(\underset{\sim}{x}) = \underset{\sim}{0}$, $\underset{\sim}{x} = T(\underset{\sim}{y})$, for some $\underset{\sim}{y} \in V$.

$\Rightarrow T(T(\underset{\sim}{y})) = \underset{\sim}{0}$

$\Rightarrow T^2(\underset{\sim}{y}) = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{y} \in \ker T^2 = \ker T$

$\Rightarrow T(\underset{\sim}{y}) = \underset{\sim}{0} \Rightarrow \underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow \ker T \cap \text{Range}\,T = \{\underset{\sim}{0}\}$.

2. Determine the linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ of $\mathbb{R}^3$ to $(1,1,1)$, $(1,1,1)$, $(1,1,1)$ respectively. Verify that $\dim \ker T + \dim \text{Im}\,T = 3$.

Solution:— $\underset{\sim}{e} = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$; c_i are unique scalars.

$c_2 + c_3 = x$, $c_1 + c_3 = y$, $c_1 + c_2 = z$.

solving $c_1 = \dfrac{y+z-x}{2}$, $c_2 = \dfrac{z+x-y}{2}$, $c_3 = \dfrac{x+y-z}{2}$.

Since T is linear,

$$T(\underset{\sim}{\xi}) = c_1 T(0,1,1) + c_2 T(1,0,1) + c_3 T(1,1,0)$$

$$= c_1(1,1,1) + c_2(1,1,1) + c_3(1,1,1)$$

$$= (c_1+c_2+c_3,\ c_1+c_2+c_3,\ c_1+c_2+c_3)$$

$$= \left(\frac{x+y+z}{2}, \frac{x+y+z}{2}, \frac{x+y+z}{2}\right);$$

$$\therefore T(x,y,z) = \left(\frac{x+y+z}{2}, \frac{x+y+z}{2}, \frac{x+y+z}{2}\right);\ (x,y,z) \in \mathbb{R}^3.$$

To find $\ker T$, let $T(x,y,z) = (0,0,0)$;

$x+y+z = 0$, let, $y = c$, $z = d$, $x = -(c+d)$.

$\therefore (x,y,z) = (-c-d, c, d) = c(-1,1,0) + d(-1,0,1)$; $c, d \in \mathbb{R}$.

Hence $\ker T = L\{(-1,1,0), (-1,0,1)\}$ and since $(-1,1,0)$ and $(-1,0,1)$ are LIN. $\dim \ker T = 2$.

$\operatorname{Im}T$ is the linear span of the vectors $T(\alpha), T(\beta), T(\gamma)$, where $\{\alpha, \beta, \gamma\}$ is any basis of the domain space $\mathbb{R}^3$.

since $(0,1,1), (1,0,1), (1,1,0)$ is a basis of $\mathbb{R}^3$,

$\operatorname{Im}T = L\{(1,1,1)\}$. Hence $\dim \operatorname{Im}T = 1$.

$\therefore \dim \ker T + \dim \operatorname{Im}T = 2+1 = 3$.

3. Determine the linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $(0,1,1), (1,0,1), (1,1,0)$ of $\mathbb{R}^3$ to the vectors $(2,0,0), (0,2,0), (0,0,2)$ respectively. Find $\ker T$ and $\operatorname{Im}T$. Verify that $\dim \ker T + \dim \operatorname{Im}T = 3$.

Solution:–

$$(x,y,z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0) \text{ ; } c_i \text{ are unique scalars.}$$

$$c_2 + c_3 = x, \quad c_3 + c_1 = y, \quad c_1 + c_2 = z.$$

solving, we have $c_1 = \frac{y+z-x}{2}$, $c_2 = \frac{z+x-y}{2}$, $c_3 = \frac{x+y-z}{2}$.

since T is linear,

$$T(x,y,z) = c_1T(0,1,1) + c_2T(1,0,1) + c_3T(1,1,0)$$

$$= \frac{y+z-x}{2}(2,0,0) + \frac{z+x-y}{2}(0,2,0) + \frac{x+y-z}{2}(0,0,2)$$

$$= (y+z-x, z+x-y, x+y-z) \text{ ; } (x,y,z) \in \mathbb{R}^3.$$

$$T(x,y,z) = (0,0,0) = (y+z-x, z+x-y, x+y-z)$$

$$\therefore x = y = z = 0.$$

$$\ker T = \{\underset{\sim}{0}\}, \ \dim \ker T = 0.$$

$\operatorname{Im}T$ is the linear span of the vectors $T(\alpha_1), T(\alpha_2), T(\alpha_3)$, where $\{\alpha_1, \alpha_2, \alpha_3\}$ is any basis of the domain space $\mathbb{R}^3$.

since $\{(0,1,1), (1,0,1), (1,1,0)\}$ is a basis of $\mathbb{R}^3$,

$$\operatorname{Im}T = L\{(2,0,0), (0,2,0), (0,0,2)\}.$$

$$\therefore \dim(\operatorname{Im}T) = 3.$$

Hence, $\dim(\ker T) + \dim(\operatorname{Im}T) = 0+3 = 3$.

4. Find a linear operator T on $\mathbb{R}^3 \ni \ker T$ is the subspace $U = \{(x,y,z) \in \mathbb{R}^3 : x+y+z=0\}$ of $\mathbb{R}^3$.

Solution:– Let $\underset{\sim}{\xi} = (a,b,c)$ be a vector in the subspace U.

Then $a+b+c=0$.

$$\underset{\sim}{\xi} = (a,b,-a-b) = a(1,0,-1) + b(0,1,-1) \text{ ; } a,b \in \mathbb{R}.$$

This shows that $\underset{\sim}{\xi} \in L\{(1,0,-1), (0,1,-1)\}$

$\therefore \{(1,0,-1), (0,1,-1)\}$ is a basis of U.

$U = \ker T$, $T(1,0,-1) = (0,0,0)$, $T(0,1,-1) = (0,0,0)$

The basis $\{(1,0,-1),(0,1,-1)\}$ of U can be extended to a basis $\{(1,0,-1),(0,1,-1),(1,0,0)\}$ of $\mathbb{R}^3$.

Let T be the linear operator on $\mathbb{R}^3 \ni T(1,0,-1)=(0,0,0)$, $T(0,1,-1)=(0,0,0)$, $T(1,0,0)=(1,0,0)$, then $\ker T = U$.

Let $(x,y,z)\in\mathbb{R}^3$, $(x,y,z)=c_1(1,0,-1)+c_2(0,1,-1)+c_3(1,0,0)$

$$c_1=-y-z,\ c_2=y,\ c_3=x+y+z.$$

$$\begin{aligned}T(x,y,z) &= (-y-z)T(1,0,-1)+yT(0,1,-1)+(x+y+z)T(1,0,0)\\ &= (-y-z)(0,0,0)+y(0,0,0)+(x+y+z)(1,0,0)\\ &= (x+y+z,0,0);\ (x,y,z)\in\mathbb{R}^3.\end{aligned}$$

<u>Note:-</u> As the image of the basis vector $(1,0,0)$ can be chosen arbitrarily (other than $(0,0,0)$), T is not unique.

5. Find a linear mapping $T:\mathbb{R}^3\to\mathbb{R}^3 \ni \operatorname{Im}T$ is the subspace $U=\{(x,y,z)\in\mathbb{R}^3 : x+y-z=0\}$ of $\mathbb{R}^3$.

<u>Solution:-</u> Let $\xi=(a,b,c)$ be a vector in the subspace U. Then $a+b-c=0$.

$\xi=(a,b,a+b)=a(1,0,1)+b(0,1,1)$.

$\therefore \xi \in L\{(1,0,1),(0,1,1)\}$.

$\therefore \{(1,0,1),(0,1,1)\}$ is a basis of U as $(1,0,1)$ and $(0,1,1)$ are LIN.

$\operatorname{Im}T$ is generated by the images of the vectors of a basis. Let us take the standard basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ of $\mathbb{R}^3$ and let

$T(1,0,0)=(1,0,1)$, $T(0,1,0)=(0,1,1)$, $T(0,0,1)=(0,1,1)$.

Then $\operatorname{Im}T = L\{(1,0,1),(0,1,1)\}=U$.

Let $(x,y,z)\in\mathbb{R}^3$.

$$\begin{aligned}T(x,y,z) &= xT(1,0,0)+y(T(0,1,0))+zT(0,0,1),\ \text{since } T \text{ is linear.}\\ &= x(1,0,1)+y(0,1,1)+z(0,1,1)\\ &= (x,y+z,x+y+z),\ (x,y,z)\in\mathbb{R}^3.\end{aligned}$$

<u>Note:-</u> As the image of the basis vectors $(0,0,1)$ can be chosen arbitrarily (as a scalar multiple of $(0,1,1)$, or as a scalar multiple of $(1,0,1)$), T is not unique.

<u>Linear Operator (Definition):-</u> Let V be a vector space. A linear operator on V is a function T from V into $V \ni$

$$T(a\alpha+b\beta)=aT(\alpha)+bT(\beta)\quad \forall\ \alpha,\beta \text{ in } V \text{ and } \forall\ a,b\in F.$$

Thus T is a linear operator on V if T is a linear transformation from V into V itself.

Composition of linear mapping :—

Let V, W and U be vector spaces over a field F and let $T: V \to W$; $S: W \to U$ be linear mapping. The composite mapping $S \circ T : V \to U$ is defined by $S \circ T(\alpha) = S\{T(\alpha)\}$, $\alpha \in V$.
the composite $S \circ T$ is generally denoted by ST and it is also said to be the product mapping ST.

Theorem:– Let V, W and U be vector spaces over a field F and $T: V \to W$, $S: W \to U$ be linear mapping. then the composite mapping $ST : V \to U$ is linear.

Proof:– Let α, β in V and $c \in F$.

$$ST(\alpha+\beta) = S[T(\alpha+\beta)]$$
$$= S[T\alpha + T\beta] \text{ , since } T \text{ is linear.}$$
$$= S(T\alpha) + S(T\beta) \text{ , since } S \text{ is linear.}$$
$$= ST(\alpha) + ST(\beta).$$
$$ST(c\alpha) = S[T(c\alpha)]$$
$$= S[cT(\alpha)] \text{ , since } T \text{ is linear.}$$
$$= cS[T(\alpha)] \text{ , since } S \text{ is linear.}$$
$$= cST(\alpha).$$

this proves that ST is linear.

Note:– The unique inverse of the mapping T is denoted by T^{-1}.

Theorem:– Let V and W be vector spaces over a field F. A linear mapping $T: V \to W$ is invertible if and only if T is one-to-one and onto.

Theorem:– Let V and W be vector spaces over a field F. If a linear mapping $T: V \to W$ be invertible, then the inverse mapping $T^{-1}: W \to V$ is linear.

Proof:– Let $\alpha', \beta' \in W$ and $T^{-1}(\alpha') = \alpha$, $T^{-1}(\beta') = \beta$.
Then $\alpha, \beta \in V$ and $T(\alpha) = \alpha'$, $T(\beta) = \beta'$
Since T is linear, $T(\alpha+\beta) = T(\alpha) + T(\beta) = \alpha' + \beta'$,
therefore $T^{-1}(\alpha'+\beta') = \alpha+\beta = T^{-1}(\alpha') + T^{-1}(\beta')$
since T is linear, $T(c\alpha) = cT(\alpha)$, $c \in F$.
$= c\alpha'$.
Therefore, $T^{-1}(c\alpha') = c\alpha = cT^{-1}(\alpha')$ $\forall$ $c \in F$.
This proves T^{-1} is linear.

Definition:- A linear mapping $T: V \to W$ is said to be non-singular if T is invertible.

Isomorphism:- Let V and W be vector spaces over a field F. A linear mapping $T: V \to W$ is said to be an isomorphism if T is both one-to-one and onto.

Since T is both one-to-one and onto, T is invertible and $T^{-1}: W \to V$ is also a linear mapping which is both one-to-one and onto.

Thus the existance of an isomorphism $T: V \to W$ implies the existance of another isomorphism $T^{-1}: W \to V$. In this case the vector spaces V and W are said to be isomorphic.

Theorem:- Two finite dimensional vector spaces V and W over a field F are isomorphic if and only if $\dim V = \dim W$.

Isomorphisms from V to F^n.

Theorem:- Let V be a vector space of dimension n over a field F. Then V is isomorphism of F^n.

Proof:- An isomorphism between V and F^n can be established in many ways.

Let $(\beta_1, \beta_2, \ldots, \beta_n)$ be an ordered basis of V. Then any vector ξ of V can be expressed as

$\xi = c_1\beta_1 + c_2\beta_2 + \cdots + c_n\beta_n$, where $c_1, c_2, \ldots, c_n$ are scalars in F.

Let us define a mapping $\phi: V \to F^n$ by

$$\phi(\xi) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \text{ where } \xi = (c_1\beta_1 + c_2\beta_2 + \cdots + c_n\beta_n) \in V.$$

Let $\alpha = a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n \in V$,

$\beta = b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n \in V$.

Then $\alpha + \beta = (a_1+b_1)\beta_1 + (a_2+b_2)\beta_2 + \cdots + (a_n+b_n)\beta_n \in V$.

$$\phi(\alpha) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \phi(\beta) = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \text{ and}$$

$$\phi(\alpha+\beta) = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \phi(\alpha) + \phi(\beta) \quad \cdots\cdots\cdots (i)$$

$$\phi(p\alpha) = \begin{pmatrix} pa_1 \\ pa_2 \\ \vdots \\ pa_n \end{pmatrix} = p\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = p\,\phi(\alpha) \quad \cdots\cdots (ii)$$

From (i) and (ii) ϕ is a ~~homomorphism~~ linear mapping.

To prove that ϕ is one-to-one, let $\alpha, \beta \in V$ $\ni$.

$$\phi(\alpha) = \phi(\beta)$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\Rightarrow a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$$

$$\Rightarrow \underset{\sim}{\alpha} = \underset{\sim}{\beta}$$

so, ϕ is one-to-one.

To prove that ϕ is onto, let $\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$ be an element in F^n.

Then $r_1\beta_1 + r_2\beta_2 + \cdots + r_n\beta_n \in V$.

and $\phi(r_1\beta_1 + r_2\beta_2 + \cdots + r_n\beta_n) = \begin{pmatrix} b_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$

so, ϕ is onto.

Since ϕ is both one-to-one and onto, so ϕ is an isomorphism.

Since ϕ is isomorphism, V is isomorphic to F^n.

Ex(1):- A linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x,y,z) = (x-y, x+2y, y+3z)$, $(x,y,z) \in \mathbb{R}^3$. Show that T is non-singular and determine T^{-1}.

Solution:- T is a linear mapping. Let us find $\ker T$.

Let $(a,b,c) \in \ker T$. Then $(a-b, a+2b, b+3c) = (0,0,0)$

Therefore $a-b=0$, $a+2b=0$, $b+3c=0$.

This gives $a=b=c=0$.

$\ker T = \{\underset{\sim}{0}\}$. $\therefore$ Therefore T is one-to-one.

Here $V = \mathbb{R}^3$, $W = \mathbb{R}^3$ and therefore $\dim V = \dim W$.

Since $T: V \to W$ is one-to-one, T is onto. Since T is one-to-one and onto, so it is non-singular.

Let $T^{-1}(x,y,z) = (a,b,c)$

Then $(x,y,z) = T(a,b,c) = (a-b, a+2b, b+3c)$

therefore, $a-b = x$, $a+2b = y$, $b+3c = z$.

$$\Rightarrow a = \frac{1}{3}(2x+y),\ b = \frac{1}{3}(-x+y),\ c = \frac{1}{9}(x-y+3z)$$

therefore $T^{-1}: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T^{-1}(x,y,z) = \left(\frac{2}{3}x + \frac{1}{3}y,\ -\frac{1}{3}x + \frac{1}{3}y,\ \frac{1}{9}x - \frac{1}{9}y + \frac{1}{3}z\right);$$

$(x,y,z) \in \mathbb{R}^3$.

Ex.(2):→ A linear mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ maps the vectors $(1,2,3)$, $(2,3,1)$, $(3,1,2)$ to the vectors $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ respectively. Show that ϕ is an isomorphism.

Solution:— The set of vectors $\{(1,2,3), (2,3,1), (3,1,2)\}$ is LIN set in $\mathbb{R}^3$, a vector space of dimension 3. Therefore it is a basis of $\mathbb{R}^3$.

The set of vectors $\{(0,1,1), (1,0,1), (1,1,0)\}$ is LIN set in $\mathbb{R}^3$, a vector space of dimension 3. Therefore it is a basis of $\mathbb{R}^3$.

The domain space and the co-domain space of the linear mapping ϕ are of the same dimension and ϕ maps a basis of the domain space to a basis of the co-domain space. So, ϕ is an isomorphism.

Ex.(3):→ A linear mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ maps the vectors $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ to the vectors $(1,1,-1)$, $(1,-1,1)$, $(1,0,0)$ respectively. Show that ϕ is not an isomorphism.

Solution:— The set of vectors $\{(0,1,1), (1,0,1), (1,1,0)\}$ is a LIN set in $\mathbb{R}^3$. Therefore it is a basis of $\mathbb{R}^3$. The images of this basis vectors are the vectors $(1,1,-1)$, $(1,-1,1)$, $(1,0,0)$ respectively. The set $\{(1,1,-1), (1,-1,1), (1,0,0)\}$ is an LIN set in $\mathbb{R}^3$ and ~~the~~ therefore it is not a basis of $\mathbb{R}^3$.

The linear mapping ϕ maps a basis of $\mathbb{R}^3$ to a set which is not a basis of $\mathbb{R}^3$. So ϕ is not an isomorphism.

Ex.(4):→ Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by $T(x,y,z) = (x+y, x-z)$; then the dimension of the null space of T is

(A) 0 (B) 1 (C) 2 (D) 3.

Solution:—

(B) Null space of $T = \{(x,y,z) : T(x,y,z) = (0,0)\}$

$\therefore (x+y, x-z) = (0,0)$

$\Rightarrow x+y=0, \quad x-z=0$

$\Rightarrow x = -y = z$

Let, $x = t$, $\therefore (x,y,z) = t(1,-1,1)$

$\therefore$ Null space of T is $= \{t(1,-1,1) : t \in \mathbb{R}\}$

$\therefore \dim N(T) = 1$.

Matrix representation of a linear mapping :-

T is completely determined by the images $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$. Each $T(\alpha_i)$ in W is a linear combination of the vectors $\beta_1, \beta_2, \ldots, \beta_m$.

Let
$$T(\alpha_1) = a_{11}\beta_1 + a_{21}\beta_2 + \cdots + a_{m1}\beta_m$$
$$T(\alpha_2) = a_{12}\beta_1 + a_{22}\beta_2 + \cdots + a_{m2}\beta_m$$
$$\vdots$$
$$T(\alpha_n) = a_{1n}\beta_1 + a_{2n}\beta_2 + \cdots + a_{mn}\beta_m,$$

[similar as — $Y = AX$, where A is the associated matrix.]

where a_{ij} are unique scalars in F determined by the ordered basis $(\beta_1, \beta_2, \ldots, \beta_m)$.

The co-ordinate vector of $T(\alpha_j)$ relative to the ordered basis $(\beta_1, \beta_2, \ldots, \beta_m)$ is given by the j^{th} column of A.

The matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ is said to be the matrix associated with the linear mapping T relative to the chosen ordered bases of V and W. A is also called the matrix of T relative to the chosen ordered bases.

WORKED EXAMPLES :-

1. A linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_1 - 3x_2 - 2x_3)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
Find the matrix of T relative to the ordered bases
(i) $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$ and $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$;
(ii) $\{(0,1,0), (1,0,0), (0,0,1)\}$ of $\mathbb{R}^3$ and $\{(0,1), (1,0)\}$ of $\mathbb{R}^2$;
(iii) $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$;

Solution:— (i)
$$T(1,0,0) = (3,1) = 3(1,0) + 1(0,1);$$
$$T(0,1,0) = (-2,-3) = -2(1,0) - 3(0,1)$$
$$T(0,0,1) = (1,-2) = 1(1,0) - 2(0,1)$$

$\therefore$ The matrix of $T = \begin{pmatrix} 3 & -2 & 1 \\ 1 & -3 & -2 \end{pmatrix}$.

(ii)
$$T(0,1,0) = (-2,-3) = -3(0,1) - 2(1,0)$$
$$T(1,0,0) = (3,1) = 1(0,1) + 3(1,0)$$
$$T(0,0,1) = (1,-2) = -2(0,1) + 1(1,0)$$

$\therefore$ The mtx of $T = \begin{pmatrix} -3 & 1 & -2 \\ -2 & 3 & 1 \end{pmatrix}$.

(iii)
$$T(0,1,1) = (-1,-5) = -1(1,0) - 5(0,1)$$
$$T(1,0,1) = (4,-1) = 4(1,0) - 1(0,1)$$
$$T(1,1,0) = (1,-2) = 1(1,0) - 2(0,1)$$

$\therefore$ The mtx of $T = \begin{pmatrix} -1 & 4 & 1 \\ -5 & -1 & -2 \end{pmatrix}$.

2. Let $(\alpha_1, \alpha_2, \alpha_3)$, (β_1, β_2) be ordered bases of the real vector spaces V and W respectively. A linear mapping $T: V \to W$ maps the basis vectors as $T(\alpha_1) = \beta_1 + \beta_2$, $T(\alpha_2) = 3\beta_1 - \beta_2$, $T(\alpha_3) = \beta_1 + 3\beta_2$.

Find the matrix of T relative to the ordered bases
(i) $(\alpha_1, \alpha_2, \alpha_3)$ of V and (β_1, β_2) of W;
(ii) $(\alpha_1 + \alpha_2, \alpha_2, \alpha_3)$ of V and $(\beta_1, \beta_1 + \beta_2)$ of W;.

Solution:– (i) $T(\alpha_1) = \beta_1 + \beta_2$; $T(\alpha_2) = 3\beta_1 - \beta_2$; $T(\alpha_3) = \beta_1 + 3\beta_2$

$\therefore$ the matrix of $T = \begin{pmatrix} 1 & 3 & 1 \\ 1 & -1 & 3 \end{pmatrix}$.

(ii) $T(\alpha_1 + \alpha_2) = T(\alpha_1) + T(\alpha_2) = 4\beta_1 = 4\beta_1 + 0(\beta_1 + \beta_2)$;

$T(\alpha_2) = (3\beta_1 - \beta_2) = 4\beta_1 - 1(\beta_1 + \beta_2)$;

$T(\alpha_3) = \beta_1 + 3\beta_2 = -2\beta_1 + 3(\beta + \beta_2)$

$\therefore$ the mtx of $T = \begin{pmatrix} 4 & 4 & -2 \\ 0 & -1 & 3 \end{pmatrix}$.

3. The matrix of a linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered bases $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0), (1,1)\}$ of $\mathbb{R}^2$ is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Find T. Also find the matrix of T relative to the ~~ordered bases $\{(0,1,0), (1,0,1), (0,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,1), (0,1)\}$ of $\mathbb{R}^2$~~

~~Solution:– $T(0,1,1) = 1(1,0) + 2(1,1)$~~

ordered bases $\{(1,1,0), (1,0,1), (0,1,1)\}$ of $\mathbb{R}^3$ and $\{(1,1), (0,1)\}$ of $\mathbb{R}^2$.

Solution:–
$$T(0,1,1) = 1(1,0) + 2(1,1) = (3,2)$$
$$T(1,0,1) = 2(1,0) + 1(1,1) = (3,1)$$
$$T(1,1,0) = 4(1,0) + 0(1,1) = (4,0)$$

Let $(x,y,z) \in \mathbb{R}^3$ and let $(x,y,z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$; where c_i are scalars in $\mathbb{R}$.

Then $c_2 + c_3 = x$, $c_1 + c_3 = y$, $c_1 + c_2 = z$

$\therefore$ $c_1 = \frac{1}{3}(y + z - x)$, $c_2 = \frac{1}{2}(z + x - y)$, $c_3 = \frac{1}{2}(x + y - z)$.

$$T(x,y,z) = c_1 T(0,1,1) + c_2 T(1,0,1) + c_3 T(1,1,0)$$
$$= c_1(3,2) + c_2(3,1) + c_3(4,0)$$
$$= (3c_1 + 3c_2 + 4c_3, 2c_1 + c_2)$$
$$= \left(2x + 2y + z, \tfrac{1}{2}(-x + y + 3z)\right), \quad (x,y,z) \in \mathbb{R}^3,$$

2nd Part:–
$$T(1,1,0) = (4,0) = 4(1,1) - 4(0,1)$$
$$T(1,0,1) = (3,1) = 3(1,1) - 2(0,1)$$
$$T(0,1,1) = (3,2) = 3(1,1) - 1(0,1)$$

The matrix of $T = \begin{pmatrix} 4 & 3 & 3 \\ -4 & -2 & -1 \end{pmatrix}$.

