UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probablity and Random Processes

Solutions 9
Spring 2006

Exam format

- The second midterm will be held in class on Thursday, April 13. You are permitted to bring a calculator, and two sides of hand-written notes on $8.5 \times 11''$ paper. (E.g., 1 sheet, both sides; or 2 sheets, 1 side only).
- Questions on the exam can be based on any material from Chapters 1 through to the END of chapter 4, as discussed in lectures #1 through #20, covered in homeworks #1 through #8, and discussion sections through Wednesday/Thursday, April 5/6, 2006.

Review problems

Problem 9.1
For X, a random variable uniformly distributed between -1 and 1, find the density function of Y where:

(a) $Y = \sqrt{|X|}$

(b) $Y = -\ln|X|$

(c) $Y = \sin(2\pi X)$

Solution:
We are given that x is a uniformly distributed random variable which takes values between -1 and 1. Because the area under the PDF of x must equal one, the height of the PDF must be $\frac{1}{2}$.

$$f_x(x_o) = \begin{cases} 0.5 & \text{if } -1 \leq x_o < 1 \\ 0 & \text{otherwise} \end{cases}$$

a) $y = \sqrt{|x|}$. The first step is to find the CDF of y, $P_{y\leq}(y_o)$. For a given value of y_o, we find that:

$$\begin{aligned} y = \sqrt{|x|} &\leq y_o \\ |x| &\leq {y_o}^2 \\ -{y_o}^2 \leq x &\leq {y_o}^2 \end{aligned}$$

So, $P_{y\leq}(y_o) = P_x(-{y_o}^2 \leq x_o \leq {y_o}^2)$. For values of $y_o \leq 0$ the CDF is 0 because the radical sign is, by convention, always the **positive** square root of the radicand. Thus, no real value of x can generate a negative value of y, and the probability of such an event is 0. For values of y_o between 0 and 1, the CDF of y is the integral of the PDF of x from $-{y_o}^2$ to ${y_o}^2$:

$$P_{y\le}(y_o) = \int_{-{y_o}^2}^{{y_o}^2} 0.5 dx_o = {y_o}^2.$$

For values of y_o greater than one, the above integral exhausts the entire PDF of x, so the value of the CDF is simply 1. Summarizing, then differentiating with respect to y_o:

$$P_{y\le}(y_o) = \begin{cases} 0 & y_o < 0 \\ {y_o}^2 & 0 \le y_o < 1 \\ 1 & 1 \le y_o \end{cases}$$

$$f_y(y_o) = \frac{d}{dy_o} P_{y\le}(y_o) = \begin{cases} 0 & y_o < 0 \\ 2y_o & 0 \le y_o < 1 \\ 0 & 1 \le y_o \end{cases}$$

b) $y = -\ln|x|$. Following the method in part a), we find that:

$$\begin{aligned} y = -\ln|x| &\le y_o \\ \ln|x| &\ge -y_o \\ |x| &\ge e^{-y_o}. \end{aligned}$$

$$\begin{aligned} P_{y\le}(y_o) &= P_x[(x_o \le -e^{-y_o}) \cup (x_o \ge e^{-y_o})] \\ &= 1 - P_x(-e^{-y_o} \le x_o \le e^{-y_o}) \\ &= 1 - \int_{-e^{-y_o}}^{e^{-y_o}} f_x(x_o) dx_o \end{aligned}$$

For values of $y_o \le 0$, $e^{-y_o} \ge 1$, so we are integrating over the entire non-zero range of $f_x(x_o)$, and the above integral takes a value of 1. The CDF at y_o is then 0. For values of $y_o > 0$, $e^{-y_o} < 1$, so no such shortcut is available. We get:

$$\begin{aligned} P_{y\le}(y_o) &= 1 - \int_{-e^{-y_o}}^{e^{-y_o}} f_x(x_o) dx_o \\ &= 1 - (0.5)[e^{-y_o} - (-e^{-y_o})] \\ &= 1 - e^{-y_o}. \end{aligned}$$

Summarizing, then differentiating with respect to y_o:

$$P_{y\le}(y_o) = \begin{cases} 0 & y_o < 0 \\ 1 - e^{-y_o} & 0 \le y_o \end{cases}$$

$$f_y(y_o) = \frac{d}{dy_o} P_{y\le}(y_o) = \begin{cases} 0 & y_o < 0 \\ e^{-y_o} & 0 \le y_o. \end{cases}$$

c) This problem was solved in Discussion Section 6

Problem 9.2

The receiver is an optical communications system uses a photodetector that counts the number of photons that arrive during the communication session. (The time of the communication session is 1 time unit.) The sender conveys information by either transmitting or not transmitting photons to the photodetector. The probability of transmitting is p. If she transmits, the number of photons X that she transmits during the session has a Poisson PMF with mean λ per time unit. If she does not transmit, she generates no photons.

