Properties of Regular Stochastic Matrix :-

Regular stochastic matrix :- A stochastic matrix P is said to be regular stochastic matrix if all the entries of some power "P^n" are positive.

Ex :- $A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

Consider ; $A^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$

$\therefore$ A is Regular stochastic matrix, $n = 3$

Properties of Regular Stochastic matrix :-

① P has a unique fixed point, $x = (x_1, x_2, \dots x_n)$ such that ;

$xP = x$

② P has a finite unique fixed probability vector, $v = (v_1, v_2, \dots v_n)$ such that ; $vP = v$, where, $v_i = \sum_{i=1}^{n} x_i$.

③ $P^2, P^3, \dots$ approaches the matrix V, ~~(then the sequence of vectors $vP, vP^2, \dots$)~~ whose each rows are fixed probability vector. v.

④ If u is any probability vector, then the sequence of vectors; $uP, uP^2, \dots$ approaches the unique fixed probability vector v.

Problems :-

1) If $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is a stochastic matrix & $v = [v_1, v_2]$ is a probability vector, show that vA is also probability vector.

Soln :- By data : $a_1 + a_2 = 1$, $b_1 + b_2 = 1$, $v_1 + v_2 = 1$

$\therefore \quad vA = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \begin{bmatrix} v_1a_1 + v_2b_1 & v_1a_2 + v_2b_2 \end{bmatrix}$

We have to proove that ; $(v_1a_1 + v_2b_1) + (v_1a_2 + v_2b_2) = 1$.

LHS $= v_1(a_1+a_2) + v_2(b_1+b_2) = v_1 \cdot 1 + v_2 = v_1 + v_2 = 1$

Thus, vA is also a "probability vector."

(2) P.T with reference to 2 second order stochastic matrices that their product is also a stochastic matrix.

Soln :- Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ & $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ be a stochastic matrix.

Hence, we have ; $\left.\begin{matrix} a_{11}+a_{12}=1, & b_{11}+b_{12}=1 \\ a_{21}+a_{22}=1 & b_{21}+b_{22}=1. \end{matrix}\right\} - (1).$

$\therefore A.B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

We have to S.T ; $a_{11}b_{11} + a_{12}b_{21} + a_{11}b_{12} + a_{12}b_{22} = 1 \quad \sim (2)$

& $a_{21}b_{11} + a_{22}b_{21} + a_{21}b_{12} + a_{22}b_{22} = 1 \quad - (3).$

LHS of (2); can be written as ;

$\Rightarrow a_{11}(b_{11}+b_{12}) + a_{12}(b_{21}+b_{22})$

$\Rightarrow a_{11} \cdot 1 + a_{12} \cdot 1 = a_{11} + a_{12} = 1$, by using (1).

LHS of (3) $\Rightarrow$

$a_{21}(b_{11}+b_{12}) + a_{22}(b_{21}+b_{22})$

$\Rightarrow a_{21} \cdot 1 + a_{22} \cdot 1 = 1. \Rightarrow a_{21} + a_{22} = 1$

Thus, AB is a "stochastic matrix."

3. Find the unique fixed probability vector of the regular stochastic matrix; $A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$

Soln :- $A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$, $v = [x, y]$

We have to find; $v = (x, y)$ where; $x + y = 1$; $\ni$: $\underline{vA = v}$

ie, $[x, y]\begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = [x, y]$

$\Rightarrow \left[\frac{3}{4}x + \frac{1}{2}y \quad \frac{1}{4}x + \frac{1}{2}y\right] = [x, y]$

$\Rightarrow \frac{3}{4}x + \frac{1}{2}y = x$ — (1) & $\frac{1}{4}x + \frac{1}{2}y = y$. (2)

We can solve either of 2 equations, by using; $y = 1 - x$.