Theorem:- Let $T: V \to V'$ be a linear transformation $\ni$ $\dim V = \dim V'$, then the followings are equivalent:

(a) T is non-singular;

(b) T is invertible;

(c) T is one-to-one;

(d) T is onto;

PROBLEMS:-

1. The matrix of a linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ w.r.t. the ordered basis $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ is given by

$$\begin{pmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{pmatrix}.$$

Find T. Find the matrix of T relative to the ordered basis $\{(2,1,1), (1,2,1), (1,1,2)\}$ of $\mathbb{R}^3$.

Solution:→

$$T(0,1,1) = 0(0,1,1) + 2(1,0,1) + 2(1,1,0) = (4,2,2)$$
$$T(1,0,1) = 3(0,1,1) + 3(1,0,1) - 1(1,1,0) = (2,2,6)$$
$$T(1,1,0) = 0(0,1,1) - 2(1,0,1) + 2(1,1,0) = (0,2,-2)$$

Let $(x,y,z) \in \mathbb{R}^3$ and let $(x,y,z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$ where $c_i \in \mathbb{R}$. Then

$$c_1 = \frac{-x+y+z}{2}, \quad c_2 = \frac{x-y+z}{2}, \quad c_3 = \frac{x+y-z}{2}.$$

$$T(x,y,z) = c_1 T(0,1,1) + c_2 T(1,0,1) + c_3 T(1,1,0)$$
$$= c_1(4,2,2) + c_2(2,2,6) + c_3(0,2,-2)$$
$$= (-x+y+3z,\ x+y+z,\ x-3y+5z).$$

$$T(2,1,1) = (2,4,4); \quad T(1,2,1) = (4,4,0); \quad T(1,1,2) = (6,4,8);$$

Let $(2,4,4) = c_1(2,1,1) + c_2(1,2,1) + c_3(1,1,2)$.
Then $c_1+c_2+c_3 = 5/2$ and therefore $c_1 = -1/2$, $c_2 = 3/2$, $c_3 = \frac{3}{2}$.

Let $(4,4,0) = c_1(2,1,1) + c_2(1,2,1) + c_3(1,1,2)$
Then $c_1+c_2+c_3 = 2$ and therefore $c_1 = 2$, $c_2 = 2$, $c_3 = -2$.

Let $(6,4,8) = c_1(2,1,1) + c_2(1,2,1) + c_3(1,1,2)$
then $c_1+c_2+c_3 = 9/2$ and therefore $c_1 = 3/2$, $c_2 = -1/2$, $c_3 = \frac{7}{2}$.

$$\therefore\ m(T) = \begin{pmatrix} -1/2 & 2 & 3/2 \\ 3/2 & 2 & -1/2 \\ -3/2 & -2 & 7/2 \end{pmatrix}.$$

The Transpose of a Linear Transformation :-

1. Let V and W be vector space over the field F. For each linear transformation T from V into W, there is a unique linear transformation T^t from W^* into V^* such that $(T^t g)(\alpha) = g(T\alpha)$ for every $g \in W^*$ and $\alpha \in V$.

2. Let V and W be vector spaces over the field F, and let T be a linear transformation from V into W. the null space of T^t is the annihilator of the range of T. then V and W are finite dimensional, then

(i) $\text{Rank}(T^t) = \text{rank}(T)$

(ii) the range of T^t is annihilator of the null space of T.

WORKED EXAMPLES :-

1. Let $(\alpha_1, \alpha_2, \alpha_3)$ be an ordered basis of a real vector space V and a linear mapping $T : V \to V$ is defined by

$$T(\alpha_1) = \alpha_1 + \alpha_2 + \alpha_3, \quad T(\alpha_2) = \alpha_1 + \alpha_2, \quad T(\alpha_3) = \alpha_1.$$

Show that T is non-singular. Find the matrix of T^{-1} relative to the ordered basis $(\alpha_1, \alpha_2, \alpha_3)$.

Solution :- Let $m(T)$ be the matrix of T relative to the ordered basis $(\alpha_1, \alpha_2, \alpha_3)$.

then $m(T) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$m(T)$ is non-singular and therefore T is non-singular.

$$T^{-1}(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_1 \Rightarrow T^{-1}(\alpha_1) + T^{-1}(\alpha_2) + T^{-1}(\alpha_3) = \alpha_1$$

$$T^{-1}(\alpha_1 + \alpha_2) = \alpha_2 \Rightarrow T^{-1}(\alpha_1) + T^{-1}(\alpha_2) = \alpha_2$$

$$T^{-1}(\alpha_1) = \alpha_3 \Rightarrow T^{-1}(\alpha_1) = \alpha_3$$

; since T^{-1} is also linear.

∴ We have $T^{-1}(\alpha_1) = \alpha_3$, $T^{-1}(\alpha_2) = \alpha_2 - \alpha_3$, $T^{-1}(\alpha_3) = \alpha_1 - \alpha_2$.

Therefore, $m(T^{-1}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$.

2) Let T be the linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$.
If $\beta = \{(1,0,-1), (1,1,1), (1,0,0)\}$, $\beta' = \{(0,1), (1,0)\}$ be ordered bases of $\mathbb{R}^3, \mathbb{R}^2$, respectively. Then find the matrix of T relative to β, β', also find rank(T) and nullity(T).

Solution:- (i)

$$T(1,0,-1) = (1,-3) = -3(0,1) + 1(1,0)$$
$$T(1,1,1) = (2,1) = 1(0,1) + 2(1,0)$$
$$T(1,0,0) = (1,-1) = -1(1,0) + 1(0,1)$$

$\therefore$ the matrix of T is $\begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

(ii) Let $(x_1, x_2, x_3) \in \ker T$.

$$T(x_1, x_2, x_3) = (0,0) = (x_1 + x_2, 2x_3 - x_1)$$
$$\therefore x_1 + x_2 = 0, \quad 2x_3 - x_1 = 0$$
$$\Rightarrow x_1 = -x_2, \quad x_1 = 2x_3$$
$$\therefore x_1 = -x_2 = 2x_3, \text{ let } x_3 = t,$$
$$\therefore \ker(T) = \{t(2,-2,1) : t \in \mathbb{R}\}$$
$$\therefore \dim \ker T = 1.$$
i.e. nullity$(T) = 1$.

We know, $\text{rank}(T) + \text{nullity}(T) = \dim \mathbb{R}^3 = 3$

$$\Rightarrow \text{rank}(T) = 3 - 1 = 2.$$

3) Show that $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z) = (y,x)$ is a linear transformation. For the basis sets $S_1 = (e_1, e_2, e_3)$ of $\mathbb{R}^1$ and $S_2 = \{(1,0), (2,3)\}$ of $\mathbb{R}^2$. Find the mtx of T.

Solution:-

$$T(\alpha u + \beta v) = (\alpha y_1 + \beta y_2, \alpha x_1 + \alpha x_2); \quad u = (x_1, y_1, z_1), \; v = (x_2, y_2, z_2) \in \mathbb{R}^3.$$
$$= \alpha(y_1, x_1) + \beta(y_2, x_2)$$
$$= \alpha T(u) + \beta T(v).$$

Hence T is a linear transformation.

$$T(e_1) = (0,1), \; T(e_2) = (1,0), \; T(e_3) = (0,0)$$

$$T(e_1) = (0,1) = -\tfrac{2}{3}(1,0) + \tfrac{1}{3}(2,3)$$
$$T(e_2) = (1,0) = 1(1,0) + 0(2,3)$$
$$T(e_3) = (0,0) = 0(1,0) + 0(2,3)$$

$\because (0,1) = x\alpha_1 + y\alpha_2$
$= x(1,0) + y(2,3)$
$= (x + 2y, 3y)$
$\therefore x = -2/3, \; y = 1/3$

The matrix representation of T is $\begin{bmatrix} -2/3 & 1 & 0 \\ 1/3 & 0 & 0 \end{bmatrix}$.

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PROBLEMS:—

1. Let T be an invertible linear operator on $V_3(\mathbb{R})$ defined by $T(x,y,z) = (3x,\ x-y,\ 2x+y+z)\ \forall\ (x,y,z) \in V_3(\mathbb{R})$. Then $T^{-1}(3,0,4)$ is equal to

(A) $(4,0,3)$ (B) $(1,1,1)$ (C) $(1,2,3)$ (D) $(0,0,0)$

Sol. (B) If $T(x,y,z) = (3,0,4)$

then $T^{-1}(3,0,4) = (x,y,z)$

Now, $T(x,y,z) = (3,0,4)$

$\Rightarrow (3x,\ x-y,\ 2x+y+z) = (3,0,4)$

$\Rightarrow 3x = 3,\ x-y=0,\ 2x+y+z=4$

$\Rightarrow x=1,\ y=1,\ z=1$

$\therefore T^{-1}(3,0,4) = (1,1,1)$

2. The linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined as $T(x_1,x_2,x_3) = (x_1-x_2+2x_3,\ 2x_1+x_2-x_3,\ -x_1-2x_2)$. Then the null space of T is

(A) $\{(0,0,-2),(-1,0,2)\}$ (B) $\{(0,0,0)\}$ (C) $\{(1,0,0),(0,0,0)\}$ (D) None

Sol. $T(x_1,x_2,x_3) = (0,0,0) = (x_1-x_2+2x_3,\ 2x_1+x_2-x_3,\ -x_1-2x_2)$

$\Rightarrow x_1-x_2+2x_3=0,\quad 2x_1+x_2-x_3=0,\quad -x_1-2x_2=0$

Coefficient mtx is $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 1 & -2 & 0 \end{bmatrix} = A$

$|A| = -9 \neq 0$

$\therefore \text{Rank}(A) = 3$

Hence the equations have the only trivial solution $x_1=x_2=x_3=0$.

$\therefore (0,0,0)$ is the only vector which belongs to the null space of T.

3. The linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined as $T(x,y,z) = (3x-2y+z,\ x-3y-2z)$, then the range of T is

(A) $\{(3,-2,1),(1,-3,-2)\}$ (B) $\{(3,1),(-2,-3),(1,-2)\}$
(C) $\{(1,0,0),(0,1,0),(0,0,1)\}$ (D) None of the above.

Solution:— (B) $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for $\mathbb{R}^3$.

$\therefore T(1,0,0) = (3,1)$

$T(0,1,0) = (-2,-3)$

$T(0,0,1) = (1,-2)$

4. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$. Then which of the following is not a null space of T

(A) $\{(3,1,-1)\}$ (B) $\{(6,-4,2)\}$ (C) $\{(2,-2/3,2/3)\}$, (D) $\{(6,-2,2)\}$

Sol. → (D)

$$T(3,1,-1) = (0,0,0)$$
$$T(6,-4,2) = (0,0,0)$$
$$T(2,-2/3,2/3) = (0,0,0)$$
But $T(6,-2,2) \neq (0,0,0)$.

5. With respect to the standard basis vectors a linear transformation $T:\mathbb{R}^4 \to \mathbb{R}^3$ is given by the matrix $\begin{bmatrix} 3 & -1 & -1 & 1 \\ -2 & 2 & -2 & -2 \\ -1 & 1 & 3 & -1 \end{bmatrix}$

Then dimension of $\ker(T)$ is

(A) 1 (B) 2 (C) 3 (D) 4

Sol. (A)

$$\text{dimension}(\ker T) = \dim(\mathbb{R}^4) - \text{rank}(T)$$
$$= 4 - 3$$
$$= 1.$$

6. The coordinate of the vector $(2,1,-6)$ relative to the basis $\{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\alpha}_3\}$, where $\underset{\sim}{\alpha}_1 = (1,1,2)$, $\underset{\sim}{\alpha}_2 = (3,-1,0)$, $\underset{\sim}{\alpha}_3 = (2,0,-1)$ is

(A) $(-7/8, -15/8, 17/4)$ (B) $(-9/4, 13/8, 17/4)$ (C) $(-7/2, -9/4, -15/8)$

(D) None of these.

Sol. Let a_i, $i = 1(1)3$ are scalars in $\mathbb{R}$.

$$(2,1,-6) = a_1\underset{\sim}{\alpha}_1 + a_2\underset{\sim}{\alpha}_2 + a_3\underset{\sim}{\alpha}_3$$
$$\Rightarrow (2,1,-6) = a_1(1,1,2) + a_2(3,-1,0) + a_3(2,0,-1)$$
$$\Rightarrow (2,1,-6) = (a_1+3a_2+2a_3,\ a_1-a_2,\ 2a_1-a_3)$$
$$\Rightarrow a_1+3a_2+2a_3 = 2,\quad a_1-a_2 = 1,\quad 2a_1-a_3 = -6$$

Solving these equations, we get

$$\Rightarrow a_1 = -7/8,\quad a_2 = -15/8,\quad a_3 = 17/4.$$

7. A linear transformation $T: V \to W$ is defined as $T(x,y,z) = (2x, 4y, 5z)$. Then find the mtx of T w.r.t. the basis $(2/3,0,0)$, $(0,1/2,0)$, $(0,0,1/4)$.

Sol.

$$T(2/3,0,0) = (4/3,0,0) = 2(2/3,0,0)$$
$$T(0,1/2,0) = (0,2,0) = 4(0,1/2,0)$$
$$T(0,0,1/4) = (0,0,5/4) = 5(0,0,1/4)$$

Hence the required mtx is $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

8. Let T be a linear operator on $\mathbb{R}^2$, the mtx of which in the ~~a~~ standard ordered basis is $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$

Prove that the only subspaces of $\mathbb{R}^2$ invariant under T are $\mathbb{R}^2$ and zero subspaces.

Sol. Ch. polynomial of A (or T) is det $\begin{bmatrix} x-1 & 1 \\ -2 & x-2 \end{bmatrix} = x^2-3x+4 = 0$,

whose roots are not real.

$\therefore$ Eigen values of A (or T) do not exist in $\mathbb{R}$. If W is an invariant subspace of $\mathbb{R}^2$ s.t. $W \neq 0$, $\mathbb{R}^2$ with dim $W = 1$.

Let W be spanned by v. Then $T(v) \in W \Rightarrow Tv = \alpha v$, $v \neq 0$

$\Rightarrow \alpha$ is an eigen value of T ($\alpha \in \mathbb{R}$), a contradiction.

$\therefore$ 0 and $\mathbb{R}^2$ are the only invariant subspaces of $\mathbb{R}^2$.

9. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(x,y,z) = (x+y+z,\ y+z,\ z)$. Then $T^n(x,y,z)$ is ? $(n \geq 1)$.

Sol. $T^2(x,y,z) = (x+2y+3z,\ y+2z,\ z)$

$T^3(x,y,z) = (x+3y+6z,\ y+3z,\ z)$

$\vdots$

$T^n(x,y,z) = \left(x+ny+\frac{n(n+1)}{2}z,\ y+nz,\ z\right)$

10. Let $V = \mathbb{R}^3$ and $T: V \to V$ is a linear map $\ni$ the mtx of T w.r.t. the standard basis is $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Show that T is an isomorphism of V onto V'. Find the mtx of T^{-1} w.r.t. the same basis and verify that it is the inverse of A.

Sol. $T(x,y,z) = \underset{\sim}{0}$,

Then, $(x+z,\ y+z,\ x+y) = (0,0,0)$

$\Rightarrow x = y = z = 0$.

Hence T is one-to-one, so it is an isomorphism.

Let $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ be the standard basis.

Since $T(\underset{\sim}{e}_1) = \underset{\sim}{e}_1 + \underset{\sim}{e}_3$, $T(\underset{\sim}{e}_2) = \underset{\sim}{e}_2 + \underset{\sim}{e}_3$, $T(\underset{\sim}{e}_3) = \underset{\sim}{e}_1 + \underset{\sim}{e}_2$

$\Rightarrow \underset{\sim}{e}_1 = \frac{1}{2}\left[T(\underset{\sim}{e}_1) + T(\underset{\sim}{e}_3) - T(\underset{\sim}{e}_2)\right] = \frac{1}{2} T(\underset{\sim}{e}_1 + \underset{\sim}{e}_3 - \underset{\sim}{e}_2)$ [As T is linear]

$\therefore T^{-1}(\underset{\sim}{e}_1) = \frac{1}{2}(\underset{\sim}{e}_1 + \underset{\sim}{e}_3 - \underset{\sim}{e}_2)$

Similarly, $T^{-1}(\underset{\sim}{e}_2) = \frac{1}{2}(\underset{\sim}{e}_2 + \underset{\sim}{e}_3 - \underset{\sim}{e}_1)$

and $T^{-1}(\underset{\sim}{e}_3) = \frac{1}{2}(\underset{\sim}{e}_1 + \underset{\sim}{e}_2 - \underset{\sim}{e}_3)$

Hence the mtx of T^{-1} is $\frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = M$.

Now, $AM = I_3$, so we get that it is also the inverse of A.

SOME PROBLEMS ON LINEAR ALGEBRA

1. $A^{n\times n}$; $a_{ij} = 1, \ \forall\, i=j$
$\qquad\qquad = \rho \ \ \forall\, i \neq j$

Find the determinant of A.

Solution:-

$$\begin{pmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \rho & \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \vdots & & \vdots \\ \rho & \rho & \rho & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+\overline{n-1}\rho & \rho & \cdots & \rho \\ 1+\overline{n-1}\rho & \cancel{\rho}1 & \cdots & \rho \\ \rho & \rho & \cdots & \rho \\ \vdots & \vdots & & \vdots \\ 1+\overline{n-1}\rho & \rho & \cdots & 1 \end{pmatrix}$$

$$= (1+\overline{n-1}\rho) \begin{pmatrix} 1 & \rho \\ 1 & \\ \vdots & \\ 1 & \end{pmatrix}$$

PROBLEMS ON LINEAR ALGEBRA [FROM CU & ISI PAPERS]

Ex. 1. Let A and B be two invertible $n\times n$ matrices. Assume that $A+B$ is invertible. Then show that $A^{-1}+B^{-1}$ is invertible.

Sol. $|A(A^{-1}+B^{-1})B| = |A+B|$

$$\Rightarrow A^{-1}+B^{-1} = \frac{|A+B|}{|A||B|}$$

Ex. 2. Let A and B be $n\times n$ real matrices $\ni A^2 = A$, $B^2 = B$. Let $I-(A+B)$ is invertible. Show that $\text{rank}(A) = \text{rank}(B)$.

Sol.
$$A[I-A-B] = A - A^2 - AB$$
$$= A^2 - A^2 - AB$$
$$= -AB$$

$$\& \ [I-A-B]B = B - AB - B^2$$
$$= -AB$$

$\therefore$ rank (A) = rank $[A(I-A-B)]$ = rank $(-AB)$ = rank (B).

$\therefore r(A) = r(B)$.

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Cayley Hamilton Theorem:— Let $A^{n\times n}$ be a matrix satisfies the matrix equation $f(A)=0$ of degree n, the ch. equation is $f(\lambda)=0$.

Ex. 3. $A^{2\times 2} \ni A^2=0$

$$\Rightarrow \lambda^2=0$$
$$\Rightarrow \lambda=0,0$$

Now, $(I+A)\underset{\sim}{x} = \underset{\sim}{x}+A\underset{\sim}{x}$
$= \underset{\sim}{x}+\lambda\underset{\sim}{x}$
$= \underset{\sim}{x}+0.\underset{\sim}{x}$
$= 1.\underset{\sim}{x}$, $\underset{\sim}{x}\neq 0$

Now, $\lambda=1,1$, then $|I+A|=1.1=1$.

Ex. 4 Let $A^{n\times n}$ be an orthogonal matrix and n is even. Let $|A|=-1$. Find $|I-A|$.

Sol. A is a mtx with charcteristic root λ.

$A'=A^{-1}$, as A is orthogonal.

$\Rightarrow \lambda=\frac{1}{\lambda}$ $\quad |A|=-1=\lambda_1\lambda_2\cdots\lambda_n$

$\Rightarrow \lambda=\pm 1$; $\quad \Rightarrow$ at least one λ_i's must be 1.

ch. equation:- $|\lambda I-A|=0$, set $\lambda=1$

$\Rightarrow |I-A|=0$.

Ex. 5. Let $A=\begin{pmatrix}1 & -1\\ 2 & -2\end{pmatrix}$. Find $A^{100}+A^5$.

Sol. $0=|A-\lambda I_2| = \begin{vmatrix}1-\lambda & -1\\ 2 & -2-\lambda\end{vmatrix} = \lambda^2+\lambda$

Using C-H Theorem, $A^2+A=0$

$A^5=(A^2)^2.A=(-A)^2.A=A^2.A=-A.A=A$

$A^{100}=(A^5)^{20}=(A^5)^4=(A^2)^2=(-A)^2=A^2=-A$

$\therefore A^{100}+A^5=0$.

Ex. 6 Find $\begin{vmatrix}1+x_1y_1 & 1+x_1y_2 & \cdots & 1+x_1y_n\\ 1+x_2y_1 & 1+x_2y_2 & \cdots & 1+x_2y_n\\ \vdots & \vdots & & \vdots\\ 1+x_ny_1 & 1+x_ny_2 & \cdots & 1+x_ny_n\end{vmatrix}$

Sol. $\Delta=\begin{vmatrix}1 & x_1 & 0 & \cdots & 0\\ 1 & x_2 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & x_n & 0 & \cdots & 0\end{vmatrix} \times \begin{vmatrix}1 & 1 & \cdots & 1\\ y_1 & y_2 & \cdots & y_n\\ \vdots & \vdots & & \vdots\\ 0 & 0 & & 0\end{vmatrix} = 0\times 0=0$.

Ex. Let P be a ~~[illegible]~~ matrix of order > 1 and entries are +ve integers. Suppose P^{-1} exists and ~~[illegible]~~ has integer entries, then what are the set of possible values of $|P|$?

(ISI)

Sol. P has integer entries.

$$\Rightarrow \lambda_1+\lambda_2+\cdots+\lambda_n = \text{trace}(P) = \text{integer}$$

$$\Rightarrow \sum_{i<j} \lambda_i\lambda_j = \text{Sum of minors of order}$$

$$\vdots$$

$$\prod_{i=1}^{n} \lambda_i = |P| = \text{integer}.$$

Then the eigenvalues of P^{-1} are $\frac{1}{\lambda_i}$ and they are also integers.

$$\Rightarrow \lambda_i = \frac{1}{\lambda_i}$$

$$\Rightarrow \lambda_i = \pm 1$$

$$\Rightarrow |P| = \prod_{i=1}^{n} \lambda_i = \pm 1.$$

Ex. Let XY be a bivariate normal vector $\ni$ $E(X)=E(Y)=0$ & $V(X)=V(Y)=1$. Let S be a subset of $\mathbb{R}^2$ and defined by

$$S = \{(a,b) : (ax+by) \text{ is independent of } y\}$$

(i) S.T. S be a subspace (ii) Find of its dimension.

(ISI)

Sol. $S = \{(a,b) : (ax+by) \text{ is independent of } Y\}$

(i) $(a_1,b_1), (a_2,b_2) \in S$.

Then a_1x+b_1y is independent of Y.

a_2x+b_2y is independent of Y.

$\Rightarrow (\alpha a_1+\beta a_2)x + (\alpha b_1+\beta b_2)y$ is independent of Y.

$\Rightarrow (\alpha a_1+\beta a_2, \alpha b_1+\beta b_2) \in S$

$\Rightarrow \alpha(a_1,b_1) + \beta(a_2,b_2) \in S \ \forall (\alpha,\beta) \in \mathbb{R}$

Hence, S is a subspace.

(ii) $(a,b) \in S$

$\Rightarrow ax+by$ is indep. of Y.

