Exercise 1.1

Question 1:

Determine whether each of the following relations are reflexive, symmetric and transitive:

(i) Relation R in the set $A = \{1, 2, 3 \ldots 13, 14\}$ defined as

$R = \{(x, y): 3x - y = 0\}$

(ii) Relation R in the set **N** of natural numbers defined as

$R = \{(x, y): y = x + 5 \text{ and } x < 4\}$

(iii) Relation R in the set $A = \{1, 2, 3, 4, 5, 6\}$ as

$R = \{(x, y): y \text{ is divisible by } x\}$

(iv) Relation R in the set **Z** of all integers defined as

$R = \{(x, y): x - y \text{ is as integer}\}$

(v) Relation R in the set A of human beings in a town at a particular time given by

(a) $R = \{(x, y): x \text{ and } y \text{ work at the same place}\}$

(b) $R = \{(x, y): x \text{ and } y \text{ live in the same locality}\}$

(c) $R = \{(x, y): x \text{ is exactly 7 cm taller than } y\}$

(d) $R = \{(x, y): x \text{ is wife of } y\}$

(e) $R = \{(x, y): x \text{ is father of } y\}$

Answer 1:

(i) $A = \{1, 2, 3 \ldots 13, 14\}$

$R = \{(x, y): 3x - y = 0\}$

$\therefore R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$

R is not reflexive since $(1, 1), (2, 2) \ldots (14, 14) \notin R$.

Also, R is not symmetric as $(1, 3) \in R$, but $(3, 1) \notin R$. $[3(3) - 1 \neq 0]$

Also, R is not transitive as (1, 3), (3, 9) ∈ R, but (1, 9) ∉ R. $[3(1) - 9 \neq 0]$

Hence, R is neither reflexive, nor symmetric, nor transitive.

(ii) R = {(x, y): $y = x + 5$ and $x < 4$} = {(1, 6), (2, 7), (3, 8)}

It is clear that (1, 1) ∉ R.

∴ R is not reflexive.

(1, 6) ∈ R But, (1, 6) ∉ R.

∴ R is not symmetric.

Now, since there is no pair in R such that (x, y) and (y, z) ∈ R, then (x, z) cannot belong to R.

∴ R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(iii) A = {1, 2, 3, 4, 5, 6}

R = {(x, y): y is divisible by x}

We know that any number (x) is divisible by itself.

So, (x, x) ∈ R

∴ R is reflexive.

Now,

(2, 4) ∈ R [as 4 is divisible by 2]

But, (4, 2) ∉ R. [as 2 is not divisible by 4]

∴ R is not symmetric.

Let (x, y), (y, z) ∈ R. Then, y is divisible by x and z is divisible by y.

∴ z is divisible by x.

⇒ (x, z) ∈ R

$\therefore$ R is transitive.

Hence, R is reflexive and transitive but not symmetric.

(iv) R = {(x, y): $x - y$ is an integer}

Now, for every $x \in \mathbf{Z}$, $(x, x) \in$ R as $x - x = 0$ is an integer.

$\therefore$ R is reflexive.

Now, for every $x, y \in \mathbf{Z}$, if $(x, y) \in$ R, then $x - y$ is an integer.

$\Rightarrow -(x - y)$ is also an integer.

$\Rightarrow (y - x)$ is an integer.

$\therefore (y, x) \in$ R

$\therefore$ R is symmetric.

Now,

Let (x, y) and $(y, z) \in$ R, where $x, y, z \in \mathbf{Z}$.

$\Rightarrow (x - y)$ and $(y - z)$ are integers.

$\Rightarrow x - z = (x - y) + (y - z)$ is an integer.

$\therefore (x, z) \in$ R

$\therefore$ R is transitive.

Hence, R is reflexive, symmetric, and transitive.

(v)

(a) R = {(x, y): x and y work at the same place}

$\Rightarrow (x, x) \in$ R [as x and x work at the same place]

$\therefore$ R is reflexive.

If $(x, y) \in$ R, then x and y work at the same place.

$\Rightarrow y$ and x work at the same place.

$\Rightarrow (y, x) \in$ R.