Unfortunately, regardless of whether or not she transmits, there may still be photons arriving at the photodetector because of a phenomenon called shot noise. The number N of photons that arrive because of the shot noise has a Poisson PMF with mean μ. N and X are independent. The total number of photons counted by the photodetector is equal to the sum of the transmitted photons and the photons generated by the shot noise effect.

(a) What is the probability that the sender transmitted if the photodetector counted k photons?

(b) Before you know anything else about a particular communication session, what is your least squares estimate of the number of photons transmitted?

(c) What is the least squares estimate of the number of photons transmitted by the sender if the photodetector counted k photons?

(d) What is the best *linear* predictor for the number of photons transmitted by the sender as a function of k, the number of the detected photons?

Solution:

1. Let A be the event that the sender transmitted, and K be the number of photons counted by the photodetector. Using Bayes rule,

$$P(A \mid K=k) = \frac{p_{K|A}(k)P(A)}{p_K(k)} = \frac{p_{X+N}(k) \cdot p}{p_N(k) \cdot (1-p) + p_{X+N}(k) \cdot p}$$

The discrete random variables X and N are given by the following PMF's:

$$p_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x \geq 0$$

$$p_N(n) = \frac{\mu^n e^{-\mu}}{n!}, \quad n \geq 0$$

The sum of two independent Poisson random variables is also Poisson, with mean equal to the sum of the means of each of the random variables. This fact can be derived by looking at the product of the transforms of X and N. Therefore:

$$p_{X+N}(k) = \frac{(\lambda+\mu)^k e^{-(\lambda+\mu)}}{k!}, \quad k \geq 0$$

Thus,

$$P(A \mid K=k) = \frac{p \cdot \frac{(\lambda+\mu)^k e^{-(\lambda+\mu)}}{k!}}{p \cdot \frac{(\lambda+\mu)^k e^{-(\lambda+\mu)}}{k!} + (1-p) \cdot \frac{\mu^k e^{-\mu}}{k!}} = \frac{1}{1 + \frac{1-p}{p}\left(\frac{\mu}{\lambda+\mu}\right)^k e^{\lambda}}$$

2. Let S be the number of photons transmitted by the sender. Then with probability p, $S = X$, and with probability $1-p$, $S = 0$. The least squares estimate of the number of photons transmitted by the sender is simply the mean, in the absence of any additional information:
$$\hat{S}_1 = E[S] = p \cdot \lambda + (1-p) \cdot 0 = p\lambda.$$

