Additional Mathematics I.

2. Differential calculus

	$y = f(x)$	$y_n = D^n(y)$.
1.	$y = e^{ax}$	$y_n = a^n e^{ax}$
2.	$y = a^{mx}$	$y_n = (m\log a)^n \cdot a^{mx}$
3.	$y = (ax+b)^m$, $m > n$	$y_n = m(m-1)(m-2)\cdots[m-(n-1)] \cdot a^n (ax+b)^{m-n}$
4.	$y = \frac{1}{ax+b}$	$y_n = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$
5.	$y = \log(ax+b)$	$y_n = \frac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$.
6.	$y = \sin(ax+b)$	$y_n = a^n \sin\left(\frac{n\pi}{2} + ax+b\right)$
7.	$y = \cos(ax+b)$	$y_n = a^n \cos\left(\frac{n\pi}{2} + ax+b\right)$
8.	$y = e^{ax} \cdot \sin(bx+c)$	$y_n = \left(\sqrt{a^2+b^2}\right)^n e^{ax} \sin\left(n\tan^{-1}(b/a) + bx+c\right)$
9.	$y = e^{ax} \cos(bx+c)$	$y_n = \left(\sqrt{a^2+b^2}\right)^n e^{ax} \cos\left(n\tan^{-1}(b/a) + bx+c\right)$.

To Find n^{th} derivative.

1) Find the n^{th} derivative of $\cosh^3 2x$

Sol^n: Let $y = \cosh^3 2x = \left(\frac{e^{2x}+e^{-2x}}{2}\right)^3$ $\quad | \because (a+b)^3 = a^3+b^3+3ab(a+b)$.

$e^0 = 1$

$$y = \frac{1}{8}\left\{(e^{2x})^3 + (e^{-2x})^3 + 3(e^{2x} \cdot e^{-2x})(e^{2x}+e^{-2x})\right\}$$

$$y = \frac{1}{8}\left\{e^{6x} + e^{-6x} + 3e^{2x} + 3e^{-2x}\right\} \quad | \because y = e^{ax},\ y_n = a^n e^{ax}.$$

$$\therefore y_n = \frac{1}{8}\left\{6^n e^{6x} + (-6)^n e^{-6x} + 3\cdot 2^n e^{2x} + 3(-2)^n e^{-2x}\right\}.$$

2) $y = e^{-x} \cdot \sinh 3x \cdot \cosh 2x.$

$$y = e^{-x}\left(\frac{e^{3x}-e^{-3x}}{2}\right)\left(\frac{e^{2x}+e^{-2x}}{2}\right).$$

$$y = \frac{1}{4}e^{-x}\left(e^{5x}+e^{x}-e^{-x}-e^{-5x}\right) = \frac{1}{4}\left(e^{4x}+1-e^{-2x}-e^{-6x}\right)$$

$$y_n = \frac{1}{4}\left\{4^n e^{4x} - (-2)^n e^{-2x} - (-6)^n e^{-6x}\right\}.$$

3) $y = \log(4x^2+8x+3)^{1/2}$ or $\log\sqrt{4x^2+8x+3}$

$\because$
$4x^2+8x+3$
$4x^2+6x+2x+3$
$2x(2x+3)+(2x+3)$
$(2x+3)(2x+1).$

$12x^2$
$6x \quad 2x$

$$y = \frac{1}{2}\log(4x^2+8x+3)$$

$$y = \frac{1}{2}\log(2x+3)(2x+1) = \frac{1}{2}\left[\log(2x+3) + \log(2x+1)\right]$$

$$y_n = \frac{1}{2}\left[\frac{(-1)^{n-1}(n-1)!\,2^n}{(2x+3)^n} + \frac{(-1)^{n-1}(n-1)!\,2^n}{(2x+1)^n}\right] = (-1)^{n-1}(n-1)!\,2^{n-1}\left[\frac{1}{(2x+3)^n} + \frac{1}{(2x+1)^n}\right]$$

4) Find the n^{th} differential coefficient of, $\log_{10}\sqrt{\frac{(3x+5)^2}{(x+1)^6}}$

$$y = \log_{10}\left(\frac{(3x+5)^2}{(x+1)^6}\right)^{1/2} \quad | \because \log_{10} x = \frac{\log_e x}{\log_e 10}$$

$$y = \frac{1}{\log_e 10}\log_e\left(\frac{(3x+5)^2}{(x+1)^6}\right)^{1/2} = \frac{1}{\log_e 10}\cdot\frac{1}{2}\left\{2\log(3x+5) - 6\log(x+1)\right\}.$$

$$\therefore y_n = \frac{1}{2\log_e 10}\left\{\frac{2(-1)^{n-1}(n-1)!\,3^n}{(3x+5)^n} - 6\,\frac{(-1)^{n-1}(n-1)!\,1^n}{(x+1)^n}\right\} = \frac{(-1)^{n-1}(n-1)!}{2\cdot\log_e 10}\left\{\frac{2\cdot 3^n}{(3x+5)^n} - \frac{6}{(x+1)^n}\right\}.$$

5) $y = \log\left[(3x+2)e^{5x+6}\right] = \log(3x+2) + (5x+6)\cdot\log_e e = \log(3x+2) + (5x+6)$

$$y_n = \frac{(-1)^{n-1}(n-1)!\,3^n}{(3x+2)^n} + 0. \quad | \text{where } n > 1.$$

(2) Q) Find the n^{th} derivative of , $\cos x \cdot \cos 3x \cdot \cos 5x$. | $\cos A \cdot \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$.

Solⁿ: Let $y = \cos x \cdot \cos 3x \cdot \cos 5x$.

$y = \frac{1}{2}(\cos 4x + \cos 2x) \cdot \cos 5x. = \frac{1}{2}(\cos 5x \cdot \cos 4x + \cos 5x \cdot \cos 2x)$.

$y = \frac{1}{2}\left\{\frac{1}{2}(\cos 9x + \cos x) + \frac{1}{2}(\cos 7x + \cos 3x)\right\}. = \frac{1}{4}[\cos x + \cos 3x + \cos 7x + \cos 9x]$

$y_n = \frac{1}{4}\left[1^n \cos\left(\frac{n\pi}{2} + x\right) + 3^n \cos\left(\frac{n\pi}{2} + 3x\right) + 7^n \cos\left(\frac{n\pi}{2} + 7x\right) + 9^n \cdot \cos\left(\frac{n\pi}{2} + 9x\right)\right]$.

Exam 15DIP31 | $\because \sin A \cdot \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$.

Q) Find n^{th} derivative of $y = e^x \cdot \sin 4x \cdot \cos x$.

Solⁿ: $y = e^x \cdot \frac{1}{2}[\sin 5x + \sin 3x] = \frac{1}{2}[e^x \cdot \sin 5x + e^x \cdot \sin 3x]$.

$y_n = \frac{1}{2}\left\{(\sqrt{26})^n \cdot e^x \cdot \sin(n\tan^{-1}(5/1) + 5x) + (\sqrt{10})^n e^x \cdot \sin(n\tan^{-1}(3/1) + 3x)\right\}$.

$y_n = \frac{e^x}{2}\left\{(\sqrt{26})^n \cdot \sin(n\tan^{-1}(5/1) + 5x) + (\sqrt{10})^n \sin(n\tan^{-1}(3/1) + 3x\right\}$.

$\because \cos 2x = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{1+\cos 2x}{2}$.

$\cos A \cdot \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

8) $y = e^{2x} \cdot \cos^2 x \cdot \sin x$.

$y = e^{2x}\left(\frac{1+\cos 2x}{2}\right) \cdot \sin x = \frac{e^{2x}}{2}(\sin x + \cos 2x \cdot \sin x) = \frac{e^{2x}}{2}\left(\sin x + \frac{1}{2}(\sin 3x - \sin x)\right)$

$y = \frac{e^{2x}}{4}(2\sin x + \sin 3x - \sin x) = \frac{e^{2x}}{4}(\sin x + \sin 3x). = \frac{1}{4}(e^{2x}\sin x + e^{2x} \cdot \sin 3x)$

$\therefore y_n = \frac{1}{4}\left\{(\sqrt{5})^n e^{2x} \cdot \sin(n\tan^{-1}(1/2) + x) + (\sqrt{13})^n e^{2x} \sin(n\tan^{-1}(3/2) + 3x)\right\}$.

$y_n = \frac{e^{2x}}{4}\left\{(\sqrt{5})^n \cdot \sin(n\tan^{-1}(1/2) + x) + (\sqrt{13})^n \sin(n\tan^{-1}(3/2) + 3x)\right\}$.

9) $y = e^{-3x} \cdot \cos^3 x = \frac{1}{4}(e^{-3x} \cdot 3\cos x + e^{-3x} \cdot \cos 3x)$ | $\because \cos 3x = 4\cos^3 x - 3\cos x$

$\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x)$.

