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ENGINEERING MATHEMATICS - III

- 17MAT31

GEODESICS ON A PLANE

Let $y = y(x)$ be the curve joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the XY plane.

The length element ds in the XY plane is given by $ds = \sqrt{(dx)^2 + (dy)^2}$

$\therefore$ The total length of the curve is given by

$$I = \int_{x_1}^{x_2} ds.$$

$$I = \int_{x_1}^{x_2} \sqrt{(dx)^2 + (dy)^2}$$

$$I = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$I = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx.$$

$f = \sqrt{1 + y'^2}$ is independent of x & y from 3rd particular case of Euler Equation $y'' = 0$. $y = ax + b$ = Straight line.

⇒ GEODESICS ON A RIGHT CIRCULAR CYLINDER

The arc length element in cylindrical polar co-ordinates (R, ϕ, z) where $x = \phi$

$y = z \Rightarrow y' = z'$

$$ds = \sqrt{(dR)^2 + (R\,d\phi)^2 + (dz)^2}$$

where, R = radius $= a$ $R = a$ $dR = 0$.

$$ds = \sqrt{a^2(d\phi)^2 + (dz)^2}$$

$$I = \int_{\phi_1}^{\phi_2} \sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2}\, d\phi$$

$$I = \int_{\phi_1}^{\phi_2} \sqrt{a^2 + z'^2}\, d\phi$$

$f = \sqrt{a^2 + z'^2}$ f is independent of ϕ & z

from 3rd particular case $z'' = 0$.

$$\Rightarrow z = C_1\phi + C_2$$

⇒ GEODESICS ON A SPHERE

The arc length element ds in spherical polar coordinates (r, θ, ϕ).

$$ds = \sqrt{(dr)^2 + (r d\theta)^2 + (r \sin\theta \, d\phi)^2}$$

$$r = a \quad , \quad dr = 0.$$

$$ds = \sqrt{0 + a^2 (d\theta)^2 + a^2 (\sin\theta)^2 (d\phi)^2}$$

$$I = a \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2} \, d\theta$$

$$I = a \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta \, \phi'^2} \, d\theta.$$

$f = \sqrt{1 + \sin^2\theta \, \phi'^2}$, f is independent of ϕ from

2nd particular case $= \dfrac{\partial f}{\partial y'} = \dfrac{\partial y}{\partial \delta'} = C$

$$\Rightarrow \frac{\partial f'}{\partial \phi'} \Rightarrow \frac{\sin^2\theta \, \cancel{2} \phi}{\cancel{2} \sqrt{1 + \sin^2\theta \, \phi'^2}} = C.$$

$$\Rightarrow \sin^2\theta \, \phi' = C \left(\sqrt{1 + \sin^2\theta \, \phi'^2} \right)$$

$$\Rightarrow \sin^4\theta . \, \phi'^2 = C^2 \left[1 + \sin^2\theta \, \phi'^2 \right]$$

⑨

$$\sin^4\theta \cdot \phi'^2 - c^2 \sin^2\theta \cdot \phi'^2 = c^2$$

$$\phi'^2 [\sin^4\theta - c^2 \sin^2\theta] = c^2$$

$$\phi'^2 \sin^2\theta [\sin^2\theta - c^2] = c^2$$

$$\phi'^2 = \frac{c^2}{\sin^2\theta [\sin^2\theta - c^2]}$$

$$\left(\frac{d\phi}{d\theta}\right)^2 = \frac{c^2 \operatorname{cosec}^2\theta}{\sin^2\theta \cdot c^2}$$

$$\frac{d\phi}{d\theta} = \frac{c \operatorname{cosec}\theta}{\sqrt{\sin^2\theta - c^2}}$$ Sq both sides

$$d\phi = \frac{c \operatorname{cosec}\theta}{\sin\theta\sqrt{1 - c^2 \operatorname{cosec}^2\theta}} d\theta.$$

$$d\phi = \frac{c \operatorname{cosec}^2\theta}{\sqrt{1 - c^2(1 + \cot^2\theta)}} d\theta$$

$$\phi = \int \frac{c \operatorname{cosec}^2\theta}{\sqrt{1 - c^2 - c^2\cot^2\theta}} d\theta$$