3. The least squares predictor has a form
$$\hat{S}_2(k) = E[S \mid K = k].$$
Using Bayes rule,
$$\hat{S}_2(k) = \sum_{s=0}^{k} s p_{S|K}(s|k) = \sum_{s=0}^{k} s \frac{p_{K|S}(k|s) p_S(s)}{p_K(k)} = \frac{1}{p_K(k)} \sum_{s=0}^{k} s p_{K|S}(k|s) p_S(s).$$
From the definitions of S and K, the following are true:
$$\begin{aligned} p_S(s) &= \begin{cases} (1-p) + pe^{-\lambda}, & s = 0 \\ p\frac{\lambda^s e^{-\lambda}}{s!}, & s = 1, 2, \ldots \end{cases} \\ p_{K|S}(k|s) &= p_N(k-s) = \frac{\mu^{(k-s)} e^{-\mu}}{(k-s)!} \\ p_K(k) &= p\frac{(\lambda+\mu)^k e^{-(\lambda+\mu)}}{k!} + (1-p)\frac{\mu^k e^{-\mu}}{k!} \end{aligned}$$
In order to obtain the last expression, we observe that $K = S + N$ with probability p, and $K = N$ with probability $1-p$. Substituting into the formula above,
$$\begin{aligned} \hat{S}_2(k) &= \frac{1}{p_K(k)} \left[0 \cdot (1-p) \frac{\mu^{(k-0)} e^{-\mu}}{(k-0)!} + \sum_{s=0}^{k} s p \frac{\lambda^s e^{-\lambda}}{s!} \frac{\mu^{(k-s)} e^{-\mu}}{(k-s)!} \right] \\ &= \frac{1}{p_K(k)} p e^{-\lambda} e^{-\mu} \frac{(\lambda+\mu)^k}{k!} \sum_{s=0}^{k} s \frac{k!}{s!(k-s)!} \left(\frac{\lambda}{\lambda+\mu}\right)^s \left(\frac{\mu}{\lambda+\mu}\right)^{(k-s)} \\ &= \frac{1}{p_K(k)} p e^{-\lambda} e^{-\mu} \frac{(\lambda+\mu)^k}{k!} k \left(\frac{\lambda}{\lambda+\mu}\right) = k \left(\frac{\lambda}{\lambda+\mu}\right) \frac{pe^{-(\lambda+\mu)}(\lambda+\mu)^k}{k! p_K(k)} \\ &= k \left(\frac{\lambda}{\lambda+\mu}\right) \frac{1}{1 + \frac{1-p}{p}\left(\frac{\mu}{\lambda+\mu}\right)^k e^{\lambda}}. \end{aligned}$$
Thus
$$\hat{S}_2(k) = \frac{k\lambda}{\lambda+\mu} \frac{1}{1 + \frac{1-p}{p}\left(\frac{\mu}{\lambda+\mu}\right)^k e^{\lambda}}.$$
Note that as k increases, the estimator can be approximated by $\frac{k\lambda}{\lambda+\mu}$.

4. The linear least squares predictor has the form

$$\hat{S}_3(k) = E[S] + \frac{\mathrm{Cov}(S,K)}{\sigma_K^2}(k - E[K]) \tag{1}$$

Note that since X and N are independent, S and N are also independent.

$$\begin{aligned}
E[S] &= p\lambda \\
E[S^2] &= pE[X^2] + (1-p)(0) = p(\lambda^2 + \lambda) \\
\sigma_S^2 &= E[S^2] - (E[S])^2 = p(\lambda^2+\lambda) - (p\lambda)^2 = p(1-p)\lambda^2 + p\lambda. \\
\Rightarrow \quad E[K] &= E[S] + E[N] = p\lambda + \mu \\
\sigma_K^2 &= \sigma_S^2 + \sigma_N^2 \\
&= p(1-p)\lambda^2 + p\lambda + \mu.
\end{aligned}$$

Finally, we need to find $\mathrm{Cov}(S,K)$.

$$\begin{aligned}
\mathrm{Cov}(S,K) &= E[(S-E[S])(K-E[K])] \\
&= E[(S-E[S])(S-E[S]+N-E[N])] \\
&= E[(S-E[S])(S-E[S])] + E[(S-E[S])(N-E[N])] \\
&= \sigma_S^2 + E[(S-E[S])(N-E[N])] \\
&= \sigma_S^2 \\
&= p(1-p)\lambda^2 + p\lambda.
\end{aligned}$$

Note that we have used the fact that $(S-E[S])$ and $(N-E[N])$ are independent, and $E[(S-E[S])] = 0 = E[(N-E[N])]$.

Therefore, substituting all the numbers into the equation above, we get the linear predictor:

$$\hat{S}_3(k) = p\lambda + \frac{p(1-p)\lambda^2 + p\lambda}{p(1-p)\lambda^2 + p\lambda + \mu}\left(k - p\lambda - \mu\right).$$

The graph bellow illustrates the three estimators for a particular setting of the parameters $\lambda = 30, \mu = 5, p = 0.2$:

Problem 9.3

Computers have subroutines that can generate experimental values of a random variable X that is uniformly distributed in the interval $[0,1]$. Such a subroutine can be used to generate experimental values of a continuous random variable with given CDF $F(y)$ as follows: Each time X takes a value $x \in (0,1)$, we generate the unique value y for which $F(y) = x$. (We can neglect the zero probability event that X takes the value 0 or 1.)

