CHAPTER 3

Chapter 3

3. FORWARD KINEMATICS AND INVERSE KINEMATICS

In this section of the thesis the forward kinematics and the inverse kinematics of the 5-DOF and 7-DOF redundant manipulator is discussed. The Denavit-Hartenberg (D-H) notation for these two manipulators is discussed with steps used for deriving the forward kinematics is presented. Then this chapter is concluded with the solution of inverse kinematics for the 5-DOF redundant manipulator is given.

The forward kinematics is concerned with the relationship between the individual joints of the robot manipulator and the position (x,y, and z) and orientation (ϕ) of the end-effector. Stated more formally, the forward kinematics is to determine the position and orientation of the end-effector, given the values for the joint variables ($\theta_i, a_i, d_i, \alpha_i$) of the robot. The joint variables are the angles between the links in the case of revolute or rotational joints, and the link extension in the case of prismatic or sliding joints. The forward kinematics is to be contrasted with the inverse kinematics, which will be studied in the next section of this chapter, and which is concerned with determining values for the joint variables that achieve a desired position and orientation for the end-effector of the robot. The above mention theory is explained diagrammatically in figure 4.

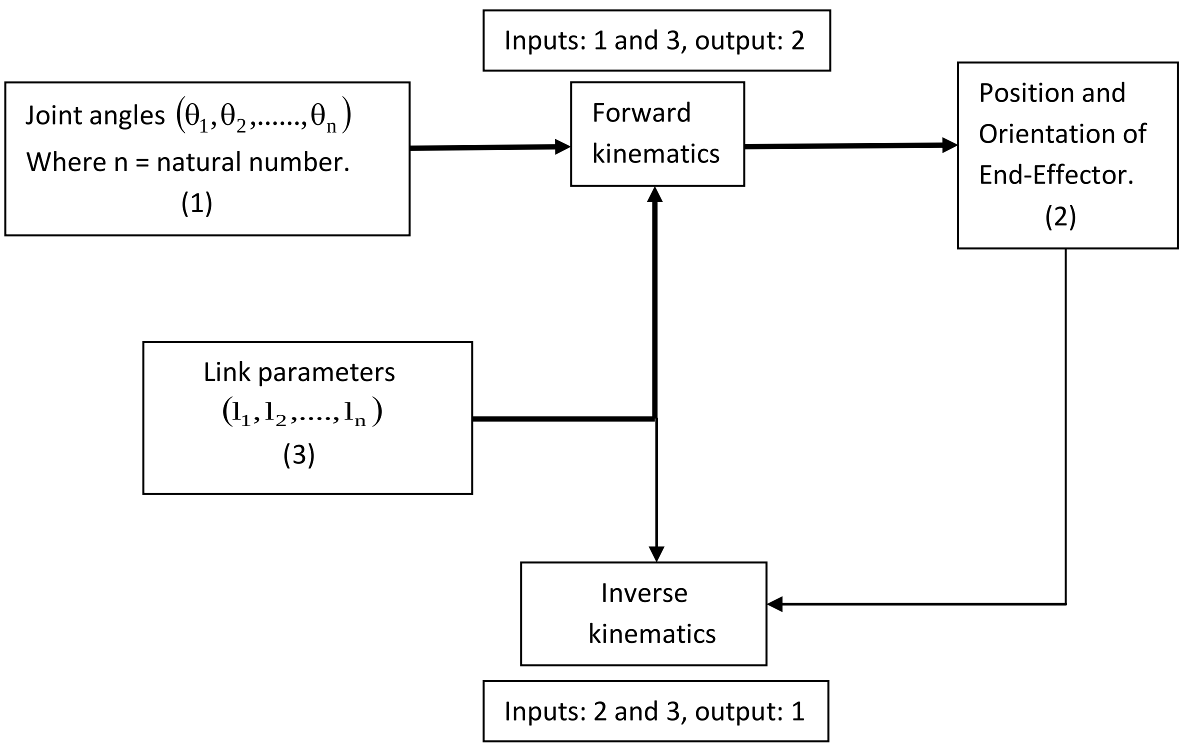

Figure 4. Forward and Inverse kinematics scheme

3.1. Denavit-Hartenberg Notation (D-H notation)

A Robot manipulator with n joints (from 1 to n) will have $n+1$ links (from 0 to n, starting from base), since each joint connect to two links. By this convention, joint i connect link $i-1$ to link i. It is considered that the location of the joint i to be fixed with respect to link $i-1$. Each link of the robot manipulator is considered to be rigidly attached to a coordinate frame for performing the kinematics analysis. In particular, link i is attached to $o_i x_i y_i z_i$. It implies that whenever the robot executes motion, the coordinate of each point on the link i are constant when expressed in the i^{th} coordinate frame. Furthermore when joint i actuate, link i and its attached frame $o_i x_i y_i z_i$, experience a resulting motion. The frame $o_0 x_0 y_0 z_0$ is a inertial frame as it attached to the robot base.

Now suppose, A_i is the homogeneous transformation matrix that express the position and orientation of $o_i x_i y_i z_i$ with respect to $o_{i-1} x_{i-1} y_{i-1} z_{i-1}$, where matrix A_i is not constant but varies as the configuration of the robot changes. Again the homogeneous transformation matrix that expresses the position and orientation of $o_j x_j y_j z_j$ with respect to $o_i x_i y_i z_i$ is called, by convention, a global transformation matrix [55] and denoted by T_j^i.

$$\text{Where, } T_j^i = A_{i+1} A_{i+2} \ldots A_{j-1} A_j \text{ if } i < j$$

$$T_j^i = I \text{ if } i = j$$

$$T_j^i = \left(T_i^j\right)^{-1} \text{if } j > i$$

As the links are rigidly attached to the corresponding frame, it concludes that the position of any point on the end-effector, when expressed in the frame n, is a constant independent of the configuration of the robot. Hence the transformation matrix gives the position and orientation of the end-effector with respect to the inertial frame. So D-H notation of the joint is introduced with some convention to solve this matrix. The convention and steps for D-H notation is represented as follows [56].

The following steps based on D-H notation are used for deriving the forward kinematics,

Step 1: Joint axes $z_0, \ldots, z_{n-1}$ are located and labelled.

Step 2: Base frame is assigned. Set the origin anywhere on the z_0-axis. The x_0 and y_0 axes are chosen conveniently to form a right-hand frame.

Step 3: The origin o_i is located, where the common normal to z_i and z_{i-1} intersects at z_i. If z_i intersects z_{i-1}, a_i located at this intersection. If z_i and z_{i-1} are parallel, locate o_i in any convenient position along z_i.

Step 4: x_i is considered along the common normal between z_{i-1} and z_i through o_i, or in thedirection normal to the $z_{i-1} - z_i$ plane if z_{i-1} and z_i intersect.

Step 5: y_i is established to complete a right-hand frame.

Step 6: The end-effector frame is assigned as $o_n x_n y_n z_n$. Assuming the n^{th} joint is revolute, set $z_n = a$ along the direction z_{n-1}. The origin on is taken conveniently along z_n direction, preferably at the centre of the gripper or at the tip of any tool that the manipulator may be carrying.

Step 7: All the link parameters $\theta_i, a_i, d_i, \alpha_i$ are tabulated.