$y_n = \frac{1}{4}\left\{3(\sqrt{10})^n e^{-3x} \cdot \cos(n\tan^{-1}(1/-3) + x) + (\sqrt{18})^n \cdot e^{-3x} \cos(n\tan^{-1}(3/-3) + 3x)\right\}$.

$y_n = \frac{e^{-3x}}{4}\left\{3(\sqrt{10})^n \cdot \cos(-n\tan^{-1}(1/3) + x) + (\sqrt{18})^n \cdot \cos(-n\pi/4 + 3x)\right\}$ [$\because \tan^{-1}(-1) = -\pi/4$.

Proper — Dr power should be more than Nr if not then improper (if powers of Nr & Dr are equal then also improper).

(2) Exam

10) Find the n^{th} derivative of $\frac{x}{(x-1)(2x+3)}$.

Solⁿ: Let $y = \frac{x}{(x-1)(2x+3)} = \frac{A}{x-1} + \frac{B}{2x+3}$

$x = A(2x+3) + B(x-1) \rightarrow (1)$.

Take $x-1=0$ & $2x+3=0 \Rightarrow x=1,\ x=-3/2$

put $x=1$ in eq^n ①, $1 = A(2+3) = 5A \Rightarrow A = 1/5$

put $x=-3/2$ in eq^{n}①, $\frac{-3}{2} = 0 + B\left(-\frac{3}{2}-1\right) = B\left(\frac{-5}{2}\right) \Rightarrow B = 3/5$

Thus, $y = \frac{1}{5}\cdot\frac{1}{x-1} + \frac{3}{5}\cdot\frac{1}{2x+3}$

$$y_n = \frac{1}{5}\frac{(-1)^n n!\, 1^n}{(x-1)^{n+1}} + \frac{3}{5}\frac{(-1)^n n!\, 2^n}{(2x+3)^{n+1}} = \frac{(-1)^n n!}{5}\left\{\frac{1}{(x-1)^{n+1}} + \frac{3\cdot 2^n}{(2x+3)^{n+1}}\right\}$$

11) $y = \frac{x}{(x-1)(x^2+x-2)}$

x^2+x-2
$x^2+2x-x-2$
$x(x+2)-1(x+2)$
$(x+2)(x-1)$

$-2x^2$, $2x$, $-1x$

$$y = \frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \longrightarrow ①$

put $x=1$, in ①, $1 = B(3) \Rightarrow B = 1/3$.

put $x=-2$ in ①, $-2 = C(-3)^2 \Rightarrow C = -2/9$

by equating coefficient of x^2, $0 = A+C \Rightarrow A = 2/9$.

Thus, $y = \frac{2}{9}\cdot\frac{1}{x-1} + \frac{1}{3}\cdot\frac{1}{(x-1)^2} - \frac{2}{9}\cdot\frac{1}{x+2}$.

$$y_n = \frac{2}{9}\left\{\frac{(-1)^n n!\,(1)^n}{(x-1)^{n+1}}\right\} + \frac{1}{3}\left[(-2)(-3)\cdots(-(n+1))\ 1^n (x-1)^{-2-n}\right] - \frac{2}{9}\frac{(-1)^n n!\ 1^n}{(x+2)^{n+1}}$$

$$y_n = \frac{2}{9}\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{1}{3}\left[\frac{(-1)^n(n+1)!}{(x-1)^{n+2}}\right] - \frac{2}{9}\frac{(-1)^n n!}{(x+2)^{n+1}}$$

$$y_n = \frac{(-1)^n n!}{3}\left[\frac{2}{3(x-1)^{n+1}} + \frac{(n+1)}{(x-1)^{n+2}} - \frac{2}{3(x+2)^{n+1}}\right].$$

12) $y = \frac{x^2}{2x^2+7x+6} = \frac{1}{2}\cdot\frac{2x^2}{2x^2+7x+6}$

$2x^2+7x+6 \,\big)\, 2x^2 \,\big(\, 1$
$2x^2+7x+6$
$-7x-6$

$$y = \frac{1}{2}\left[1 - \frac{7x+6}{2x^2+7x+6}\right] = \frac{1}{2} - \frac{1}{2}\frac{7x+6}{2x^2+7x+6}$$

$2x^2+7x+6$
$2x^2+3x+4x+6$
$x(2x+3)+2(2x+3)$
$(2x+3)(x+2)$

$12x^2$, $3x$, $4x$

$$y_n = 0 - \frac{1}{2}D^n\left[\frac{7x+6}{2x^2+7x+6}\right] \longrightarrow ①$$

Let $\frac{7x+6}{2x^2+7x+6} = \frac{7x+6}{(2x+3)(x+2)} = \frac{A}{2x+3} + \frac{B}{x+2}$

(4) $7x+6 = A(x+2) + B(2x+3) \longrightarrow (2)$.

Put $x=-2$, $7(-2)+6 = B(-4+3) \Rightarrow -8 = -B \Rightarrow \boxed{B=8}$

Put $x=-3/2$, $7\left(\frac{-3}{2}\right)+6 = A\left(\frac{-3}{2}+2\right) \Rightarrow \frac{-21+12}{2} = A\left(\frac{-3+4}{2}\right) \Rightarrow -\frac{9}{\cancel{2}} = \frac{A}{\cancel{2}} \Rightarrow \boxed{A=-9}$

Thus, $y_n = -\frac{1}{2}\left[-9\frac{(-1)^n \cdot n!\, 2^n}{(2x+3)^{n+1}} + 8\frac{(-1)^n n!\, 1^n}{(x+2)^{n+1}}\right] = \frac{(-1)^{n+1} n!}{2}\left[\frac{-9\cdot 2^n}{(2x+3)^{n+1}} + \frac{8}{(x+2)^{n+1}}\right]$

Exam 6) Find the n^{th} derivative of $\cos 2x \cdot \cos 3x$.

Solⁿ: Let $y = \cos 2x \cdot \cos 3x$

$y = \frac{1}{2}[\cos 5x + \cos x]$ $\quad \because \cos(-\theta) = \cos\theta$.

$y_n = \frac{1}{2}\left[5^n \cdot \cos\left(\frac{n\pi}{2}+5x\right) + 1^n \cdot \cos\left(\frac{n\pi}{2}+x\right)\right]$

Leibnitz theorem: If u & v are functions of x then,

$D^n(uv) = (uv)_n = uv_n + {}^nC_1 u_1 v_{n-1} + {}^nC_2 u_2 v_{n-2} + \dots + u_n v$.

where ${}^nC_1 = n$, ${}^nC_2 = \frac{n(n-1)}{2}$, ${}^nC_3 = \frac{n(n-1)(n-2)}{1\times 2\times 3}$.

[Working rule & NOTE].

NOTE: ${}^nC_n = {}^nC_0 = 1$.

Exam (15MATDIP31)

1) If $y = a\cos(\log x) + b\sin(\log x)$ show that $x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.

Solⁿ: diff w.r.t x.

$y_1 = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\frac{1}{x}$

$xy_1 = -a\sin(\log x) + b\cos(\log x)$.

Diff w.r.t x.

$xy_2 + y_1 = -a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$.

$x^2y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)] = -y$

$x^2y_2 + xy_1 + y = 0$.

Now diff n times,

$D^n(x^2y_2) + D^n(xy_1) + D^n(y) = 0$.

Apply Leibnitz theorem,

$$x^2 y_{n+2} + n\cdot 2x y_{n+1} + \frac{n(n-1)}{\cancel{2}}\cdot \cancel{2} y_n + x y_{n+1} + n(1) y_n + y_n = 0.$$

$$x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2 - \cancel{n} + \cancel{n} + 1) y_n = 0.$$

$$x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0.$$

301 Exam

2) If $y = \sin^{-1}x$, prove that $(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - n^2 y_n = 0.$

Sol^n: $y = \sin^{-1} x$

diff w.r.t x

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = 1$$

squaring on b.s

$$(1-x^2) y_1^2 = 1$$

$$(1-x^2) 2 y_1 y_2 + (-2x) y_1^2 = 0.$$

$\div$ by $2y_1$, $\quad (1-x^2) y_2 - x y_1 = 0.$

Diff n times,

$$D^n[(1-x^2) y_2] - D^n[x y_1] = 0.$$

Apply Leibnitz theorem,

$$(1-x^2) y_{n+2} + n(-2x) y_{n+1} + \frac{n(n-1)}{\cancel{2}} (-\cancel{2}) y_n - x y_{n+1} - n(1) y_n = 0.$$

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2 - \cancel{n} + \cancel{n}) y_n = 0$$

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - n^2 y_n = 0$$

301 Exam

3) If $y = e^{a\sin^{-1}x}$ then prove that $(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2+a^2) y_n = 0.$

Sol^n: diff w.r.t x

$$y_1 = e^{a\sin^{-1}x} \cdot a \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = a \cdot y \qquad \left| \because y = e^{a\sin^{-1}x} \right.$$

squaring on b.s

$$(1-x^2) y_1^2 - a^2 \cdot y^2 = 0.$$

diff w.r.t x.