(a) Show that the CDF $F_Y(y)$ of the random variable Y thus generated is indeed equal to the given $F(y)$.

(b) Describe how the procedure can be used to simulate an exponential random variable with parameter λ.

Solution:

1. For the random variable Y thus generated, we have

$$F_Y(y) = P(Y \le y) = P(X \le F(y)).$$

But since X is uniformly distributed in $[0,1]$, we have that $P(X \le F(y))$ is equal to $F(y)$.

2. The exponential PDF has the form $F(y) = 1 - e^{-\lambda y}$ for $y \ge 0$. Thus to generate values of Y, we should generate values $x \in (0,1)$ of a uniformly distributed random variable X, and set y to the value for which $1 - e^{-\lambda y} = x$, or $y = -\ln(1-x)/\lambda$.

Problem 9.4

Your friend Bob took EE 126 last fall and now supplements his income by gambling. When he visits the casino, he plays blackjack for Y hours, where Y is an exponentially-distributed random variable with mean 1. Dealers are changed every hour after he starts play and, for a given dealer, the rate of his earnings (in thousands of dollars per hour) is described by a Gaussian random variable with mean μ and variance σ^2.

Let us denote the integer part of a real number y by $\lfloor y \rfloor$ so, e.g., $\lfloor 2.8 \rfloor = 2$. Then, Bob's total earnings X is given by

$$X = \sum_{k=1}^{S} Z_k + Z_{S+1}T,$$

where $\{Z_k\}$ is an i.i.d. sequence of Gaussian random variables with mean μ and variance σ^2 (independent of Y), $S = \lfloor Y \rfloor$, and $T = Y - \lfloor Y \rfloor$.

You would like to know whether you should follow Bob's footsteps, so you conduct a probabilistic analysis of his earnings.

(a) Find $\mathbb{E}[X]$.

(b) Find $p_S(k)$, the PMF of S, for $k = 0, 1, \ldots$.

(c) Find $f_{T|S}(t|k)$, the conditional PDF of T given S, for $k = 0, 1, \ldots$.

You realize that, although it is difficult to obtain a description of the exact distribution of X, you can bound X by

$$U = \sum_{k=1}^{S+1} Z_k$$

and

$$V = \sum_{k=1}^{S} Z_k$$

from above and below, respectively.

(d) Find the PDFs or transforms of U and V.

Solution:

(a)

$$\begin{aligned}
E[X] &= E\left[\sum_{k=1}^{S} Z_k + Z_{S+1}T\right] \\
&= E\left[\sum_{k=1}^{S} Z_k\right] + E[Z_{S+1}]E[T] \\
&= E[Z_1]E[S] + E[Z_1]E[T] = E[Z_1](E[S] + E[T]) \\
&= E[Z_1]E[S+T] = E[Z_1]E[Y] \\
&= \mu.
\end{aligned}$$

(b)

$$\begin{aligned}
p_S(k) &= P(k \le Y < k+1) = F_Y(k+1) - F_Y(k) = (1 - e^{-(k+1)}) - (1 - e^{-k}) \\
&= e^{-k} - e^{-(k+1)} = e^{-k}(1 - e^{-1}).
\end{aligned}$$

(c) We have

$$f_Y(t|k) = f_{Y|S}(t+k|k) = \begin{cases} \frac{f_{T|S}(t+k)}{p_S(k)}, & \text{if } k \le t+k < k+1 \\ 0, & \text{otherwise} \end{cases}$$

Now, for $k \le t+k < k+1$ or, equivalently, $0 \le t < 1$,

$$\frac{f_{T|S}(t+k)}{p_S(k)} = \frac{e^{-(t+k)}}{e^{-k}(1-e^{-1})} = \frac{e^{-t}}{1-e^{-1}}$$

(d) We have

$$M_S(s) = \sum_{k=0}^{\infty} e^{sk} e^{-k}(1 - e^{-1}) = (1 - e^{-1}) \sum_{k=0}^{\infty} e^{(s-1)k} = (1 - e^{-1}) \frac{1}{1 - e^{s-1}}$$

and

$$M_{S+1}(s) = E[e^{s(S+1)}] = e^s M_S(s) = (1 - e^{-1}) \frac{e^s}{1 - e^{s-1}}.$$