Put $c \cot\theta = t$

$$-c \operatorname{cosec}^2\theta \, d\theta = dt$$

$$\theta = \int \frac{-dt}{\sqrt{(1-c^2)-t^2}}$$

$$= \int \frac{-dt}{\sqrt{(\sqrt{1-c^2})^2 - t^2}}$$

$$\phi = \cos^{-1} \frac{t}{\sqrt{1-c^2}} + b$$

$$\phi = \cos^{-1} \left[\frac{c \cot\theta}{\sqrt{1-c^2}}\right] + b$$

To show that geodesics on a sphere are arc of its great circle.

$$\boxed{\cos(A-B) = \cos A \cos B + \sin A \sin B.}$$

$$\cos(\phi - b) = \frac{c \cot\theta}{\sqrt{1-c^2}}$$

$$\cos\phi \cdot \cos b + \sin b \cdot \sin\phi = k \cdot \cot\theta.$$

$$\cot\theta = A\cos\phi + B \sin\phi$$

$$\cos\theta = A\cos\phi \cdot \sin\theta + B \sin\phi \sin\theta$$

$$a\cos\theta = A\, \underbrace{a\cos\phi \cdot \sin\theta} + B \cdot \underbrace{a \sin\phi \cdot \sin\theta}$$

In spherical PC system the parametric equations are

$$x = a\sin\theta \cos\phi$$

$$y = a \sin\theta \sin\phi$$

$$z = a\cos\theta.$$

(6)

$$Z = Ax + By$$

represents a plane through the origin

$\therefore$ Geodesics on a sphere are arc of a great circle.

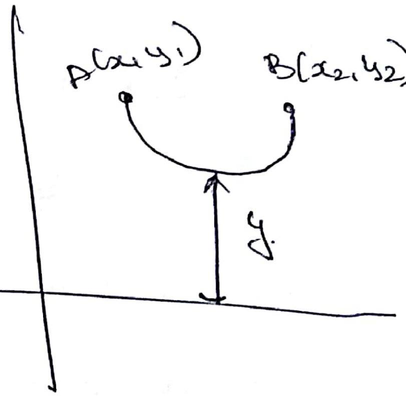
⇒) HANGING CHAIN PROBLEM

Catenary: Catenary is a curve which minimizes the gravitational potential energy or which minimizes the surface area of revolution.

$$y = C \cosh \left[\frac{x+a}{c}\right]$$

Show that the shape of a hanging chain or cable is a catenary.

Proof: Let $A(x_1 y_1)$ & $B(x_2 y_2)$ are two fixed points of the hanging cable. If ds is the elementary arc length & ρ is the density of the cable then mass of the element $= \rho\, ds$.



The potential energy of this element $= mgh$

$m =$ mass

$g =$ gravity acceleration

$h =$ height

$\therefore = \rho\, ds\, g y.$

∴ The total potential energy of the cable is given as

$$\int_A^B \rho \, ds \, g y$$

$$T = \int_A^B \rho g y \sqrt{(dx)^2 + (dy)^2}$$

$$= \rho g \int_{x_1}^{x_2} y \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \rho g \int_{x_1}^{x_2} y \sqrt{1+y'^2} \, dx$$

$f = y\sqrt{1+y'^2}$ independent of x. from particular case only case 1 of E.Eⁿ

$$f - y' \frac{\partial f}{\partial y'} = c$$

$$y\sqrt{1+y'^2} - y' \frac{y \, 2y'}{2\sqrt{1+y'^2}} = c.$$

$$y(1+y'^2) - yy'^2 = c\sqrt{1+y'^2}$$

$$y + yy'^2 - yy'^2 = c\sqrt{1+y'^2}$$

$$y^2 = c^2(1+y'^2)$$

$$y^2 - c^2 = c^2 y'^2$$

$$y'^2 = \frac{y^2-c^2}{c^2}$$

$$y' = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{\sqrt{y^2-c^2}} = \frac{1}{c}\,dx$$

$$\int \frac{dy}{\sqrt{y^2-c^2}} = \int \frac{1}{c}\,dx$$

$$\cosh^{-1}\left(\frac{y}{c}\right) = \frac{1}{c}x + k$$

$$\frac{y}{c} = \cosh\left[\frac{x+kc}{c}\right]$$

$$y = c.\cosh\left[\frac{x+a}{c}\right]$$

Hence proved//