

Properties of Regular Stochastic Matrix :-

Regular stochastic matrix :- A stochastic matrix P is said to be regular stochastic matrix if all the entries of some power " P^n " are positive.

Ex :- $A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

Consider ; $A^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$

$\therefore A$ is Regular stochastic matrix, $n=3$

Properties of Regular Stochastic matrix :-

① P has a unique fixed point, $x = (x_1, x_2, \dots, x_n)$ such that ;

$xP = x$

② P has a finite unique fixed probability vector, $v = (v_1, v_2, \dots, v_n)$ such that ; $vP = v$, where, $v_i = \sum_{i=1}^n x_i$.

③ P^2, P^3, \dots approaches the matrix V^x (then the sequence of vectors vP, vP^2, \dots) whose each rows are fixed probability vector. v .

④ If u is any probability vector, then the sequence of vectors ; uP, uP^2, \dots approaches the unique fixed probability vector v .

Problems :-

1) If $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is a stochastic matrix & $v = [v_1, v_2]$ is a probability vector, show that vA is also probability vector.

Soln :-

By data : $a_1 + a_2 = 1$, $b_1 + b_2 = 1$, $v_1 + v_2 = 1$

$$\therefore vA = [v_1 \ v_2] \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = [v_1 a_1 + v_2 b_1 \quad v_1 a_2 + v_2 b_2]$$

We have to prove that ; $(v_1 a_1 + v_2 b_1) + (v_1 a_2 + v_2 b_2) = 1$.

$$\text{LHS} = v_1(a_1 + a_2) + v_2(b_1 + b_2) = v_1 \cdot 1 + v_2 \cdot 1 = v_1 + v_2 = 1$$

Thus, vA is also a "probability vector."

② P.T with reference to 2 second order stochastic matrices that their product is also a stochastic matrix.

Soln :- let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ & $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ be a stochastic matrix.

$$\text{Hence, we have; } \left. \begin{array}{l} a_{11} + a_{12} = 1, \quad b_{11} + b_{12} = 1 \\ a_{21} + a_{22} = 1, \quad b_{21} + b_{22} = 1. \end{array} \right\} \text{--- (1)}$$

$$\therefore A \cdot B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$\text{We have to s.T; } a_{11}b_{11} + a_{12}b_{21} + a_{11}b_{12} + a_{12}b_{22} = 1 \quad \sim \text{--- (2)}$$

$$\& a_{21}b_{11} + a_{22}b_{21} + a_{21}b_{12} + a_{22}b_{22} = 1 \quad \sim \text{--- (3)}$$

LHS of (2); can be written as ;

$$\Rightarrow a_{11}(b_{11} + b_{12}) + a_{12}(b_{21} + b_{22}) = 1$$

$$\Rightarrow a_{11} \cdot 1 + a_{12} \cdot 1 = \underline{a_{11} + a_{12} = 1}, \text{ by using (1).}$$

LHS of (3) \Rightarrow

$$a_{21}(b_{11} + b_{12}) + a_{22}(b_{21} + b_{22})$$

$$\Rightarrow a_{21} \cdot 1 + a_{22} \cdot 1 = 1 \Rightarrow \underline{a_{21} + a_{22} = 1}$$

Thus, AB is a "stochastic matrix".

③ Find the unique fixed probability vector of the regular stochastic matrix; $A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$

Soln :- $A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$, $v = [x, y]$

We have to find; $v = (x, y)$ where; $x + y = 1$; $\Rightarrow vA = v$

ie, $[x, y] \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = [x, y]$

$\Rightarrow [3/4x + 1/2y \quad 1/4x + 1/2y] = [x, y]$

$\Rightarrow 3/4x + 1/2y = x$ (1) & $1/4x + 1/2y = y$ (2)

We can solve either of 2 equations, by using; $y = 1 - x$.

Using; $y = 1 - x$ in Eqn (1)

We have; $3/4x + 1/2(1 - x) = x \Rightarrow 3x + 2 - 2x = 4x$
 $\therefore x = 2/3$

Hence, $y = 1 - x = 1/3 = y$

$\therefore v = (x, y) = (2/3, 1/3)$

Thus, $(2/3, 1/3)$ is the unique fixed probability vector.