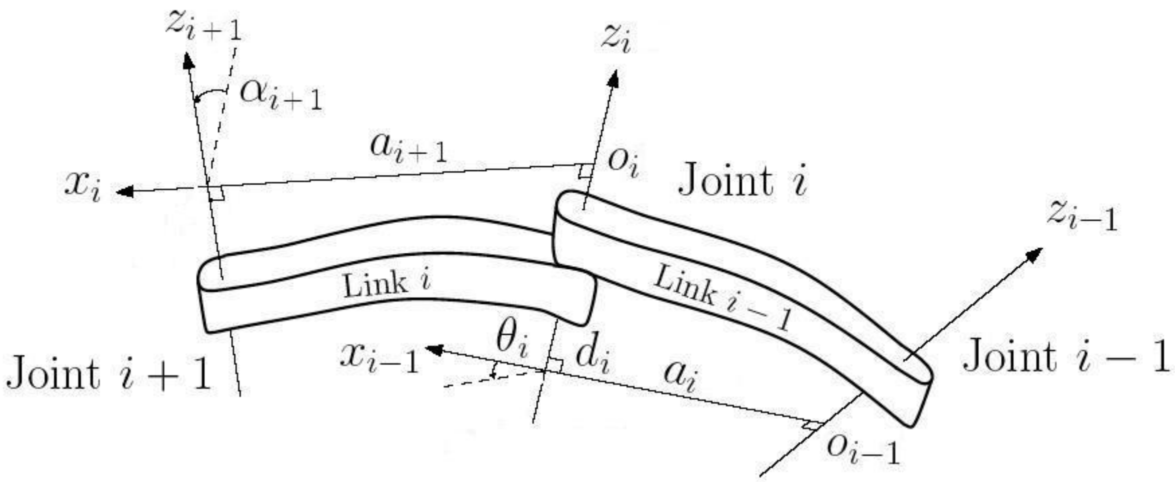

Figure 5. D-H parameters of a link i.e. $\theta_i, a_i, d_i, \alpha_i$

Step 8: The homogeneous transformation matrices A_i is determined by substituting the above parameters as shown in equation (1).

Step 9: Then the global transformation matrix ${}^0T_{End}$ is formed, as shown in equation (2).

This then gives the position and orientation of the tool frame expressed in base coordinates.

In this convention, each homogeneous transformation matrix A_i is represented as a product of four basic transformations:

$$A_i = Rot(z, \theta_i) Trans(z, d_i) Trans(x, a_i) Rot(x \ , \alpha_i)$$

$$= \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

Where four quantities $\theta_i, a_i, d_i, \alpha_i$ are parameter associate with link i and joint i. The four parameters $\theta_i, a_i, d_i, \alpha_i$ in the above equation are generally given name as joint angle, link length, link offset, and link twist.

