Module-03. Integral Calculus.

Reduction formulae: It is basically a recurrence relation which reduces integral of functions of higher degree to lower degree.

1. $\int \sin^n x\, dx = -\frac{\sin^{n-1}x \cdot \cos x}{n} + \frac{n-1}{n} I_{n-2}$.

2. $\int \cos^n x\, dx = \frac{\cos^{n-1}x \cdot \sin x}{n} + \frac{n-1}{n} I_{n-2}$.

3. $\int \sin^m x \cdot \cos^n x\, dx = -\frac{\sin^{m-1}x \cdot \cos^{n+1}x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$

4. $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \times K$.

   where $K = \pi/2$ only when n is even.

5. $\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{[(m-1)(m-3)\cdots][(n-1)(n-3)\cdots]}{(m+n)(m+n-2)\cdots} \times K$.

   where $K = \pi/2$ only when m & n are even & $K=1$ otherwise.

Problems:

15MATDIP31

1. Evaluate $\int_0^{\pi} \sin^5(x/2)\, dx$

put $x/2 = t \Rightarrow x = 2t$, $dx = 2\,dt$.

at $x=0$, $t=0$; $x=\pi$, $t=\pi/2$.

sol^n: $= 2\int_0^{\pi/2} \sin^5 t\, dt$.

$= 2 \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{16}{15}$ | ∵ reduction formula.

15MATDIP31

2. Evaluate $\int_0^{\pi/6} \sin^6 3x\, dx$ using reduction formula.

put $3x = t \Rightarrow 3\,dx = dt$, $dx = \frac{1}{3}dt$.

at $x=0$, $t=0$; $x=\pi/6$, $t=\pi/2$.

sol^n: $= \frac{1}{3}\int_0^{\pi/2} \sin^6 t\, dt$

$= \frac{1}{\cancel{3}} \cdot \frac{5}{6} \cdot \frac{\cancel{3}}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$ | ∵ reduction formula.

$= \frac{5\pi}{96}$

15MATDIP31

3. Evaluate $\int_0^{\pi} x \cos^6 x\, dx$

sol^n: Let $I = \int_0^{\pi} x \cos^6 x\, dx$

$= \int_0^{\pi} (\pi - x) \cos^6(\pi - x)\, dx$

| ∵ property, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(2)

$$I = \int_0^{\pi} (\pi - x) \cos^6 x \, dx$$ | ∵ Allied angles (or) II quadrant. | ∵ $\cos^6(-x) = \cos^6 x$.

$$I = \pi \int_0^{\pi} \cos^6 x \, dx - \int_0^{\pi} x \cdot \cos^6 x \, dx$$

$$I = \pi \cdot 2 \int_0^{\pi/2} \cos^6 x \, dx - I$$ | ∵ property. $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ & $\int_0^{\pi} x \cos^6 x \, dx = I$.

$$\cancel{2} I = \cancel{2} \pi \int_0^{\pi/2} \cos^6 x \, dx$$

$$I = \pi \frac{5}{\cancel{6}_2} \cdot \frac{\cancel{3}}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5\pi^2}{32}$$

15MATDIP31

4) Evaluate $\int_0^{\pi/2} \sin^3 x \cdot \cos^7 x \, dx$.

Sol^n:
$$= \frac{[(\cancel{2})][(\cancel{6})(\cancel{4})(\cancel{2})]}{10 \times \cancel{8}_4 \times \cancel{6} \times \cancel{4} \times \cancel{2}} = \frac{1}{40}$$ | ∵ reduction formula.

15MATDIP31

5) Evaluate $\int_0^1 x^6 \sqrt{1-x^2} \, dx$ using reduction formula.