(6). $(1-x^2)2y_1y_2+(-2x)y_1^2-a^2 2yy_1=0$

$\div$ by $2y_1$, $(1-x^2)y_2-xy_1-a^2y=0.$

Diff n times.

$$D^n[(1-x^2)y_2]-D^n[xy_1]-a^2D^n(y)=0.$$

Apply Leibnitz theorem,

$$(1-x^2)y_{n+2}+\underline{n(-2x)y_{n+1}}+\frac{n(n-1)}{\cancel{2}}(-\cancel{2})y_n-\underline{xy_{n+1}}-ny_n-a^2y_n=0.$$

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-\cancel{n}+\cancel{n}+a^2)y_n=0.$$

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0.$$

15MATDIP31

4) If $y=\cos(m\log x)$ then prove that $x^2y_{n+2}+(2n+1)xy_{n+1}+(m^2+n^2)y_n=0$.

Soln: diff w.r.t x.

$y_1=-\sin(m\log x)\cdot m\frac{1}{x}.$

$xy_1=-m\cdot\sin(m\log x)$

diff w.r.t x.

$xy_2+y_1=-m\cos(m\log x)\cdot m\cdot\frac{1}{x}$

$x^2y_2+xy_1=-m^2y \quad | \because y=\cos(m\log x).$

$x^2y_2+xy_1+m^2y=0.$

Diff n times,

$$D^n[x^2y_2]+D^n[xy_1]+m^2D^n[y]=0$$

Apply Leibnitz theorem,

$$x^2y_{n+2}+n(2x)y_{n+1}+\frac{n(n-1)}{2}(2)y_n+xy_{n+1}+ny_n+m^2y_n=0.$$

$$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2-\cancel{n}+\cancel{n}+m^2)y_n=0.$$

$$x^2y_{n+2}+(2n+1)xy_{n+1}+(m^2+n^2)y_n=0.$$

Working rule:

i) We first establish a relation involving y_2, y_1 (or) y_1, y.

ii) Next establish a relation involving y_{n+2}, y_{n+1} & y_n

iii) Differentiate n times & apply Leibnitz theorem.

NOTE: $D^n(y_2)=y_{n+2}$, $D^{n-1}(y_2)=y_{n+1}$, $D^{n-2}(y_2)=y_n$, $D^{n+1}(y)=y_{n+1}$, $D^{(n+1)}(y_1)=y_{n+2}$ etc.

5) If $y=\sin\log(x^2+2x+1)$, show that $(x+1)^2y_{n+2}+(2n+1)(x+1)y_{n+1}+(n^2+4)y_n=0$.

Soln: $y=\sin[2\log(x+1)]$ | ∵ x^2+2x+1
$x^2+x+x+1$
$x(x+1)+1(x+1)$
$(x+1)(x+1)=(x+1)^2$.

diff w.r.t x.

$$y_1=\cos[2\log(x+1)]\cdot 2\cdot\frac{1}{(x+1)}$$

$$(x+1)y_1=2\cdot\cos[2\log(x+1)]$$

diff w.r.t x.

$$(x+1)y_2+1\cdot y_1=-2\sin[2\log(x+1)]\cdot 2\,\frac{1}{x+1}$$

$$(x+1)^2y_2+(x+1)y_1=-4y \quad | \because y=\sin[2\log(x+1)].$$

$$(x+1)^2y_2+(x+1)y_1+4y=0.$$

Diff n times.

$$D^n[(x+1)^2y_2]+D^n[(x+1)y_1]+4D^n(y)=0.$$

Apply Leibnitz theorem,

$$(x+1)^2\cdot y_{n+2}+n2(x+1)y_{n+1}+\frac{n(n-1)}{\cancel{2}}\cancel{(2)}y_n+(x+1)y_{n+1}+ny_n+4y_n=0.$$

$$(x+1)^2y_{n+2}+(2n+1)(x+1)y_{n+1}+(n^2-\cancel{n}+\cancel{n}+4)y_n=0.$$

$$(x+1)^2y_{n+2}+(2n+1)(x+1)y_{n+1}+(n^2+4)y_n=0.$$

Polar curves: ~~Angle~~ - $P(x,y)$ → cartesian coordinates $x=r\cos\theta$, $y=r\sin\theta$
$P(r,\theta)$ → polar coordinates. $r=f(\theta)$ (polar curve).

15MATDTP31 (2).

Angle b/w radius vector & tangent. (Q) With usual notation prove that $\tan\phi=r\frac{d\theta}{dr}$

Let $P(r,\theta)$ be any point on the curve $r=f(\theta)$.

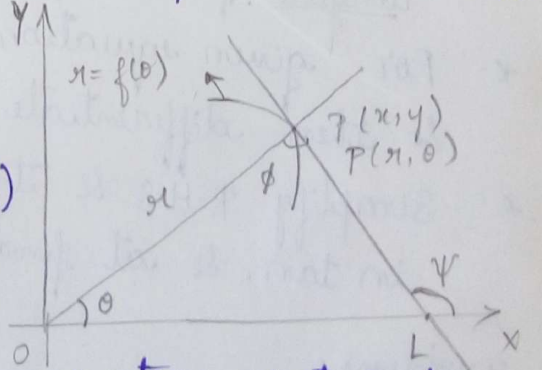
∴ $X\hat{O}P=\theta$, $OP=r$.

PL = tangent to the curve at P.

ψ - +ve direction angle of the initial line (x-axis)

ϕ - angle b/w radius vector OP & tangent PL.

i.e., $O\hat{P}L=\phi$.



From diagram, $\psi=\phi+\theta$ | ∵ exterior angle = sum of interior opposite angles.

$$\Rightarrow \tan\psi=\tan(\phi+\theta)$$

$$\tan\psi=\frac{\tan\phi+\tan\theta}{1-\tan\phi\cdot\tan\theta} \longrightarrow (1).$$

(8.) Let (x,y) be the cartesian coordinate of P,

$x = r\cos\theta$, $y = r\sin\theta$ | ∵ parametric eqⁿˢ in terms of θ.

W.K.T, $\tan\psi = \frac{dy}{dx}$ = slope of the tangent PL. | ∵ geometrical meaning of the derivative.

$\tan\psi = \frac{dy/d\theta}{dx/d\theta}$ | ∵ x & y are functions of θ.

$$\tan\psi = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta} \quad \text{where } r' = \frac{dr}{d\theta}.$$

÷ both Nr & Dr by $r'\cos\theta$ | ∵ to corelate eqⁿ ①,

$$\tan\psi = \frac{\frac{r}{r'} + \tan\theta}{1 - \frac{r}{r'}\tan\theta} \longrightarrow ②$$

comparing eqⁿ ① & ②, $\tan\phi = \frac{r}{r'} = \frac{r}{\frac{dr}{d\theta}} \Rightarrow \boxed{\tan\phi = r\left(\frac{d\theta}{dr}\right)}$.

② , $\boxed{\cot\phi = \frac{1}{r}\left(\frac{dr}{d\theta}\right)}$.

NOTE:

* Length of the perpendicular from the pole to the tangent. $\boxed{p = r\sin\phi}$.
* Angle of intersection of two polar curves is $= |\phi_2 - \phi_1|$.
* If two curves intersect orthogonally then, $|\phi_2 - \phi_1| = \pi/2$ (or) $\tan\phi_1 \cdot \tan\phi_2 = -1$.

* $\tan(\pi/4 + \theta) = \frac{1+\tan\theta}{1-\tan\theta}$ * $\cot(\pi/4+\theta) = \frac{1-\tan\theta}{1+\tan\theta}$.

· Working rule to find the angle b/w the radius vector & the tangent for the ~~follows~~ polar curves:

* For given equation $r = f(\theta)$, first take logarithm on both side & then differentiate w.r.t θ.
* Simplify R.H.S & try to put in terms of cot, if not then put in tan. & at final $\phi = \tan^{-1}\alpha$. (α is not simplified term)

problems:

MAT101P301

1) Find the angle between the radius vector & the tangent to the curve $r = a(1-\cos\theta)$ at the point $\theta = \pi/3$.

Solⁿ: $r = a(1-\cos\theta)$

Take log on b.s.