3.2. The forward kinematics of a 5-DOF and 7-DOF Redundant manipulator.

3.2.1. Coordinate frame of a 5-DOF Redundant manipulator.

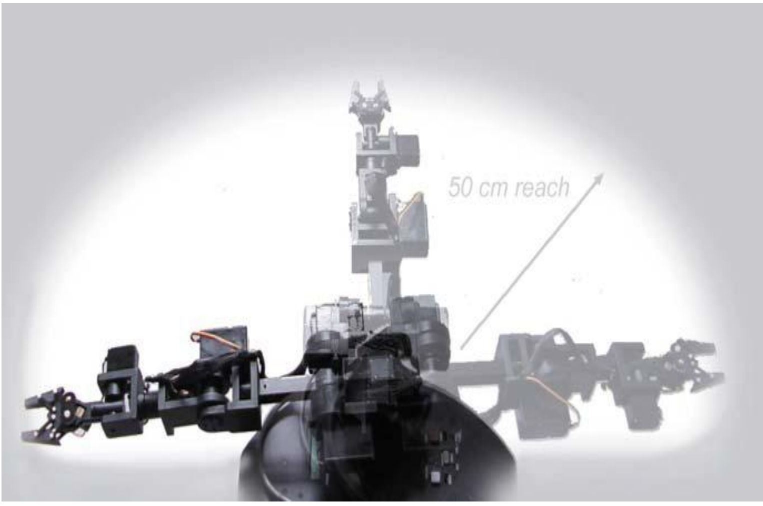

Figure 6. A Pioneer Arm Redundant manipulator

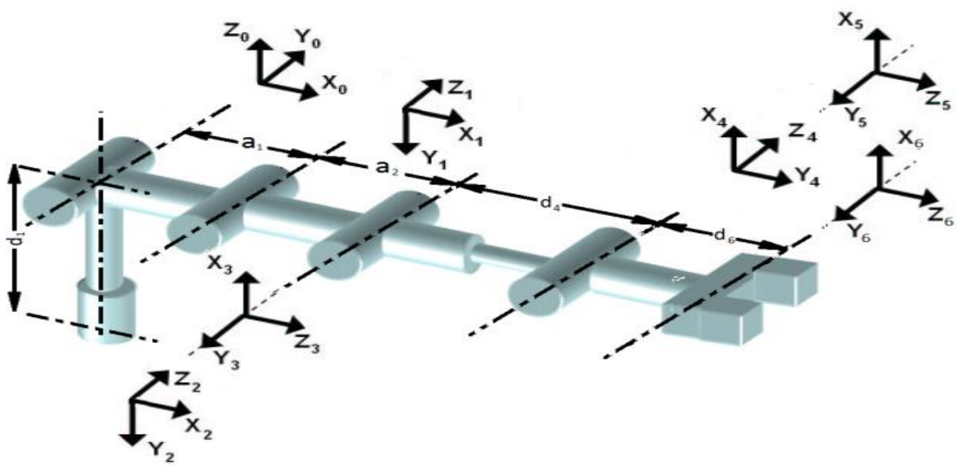

Figure 7.Coordinate frame for the 5-DOF Redundant manipulator

Table 1. Angle of rotation of joints

Types of Joint	Range of Rotation
Rotating base/ shoulder (θ_1)	$0^0 - 180^0$
Rotating elbow (θ_2)	$0^0 - 150^0$
Pivoting elbow (θ_3)	$0^0 - 150^0$
Rotating wrist (θ_4)	$0^0 - 85^0$
Pivoting wrist (θ_5)	$15^0 - 45^0$

3.2.2. Forward kinematics calculation of the 5-DOF Redundant manipulator.

Robot control actions are executed in the joint coordinates while robot motions are specified in the Cartesian coordinates. Conversion of the position and orientation of a robot manipulator end-effecter from Cartesian space to joint space is called as inverse kinematics problem, which is of fundamental importance in calculating desired joint angles for robot manipulator design and control. The Denavit-Hartenberg (DH) notation and methodology [56] is used to derive the kinematics of the 5-DOF Redundant manipulator. The coordinates frame assignment and the DH parameters are depicted in Figure 2 and listed in Table 2 respectively where (x_4, y_4, z_4) represents the local coordinate frames at the five joints respectively, (x_5, y_5, z_5) represents rotation coordinate frame at the end-effector where θi represents rotation about the Z-axis and transition on about the X- axis, d_i transition along the Z-axis, and a_i transition along the X-axis.

Table 2. The D-H parameters of the 5-DOF Redundant manipulator.

Frame	θ_i (degree)	d_i (mm)	a_i (mm)	α_i (degree)
O_0 - O_1	θ_1	$d_1 = 130$	a_1 = 70	-90
$O_1 - O_2$	θ_2	0	a_2 = 160	0