Using; $\underline{y = 1 - x}$ in Eqn (1)

we have; $\frac{3}{4}x + \frac{1}{2}(1 - x) = x \Rightarrow 3x + 2 - 2x = 4x$

$\therefore \boxed{x = 2/3}$

Hence, $y = 1 - x = \boxed{1/3 = y}$

$\therefore \boxed{v = (x, y) = (2/3, 1/3)}$

Thus, $(2/3, 1/3)$ is the unique "$\underline{\text{fixed probability vector}}$" for the regular.

4. Find the unique fixed probability vector for the regular stochastic matrix: $A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$

Soln :- We have to find $v = (x, y, z)$, where ; $x + y + z = 1$

$\ni$: $\underline{vA = v}$

$\Rightarrow [x, y, z]\begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [x, y, z]$

$\Rightarrow \left[y/6 \, . \, x + y/2 + \frac{2z}{3} \, . \, \frac{y}{3} + \frac{z}{3}\right] = [x \, . \, y \, . \, z]$

$\Rightarrow \frac{y}{6} = x, \quad x + \frac{y}{2} + \frac{2z}{3} = y, \quad \frac{y}{3} + \frac{z}{3} = z$

ie, $\underline{y = 6x}, \quad 6x + \underline{3y + 4z} = 6y, \quad y + \underline{z} = 3z$

$\underline{y = 6x} \qquad 6x - 3\underline{y + 4z} = 0 \qquad y - \underline{2z} = 0$

Using $\underline{y = 6x}$ & $z = 1 - x - y = 1 - x - 6x = 1 - 7x$ in : $6x - 3y + 4z = 0$

We have; $\quad 6x - 18x + 4 - 28x = 0.$

$\therefore \boxed{x = 1/10}$

Hence, $\boxed{y = 6/10} \quad \boxed{z = 3/10}$

Thus, the required unique fixed probability vector v is given by;

$\boxed{v = (1/10, 6/10, 3/10)}$

(5) Show that $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is Regular stochastic matrix.

Soln :- $P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$

$P^3 = P^2 . P = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$

$P^4 = P^3 . P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$

$P^5 = P^4 . P = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$

$\therefore$ We observe that in: P^5 all entries are "positive"

Thus, P is a regular "stochastic matrix."

MARKOV CHAINS :-

A stochastic process which is such that the generation of the probability distribution depend only on the present state is called "Markov chain".

Transition matrix :- A square matrix of order "m", in which all the entries, (probabilities (p_{ij})) which are non-zero real numbers is a transition matrix.

Higher transition probability / n-step transition matrix. :-
The probability that the system changes from the state a_i to state a_j in exactly "n" steps is denoted by "$P_{ij}^{(n)}$", the matrix formed by probabilities ; $P_{ij}^{(n)}$ is called "n-step transition matrix".

Stationary distribution of regular Markov chains :-
A markov chain is said to be "Regular", if the associated transition probability matrix P is "Regular".

Absorbing state of a Markov chain :-
In a markov chain, if process reaches to a certain state after which it continues to remain in the same state is "Absorbing state of a Markov chain."

Problems :-

1) The transition matrix P of a markov chain is Given by;
$\begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$ with the initial probability distribution : $p^{(0)} = (1/4, 3/4)$

Define & find the following :- ① $P_{21}^{(2)}$ ② $P_{12}^{(2)}$ ③ $p^{(2)}$ ④ $P_1^{(2)}$

Soln :-

① $P_{21}^{(2)}$ is the probability of moving from state a_2 to state a_1 in 2 steps; this can be obtained from 2-step transition matrix P^2.