Scanned by CamScanner

$$\log r = \log a + \log(1-\cos\theta)$$

diff w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1-\cos\theta} \qquad \Big|\because \frac{1}{r}\frac{dr}{d\theta} = \cot\phi.$$

$\Big|\because \sin\theta = 2\sin\theta/2 \cdot \cos\theta/2$

$\Big|\because \cos\theta = 1 - 2\sin^2\theta/2$

$1-\cos\theta = 2\sin^2\theta/2.$

$$\cot\phi = \frac{\cancel{2}\sin\cancel{\theta/2}\cdot\cos\theta/2}{\cancel{2}\sin^{\cancel{2}}\theta/2} = \cot\theta/2$$

$\Rightarrow \cot\phi = \cot\theta/2$

Thus $\phi = \theta/2$. at $\theta = \pi/3$, $\phi = \frac{\pi}{6}$

4) Find the angle between the radius vector & the tangent for the polar curve, $r^2 \cdot \cos 2\theta = a^2$.

Solⁿ: $r^2\cos 2\theta = a^2$.

Take log on both side

$$2\log r + \log\cos 2\theta = 2\log a$$

diff w.r.t θ

$$2\cdot\frac{1}{r}\frac{dr}{d\theta} + \frac{1}{\cos 2\theta}(-2\sin 2\theta) = 0.$$

$$\cancel{2}\cdot\frac{1}{r}\frac{dr}{d\theta} = \cancel{2}\tan 2\theta \quad \Big|\because \frac{1}{r}\frac{dr}{d\theta} = \cot\phi \quad \Big|\because \cot(\pi/2-\theta) = \tan\theta.$$

$$\cot\phi = \tan 2\theta = \cot(\pi/2 - 2\theta)$$

Thus, $\phi = \pi/2 - 2\theta$.

5) Find the angle between the radius vector & the tangent & also find the slope of the tangent for the curve, $r = a(1+\cos\theta)$ at $\theta = \pi/3$.

Solⁿ: $r = a(1+\cos\theta)$

Take log on both side.

$$\log r = \log a + \log(1+\cos\theta)$$

Diff w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{1+\cos\theta}$$

$$\cot\phi = \frac{-\cancel{2}\sin\theta/2\cdot\cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2} = -\tan\theta/2.$$

$$\cot\phi = \cot(\pi/2 + \theta/2) \quad \Big|\because \cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right) = -\tan\frac{\theta}{2}.$$

$\phi = \frac{\pi}{2} + \frac{\theta}{2}$. at $\theta = \frac{\pi}{3}$, $\phi = \frac{\pi}{2} + \frac{\pi}{6} = \frac{\cancel{4}^2\pi}{\cancel{6}3} = \frac{2\pi}{3}$ (or) 120°.

W.K.T $\psi = \theta + \phi = \frac{\pi}{3} + \frac{2\pi}{3} = \frac{\cancel{3}\pi}{\cancel{3}} = \pi = 180^\circ$.

Thus the slope of the tangent $= \tan\psi = \tan 180^\circ = 0$.

4) Find the angle b/w the radius vector & the tangent for the curve $r\cos^2(\theta/2) = a$ at $\theta = 2\pi/3$.

Soln: $r\cos^2(\theta/2) = a$.

Take log on b.s

$$\log r + 2\log\cos\theta/2 = \log a.$$

$$\frac{1}{r}\frac{dr}{d\theta} + \cancel{2}\frac{-\sin\theta/2 \cdot \frac{1}{\cancel{2}}}{\cos\theta/2} = 0.$$

$$\cot\phi = \tan\theta/2 = \cot(\pi/2 - \theta/2). \quad | \because \cot(\pi/2 - \theta) = \tan\theta.$$

$$\phi = \frac{\pi}{2} - \frac{\theta}{2} = \frac{\pi}{2} - \frac{\cancel{2}\pi/3}{\cancel{2}} = \frac{\pi}{2} - \frac{\pi}{3} = 90^\circ - 60^\circ = 30^\circ \text{ or } \frac{\pi}{6}.$$

5) ~~Also~~ Find the angle b/w the radius vector & the tangent for the curve $r = a(1+\sin\theta)$ at $\theta = \pi/2$.

Soln: $r = a(1+\sin\theta)$

Take log on b.s.

$$\log r = \log a + \log(1+\sin\theta)$$

diff w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\cos\theta}{1+\sin\theta}$$

$$\cot\phi = \frac{\cos\theta}{1+\sin\theta} \quad \text{at } \theta = \frac{\pi}{2}, \ \cot\phi = \frac{0}{1+1} = 0.$$

$$\Rightarrow \cot\phi = 0$$

$$\phi = \cot^{-1}(0) = \pi/2.$$

6) Find the angle b/w the radius vector & the tangent for the polar curve, $r^m = a^m(\cos m\theta + \sin m\theta)$.

Soln: $r^m = a^m(\cos m\theta + \sin m\theta)$

Take log on both side.

$$\cancel{m\,r^{m-1}\frac{dr}{d\theta}} \quad m\log r = m\log a + \log(\cos m\theta + \sin m\theta)$$

diff w.r.t θ

$$m \cdot \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{(-m\sin m\theta + m\cos m\theta)}{\cos m\theta + \sin m\theta}$$

$$\cancel{m} \cdot \cot\phi = \frac{\cancel{m}(\cos m\theta \mp \sin m\theta)}{\cos m\theta + \sin m\theta}.$$

$$\cot\phi = \frac{\cos m\theta\,(1 - \tan m\theta)}{\cos m\theta\,(1+\tan m\theta)}$$

$$\cot\phi = \cot(\pi/4 + m\theta) \quad \Big|\because \frac{1-\tan\theta}{1+\tan\theta} = \cot\left(\frac{\pi}{4}+\theta\right)$$

Thus, $\phi = \frac{\pi}{4} + m\theta$.

7) Show that the two curves $r = a(1+\cos\theta)$ & $r = a(1-\cos\theta)$ cut each other orthogonally.

Sol^n: $r = a(1+\cos\theta)$

Take log on b.s

$\log r = \log a + \log(1+\cos\theta)$

diff w.r.t θ.

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{0-\sin\theta}{1+\cos\theta}$$

$$\cot\phi_1 = \frac{-\cancel{2}\sin\theta/2\cdot\cos\theta/2}{\cancel{2}\cos^2\theta/2}$$

$$\cot\phi_1 = -\tan\theta/2 = \cot(\pi/2 + \theta/2)$$

$$\phi_1 = \frac{\pi}{2} + \frac{\theta}{2}.$$

$r = a(1-\cos\theta)$.

Take log on b.s

$\log r = \log a + \log(1-\cos\theta)$

diff w.r.t θ

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{0+\sin\theta}{1-\cos\theta}$$

$$\cot\phi_2 = \frac{\cancel{2}\sin\theta/2\cdot\cos\theta/2}{\cancel{2}\sin^2\theta/2}$$

$$\cot\phi_2 = \cot\theta/2.$$

$$\phi_2 = \theta/2.$$

$$\therefore \text{angle of intersection} = |\phi_2 - \phi_1| = |\theta/2 - \pi/2 - \theta/2| = \pi/2$$

Thus the curve cut each other othogonally.

8) Show that the pair of curves ~~intersect~~ $r = a(1+\sin\theta)$ & $r = a(1-\sin\theta)$ intersect each other orthogonally.

Sol^n: $r = a(1+\sin\theta)$

Take log on b.s

$\log r = \log a + \log(1+\sin\theta)$

diff w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\cos\theta}{1+\sin\theta}$$

$$\cot\phi_1 = \frac{\cos\theta}{1+\sin\theta}$$

$$\tan\phi_1 = \frac{1+\sin\theta}{\cos\theta}$$

$r = a(1-\sin\theta)$

Take log on b.s

$\log r = \log a + \log(1-\sin\theta)$

diff w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{0-\cos\theta}{1-\sin\theta}$$

$$\cot\phi_2 = \frac{-\cos\theta}{1-\sin\theta}$$

$$\tan\phi_2 = \frac{1-\sin\theta}{-\cos\theta}.$$

$$\therefore \tan\phi_1\cdot\tan\phi_2 = \frac{(1+\sin\theta)(1-\sin\theta)}{-\cos^2\theta} = \frac{1-\sin^2\theta}{-\cos^2\theta} = \frac{\cancel{\cos^2\theta}}{-\cancel{\cos^2\theta}} = -1.$$

Thus, the curves intersect each other orthogonally.

(12) 9) Find the angle between the pair of curves, $r = \sin\theta + \cos\theta$, $r = 2\sin\theta$.