Sol^n:
let $I = \int_0^1 x^6 \sqrt{1-x^2} \, dx$

| put $x = \sin\theta$
$dx = \cos\theta \, d\theta$
at $x = 0$, $\theta = 0$
$x = 1$, $\theta = \pi/2$

$$I = \int_0^{\pi/2} \sin^6\theta \cdot \cos\theta \cdot \cos\theta \, d\theta$$

| ∵ $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$.

$$I = \int_0^{\pi/2} \sin^6\theta \cdot \cos^2\theta \, d\theta$$

$$I = \frac{[(5)(\cancel{3})(1)][(1)]}{8 \times \cancel{6}_2 \times 4 \times 2} \cdot \frac{\pi}{2} = \frac{5\pi}{128}$$

15MATDIP31

6) Evaluate $\int_0^{2a} x^2 \sqrt{2ax - x^2} \, dx$

Sol^n:
let $I = \int_0^{2a} x^2 \sqrt{2ax - x^2} \, dx$

Put $x = 2a \sin^2\theta$
$dx = 4a \sin\theta \cdot \cos\theta \, d\theta$.
at $x = 0$, $\theta = 0$
$x = 2a$, $\theta = \pi/2$

| $\sqrt{2ax - x^2} = \sqrt{4a^2 \sin^2\theta - 4a^2 \sin^4\theta}$
$= \sqrt{4a^2 \sin^2\theta (1 - \sin^2\theta)}$
$= \sqrt{4a^2 \sin^2\theta \cdot \cos^2\theta}$
$= 2a \sin\theta \cdot \cos\theta$.

$$I = \int_0^{\pi/2} 4a^2 \sin^4\theta \cdot 2a \sin\theta \cdot \cos\theta \cdot 4a \sin\theta \cdot \cos\theta \, d\theta .$$

$$I = 32a^4 \int_0^{\pi/2} \sin^6\theta \cdot \cos^2\theta \, d\theta$$

$$I = \cancel{32}^{\,4} a^4 \frac{[(5)(\cancel{3})(1)][1]}{\cancel{8} \times \cancel{6}_{2} \times \cancel{4} \times 2} \cdot \frac{\pi}{2} = \frac{5\pi a^4}{8}$$

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7) Evaluate $\int_0^a \frac{x^3 dx}{\sqrt{a^2-x^2}}$

Sol^n: Let $I = \int_0^a \frac{x^3 dx}{\sqrt{a^2-x^2}}$

Put $x = a\sin\theta$

$dx = a\cos\theta \, d\theta$

at $x=0, \quad \theta = 0$

$x = a, \quad \theta = \pi/2$

$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta}$

$= \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta}$

$= a\cos\theta .$

$$I = \int_0^{\pi/2} \frac{a^3 \sin^3\theta}{\cancel{a\cos\theta}} \cdot \cancel{a\cos\theta} \, d\theta$$

$$I = a^3 \int_0^{\pi/2} \sin^3\theta \, d\theta = a^3 \cdot \frac{2}{3} = \frac{2a^3}{3} .$$

301

8) Evaluate $\int_0^2 \frac{x^4}{\sqrt{4-x^2}} dx .$

Sol^n: Let $I = \int_0^2 \frac{x^4}{\sqrt{4-x^2}} dx .$

Put $x = 2\sin\theta .$

$dx = 2\cos\theta \, d\theta$

at $x = 0, \quad \theta = 0$

$x = 2, \quad \theta = \pi/2$

$\sqrt{4-x^2} = \sqrt{4 - 4\sin^2\theta}$

$= \sqrt{4(1-\sin^2\theta)} = \sqrt{4\cos^2\theta}$

$= 2\cos\theta .$

$$I = \int_0^{\pi/2} \frac{2^4 \sin^4\theta}{\cancel{2\cos\theta}} \cdot \cancel{2\cos\theta} \, d\theta$$

$$= 2^4 \int_0^{\pi/2} \sin^4\theta \, d\theta = \cancel{16}^{\,4} \times \frac{3}{\cancel{4}} \times \frac{1}{\cancel{2}} \times \frac{\pi}{\cancel{2}} = 3\pi .$$

301

9) Evaluate $\int_0^4 x^{3/2} \cdot (4-x)^{5/2} dx .$

4.