Consider; $P^2 = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} = \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix} = \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} \end{bmatrix}$

$\therefore P_{21}^{(2)} = 9/16 = \boxed{9/16 = P_{21}^{(2)}}$

② $P_{12}^{(2)}$ is the probability of moving from state a_1 to a_2 in 2 steps : $\boxed{P_{12}^{(2)} = 3/8.}$

③ $p^{(2)}$ is the probability distribution of system after 2 steps :-

$\therefore p^{(2)} = p^{(0)} P^2 = [1/4, 3/4] \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix} = \begin{bmatrix} 37/64 & 27/64 \end{bmatrix}$

ie; $p^{(2)} = [37/64 \quad 22/64] = [P_1^{(2)} \quad P_2^{(2)}]$

④ $P_1^{(2)}$ is the probability that process is in the state a_1 after 2 steps.

ie; $\boxed{P_1^{(2)} = 37/64}$

② The "transition probability matrix" (tpm) of a Markov chain is given by; $P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$ and the initial probability distribution is:

$$p(0) = [1/2, 1/2, 0]$$

Find: ① $P_{13}^{(2)}$ ② $P_{23}^{(2)}$ ③ $p^{(2)}$ ④ $p_1^{(2)}$.

Soln:- First, we find the 2 step transition matrix; P^2.

$$\therefore P^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} P_{11} & P_{12} & P_{13} \end{array} \\ \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix} \end{array} = \begin{bmatrix} P_{11}^2 & P_{12}^{(2)} & P_{13}^{(2)} \\ P_{21}^2 & P_{22}^2 & P_{23}^2 \\ P_{31}^2 & P_{32}^2 & P_{33}^2 \end{bmatrix}$$

$$\therefore ① \Rightarrow \boxed{P_{13}^{(2)} = 3/8}$$

$$② \Rightarrow \boxed{P_{23}^{(2)} = 1/2}$$

Consider; $p^{(2)} = p^{(0)} \cdot P^2 = [1/2 \;\; 1/2 \;\; 0] \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix}$

$$\therefore \boxed{p^{(2)} = [7/16 \;\; 1/8 \;\; 7/16]}$$

$$\therefore \boxed{p_1^{(2)} = 7/16}$$

③. P.T. the Markov chain whose "tpm" is: $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is irreducible.

Soln:- We shall show that; P is a regular stochastic matrix. For convenience, we shall write the given matrix in the form;

$$P = \frac{1}{6} \begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}$$

Consider ; $P^2 = \frac{1}{36}\begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 18 & 6 & 12 \\ 9 & 21 & 6 \\ 9 & 12 & 15 \end{bmatrix}$.

Since, all the entries in P^2 are "positive"

We conclude that "tpm" is Regular.

Hence, the markov chain having "t.p.m. P is "irreducible".

Word problems :-

① A student's study habits are as follows, If he studies one night, he is 70% sure not to study next night, On the other hand if he doesnot study one night, he is 60% sure not to study the next night, In the long run how often does he study ?

Soln :- The state space of the system is : {A, B}

where ; A : studying , B : Not studying

The associated transition matrix P is as follows ;

$$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

$70\% = \frac{70}{100} = 0.7$

$60\% = \frac{60}{100} = 0.6.$

$(100-70)\% = 30\% = \frac{30}{100} = 0.3$

$(100-60)\% = 40\% = \frac{40}{100} = 0.4$,

In order to find the happening in the long run we have to find the unique fixed probability vector v of P,

ie, to find : $v = (x, y) \ni : vP = v$

where ; $x + y = 1$.

ie, $[x, y]\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [x, y]$

$$\Rightarrow \begin{bmatrix} 0.3x+0.4y & 0.7x+0.6y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\Rightarrow 0.3x+0.4y = x \quad \& \quad 0.7x+0.6y = y.$$

Using : $y = 1-x$ in the 1st of the equation, we have ;

$\Rightarrow 0.3x + 0.4(1-x) = x$ (or) $1.1x = 0.4$

$$\therefore \boxed{x = 4/11}$$

Since, $\boxed{x = 4/11}$, $\boxed{y = 7/11}$, $v = (4/11 \ \& \ 7/11) = (P_A, P_B)$.

Then, we Conclude that in the long run, the students will study $4/11$ of the time (or) 36.36% of the time ..