$\Rightarrow \text{cov}[ax+by, y] = 0$

$\Rightarrow a\,\text{cov}[x,y] + b\,\text{cov}[y,y] = 0$

$\Rightarrow a\rho + b.1 = 0$ since $\text{cov}[x,y] = \rho$

$\Rightarrow b = -a\rho$ & $\text{cov}[y,y] = \text{Var}(Y) = 1$, as $E(X)=0=E(Y)$ & $V(X)=1=V(Y)$

$\therefore (a,b) = a(1,-\rho),\ a \in \mathbb{R}$

$\therefore S = \{(a,b) : (a,b) = a(1,-\rho),\ a \in \mathbb{R}\}$

$\therefore \dim(S) = 1.$

Characteristic Roots & Ch. vectors:-

In mathematics and social sciences we may be requested in the solution of $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$.
We are looking for non-trivial solutions, i.e. for vectors $\underset{\sim}{x} \neq \underset{\sim}{0}$ ∋

$$A\underset{\sim}{x} = \lambda\underset{\sim}{x}$$
$$\Rightarrow (A - \lambda I_n)\underset{\sim}{x} = \underset{\sim}{0} \quad \cdots\cdots\cdots (1)$$

There will be a non-trivial solution $\underset{\sim}{x} \neq \underset{\sim}{0}$,
iff $r(A - \lambda I_n) < n$
iff $|A - \lambda I_n| = 0$.
iff
$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad \cdots\cdots\cdots (2)$$

Note that, $f(\lambda) = |A - \lambda I_n|$
$$= (-\lambda)^n + b_{n-1}(-\lambda)^{n-1} + \cdots + b_0;$$ a polynomial in λ, is called the characteristic polynomial of the matrix $A^{n\times n}$.
The equation (2), i.e. $|A - \lambda I_n| = 0$ or $f(\lambda) = 0$ is called the characteristic equation of the mtx A.

The n^{th} degree polymial equation $f(\lambda) = 0$ has n roots, roots may be real or imaginary and all roots need not to be different.
The roots, say $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the equation (2), i.e.,
$|A - \lambda I_n| = 0$ are called the characteristic roots or eigen value or latent roots of the matrix A.
When $\lambda \neq \lambda_i$, $i = 1(1)n$, then $|A - \lambda I_n| \neq 0$ and $(A - \lambda I_n)\underset{\sim}{x} = \underset{\sim}{0}$ has the only solution $\underset{\sim}{x} = \underset{\sim}{0}$.
When $\lambda = \lambda_i$, $i = 1(1)n$, then $|A - \lambda_i I_n| = 0$ and $(A - \lambda_i I_n)\underset{\sim}{x} = \underset{\sim}{0}$ has at least one ~~sol~~ non-null solution $\underset{\sim}{x}$.
The maximum number of LIN vectors $\underset{\sim}{x} \neq \underset{\sim}{0}$ which satisfy $[A - \lambda_i I_n]\underset{\sim}{x} = \underset{\sim}{0}$ is $\dim\{N(A - \lambda_i I_n)\}$.
The vectors $\underset{\sim}{x} \neq \underset{\sim}{0}$ which satisfy $(A - \lambda_i I_n)\underset{\sim}{x} = \underset{\sim}{0}$ are called the characteristic vector or eigen vectors corresponding to the eigen values λ_i of A.

Note that $f(\lambda) = |A - \lambda I_n|$
$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda).$$

Ex. 9. Let A be an upper triangular matrix with $d_1, \ldots, d_n$ as the diagonal elements, show that the eigen values of A are $d_1, d_2, \ldots, d_n$.

Solution:- Characteristic equation gives

$$|A-\lambda I_n|=0 = \begin{vmatrix} d_1-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & d_2-\lambda & a_{23} & \cdots & a_{2n} \\ 0 & 0 & d_3-\lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n-\lambda \end{vmatrix}$$

$$= \prod_{i=1}^{n} (d_i - \lambda)$$

$$\Rightarrow \lambda = d_1, \ldots, d_n$$

Hence $d_1, d_2, \ldots, d_2$ are the eigen values.

Ex. 10. Find the characteristic roots of $A = \alpha\alpha^T$, $\underset{\sim}{\alpha} \neq \underset{\sim}{0}$.

Solution:- Characteristic equation is

$$0 = |\underset{\sim}{\alpha}\underset{\sim}{\alpha}^T - \lambda I_n|$$

$$= (-\lambda)^n \left| I_n - \frac{1}{\lambda} \cdot \underset{\sim}{\alpha}\underset{\sim}{\alpha}^T \right|$$

$$= (-\lambda)^n \left(1 - \frac{1}{\lambda} \cdot \underset{\sim}{\alpha}'\underset{\sim}{\alpha} \right)$$

$$= (-1)^n \lambda^{n-1} (\lambda - \underset{\sim}{\alpha}'\underset{\sim}{\alpha})$$

Hence the ch. roots are $\lambda_1 = \underset{\sim}{\alpha}'\underset{\sim}{\alpha}$ and $\lambda_2 = 0$ with multiplicity $(n-1)$.

Ex. 11. Find the ch. roots of $A = (I_n - \underset{\sim}{\alpha}\underset{\sim}{\alpha}^T)$.

Solution:- Characteristic equation is

$$0 = |I_n - \underset{\sim}{\alpha}\underset{\sim}{\alpha}^T - \lambda I_n|$$

$$\Rightarrow |(1-\lambda) I_n - \underset{\sim}{\alpha}\underset{\sim}{\alpha}^T| = 0$$

$$\Rightarrow (1-\lambda)^n \left| I_n - \frac{1}{1-\lambda} \underset{\sim}{\alpha}\underset{\sim}{\alpha}^T \right| = 0$$

$$\Rightarrow (1-\lambda)^n \left(1 - \frac{1}{1-\lambda} \underset{\sim}{\alpha}^T\underset{\sim}{\alpha} \right) = 0$$

$$\Rightarrow (1-\lambda)^{n-1} \{ (1-\lambda) - \underset{\sim}{\alpha}^T\underset{\sim}{\alpha} \} = 0$$

Rank $(I_n - \underset{\sim}{\alpha}\underset{\sim}{\alpha}')$
$=$ trace (I_n) $-$ trace $(\underset{\sim}{\alpha}\underset{\sim}{\alpha}')$
$= n - 1$.

$$\Rightarrow \lambda = 1 - \underset{\sim}{\alpha}^T\underset{\sim}{\alpha} \text{ or } 1 \text{ with multiplicity } (n-1).$$

Ex. 12 (a) If $AA' = I_n$, then find the eigen value of A.

Sol. $A^T\underset{\sim}{x} = \lambda\underset{\sim}{x}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$ [$\because |A - \lambda I_n| = 0 \Rightarrow |A^T - \lambda I_n| = 0$
$\Rightarrow A$ and A^T have the same set of eigen values]

$\therefore A\underset{\sim}{x} = \lambda\underset{\sim}{x}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$

$$\therefore A^T\underset{\sim}{x} = \lambda\underset{\sim}{x}$$

$$\Rightarrow AA^T\underset{\sim}{x} = \lambda A\underset{\sim}{x} = \lambda^2\underset{\sim}{x}$$

$$\Rightarrow I_n\underset{\sim}{x} = \lambda^2\underset{\sim}{x}$$

$$\Rightarrow (1-\lambda^2)\underset{\sim}{x} = \underset{\sim}{0}, \ \underset{\sim}{x} \neq \underset{\sim}{0}$$

$$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1.$$

(b) If $AA^T = n \cdot I_n$ then find the eigen values of A.

Sol. $AA^T \underset{\sim}{x} = \lambda^2 \underset{\sim}{x}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow n \cdot I_n \underset{\sim}{x} = \lambda^2 \underset{\sim}{x}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow \lambda^2 = n$

$\Rightarrow \lambda = \pm\sqrt{n}$.

Ex.13. Find the eigen value of $A = \begin{bmatrix} 1 & p & p \cdots p \\ p & 1 & p \cdots p \\ \vdots & \vdots & \vdots \\ p & p & p \cdots 1 \end{bmatrix} = (1-p)I_n + p\underset{\sim}{1}\cdot\underset{\sim}{1}^T$

Sol.

$$A = (1-p)I_n + p\underset{\sim}{1}\cdot\underset{\sim}{1}^T$$

characteristic equation is:

$$|A - \lambda I_n| = |(1-p)I_n + p\underset{\sim}{1}\cdot\underset{\sim}{1}^T - \lambda I_n| = 0$$

$$\Rightarrow |(1-p-\lambda)I_n + p\underset{\sim}{1}\cdot\underset{\sim}{1}^T| = 0$$

$$\Rightarrow (1-p-\lambda)^n \left| I_n + \frac{p\,\underset{\sim}{1}\cdot\underset{\sim}{1}^T}{(1-p-\lambda)} \right| = 0$$

$$\Rightarrow (1-p-\lambda)^n \left\{ 1 + \frac{p}{1-p-\lambda}\underset{\sim}{1}^T\cdot\underset{\sim}{1} \right\} = 0$$

$$\Rightarrow (1-p-\lambda)^{n-1} \left\{ 1-p-\lambda + p\underset{\sim}{1}^T\underset{\sim}{1} \right\} = 0$$

$\Rightarrow \lambda = 1-p+p\underset{\sim}{1}^T\underset{\sim}{1}$, $(1-p)$ with multiplicity $(n-1)$.

Ex.14. If A is an idempotent matrix, then find the eigen values of A and hence show that rank(A) = trace(A).

Sol. Let λ be an eigen value of A.

By definition, $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow A^2\underset{\sim}{x} = \lambda\underset{\sim}{x}$

$\Rightarrow A(A\underset{\sim}{x}) = \lambda\underset{\sim}{x}$

$\Rightarrow \lambda(A\underset{\sim}{x}) = \lambda\underset{\sim}{x}$

$\Rightarrow \lambda(\lambda\underset{\sim}{x}) = \lambda\underset{\sim}{x}$

$\Rightarrow \lambda^2\underset{\sim}{x} = \lambda\underset{\sim}{x}$

$\Rightarrow \lambda(\lambda-1)\underset{\sim}{x} = \underset{\sim}{0}$ $\forall\, \underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow \lambda = 0, 1$.

Now, we can find an orthogonal mtx Q such that

$$Q'AQ = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

$\therefore$ Rank(A) = rank$\{Q'AQ\}$, since $|Q| = \pm 1$, i.e. Q is non-singular

$= r\{\text{diag}(\lambda_1, \ldots, \lambda_n)\}$

$=$ No. of non-zero λ_i's

$= \sum_{i=1}^{n} \lambda_i$, since $\lambda_i = 0, 1$.

$=$ trace$\{\text{diag}(\lambda_1, \ldots, \lambda_n)\}$

$=$ trace$(Q'AQ)$ [$A = Q^T$, $B = AQ$

$=$ trace(AQQ') $\Rightarrow$ trace(AB) = tr(BA)]

$=$ trace(A).

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Ex. 15. Let λ be an eigen value of an n.s. mtx A. Show that $\frac{1}{\lambda}$ is an eigen value of A^{-1}. Further show that A and A^{-1} have the same set of eigen values. Can A have zero as eigen value?

Sol. Let $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow A^{-1} A \underset{\sim}{x} = \lambda A^{-1} \underset{\sim}{x}$ [multiplying by A^{-1}]

$\Rightarrow \frac{1}{\lambda} A^{-1} A \underset{\sim}{x} = A^{-1} \underset{\sim}{x}$

$\Rightarrow \frac{1}{\lambda} \underset{\sim}{x} = A^{-1} \underset{\sim}{x}$ [$\because$ as $A^{-1}A = I$]

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of A^{-1}. (Proved)

If possible let $\lambda = 0$,

Then $A\underset{\sim}{x} = \underset{\sim}{0}$ $\forall \underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow$ rank(A) $< n$, if $A^{n\times n}$ mtx.

$\Rightarrow |A| = 0$ which is a contradiction to the fact that A is non-singular.

Hence $\lambda \neq \underset{\sim}{0}$.

Ex. 16. Show that eigen values of real square mtx A (not necessarily symmetric) are same as the eigen values of A^T. Are the eigen vectors of A same as the A^T?

Sol. Characteristic equation of A is

$$|A - \lambda I_n| = 0 = |A - \lambda I_n|' = |(A - \lambda I_n)^T|$$

i.e. $0 = |A^T - \lambda I_n|$ is the ch. equation of A^T.

$(A - \lambda I_n)\underset{\sim}{x} = \underset{\sim}{0}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{x} \in N(A - \lambda I_n) - \{\underset{\sim}{0}\}$

and $(A^T - \lambda I_n)\underset{\sim}{y} = \underset{\sim}{0}$, $\underset{\sim}{y} \neq \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{y} \in N(A^T - \lambda I_n) - \{\underset{\sim}{0}\}$

If A is symmetric then A and A^T will have the same set of eigen vectors.

If A is not symmetric, then A and A^T will have different set of eigen vectors.

Ex.17. If A and B are row-equivalent $m\times n$ matrices, then the homogeneous system of linear equations $AX=0$ and $BX=0$ have exactly the same solution.

Proof:- Suppose, we pass from A to B by a finite sequence of elementary row operations:

$$A=A_0 \to A_1 \to A_2 \to \cdots\cdots \to A_k=B.$$

It is enough to prove that the systems $A_jX=0$ and $A_{j+1}X=0$ have the same solution, i.e. that one elementary row operation does not disturb the set of solutions.

So, suppose that B is obtained from A by a single elementary row operation. No matter which of the types of operations are used, each equation in the system $BX=0$ will be a linear combination of the equations in the system $AX=0$. Since the inverse of an elementary row operation is an elementary row operation, each equation in $AX=0$ will be a linear combination of the equations in $BX=0$. Hence, there are two equivalent systems and have ~~the~~ the same solutions.

Ex.18. Let A and B be two matrices such that B is obtained by interchanging its 1st and 2nd rows. Find an n.s. mtx $P \ni PA=B$

Sol. Let $A=\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$ then $B=\begin{pmatrix} a_2 \\ a_1 \\ \vdots \\ a_m \end{pmatrix}$

$B=E_{12}A$, E_{12} is the elementary mtx obtained from I_n by interchanging the first and second row.

$\therefore B=PA$, (given)

$\therefore E_{12}=P$

P being an elementary mtx and non-singular because $|P|=-|I_n|$
$=-1\neq 0.$

Ex.19. Let A and B be two matrices of order $p\times q$ and $q\times p$, respectively. Show that $|AB|=0$ if $p>q$.

Sol. AB is matrix of ~~rank~~ $p\times p$.

$$\text{Rank}(AB) \le \min[\text{Rank}(A), \text{Rank}(B)] \text{ ——— } ①$$

For $q<p$, rank$(A)\le q$ and rank$(B)\le q$.

giving that Rank$(AB)\le q$.

$\Rightarrow$ Columns of $(AB)_{p\times p}$ are linearly dependent.

$\Rightarrow |AB|=0$.

Ex. 20. For two subspaces W_1 and W_2 of $V_n(F)$,
$$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

Proof:-

Let $\{\alpha_1, \ldots, \alpha_k\}$ be a basis of $W_1 \cap W_2$.
As this set is linearly independent and contains vectors from W_1, it can be extended to form a basis

$$\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m\} \text{ of } W_1.$$

and for the similar reason

$$\{\alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_n\} \text{ of } W_2.$$

The subspace $W_1 + W_2$ is spanned by the vectors

$$\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n$$

and these vectors form an independent set:

For suppose $\sum x_i \alpha_i + \sum y_j \beta_j + \sum z_r \gamma_r = 0$.

Then $-\sum z_r \gamma_r = \sum x_i \alpha_i + \sum y_j \beta_j$ which shows that $\sum z_r \gamma_r$ belongs to W_1. As $\sum z_r \gamma_r$ also belong to W_2 it follows that

$$\sum z_r \gamma_r = \sum c_i \alpha_i,$$

for certain scalars $c_1, \ldots, c_k$. Because the set

$\{\alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_n\}$ is independent, each of the scalars $z_r = 0$. Thus

$$\sum x_i \alpha_i + \sum y_j \beta_j = 0$$

and since $\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m\}$ is also an independent set, each $x_i = 0$ and each $y_j = 0$. Thus,

$$\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n\}$$

is a basis for $W_1 + W_2$. Finally

$$\begin{aligned}
\dim(W_1) + \dim(W_2) &= (k+m) + (k+n) \\
&= k + (k+m+n) \\
&= \dim(W_1 \cap W_2) + \dim(W_1 + W_2)
\end{aligned}$$

$$\therefore \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

EX. 21. Let A be a square matrix of order p $\ni$ for some non-null vector $\underset{\sim}{\alpha}$, $A\underset{\sim}{\alpha} = 0$. Show that for any square matrix B of order p, $|AB| = 0$.

Sol. $A\underset{\sim}{\alpha} = \underset{\sim}{0}$ for $\underset{\sim}{\alpha} \neq \underset{\sim}{0}$

Nullity of $A \geq 1$

$R(A) \leq n-1$

$\therefore$ A is non-singular.

$\therefore |A| = 0$

$\therefore |AB| = |A||B| = 0$.

EX. 22. Find the dimension of the subspace of E^3;

i) $S_1 = \{(x_1, x_2, 0) : x_1, x_2 \in R\}$

ii) $S_2 = \{(x_1, x_2, x_3) : x_1 - x_2 = x_3\}$

iii) $S_3 = \{\lambda(1,1,1) : \lambda \in R\}$

Sol. (i) Let $\underset{\sim}{x} \in S_3$ be an arbitrary vector.

Then $\underset{\sim}{x} = (x_1, x_2, 0)$, $x_1, x_2 \in R$

$= x_1 \underset{\sim}{e}_1 + x_2 \underset{\sim}{e}_2$.

Clearly, $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2\}$ spans S_3 since it is LIN set of vectors.

$\therefore \{e_1, e_2\}$ forms a basis for S_3;

Hence, $\dim(S_3)$ = No. of vectors in a basis = 2.

(ii) Let $\underset{\sim}{x} \in S_3$ be an arbitrary vector.

Then $\underset{\sim}{x} = (x_1, x_2, x_3)$: $x_1 - x_2 = x_3$

$= (x_1, x_2, x_1 - x_2)$

$= x_1(1,0,1) + x_2(0,1,-1)$

Clearly $\{(1,0,1), (0,1,-1)\}$ spans S_3 and they are LIN, i.e. they form a basis for S_3. Hence, $\dim(S_3)$ = No. of vectors in a basis = 2.

Remark:- It is also intuitively appealing to takes the dimension of a vector space as the number of independent components in a vector of the vector space.

Clearly, $S_3 = \{(x_1, x_2, x_3)) : x_1, x_2 \in R\}$ has 2 independent components in each vector, $\dim(S_3) = 2$.

Ex.23. For any square mtx, the determinant of the transposed matrix is the same as the determinant of the matrix itself.

Solution:- Let the square matrix be of order n. Let $A=(a_{ij})$ then the determinant of A can be written as

$$|A| = \sum \mp a_{1k_1} a_{2k_2} a_{3k_3} \cdots\cdots a_{nk_n} \cdots\cdots\cdots (1)$$

where, the sum is taken over all permutations $(k_1, \ldots, k_n)$ of first n natural numbers and the sign attached to every term of the sum is minus or plus if the corresponding permutation is odd or even.

Let us re-arrange the factors of each term so as to bring the secondary suffixes in the natural order $(1, 2, \ldots, n)$ and modifying the primary suffixes accordingly. The term $a_{1k_1} a_{2k_2} a_{3k_3} \cdots\cdots a_{nk_n}$ then taken as the form $a_{l_1 1} a_{l_2 2} a_{l_3 3} \cdots\cdots a_{l_n n}$ and the permutation $(l_1, \ldots, l_n)$ is odd or even according to the permutation $(k_1, \ldots, k_n)$ is odd or even.

Thus, $$|A| = \sum \mp a_{l_1 1} a_{l_2 2} \cdots\cdots a_{l_n n} \cdots\cdots (2)$$

If A' be the transpose of A, a'_{ij} be the (i,j)th term of A' then $a'_{ij} = a_{ji}$ $\forall\, i, j$.

So, $$|A'| = \sum \mp a'_{1l_1} a'_{2l_2} \cdots\cdots a'_{nl_n}$$

or $$|A'| = \sum \mp a_{l_1 1} a_{l_2 2} \cdots\cdots a_{l_n n} \cdots\cdots\cdots (3)$$

From (2) & (3), $|A| = |A'|$.

Ex.24. If the two rows of a determinant $|A|$ are equal, then $|A| = 0$.

Solution:-

Since the two rows of the determinant $|A|$ are identical, if we interchange the identical rows, then its value remains $|A|$. Further, due to the interchange of two rows, the sign of the determinant gets changed. Thus, we get,

$$|A| = -|A|$$
$$\therefore |A| = 0.$$

Ex. 25. $A\underset{\sim}{x} = \underset{\sim}{0}$ $\forall\ \underset{\sim}{x} \neq \underset{\sim}{0}$, What can you say about A?

Sol. $A\underset{\sim}{x} = \underset{\sim}{0}$ $\forall\ \underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow$ The rows of A are linearly dependent.

$\therefore$ A is singular.

Ex. 26. Suppose $AA' = pI_p$, A $p\times p$ matrix, then What can you say about $A'A$?

Sol. As, $AA' = pI_p$

$$\Rightarrow \frac{A}{\sqrt{p}} \cdot \frac{A'}{\sqrt{p}} = I_p$$

$$\Rightarrow BB' = I_p \quad \text{, where } B = \frac{A}{\sqrt{p}},\ B' = \frac{A'}{\sqrt{p}}$$

$\therefore$ B is an orthogonal matrix.

$$\Rightarrow B'B = I_p$$

$$\Rightarrow \frac{A'}{\sqrt{p}} \cdot \frac{A}{\sqrt{p}} = I_p$$

$$\Rightarrow A'A = pI_p$$

And $\det(A'A) = \det(pI_p) = p^p \neq 0$ for $p \neq 0$.

$\therefore (A'A)^{-1}$ exists and $A'A$ is non-singular.

Ex. 27. Give an example of an idempotent matrix whose rank is 1.

Sol.

$$A = \begin{pmatrix} 1/n & 1/n & \cdots & 1/n \\ 1/n & 1/n & \cdots & 1/n \\ \vdots & \vdots & & \vdots \\ 1/n & 1/n & \cdots & 1/n \end{pmatrix} \qquad \underline{\text{e.g.}}\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$A^2 = A$.. Rank of A mtx = trace of A mtx.

Rank$(A) = 1$.

Note:- If

$$A = \begin{pmatrix} K/n & 1/n & \cdots & 1/n \\ 1/n & K/n & \cdots & 1/n \\ \vdots & \vdots & & \vdots \\ 1/n & 1/n & \cdots & K/n \end{pmatrix}$$

Here, Rank $(A) = K$.

Ex. 28. Consider two subspaces S_n' and S_n'' of V_n. Define

$$S_n' + S_n'' = \{ \underset{\sim}{a} + \underset{\sim}{b} : \underset{\sim}{a} \in S_n', \underset{\sim}{b} \in S_n'' \}.$$

Prove that $S_n' + S_n''$ is a vector subspace of V_n. If $S_2' = \{ \lambda(2,1) : \lambda \in R \}$ and $S_2'' = \{ \alpha(1,3) : \alpha \in R \}$, show that $S_2' + S_2'' = V_2$. Illustrate geometrically.

Solution:- Let $\underset{\sim}{x}, \underset{\sim}{y} \in S_n' + S_n''$

Then $\underset{\sim}{x} = \underset{\sim}{a}_1 + \underset{\sim}{b}_1$
$\underset{\sim}{y} = \underset{\sim}{a}_2 + \underset{\sim}{b}_2$ where $\underset{\sim}{a}_1, \underset{\sim}{a}_2 \in S_n'$
$\underset{\sim}{b}_1, \underset{\sim}{b}_2 \in S_n''$

Now $\underset{\sim}{x} + \underset{\sim}{y} = (\underset{\sim}{a}_1 + \underset{\sim}{a}_2) + (\underset{\sim}{b}_1 + \underset{\sim}{b}_2)$
$= \underset{\sim}{a}^* + \underset{\sim}{b}^*$; $\underset{\sim}{a}^*, \underset{\sim}{b}^* \in S_n', S_n''$.

As S_n' and S_n'' are closed under vector addition,

$\underset{\sim}{x} + \underset{\sim}{y} \in S_n' + S_n''$.

Again, for any $\alpha \in R$, $\alpha \underset{\sim}{x} = \alpha \underset{\sim}{a}_1 + \alpha \underset{\sim}{b}_1$
$= \underset{\sim}{a}' + \underset{\sim}{b}'$

$\therefore \alpha \underset{\sim}{x} \in S_n' + S_n''$ for any $\alpha \in R$.