Sol^n: $r = \sin\theta + \cos\theta$

Take log on b.s

$\log r = \log(\sin\theta + \cos\theta)$

diff w.r.t θ

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}$

$\cot\phi_1 = \frac{\cancel{\cos\theta}(1-\tan\theta)}{\cancel{\cos\theta}(1+\tan\theta)}$

$\cot\phi_1 = \cot(\frac{\pi}{4}+\theta)$

$\phi_1 = \pi/4 + \theta$

$r = 2\sin\theta$

Take log on b.s

$\log r = \log 2 + \log\sin\theta$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{\cos\theta}{\sin\theta}$

$\cot\phi_2 = \cot\theta$

$\phi_2 = \theta$.

$\therefore |\phi_1 - \phi_2| = |\frac{\pi}{4} + \theta - \theta| = \pi/4$.

Thus the angle of intersection is $\pi/4$.

NOTE: i) After solving $|\phi_1 - \phi_2|$ (or) $|\phi_2 - \phi_1|$ if this contains θ then we have to find θ by solving the pair of given equations.

ii) Suppose we are not able to obtain ϕ_1 & ϕ_2 explicitly then write the expressions for $\tan\phi_1$, $\tan\phi_2$ & use formula $\tan(\phi_1 - \phi_2)$.

iii) If $\tan(\phi_1 - \phi_2) = \alpha$ then the angle of intersection is $\tan^{-1}\alpha$.

10) Find the angle between the curves $r^2\sin 2\theta = 4$ & $r^2 = 16\sin 2\theta$.

Sol^n: $r^2\sin 2\theta = 4$

$2\log r + \log\sin 2\theta = \log 4$

diff w.r.t θ

$2\cdot\frac{1}{r}\frac{dr}{d\theta} + \frac{2\cos 2\theta}{\sin 2\theta} = 0$

$\cancel{2}\cot\phi_1 = -\cancel{2}\cot 2\theta$

$\cot\phi_1 = \cot(-2\theta)$

$\Rightarrow \phi_1 = -2\theta$

$r^2 = 16\sin 2\theta$

$2\log r = \log 16 + \log\sin 2\theta$.

diff w.r.t θ

$2\cdot\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{2\cos 2\theta}{\sin 2\theta}$.

$\cancel{2}\cot\phi_2 = \cancel{2}\cot 2\theta$.

$\Rightarrow \phi_2 = 2\theta$.

$\therefore |\phi_1 - \phi_2| = |-2\theta - 2\theta| = 4\theta \longrightarrow (1)$.

Since $|\phi_1 - \phi_2|$ contains θ, to find θ solve the pair of given eq^{ns},

$r^2 = \frac{4}{\sin 2\theta}$ & $r^2 = 16\sin 2\theta$.

Equating the R.H.S, $\frac{\cancel{4}}{\sin 2\theta} = \cancel{16}^{4}\sin 2\theta$. $\Rightarrow 4\sin^2 2\theta = 1$.

$\sin^2 2\theta = \frac{1}{4} \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \sin^{-1}\left(\frac{1}{2}\right)$ $\quad | \because \sin\frac{\pi}{6} = \frac{1}{2}$

$2\theta = 30^\circ$ $\quad$ (or) $\sin 30^\circ = \frac{1}{2}$.

$\Rightarrow \theta = 15^\circ$.

Put $\theta = 15^\circ$ in eq^n (1),

$|\phi_1 - \phi_2| = 4(15^\circ) = 60^\circ$ (or) $\frac{\pi}{3}$.

Thus the angle of intersection $= \pi/3$ (or) 60°.

11) Find the angle of intersection b/w the given pair of curves,

$r = a\log\theta$ & $r = \frac{a}{\log\theta}$

Sol^n:

$r = a\log\theta$

Take log on b.s

$\log r = \log a + \log(\log\theta)$

diff w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\log\theta}\cdot\frac{1}{\theta}$$

$$\cot\phi_1 = \frac{1}{\theta\cdot\log\theta}$$

$r = \frac{a}{\log\theta}$

Take log on b.s

$\log r = \log a - \log(\log\theta)$.

diff w.r.t θ.

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 - \frac{1}{\log\theta}\cdot\frac{1}{\theta}.$$

$$\cot\phi_2 = -\frac{1}{\theta\cdot\log\theta}$$

NOTE: We cannot find ϕ_1 & ϕ_2 explicitly,

$\therefore \tan\phi_1 = \theta\log\theta$ $\quad , \quad \tan\phi_2 = -\theta\log\theta$.

Now consider, $\tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \cdot \tan\phi_2} = \frac{\theta\log\theta + \theta\log\theta}{1 - (\theta\log\theta)^2}$

$$\tan(\phi_1 - \phi_2) = \frac{2\theta\log\theta}{1 - (\theta\log\theta)^2} \rightarrow (1)$$

Now, Find θ by solving the given pair of curves, equate R.H.S

$$\cancel{a}\log\theta = \frac{\cancel{a}}{\log\theta}$$

$(\log\theta)^2 = 1$ (or) $\log\theta = 1 \Rightarrow \theta = 1$ $\quad | \because \log_e^e = 1$.

Put $\theta = 1$ in eq^n (1), $\tan(\phi_1 - \phi_2) = \frac{2e}{1 - e^2}$

$\therefore$ The angle of intersection, $\phi_1 - \phi_2 = \tan^{-1}\left(\frac{2e}{1 - e^2}\right)$ //.

(14). **Pedal equation of a polar curve**: The equation of the given curve $r=f(\theta)$ expressed in terms of p & r is called as the pedal equation.

Working rule:

i) obtain ϕ value for the given $r=f(\theta)$. (as previous procedure).

ii) Substitute ϕ value in $p=r\sin\phi$. ($p=r\,g(\theta)$)

iii) Eliminate θ from $r=f(\theta)$ & $p=r\,g(\theta)$ to obtain pedal equation (equation will be in terms of p & r).

Problems.

(2) 15MATDIP; MAT DIP. 30;

1) Find the pedal equation of $r=a(1+\cos\theta)$.

Solⁿ: $r=a(1+\cos\theta)$

Take log on b.s.

$$\log r=\log a+\log(1+\cos\theta)$$

diff w.r.t θ.

$$\frac{1}{r}\frac{dr}{d\theta}=0+\frac{-\sin\theta}{1+\cos\theta}$$

$$\cot\phi=\frac{-2\sin\theta/2\cdot\cos\theta/2}{2\cos^2\theta/2}=-\tan\theta/2=\cot(\pi/2+\theta/2)\quad\left[\because\cot(\pi/2+\theta)=-\tan\theta\right.$$

$$\Rightarrow\phi=\frac{\pi}{2}+\frac{\theta}{2}.$$

put ϕ value in $p=r\sin\phi$

$$p=r\sin(\pi/2+\theta/2)=r\cos\theta/2.$$

Eliminate θ from $r=a(1+\cos\theta)$ & $p=r\cos\theta/2$.

$$r=a(2\cos^2\theta/2)\quad\Big|\quad\frac{p}{r}=\cos\theta/2$$

$$r=2a\cdot\frac{p^2}{r^2}$$

$$r^3=2ap^2$$ is the required pedal equation.

15MATDIP.

2) Find the pedal equation of $r=a(1-\cos\theta)$.

solⁿ: $r=a(1-\cos\theta)$

Take log on b.s.

$$\log r=\log a+\log(1-\cos\theta)$$

diff w.r.t θ.

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{a\sin\theta}{1-\cos\theta}$$

$$\cot\phi = \frac{\cancel{2}\sin\theta/2\cos\theta/2}{\cancel{2}\sin^2\theta/2} = \cot\theta/2 \Rightarrow \phi = \theta/2$$

Put ϕ value in $p = r\sin\phi$.

$p = r\sin\theta/2$

Eliminate θ from, $r = a(1-\cos\theta)$ & $p = r\sin\theta/2$.

$r = a\,2\sin^2\theta/2$ | $\frac{p}{r} = \sin\theta/2$.

$r = 2a\frac{p^2}{r^2}$

$r^3 = 2ap^2$ is the required pedal equation.

3) obtain the pedal equation of the curve, $r^n = a^n\cos n\theta$.

Sol^n: $r^n = a^n\cos n\theta$.

Take log on b.s

$n\log r = n\log a + \log\cos n\theta$.

$$n\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin n\theta\cdot n}{\cos n\theta}$$

$\cancel{n}\cot\phi = -\cancel{n}\tan n\theta = \cot(\pi/2 + n\theta)$

$\phi = \frac{\pi}{2} + n\theta$.

Substitute ϕ value in $p = r\sin\phi. = r\sin(\frac{\pi}{2}+n\theta) = r\cos n\theta$.

Eliminate θ from $r^n = a^n\cos n\theta$ & $p = r\cos n\theta$.

$r^n = a^n\left(\frac{p}{r}\right)$ | $\frac{p}{r} = \cos n\theta$.

$r^{n+1} = pa^n$ is the required pedal equation.

4) Find the pedal equation for the curve $r^m\cos m\theta = a^m$.