Sol^n: Let $I = \int_0^4 x^{3/2} \cdot (4-x)^{5/2} dx$.

put $x = 4\sin^2\theta$

$dx = 4(2\sin\theta \cdot \cos\theta) d\theta = 8\sin\theta \cdot \cos\theta\, d\theta$

at $x=0, \theta = 0$

$x = 4, \theta = \pi/2$

$(4-x)^{5/2} = (4-4\sin^2\theta)^{5/2}$
$= (4(1-\sin^2\theta))^{5/2}$
$= (2^2 \cdot \cos^2\theta)^{5/2}$
$= 2^5 \cos^5\theta$

& $x^{3/2} = (4\sin^2\theta)^{3/2}$
$= (2^2 \cdot \sin^2\theta)^{3/2} = 2^3 \cdot \sin^3\theta$.

$I = \int_0^{\pi/2} 2^3 \cdot \sin^3\theta \cdot 2^5 \cdot \cos^5\theta \cdot 8\sin\theta \cdot \cos\theta\, d\theta$

$= 2^3 \times 2^5 \times 8 \int_0^{\pi/2} \sin^4\theta \cdot \cos^6\theta\, d\theta$.

$= \cancel{8}^{4} \times \cancel{32}^{16} \times \cancel{8} \times \frac{[(3)(\cancel{1})] [(5)(\cancel{3})(1)]}{\cancel{2} 10 \times \cancel{8} \times \cancel{6} \times \cancel{4} \times \cancel{2}} \times \frac{\pi}{2} = \frac{\cancel{16} \times 3 \times \cancel{8}}{\cancel{10}} \times \frac{\pi}{\cancel{2}} = 12\pi$

10) Evaluate $\int_0^\infty \frac{x^2}{(1+x^2)^{7/2}} dx$

Sol^n: Let $I = \int_0^\infty \frac{x^2}{(1+x^2)^{7/2}} dx$.

$(1+x^2)^{7/2} = (1+\tan^2\theta)^{7/2} = (\sec^2\theta)^{7/2} = \sec^7\theta$.

put $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$.

at $x = 0, \theta = 0$.

$x = \infty, \theta = \pi/2$

$\therefore I = \int_0^{\pi/2} \frac{\tan^2\theta \cdot \cancel{\sec^2}\theta}{\sec^{\cancel{7}}\theta} d\theta = \int_0^{\pi/2} \frac{\tan^2\theta}{\sec^5\theta} d\theta = \int_0^{\pi/2} \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^5\theta}} d\theta$.

$I = \int_0^{\pi/2} \sin^2\theta \cdot \cos^3\theta\, d\theta = \frac{(1)[2]}{5 \times 3 \times 1} = \frac{2}{15}$ [$\because$ reduction formula.

11) Evaluate. $\int_{-\pi/2}^{\pi/2} \cos^8 x\, dx$.

Sol^n: Let $I = \int_{-\pi/2}^{\pi/2} \cos^8 x\, dx$

property: $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ if $f(-x) = f(x)$

[$\because \cos^8(-x) = \cos^8 x$ $\therefore$ even.

$I = 2\int_0^{\pi/2} \cos^8 x\, dx$.

$I = \cancel{2} \times \frac{7}{8} \times \frac{5}{\cancel{6}_2} \times \frac{\cancel{3}}{4} \times \frac{1}{2} \times \frac{\pi}{\cancel{2}} = \frac{35\pi}{16 \times 8} = \frac{35\pi}{128}$

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Double integrals: It can be evaluated by expressing them in terms of two single integrals.