$O_2 - O_3$	$-90 + \theta_3$	0	0	-90
$O_3 - O_4$	θ_4	$d_4 = 140$	0	90
$O_4 - O_5$	θ_5	0	0	-90
$O_5 - O_6$	0	$d_6 = 120$	0	0

The transformation matrix Ai between two neighbouring frames O_{i-1} and O_i is expressed in equation (1) as,

$$A_i = Rot(z, \theta_i)Trans(z, d_i)Trans(x, a_i)Rot(x\ , \alpha_i)$$

$$= \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

By substituting the D-H parameters in Table 1 into equation (1), it can be obtained the individual transformation matrices A_1 to A_6 and the general transformation matrix from the first joint to the last joint of the 5-DOF Redundant manipulator can be derived by multiplying all the individual transformation matrices ($^0T_6$).

$$^0T_6 = A_1A_2A_3A_4A_5A_6 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

where (p_x, p_y, p_z) are the positions and $\{(n_x, n_y, n_z), (o_x, o_y, o_z), (a_x, a_y, a_z)\}$ are the orientations of the end-effector. The orientation and position of the end-effector can be calculated in terms of joint angles and the D-H parameters of the manipulator are shown in following matrix as:

$$\begin{bmatrix} c_1s_{23}c_4c_5 + s_1s_4c_5 + c_1c_{23}s_5 & -c_1s_{23}s_4 + s_1c_4 & -c_1s_{23}c_4s_5 - s_1s_4s_5 + c_1c_{23}c_5 & -d_6c_1s_{23}c_4s_5 - d_6s_1s_4s_5 + d_6c_1c_{23}c_5 + d_4c_1c_{23} + a_2c_1c_2 + a_1c_1 \\ s_1s_{23}c_4c_5 - c_1s_4c_5 + s_1c_{23}s_5 & -s1s_{23}s_4 - c_1c_4 & -s_1s_{23}c_4s_5 + c_1s_4s_5 + s_1c_{23}c_5 & -d_6s_1s_{23}c_4s_5 + d_6c_1s_4s_5 + d_6s_1c_{23}c_5 + d_4s_1c_{23} + a_2s_1c_2 + a_1s_1 \\ c_{23}c_4c_5 - s_{23}s_5 & -c_{23}s_4 & -c_{23}c_4s_5 - s_{23}c_5 & -d_6c_{23}c_4s_5 - d_6s_{23}c_5 - d_4s_{23} - a_2s_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

where $c_i = \cos(\theta_i), s_i = \sin(\theta_i), c_{23} = \cos(\theta_2 + \theta_3)$ and $s_{23} = \sin(\theta_2 + \theta_3)$