④ Find the unique fixed probability vector for the regular stochastic matrix: $A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$

Soln :- we have to find $v = (x, y, z)$, where; $x + y + z = 1$

$\Rightarrow vA = v$

$\Rightarrow [x, y, z] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [x, y, z]$

$\Rightarrow [y/6 \cdot x + y/2 + \frac{2z}{3}, \frac{y}{3} + \frac{z}{3}] = [x, y, z]$

$$\Rightarrow \frac{y}{6} = x, \quad x + \frac{y}{2} + \frac{2z}{3} = y, \quad \frac{y}{3} + \frac{z}{3} = z$$

$$\text{ie, } \underline{y = 6x}, \quad 6x + 3y + 4z = 6y, \quad y + z = 3z$$

$$\underline{y = 6x} \quad 6x - 3y + 4z = 0 \quad y - 2z = 0$$

Using $\underline{y = 6x}$ & $z = 1 - x - y = 1 - x - 6x = 1 - 7x$ in: $6x - 3y + 4z = 0$

We have; $6x - 18x + 4 - 28x = 0$.

$$\therefore \boxed{x = 1/10}$$

$$\text{Hence, } \boxed{y = 6/10} \quad \boxed{z = 3/10}$$

Thus, the required unique fixed probability vector v is given by;

$$\boxed{v = (1/10, 6/10, 3/10)}$$

⑤ Show that $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is Regular stochastic matrix.

$$\underline{\text{Soln}} :- P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = P^2 \cdot P = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$P^4 = P^3 \cdot P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$P^5 = P^4 \cdot P = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

\therefore We observe that in: P^5 all entries are positive.

Thus, P is a regular stochastic matrix.

MARKOV CHAINS:-

A stochastic process which is such that the generation of the probability distribution depend only on the present state is called "Markov chain".

Transition matrix:- A square matrix of order " m ", in which all the entries, (probabilities (p_{ij})) which are non-zero real numbers is a transition matrix.

Higher transition probability / n -step transition matrix:-

The probability that the system changes from the state a_i to state a_j in exactly " n " steps is denoted by " $P_{ij}^{(n)}$ ", the matrix formed by probabilities $P_{ij}^{(n)}$ is called " n -step transition matrix".

Stationary distribution of regular Markov chains:-

A markov chain is said to be "Regular", if the associated transition probability matrix P is "Regular".

Absorbing state of a Markov chain:-

In a markov chain, if process reaches to a certain state after which it continues to remain in the same state is "Absorbing state of a Markov chain".

Problem :-

- 1) The transition matrix P of a Markov chain is given by;
$$\begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$$
 with the initial probability distribution: $P^{(0)} = [1/4, 3/4]$
Define & find the following :-
- ① $P_{21}^{(2)}$
 - ② $P_{12}^{(2)}$
 - ③ $P^{(2)}$
 - ④ $P_1^{(2)}$

Soln :-

① $P_{21}^{(2)}$ is the probability of moving from state a_2 to state a_1 in 2 steps; This can be obtained from 2-step transition matrix P^2 .

$$\text{Consider; } P^2 = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} = \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix} = \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} \end{bmatrix}$$

$$\therefore P_{21}^{(2)} = 9/16 = \boxed{3/8 = P_{21}^{(2)}}$$

② $P_{12}^{(2)}$ is the probability of moving from state a_1 to a_2 in 2 steps: $\boxed{P_{12}^{(2)} = 3/8}$

③ $P^{(2)}$ is the probability distribution of system after 2 steps :-
$$\therefore P^{(2)} = P^{(0)} P^2 = [1/4, 3/4] \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix} = \begin{bmatrix} 37/64 & 27/64 \end{bmatrix}$$

$$\text{ie; } P^{(2)} = \begin{bmatrix} 37/64 & 27/64 \end{bmatrix} = \begin{bmatrix} P_1^{(2)} & P_2^{(2)} \end{bmatrix}$$

④ $P_1^{(2)}$ is the probability that process is in the state a_1 after 2 steps. \therefore ie; $\boxed{P_1^{(2)} = 37/64}$

② The "transition probability matrix" of a Markov chain is given by; $P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$ and the initial probability distribution is:

$$P^{(0)} = [1/2, 1/2, 0]$$

Find: ① $P_{13}^{(2)}$ ② $P_{23}^{(2)}$ ③ $P^{(2)}$ ④ $P_1^{(2)}$.