If the region R is bounded by curves $x = x_1, x = x_2$ & $y = y_1, y = y_2$

then $I = \iint_R f(x,y)\,dx.dy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx.dy.$

case i) Let x_1, x_2 & y_1, y_2 be constants, then we can first integrate w.r.t x & then w.r.t y or vice versa.

case ii) Let x_1, x_2 be constants & y_1, y_2 be functions of x, then we first integrate w.r.t y treating x as constant.

$$\text{i.e., } I = \int_{x_1}^{x_2}\left[\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\right]dx.$$

case iii) Let y_1, y_2 be constants & x_1, x_2 be functions of y, then we first integrate w.r.t x treating y as constant.

$$\text{i.e, } I = \int_{y_1}^{y_2}\left[\int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\right]dy.$$

Problems.

15MATDIP31

1) Evaluate $\int_0^1\int_x^{\sqrt{x}} xy\,dy\,dx.$

Soln: Let $I = \int_0^1\int_x^{\sqrt{x}} xy\,dy\,dx = \int_0^1 x\,\frac{y^2}{2}\Big|_x^{\sqrt{x}}dx = \frac{1}{2}\int_0^1 x(x - x^2)\,dx.$

$$I = \frac{1}{2}\int_0^1 (x^2 - x^3)\,dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}\left[\left(\frac{1}{3} - \frac{1}{4}\right) - 0\right]$$

$$I = \frac{1}{2}\left[\frac{4-3}{12}\right] = \frac{1}{24}$$

15MATDIP31

2) Evaluate $\int_0^1\int_0^{\sqrt{1-y^2}} x^3 y\,dx\,.dy.$

Soln: Let $I = \int_0^1\int_0^{\sqrt{1-y^2}} x^3 y\,dx\,dy. = \int_0^1 y\cdot\frac{x^4}{4}\Big|_0^{\sqrt{1-y^2}}dy = \frac{1}{4}\int_0^1 \left(y\left(\sqrt{1-y^2}\right)^4 - 0\right)dy$

$$I = \frac{1}{4}\int_0^1 y\left(\left(\cancel{\sqrt{1-y^2}}\right)^2\right)^2 dy = \frac{1}{4}\int_0^1 y(1 + y^4 - 2y^2)\,dy.$$

⑥
$$I = \frac{1}{4}\int_0^1 (y + y^5 - 2y^3)\,dy$$
$$I = \frac{1}{4}\left[\frac{y^2}{2} + \frac{y^6}{6} - \cancel{2}\frac{y^4}{\cancel{4}2}\right]_0^1$$
$$I = \frac{1}{4}\left[\left(\frac{1}{2} + \frac{1}{6} - \frac{1}{2}\right) - 0\right] = \frac{1}{24}$$

15MATDIP31

3) Evaluate $\int_1^2\int_0^{2-y} xy\,dx\,dy$.

Sol^n: let $I = \int_1^2\int_0^{2-y} xy\,dx\,dy = \int_1^2 y\frac{x^2}{2}\Big|_0^{2-y} dy = \frac{1}{2}\int_1^2 y\left[(2-y)^2 - 0\right]dy.$

$$I = \frac{1}{2}\int_1^2 y(4 + y^2 - 4y)\,dy = \frac{1}{2}\int_1^2 (4y + y^3 - 4y^2)\,dy.$$
$$I = \frac{1}{2}\left[\cancel{4}^2\frac{y^2}{\cancel{2}} + \frac{y^4}{4} - 4\frac{y^3}{3}\right]_1^2 = \frac{1}{2}\left[\left(8 + \frac{\cancel{16}^4}{\cancel{4}} - \frac{32}{3}\right) - \left(2 + \frac{1}{4} - \frac{4}{3}\right)\right]$$
$$I = \frac{1}{2}\left[12 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right] = \frac{1}{2}\left[10 - \frac{28}{3} - \frac{1}{4}\right]$$
$$I = \frac{1}{2}\left[\frac{120 - 112 - 3}{12}\right] = \frac{1}{2}\left[\frac{120 - 115}{12}\right] = \frac{1}{2}\left[\frac{5}{12}\right] = \frac{5}{24}.$$