By equalizing the matrices in equations (2) and (3), the following equations are derived

$$p_x = -d_6c_1s_{23}c_4s_5 - d_6s_1s_4s_5 + d_6c_1c_{23}c_5 + d_4c_1c_{23} + a_2c_1c_2 + a_1c_1 \tag{4}$$

$$p_y = -d_6 s_1 s_{23} c_4 s_5 + d_6 c_1 s_4 s_5 + d_6 s_1 c_{23} c_5 + d_4 s_1 c_{23} + a_2 s_1 c_2 + a_1 s_1 \quad (5)$$

$$p_z = -d_6 c_{23} c_4 s_5 - d_6 s_{23} c_5 - d_4 s_{23} c_5 - d_4 s_{23} - a_2 s_2 + d_1 \quad (6)$$

$$n_x = c_1 s_{23} c_4 c_5 + s_1 s_4 c_5 + c_1 c_{23} s_5 \quad (7)$$

$$n_y = s_1 s_{23} c_4 c_5 - c_1 s_4 c_5 + s_1 c_{23} s_5 \quad (8)$$

$$n_z = c_{23} c_4 c_5 - s_{23} s_5 \quad (9)$$

$$o_x = -c_1 s_{23} s_4 + s_1 c_4 \quad (10)$$

$$o_y = -s_1 s_{23} s_4 - c_1 c_4 \quad (11)$$

$$o_z = -c_{23} s_4 \quad (12)$$

$$a_x = -c_1 s_{23} c_4 s_5 - s_1 s_4 c_5 + c_1 c_{23} c_5 \quad (13)$$

$$a_y = -s_1 s_{23} c_4 s_5 + c_1 s_4 s_5 + s_1 c_{23} c_5 \quad (14)$$

$$a_z = -c_{23} c_4 s_5 - s_{23} c_5 \quad (15)$$

From equation (4) to (15), the position and orientation of the 5-DOF Redundant manipulator end-effector can be calculated if all the joint angles are given. This is the solution to the forward kinematics.

3.2.3. Work space for the 5-DOF Redundant manipulator.

Considering all the D-H parameters, the x, y and z coordinates are calculated for 5-DOF Redundant manipulator End-effector using forward kinematics equation shown in equations 4-15. For solving the forward kinematics equations, the angles of rotation of the joints are taken as tabulated in Table 1. Figure 4 shows the workspace for 5-DOF Redundant manipulator.