Hence, $S_n' + S_n''$ is a subspace of V_n.

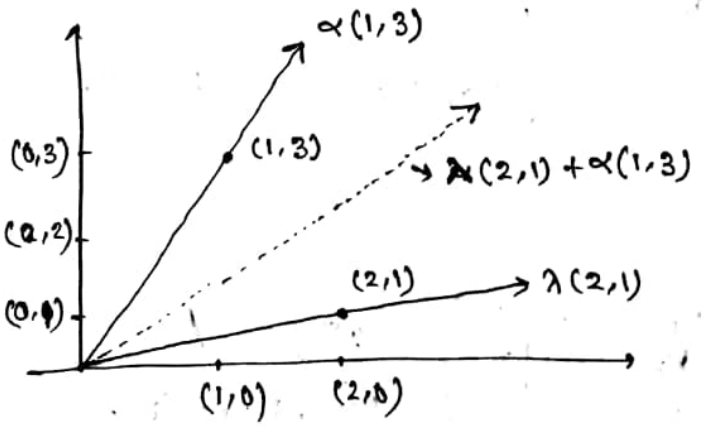
Here, $S_2' + S_2'' = \{ \lambda(2,1) + \alpha(1,3) : \lambda, \alpha \in R \}$

Note that $(2,1)$ and $(1,3)$ are LIN from V_2, because $(2,1)$ can't be written as a linear combination of $(1,3)$.

Hence every vector in V_2 can be written as a linear combination of $(2,1)$ and $(1,3)$.

i.e. $V_2 = \{ \lambda(2,1) + \alpha(1,3) : \lambda, \alpha \in R \}$

$\therefore S_2' + S_2'' = V_2$



For different (λ, α), we get different vector in V_2. Geometrically, every vector in V_2 can be obtained as $\lambda(2,1) + \alpha(1,3)$.

MATRIX ALGEBRA

Elementary Row Operation :—

i) Multiplication of one row of A by a non-zero scalar 'c'.
ii) A replacement of r^{th} row of A by row 'r' plus c times row 's', c be any scalar and $r \neq s$.
iii) Interchange of two rows of A.

Row-equivalent Matrix :—

Definition:— For two matrix A and B of order $m \times n$ over the field F, we say that B is row-equivalent to A if B can be obtained from A by a finite sequence of elementary row operations.

Result :— If A and B are row-equivalent, then the homogeneous system of equations $A\underset{\sim}{x} = 0$ & $B\underset{\sim}{x} = 0$ have exactly the same solution.

Row-reduced Matrix :—

Definition :— An $m \times n$ matrix R is called Row-reduced matrix if
i) the first non-zero entry in each non-zero row of R is 1, &
ii) each column that contains the leading non-zero entry of some non-zero row has all its other entries 0.

e.g. A row reduced matrix is : $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$

Result :— Every $m \times n$ matrix is row-equivalent to a row-reduced matrix.

Echelon Matrix :—

Definition :— An $m \times n$ matrix A is called an echelon matrix if
i) the 1st k (≥ 0) rows are non-zero, and the remaining $(m-k)$ rows are zero.
ii) in the i^{th} row $i = 1, 2, \dots, k$ (if $k \geq 1$), the first non-zero element is 1, and
iii) the arrangement of the rows is such that $c_1 < c_2 < \dots < c_k$, where c_i is the column in which the leading non-zero element 1 of row i occurs.

Ex :— $H = \begin{pmatrix} 0 & 1 & h_{13} & h_{14} & h_{15} & h_{16} \\ 0 & 0 & 0 & 1 & h_{25} & h_{26} \\ 0 & 0 & 0 & 0 & 1 & h_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is an echelon matrix.

e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$

Result:- i) If $(A+B)$ is defined then $(A+B)' = A'+B'$.
ii) If AB is defined then $(AB)' = B'A'$.

Proof:- i) (i,j) th element of $A'+B'$
$= (i,j)$ th element of $A' + (i,j)$ th element of B'
$= (j,i)$ th element of $A + (j,i)$ th element of B.
$= (j,i)$ th element of $(A+B)$
$= (i,j)$ th element of $(A+B)'$ $\forall\ i,j$.

ii) (i,j) th element of $B'A'$
= scalar product of i th row of B' and j th column of A'
= scalar product of i th column of B and j th ~~column~~ row of A
$= (j,i)$ th element of AB
$= (i,j)$ th element of $(AB)'$.

Elementary Matrices:- A square matrix $E_{m\times m}$ is said to be an elementary matrix if it can be obtained from the identity matrix I_m by means of a single elementary row operations. So, elementary matrices are of three distinct type corresponding to the three types of elementary row operation.

(i) $(E_{ij})_{m\times m}$ is the elementary matrix obtained by interchanging the i th and j th rows of the identity matrix I_m.

Example: $E_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ obtained from $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(ii) $E_i(\lambda)$ is the matrix obtained by multiplying the i th row of I_m, by $\lambda (\neq 0)$.

$$E_2(3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(iii) $E_i(\lambda|j)$ is the matrix obtained from I_m by adding the i th row $\lambda(\neq 0)$ times the j th row.

$$E_2(5|3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}.$$

Result:- Let A and B are $m\times n$ matrices over the field F. then B is row equivalent to A if and only if $B = PA$, where P is a product of some $m\times m$ elementary matrices.

Rank of a Matrix:— Let A be a matrix of order $m\times n$. We may look upon A as an ordered set of row vectors as

$$A = \begin{pmatrix} a_1' \\ a_2' \\ \vdots \\ a_m' \end{pmatrix}, \text{ where } a_i'\text{s being the } i^{th} \text{ row of } A.$$

The set spanned by these m rows, a_i' is a sub-space of $V_n(F)$ and is called the row-space of the matrix A. Again, each of the n columns of $A = (b_1, \dots, b_n)$ consist of m elements and is an m-vector belonging to $V_m(F)$. The set spanned by the n-columns $b_1, \dots, b_n$ is a subspace of $V_m(F)$ and is called the column space of A.

The dimension of the row-space (column space) of A is is called row rank (column rank) of the mtx A. Since row rank and column rank are the same, so it simply called rank of A.

Result:— For two matrices A and B of the same order, show that

$$\text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B)$$

Proof:— Let A and B be of the same order $m\times n$ and $C = A+B$, Considering these matrices as sequences of column vectors as $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n)$ and $C = (c_1, \dots, c_n)$, where $c_j = a_j + b_j \ \forall\ j = 1(1)n$.

Denoting by $S(A), S(B)$ and $S(C)$ as the column spaces of A, B, C respectively, we get —

$$S(C) \subseteq S(A) + S(B) \quad \text{——— (1)}$$

Further, we know that for two subspaces W_1 and W_2,

$$\dim(W_1 + W_2) \le \dim(W_1) + \dim(W_2) \quad \text{——— (2)}$$

$$\dim(S(C)) \le \dim(S(A) + S(B)), \text{ due to (1)}$$

$$\le \dim(S(A)) + \dim(S(B)), \text{ due to (2)}.$$

$$\Rightarrow \text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B).$$

Result:- $\text{rank}(AB) \le \min[\text{rank}(A), \text{rank}(B)]$

Proof:- $A_{m\times p}$, $B_{p\times n}$ mtxs and $C = AB_{m\times n}$.

a_j, b_j, c_j be the columns of A, B and C respectively

Then $c_{j\,m\times 1} = j^{th}$ column of $C = A_{m\times p}\, b_{j\,p\times 1}$, where $b_j = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{pmatrix}$

$= b_{1j}a_1 + b_{2j}a_2 + \dots + b_{pj}a_j$, where →

$\Rightarrow$ Each of the m columns of C is a linear combination of the p columns of A. Consequently, Column space $(AB) \subseteq$ Column space (A),

$$\Rightarrow \text{rank}(AB) \le \text{rank}(A)$$

Now, $\text{rank}(AB) = \text{rank}[(AB)'] = \text{rank}(B'A') \le \text{rank}(B') = \text{rank}(B)$

$\therefore$ We have $\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$

Theorem:- For any matrix $A_{m\times n} = (a_{ij})$, show that
Row rank (A) = Column rank (A).

Proof:- Let the row space of A has dimension k and the row-vectors $v_1, v_2, \ldots, v_k$ form a basis of the row-space of A. Then each of the rows of A, denoted by $R_1, R_2, \ldots$, etc. can be expressed as a linear combination of $v_1, v_2, \ldots, v_k$, implying that $\exists$ scalars $c_{ij} \ni$

$$\left.\begin{aligned} R_1 &= c_{11}v_1 + c_{12}v_2 + \cdots + c_{1k}v_k \\ R_2 &= c_{21}v_1 + c_{22}v_2 + \cdots + c_{2k}v_k \\ &\vdots \\ R_m &= c_{m1}v_1 + c_{m2}v_2 + \cdots + c_{mk}v_k \end{aligned}\right\} \text{——— (1)}$$

By collecting the j^{th} component from both sides of each equation in (1), with the notation $v_i = (v_{i1}, v_{i2}, \ldots, v_{in})$, we get $\longrightarrow$

$$\left.\begin{aligned} a_{1j} &= c_{11}v_{1j} + c_{12}v_{2j} + \cdots + c_{1k}v_{kj} \\ a_{2j} &= c_{21}v_{1j} + c_{22}v_{2j} + \cdots + c_{2k}v_{kj} \\ &\vdots \\ a_{mj} &= c_{m1}v_{1j} + c_{m2}v_{2j} + \cdots + c_{mk}v_{kj} \end{aligned}\right\}$$

or,
$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} = v_{1j}\begin{pmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{pmatrix} + v_{2j}\begin{pmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{m2} \end{pmatrix} + \cdots + v_{kj}\begin{pmatrix} c_{1k} \\ c_{2k} \\ \vdots \\ c_{mk} \end{pmatrix} \text{——— (2)}$$

The L.H.S. of (2) is the j^{th} column of A, and thus each column of A lies in the subspace spanned by the k vectors on the RHS of (2). So, the dimension of the column space of $A \leq k$.

As, k = dimension of the row space of A,
dim (column space of A) $\leq$ dim (row space of A) ——— (3)

Further, as the matrix A is arbitrary, inequality (3) is true for A', i.e.,
dim(column space of A') $\leq$ dim (row space of A')
or, dim(row space of A) $\leq$ dim (column space of A) ——— (4)

Inequalities (3) and (4) give —
dim(columnspace of A) = dim(row space of A).

i.e. $\boxed{\text{Column rank}(A) = \text{Row rank}(A)}$

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Theorem:- Let $H_{m\times n}$ be a non-zero echelon matrix having k non-zero rows. Then the non-zero rows form a basis of the row-space of H, giving rank$(H) = k$.

Proof:- Let $h_1', h_2', h_3', \ldots, h_k'$ be k non-zero rows of H with $h_i' = (h_{i1}, h_{i2}, \ldots, h_{in})\ \forall\ i = 1, 2, 3, \ldots, k$.

Certainly, these k non-zero rows span the row-space of H. So to show that h_i', $i = 1, 2, \ldots, k$, form a basis of the row space of H, we need to show that these k-vectors are LIN. For this, we consider the following equations with unknown scalars λ_i $(i = 1, 2, \ldots, k)$.

$$\lambda_1 h_1' + \lambda_2 h_2' + \cdots + \lambda_k h_k' = \underset{\sim}{0}' \text{ ——— } (1)$$

As, H is an echelon matrix, row h_i' has the element leading 1 in column C_i (say), $i = 1, 2, \ldots, k$. The elements of column C_i below row i, all are zero. Hence, equation (1) implies —

$$\lambda_1 = 0$$
$$\lambda_1 h_{1C_2} + \lambda_2 = 0$$
$$\lambda_1 h_{1C_3} + \lambda_2 h_{2C_3} + \lambda_3 = 0$$
etc.....

Solving these equations successively, we have then
$$\lambda_1 = \lambda_2 = \cdots\cdots = \lambda_k = 0.$$

Thus, the non-zero rows of H spans the row-space and are linearly independent implying the rows $h_1', h_2', \ldots, h_k'$ form a basis of the row-space of H.

Cor:- Let $H_{m\times n}$ be an echelon mtx row-equivalent to $A_{m\times n}$, then the non-zero rows of H form a basis of the row-space of A.

Proof:- Let $a_1, a_2, \ldots, a_m$ be the m-rows of the matrix A and $h_1, h_2, \ldots, h_k$ $(k \le m)$ be the non-zero rows of H.

We can write the row space of A as —

$$\text{Row space}(A) = \{\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m \mid \lambda_1, \ldots, \lambda_m \in F\}$$

As H is row-equivalent to A, then A is also row-equivalent to H. Then a_i is a linear combination of $h_1, h_2, \ldots, h_k$ $\forall\ i = 1(1)m$.

Therefore, $\text{Row space of } A = \{\gamma_1 h_1 + \gamma_2 h_2 + \cdots + \gamma_k h_k \mid \gamma_1, \ldots, \gamma_k \in F\}$

$\Rightarrow h_1, \ldots, h_k$ generate the row-space of A.

Further, we know that the non-zero rows $h_1, \ldots, h_k$ of H are linearly independent. Hence the result.

Result:- If H is an echelon matrix row-equivalent to A, the rank(A) = rank(H).

Rank-Factorization theorem:—

Statement:— Let A be an $m\times n$ matrix of rank r. Then $\exists$ two matrices $B_{m\times r}$ and $C_{r\times n}$ such that $A = BC$ and $\text{rank}(B)=r$ and $\text{rank}(C)=r$.

Proof:- $\text{Rank}(A)=r$

Let us suppose that $\{\underset{\sim}{b_1}, \underset{\sim}{b_2}, \dots, \underset{\sim}{b_r}\}$ forms a basis of the column space of A.

Let $\underset{\sim}{\beta_j}$ denote the j^{th} column vector of A. Then $\exists$ real scalars $\ni$ $\underset{\sim}{\beta_j}$ can be expressed in terms of $\underset{\sim}{b_1}, \underset{\sim}{b_2}, \dots, \underset{\sim}{b_r}$ as follows

$$\underset{\sim}{\beta_j} = C_{1j}\underset{\sim}{b_1} + C_{2j}\underset{\sim}{b_2} + \dots + C_{rj}\underset{\sim}{b_r}$$

$$\text{or, } \underset{\sim}{\beta_j} = \left(\underset{\sim}{b_1}\ \underset{\sim}{b_2}\ \cdots\ \underset{\sim}{b_r}\right)\begin{pmatrix} C_{1j} \\ C_{2j} \\ \vdots \\ C_{rj} \end{pmatrix} = B_{m\times r}\, C_{j\, r\times 1} \quad \forall\, j = 1(1)n.$$

Now, $A = (\underset{\sim}{\beta_1} \cdots\cdots \underset{\sim}{\beta_n})$

$= (B\underset{\sim}{C_1}\ BC_2 \cdots\cdots BC_n)$

$= B_{m\times r}C_{r\times n}$ [Proved]

Now, we also have

$\text{rank}(BC) \le \min\{r(B), r(C)\}$

i.e. $\text{rank}(BC) \le \text{rank}(B)$

$\le \text{rank}(C)$

i.e. $\text{rank}(A) \le \text{rank}(B)$ & $\text{rank}(A) \le \text{rank}(C)$

$\therefore r \le \text{rank}(B)$ & $r \le \text{rank}(C)$

But B has r column vectors and C has r row vectors.

& $\text{rank}(B) \le r$ and $\text{rank}(C) \le r$

$\therefore$ Finally we get. $\text{rank}(B) = \text{rank}(C) = r$.

RANK RELATED PROBLEMS:—

1. Let A be an $m\times n$ mtx with rank m and S be a $r\times m$ mtx with rank r, then show that $\text{rank}(SA) = r$.

Solution:— $\text{rank}(A) + \text{rank}(S) - m \le \text{rank}(SA) \le \min\{\text{rank}(S), \text{rank}(A)\}$

$\Rightarrow r + m - m \le \text{rank}(SA) \le \min\{r, m\} = r$

i.e. $\text{rank}(SA) = r \le m$.

2. If $\text{rank}(A-I) = p$ and $\text{rank}(B-I) = q$, then S.T. $\text{rank}(AB-I) \le p+q$

Solution:- $AB - I = A(B-I) + (A-I)$

$\text{Rank}(AB-I) \le \text{Rank}[A(B-I)] + \text{Rank}(A-I)$

$\le \text{Rank}(B-I) + \text{Rank}(A-I)$

$\le p+q$.

3) If A be an $m \times n$ mtx and B be an $s \times n$ mtx $\ni AB' = 0$, then $\text{rank}(A'A + B'B) = r(A) + r(B)$.

Sol. Let us define, $C = \left(\underset{n\times m}{A'} \;\vdots\; \underset{n\times s}{B'} \right)$

Then $CC' = \left(A' \;\vdots\; B' \right) \begin{pmatrix} A \\ \hdashline B \end{pmatrix}$

$= A'A + B'B$.

and $C'C = \begin{pmatrix} A \\ \hdashline B \end{pmatrix} \left(A' \;\vdots\; B' \right) = \left(\begin{array}{c:c} AA' & AB' \\ \hdashline BA' & BB' \end{array} \right)$

$= \left(\begin{array}{c:c} AA' & 0 \\ \hdashline 0 & BB' \end{array} \right)$

Since $\text{rank}(CC') = \text{rank}(C'C)$

$\therefore \text{rank}(A'A + B'B) = \text{rank} \left(\begin{array}{c:c} AA' & 0 \\ \hdashline 0 & BB' \end{array} \right)$

$= \text{rank}(AA') + \text{rank}(BB')$

$= \text{rank}(A) + \text{rank}(B)$.

PROBLEM:- Determine all the idempotent diagonal matrices of order n.

Solution:- Let $A = \text{diagonal}\,[d_1, \cdots, d_n]$ be an idempotent mtx.

Then $A^2 = A$.

$A = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$, $A^2 = \begin{bmatrix} d_1^2 & 0 & \cdots & 0 \\ 0 & d_2^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^2 \end{bmatrix}$

$A^2 = A$ gives $\quad d_1^2 = d_1,\ d_2^2 = d_2, \ldots\ldots, d_n^2 = d_n$

$\therefore d_1 = 0, 1;\ d_2 = 0, 1;\ \cdots\cdots;\ d_n = 0, 1.$

Hence the required idempotent diagonal mtx is

$\text{diag}\,[d_1, \cdots, d_n]$ with $d_i = 0, 1 \;\; \forall\, i = 1(1)n$.

SOME BASIC THEOREMS :-

Theorem:-

1. If $AB = A$ and $BA = B$. S.T. A and B are idempotent.

Solution:- We have $AB = A$

$A(AB) = A$, $[\because BA = A]$

$\Rightarrow (AB)A = A$

or, $A^2 = A \Rightarrow A$ is idempotent.

Similar for the later.

2. If B is an idempotent matrix, S.T. $A = I - B$ is also idempotent and that $AB = BA = 0$.

Solution:- As B is idempotent, $\therefore B^2 = B$.

$A^2 = (I-B)^2 = (I-B)(I-B)$

$= I - IB - BI + B^2$

$= I - B - B + B$

$= I - B$

$= A$

$\therefore$ A is idempotent.

$\therefore AB = (I-B)B = B - B^2 = 0$.

$\therefore BA = B(I-B) = B - B^2 = 0$.

3. A matrix is said to be involutory if $A^2 = I$. Show that A is involutory iff $(I+A)(I-A) = 0$.

Solution:- Let A be an involutory matrix of order n.

Then $A^2 = I$.

$\therefore I - A^2 = 0$

$\therefore (I+A)(I-A) = 0$.

Conversely, if $(I+A)(I-A) = 0$

then $I^2 - IA + AI - A^2 = 0$

or, $I - A^2 + AI - AI = 0$

or, $I - A^2 = 0$

or, $A^2 = I$.

$\therefore$ A is involutory.

IDEA ABOUT INVERSE :

Definition 1:- An $m \times n$ matrix A is said to be of full row rank if its rank is m, i.e., if its rows are linearly independent. Similarly A is said to be of full column rank if its columns are linearly independent.

Definition 2:- A left inverse of a matrix A is any matrix B such that $BA = I$. A right inverse of A is any matrix C such that $AC = I$. A mtx B is said to be an inverse of A if it is both a left inverse and a right inverse of A.

PROPERTIES OF INVERSE:- A square matrix A is said to be non-singular if it has an inverse. A square matrix which does not possess an inverse is said to be singular.

Theorem 1:- If A is non-singular, then A^{-1} and A^T are also non-singular

$$(A^{-1})^{-1} = A \quad \text{and} \quad (A^T)^{-1} = (A^{-1})^T.$$

Proof:- Since $(A^{-1})A = I$, it follows that A is a right inverse and so the inverse of A^{-1} $\Rightarrow A = (A^{-1})^{-1}$

Similarly, $A^T (A^{-1})^T = (A^{-1}A)^T = I$

$$\Rightarrow (A^T)^{-1} = (A^{-1})^T.$$

Theorem 2:- Let A and B be square matrices of the same order. Then AB is non-singular iff both A and B are non-singular. Also then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof:- If AB is non-singular and $C = (AB)^{-1}$, then $ABC = I$, so BC is an inverse of A and A is non-singular. Also, $CAB = I$, so, B is non-singular.

Conversely, Let A and B be both non-singular. Then

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I.$$

So, $B^{-1}A^{-1}$ is the inverse of AB and AB is non-singular.

By repeated application of the preceeding theorem, it can be shown that if $A_1, A_2, \dots, A_K$ are non-singular matrices of the same order, then $(A_1 A_2 \cdots A_K)^{-1} = A_K^{-1} A_{K-1}^{-1} \cdots A_1^{-1}$.

Note:- ①. If $A = I$, the identity matrix, then $A^{-1} = I$.

②. If $A = \operatorname{diag}(d_1, \dots, d_k)$, a diagonal mtx, then

$$A^{-1} = \operatorname{diag}\left(\frac{1}{d_1}, \dots, \frac{1}{d_k}\right).$$

Theorems on the inverse of Elementary Matrices :-

Theorem:- We have the followings :

(a) $E_{ij}^{-1} = E_{ij}$ (b) $E_i^{-1}(\lambda|j) = E_i(-\lambda|j)$ (c) $E_i^{-1}(\lambda) = E_i(1/\lambda)$

Proof:-

(a) E_{ij} is obtained from the identity matrix I of the same order by interchanging the i^{th} & the j^{th} rows (or, columns) of I. As such, if the i^{th} and j^{th} rows (or, columns) of E_{ij} itself are interchanged, which will mean post - multiplication or pre-multiplication of E_{ij} by E_{ij} then one gets back the identity matrix.

Hence , $E_{ij}E_{ij} = I$ implying that $E_{ij}^{-1} = E_{ij}$.

(b) $E_i(\lambda|j)$ is obtained from the identity matrix of the same order by adding to the i^{th} row (j^{th} column) of the latter λ times the j^{th} row (i^{th} column).

Hence, if we subtract from the i^{th} row (j^{th} column) of $E_i(\lambda|j)$ λ times the j^{th} row (i^{th} column), we get back the identity matrix. Since this means premultiplication (post multiplication) of $E_i(\lambda|j)$ by $E_i(-\lambda|j)$, we ~~have~~ get the identity matrix back .

$$E_i(-\lambda|j)\,E_i(\lambda|j) = I \Rightarrow E_i^{-1}(\lambda|j) = E_i(-\lambda|j).$$

(c) $E_i(\lambda)$ is obtained from the identity matrix I of the same order by multiplying the i^{th} row or column by the non-zero scalar λ. Hence, if we post multiply $E_i(\lambda)$ by $E_i(1/\lambda)$ or premultiply $E_i(\lambda)$ by $E_i(1/\lambda)$, this will have the effect of multiplying of the i^{th} row or column of $E_i(\lambda)$ by $1/\lambda$. Thus,

$$E_i(1/\lambda)\,E_i(\lambda) = E_i(\lambda)\,E_i(1/\lambda) = I,$$

implying that $E_i^{-1}(\lambda) = E_i(1/\lambda)$

NORMAL FORM OF A MATRIX :-

<u>Definition:-</u> A matrix is said to be in normal form if it is $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ for some r. We use the conversion that it denotes a null mtx if $r=0$. Clearly the rank of this mtx is r.

<u>Theorem:-</u> Every matrix can be reduced to a matrix in normal form by elementary (row and column) operation.

