USN 1CR13CV400

Third Semester B.E. Degree Examination, Dec.2013/Jan.2014

Advanced Mathematics – I

Time: 3 hrs. Max. Marks:100

Note: ***Answer any FIVE full questions.***

1 a. Express the complex number $\frac{(1+i)(1+3i)}{1+5i}$ in the form x + iy. **(06 Marks)**

b. Find the modulus and amplitude of $\frac{(3-\sqrt{2}i)^2}{1+2i}$. **(07 Marks)**

c. Expand $\cos^8\theta$ in a series of cosines multiples of θ. **(07 Marks)**

2 a. Find the n^{th} derivative of $\sin(ax+b)$. **(06 Marks)**

b. If $y = (\sin^{-1} x)^2$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$. **(07 Marks)**

c. Find the nth derivative of $\left[\frac{1}{5(x-1)} + \frac{-3/2}{\left(\frac{-3}{2}-1\right)(2x+3)}\right]$. **(07 Marks)**

3 a. Using Taylor's theorem, express the polynomial $2x^3 + 7x^2 + x - 6$ in powers of $(x-1)$. **(06 Marks)**

b. Using Maclaurin's series, expand tan x upto the term containing x^5. **(07 Marks)**

c. If $Z = x^3 + y^3 - 3axy$ then prove that $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$. **(07 Marks)**

4 a. If $u = x\log xy$ where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$. **(06 Marks)**

b. If $z = f(x, y)$ and $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \cdot \frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$. **(07 Marks)**

c. If $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$, find the value of $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at (1, –1, 0). **(07 Marks)**

5 a. Obtain the reduction formula for $\int \sin^n x\, dx$. **(06 Marks)**

b. Evaluate $\int_0^a \frac{x^7 dx}{\sqrt{a^2-x^2}}$. **(07 Marks)**

c. Evaluate $\int_1^2\int_3^4 (xy + e^y)dydx$. **(07 Marks)**

6 a. Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z}dxdydz$. **(06 Marks)**

b. Find the value of $\Gamma\left(\frac{1}{2}\right)$. **(07 Marks)**

c. Prove that $\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$. **(07 Marks)**

7 a. Solve $\frac{dy}{dx} = e^{3x-2y} + x^2 \cdot e^{-2y}$. **(06 Marks)**

b. Solve $\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$ which is homogeneous in x and y. **(07 Marks)**

c. Solve $\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1)$. **(07 Marks)**

8 a. Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^x$. **(06 Marks)**

b. Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin 2x$. **(07 Marks)**

c. Solve $(D^2 - 1)y = x \sin 3x + \cos x$. **(07 Marks)**

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1. Express the complex no. $\frac{(1+i)(1+3i)}{1+5i}$ in the form $x+iy$

$$\frac{(1+i)(1+3i)}{1+5i} = \frac{1+4i-3}{1+5i} = \frac{-2+4i}{1+5i}$$

$$\frac{(-2+4i)}{(1+5i)} \frac{(1-5i)}{(1-5i)} = \frac{14i+18}{1+25}$$

$$= \frac{7i+9}{13} = \frac{9}{13} + \frac{7}{13} i$$

2. Find the modulus and amplitude of $\frac{(3-\sqrt{2}i)^2}{1+2i}$

Let $z = \frac{(3-\sqrt{2}i)^2}{1+2i} = \frac{9+(\sqrt{2}i)^2 - 6\sqrt{2}i}{1+2i}$

$$= \frac{7-6\sqrt{2}i}{1+2i} \frac{1-2i}{1-2i}$$

$$= \underbrace{\frac{(7-12\sqrt{2})}{5}}_{a} - i\underbrace{\left(\frac{14+6\sqrt{2}}{5}\right)}_{b}$$

modulus $= \sqrt{a^2+b^2}$

$$= \left[\left(\frac{7-12\sqrt{2}}{5}\right)^2 + \left(-\left(\frac{14+6\sqrt{2}}{5}\right)\right)^2\right]^{1/2}$$