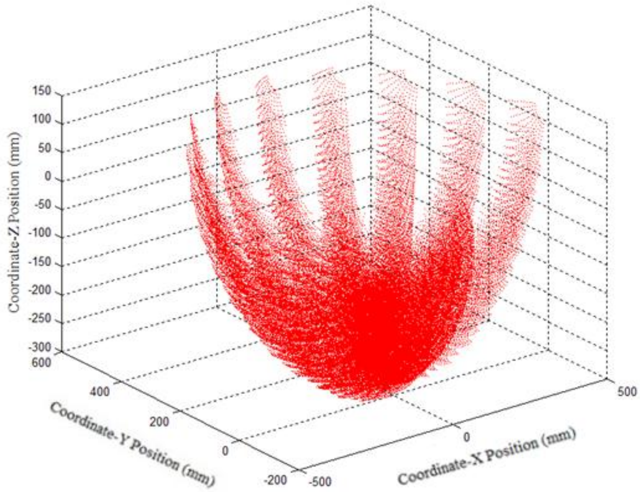

Figure 8. Work space for 5-DOF Redundant manipulator

3.2.4. Coordinate frame of a 7-DOF Redundant manipulator.

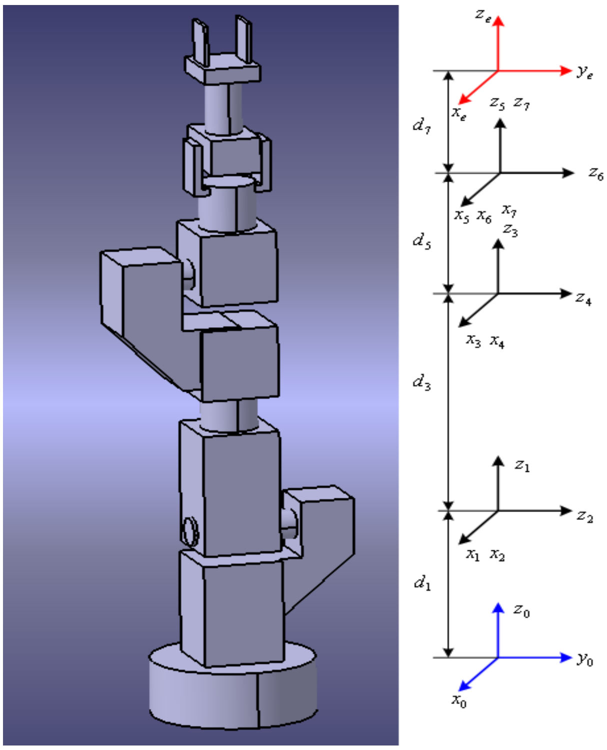

Figure 9. Coordinate frame for a 7-DOF Redundant manipulator

3.2.5. Forward kinematics calculation of the 7-DOF Redundant manipulator.

The D-H parameter for the 7-DOF Redundant manipulator is tabulated in Table 2.

Table 3. The D-H parameters of the 7-DOF Redundant manipulator

frame	Link	θ_i (degree)	d_i (cm)	a_i (cm)	α_i (degree)
$o_0 - o_1$	1	$\theta_1 = -270 \text{ to } 270$	$d_1 = 30$	0	0

$o_1 - o_2$	2	$\theta_2 = -110 \text{ to } 110$	0	0	-90
$o_2 - o_3$	3	$\theta_3 = -180 \text{ to } 180$	$d_3 = 35$	0	90
$o_3 - o_4$	4	$\theta_4 = -110 \text{ to } 110$	0	0	-90
$o_4 - o_5$	5	$\theta_5 = -180 \text{ to } 180$	$d_5 = 31$	0	90
$o_5 - o_6$	6	$\theta_6 = -90 \text{ to } 90$	0	0	-90
$o_6 - o_7$	7	$\theta_7 = -270 \text{ to } 270$	0	0	90
$o_7 - \text{End}$	End	_	$d_7 = 42$	0	0

By substituting the D-H parameters in Table 2 into equation (1), the individual transformation matrices A_1 to A_{End} can be obtained and the global transformation matrix $({}^0T_{End})$ from the first joint to the last joint of the 7-DOF Redundant manipulator can be derived by multiplying all the individual transformation matrices. So,

$$ {}^0T_{End} = A_1A_2A_3A_4A_5A_6A_7A_{End} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{16} $$

Where p_x, p_y, p_z are the positions and $\{(n_x, n_y, n_z), (o_x, o_y, o_z), \text{and } (a_x, a_y, a_z)\}$ are the orientations of the end-effector. The orientation and position of the end-effector can be calculated in terms of joint angles and the D-H parameters of the manipulator are shown in following equations:

$$ \begin{aligned} n_x = {} & (c_7c_6c_5 - s_7s_5)\{c_3c_4(c_1c_2 - s_1s_2) - s_4(c_1s_2 + s_1c_2)\} + (c_7s_5c_6 + s_7c_5)\{s_3(s_1s_2 - c_1c_2)\} \\ & + s_6c_7\{c_3s_4(s_1s_2 - c_1c_2) - c_4(c_1s_2 + s_1c_2)\} \end{aligned} \tag{17} $$

$$ \begin{aligned} n_y = {} & (c_7c_6c_5 - s_7s_5)\{c_3c_4(s_1c_2 + c_1s_2) + s_4(c_1c_2 - s_1s_2)\} + (c_7s_5c_6 + s_7c_5)\{-s_3(s_1c_2 + c_1s_2)\} \\ & + s_6c_7\{-c_3s_4(s_1c_2 + c_1s_2) + c_4(c_1c_2 - s_1s_2)\} \end{aligned} \tag{18} $$