<u>Proof:-</u> If $A=0$, then A itself is in normal form. So let $A \neq 0$. Then by interchanging the first row with another row and first column with another column, if necessary, we make $a_{11} \neq 0$. Then we sweep out the first column and the first row using a_{11} as the pivot. At this stage the matrix is of the form $A = \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}$. If $B=0$, the current matrix is in normal form. If $B \neq 0$, by interchanging the second row with some later row and the second column with some later column, if necessary, we make $a_{22} \neq 0$ & sweep out the 2nd column & the 2nd row with a_{22} as the pivot. Note that these operations do not disturb the first row and the first column. Now the matrix is of the form $A = \begin{bmatrix} I_2 & 0 \\ 0 & C \end{bmatrix}$

If $C=0$, then the work is done; otherwise we proceed as before until we arrive at a matrix in normal form.

<u>NOTE:-</u> Since row operations amount to premultiplication and column operations amount to post multiplication by non-singular matrices, we have the following theorem:-

<u>Theorem:-</u> Let A be an $m \times n$ matrix of rank r. Then $\exists$ n.s. matrices P of order m and Q of order n $\ni$

$$\longrightarrow \qquad PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

By taking $R = P^{-1}$ and $S = Q^{-1}$, we can rewrite the above as $A = R \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} S$

This representation is often more useful than earlier. For instance, we can readily get a rank-factorization of A from the later. Partition

$$R = [R_1 \,\vdots\, R_2]$$

$$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

where, R_1 is of order $m \times r$ and S_1 is of order $r \times n$. Then $A = R_1 S_1$, So, (R_1, S_1) is a rank factorization of $\mathbf{A}$.

Problem:- Let A be a skew-symmetric matrix and $(I+A)$ is non-singular matrix. Show that $B = (I-A)(I+A)^{-1}$ is orthogonal.

Solution:- Since A is skew-symmetric, $A' = -A$ and $(I+A)$ is a non-singular matrix, i.e., $|I+A| \neq 0$.

$$B = (I-A)(I+A)^{-1}$$

$$B^T = \{(I-A)(I+A)^{-1}\}^T$$

$$= \{(I+A)^{-1}\}^T (I-A)^T$$

$$= (I+A^T)^{-1}(I+A)$$

$$= (I-A)^{-1}(I+A).$$

Thus, $B^TB = (I-A)^{-1}(I+A)(I-A)(I+A)^{-1}$

$$= I.$$

$$BB^T = I$$

So, it follows that B is orthogonal.

* ———————————— *

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DETERMINANTS

DEFINITION:- Consider any permutation, say $p = (k_1, k_2, \ldots, k_n)$; of the first n natural numbers. Such a permutation has some or no inversions, an inversion being a derivation from the natural order of two positive integers, the bigger integer preceeding the smaller. A permutation will be said to be an even or odd permutation according as the number of inversions in it is even or odd.

Consider now a square matrix of order n, say the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \quad \text{------ (i)}$$

By definition, the determinant of A, denoted by $|A|$ is given by,

$$|A| = \sum \mp a_{1k_1} a_{2k_2} \ldots a_{nk_n} \quad \text{------ (ii)}$$

where, the sum is taken over all the $n!$ permutations of the first n natural numbers and the sign attached to any given term is a plus $(+)$ or a minus $(-)$ sign according as the permutation is even or odd. This rule is to be followed irrespective of whether the value of the product $a_{1k_1} a_{2k_2} \ldots a_{nk_n}$ itself is +ve or −ve.

Example:- Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Here $n = 3$ & we are to consider all the $3! = 6$ permutations of the first 3 natural numbers 1, 2 & 3. These are listed below:-

$(1, 2, 3)$:- Even permutation, the # of inversions being 0
$(1, 3, 2)$:- Odd " , " " " " 1
$(2, 1, 3)$:- Odd " , " " " " 1
$(2, 3, 1)$:- Even " , " " " " 2
$(3, 1, 2)$:- " " , " " " " 2
$(3, 2, 1)$:- Odd " , " " " " 3

As such, we have $|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$

Thus, $|A| = |a_{ij}| = ((a_{ij}))$

Adjoint of a matrix:- Given any square matrix $A = (a_{ij})$, not necessary non-singular, the adjoint of A, denoted by Adj A, is the matrix of the same order where $(i,j)^{th}$ element, is A_{ij}, the cofactor of a_{ji} in $|A|$.

Thus, $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ while $\text{Adj}A = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & & \vdots \\ A_{1n} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$

Theorem:- If A be any non-singular matrix then $A^{-1} = \dfrac{\text{Adj}A}{|A|}$.

Proof:- We have, [redacted]

$$\sum_{j=1}^{n} a_{ij} a'_{ij} = |A| \quad \text{if } i = i'$$
$$= 0 \quad \text{if } i \neq i$$

Again, considering the columns of A, we have,

$$\sum_{i=1}^{n} a_{ij} A_{ij'} = |A| \quad \text{if } j = j'$$
$$= 0 \quad \text{if } j \neq j'.$$

As such,

$$(\text{adj}A)\, A = \begin{pmatrix} \sum_{i=1}^{n} a_{i1}A_{i1} & \sum_{i=1}^{n} a_{i2}A_{i1} & \cdots & \sum_{i=1}^{n} a_{i1}A_{i1} \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^{n} a_{i1}A_{in} & \sum_{i=1}^{n} a_{i2}A_{in} & \cdots & \sum_{i=1}^{n} a_{in}A_{in} \end{pmatrix}$$

$$= \begin{pmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & |A| \end{pmatrix} = |A| \cdot I \quad \text{-----} \ (1)$$

$= |A|$, being a scalar.

Again,

$$A(\text{Adj}A) = \begin{pmatrix} \sum_{j=1}^{n} a_{ij}A_{ij} & \sum_{j=1}^{n} a_{ij}A_{2j} & \cdots & \sum_{j=1}^{n} a_{ij}A_{nj} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{n} a_{nj}A_{ij} & \sum_{j=1}^{n} a_{nj}A_{2j} & \cdots & \sum_{j=1}^{n} a_{nj}A_{nj} \end{pmatrix}$$

$$= \begin{pmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & |A| \end{pmatrix}$$

$$= |A| \quad \text{-----} \ (2)$$

From (1) & (2),

$$\left(\frac{\text{Adj}A}{|A|}\right) A = A \left(\frac{\text{Adj}A}{|A|}\right) = I.$$

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consequently, the reciprocal matrix of A is obtained as

$$A^{-1} = \frac{1}{|A|}(\mathrm{Adj}A)$$

$$= \begin{pmatrix} \frac{A_{11}}{|A|} & \frac{A_{21}}{|A|} & \cdots\cdots & \frac{A_{n1}}{|A|} \\ \frac{A_{12}}{|A|} & \frac{A_{22}}{|A|} & \cdots\cdots & \frac{A_{n2}}{|A|} \\ \vdots & \vdots & & \vdots \\ \frac{A_{1n}}{|A|} & \frac{A_{2n}}{|A|} & \cdots\cdots & \frac{A_{nn}}{|A|} \end{pmatrix}$$

In other words, if we write $A^{-1} = (a^{ij})$, then $a^{ij} = \frac{A_{ji}}{|A|} \forall j, i$.

Theorem:- Prove that any square matrix $A_{n\times n}$ possesses an inverse iff $|A_{n\times n}| \neq 0$.

Proof:- Only if Part:- Let the matrix $A_{n\times n}$ be invertible and B be the inverse of A then $AB = I_n$

$\therefore |AB| = 1$

or, $|A||B| = 1$

since the product of two determinants is one, $|A_{n\times n}|$ must be non-zero, so, $|A_{n\times n}| \neq 0$ is the necessary condition.

If Part:- Suppose that the condition $|A_{n\times n}| \neq 0$ holds. Then let us define a matrix B by the relation $B = \frac{\mathrm{Adj}A}{|A|}$

Then $AB = \frac{1}{|A|} A\,\mathrm{Adj}A$

$BA = \frac{1}{|A|} \mathrm{Adj}A.A$

But we proved that $A(\mathrm{Adj}A) = (\mathrm{Adj}A)A = |A|\cdot I_n$

Here $AB = BA = I_n$.

so, A is an invertible matrix and B is the inverse of A.

Problem:- If A be an $n\times n$ matrix, prove that $|\mathrm{Adj}A| = |A|^{n-1}$.

Solution:- We have $A.(\mathrm{Adj}A) = |A|.I_n$

$\therefore |A(\mathrm{Adj}A)| = |A|^n |I_n|$ $[\because |AB| = |A||B|$ & $|kA| = k^n|A|]$

$\Rightarrow |A||\mathrm{Adj}A| = |A|^n$

$\Rightarrow |\mathrm{Adj}A| = |A|^{n-1}$ [If $|A| \neq 0$]

If $|A| = 0$, then $|\mathrm{Adj}A| = 0$.

Hence the result.

Problem:- If A be an $n\times n$ matrix, show that the rank of $\operatorname{Adj}A$ is n, 1 or 0 according as the rank of A is $n, n-1$ or less than $n-1$.

Solutions:- (i) Let A be an $n\times n$ matrix. Then

$$A(\operatorname{Adj}A) = |A|\cdot I_n$$

$$\therefore |A|\,|\operatorname{Adj}A| = |A|^n\,|I_n|$$

$$\therefore |A|\,|\operatorname{Adj}A| = |A|^n$$

Now, since there is $\operatorname{rank}(A)=n$

therefore $|A|\neq 0$

$\therefore |A|\,|\operatorname{Adj}A| = |A|$ gives $|\operatorname{Adj}A| = |A|^{n-1}\neq 0$.

$\therefore$ The matrix $|\operatorname{Adj}A|\neq 0$ hence it is of full rank, i.e. rank n.

(ii) If the rank of A is $n-1$, then at least one minor of order $(n-1)$ of the matrix is not equal to zero, therefore the matrix $\operatorname{Adj}A$ will be a non-zero matrix and the rank of the matrix $\operatorname{Adj}A$ will be greater than zero.

Again, the rank of A is $n-1$, therefore, $|A|=0$.

$\therefore A(\operatorname{Adj}A)$ is a zero matrix and hence is of rank zero.

Hence, by sylvester's inequality,

$$\operatorname{rank}(A\,\operatorname{adj}A) \geq \operatorname{rank}(A)+\operatorname{rank}(\operatorname{adj}A)-n$$

or, $\operatorname{rank}(A)+\operatorname{rank}(\operatorname{adj}A)-n\leq 0$

or, $\operatorname{rank}(\operatorname{adj}A)\leq 1$.

But we have shown that $\operatorname{rank}(\operatorname{adj}A)>0$

Hence, $\operatorname{rank}(\operatorname{adj}A)=1$.

(iii) If the rank of A is less than $n-1$, then all the minors order $n-1$ of the matrix A will be zero. Therefore, the matrix $\operatorname{adj}A$ will be a zero matrix & hence $\operatorname{rank}(\operatorname{adj}A)=0$.

INVERSE OF A PARTITIONED MATRIX :-

Let A be a square matrix of order n, written in the partitioned form

$$A_{n\times n} = \begin{pmatrix} \underset{m\times m}{A_{11}} & \underset{m\times(n-m)}{A_{12}} \\ \underset{(n-m)\times m}{A_{21}} & \underset{(n-m)\times(n-m)}{A_{22}} \end{pmatrix}$$

Theorem:- If A be of the above structure and A_{11} is non-singular then $|A| = |A_{11}|\,|A_{22} - A_{21}A_{11}^{-1}A_{12}|$

Proof:- Consider together with A, the following $(n\times n)$ matrices,

$$B_{n\times n} = \begin{pmatrix} I_m & 0_{m\times(n-m)} \\ -A_{21}A_{11}^{-1}{}_{(n-m)\times m} & I_{(n-m)} \end{pmatrix}$$

and $$C_{n\times n} = \begin{pmatrix} I_m & -A_{11}^{-1}A_{12}\,{}_{m\times(n-m)} \\ 0_{(n-m)\times m} & I_{(n-m)} \end{pmatrix}$$

Then $|B| = |I_m|\,|I_{(n-m)}| = 1$ and $|C| = |I_m|\,|I_{n-m}| = 1$.

Again, $$BAC = \begin{pmatrix} I_m & 0 \\ -A_{21}A_{11}^{-1} & I_{n-m} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} I_m & -A_{11}^{-1}A_{12} \\ 0 & I_{n-m} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$

Hence taking determinants on both sides, we get,

$$|BAC| = |A| = |A_{11}|\,|A_{22} - A_{21}A_{11}^{-1}A_{12}|.$$

Theorem:- If A has the same structure as above and is non-singular Then

$$A^{-1} = \begin{pmatrix} A_{11}^{-1}(I_m + A_{12}F^{-1}A_{21}A_{11}^{-1}) & -A_{11}^{-1}A_{12}F^{-1} \\ -F^{-1}A_{21}A_{11}^{-1} & F^{-1} \end{pmatrix}$$

where, $F = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

Proof:- Since A is non-singular, A_{11} is also non-singular so that A_{11}^{-1} exists. Now, let $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ and $AB = I$.

i.e. $$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} I_m & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & I_{(n-m)} \end{pmatrix}$$

We have the following equations $\longrightarrow$

$$A_{11}B_{11} + A_{12}B_{21} = I_m \quad \text{——— (i)}$$

$$A_{11}B_{12} + A_{12}B_{22} = 0_{m\times(n-m)} \quad \text{——— (ii)}$$

$$A_{21}B_{11} + A_{22}B_{21} = 0_{(n-m)\times m} \quad \text{——— (iii)}$$

$$A_{21}B_{12} + A_{22}B_{22} = I_{(n-m)} \quad \text{——— (iv)}$$

Equation (i) gives $B_{11} + A_{11}^{-1} A_{12} B_{21} = A_{11}^{-1}$ ——— (v)

~~Post~~ Premultiplying by A_{21} and subtracting from (iii)

$$A_{22} B_{21} - A_{21} A_{11}^{-1} A_{12} B_{21} = -A_{21} A_{11}^{-1}$$

or, $(A_{22} - A_{21} A_{11}^{-1} A_{12}) B_{21} = -A_{21} A_{11}^{-1}$

or, $F B_{21} = -A_{21} A_{11}^{-1}$

or, $B_{21} = -F^{-1} A_{21} A_{11}^{-1}$

So, from (v),

$$B_{11} - A_{11}^{-1} A_{12} F^{-1} A_{21} A_{11}^{-1} = A_{11}^{-1}$$

$$B_{11} = A_{11}^{-1} (I_m + A_{12} F^{-1} A_{21} A_{11}^{-1})$$

Premultiplying equation (ii) by ~~[illegible]~~ $A_{21} A_{11}^{-1}$ & subtracting from (iv)

$$A_{21} B_{12} + A_{22} B_{22} - A_{21} B_{12} - A_{21} A_{11}^{-1} A_{12} B_{22} = I_{(n-m)}.$$

or, $(A_{22} - A_{21} A_{11}^{-1} A_{12}) B_{22} = I_{(n-m)}$.

or, $F B_{22} = I_{(n-m)}$

or, $B_{22} = F^{-1}$.

~~Subtracting from equation~~

Substituting in equation (ii)

$$B_{12} = -A_{11}^{-1} A_{12} F^{-1}$$

Hence the result.

<u>Result:</u>- Let A be an n.s. mtx of order n and $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be two n-component column vectors. Then show that $|A + \underset{\sim}{u}\underset{\sim}{v}'| \neq 0$ iff $1 + \underset{\sim}{v}' A^{-1} \underset{\sim}{u} \neq 0$. Then also show that

$$(A + \underset{\sim}{u}\underset{\sim}{v}')^{-1} = A^{-1} - \frac{(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}' A^{-1})}{1 + \underset{\sim}{v}' A^{-1} \underset{\sim}{u}}.$$

<u>ANSWER:</u>- $1 + \underset{\sim}{v}' A^{-1} \underset{\sim}{u} \neq 0$

Let us define, $M = \begin{pmatrix} A & -\underset{\sim}{u} \\ \underset{\sim}{v}' & 1 \end{pmatrix}$

$\therefore |M| = |A + \underset{\sim}{u}\underset{\sim}{v}'|$

Since A is non-singular, A^{-1} exists.

Now, again, $|M| = |A| (1 + \underset{\sim}{v}' A^{-1} \underset{\sim}{u})$

So, $1 + \underset{\sim}{v}' A^{-1} \underset{\sim}{u} \neq 0 \Leftrightarrow |M| \neq 0$

$\Leftrightarrow |A + \underset{\sim}{u}\underset{\sim}{v}'| \neq 0$

Thus, $|A + \underset{\sim}{u}\underset{\sim}{v}'| \neq 0$ iff $1 + \underset{\sim}{v}' A^{-1} \underset{\sim}{u} \neq 0$

Now, $(A+\underset{\sim}{u}\underset{\sim}{v}')\left(A^{-1}-\dfrac{(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}'A^{-1})}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}\right)$

$$= AA^{-1}+\underset{\sim}{u}\underset{\sim}{v}'A^{-1}-\frac{A(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}'A^{-1})}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}-\frac{\underset{\sim}{u}\underset{\sim}{v}'(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}'A^{-1})}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}$$

$$= I+\underset{\sim}{u}\underset{\sim}{v}'A^{-1}-\frac{(AA^{-1})(\underset{\sim}{u}\underset{\sim}{v}'A^{-1}}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}-\frac{\underset{\sim}{u}(\underset{\sim}{v}'A^{-1}\underset{\sim}{u})\underset{\sim}{v}'A^{-1}}{1+\underset{\sim}{v}A^{-1}\underset{\sim}{u}}$$

$$= I+\underset{\sim}{u}\underset{\sim}{v}'A^{-1}-\frac{\underset{\sim}{u}\underset{\sim}{v}'A^{-1}}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}-\frac{(\underset{\sim}{v}'A^{-1}\underset{\sim}{u})\underset{\sim}{u}\underset{\sim}{v}'A^{-1}}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}$$

$$= I+u\underset{\sim}{v}'A^{-1}-\frac{(1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u})\underset{\sim}{u}\underset{\sim}{v}'A^{-1}}{(1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u})}$$

$$= I.$$

Further, $\left(A^{-1}-\dfrac{(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}'A^{-1})}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}\right)(A+\underset{\sim}{u}\underset{\sim}{v}')$

$$= A^{-1}A-\frac{(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}'A^{-1})A}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}+A^{-1}\underset{\sim}{u}\underset{\sim}{v}'-\frac{(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}'A^{-1})\underset{\sim}{u}\underset{\sim}{v}'}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}$$

$$= I-\frac{(A^{-1}\underset{\sim}{u}\underset{\sim}{v}')(A^{-1}A)}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}+A^{-1}\underset{\sim}{u}\underset{\sim}{v}'-\frac{(A^{-1}\underset{\sim}{u}\underset{\sim}{v}')(\underset{\sim}{v}'A^{-1}\underset{\sim}{u})}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}$$

$$= I-\frac{A^{-1}\underset{\sim}{u}\underset{\sim}{v}'}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}+A^{-1}\underset{\sim}{u}\underset{\sim}{v}'-\frac{(\underset{\sim}{v}A^{-1}\underset{\sim}{u})(A^{-1}\underset{\sim}{u}\underset{\sim}{v}')}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}$$

$$= I+A^{-1}\underset{\sim}{u}\underset{\sim}{v}'-\frac{(A^{-1}\underset{\sim}{u}\underset{\sim}{v}')(1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u})}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}$$

$$= I.$$

Hence, $(A+\underset{\sim}{u}\underset{\sim}{v}')^{-1}=A^{-1}-\dfrac{(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}'A^{-1})}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}.$

Problem:- Find the inverse of the matrix $A = \begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{pmatrix}_{n\times n}$.

ANSWER:-

The given matrix $A = \begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ \vdots & \vdots & \vdots & & \\ b & b & b & \cdots & a \end{pmatrix}$

$$= \begin{pmatrix} a-b & 0 & 0 & \cdots & 0 \\ 0 & a-b & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & a-b \end{pmatrix} + \begin{pmatrix} b & b & \cdots & b \\ b & b & \cdots & b \\ \vdots & \vdots & & \vdots \\ b & b & \cdots & b \end{pmatrix}$$

$= (a-b)I_n + bJ_n$ $\quad$ I_n : Identity mtx of order n. J_n : Sum mtx of order n.

$= (a-b)I_n + b\underset{\sim}{1}\underset{\sim}{1}'$

$= (a-b)\left[I_n + \frac{b}{a-b}\underset{\sim}{1}\underset{\sim}{1}'\right]$; $\underset{\sim}{1}_{n\times 1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$= (a-b)(C + \underset{\sim}{u}\underset{\sim}{v}')$, $C = I_n$, $\underset{\sim}{u} = \frac{b}{a-b}\underset{\sim}{1}$

$v' = \underset{\sim}{1}'$

Here $|C + \underset{\sim}{u}\underset{\sim}{v}'| \neq 0$.

$$\therefore A^{-1} = \frac{1}{(a-b)}\left[C^{-1} - \frac{(C^{-1}\underset{\sim}{u})(\underset{\sim}{v}'C^{-1})}{1 + \underset{\sim}{v}'C^{-1}\underset{\sim}{u}}\right]$$

$$= \frac{1}{(a-b)}\left[I_n - \frac{\left\{\left(\frac{b}{a-b}\right)\underset{\sim}{1}\right\}(\underset{\sim}{1}')}{1 + \underset{\sim}{1}'\underset{\sim}{1}}\right]$$

$$= \frac{1}{(a-b)}\left[I_n - \frac{\frac{b}{a-b}J_n}{n+1}\right] \quad (\text{Ans})$$

Problem:- Find the inverse of the following matrix

$$A = \begin{pmatrix} np_1(1-p_1) & -np_1p_2 & \cdots & -np_1p_{k-1} \\ -np_1p_2 & np_2(1-p_2) & \cdots & -np_2p_{k-1} \\ \vdots & \vdots & & \vdots \\ -np_1p_{k-1} & -np_2p_{k-1} & \cdots & np_{k-1}(1-p_{k-1}) \end{pmatrix}_{(k-1)\times(k-1)}$$

where $\sum_{i=1}^{k-1} p_i < 1$.