(2) Three boys : A, B, C are throwing ball to each other, "A" always throws the ball to B & B always throws the ball to C, C is just likely to throw the ball to B as to A. If C was the 1st person to throw the ball, find the probabilities that after 3 throws : (1) "A" has the ball (2) "B" has the ball (3) "C" has the ball.

Soln:- State space = {A, B, C} and the associated "t.p.m" is as follows :-

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & 1/2 & 1/2 & 0 \end{array}$$

1st throw : $C \to B \to 1$ (B)
$B \not\to A \to (0, 0)$ (A B)

2nd throw : $B \to C \to 1$ (C)

3rd throw : $C \to B$ (or) A.
$1/2 \quad 1/2$

Initially, if C has the ball, the associated initial probability vector is given by ;

$$p^{(0)} = (0, 0, 1)$$

Since, the probabilities are desired after 3 throws, we find : $p^{(3)} = p^{(0)} \cdot p^{(3)}$

$$\therefore\ P^2 = P.P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$\therefore\ P^3 = P^2.P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$\text{WKT; } P^{(3)} = P^{(0)}.P^3 = \begin{bmatrix} \overset{0}{\cancel{1/4}} & \overset{0}{\cancel{1/4}} & \overset{1}{\cancel{1/2}} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$\therefore\ \boxed{P^{(3)} = \left[1/4, 1/4, 1/2\right]} = \left[P_A^{(3)}, P_B^{(3)}\ P_C^{(3)}\right]$$

Thus, after 3 throws, the probability that, the ball is with A is : $1/4$, B is : $1/4$ & C is : $1/2$

③ A gambler's luck follows a pattern, If he wins a game, the possibility of winning the next game is : 0.6, However if he loses a game, the probability of losing the next game is 0.7 There is an even chance of gambler winning the 1st game, If so:

ⓐ What is the probability of he winning 2nd game.?

ⓑ what is the probability of he winning 3rd game ?

ⓒ In the long run, how often he will win ?

Soln :- State space = { win (W), lose (L) } and the associated t.p.m is as follows :-

$$P = \begin{array}{c c} & \begin{array}{cc} W & L \end{array} \\ \begin{array}{c} W \\ L \end{array} & \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} \end{array} = \frac{1}{10}\begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix}$$

Probability of winning of 1st game is : $1/2$

$\therefore$ initial probability vector : $P^{(0)} = (1/2 \;\; 1/2)$

ⓐ Now, $P^{(1)} = P^{(0)} \cdot P = \frac{1}{2}[1, 1] \cdot \frac{1}{10} \cdot \begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix}$

$$= \frac{1}{20}[9, 11]$$

Hence, $P^{(1)} = [9/20 \;\; 11/20] = [P^{(W)}, P^{(L)}]$

Thus, the probability that, winning of 2nd game is : $9/20$

ⓑ $P^{(2)} = P^{(1)} \cdot P = \frac{1}{20}[9 \;\; 11] \cdot \frac{1}{10}\begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix}$

$$= \frac{1}{200}[87, 113]$$

Hence, $P^{(2)} = [87/200, \; 113/200] = [P^{(W)}, P^{(L)}]$

Thus, the probability of winning 3rd game is : $87/200$

ⓒ we shall find fixed probability vector ;

$v = (x, y) \ni : vP = v$, where, $x + y = 1$

ie; $[x, y] \cdot \frac{1}{10}\begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix} = [x, y]$

$\Rightarrow 6x + 3y = 10x, \quad 4x + 7y = 10y.$

(or) $3y = 4x$ & by using ; $y = 1 - x$ we get ...

$\Rightarrow 3(1 - x) = 4x \quad \Rightarrow \boxed{x = 3/7}$

$\boxed{y = 4/7}$

Hence, $v = [3/7 \;\; 4/7] = [P^{(W)}, P^{(L)}]$

Thus, in the long run, he wins : $3/7$ of the time.