Soln:- First, we find the 2 step transition matrix; P^2 .

$$\therefore P^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} = \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 7/16 & 1/8 & 3/16 \end{bmatrix} = \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} & P_{13}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} & P_{23}^{(2)} \\ P_{31}^{(2)} & P_{32}^{(2)} & P_{33}^{(2)} \end{bmatrix}$$

$$\textcircled{1} \Rightarrow \boxed{P_{13}^{(2)} = 3/8}$$

$$\textcircled{2} \Rightarrow \boxed{P_{23}^{(2)} = 1/2}$$

$$\text{Consider; } P^{(2)} = P^{(0)} \cdot P^2 = \begin{bmatrix} 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 7/16 & 1/8 & 3/16 \end{bmatrix}$$

$$\therefore \boxed{P^{(2)} = \begin{bmatrix} 7/16 & 1/8 & 7/16 \end{bmatrix}}$$

$$\therefore \boxed{P_1^{(2)} = 7/16}$$

③. P.T. the Markov chain whose "tpm" is: $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is irreducible.

Soln:- We shall show that; P is a regular stochastic matrix, For convenience, we shall write the given matrix in the form;

$$P = \frac{1}{6} \begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$\text{Consider ; } P^2 = \frac{1}{36} \begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 18 & 6 & 12 \\ 9 & 21 & 6 \\ 9 & 12 & 15 \end{bmatrix}$$

Since, all the entries in P^2 are "positive"

We conclude that "t.p.m" is Regular.

Hence, the markov chain having "t.p.m. P is "irreducible".

Word problems :-

① A student's study habits are as follows, If he studies one night, he is 70% sure not to study next night, On the other hand if he doesnot study one night, he is 60% sure not to study the next night, In the long run how often does he study?

Soln :- The state space of the system is : $\{A, B\}$

where; A : Studying , B : Not studying

The associated transition matrix P is as follows;

$$P = \begin{matrix} & \begin{matrix} A & B \end{matrix} \\ \begin{matrix} A \\ B \end{matrix} & \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \end{matrix}$$

In order to find the happening in the long run we have to find the unique fixed probability vector v of P ,

ie, to find: $v = (x, y) \Rightarrow vP = v$

where; $x+y=1$.

$$\text{ie, } [x, y] \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [x, y]$$

$$\left| \begin{array}{l} 70\% = \frac{70}{100} = 0.7 \\ 60\% = \frac{60}{100} = 0.6 \\ (100-70)\% = 30\% = \frac{30}{100} = 0.3 \\ (100-60)\% = 40\% = \frac{40}{100} = 0.4 \end{array} \right.$$

$$\Rightarrow [0.3x + 0.4y \quad 0.7x + 0.6y] = [x \quad y]$$

$$\Rightarrow 0.3x + 0.4y = x \quad \& \quad 0.7x + 0.6y = y.$$

Using: $y = 1 - x$ in the 1st of the equation, we have;

$$\Rightarrow 0.3x + 0.4(1-x) = x \quad (\text{or}) \quad 1-x = 0.4$$

$$\therefore \boxed{x = 4/11}$$

Since, $\boxed{x = 4/11}$, $\boxed{y = 7/11}$, $v = (4/11 \quad \& \quad 7/11) = (P_A, P_B)$.