ANSWER:-

$$A = n\begin{pmatrix} p_1(1-p_1) & -p_1p_2 & \cdots & -p_1p_{k-1} \\ -p_1p_2 & p_2(1-p_2) & \cdots & -p_2p_{k-1} \\ \vdots & \vdots & & \vdots \\ -p_1p_{k-1} & -p_2p_{k-1} & \cdots & p_{k-1}(1-p_{k-1}) \end{pmatrix}.$$

$$= n\left[\begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & p_{k-1} \end{pmatrix} - \begin{pmatrix} p_1^2 & p_1p_2 & p_1p_3 & \cdots & p_1p_{k-1} \\ p_1p_2 & p_2^2 & p_2p_3 & \cdots & p_2p_{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ p_1p_{k-1} & p_2p_{k-1} & p_3p_{k-1} & \cdots & p_{k-1}^2 \end{pmatrix}\right]$$

$$= n\left[\mathrm{diag}(p_1, \ldots, p_{k-1}) + \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{k-1} \end{pmatrix}(-p_1, -p_2, \ldots, -p_{k-1})\right]$$

$$= n\left[C + \underset{\sim}{u}\underset{\sim}{v}'\right], \quad C = \mathrm{diag}(p_1, \ldots, p_{k-1}), \ \underset{\sim}{u} = \begin{pmatrix} p_1 \\ \vdots \\ p_{k-1} \end{pmatrix},$$

$$\underset{\sim}{v} = -\underset{\sim}{u}.$$

Now, $A^{-1} = \frac{1}{n}\left(C + \underset{\sim}{u}\underset{\sim}{v}'\right)^{-1}$

$$= \frac{1}{n}\left[C^{-1} - \frac{(C^{-1}\underset{\sim}{u})(\underset{\sim}{v}'C^{-1})}{1 + \underset{\sim}{v}'C^{-1}\underset{\sim}{u}}\right].$$

Now, $$C^{-1}\underset{\sim}{u} = \begin{pmatrix} 1/p_1 & 0 & \cdots & 0 \\ 0 & 1/p_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/p_{k-1} \end{pmatrix}\begin{pmatrix} p_1 \\ \vdots \\ p_{k-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \underset{\sim}{1}_{(k-1)}.$$

and $$\underset{\sim}{v}'C^{-1} = \begin{pmatrix} -p_1 \\ -p_2 \\ \vdots \\ -p_{k-1} \end{pmatrix}\begin{pmatrix} 1/p_1 & 0 & \cdots & 0 \\ 0 & 1/p_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/p_{k-1} \end{pmatrix} = (-1 \ -1 \ \cdots \ -1)$$
$$= -\underset{\sim}{1}'_{(k-1)}.$$

and, $$\underset{\sim}{v}'C^{-1}\underset{\sim}{u} = -\underset{\sim}{1}'_{k-1}\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{k-1} \end{pmatrix} = -\sum_{i=1}^{k-1} p_i$$

$$1 + \underset{\sim}{v}'C^{-1}\underset{\sim}{u} = 1 - \sum_{i=1}^{k-1} p_i$$

$$\therefore A^{-1} = \frac{1}{n}\left[\mathrm{diag}(1/p_1, 1/p_2, \ldots, 1/p_{k-1}) + \frac{\underset{\sim}{1}_{k-1}\underset{\sim}{1}'_{k-1}}{1 - \sum_{i=1}^{k-1} p_i}\right]$$

$$= \frac{1}{n}\left[\mathrm{diag}(1/p_1, \ldots, 1/p_{k-1}) + \frac{J_{k-1}}{1 - \sum_1^{k-1} p_i}\right], \text{ define, } p_k = 1 - \sum_1^{k-1} p_i$$

$$A^{-1} = \frac{1}{n}\left[\mathrm{diag}(1/p_1, \ldots, 1/p_{k-1}) + \frac{J_{k-1}}{p_k}\right]$$

$$= \frac{1}{n}\begin{pmatrix} 1/p_1 + 1/p_k & 1/p_k & 1/p_k & \cdots & 1/p_k \\ 1/p_k & 1/p_2 + 1/p_k & 1/p_k & \cdots & 1/p_k \\ \vdots & \vdots & \vdots & & \vdots \\ 1/p_k & 1/p_k & 1/p_k & \cdots & 1/p_{k-1} + 1/p_k \end{pmatrix}_{(k-1)\times(k-1)}.$$

——————

- Evaluate the value of the determinant:

$$\begin{vmatrix} a & b & b & b & \cdots & b \\ b & a & c & c & \cdots & c \\ b & c & a & c & \cdots & c \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ b & c & c & c & \cdots & a \end{vmatrix}$$

<u>Solution:-</u>

$$\begin{vmatrix} a & b & b & b & \cdots & b \\ b & a & c & c & \cdots & c \\ b & c & a & c & \cdots & c \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ b & c & c & c & \cdots & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & b & b & \cdots & b \\ 0 & a' & c' & c' & \cdots & c' \\ 0 & c' & a' & c' & \cdots & c' \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & c' & c' & c' & \cdots & a' \end{vmatrix} \qquad \begin{aligned} R_i' &= R_i - \frac{b}{a}R_1 \ \forall i>1 \\ a' &= a - \frac{b^2}{a} \\ c' &= c - \frac{b^2}{a} \end{aligned}$$

$$= a \begin{vmatrix} a' & c' & c' & \cdots & c' \\ c' & a' & c' & \cdots & c' \\ \vdots & \vdots & \vdots & & \vdots \\ c' & c' & c' & \cdots & a' \end{vmatrix}$$

$$= a\left[a' + (n-2)c'\right](a'-c')^{n-2}.$$

MATRIX AND DETERMINANTS

A **Matrix** of order $m \times n$ is rectangular array of mn elements in m rows and n columns. It is usually denoted by an upper case letter.

If a_{ij} be the element in the i^{th} row and j^{th} column of the matrix A, we write

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$= (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n), \quad \underset{\sim}{\alpha}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}_{m \times 1}$$

$$= \begin{pmatrix} \underset{\sim}{\beta}_1 \\ \underset{\sim}{\beta}_2 \\ \vdots \\ \underset{\sim}{\beta}_m \end{pmatrix}, \quad \underset{\sim}{\beta}_i = (a_{i1}, a_{i2}, \ldots, a_{in})_{1 \times n}$$

$$= ((a_{ij}))_{\substack{i=1(1)m \\ j=1(1)n}}$$

Square Matrix :- A matrix is said to be a square matrix if no of rows of the matrix is equal to the no of columns.

A square matrix A of $m \times n$ is denoted by A_n, as $m = n$, so $A_n = ((a_{ij}))$

then a_{ii}'s are called diagonal elements of matrix A_n and a_{ij}'s are called off-diagonal elements of A_n.

Diagonal Matrix :- If all the off diagonal ~~matrix~~ elements of a square matrix vanish then it reduces to a diagonal matrix.

A diagonal matrix of order $n \times n$ consisting of elements $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is denoted by $diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Scalar Matrix :- If all diagonal elements of a diagonal matrix became equal then it reduces to a scalar matrix.

$$A = diag(\lambda, \lambda, \ldots, \lambda)$$

if $\lambda = 1$, then A is said to be an identity matrix.

$$I_n = ((\delta_{ij})), \quad \delta_{ij} = \text{kronecker's - } '\delta'.$$

Triangular matrix :- Let A be a square matrix of order n and $A = ((a_{ij}))$

Then A is said to be upper triangular matrix if $a_{ij} = 0$, $\forall i > j$.

And A is " " " lower " " " " " $\forall i < j$.

Null Matrix :- A matrix is said to be null mtx if all of its elements vanish, denoted by $O_{m \times n}$.

Sum Matrix :- A matrix is said to be a sum matrix if all of its elements become unity.

Matrix Addition:- Two matrices are said to be conformable for addition if they are of same order.

Let, $A = ((a_{ij}))_{m\times n}$ and $B = ((b_{ij}))_{m\times n}$

then $A + B = ((a_{ij} + b_{ij}))_{m\times n}$.

Matrix addition is commutative and associative.

$$A+B = B+A.$$

$$(A+B)+C = A+(B+C).$$

Multiplication by a Scalar:- Let $A = ((a_{ij}))_{m\times n}$, λ = Scalar

$$\lambda A = ((\lambda a_{ij}))_{m\times n}$$

Combining the above two operations, we have

$$\alpha A + \beta B = ((\alpha a_{ij} + \beta b_{ij}))$$

$$A = ((a_{ij})), \ B = ((b_{ij}))$$

If we choose $\alpha = 1$ and $\beta = -1$, we get matrix subtraction.

Matrix Multiplication:- Two matrices A and B are said to be conformable for matrix multiplication in the given order if no. of columns of A = no. of columns of B.

Let $A = \begin{pmatrix} \underset{\sim}{\alpha}_1 \\ \underset{\sim}{\alpha}_2 \\ \vdots \\ \underset{\sim}{\alpha}_m \end{pmatrix}_{m\times n}$, $B = (\underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2, \ldots, \underset{\sim}{\beta}_n)_{n\times n}$

then AB is defined as $AB = ((\alpha_i \beta_j))_{m\times n}$

$$= \left(\left(\sum_{k=1}^{n} a_{ik} b_{kj}\right)\right) \text{ if}$$

$$A = ((a_{ij})), \ B = ((b_{ij}))$$

If AB is defined well, BA may not be so and if AB and BA are both defined then they may not be equal.

Idempotent matrix:- A is said to be an idempotent matrix if

$$A^2 = A.$$

Now, $\left(I_n - \frac{1_n 1_n'}{n}\right)\left(I_n - \frac{1_n 1_n'}{n}\right)$

$$= I_n - \frac{1_n 1_n'}{n} - \frac{1_n 1_n'}{n} + \frac{1_n (1_n' 1_n) 1_n'}{n^2}$$

$$= I_n - \frac{1_n 1_n'}{n}.$$

Trace over matrix:- Trace over a matrix defined as the sum of the diagonal elements of A and denoted by $tr(A)$.

Properties:- $tr(A+B) = tr(A) + tr(B)$

$$tr(AB) = tr(BA)$$

$$tr\left(I_n - \frac{1_n 1_n'}{n}\right) = tr(I_n) - \frac{1}{n} tr(\underset{\sim}{1}_n \underset{\sim}{1}_n')$$

$$= n - \frac{1}{n} \times n$$

$$= n - 1.$$

Transpose of a Matrix:- Transpose of a matrix A is denoted by A'/A^T and it is defined as a matrix obtained by replacing rows of A by columns of A or columns by rows.

Properties:- $(A')' = A$, $(AB)' = B'A'$, $(A+B)' = A'+B'$.

Symmetric and Skew-symmetric Matrix:- A square matrix A is said to be a symmetric matrix if $A' = A$ and will be skew-symmetric if $A' = -A$.

$$A' = A \Leftrightarrow a_{ij} = a_{ji} \ \forall (i,j)$$
$$A' = -A \Leftrightarrow a_{ij} = -a_{ji} \ \forall (i,j) \text{ and } a_{ii} = 0 \ \forall i.$$

Here, AA', $A'A$ and $(A \pm A')$ are also symmetric matrices. (Check)

Note:- Any square matrix can uniquely be written as a sum of symmetric and skew-symmetric matrix.

$$A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A').$$

DETERMINANTS:-

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{23}a_{31}$$

$$= \sum_{1 \le i_1 \ne i_2 \ne i_3 \le 3} (-1)^{N(i_1,i_2,i_3)} a_{1i_1}a_{2i_2}a_{3i_3}$$

$$= \sum_{1 \le i_1 \ne i_2 \ne i_3 \le 3} (-1)^{N(i_1,i_2,i_3)} a_{i_1 1}a_{i_2 2}a_{i_3 3}$$

$$= ((a_{ij}))_{n \times n}.$$

$$\therefore \ |A| = \det(A) = \sum_{1 \le i_1 \ne i_2 \ne \cdots \ne i_n \le n} (-1)^{N(i_1,i_2,i_3,\ldots,i_n)} \prod_{j'=1}^{n} a_{i_{j'} j},$$

where $N(i_1,\ldots,i_n)$ = No. of inversions in $(i_1,\ldots,i_n)$.

Summation is taken over all possible permutations of $(1,2,\ldots,n)$, i.e. no. of terms under the summation is $n!$.

Properties : 1. $\operatorname{Det}(A) = \operatorname{Det}(A')$

Sol. Let $A = ((a_{ij}))_{n\times n}$, where $B = ((b_{ij}))_{n\times n}$, $B = A'$.

clearly, $a_{ji} = b_{ij} \ \forall (i,j)$

$$|A'| = |B| = \sum_{1\le i_1\ne i_2\ne \cdots \ne i_n \le n} (-1)^{N(i_1,\dots,i_n)} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n}$$

$$= |A| .$$

P : 2. $\det(\lambda A) = \lambda^n \det(A)$, where A is of order n.

Sol. Let $A = ((a_{ij}))$, $B = ((b_{ij}))$

$B = \lambda A$, clearly, $b_{ij} = \lambda a_{ij} \ \forall (i,j)$

$$|B| = \sum_{1\le i_1\ne i_2\ne \cdots \ne i_n \le n} (-1)^{N(i_1,\dots,i_n)} b_{1i_1} b_{2i_2} \cdots b_{ni_n}$$

$$= \sum_{1\le i_1\ne i_2\ne \cdots \ne i_n \le n} (-1)^{N(i_1,\dots,i_n)} \lambda a_{1i_1} \lambda a_{2i_2} \cdots \lambda a_{ni_n} .$$

$$= \lambda^n |A| .$$

P.3. Let $A = \begin{pmatrix} \underset{\sim}{\alpha}_1' \\ \vdots \\ \underset{\sim}{\alpha}_m' \end{pmatrix} = ((a_{ij}))$ and $B = \begin{pmatrix} \underset{\sim}{\alpha}_1' + \sum_{j\ne 1} \lambda_j \underset{\sim}{\alpha}_i' \\ \underset{\sim}{\alpha}_2' \\ \vdots \\ \underset{\sim}{\alpha}_m' \end{pmatrix}$ then $|B| = |A|$

Sol. Let $B = ((b_{ij}))_{m\times m}$

$$b_{1j} = a_{1j} + \sum_{k\ne 1} \lambda_k a_{kj} , \quad b_{ij} = a_{ij} + \sum_{k\ne 1} \lambda_k a_{kj}$$

$$|B| = \sum_{1\le i_1\ne i_2\ne \cdots \ne i_m \le m} (-1)^{N(i_1,\dots,i_m)} b_{1i_1} b_{2i_2} \cdots b_{mi_m}$$

$$= \sum_{1\le i_1\ne i_2\ne \cdots \ne i_m \le m} (-1)^{N(i_1,\dots,i_m)} \left(a_{1i_1} + \sum_{k=2}^{m} \lambda_k a_{ki_1}\right) a_{2i_2} \cdots a_{mi_m}$$

$$= \sum_{1\le i_1\ne i_2\ne \cdots \ne i_m \le m} (-1)^{N(i_1,\dots,i_m)} a_{1i_1} a_{2i_2} \cdots a_{mi_m} + 0 = |A| .$$

P.4 :- $|AB| =$ [redacted] $|A||B|$

Let $A = ((a_{ij}))_{n\times n}$ $\qquad AB = ((\sum a_{ik} b_{kj}))_{n\times n}$

$B = ((b_{ij}))_{n\times n}$

$$|AB| = \sum_{1\le i_1\ne i_2\ne \cdots \ne i_n \le n} (-1)^{N(i_1,\dots,i_n)} \sum_{k=1}^{n} a_{1k} b_{ki_1} \sum_{k=1} a_{2k} b_{ki_2} \cdots \sum_{k=1}^{n} a_{nk} b_{ki_n} .$$

$$= \sum_{1\le k_1, k_2, \dots, k_n \le n} a_{1k_1} a_{2k_2} \cdots a_{nk_n} \sum_{1\le i_1\ne i_2\ne \cdots \ne i_n \le n} (-1)^{N(i_1,\dots,i_n)} b_{k_1 i_1} b_{k_2 i_2} \cdots b_{k_n i_n}$$

Rewrite B as $B = B(1, 2, \dots, n)$

$B(k_1, \dots, k_n)$ = matrix obtained from B replacing the [redacted] 1st row by k_1th row and 2nd row by k_2th row and so on, $\forall i = 1, \dots, n \ \forall i$.

$$|AB| = \sum_{1\le k_1, k_2, \dots, k_n \le n} a_{1k_1} a_{2k_2} \cdots a_{nk_n} |B(k_1, \dots, k_n)|$$

$$= \sum_{1\le k_1\ne k_2\ne \cdots \ne k_n\le n} (-1)^{N(k_1,\dots,k_n)} |B(1,2,\dots,n)| a_{1k_1} a_{2k_2} \cdots a_{nk_n}$$, if at least two k_i's are equal then $|B(k_1, \dots, k_n)|$ vanishes.

$$= |B| \sum_{1\le k_1\ne \cdots \ne k_n\le n} (-1)^{N(k_1,\dots,k_n)} a_{1k_1} a_{2k_2} \cdots a_{nk_n}$$

$$= |A||B| .$$

Minor • Cofactor • Inverse :-

Let A be a square matrix of order $n\times n$, $A = ((a_{ij}))$, then minor of a_{ij} is determinant of the matrix obtained from A omitting the ith row and jth column.

Cofactor of $a_{ij} = A_{ij}$, say

$$= (-1)^{i+j} \times \text{minor of } a_{ij} .$$

Adjoint of $A = A^*$ or $Adj(A)$

$$= ((A_{ij}))^T$$

Now, $A = ((a_{ij}))_{n\times n}$; A_{ij} = Cofactor of a_{ij}

$$\sum a_{ij} A_{i'j} = \begin{cases} |A| & \text{if } i = i' \\ 0 & \text{if } i \ne i', \forall j . \end{cases}$$

or, $$\sum a_{ij} A_{ij'} = \begin{cases} |A| & \text{if } j = j' \\ 0 & \text{if } j \ne j' \ \forall i . \end{cases}$$

Result:- $A = ((a_{ij}))_{n\times n}$, A_{ij} = Co-factor of a_{ij}, $A^* = Adj(A)$.

Then $AA^* = A^*A = |A| I_n$.

Proof:-

$$AA^* = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{n1} \\ A_{21} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \sum a_{1k} A_{1k} & \sum a_{2k} A_{2k} & \cdots & \sum a_{1k} A_{nk} \\ \sum a_{2k} A_{1k} & \sum a_{2k} A_{2k} & \cdots & \sum a_{2k} A_{nk} \\ \vdots & \vdots & & \vdots \\ \sum a_{nk} A_{1k} & \sum a_{nk} A_{2k} & \cdots & \sum a_{nk} A_{nk} \end{pmatrix}$$

$$= \text{diag}(|A|, |A|, \dots, |A|)$$

$$= |A| \cdot I_n .$$

Singular and Non-singular Matrices :—

A matrix (square) is said to be a non-singular matrix if $|A| \neq 0$ otherwise it is said to be singular.

Let us assume that A is non-singular, i.e. $|A| \neq 0$

$$A\left(\frac{1}{|A|} \cdot A^*\right) = \left(\frac{1}{|A|} \cdot A^*\right) A = I_n \qquad \text{[SEE P.T.O.]}$$

if $B = \frac{1}{|A|} \cdot A^*$, then $AB = BA = I_n$. $[\because |A| \neq 0]$

[Analogous to the feature of the real numbers (non-zero), $x \times \frac{1}{x} = \frac{1}{x} \times x = 1$.]

Here B is the inverse matrix of A and usually denoted by A^{-1}, then $AA^{-1} = A^{-1}A = I$.

Result:- Inverse of a square matrix A exists if and only if A is non-singular.

Proof:- **If Part:-** Let A be a non-singular matrix, $|A| \neq 0$.

Then $AA^* = A^*A = |A| I$

$$\Rightarrow A\left(\frac{1}{|A|} A^*\right) = \left(\frac{1}{|A|} A^*\right) A = I$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} A^* \text{ exists if } |A| \neq 0.$$

Only if Part:- Let A^{-1} exists.

$$AA^{-1} = A^{-1}A = I$$

$$\Rightarrow |AA^{-1}| = 1$$

$$\Rightarrow |A||A^{-1}| = 1$$

$$\Rightarrow |A| \neq 0.$$

Properties 1:- Inverse of a square matrix is unique.

$\Rightarrow$ If possible let B and C be the inverse of A. Then

$$AB = BA = I \text{ ——— ① } \times C \text{ (Pre-multiply)}$$

$$AC = CA = I \text{ ——— ② } \times B \text{ (Post-multiply)}$$

$$CAB = CBA = C$$

$$ACB = CAB = B$$

$$\therefore B = C.$$

P-2:- $(AB)^{-1} = B^{-1}A^{-1}$.

Sol.

$$(AB) B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I$$

$$(AB)^{-1} AB B^{-1} A^{-1} = (AB)^{-1} I = (AB)^{-1}$$

$$\Rightarrow IB^{-1}A^{-1} = (AB)^{-1}$$

$$\therefore (AB)^{-1} = B^{-1}A^{-1}.$$

Result:- Let $A_{p\times p}$ be a non-singular matrix and $B_{p\times q}$ and $C_{q\times p}$ such that $A+BC$ is also non-singular then,

$$[A_{p\times p} + B_{p\times q}C_{q\times p}]^{-1} = A^{-1} - A^{-1}B(I_p + CA^{-1}B)CA^{-1},$$

when the inverses exist.

Proof:-

$$D = A+BC$$

$$DA^{-1} = (A+BC)A^{-1}$$
$$= AA^{-1} + BCA^{-1}$$
$$= I_p + BCA^{-1}$$

$$DA^{-1}B = (I_p + BCA^{-1})B$$
$$= B + BCA^{-1}B$$
$$= B(I_q + CA^{-1}B)$$

$$\Rightarrow DA^{-1}B(I_q + CA^{-1}B)^{-1} = B.$$

$$\Rightarrow DA^{-1}B(I_q + CA^{-1}B^{-1})C = BC$$

$$\Rightarrow A + DA^{-1}B(I_q + CA^{-1}B^{-1})C = A + BC = D$$

$$\Rightarrow D(I_p - A^{-1}B(I_q + CA^{-1}B)^{-1}C) = A$$

$$\Rightarrow D(I_p - A^{-1}B(I_q + CA^{-1}B)^{-1}C)A^{-1} = I_p$$

$$\Rightarrow D^{-1}D(A^{-1} - A^{-1}B(I_q + CA^{-1}B)^{-1}CA^{-1}) = D^{-1}I_p$$

$$D^{-1} = (A+BC)^{-1} = A^{-1} - A^{-1}B(I_q + CA^{-1}B)^{-1}CA^{-1}.$$

General Case:- $A_{p\times p}$, $C_{q\times q}$ are non-singular matrices
$B_{p\times q}$, $D_{q\times p}$ are non-singular matrices

Then $\ni$ $(A+BCD)$, a non-singular matrix then

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B(I_q + CDA^{-1}B)^{-1}CDA^{-1}$$

$$= A^{-1} - A^{-1}B(C^{-1}(I_q + CDA^{-1}B))^{-1}DA^{-1}$$

Particular Case:- $(A + \underset{\sim}{\alpha}\underset{\sim}{\beta}')^{-1} = A^{-1} - A^{-1}\underset{\sim}{\alpha}(1 + \underset{\sim}{\beta}'A^{-1}\underset{\sim}{\alpha})^{-1}\underset{\sim}{\beta}'A^{-1}$

$$= A^{-1} - \frac{1}{1+\underset{\sim}{\beta}'A^{-1}\underset{\sim}{\alpha}}A^{-1}\underset{\sim}{\alpha}\underset{\sim}{\beta}'A^{-1}.$$

Partitioning and Augmenting :-

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1s} & a_{1\,\overline{s+1}} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2s} & a_{2\,\overline{s+1}} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{ms} & a_{m\,\overline{s+1}} & \cdots & a_{mn} \\ a_{\overline{m+1}\,1} & a_{\overline{m+1}\,2} & \cdots & a_{\overline{m+1}\,s} & a_{\overline{m+1}\,\overline{s+1}} & \cdots & a_{\overline{m+1}\,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ns} & a_{n\,\overline{s+1}} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11\,m\times s} & A_{12\,m\times\overline{n-s}} \\ A_{21\,\overline{n-m}\times s} & A_{22\,\overline{n-m}\times\overline{n-s}} \end{bmatrix}$$

Here A is partitioned into sub-matrices $A_{11}, A_{12}, A_{21}, A_{22}$. In particular, the submatrices may reduce to vectors or scalars.

Example:-

$$\left(\begin{array}{c|ccc} 2 & 3 & 4 & 6 \\ \hline 7 & 9 & 1 & 2 \\ 4 & 5 & 6 & 3 \end{array}\right) = \begin{pmatrix} 2 & \underset{\sim}{\alpha}'_{1\times 3} \\ \underset{\sim}{\beta}'_{2\times 1} & C_{2\times 3} \end{pmatrix}$$

Again $A_{m\times n}$ is augmented by $B_{m\times n}$ may be of the form $(A/B)_{m\times 2n}$ or $\left(\frac{A}{B}\right)_{2m\times n}$.

Problem:- 1. $\begin{vmatrix} A_{n\times n} & O_{n\times m} \\ O_{m\times n} & I_m \end{vmatrix} = |A|.$

Proof:-

$$\begin{vmatrix} A_{n\times n} & O_{n\times m} \\ O_{m\times n} & I_m \end{vmatrix} = 1 \begin{vmatrix} A_{n\times n} & O_{n\times\overline{m-1}} \\ O_{\overline{m-1}\times n} & I_{m-1} \end{vmatrix}$$

$$= 1 \begin{vmatrix} A_{n\times n} & O_{n\times\overline{m-2}} \\ O_{\overline{m-2}\times n} & I_{m-2} \end{vmatrix}$$

$$= 1 \begin{vmatrix} A_{n\times n} & O_{n\times 1} \\ O_{1\times n} & 1 \end{vmatrix}$$

$$= |A|.$$

Problem:- 2. $\begin{vmatrix} A_{m\times m} & B_{m\times n} \\ O_{n\times m} & C_{n\times n} \end{vmatrix} = |A|\,|C|$

Proof:-

$$= \begin{vmatrix} I & O \\ O & C \end{vmatrix} \times \begin{vmatrix} A & B \\ O & I \end{vmatrix}$$

$$= |C| \times |A|$$

$$= |A|\,|C|.$$

Problem:-3:- $\begin{vmatrix} A_{m\times n} & O_{m\times n} \\ O_{m\times n} & B_{n\times n} \end{vmatrix} = |A|\,|B|$

Proof:-

$$\begin{pmatrix} A_{m\times m} & O_{m\times n} \\ O_{n\times m} & I_n \end{pmatrix} \begin{pmatrix} I_n & O_{m\times n} \\ O_{n\times m} & B_{n\times m} \end{pmatrix}$$

$$= \begin{pmatrix} A_{m\times n} & O_{m\times n} \\ O_{n\times n} & B_{n\times n} \end{pmatrix} = |A|\,|B| .$$

Result:- Show that —

$$\det \begin{bmatrix} \underset{m\times m}{A_{11}} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}_{n\times n} = |A_{11}|\,|A_{22} - A_{21}A_{11}^{-1}A_{12}| \cdots\cdots (1)$$

$$= |A_{22}|\,|A_{11} - A_{12}A_{22}^{-1}A_{21}| \cdots\cdots (2)$$

if A_{11} is an non-singular matrix.