In addition,

$$M_Z(s) = e^{(\sigma^2 s^2/2) + \mu s}.$$

Therefore,

$$M_U(s) = (1 - e^{-1}) \frac{e^{(\sigma^2 s^2/2) + \mu s}}{1 - e^{(\sigma^2 s^2/2) + \mu s - 1}}$$

and

$$M_V(s) = (1 - e^{-1}) \frac{1}{1 - e^{(\sigma^2 s^2/2) + \mu s - 1}}.$$

Problem 9.5

An absent-minded professor schedules two student appointments for the same time; unfortunately, the professor is only able to meet with one student at a time. The appointment durations are independent and exponentially distributed with mean thirty minutes. Suppose that the first student arrives on time, but the second student arrives 5 minutes late. Determine the expected value and the variance of the time between the arrival of the first student and the departure of the second student (**Note:** The expected value is NOT 60 minutes nor 65 minutes.)

Solution:

Let t be the time between the arrival of the first student and the departure of the second student, t_1 be the time between the arrival of the first student and the beginning of the second appointment, and t_2 be the time between the beginning and end of the second appointment. Then, since t_2 is exponentially distributed with mean 30 minutes, we have

$$\mathrm{E}[t] = \mathrm{E}[t_1 + t_2] = \mathrm{E}[t_1] + \mathrm{E}[t_2] = \mathrm{E}[t_1] + 30.$$

and similarly for the variance (because t_1 and t_2 are independent)

$$\sigma_t^2 = \sigma_{t_1}^2 + \sigma_{t_2}^2 = \sigma_{t_1}^2 + 900$$

Let A be the event the 1st appointment takes less than 5 minutes. Then

$$\mathrm{E}[t_1] = P\{A\}\mathrm{E}[t_1|A] + P\{A'\}\mathrm{E}[t_1|A'].$$

If we're given that the first appointment takes more than 5 minutes, the *remaining* length of the appointment after those 5 minutes is exponentially distributed. Thus $\mathrm{E}[t_1|A'] = 5 + 30$ and

$$\begin{aligned}
\mathrm{E}[t_1] &= \left(\int_0^5 \frac{1}{30}e^{-t/30}dt\right)(5) + \left(\int_5^\infty \frac{1}{30}e^{-t/30}dt\right)(35) \\
&= \left(1 - e^{-5/30}\right)(5) + e^{-5/30}(35) = 30.394.
\end{aligned}$$

So $\mathrm{E}[t] = \boxed{60.394}$.

We can use the variance formula from the iterated expectations handout to compute the variance of t_1. For notational convenience, we define a new random variable a. We let a be 1 when event A is true and 0 otherwise.

$$\begin{aligned}
\mathrm{Var}(t_1) &= \mathrm{Var}(E[t_1|a]) + E[\mathrm{Var}(t_1|a)] \\
&= (35 - 30.394)^2(e^{-5/30}) + (5 - 30.394)^2(1 - e^{-5/30}) + E[\mathrm{Var}(t_1|a)] \\
&= 116.955 + E[\mathrm{Var}(t_1|a)] \\
&= 116.955 + (e^{-5/30})(900) + (1 - e^{-5/30})(0) \\
&= 878.79
\end{aligned}$$

So, $\sigma_t^2 = \boxed{1778.79}$.

Problem 9.6

Every night at closing time, Virginie the chef leaves her restaurant and walks in a straight line down the street such that, at time t after she starts walking, her position X_t has a Gaussian distribution with mean vt and variance $\sigma_X^2 t$. On most nights, Virginie is accompanied by her cat, whose position at time t, Z_t, is such that $Z_t - X_t = Y_t$, where Y_t is a Gaussian-distributed random variable with mean 0 and variance σ_Y^2. The random variables X_t and Y_t are independent.

(a) Find the PDF of the cat's position Z_t as a function of t. What is the correlation coefficient between the cat's position and Virginie's position as a function of t?

(b) On one night, at time τ after closing time, Virginie's cat is observed at position z. What is the linear least-squares estimate of Virginie's position at time τ given that $Z_\tau = z$?

(c) Show that the Bayes least-squares estimate of Virginie's position at time τ given $Z_\tau = z$ is the same as the linear least-squares estimate you obtained in part (b).