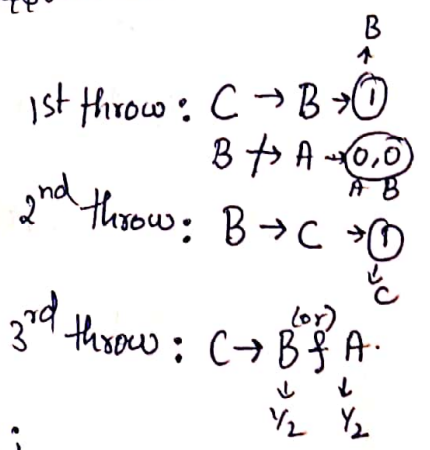
Thus, we Conclude that in the long run, the students will study 4/11 of the time (or) 36.36% of the time

- ② Three boys: A, B, C are throwing ball to each other, "A" always throws the ball to B & B always throws the ball to C, C is just likely to throw the ball to B as to A. If C was the 1st person to throw the ball, find the probabilities that after 3 throws:
- ① "A" has the ball
 - ② "B" has the ball
 - ③ "C" has the ball.

Soln:- State space = {A, B, C} and the associated "t.p.m".

is as follows:-

$$P = \begin{matrix} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{matrix}$$



Initially, if C has the ball, the associated initial probability vector is given by;

$$p^{(0)} = (0, 0, 1)$$

Since, the probabilities are desired after 3 throws,

we find: $p^{(3)} = p^{(0)} \cdot p^{(3)}$

$$\therefore P^2 = P \cdot P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$\therefore P^3 = P^2 \cdot P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$\text{WKF; } P^{(3)} = P^{(0)} \cdot P^3 = \begin{bmatrix} 0 & 0 & 1 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$\therefore P^{(3)} = \begin{bmatrix} 1/4 & 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} P_A^{(3)} & P_B^{(3)} & P_C^{(3)} \end{bmatrix}$$

Thus, after 3 throws, the probability that, the ball is with
A is : 1/4 , B is : 1/4 & C is : 1/2

- ③ A gambler's luck follows a pattern, If he wins a game, the probability of winning the next game is : 0.6, However if he loses a game, the probability of losing the next game is 0.7. There is an even chance of gambler winning the 1st game, If so:
- What is the probability of he winning 2nd game?
 - What is the probability of he winning 3rd game?
 - In the long run, how often he will win?

Soln :- State space = { win (W), lose (L) } and the associated t.p.m is as follows :-

$$P = \begin{matrix} & \begin{matrix} W & L \end{matrix} \\ \begin{matrix} W \\ L \end{matrix} & \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} \end{matrix} = \frac{1}{10} \begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix}$$

Probability of winning of 1st game is 1/2

\therefore initial probability vector : $P^{(0)} = \underline{\underline{(\frac{1}{2} \ \frac{1}{2})}}$

$$\textcircled{a} \text{ Now, } P^{(1)} = P^{(0)} \cdot P = \frac{1}{2} [1, 1] \cdot \frac{1}{10} \begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix} \\ = \frac{1}{20} [9, 11]$$

Hence, $P^{(1)} = [9/20 \ 11/20] = [P^{(w)}, P^{(L)}]$

Thus, the probability that, winning of 2nd game is : 9/20

$$\textcircled{b} \ P^{(2)} = P^{(1)} \cdot P = \frac{1}{20} [9 \ 11] \cdot \frac{1}{10} \begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix} \\ = \frac{1}{200} [87, 113]$$

Hence, $P^{(2)} = [87/200, 113/200] = [P^{(w)}, P^{(L)}]$

Thus, the probability of winning 3rd game is : 87/200

\textcircled{c} we shall find fixed probability vector ;

$$v = (x, y) \text{ s.t. } \underline{\underline{vP = v}}, \text{ where, } x + y = 1$$

$$\text{i.e. } [x, y] \cdot \frac{1}{10} \begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix} = [x, y]$$

$$\Rightarrow 6x + 3y = 10x, \quad 4x + 7y = 10y.$$

$$\text{(or)} \quad 3y = 4x \text{ \& by using ; } y = 1 - x \text{ we get ...}$$

$$\Rightarrow 3(1 - x) = 4x \Rightarrow \boxed{x = 3/7}$$

$$\boxed{y = 4/7}$$

Hence, $v = \underline{\underline{[3/7 \ 4/7]}} = [P^{(w)}, P^{(L)}]$

Thus, in the long run, he wins : 3/7 of the time.