Proof:-

$$\begin{bmatrix} I_m & O_{m\times \overline{n-m}} \\ -A_{21}A_{11}^{-1} & I_{\overline{n-m}} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

$$\therefore \det \begin{bmatrix} I_m & O_{m\times \overline{n-m}} \\ -A_{21}A_{11}^{-1} & I_{\overline{n-m}} \end{bmatrix} \det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$= \det \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

$$= |A_{11}|\,|A_{22} - A_{21}A_{11}^{-1}A_{12}| \cdots\cdots (1)$$

$$\begin{bmatrix} I_m & -A_{12}A_{22}^{-1} \\ O_{\overline{n-m}\times m} & I_{n-m} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

$$\therefore \det \begin{bmatrix} I_m & -A_{12}A_{22}^{-1} \\ O_{\overline{n-m}\times m} & I_{n-m} \end{bmatrix} \det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$= \det \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

$$= |A_{22}|\,|A_{11} - A_{12}A_{22}^{-1}A_{21}| \cdots\cdots (2)$$

Rank of a Matrix :-

$V_C(A)$: Column space of A (Vector space generated by the columns of A).

$V_R(A)$: Row-space of A.

Column Rank of A is defined as $\dim\{V_C(A)\}$ or number of LIN columns of A.

Row rank of A = Column rank of A = $R(A) \le \min(m,n)$, where $A_{m\times n} = A$.

Definition:- Rank of the matrix A is the order of highest order non-vanishing minor of A.

Some Useful Results:-

1. $\text{Rank}(AB) \le \min[\text{Rank}(A), \text{Rank}(B)]$

Proof:- Let A be a matrix of order $m\times n$ and B be a matrix of order $n\times p$.

suppose $A = ((a_{ij}))$

$B = ((b_{ij}))$

$$A = (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \dots, \underset{\sim}{\alpha}_n)$$

$$AB = (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \dots, \underset{\sim}{\alpha}_n)\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{pmatrix}$$

$$= \left(\sum_{k=1}^{n} b_{k1}\underset{\sim}{\alpha}_k, \sum_{k=1}^{n} b_{k2}\underset{\sim}{\alpha}_k, \dots, \sum_{k=1}^{n} b_{kp}\underset{\sim}{\alpha}_k\right)$$

Columns of AB are linear combination of columns of A.

$$V_C(AB) \subseteq V_C(A)$$

$$\Rightarrow \dim V_C(AB) \le \dim V_C(A)$$

$$\Rightarrow R(AB) \le R(A) \quad \dots\dots (i)$$

Similarly we can show that $R(AB) \le R(B)$ ......(ii)

Combining (i) & (ii) we have, $R(AB) \le \min[R(A), R(B)]$.

2. If A be an idempotent matrix then Rank (A) = Trace(A).

Proof:- Let A be a matrix of order $n \times n$, such that $A^2 = A$ and rank (A) = r.

By rank-factorisation theorem, we have

$$A = B_{n\times r} C_{r\times n} \text{ where } \operatorname{rank}(B) = \operatorname{rank}(C) = r.$$

$$A^2 = A$$

$$\Rightarrow BCBC = BC$$

$$\Rightarrow B'BCBCC' = B'BCC'$$

We know, $R(B'B) = \operatorname{Rank}(B) = r$, where BB' is of order r.

i.e. BB' is non-singular $\Rightarrow (BB')^{-1}$ exists.

Similarly, $(CC')^{-1}$ exists.

$$(B'B)^{-1}B'BCBCC'(CC')^{-1}$$

$$= (B'B)^{-1}B'BCC'(CC')^{-1}$$

$$\Rightarrow CB = I_r$$

$\therefore$ Trace (CB) = trace(BC) = trace(A) = trace(I_r) = r = rank(A)

3. Rank of a matrix remain unaltered if it is premultiplied or post multiplied by a non-singular matrix,

$$R(PA) = R(AQ) = R(PAQ) = R(A), \text{ where}$$

$P_{m\times n}$, $Q_{n\times n}$ are non-singular matrices.

Proof:-

$$R(PA) \le R(A) \text{ ..... } (1)$$

Again, $R(A) = R(I_m A)$

$= R(P^{-1}PA)$ [P^{-1} exists, since P is n.s.]

$\le R(PA)$ .... (2)

(1) & (2) gives, $R(A) = R(PA)$.

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Null Space & Nullity :-

Let A be a matrix of order $m \times n$.

Define, $N(A) = \{ \underset{\sim}{x} : A\underset{\sim}{x} = \underset{\sim}{0} \}$

$N(A)$ constitutes a vector space.

Suppose, $A = \begin{pmatrix} \underset{\sim}{\alpha}_1' \\ \underset{\sim}{\alpha}_2' \\ \vdots \\ \underset{\sim}{\alpha}_m' \end{pmatrix}$

$$A\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \begin{pmatrix} \underset{\sim}{\alpha}_1'\underset{\sim}{x} \\ \underset{\sim}{\alpha}_2'\underset{\sim}{x} \\ \vdots \\ \underset{\sim}{\alpha}_m'\underset{\sim}{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Rightarrow$ i.e. $\underset{\sim}{x}$ is orthogonal to any row of A.

$N(A)$ and $V_R(A)$ are orthogonal spaces.

i.e. $N(A) \perp V_R(A)$.

Thus, if $R(A) = r$,

i.e. if $\dim V_R(A) = r$

then $\dim N(A) = n - r$

$$\left[N(A) + V_R(A) = E_n \ \& \ N(A) \cap V_R(A) = \{\underset{\sim}{0}\} \right]$$

Here $N(A)$ is termed as Null space of the matrix A and $\dim(N(A))$ is the Nullity of A.

Result:- S.T. $\dim \left[\underset{m \times n}{N(A)} \right] = n - R(A)$.

Proof:- Let $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_s\}$ be a basis of $V_R(A)$

and $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \dots, \underset{\sim}{y}_t\}$ be a basis of $N(A)$.

$\therefore$ By definition of $N(A)$, we have,

$$\underset{\sim}{x}_i' \underset{\sim}{y}_j = 0 \ \forall\, i, j.$$

$\therefore$ $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_s, \underset{\sim}{y}_1, \underset{\sim}{y}_2, \dots, \underset{\sim}{y}_t\}$ be a set of LIN vectors

n-component vectors can't contain more than n vectors.

$$s + t \le n$$

if possible, let $s + t < n$.

$\exists$ at least one vector which is orthogonal to any one of

$\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_s, \underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_t$

Let $\underset{\sim}{\gamma}$ be such a vector $\ni$ $\underset{\sim}{x}_i'\underset{\sim}{\gamma} = \underset{\sim}{0}$ $\forall i = 1, 2, \ldots, s$.

$\Rightarrow \underset{\sim}{\gamma} \in N(A)$

$\Rightarrow \underset{\sim}{\gamma} = \sum_{i=1}^{t} \theta_i \underset{\sim}{y}_i$ for some θ_i's.

$\underset{\sim}{y}_i'\underset{\sim}{\gamma} = \underset{\sim}{0}$ $\forall i = 1(1)t$

$s+t \not< n$.

$\therefore \dim(V_R(A)) + \dim(N(A)) = n$

$\therefore \dim(N(A)) = n - R(A)$.

Result:- $R(AA') = R(A) = R(A') = R(AA')$.

Proof:- Let $x \in N(A'A)$

$\Rightarrow A'A\underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{x}'A'A\underset{\sim}{x} = \underset{\sim}{0}$

i.e. $\underset{\sim}{y}'\underset{\sim}{y} = \sum y_i^2 = 0$, $\underset{\sim}{y} = A\underset{\sim}{x}$

$\Rightarrow y_i = 0 \; \forall i$.

i.e. $\underset{\sim}{y} = \underset{\sim}{0}$

$A\underset{\sim}{x} = \underset{\sim}{0}$, $\underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$, assuming A has order $m \times n$.

$\Rightarrow \underset{\sim}{x} \in N(A)$

$N(A'A) \subseteq N(A)$ ------- (1)

Let $\underset{\sim}{x} \in N(A)$

$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow A'A\underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{x} \in N(A'A)$; $N(A) \subseteq N(A'A) \Rightarrow R(A) = R(A'A)$.

Now, let $A' = B$, so for the matrix B, rank $(B'B) = r(B)$.

i.e. $R(AA') = R(A') = R(A) = R(A'A)$.

Reducing a matrix into Normal form:—

Let A be a matrix of order $m \times n$, suppose $R(A) = r < \min(m,n)$.
Then $\exists$ non-singular matrices P and Q such that

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof:- Since $R(A) = r < \min(m,n)$

$\exists$ a non-singular matrix P_1 such that $P_1 A = E$, an echelon matrix with r non-null rows.

Clearly, $E = \begin{pmatrix} E_1 \\ 0 \end{pmatrix}_{m \times n}$, E_1 is also an echelon matrix.

Now $\text{Rank}(E_1) = r =$ Column rank of E_1.

E_1 has $n-r$ columns each of which is LD on the rest r LIN columns. Hence through column operations those dependent columns can be reduced to null columns.

i.e. $E_1 \sim (\underset{r\times r}{E_2} \quad \underset{r \times n}{0})$, where E_2 has full rank.

$\therefore \exists$ a non-singular matrix $P_2 \ni$

$$P_1 A P_2 = \begin{pmatrix} E_2 & 0 \\ 0 & 0 \end{pmatrix}$$

As E_2 is a square matrix of full rank through row column operation it can be reduced to I_r.

Hence $\exists$ non-singular matrices P_3 and P_4

$$P_3 P_1 A P_2 P_4 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m \times n}$$

Let $P_3 P_1 = P$ and $P_2 P_4 = Q$

Clearly, P and Q are non-singular matrices as they are product of non-singular matrices.

A square matrix of full rank is always non-singular.

Let A be a square matrix of order n possesing full rank then $\exists$ non-singular matrices P and Q $\ni$

$$PAQ = I_n.$$

$$|P||A||Q| = 1$$

$$\therefore |A| \neq 0.$$

- **Sylvester's Inequality :-**

$$R(A_{n\times n} B_{n\times n}) \geq R(A) + R(B) - n,$$

Proof:- Let $R(A) = r < n$

$\exists$ n.s. matrices P and Q such that

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Define a square matrix C of the order $n\times n$ such that

$$PCQ = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}$$

Clearly, $P(A+C)Q = I_n$

Therefore, $|P(A+C)Q| = |I_n| = 1$

$\Rightarrow |P|\,|A+C|\,|Q| = 1.$

$\Rightarrow A+C$ is non singular.

Hence, $R(B) = R((A+C)B)$

$= R(AB+CB)$

$\leq R(AB) + R(CB)$

$\leq R(AB) + R(C)$

$= R(AB) + n - r$

$= R(AB) + n - R(A)$

$\therefore R(AB) \geq R(A) + R(B) - n.$

Orthogonal Transformation:- A matrix transformation $\underset{\sim}{Y} = A_{n\times n}\underset{\sim}{x}$ is said to be orthogonal iff A is orthogonal,

Here, $AA^T = I_n$

$\Rightarrow |A| = \pm 1.$

$\Rightarrow$ A is singular.

Now, A matrix transformation is said to be non-singular iff A is non-singular.

i.e. $\underset{\sim}{y} = A\underset{\sim}{x} \Rightarrow \underset{\sim}{x} = A^{-1}\underset{\sim}{y}$, i.e. N.S. transformation is one-one

An orthogonal transformation is an n.s. transformation & hence it is one-to-one.

Note that, $\underset{\sim}{y}^T\underset{\sim}{y} = (A\underset{\sim}{x})^T(A\underset{\sim}{x})$

$\Rightarrow \underset{\sim}{x}^T A^T A \underset{\sim}{x} = \underset{\sim}{x}^T\underset{\sim}{x}$

i.e. $\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} x_i^2$

i.e. the length of the vector is preserved under orthogonal transformation.

Helmert's Transformation :—

$(x_1, x_2, \ldots, x_n) \longrightarrow (y_1, y_2, \ldots, y_n) \ni$

$$y_1 = \frac{x_1}{\sqrt{n}} + \frac{x_2}{\sqrt{n}} + \cdots + \frac{x_n}{\sqrt{n}}$$

$$y_2 = \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}}$$

$$y_3 = \frac{x_1}{\sqrt{6}} + \frac{x_2}{\sqrt{6}} - \frac{2x_3}{\sqrt{6}}$$

$$\vdots$$

$$y_n = \frac{x_1}{\sqrt{n(n-1)}} + \frac{x_2}{\sqrt{n(n-1)}} + \cdots + \frac{x_{n-1}}{\sqrt{n(n-1)}} - \frac{(n-1)x_n}{\sqrt{n(n-1)}}$$

i.e. $\underset{\sim}{y} = P\underset{\sim}{x}$, where P is an orthogonal matrix.

$\Rightarrow \underset{\sim}{y}'\underset{\sim}{y} = \underset{\sim}{x}'P'P\underset{\sim}{x} = \underset{\sim}{x}' I_n \underset{\sim}{x} = \underset{\sim}{x}'\underset{\sim}{x}$

$\Rightarrow \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} x_i^2$.

Example:-

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$ns^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=2}^{n} y_i^2$$

Note that, $y_1 = \frac{\sum x_i}{\sqrt{n}} = \frac{n\bar{x}}{\sqrt{n}} = \sqrt{n}\,\bar{x}$

Now, $ns^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$

$$= \sum_{i=1}^{n} y_i^2 - y_1^2$$

$$= \sum_{i=2}^{n} y_i^2$$

$$ns^2 = \sum_{i=2}^{n}\frac{i}{i-1}(x_i - \bar{x}_i)^2 \; ; \quad \bar{x}_i = \sum_{j=1}^{i} x_j/i \; , \; i = 1(1)n.$$

$$\Rightarrow ns^2 = \sum_{i=2}^{n}\left\{\sqrt{\frac{i}{i-1}}\left(\frac{i-1}{i}x_i - \frac{x_1}{i} - \frac{x_2}{i} - \cdots - \frac{x_i}{i}\right)\right\}^2$$

$$= \sum_{i=2}^{n}\left\{\frac{x_1}{\sqrt{i(i-1)}} + \frac{x_2}{\sqrt{i(i-1)}} + \cdots + \frac{x_{i-1}}{\sqrt{i(i-1)}} - \frac{(i-1)x_i}{\sqrt{i(i-1)}}\right\}^2$$

$$= \sum_{i=2}^{n} y_i^2$$

[CU]

- **Problem:-** Show that the SD s of a set of observation $x_1, x_2, \dots, x_n$ is given by $n.s^2 = \sum_{i=2}^{n}\frac{i}{i-1}(x_i - \bar{x}_i)^2$, where $\bar{x}_i = \sum_{j=1}^{i} x_j/i$, $i = 2(1)n$.

Solution:- Note that, $\sqrt{\frac{i}{i-1}}(\bar{x}_i - x_i)$

$$= \sqrt{\frac{i}{i-1}}\left(\frac{x_1 + \cdots + x_{i-1} + x_i - i.x_i}{i}\right)$$

$$= \frac{x_1 + \cdots + x_{i-1} - (i-1)x_i}{\sqrt{i(i-1)}}, \; i = 2(1)n.$$

Let,
$$y_1 = \frac{x_1 + x_2 + \cdots + x_n}{\sqrt{n}},$$
$$y_i = \frac{x_1 + \cdots + x_{i-1} - (i-1)x_i}{\sqrt{i(i-1)}}, \quad i = 2(1)n.$$

$$y_1 = \frac{x_1}{\sqrt{n}} + \frac{x_2}{\sqrt{n}} + \cdots + \frac{x_n}{\sqrt{n}}$$
$$y_2 = \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}} = \frac{x_1 - x_2}{\sqrt{1 \cdot 2}}$$
$$y_3 = \frac{x_1}{\sqrt{6}} + \frac{x_2}{\sqrt{6}} - \frac{2x_3}{\sqrt{6}} = \frac{x_1 + x_2 - 2x_3}{\sqrt{2 \cdot 3}}$$
$$y_4 = \frac{x_1}{\sqrt{12}} + \frac{x_2}{\sqrt{12}} + \frac{x_3}{\sqrt{12}} - \frac{3x_4}{\sqrt{12}} = \frac{x_1 + x_2 + x_3 - 3x_4}{\sqrt{3 \cdot 4}}$$
$$\vdots$$
$$y_{n-1} = \frac{x_1 + \cdots + x_{n-1} - (n-2)x_{n-1}}{\sqrt{(n-2)(n-1)}}$$
$$y_n = \frac{x_1 + \cdots + x_{n-1} - (n-1)x_n}{\sqrt{(n-1)n}}$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1/\sqrt{n} & 1/\sqrt{n} & 1/\sqrt{n} & \cdots & 1/\sqrt{n} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & \cdots & 0 \\ 1/\sqrt{6} & +1/\sqrt{6} & -2/\sqrt{6} & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1/\sqrt{n(n-1)} & \frac{1}{\sqrt{(n-1)n}} & \cdots & \cdots & -\frac{n-1}{\sqrt{(n-1)n}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$\Rightarrow \underset{\sim}{y} = A\underset{\sim}{x}$, where A is an orthogonal matrix.

$$\underset{\sim}{y}'\underset{\sim}{y} = (A\underset{\sim}{x})'(A\underset{\sim}{x})$$
$$= \underset{\sim}{x}'A'A\underset{\sim}{x}$$
$$= \underset{\sim}{x}'\underset{\sim}{x}$$
$$\Leftrightarrow \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} x_i^2$$
$$\Rightarrow \sum_{i=1}^{n} x_i^2 - y_1^2 = \sum_{i=2}^{n} y_i^2$$
$$\Rightarrow \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 = \sum_{i=2}^{n} y_i^2$$
$$\Rightarrow \sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=2}^{n} \frac{i}{i-1}(x_i - \bar{x}_i)^2$$
$$\Rightarrow ns^2 = \sum_{i=2}^{n} \frac{i}{i-1}(x_i - \bar{x})^2.$$

• [CU] For $n \geq 2$, suppose $x_1, x_2, \ldots, x_n$ are not all equal. Consider the orthogonal transformation of $(y_1, y_2, \ldots, y_n)$ to $(z_1, z_2, \ldots, z_n)$ such that

$$\underset{\sim}{z}_{n\times1} = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots\cdots & \frac{1}{\sqrt{n}} \\ \frac{x_1-\bar{x}}{\sqrt{\sum_i (x_i-\bar{x})^2}} & \frac{x_2-\bar{x}}{\sqrt{\sum_i (x_i-\bar{x})^2}} & \cdots\cdots & \frac{x_n-\bar{x}}{\sqrt{\sum_i (x_i-\bar{x})^2}} \\ a_{31} & a_{32} & \cdots\cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{pmatrix}_{n\times n} \cdot \underset{\sim}{y}_{n\times1}$$

show that $\sum_{i=3}^{n} z_i^2 = \left\{\sum_{i=1}^{n}(y_i-\bar{y})^2\right\}\left\{1-r_{xy}^2\right\}$,

where r_{xy} is the correlation coefficient between x and y.

Solution:- Here, $\underset{\sim}{z}^{n\times1} = A^{n\times n} \cdot \underset{\sim}{y}^{n\times1}$

where, A is the given orthogonal matrix.

By property, $\underset{\sim}{z}'\underset{\sim}{z} = \underset{\sim}{y}'\underset{\sim}{y}$

$$\Rightarrow \sum_{i=1}^{n} z_i^2 = \sum_{i=1}^{n} y_i^2$$

$$\Rightarrow \sum_{i=3}^{n} z_i^2 = \sum_{i=1}^{n} y_i^2 - z_1^2 - z_2^2$$

Here, $z_1 = \frac{y_1+y_2+\cdots+y_n}{\sqrt{n}} = \sqrt{n}\,\bar{y}$.

$$z_2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})y_i}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}}$$

$$= \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}}$$

$$= r_{xy} \cdot \sqrt{\sum_{i=1}^{n}(y_i-\bar{y})^2}$$

$$\therefore \sum_{i=3}^{n} z_i^2 = \left(\sum_{i=1}^{n} y_i^2 - n\bar{y}^2\right) - r_{xy}^2 \cdot \sum_{i=1}^{n}(y_i-\bar{y})^2$$

$$= \sum_{i=1}^{n}(y_i-\bar{y})^2 - r_{xy}^2 \cdot \sum_{i=1}^{n}(y_i-\bar{y})^2$$

$$= \left\{\sum_{i=1}^{n}(y_i-\bar{y})^2\right\}\left(1-r_{xy}^2\right)$$

Gaussian Reduction or Gaussian Elimination for n.s. matrix:

Consider a system $A\underset{\sim}{x} = \underset{\sim}{b}$; where $A^{n\times n}$ is n.s. Let P be the product of elementary row matrices which take $(A, \underset{\sim}{b})$ into $(H, \underset{\sim}{d})$, where H is an echelon matrix.

$$\therefore A\underset{\sim}{x} = \underset{\sim}{b}$$

$$\Rightarrow PA\underset{\sim}{x} = P\underset{\sim}{b}$$

$$\Rightarrow H\underset{\sim}{x} = \underset{\sim}{d}$$

$$\Rightarrow \begin{pmatrix} 1 & h_{12} & h_{13} & \cdots & h_{1\,n-1} & h_{1n} \\ 0 & 1 & h_{23} & \cdots & h_{2\,n-1} & h_{2n} \\ 0 & 0 & 1 & \cdots & h_{3\,n-1} & h_{3n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & h_{n-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \\ d_n \end{pmatrix}$$

$$\therefore x_n = d_n$$

$$\therefore x_{n-1} + h_{n-1\,n} x_n = d_{n-1}$$

$$\vdots$$

$$\therefore x_2 + h_{23}x_3 + \cdots + h_{2n}x_n = d_2$$

$$\therefore x_1 + h_{12}x_2 + \cdots + h_{1n}x_n = d_1$$

SYSTEM OF LINEAR EQUATION

Theorem:- A consistent system $\underset{m\times n}{A}\underset{\sim}{x} = \underset{\sim}{b}$ has a unique solution iff A is of full column rank [i.e. $r(A) = n$] & at least two solution iff $r(A) = r(A \vdots b) < n$.

Proof:- [**Lemma:** The solution set S of $A\underset{\sim}{x} = \underset{\sim}{b}$ be a consistent system, can be written as $\{\underset{\sim}{u} + \underset{\sim}{v} : \underset{\sim}{v} \in N(A)\}$, where $\underset{\sim}{u}$ is a particular solution of $A\underset{\sim}{x} = \underset{\sim}{b}$.

Proof:- Let $\underset{\sim}{\omega} \in S$. Then $A\underset{\sim}{u} = \underset{\sim}{b}$ and $A\underset{\sim}{\omega} = \underset{\sim}{b}$.

$$\Rightarrow A(\underset{\sim}{\omega} - \underset{\sim}{u}) = 0$$

$$\Rightarrow \underset{\sim}{\omega}, \underset{\sim}{u} \in N(A).$$

$$\therefore \underset{\sim}{\omega} = \underset{\sim}{u} + \underset{\sim}{\omega} - \underset{\sim}{u} = \underset{\sim}{u} + \underset{\sim}{v}, \text{ where } \underset{\sim}{v} \in N(A).$$

$$\therefore \underset{\sim}{\omega} \in \{\underset{\sim}{u} + \underset{\sim}{v} : \underset{\sim}{v} \in N(A)\}$$

Let $\underset{\sim}{\omega} \in \{\underset{\sim}{u} + \underset{\sim}{v} : \underset{\sim}{u} \in N(A)\}$

Then $A\underset{\sim}{\omega} = A\underset{\sim}{u} + A\underset{\sim}{v}$

$$= \underset{\sim}{b} + \underset{\sim}{0} = \underset{\sim}{b}.$$

$$\Rightarrow \underset{\sim}{\omega} \in S.$$

Hence, $S = \{\underset{\sim}{u} + \underset{\sim}{v} : \underset{\sim}{v} \in N(A)\}$.]