301

4) Evaluate : $\int_0^{2a}\int_0^{\sqrt{2ax - x^2}} xy\,dy.dx.$

Sol^n: Let $I = \int_0^{2a}\int_0^{\sqrt{2ax-x^2}} xy\,dy.dx. = \int_0^{2a} x.\frac{y^2}{2}\Big|_0^{\sqrt{2ax-x^2}} dx.$

$$I = \frac{1}{2}\int_0^{2a}\left(x.(\sqrt{2ax - x^2})^2 - 0\right)dx = \frac{1}{2}\int_0^{2a}(2ax^2 - x^3)\,dx$$
$$I = \frac{1}{2}\left[2a\frac{x^3}{3} - \frac{x^4}{4}\right]_0^{2a} = \frac{1}{2}\left[\left(2a\frac{(8a^3)}{3} - \frac{2^4.a^4}{4}\right) - 0\right]$$
$$I = \frac{1}{2}\left[\frac{16a^4}{3} - \frac{16a^4}{4}\right] = \frac{1}{2}\left[\frac{4(16a^4) - 3(16a^4)}{12}\right]$$
$$I = \frac{1}{2}\left[\frac{64a^4 - 48a^4}{12}\right] = \frac{1}{\cancel{2}}\left(\frac{\cancel{16}^{8\,2}a^4}{\cancel{12}_3}\right) = \frac{2a^4}{3}.$$

301

5) Evaluate : $\int_1^2\int_1^3 xy^2\,dx.dy.$

Sol^n: Let $I = \int_1^2 \int_1^3 x\, y^2\, dx \cdot dy$

$$I = \int_1^2 y^2 \frac{x^2}{2}\Big|_1^3 dy = \frac{1}{2}\int_1^2 y^2(9-1)\,dy = \frac{1}{2}\int_1^2 8y^2\,dy$$

$$I = \frac{\cancel{8}^4}{\cancel{2}}\int_1^2 y^2 dy = 4\left[\frac{y^3}{3}\right]_1^2 = \frac{4}{3}[8-1] = \frac{4}{3}(7).$$

$$I = \frac{28}{3}.$$

6.) Evaluate $\int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx \cdot dy$

Sol^n: Let $I = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} 1 \cdot dx \cdot dy = \int_0^{4a} x\Big|_{y^2/4a}^{2\sqrt{ay}} dy = \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy.$

$$I = \int_0^{4a}\left(2a^{1/2}y^{1/2} - \frac{1}{4a}y^2\right)dy = 2a^{1/2}\frac{y^{3/2}}{3/2}\Big|_0^{4a} - \frac{1}{4a}\frac{y^3}{3}\Big|_0^{4a}$$

$$I = \frac{2\times 2a^{1/2}}{3}y^{3/2}\Big|_0^{4a} - \frac{1}{12a}y^3\Big|_0^{4a} = \frac{4a^{1/2}}{3}\left[(4a)^{3/2}-0\right] - \frac{1}{12a}\left[(4a)^3-0\right]$$

$$\because a^{1/2}a^{3/2} = a^{\frac{1}{2}+\frac{3}{2}} = a^{4/2}$$

$$I = \frac{4a^{1/2}}{3}(2^{\cancel{2}})^{3/\cancel{2}}a^{3/2} - \frac{\cancel{64}^{16}a^{\cancel{3}}}{{}_3\cancel{12a}} = \frac{4a^{\cancel{4/2}\,2}\,2^3}{3} - \frac{16a^2}{3}$$

$$I = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

Triple integrals: It can be evaluated by expressing it in terms of three integrals in the form,

$$I = \iiint_V f(x,y,z)\,dx\,dy\,dz = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx.$$

If i) x_1, x_2 are constants,
ii) y_1, y_2 are constants (or) functions of x,
iii) z_1, z_2 are constants (or) functions of x & y then above integral is evaluated as,
first w.r.t z keeping x & y fixed (constant), then the resulting expression is integrated w.r.t y treating x as constant finally the obtained result is integrated w.r.t x.

i.e., $\int_{x_1}^{x_2}\left[\int_{y_1(x)}^{y_2(x)}\left\{\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\right\}dy\right]dx.$

Integration is carried out from innermost bracket to the outermost bracket.