$$ n_z = c_7c_6s_3c_4s_5 - c_7c_6s_5c_3 + c_7s_6s_3c_3 + c_7s_6s_3s_4 + s_7c_3c_5 - s_7s_3c_4s_5 \tag{19} $$

$$ \begin{aligned} o_x = {} & c_6\{c_3s_4(s_1s_2 - c_1c_2) - c_4(c_1s_2 + s_1c_2)\} - s_6c_5\{c_3c_4(c_1c_2 - s_1s_2) - s_4(c_1s_2 + s_1c_2)\} \\ & - s_6s_5\{s_3(s_1s_2 - c_1c_2)\} \end{aligned} \tag{20} $$

$$o_y = c_6\{-c_3s_4(s_1c_2+c_1s_2)+c_4(c_1c_2-s_1s_2)\}-s_6c_5\{c_3c_4(s_1c_2+c_1s_2)+s_4(c_1c_2-s_1s_2)\} \\ -s_6s_5\{-s_3(s_1c_2+c_1s_2)\} \quad (21)$$

$$o_z = s_6s_3c_4c_5+s_5c_3s_6+s_3c_4c_6 \quad (22)$$

$$a_x = (s_7c_6c_5+c_7s_5)\{c_3c_4(c_1c_2-s_1s_2)-s_4(c_1s_2+s_1c_2)\}+(s_7s_5c_6-c_7c_5)\{s_3(s_1s_2-c_1c_2)\} \\ +s_6s_7\{c_3s_4(s_1s_2-c_1c_2)-c_4(c_1s_2+s_1c_2)\} \quad (23)$$

$$a_y = (s_7c_6c_5+c_7s_5)\{c_3c_4(s_1c_2+c_1s_2)+s_4(c_1c_2-s_1s_2)\}+(s_7c_6s_5-c_7c_5)\{-s_3(s_1c_2+c_1s_2)\} \\ +s_7s_6\{-c_3s_4(s_1c_2+c_1s_2)+c_4(c_1c_2-s_1s_2)\} \quad (24)$$

$$a_z = s_7c_6s_3c_4c_5-s_7c_6s_5c_3+s_7s_6s_4s_3+c_7c_4s_3s_5-c_3c_5c_7 \quad (25)$$

$$p_x = \{d_7(s_7c_6c_5+c_7s_5)\}\{c_3c_4(c_1c_2-s_1s_2)-s_4(c_1s_2+s_1c_2)\}+\{d_7(s_7s_5c_6-c_7c_5)\}\{s_3(s_1s_2-c_1c_2)\} \\ +(d_7s_7s_6+d_5)\{c_3s_4(s_1s_2-c_1c_2)-c_4(c_1s_2+s_1c_2)\}+\{-d_3(c_1s_2+s_1c_2)\} \quad (26)$$

$$p_y = \{d_7(s_7c_6c_5+s_5c_7)\}\{c_3c_4(s_1c_2+c_1s_2)+s_4(c_1c_2-s_1s_2)\}+\{d_7(s_7s_5c_6-c_7c_5)\}\{-s_3(s_1c_2+c_1s_2)\} \\ +(d_7s_7s_6+d_5)\{-c_3s_4(s_1c_2+c_1s_2)+c_4(c_1c_2-s_1s_2)\} \quad (27)$$

$$p_z = d_7s_7s_3c_6c_5c_4-d_7s_7s_5c_6c_3+d_7s_7s_6s_4s_3+d_7s_5s_3c_7c_4-d_7c_7c_5c_3+d_5s_4s_3+d_1$$

(28)

From equation (17)-(28), are the position and orientation of the 7-DOF Redundant manipulator end-effector and the exact value of these equations can be calculated if all the joint angles are given. This is the solution to the forward kinematics.

3.2.6. Work space for the 7-DOF Redundant manipulator.

Considering all the D-H parameters, the x, y and z coordinates (i.e. End-effector coordinates) are calculated for 7-DOF Redundant manipulator using forward kinematics equation as shown in equations 17-28. For solving the forward kinematics equations, the angles of rotation of the joints are taken as tabulated in Table 2. Figure 6 shows the workspace for this manipulator.