∴ R is symmetric.

Now, let $(x, y), (y, z) \in R$

⇒ x and y work at the same place and y and z work at the same place.

⇒ x and z work at the same place.

⇒ $(x, z) \in R$

∴ R is transitive.

Hence, R is reflexive, symmetric and transitive.

(b) $R = \{(x, y): x \text{ and } y \text{ live in the same locality}\}$

Clearly, $(x, x) \in R$ as x and x is the same human being.

∴ R is reflexive.

If $(x, y) \in R$, then x and y live in the same locality.

⇒ y and x live in the same locality.

⇒ $(y, x) \in R$

∴ R is symmetric.

Now, let $(x, y) \in R$ and $(y, z) \in R$.

⇒ x and y live in the same locality and y and z live in the same locality.

⇒ x and z live in the same locality.

⇒ $(x, z) \in R$

∴ R is transitive.

Hence, R is reflexive, symmetric and transitive.

(c) $R = \{(x, y): x \text{ is exactly 7 cm taller than } y\}$

Now, $(x, x) \notin R$

Since human being x cannot be taller than himself.

∴ R is not reflexive.

Now, let $(x, y) \in R$.

$\Rightarrow$ x is exactly 7 cm taller than y.

Then, y is not taller than x. [Since, y is 7 cm smaller than x]

$\therefore (y, x) \notin R$

Indeed if x is exactly 7 cm taller than y, then y is exactly 7 cm shorter than x.

$\therefore$ R is not symmetric.

Now,

Let $(x, y), (y, z) \in R$.

$\Rightarrow$ x is exactly 7 cm taller than y and y is exactly 7 cm taller than z.

$\Rightarrow$ x is exactly 14 cm taller than z .

$\therefore (x, z) \notin R$

$\therefore$ R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(d) $R = \{(x, y): x \text{ is the wife of } y\}$

Now,

$(x, x) \notin R$

Since x cannot be the wife of herself.

$\therefore$ R is not reflexive.

Now, let $(x, y) \in R$

$\Rightarrow$ x is the wife of y.

Clearly y is not the wife of x.

$\therefore (y, x) \notin R$

Indeed if x is the wife of y, then y is the husband of x.

$\therefore$ R is not transitive.

Let $(x, y), (y, z) \in R$

⇒ x is the wife of y and y is the wife of z.

This case is not possible. Also, this does not imply that x is the wife of z.

∴ $(x, z) \notin$ R

∴ R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(e) R = {(x, y): x is the father of y}

$(x, x) \notin$ R

As x cannot be the father of himself.

∴ R is not reflexive.

Now, let $(x, y) \notin$ R.

⇒ x is the father of y.

⇒ y cannot be the father of y.

Indeed, y is the son or the daughter of y.

∴ $(y, x) \notin$ R

∴ R is not symmetric.

Now, let $(x, y) \in$ R and $(y, z) \notin$ R.

⇒ x is the father of y and y is the father of z.

⇒ x is not the father of z.

Indeed x is the grandfather of z.

∴ $(x, z) \notin$ R

∴ R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

Question 2:

Show that the relation R in the set **R** of real numbers, defined as

R = $\{(a, b): a \leq b^2\}$ is neither reflexive nor symmetric nor transitive.

Answer 2:

R = $\{(a, b): a \leq b^2\}$

It can be observed that $\left(\frac{1}{2}, \frac{1}{2}\right) \notin$ R, since, $\frac{1}{2} > \left(\frac{1}{2}\right)^2$

∴ R is not reflexive.

Now, $(1, 4) \in$ R as $1 < 4^2$ But, 4 is not less than 1^2.

∴ $(4, 1) \notin$ R

∴ R is not symmetric.

Now,

$(3, 2), (2, 1.5) \in$ R [as $3 < 2^2 = 4$ and $2 < (1.5)^2 = 2.25$]

But, $3 > (1.5)^2 = 2.25$

∴ $(3, 1.5) \notin$ R

∴ R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

Question 3:

Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as R = $\{(a, b):$ $b = a + 1\}$ is reflexive, symmetric or transitive.