<u>NOTE</u> : If all the limits are constants then integration can be performed in any order.

Problems :

301

1) Evaluate: $\int_0^1\int_0^1\int_0^1 (x+y+z)\,dx\,dy\,dz$

Sol^n: Let $I = \int_0^1\int_0^1\int_0^1 (x+y+z)\,dx\,dy\,dz = \int_0^1\int_0^1 \frac{x^2}{2}+(y+z)x\Big|_0^1 dy\,dz$

$I = \int_0^1\int_0^1\left(\frac{1}{2}+(y+z)-0\right)dy.dz = \int_0^1 \frac{1}{2}y+\frac{y^2}{2}+zy\Big|_0^1 dz$

$I = \int_0^1\left(\left(\frac{1}{2}+\frac{1}{2}+z\right)-0\right)dz = \int_0^1 (1+z)\,dz \quad \Big|\because \frac{1}{2}+\frac{1}{2}=1.$

$I = z+\frac{z^2}{2}\Big|_0^1 = \left(1+\frac{1}{2}\right)-0 = \underline{\underline{\frac{3}{2}}}.$

15MATDIP31

2) Evaluate ; $\int_0^3\int_0^2\int_0^1 (x+y+z)\,dz\,dx\,dy.$

Sol^n: Let $I = \int_0^3\int_0^2\int_0^1 (x+y+z)\,dx\,dy\,dz = \int_0^3\int_0^2 \frac{x^2}{2}+(y+z)x\Big|_0^1 dy\,dz$

$I = \int_0^3\int_0^2\left(\frac{1}{2}+(y+z)-0\right)dy\,dz = \int_0^3 \frac{1}{2}y+\frac{y^2}{2}+zy\Big|_0^2 dz$

$I = \int_0^3\left(\left(\frac{\cancel{2}}{\cancel{2}}+\frac{4}{\cancel{2}}^{2}+2z\right)-0\right)dz = \int_0^3 (3+2z)\,dz = 3z+\frac{\cancel{2}z^2}{\cancel{2}}\Big|_0^3$

$I = (3(3)+9)-0 = 9+9 = \underline{\underline{18}}.$

3) Evaluate : $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz.$

Sol^n: Let $I = \int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz = \int_0\int_0\int_0 e^x e^y e^z\,dx\,dy\,dz$

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$I = \int_0^1\int_0^1 e^y e^z [e^x]_0^1 \, dy\, dz = \int_0^1\int_0^1 e^y e^z [e^1 - e^0] \, dy\, dz$ $\quad | \because e^0 = 1$

$I = (e-1)\int_0^1\int_0^1 e^y e^z \, dy\, dz = (e-1)\int_0^1 e^z [e^y]_0^1 \, dz$

$I = (e-1)\int_0^1 e^z (e^1 - e^0) \, dz = (e-1)^2 \int_0^1 e^z \, dz = (e-1)^2 [e^z]_0^1$

$I = (e-1)^2 (e^1 - e^0) = (e-1)^3$

15MATDIP31

4.) Evaluate $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z} \, dz\, dy\, dx$ (or) $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z} \, dx.dy.dz$.

Sol^n: Let $I = \int_0^a\int_0^x\int_0^{x+y} e^x e^y e^z \, dz.dy.dx = \int_0^a\int_0^x e^x e^y [e^z]_0^{x+y} \, dy\, dx$.

$I = \int_0^a\int_0^x e^x e^y [e^{x+y} - e^0] \, dy.dx = \int_0^a\int_0^x (e^{2x} e^{2y} - e^x e^y) \, dy.dx$ $\quad | \because e^{x+y} = e^x.e^y$, $e^x.e^x = e^{2x}$

$I = \int_0^a \left[e^{2x} \cdot \frac{e^{2y}}{2} - e^x e^y \right]_0^x dx$

$I = \int_0^a \left[\left(\frac{e^{2x}}{2} e^{2x} - e^x e^x \right) - \left(\frac{e^{2x}}{2} e^0 - e^x e^0 \right) \right] dx$

$I = \int_0^a \left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x \right) dx = \int_0^a \left(\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^x \right) dx$.