Sol^n: $r^m\cos m\theta = a^m$

$m\log r + \log\cos m\theta = m\log a$.

$$m\cdot\frac{1}{r}\frac{dr}{d\theta} + \frac{-m\sin m\theta}{\cos m\theta} = 0.$$

$\cancel{m}\cot\phi = m\tan m\theta = \cancel{m}\cot(\frac{\pi}{2} - m\theta) \Rightarrow \phi = \frac{\pi}{2} - m\theta$.

Put ϕ value in $p = r\sin\phi$.

$p = r\sin\left(\frac{\pi}{2} - m\theta\right) = r\cos m\theta$.

Eliminate θ from, $r^m\cos m\theta = a^m$ & $p = r\cos m\theta$.

$r^m(p/r) = a^m$ | $p/r = \cos m\theta$.

$r^{m-1}p = a^m$ (or) is the required pedal equation.

5) Find the pedal equation for $r^2 = a^2 \sec 2\theta$.

Sol^n: $r^2 = a^2 \sec 2\theta$.

Take log on b.s

$2\log r = 2\log a + \log \sec 2\theta$.

$\cancel{2}\cdot\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{\sec 2\theta \cdot \tan 2\theta \cdot \cancel{2}}{\sec 2\theta}$

$\cot\phi = \tan 2\theta = \cot(\pi/2 - 2\theta) \Rightarrow \phi = \frac{\pi}{2} - 2\theta$.

Put ϕ value in $p = r\sin\phi = r\sin(\pi/2 - 2\theta) = r\cos 2\theta$.

Eliminate θ from, $r^2 = a^2 \sec 2\theta$ & $p = r\cos 2\theta$.

$r^2 = a^2\left(\frac{r}{p}\right)$ | $\frac{p}{r} = \cos 2\theta$. (or) $\frac{r}{p} = \sec 2\theta$.

$pr = a^2$. is the required pedal equation.

Taylor's & Maclaurin's series.: Taylors theorem is a pathway to discuss the series expansion of a differentiable function referred to as Taylor's series.

Maclaurin's theorem is a particular case of Taylor's theorem that leads to Maclaurin's series.

Maclaurin's expansion: if $a = 0$,

$$\boxed{y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \cdots}$$

Working rule:

i) Evaluate at $x = 0$ for obtaining the Maclaurin's expansion.

ii) To reduce computational work we perfer successive differentiatio

3rd Exam

1) Obtain the Maclaurin's series expansion of the function $e^x \sin x$.

Sol^n: Maclaurin's expansion,

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \cdots$$

Let $y = e^x \sin x$ $\quad \therefore y(0) = e^0 \sin 0 = 0$.

$y_1 = e^x \cos x + \sin x \cdot e^x$ $\quad y_1(0) = e^0 \cdot \cos 0 + 0 = 1$

$y_1 = e^x \cos x + y$ $\quad y_2(0) = 0 + e^0 \cdot \cos 0 + 1 = 2$.

$y_2 = -e^x \sin x + e^x \cos x + y_1$

$y_2 = -y + e^x \cos x + y_1$ (or) $-y + y_1 - y + y_1 = -2y + 2y_1$

$\cancel{y_3 = -2y_1 + e^x(-\sin x) + e^x \cos x + y_2}$

$y_3(0) = -2(1) + 2(2) = -2 + 4 = 2$.

$y_3 = -2y_1 + 2y_2$ $\quad y_3(0) = -2(1)+2(2) = -2+4 = 2$... $y_4(0) = -2(2) + 2(2) = -4 + 4 = 0$.

$y_4 = -2y_2 + 2y_3$ $\quad y_5(0) = -2(2) + 2(0) = -4$

$y_5 = -2y_3 + 2y_4$

Substituting all these values in $y(x)$ expansion,

$$e^x \sin x = 0 + x(1) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(-4) + \cdots$$

$$e^x \sin x = x + \frac{x^2}{2} \cdot 2 + \frac{x^3}{6_3} 2 - \frac{x^5}{120_{30}} \times 4 + \cdots$$

$$e^x \sin x = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

3rd Exam

2) Obtain the Maclaurin's series expansion of the function, $\sqrt{1+\sin 2x}$ up to the term containing x^4.

Sol^n: Maclaurin's expansion,

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \cdots$$

Let $y = \sqrt{1+\sin 2x} = \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cdot \cos x} = \sqrt{(\sin x + \cos x)^2} = \sin x + \cos x$.

$y(0) = 0 + 1 = 1$.

$y_1(0) = 1 - 0 = 1$.

$y_1 = \cos x - \sin x$ $\quad y_2(0) = 0 - 1 = -1$.

$y_2 = -\sin x - \cos x = -(\sin x + \cos x) = -y$ $\quad y_3(0) = -1$

$y_3 = -y_1$ $\quad y_4(0) = -(-1) = 1$.

$y_4 = -y_2$

substituting all these values in $y(x)$ expansion,

$$\sqrt{1+\sin 2x} = 1 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(1) + \cdots$$

(18). $\sqrt{1+\sin 2x} = 1+x-\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}+\cdots$

ISHATDIP

3) Expand $\log_e(1+x)$ in ascending powers of x as far as the term containing x^4.

Sol^n^: Maclaurin's expansion,

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \cdots$$

$y = \log_e(1+x)$ $\qquad$ $y(0) = \log 1 = 0.$

$y_1 = \frac{1}{1+x}$ $\qquad$ $y_1(0) = \frac{1}{1} = 1.$

$(1+x)y_1 = 1$

$(1+x)y_2 + (0+1)y_1 = 0$ $\quad$ at $x=0$, $\quad$ $1 y_2(0) + y_1(0) = 0 \Rightarrow y_2(0) = -y_1$

$y_2(0) = -1$

$(1+x)y_3 + (1)y_2 + y_2 = 0$

$(1+x)y_3 + 2y_2 = 0$ $\quad$ at $x=0$, $\quad$ $y_3(0) + 2y_2 = 0 \Rightarrow y_3(0) = -2y_2 = -2(-1)$

$\cancel{2y_3(0) = -2(-1)}$

$\cancel{y_3(0) = \frac{2}{2}}$ $\quad y_3(0) = 2.$

$(1+x)y_4 + (1)y_3 + 2y_3 = 0$

$(1+x)y_4 + 3y_3 = 0$ $\quad$ at $x=0$, $\quad$ $y_4(0) + 3y_3 = 0$

$y_4(0) = -3y_3 = -3(2) = -6$

$\cancel{y_4(0) = -\frac{3}{4}}$

Substituting all these values in expansion,

$$\log_e(1+x) = 0 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6) + \cdots$$

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

4) Find the Maclaurin's series expansion of $\sec x$ upto x^4 term,

Sol^n^: Maclaurin's expansion,

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \cdots$$

$y(0) = \sec 0 = 1$

Let $y = \sec x$ $\qquad$ $y_1(0) = 1(0) = 0.$

$y_1 = \sec x \cdot \tan x = y \tan x$ $\qquad$ $y_2(0) = 1 + 0 = 1$

$y_2 = y\sec^2 x + \tan x \cdot y_1$ $\qquad$ $y_3(0) = 4(0) + 0 = 0.$

$y_3 = \boxed{y\, 2\sec x \cdot \sec x \tan x} + y_1 \sec^2 x + \tan x \cdot y_2 + y_1 \sec^2 x$

$y_3 = 2y_1 \sec^2 x + 2y_1 \sec^2 x + \tan x\, y_2 = 4y_1 \sec^2 x + \tan x \cdot y_2$

$y_4 = 4y_1(2\sec^2x\tan x) + 4\sec^2x\cdot y_2 + \tan x\, y_3 + y_2\sec^2x.$

$y_4 = 8y_1\sec^2x\cdot\tan x + 5y_2\sec^2x + \tan x\cdot y_3$ $\quad y_4(0) = 0+5(1)(1)+0 = 5.$

Substituting all these values in expansion,

$\sec x = 1 + 0 + \frac{x^2}{2!}(1) + 0 + \frac{x^4}{4!}(5) + \cdots$

$\sec x = 1 + \frac{x^2}{2} + \frac{x^4}{24}\cdot 5 + \cdots$

5) Obtain Maclaurin's series for $\log(\sec x)$ upto the term containing x^6.