Solution:

(a) We have $Z_t = X_t + Y_t$, so Z_t is a Gaussian-distributed random variable with mean vt and variance $\sigma_X^2 t + \sigma_Y^2$. The correlation coe.cient between Z_t and X_t is

$$\rho(Z_t, X_t) = \frac{E[(Z_t - E[Z_t])(X_t - E[X_t])]}{\sqrt{Var(Z_t)Var(X_t)}}$$

and

$$\begin{aligned} E[(Z_t - E[Z_t])(X_t - E[X_t])] &= E[(Z_t - E[Z_t])(X_t - E[X_t])] \\ &= E[(X_t + Y_t - E[X_t] - E[Y_t])(X_t - E[X_t])] \\ &= E[(X_t - E[X_t])(X_t - E[X_t])] + E[(Y_t - E[Y_t])(X_t - E[X_t])] \\ &= Var(X_t) + Cov(X_t, Y_t) \\ &= Var(X_t) \end{aligned}$$

so

$$\rho(Z_t, X_t) = \frac{Var(X_t)}{\sqrt{Var(Z_t)Var(X_t)}} = \sqrt{\frac{\sigma_X^2 t}{\sigma_X^2 t + \sigma_Y^2}}$$

(b) This question asks for the linear least-squares estimate of X_τ given $Z_\tau = z$, which is given by

$$E[X_\tau] + \rho(Z_\tau, X_\tau)\frac{\sigma_{X_\tau}}{\sigma_{Z_\tau}}(z - E[Z_\tau]) = vt + \frac{\sigma_X^2 t}{\sigma_X^2 t + \sigma_Y^2}(z - vt) = \frac{\sigma_X^2 t}{\sigma_X^2 t + \sigma_Y^2} z + \frac{\sigma_Y^2}{\sigma_X^2 t + \sigma_Y^2} vt$$

(c) The least-squares estimate of X_τ given $Z_\tau = z$ is given by

$$E[X_\tau | Z_\tau = z] = \int_{\infty}^{\infty} x f_{X_\tau | Z_\tau}(x|z) dx$$

Now, using Bayes rule,

$$
\begin{aligned}
f_{X_\tau|Z_\tau}(x|z) &= \frac{f_{Z_\tau|X_\tau}(z|x) f_{X_\tau}(x)}{f_{Z_\tau}(z)} = \frac{f_{Y_\tau}(z-x) f_{X_\tau}(x)}{f_{Z_\tau}(z)} \\
&= \frac{\frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-\frac{(y-x)^2}{2\sigma_Y^2}} \frac{1}{\sqrt{2\pi\sigma_X^2\tau}} e^{-\frac{(x-v\tau)^2}{2\sigma_X^2\tau}}}{\frac{1}{\sqrt{2\pi(\sigma_Y^2+\sigma_X^2\tau)}} e^{-\frac{(z-v\tau)^2}{2(\sigma_Y^2+\sigma_X^2\tau)}}} \\
&= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\sigma_Y^2+\sigma_X^2\tau}{\sigma_Y^2\sigma_X^2\tau}} e^{-\frac{1}{2}\left(\frac{z^2-2zx+x^2}{\sigma_Y^2}+\frac{x^2-2xv\tau+x^2\tau^2}{\sigma_X^2\tau^2}\right)+\frac{(z-v\tau)^2}{2(\sigma_Y^2+\sigma_X^2\tau)}}
\end{aligned}
$$

simplifying the exponent we obtain

$$
f_{X_\tau|Z_\tau}(x|z) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\sigma_Y^2+\sigma_X^2\tau}{\sigma_Y^2\sigma_X^2\tau}} e^{-\frac{1}{2}\left(\frac{\sigma_Y^2+\sigma_X^2\tau}{\sigma_Y^2\sigma_X^2\tau}\left[x-\frac{\sigma_X^2 t}{\sigma_X^2 t+\sigma_Y^2}z+\frac{\sigma_Y^2}{\sigma_X^2 t+\sigma_Y^2}vt\right]^2\right)}
$$

therefore

$$
E[X_\tau|Z_\tau=z] = \frac{\sigma_X^2 t}{\sigma_X^2 t+\sigma_Y^2} z + \frac{\sigma_Y^2}{\sigma_X^2 t+\sigma_Y^2} vt
$$