$I = \left[\frac{1}{2}\left(\frac{e^{4x}}{4}\right) - \frac{3}{2}\left(\frac{e^{2x}}{2}\right) + e^x \right]_0^a = \left(\frac{1}{8}(e^{4a}) - \frac{3}{4} e^{2a} + e^a \right) - \left(\frac{1}{8} e^0 - \frac{3}{4} e^0 + e^0 \right)$

$I = \frac{1}{8} e^{4a} - \frac{3e^{2a}}{4} + e^a - \frac{1}{8} + \frac{3}{4} - 1$

$I = \frac{e^{4a} - 6e^{2a} + 8e^a - 1 + 6 - 8}{8} = \frac{1}{8}(e^{4a} - 6e^{2a} + 8e^a - 3)$

301

5.) Evaluate: $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz \, dz\, dy\, dx$.

Sol^n: Let $I = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz \, dz\, dy\, dx = \int_0^1\int_0^{\sqrt{1-x^2}} xy \frac{z^2}{2}\Big|_0^{\sqrt{1-x^2-y^2}} dy\, dx$.

(10).

$$I = \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} xy\,(1-x^2-y^2)\,dy\,dx \qquad \Big|\ \because z^2 = \left(\sqrt{1-x^2-y^2}\right)^2 = 1-x^2-y^2.$$

$$I = \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} (xy - x^3y - xy^3)\,dy\,dx = \frac{1}{2}\int_0^1 \left[x\frac{y^2}{2} - x^3\frac{y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{1-x^2}}$$

$$I = \frac{1}{2}\int_0^1 \left[\left(\frac{x}{2}(1-x^2) - \frac{x^3}{2}(1-x^2) - \frac{x}{4}\left(1-x^2\right)^2\right) - 0\right]dx \qquad \Big|\ \because y^4 = (y^2)^2 = (1-x^2)^2$$

$$\Big|\ \because (1-x^2)^2 = 1+x^4-2x^2$$

$$I = \frac{1}{2}\int_0^1 \left(\frac{x}{2} - \underline{\frac{x^3}{2} - \frac{x^3}{2}} + \underline{\frac{x^5}{2}} - \frac{x}{4} \mp \underline{\frac{x^5}{4}} + \frac{2x^3}{4}\right)dx.$$

$$\Big|\ \because -\frac{x^3}{2} - \frac{x^3}{2} = \frac{-2x^3}{2}$$

$$\frac{x^5}{2} - \frac{x^5}{4} = \frac{2x^5 - x^5}{4} = \frac{x^5}{4}.$$

$$I = \frac{1}{2}\int_0^1 \left(\frac{x}{2} - \frac{\cancel{2}x^3}{\cancel{2}} + \frac{x^5}{4} - \frac{x}{4} + \frac{2x^3}{4}\right)dx$$

$$I = \frac{1}{2}\left[\frac{x^2}{2\times2} - \frac{x^4}{4} + \frac{x^6}{4\times6} \mp \frac{x^2}{4\times2} + \frac{2x^4}{4\times4}\right]_0^1$$

$$\Big|\ 8\,\underline{|24, 8, 16}$$
$$3 \quad 1 \quad 2$$
$$\therefore LCM = \underline{48}.$$

$$I = \frac{1}{2}\left[\left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} + \frac{1}{24} - \frac{1}{8} + \frac{2}{16}\right) - 0\right]$$

$$I = \frac{1}{2}\left(\frac{2 - \cancel{6} + \cancel{6}}{48}\right) = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{48}\right) = \underline{\underline{\frac{1}{48}}}.$$

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