Sol^n: Maclaurin's expansion,

$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \frac{x^5}{5!}y_5(0) + \frac{x^6}{6!}y_6(0) + \cdots$

$y = \log(\sec x)$ $\quad y(0) = \log 1 = 0.$

$y_1 = \frac{1}{\sec x}\sec x\cdot\tan x = \tan x$ $\quad y_1(0) = 0.$

$y_2 = \sec^2 x$ $\quad y_2(0) = \sec^2(0) = 1.$

$y_3 = 2\sec x\cdot\sec x\cdot\tan x$ $\quad y_3(0) = 0.$

$y_3 = 2\sec^2x\cdot\tan x = 2y_2y_1$ $\quad y_4(0) = 2(0+1) = 2.$

$y_4 = 2(y_1y_3 + y_2^2)$ $\quad y_5(0) = 0+0 = 0.$

$y_5 = 2(y_1y_4 + \underline{y_2y_3} + \underline{2y_2y_3}) = 2y_1y_4 + 6y_2y_3$ $\quad y_6(0) = 0 + 8(1)(2) + 0$

$y_6 = 2y_1y_5 + 2y_2y_4 + 6y_2y_4 + 6y_3^2$ $\quad = 16$

$y_6 = 2y_1y_5 + 8y_2y_4 + 6y_3^2$

Substitute all these values in expansion,

$\log(\sec x) = 0 + 0 + \frac{x^2}{2!}(1) + 0 + \frac{x^4}{4!}(2) + 0 + \frac{x^6}{6!}(16) + \cdots$

$\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{\cancel{24}_{12}}\cdot\cancel{2} + \frac{x^6}{\cancel{720}_{45}}\cdot\cancel{16} + \cdots = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots$

Partial Differentiation: "It is a method to differentiate a function w.r.t 1 independent variable while treating the other variable as constant."

The partial derivative of u w.r.t x is given by,

$\frac{\partial u}{\partial x}$ (or) u_x (treating y as constant).

The partial derivative of u w.r.t y is given by.

$\frac{\partial u}{\partial y}$ (or) u_y (treating x as a constant).

(20)

$$u = f(x,y)$$

1st order partial derivative

$$u_x = \frac{\partial u}{\partial x} \qquad u_y = \frac{\partial u}{\partial y}$$

2nd order partial derivatives

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \qquad u_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$$

Mixed partial derivatives

$$u_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} \longleftrightarrow u_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \cdot \partial y}$$

NOTE: $\frac{\partial^2 u}{\partial y \cdot \partial x} = \frac{\partial^2 u}{\partial x \cdot \partial y}$ (or) $u_{yx} = u_{xy}$.

Problems.

ISNATDIT

1) If $u = \log_e\left(\frac{x^4+y^4}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

Sol^n: $u = \log(x^4+y^4) - \log(x+y)$

diff w.r.t x.

$$\frac{\partial u}{\partial x} = u_x = \frac{1}{x^4+y^4}(4x^3) - \frac{1}{x+y}(1) \quad \text{| Here } y \text{ is treated as constant,}$$

$$\frac{\partial u}{\partial y} = u_y = \frac{1}{x^4+y^4}(4y^3) - \frac{1}{x+y}(1) \quad \text{| Here } x \text{ is treated as constant.}$$

$$x \cdot u_x + y \cdot u_y = \frac{4x^4}{x^4+y^4} - \frac{x}{x+y} + \frac{4y^4}{x^4+y^4} - \frac{y}{x+y}$$

$$x \cdot u_x + y \cdot u_y = \frac{4\cancel{(x^4+y^4)}}{\cancel{x^4+y^4}} - \frac{\cancel{(x+y)}}{\cancel{(x+y)}} = 4 - 1 = 3.$$

SOIDIP

2) If $u = e^{x^3+y^3}$, then prove that $x\frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 3u \cdot \log u$.

Sol^n: $\frac{\partial u}{\partial x} = e^{x^3+y^3} \cdot (3x^2)$

$x \cdot \frac{\partial u}{\partial x} = e^{x^3+y^3}(3x^3) \rightarrow (1)$

| $\frac{\partial u}{\partial y} = e^{x^3+y^3}(3y^2)$

| $y \cdot \frac{\partial u}{\partial y} = e^{x^3+y^3}(3y^3) \rightarrow (2)$.

Add eq^n (1) & (2),

| $\because e^{x^3+y^3}(3x^3) + e^{x^3+y^3}(3y^3) = 3e^{x^3+y^3}(x^3+y^3)$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 3e^{x^3+y^3}(x^3+y^3)$$

| $\because u = e^{x^3+y^3}$

$\log_e u = x^3+y^3 \Rightarrow u = e^{x^3+y^3}$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 3 \cdot u \cdot \log u$$

3) If $u = e^{ax+by} f(ax-by)$ prove that $b\cdot\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abu$.

Sol^n: $\frac{\partial u}{\partial x} = e^{ax+by} f'(ax-by)\cdot a + a\cdot e^{ax+by} f(ax-by)$ | $\because$ product rule.

$\frac{\partial u}{\partial x} = e^{ax+by} f'(ax-by)\cdot a + a u$

$b\cdot\frac{\partial u}{\partial x} = ab\, e^{ax+by} f'(ax-by) + abu \rightarrow (1)$.

$\frac{\partial u}{\partial y} = e^{ax+by} f'(ax-by)(-b) + b e^{ax+by} f(ax-by)$ | $\because$ product rule.

$\frac{\partial u}{\partial y} = -e^{ax+by} f'(ax-by)\, b + bu$

$a\cdot\frac{\partial u}{\partial y} = -ab\, e^{ax+by} f'(ax-by) + abu \rightarrow (2)$.

Add eq^n (1) & (2), $b\cdot\frac{\partial u}{\partial x} + a\cdot\frac{\partial u}{\partial y} = ab e^{ax+by} \cancel{f'(ax-by)} + abu - ab e^{ax+by} \cancel{f'(ax-by)} + abu$

$b\cdot\frac{\partial u}{\partial x} + a\cdot\frac{\partial u}{\partial y} = 2abu$.

4) If $z = \sin(ax+by) + \cos(ax-y)$, prove that $\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$

Sol^n: $\frac{\partial z}{\partial x} = \cos(ax+by)\cdot a - \sin(ax-y)\cdot a = a(\cos(ax+by) - \sin(ax-y))$.

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = a(-\sin(ax+by)\cdot a - \cos(ax-y)\, a) = -a^2(\sin(ax+y) + \cos(ax-y))$

$\frac{\partial^2 z}{\partial x^2} = -a^2 z \rightarrow (1)$ | $\because z = \sin(ax+y) + \cos(ax-y)$.

$\frac{\partial z}{\partial y} = \cos(ax+y)\cdot 1 - \sin(ax-y)(-1) = \cos(ax+y) + \sin(ax-y)$.

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = -\sin(ax+y)\cdot 1 + \cos(ax-y)(-1) = -(\sin(ax+y) + \cos(ax-y))$

$\frac{\partial^2 z}{\partial y^2} = -z$ | $\because$ given. $\Rightarrow a^2\frac{\partial^2 z}{\partial y^2} = -a^2 z \rightarrow (2)$.

From eq^n (1) & (2), $\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$.

5) If $u = \tan^{-1}(y/x)$ verify that $\frac{\partial^2 u}{\partial y\cdot\partial x} = \frac{\partial^2 u}{\partial x\cdot\partial y}$.

Sol^n: $\frac{\partial u}{\partial x} = \frac{1}{1+(y/x)^2}\cdot\frac{\partial}{\partial x}(y/x) = \frac{\cancel{x^2}}{x^2+y^2}\left(\frac{-y}{\cancel{x^2}}\right) = \frac{-y}{x^2+y^2}$.

$\frac{\partial u}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\frac{\partial}{\partial y}(y/x) = \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$.

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$

(22). $\frac{\partial^2 u}{\partial y \cdot \partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} \rightarrow (1)$

111^{ly} $\frac{\partial^2 u}{\partial x \cdot \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{x^2+y^2(1)-x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \rightarrow (2)$

from eqⁿ (1) & (2), $\frac{\partial^2 u}{\partial x \cdot \partial y} = \frac{\partial^2 u}{\partial y \cdot \partial x}$ (or) $u_{xy} = u_{yx}$.

Total differentiation: If $u = f(x,y)$ then the total differential or the exact differential of u is defined as $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} \cdot dy$.

Differentiation of composite functions:

i) If $u = f(x,y)$ where $x = x(t)$ & $y = y(t)$ then u is said to be a composite function of the single variable 't'.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}.$$

ii) If $u = f(x,y)$ where $x = x(r,s)$ & $y = y(r,s)$ then u is a composite function of two variables r, s.

$u \rightarrow (x,y) \rightarrow (r,s) \Rightarrow u \rightarrow (r,s)$ $\nearrow \partial u/\partial r$, $\searrow \partial u/\partial s$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} \quad \& \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}.$$

Working rule:

i) Analyse the composition of the variables & write the appropriate formula.

ii) Then substitute for the possible derivatives in the formula & simplify according to the requirement.

Problem:

15MAT11P

1) Find the total derivative of $z = xy^2 + x^2y$, where $x = at^2$, $y = 2at$.

