# RODRIGUE'S FORMULA :-



The Legendre polynomials $P_n(x)$ of the form ;

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$ is known as "Rodrigue's formula."

**proof :-** Let $u = (x^2-1)^n$.

We shall 1st establish that the $n^{th}$ derivative of $u$, that is $u_n$ is the solution of Legendre's differential Equation.

$\Rightarrow \quad (1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{—} (1)$.

$\Rightarrow$ dwt $u$ wrt $x$, we have...

$$\frac{du}{dx} = u_1 = n(x^2-1)^{n-1} \cdot 2x$$

$$u_1 = 2nx \cdot \frac{(x^2-1)^n}{(x^2-1)}$$

$$(x^2-1)u_1 = 2nx\underbrace{(x^2-1)^n}$$

$$\Rightarrow \quad (x^2-1)u_1 = 2nxu.$$

$\Rightarrow$ dwt again wrt $x$, we get...

$$(x^2-1)u_2 + u_1(2x) = 2n[xu_1 + u(1)]$$

$\Rightarrow$ We shall now dwt, the result $n$ times by applying Leibnitz theorem for $n^{th}$ derivative of product given by;

$$(uv)_n = uv_n + nu_1v_{n-1} + \frac{n(n-1)}{2!}u_2v_{n-2} + \dots + u_nv.$$

$$\therefore \quad [(x^2-1)u_2]_n + 2[xu_1]_n = 2n[xu_1]_n + 2nu_n$$

$$\Big\{\underbrace{[(x^2-1)u_2]_n}_{1^{st}\ term} + n[2x]u_{n+2-1} + \frac{n(n-1)}{2!}(2(1))u_{n+2-2}\Big\} + \underbrace{2\Big[2x\,u_{1n} + n(2(1))u_{1n-1}\Big]}_{=2n[xu_{n+1}+nu_n]+2nu_n}.$$

ie, $\left[(x^2-1)u_{n+2} + n.2x.u_{n+1} + \frac{n(n-1)}{2}.2.u_n\right] + 2\left[x u_{n+1} + n.1.u_n\right]$
$= 2n\left[x u_{n+1} + n.1.u_n\right] + 2$

$\Rightarrow (x^2-1)u_{n+2} + 2nx\,u_{n+1} + (n^2-n)u_n + 2xu_{n+1} + 2nu_n = 2nx\,u_{n+1} + 2n^2$
$+ 2nu_n.$

ie, $(x^2-1)\,u_{n+2} + 2xu_{n+1} - n^2u_n - nu_n = 0.$

$(x^2-1)\,u_{n+2} + 2xu_{n+1} - nu_n(n+1) = 0.$

(or) $(1-x^2)\,u_{n+2} - 2x.u_{n+1} + n(n+1)\,u_n = 0.$

This can be put in the form;

$\Rightarrow (1-x^2)\,u_n'' - 2xu_n' + n(n+1)\,u_n = 0. \sim (2).$

Comparing (2) with (1) we conclude that, $u_n$ is a solution of Legendre's diff. Eqn. *(It may be observed that $u$ is a polynomial of degree $2n$ & hence $u_n$ will be polynomial of deg $n$)*

Also $P_n(x)$ which satisfies Legendre's dE is also a polynomial of deg. $n$. Hence $u_n$ is same as $P_n(x)$

$\Rightarrow P_n(x) = K u_n = K\left[(x^2-1)^n\right]_n.$

$P_n(x) = K\left[(x-1)^n(x-1)^n\right]_n.$

Applying Leibnitz thm for RHS;

$P_n(x) = K\left[(x-1)^n\{(x+1)^n\}_n + n.n(x-1)^{n-1}\{(x+1)^n\}_{n-1} + \frac{n(n-1)}{2!}n(n-1)(x-1)^{n-2}\{(x+1)^n\}_{n-2}\right.$
$\left. + \cdots\cdots \{(x-1)^n\}_n (x+1)^n\right] \sim (3)$

Since, $z = (x-1)^n$ then;

(9)

$z_1 = n(x-1)^{n-1}$

$z_2 = n(n-1)(x-1)^{n-2}$ ....

$z_n = n(n-1)(n-2)$ ........ $2.1\,(x-1)^{n-n}$

$z_n = n!\,(x-1)^0 = n!$

$\therefore \{(x-1)^n\}_n = n!$

putting, $\underline{x=1}$ in Eqn (3), all terms in RHS become 0, except last term, ie, $n!\,(1+1)^n = n!\,2^n$.

(3) ie $\Rightarrow$ $P_n(1) = K\,n!\,2^n$ , $P_n(1) = 1$ // By defn of $P_n(x)$.

$1 = K\,n!\,2^n$.

$\boxed{K = \frac{1}{n!\,2^n}}$

Since, $P_n(x) = K u_n$, we have; $P_n(x) = \frac{1}{n!\,2^n}\{(x^2-1)^n\}_n$.

Therefore; $\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n}$ is "Rodrigue's Formula".

Note :-

1) $P_0(x) = 1$
2) $P_1(x) = x$
3) $P_2(x) = \frac{1}{2}(3x^2-1)$
4) $P_3(x) = \frac{1}{2}(5x^3-3x)$
5) $P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$

* $1 = P_0(x)$
* $x = P_1(x)$.
* $x^2 = 2/3\,P_2(x) + 1/3\,P_0(x)$
* $x^3 = 2/5\,P_3(x) + 3/5\,P_1(x)$.
* $x^4 = 8/35\,P_4(x) + 4/7\,P_2(x) + 1/5\,P_0(x)$.