The system has a unique solution iff no. of solution in S is one.

iff $\dim\{N(A)\} = 0$

iff $n - r(A) = 0$

iff $r(A) = n$.

[S has only one solution

$\Rightarrow \underset{\sim}{u} + \underset{\sim}{0}$ is the only one solution.

$\Rightarrow \underset{\sim}{v} = \underset{\sim}{0}$ is the only member of $N(A)$.

i.e. $N(A) = \{\underset{\sim}{0}\}$]

The system has at least two solution iff the no. of solutions in S is ≥ 2

iff $\underset{\sim}{u} + \underset{\sim}{0}$ is a solution and there is a non-null $\underset{\sim}{v} \in N(A) \ni$ $\underset{\sim}{v} + \underset{\sim}{u}$ is a solution.

iff $\dim(N(A)) \geq 1$

iff $n - r(A) \geq 1$

iff $r(A) \leq n-1 < n$

$\Rightarrow r(A) = r(A \mid b) < n$.

Theorem:- Let A be a matrix of order $m \times n$, then the system $\underset{m\times n}{A}\underset{\sim}{x} = \underset{m\times 1}{\underset{\sim}{0}}$ will have a non-trivial solution iff $R(A) < n$.

Proof:- **If Part:-** Suppose $R(A) < n$

We have to show that $A\underset{\sim}{x} = \underset{\sim}{0}$ has a trivial solution.

Let $A = (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n)$

since $R(A) < n$

$\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n$ are linearly dependent.

so $\exists$ scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ at least one in non-zero $\ni$

$$\lambda_1\underset{\sim}{\alpha}_1 + \lambda_1\underset{\sim}{\alpha}_2 + \cdots + \lambda_n\underset{\sim}{\alpha}_n = \underset{\sim}{0}$$

$$\Rightarrow (\underset{\sim}{\alpha}_1 \ \underset{\sim}{\alpha}_2 \cdots \underset{\sim}{\alpha}_n)\begin{pmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_n\end{pmatrix} = \underset{\sim}{0}$$

$\Rightarrow A.\underset{\sim}{\lambda} = \underset{\sim}{0}$, where $\underset{\sim}{\lambda} = (\lambda_1\ \lambda_2 \cdots \lambda_n)' \neq 0$

Hence $\underset{\sim}{x} = \underset{\sim}{\lambda}\ (\neq 0)$ is a solution of the system $A\underset{\sim}{x} = \underset{\sim}{0}$.

Hence, we have a non-trivial solution of the system $A\underset{\sim}{x} = \underset{\sim}{0}$.

Only If Part:- Suppose $\exists$ a non-trivial solution, say, $\underset{\sim}{x} = \underset{\sim}{h}\ (\neq \underset{\sim}{0})$ of the system $A\underset{\sim}{x} = \underset{\sim}{0}$.

$\therefore A\underset{\sim}{h} = \underset{\sim}{0}$

$\Rightarrow (\underset{\sim}{\alpha}_1 \ \underset{\sim}{\alpha}_2 \cdots \underset{\sim}{\alpha}_n)\begin{pmatrix}h_1\\ h_2\\ \vdots\\ h_n\end{pmatrix} = \underset{\sim}{0}$, where $h_i \neq 0$ for at least one i.

$\Rightarrow \sum_{i=1}^{n} h_i\alpha_i = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n$ are linearly dependent.

$\Rightarrow R(A) < n$.

Theorem:- For a square matrix A, suppose $A\underset{\sim}{x} = \underset{\sim}{0}$ for some zero vector $\underset{\sim}{x}$. Explain why A is singular.

Proof:- $A = (\underset{\sim}{\alpha}_1 \; \underset{\sim}{\alpha}_2 \cdots \underset{\sim}{\alpha}_n)_{n\times n}$

$\underset{\sim}{x} = (x_1 \; x_2 \cdots x_n)'$ [$x_i \neq 0$ for atleast one i]

Now, $A\underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow \sum_i \alpha_i x_i = \underset{\sim}{0}$

$\underset{\sim}{\alpha}_i$'s are linearly dependent.

$\Rightarrow R(A) < n.$

$\Rightarrow \det(A) = 0.$

Theorem:- If $R(A) = r (<n)$ then the system of equations $A\underset{\sim}{x} = \underset{\sim}{0}$ has exactly $(n-r)$ independent non-trivial solution.

Proof:- Let $N(A) = \{\underset{\sim}{x} : A\underset{\sim}{x} = \underset{\sim}{0}\}$

The set of all solutions of the system $A\underset{\sim}{x} = \underset{\sim}{0}$.

Since $\text{Rank}(A) = r$

$\Rightarrow \dim[N(A)] = n-r.$

Hence, $N(A)$ has $n-r$ linearly independent non-trivial vectors of $\underset{\sim}{x}$.

$\therefore A\underset{\sim}{x} = \underset{\sim}{0}$ has $(n-r)$ non-trivial solution.

Theorem:- S.T. if a system of linear equation has two distinct solution then $\exists$ an infinite no. of solutions.

Proof:- Suppose $\underset{\sim}{x}_1$ & $\underset{\sim}{x}_2$ are two distinct solution of the system $A\underset{\sim}{x} = \underset{\sim}{b}$.

Then $A\underset{\sim}{x}_1 = \underset{\sim}{b}$ & $A\underset{\sim}{x}_2 = \underset{\sim}{b}$

Let us define, $\underset{\sim}{x}^* = \lambda\underset{\sim}{x}_1 + (1-\lambda)\underset{\sim}{x}_2$ for some scalar λ.

$$\therefore A\underset{\sim}{x}^* = \lambda A\underset{\sim}{x}_1 + (1-\lambda)A\underset{\sim}{x}_2$$

$$= \lambda\underset{\sim}{b} + (1-\lambda)\underset{\sim}{b}$$

$$= \underset{\sim}{b}, \; \lambda \in \mathbb{R}.$$

$\underset{\sim}{x} = \underset{\sim}{x}^*$ is also a solution $A\underset{\sim}{x} = \underset{\sim}{b}$ and as λ can be chosen in an infinite number of ways. Hence the system has an infinite no. of solutions.

PROBLEMS ON MATRICES & DETERMINANTS

1. Diagonalise the following matrix using non-singular transformation and find the non-singular matrices which diagonalise

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 2 & -1 \\ 1 & 2 & -1 \end{pmatrix}.$$

Solution:-

$$A \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 4 & 3 \\ 0 & 1 & -3 \end{pmatrix} \quad \begin{matrix} R_2' \leftrightarrow R_2 + 2R_1 \\ R_3' \leftrightarrow R_3 - R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & -15/4 \end{pmatrix} \quad R_3' \leftrightarrow R_3 - \frac{R_2}{4}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & -15/4 \end{pmatrix} \quad \begin{matrix} C_2' \leftrightarrow C_2 - C_1 \\ C_3' \leftrightarrow C_3 - 2C_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -15/4 \end{pmatrix} \quad C_3' \leftrightarrow C_3 - \frac{3}{4}C_2$$

$$= \Lambda$$

$$E_2 E_1 A E_3 E_4 = \Lambda$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/4 & 1 \end{pmatrix}$$

$$P = E_2 \times E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3/2 & -1/4 & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3/4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q = E_3 \times E_4 = \begin{pmatrix} 1 & -1 & -5/4 \\ 0 & 1 & -3/4 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore PAQ = \Lambda$

2) Find the rank of the matrix $\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{pmatrix}$.

Solution:- $A \sim \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ $\quad R_2' = R_2 - R_1$, $R_3' = R_3 - 2R_1$, $R_4' = R_4 - 3R_1$

$\sim \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad R_4' = R_4 - R_2$

Rank$(A) = 3$.

3) Find $\det \begin{pmatrix} \sigma_1^2 & p\sigma_1\sigma_2 & \cdots & p\sigma_1\sigma_n \\ p\sigma_1\sigma_2 & \sigma_2^2 & \cdots & p\sigma_2\sigma_n \\ \cdots & \cdots & \cdots & \cdots \\ p\sigma_n\sigma_1 & p\sigma_n\sigma_2 & \cdots & \sigma_n^2 \end{pmatrix}$; $\sigma_i \neq 0 \ \forall i = 1(1)n$.

Sol. $\Delta = \sigma_1^2 \sigma_2^2 \cdots \sigma_n^2 \begin{vmatrix} 1 & p & \cdots & p \\ p & 1 & \cdots & p \\ \vdots & \vdots & & \vdots \\ p & p & \cdots & 1 \end{vmatrix}$ $\quad R_i' = \frac{R_i}{\sigma_i} \ \forall i$, $C_i' = \frac{C_i}{\sigma_i} \ \forall i$

$$= \prod_{i=1}^{n} \sigma_i^2 \left\{ (1-p)^{n-1} \{1 + (n-1)p\} \right\}.$$

(Ans)

Cor.

$\begin{vmatrix} 1+x & 1 & \cdots & 1 \\ 1 & 1+x & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1+x \end{vmatrix} = (n+x) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1+x & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1+x \end{vmatrix}$ $\quad R_1' = \sum R_i$, $R_1'' = \frac{R_1}{(n+x)}$

$= (n+x) \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & x & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & x \end{vmatrix}$ $\quad C_i = C_i - C_1$

$= (n+x) \cdot x^{n-1}$

$= x^n + nx^{n-1}$.

4. Find $\begin{vmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \vdots & \vdots & \vdots & & \vdots \\ \rho & \rho & \rho & \cdots & 1 \end{vmatrix}$. OR $A^{n\times n}$, where $a_{ij} = \begin{cases} 1 & \text{if } i=j \\ \rho & \text{if } i\neq j \end{cases}$

Find $|A|$. [ISI M.STAT]

Sol.

$$\Delta = \begin{vmatrix} 1+(n-1)\rho & 1+(n-1)\rho & \cdots & 1+(n-1)\rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & & \vdots \\ \rho & \rho & \cdots & \rho \end{vmatrix} \qquad R_1' = \sum_i R_i$$

$$= [1+(n-1)\rho] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & & \vdots \\ \rho & \rho & \cdots & 1 \end{vmatrix}$$

$$= [1+(n-1)\rho] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1-\rho & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1-\rho \end{vmatrix} \qquad R_i' = R_i - \rho R_1 \ \forall i>1$$

$$= 1+(n-1)\rho\ (1-\rho)^{n-1}. \qquad \text{[expanding by its first column]}$$

5.

Find $\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{vmatrix}$.

Sol.

$$\Delta = [a+(n-1)b] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ b & a & \cdots & b \\ \vdots & \vdots & & \vdots \\ b & b & & a \end{vmatrix}$$

$$= [a+(n-1)b] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & a-b & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a-b \end{vmatrix} \qquad R_i' = R_i - bR_1 \ \forall\ i>1.$$

[expanding by the first column]

$$= [(a+(\ -1)b](a-b)^{n-1}$$
$$= (a-b)^n + nb(a-b)^{n-1}.$$

6. <u>Vandermonde's Determinant:-</u>

Find $\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}$.

Sol.

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & (a_2-a_1) & (a_3-a_1) & \cdots & (a_n-a_1) \\ 0 & a_2(a_2-a_1) & a_3(a_3-a_1) & \cdots & a_n(a_n-a_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_2^{n-2}(a_2-a_1) & a_3^{n-2}(a_3-a_1) & \cdots & a_n^{n-2}(a_n-a_1) \end{vmatrix} \quad R_i' = R_i - a_1 R_{i-1} \ \forall i>1.$$

$$= \begin{vmatrix} a_2-a_1 & a_3-a_1 & \cdots & a_n-a_1 \\ a_2(a_2-a_1) & a_3(a_3-a_1) & \cdots & a_n(a_n-a_1) \\ \vdots & \vdots & & \vdots \\ a_2^{n-2}(a_2-a_1) & a_3^{n-2}(a_3-a_1) & \cdots & a_n^{n-2}(a_n-a_1) \end{vmatrix}$$ [expanding by the first column]

$$= \prod_{i=2}^{n}(a_i-a_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_2 & a_3 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_2^{n-2} & a_3^{n-2} & \cdots & a_n^{n-2} \end{vmatrix} \quad C_i' = \frac{C_i}{a_{i+1}-a_1}$$

$$= \prod_{i=1}^{n}(a_i-a_1)\prod_{i=3}^{n}(a_i-a_2) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_3 & a_4 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_3^{n-3} & a_4^{n-3} & \cdots & a_n^{n-3} \end{vmatrix}$$

$$= \prod_{i=1}^{n}(a_i-a_1)\prod_{i=3}^{n}(a_i-a_2)\cdots\prod_{i=1}^{n}(a_i-a_{n-2}) \begin{vmatrix} 1 & 1 \\ a_{n-1} & a_n \end{vmatrix}$$

$$= \prod_{i>j}\prod(a_i-a_j)$$

$$= (-1)^{\frac{n(n-1)}{2}} \prod_{i<j}\prod(a_i-a_j).$$

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7. Find $\begin{vmatrix} 1+a_1^2 & a_1a_2 & a_1a_3 & \cdots & a_1a_n \\ a_2a_1 & 1+a_2^2 & a_2a_3 & \cdots & a_2a_n \\ a_3a_1 & a_3a_2 & 1+a_3^2 & \cdots & a_3a_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \cdots & 1+a_n^2 \end{vmatrix}$

Sol.

$$\Delta = \begin{vmatrix} 1+a_1^2 & a_1a_2 & a_1a_3 & \cdots & a_1a_n \\ a_2a_1 & 1+a_2^2 & a_2a_3 & \cdots & a_2a_n \\ a_3a_1 & a_3a_2 & 1+a_3^2 & \cdots & a_3a_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_na_1 & a_na_2 & a_na_3 & & 1+a_n^2 \end{vmatrix}$$

$$= a_1^2a_2^2\cdots a_n^2 \begin{vmatrix} \left(\frac{1}{a_1^2}+1\right) & 1 & 1 & \cdots & 1 \\ 1 & \left(\frac{1}{a_2^2}+1\right) & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & \left(1+\frac{1}{a_n^2}\right) \end{vmatrix} \quad R_i' = \frac{R_i}{a_i},\ C_i' = \frac{C_i}{a_i}\ \forall i$$

$$= \begin{vmatrix} 1+a_1^2 & a_1^2 & a_1^2 & \cdots & a_1^2 \\ a_2^2 & 1+a_2^2 & a_2^2 & \cdots & a_2^2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_n^2 & a_n^2 & a_n^2 & \cdots & 1+a_n^2 \end{vmatrix} \quad R_i' = R_i a_i^2\ \forall i$$

$$= \begin{vmatrix} 1+\sum a_i^2 & 1+\sum a_i^2 & \cdots & 1+\sum a_i^2 \\ a_2^2 & 1+a_2^2 & \cdots & a_2^2 \\ \vdots & \vdots & & \vdots \\ a_n^2 & a_n^2 & \cdots & 1+a_n^2 \end{vmatrix} \quad R_1' = \sum_i R_i$$

$$= \left(1+\sum_i a_i^2\right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_2^2 & 1+a_2^2 & \cdots & a_2^2 \\ \vdots & \vdots & & \vdots \\ a_n^2 & a_n^2 & \cdots & 1+a_n^2 \end{vmatrix}$$

$$= \left(1+\sum_i a_i^2\right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} \quad R_i' = R_i - a_i^2 R_1\ \forall i > 1.$$

$$= 1 + \sum_{i=1}^{n} a_i^2 .$$

——————*———*———

LINEAR ALGEBRA

SYSTEM OF LINEAR EQUATION

Suppose that there are m equations and n unknowns, say $x_1, x_2, \ldots, x_n$. We may put the equations in the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

We may write the equations in the matrix form —

$$A\underset{\sim}{x} = \underset{\sim}{b} \quad \text{———} (*)$$

where, $A = ((a_{ij}))_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$, and

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = (x_1, x_2, \ldots, x_n)' \quad \& \quad \underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1} = (b_1, b_2, \ldots, b_m)'$$

Definition :— $\boxed{\text{C.U.}}$

Consistent system and Inconsistent System :—

By a solution of the system $(*)$, we mean any set of values $x_1, x_2, \ldots, x_n$ that satisfies the m equations simultaneously. The system is said to be consistent if it has at least one solution, otherwise it is said to be inconsistent.

***Ex.1.** The system of equations

$$x_1 + 2x_2 = 5$$
$$2x_1 + 4x_2 = 3$$

— it is an inconsistent system since it does not have any solution. The term 'inconsistent' is appropriate to this situation, for the first equation implies $2x_1 + 4x_2 = 10$, which is incompatible with the second equation.

* **Ex. 2.** The system

$$x_1 + 2x_2 = 5$$
$$3x_1 - x_2 = 1$$

— it is a consistent system having a unique solution with $x_1 = 1, x_2 = 2$.

* **Ex. 3.** The equation

$$x_1 + 2x_2 = 5$$

— it is a consistent system having infinite number of solutions.

Now we shall go for finding conditions under which a system will be consistent.

Case of a homogeneous System :— If in a system each term in the R.H.S is zero or in the other word, $\underset{\sim}{b} = \underset{\sim}{0}$, then the system is said to be Homogeneous System.

$$A\underset{\sim}{x} = \underset{\sim}{0} \quad \text{——— (**)}$$

— A system of homogeneous equation is necessarily consistent as it posses a trivial solution $\underset{\sim}{x} = \underset{\sim}{0}$.

i.e. $\underset{\sim}{x} = \underset{\sim}{0}$ is always a soln. of (**), ~~whether~~ whatever the matrix A may be, we call the solution a trivial solution.

Thus it may be of interest to know whether the system has any non-trivial soln. (non-null) or; it possess the trivial solution as the only solution. If there is atleast one non-trivial soln. then it is of interest to find a set of maximum number of linearly independent non-null soln.

Theorem :→ The necessary and sufficient condition for a system $\underset{m\times n}{A}\ \underset{n\times 1}{\underset{\sim}{x}} = \underset{m\times 1}{\underset{\sim}{0}}$ to have a non trivial soln. is rank (A) < n ***

Proof:— Let rank(A) < n.

⟺ the columns of A, say $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n$ are linearly dependent.

⟺ the equation $x_1\underset{\sim}{a}_1 + x_2\underset{\sim}{a}_2 + \cdots + x_n\underset{\sim}{a}_n = \underset{\sim}{0}$ has at least one non-trivial solution.

⟺ $A\underset{\sim}{x} = \underset{\sim}{0}$ has at least one non trivial solution.

⟺ System is consistent.

Equivalent matrices and Sweep-out method :—

We know that the rank of a matrix is obtained by reducing A to an echelon matrix through elementary row (column) operations, i.e., by pre-multiplying (post multiplying) A by elementary matrices.

Defn. → Any matrix B obtained from a given matrix A by performing a succession of elementary (row or column) operations is said to be equivalent to A. Thus we can say that B is equivalent to A if and only if $B = PAQ$ where P and Q are the products of elementary matrices.

Since P and Q are non-singular then we may write P^{-1} and Q^{-1} exists so, we have $A = P^{-1}BQ^{-1}$. Clearly A is equivalent to B in case B is also equivalent to A.

We may say that A and B are mutually equivalent if $B = PAQ$ holds.

Result :→ If A and B are equivalent matrices then $\text{rank}(A) = \text{rank}(B)$

Proof :→ One should note that an echelon matrix H is obtained from a given matrix A by elementary row (or column) operations so that $H = AQ$ or, $H = PA$; where A and H are equivalent matrices. i.e. in our case, $B = PA$ or, $B = AQ$, where P and Q are non-singular matrices. We know – rank of a matrix is unchanged by pre or post multiplication of non-singular matrix.

Hence the result.

Result :→ If A and B are equivalent matrices then $A\underset{\sim}{x} = \underset{\sim}{0}$ and $B\underset{\sim}{x} = 0$ has the same solution space.

Definition of equivalent system :— Two systems of linear equations are equivalent if each equation in each system is a linear combination of the equations in the other system.

Remark: Equivalent system of linear equations have exactly the same solutions.

Result: Each matrix A is equivalent to some matrix B in which, for some integer $r (\geq 0)$, the first r elements on the diagonal are unity and all other elements are zero. A and B have the same rank, r.

Proof: We know that matrix A can be reduced to an echelon matrix H some of whose columns, say columns $c_1, c_2, \ldots, c_r$, are unit vectors, all but the first r rows of H being non-null vectors. Now, the columns $c_1, c_2, \ldots, c_r$, can by column ~~operations~~ interchanges be brought into the first r positions. Thus we arrive a matrix of the form

$$PAQ_1 = \begin{pmatrix} I_r & D \\ 0 & 0 \end{pmatrix},$$

Now D can be converted into a null matrix by performing suitable column operations. Thus, we have —

$$PAQ_1Q_2 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

or, taking $Q = Q_1 Q_2$,

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Taking $B = PAQ$, we may say that A has been reduced to the desired form, again, B has rank r and so has A, which is equivalent to B.

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Theorem: For a matrix A prove that $\dim(S) = n - \dim$ (row space of A) where S is the solution space of $A\underset{\sim}{x} = \underset{\sim}{0}$.

Proof: Let R be the row-reduced echelon matrix which is row-equivalent to A then the solution space of $A\underset{\sim}{x} = \underset{\sim}{0}$ will be same as the solution space of $R\underset{\sim}{x} = \underset{\sim}{0}$.

Let R has r non-zero rows. Then the system $R\underset{\sim}{x} = \underset{\sim}{0}$ simply express r of the unknowns $x_1, x_2, \ldots, x_n$ in terms of the remaining $(n-r)$ unknowns. Now suppose that the leading non-zero entries of R occur in the columns $k_1, k_2, \ldots, k_r$ respectively. Define J be the set of all indices different from $k_1, k_2, \ldots, k_r$, i.e., $J = \{1, 2, 3, \ldots, n\} - \{k_1, k_2, \ldots, k_r\}$. Then the system $R\underset{\sim}{x} = \underset{\sim}{0}$ can be written in the form —

$$\left.\begin{array}{l} x_{k_1} + \sum_{j\in J} c_{1j} x_j = 0 \\ x_{k_2} + \sum_{j\in J} c_{2j} x_j = 0 \\ \vdots \\ x_{k_r} + \sum_{j\in J} c_{rj} x_j = 0 \end{array}\right\} \longrightarrow (*)$$

, where c_{ij}'s, $i = 1(1)r$ are certain scalars.

Here each solution of $(*)$ is obtained by assigning values to those x_j's with $j \in J$ and computing the different values of $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$.

For each j in J, let $\underset{\sim}{\varepsilon_j}$ be the solution obtained by assigning $x_j = 1$ and $x_{j'} = 0$, $j \neq j' \in J$.

As ε_j has a 1 in the jth row and 0 in the other rows indexed by the entries in J. then ε_j's are linearly independent.

Now if $\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ is in the solution space then $\underset{\sim}{\alpha} = \sum_{j\in J} a_j \varepsilon_j$ is also in the solution space and is a solution space such that $x_j = a_j \; \forall j \in J$. the solution with this property is unique and therefore $\underset{\sim}{\alpha} = \underset{\sim}{a}$ and $\underset{\sim}{a}$ is in the span of $\{\varepsilon_j : j \in J\}$. Thus the set of $(n-r)$ vectors $\underset{\sim}{\varepsilon_j}$, $j \in J$ forms a basis of the solution space S.

$$\therefore \dim(S) = n - r$$
$$= n - \dim(\text{rowspace of } A).$$

Homogeneous Linear Equations:—

If $b_1 = b_2 = \cdots = b_m = 0$, the set of equations is said to be homogeneous.

For a set of homogeneous equations, the condition $\text{rank}(A \,\vdots\, b) = \text{rank}(A)$ is automatically satisfied.

If $r = n$, the only solution is $x_1 = 0 = x_2 = \cdots = x_n$ (trivial solution).

If $r < n$, the $(n-r)$ unknowns may have arbitrary values.

Consequently the necessary and sufficient condition for non-zero solutions is $r < n$, i.e., the rank is less than the no. of unknowns.

When there are n equations and n unknowns, this condition becomes $|A| = 0$, and when there are $m\,(< n)$ equations and n unknowns, there are always non-trivial solutions.