Solⁿ: $z \rightarrow (x,y) \rightarrow t \Rightarrow z \rightarrow t$ & $\frac{dz}{dt}$ is total derivative.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

$\left| \because \frac{\partial z}{\partial x} = y^2 + 2xy, \; \frac{dx}{dt} = 2at \right.$

$\frac{\partial z}{\partial y} = 2xy + x^2, \; \frac{dy}{dt} = 2a$

$\frac{dz}{dt} = (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$ $\mid \because x = at^2, y = 2at$.

$\frac{dz}{dt} = (4a^2t^2 + 4a^2t^3)(2at) + (4a^2t^3 + a^2t^4)(2a)$

$\frac{dz}{dt} = 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^4 = 16a^3t^3 + 10a^3t^4$

∴ The total derivative $\frac{dz}{dt} = 16a^3t^3 + 9a^3t^4 \to (1)$

$z = xy^2 + x^2y = (at^2)(2at)^2 + (at^2)^2(2at)$

$z = 4a^3t^4 + 2a^3t^5$

$\frac{dz}{dt} = 16a^3t^3 + 10a^3t^4 \to (2)$.

Thus, from (1) & (2) the result is verified.

301 MAT2TP

2) Find the total derivative of $z = xy^2 + x^2y$, where $x = at$, $y = 2at$ & also verify the result by direct substitution.

Sol^n: $z \to (x, y) \to t \Rightarrow z \to t$ & $\frac{dz}{dt}$ is total derivative.

$\therefore \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$.

$\frac{dz}{dt} = (y^2 + 2xy)a + (2xy + x^2)2a$. $= (8a^2t^2)(a) + (5a^2t^2)(2a)$.

$\frac{dz}{dt} = (4a^2t^2 + 4a^2t^2)a + (4a^2t^2 + a^2t^2)2a$

$\frac{dz}{dt} = 8a^3t^2 + 10a^3t^2 = 18a^3t^2 \to (1)$.

$z = xy^2 + x^2y = (at)(2at)^2 + (at)^2(2at)$

$z = 4a^3t^3 + 2a^3t^3 = 6a^3t^3 \Rightarrow z = 6a^3t^3$

$\frac{dz}{dt} = 6a^3 \cdot (3t^2) = 18a^3t^2 \to (2)$.

Thus from (1) & (2), $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$ is verified.

MAT2TP301

3) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

Sol^n: Let $u = f(r, s, t)$ where $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$.

i.e., $u \to (r, s, t) \to (x, y, z) \Rightarrow u \to (x, y, z)$.

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$.

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{1}{y} + \frac{\partial u}{\partial s} \cdot 0 + \frac{\partial u}{\partial t} \cdot \left(\frac{-z}{x^2}\right)$

x^ly b.s by x.

$x \cdot \frac{\partial u}{\partial x} = \frac{x}{y} \cdot \frac{\partial u}{\partial r} - \frac{z}{x^2} \cdot \frac{\partial u}{\partial t} \to (1)$.

$y\frac{\partial u}{\partial y} = \frac{y}{z}\frac{\partial u}{\partial s} - \frac{x}{y}\frac{\partial u}{\partial r} \to (2)$

| ∵ Symmetric condition,

$x \to y \to z \to x$ $\quad$ $r \to s \to t \to r$

$$z\frac{\partial u}{\partial z} = \frac{z}{x}\cdot\frac{\partial u}{\partial t} - \frac{y}{z}\cdot\frac{\partial u}{\partial s} \rightarrow (3).$$

Add eqn (1), (2) & (3), $\quad x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$

15MATDIP

4.) If $u = f(x-y,\, y-z,\, z-x)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

Soln: Let $u = f(r, s, t)$. where $r = x-y,\ s = y-z,\ t = z-x.$

i.e., $u \rightarrow (r,s,t) \rightarrow (x,y,z) \Rightarrow u \rightarrow (x,y,z).$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}.$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot(1) + \frac{\partial u}{\partial s}\cdot 0 + \frac{\partial u}{\partial t}\cdot(-1) \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cancel{+} - \frac{\partial u}{\partial t} \rightarrow (1).$$

$|||^{ly}$ $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r} \rightarrow (2)$ & $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} - \frac{\partial u}{\partial s} \rightarrow (3)$ $\quad$ [$\because$ symmetric

Add (1), (2), (3),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

MATDIP301

5.) If $u = f(y-z,\, z-x,\, x-y)$ prove that : $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

where $r = y-z,\ s = z-x,\ t = x-y.$

Soln: Let $u = f(r,s,t) \Rightarrow u \rightarrow (x,y,z)$

i.e., $u \rightarrow (r,s,t) \rightarrow$ ~~these~~ (x,y,z)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}.$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot 0 + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}(1)$$

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \rightarrow (1)$$

$|||^{ly}$ $\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} \rightarrow (2).$

$$\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \rightarrow (3).$$

Add eqns (1), (2) & (3),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Jacobians:

Let u & v ~~be~~ be functions of two independent variables x & y. The Jacobian (J) of u & v w.r.t x & y ~~are~~ is defined as follows,

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Similarly if u, v, w are functions of three independent variables x, y, z then, (25)

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

problems:

MATDIP301

1) If $u = x^2+y^2+z^2$, $v = xy+yz+zx$, $w = x+y+z$ find $J\left(\frac{u,v,w}{x,y,z}\right)$.

Sol^n: $$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

$$J = 2x(\cancel{x}+z-y-\cancel{x}) - 2y(\cancel{y}+z-\cancel{y}-x) + 2z(y+\cancel{z}-x-\cancel{z})$$

$$J = \cancel{2xz} - \cancel{2xy} - \cancel{2yz} + \cancel{2xy} + \cancel{2yz} - \cancel{2xz} = \underline{0}.$$

2) Find the jacobian of u, v, w w.r.t x, y, z given $u = x+y+z$, $v = y+z$, $w = z$.

Sol^n: $$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$J = 1(1-0) - 1(0-0) + 1(0-0) = 1 - 0 + 0 = \underline{1}.$$

MATDIP301

3) If $x = u(1-v)$ & $y = uv$ find $J = \frac{\partial(x,y)}{\partial(u,v)}$ & $J' = \frac{\partial(u,v)}{\partial(x,y)}$ & also verify $J \cdot J' = 1$.

$\because x = u - uv$
$y = uv$

Sol^n: $$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$J = u(1-v) + uv = u - \cancel{uv} + \cancel{uv} = u.$$

Now using given equations get the values for u & v.

Let $x = u - uv \rightarrow (1)$ & $y = uv \rightarrow (2)$.

$u = x - y$ | $\because y = uv$ | $v = \frac{y}{u} = \frac{y}{x-y}$ | $\because u = x-y$.

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ \frac{-y}{(x-y)^2} & \frac{x}{(x-y)^2} \end{vmatrix}$$

$\because \frac{\partial v}{\partial x} = y\left(\frac{-1}{(x-y)^2}\right) = \frac{-y}{(x-y)^2}$

$\because \frac{\partial v}{\partial y}$ use Quotient rule.

(26)

$$J' = \frac{x}{(x-y)^2} - \frac{y}{(x-y)^2}$$ | i.e., $\frac{\partial v}{\partial y} = \frac{(x-y)(1)-y(-1)}{(x-y)^2} = \frac{x-\cancel{y}+\cancel{y}}{(x-y)^2}$

$$J' = \frac{\cancel{(x-y)}}{(x-y)^{\cancel{2}}} = \frac{1}{x-y} = \frac{1}{u}$$ | $\because u = x-y$.

~~$\therefore J \times J' =$~~ $\therefore J \cdot J' = u \times \frac{1}{u} = \underline{1}$.

4) If $u = x+y+z$, $uv = y+z$, $uvw = z$ then show that

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v.$$

Sol^n:
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

we should have x, y, z in terms of u, v, w.

consider, $u = x+y+z \rightarrow (1)$ $uv = y+z \rightarrow (2)$ $uvw = z \rightarrow (3)$.

Put eq^n(2) in (1), $u = x+uv \Rightarrow x = u-uv$.

put eq^n(3) in (2), $uv = y+uvw \Rightarrow y = uv-uvw$. W.K.T $z = uvw$.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)\left(uv(u-uw) + u^2vw\right) + u\left(uv(v-vw) + uv^2w\right) + 0.$$

$$= (1-v)\left(u^2v - u^2\cancel{v}w + u^2\cancel{v}w\right) + u\left(uv^2 - uv^2\cancel{w} + uv^2\cancel{w}\right)$$

$$= u^2v - \cancel{u^2v^2} + \cancel{u^2v^2}$$

$$= \underline{u^2 v}.$$

Scanned by CamScanner