$$= \left[\frac{49 - \cancel{168\sqrt{2}} + 288 + 196 + \cancel{168\sqrt{2}} + 72}{25}\right]^{1/2}$$

$$= \sqrt{\frac{605}{25}} = \frac{11\sqrt{5}}{5}$$

268
288
49
605

amplitude $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$$= \tan^{-1}\left(\frac{-\left(\frac{14+6\sqrt{2}}{5}\right)}{\frac{7-12\sqrt{2}}{5}}\right)$$

$$= \tan^{-1}\left(\frac{14+6\sqrt{2}}{12\sqrt{2}-7}\right)$$

1. c) Expand $\cos^8\theta$ in a series of cosine multiples of θ.

Let $t = \cos\theta + i\sin\theta$

$$\frac{1}{t} = \cos\theta - i\sin\theta$$

$t^n = \cos n\theta + i\sin n\theta \quad \frac{1}{t^n} = \cos n\theta - i\sin n\theta$

$t + \frac{1}{t} = 2\cos\theta \; ; \; t^n + \frac{1}{t^n} = 2\cos n\theta$

$$\cos^8\theta = \left[\frac{1}{2}\left(t+\frac{1}{t}\right)\right]^8$$

$$= \frac{1}{2^8}\left(t+\frac{1}{t}\right)^8$$

$$= \frac{1}{2^8}\Big\{ t^8 + 8c_1 t^7\left(\frac{1}{t}\right) + 8c_2 t^6\left(\frac{1}{t}\right)^2 + 8c_3 t^5\left(\frac{1}{t}\right)^3 + 8c_4 t^4\left(\frac{1}{t}\right)^4 + 8c_5 t^3\left(\frac{1}{t}\right)^5 + 8c_6 t^2\left(\frac{1}{t}\right)^6 + 8c_7 t\left(\frac{1}{t}\right)^7 + 8c_8\left(\frac{1}{t}\right)^8 \Big\}$$

$$= \frac{1}{2^8}\Big\{\left(t^8+\frac{1}{t^8}\right) + 8\left(t^6+\frac{1}{t^6}\right) + 28\left(t^4+\frac{1}{t^4}\right) + 56\left(t^2+\frac{1}{t^2}\right) + 8c_4\Big\}$$

$$= \frac{1}{2^8}\Big\{2\cos 8\theta + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta) + 70\Big\}$$

$$= \frac{1}{2^8}\Big\{2\cos 8\theta + 16\cos 6\theta + 56\cos 4\theta + 112\cos 2\theta + 70\Big\}$$

2. a) Find the n^{th} derivative of $\sin(ax+b)$

Let $y = \sin(ax+b)$

$$y_1 = a\cos(ax+b)$$
$$= a\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = a \cdot a\cos\left(ax+b+\frac{\pi}{2}\right)$$
$$= a^2\cos\left(ax+b+\frac{\pi}{2}\right)$$
$$= a^2\sin\left(ax+b+2\frac{\pi}{2}\right)$$

$$y_3 = a^2\cos\left(ax+b+2\frac{\pi}{2}\right)$$
$$= a^3\sin\left(ax+b+3\frac{\pi}{2}\right)$$

$$\vdots$$

$$y_n = a^n\sin\left(ax+b+n\frac{\pi}{2}\right)$$

b) If $y = (\sin^{-1}x)^2$

S.T $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

$$y_1 = 2\sin^{-1}x \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = 2\sin^{-1}x$$

$$(1-x^2)y_1^2 = 4(\sin^{-1}x)^2$$

3.a) Using Taylors Theorem express the polynomial $2x^3+7x^2+x-6$ in powers of $x-1$.

Let $f(x) = 2x^3+7x^2+x-6$

$f(1) = 2+7+1-6 = 4$

$f'(x) = 6x^2+14x+1 \quad f'(1) = 21$

$f''(x) = 12x+14 \quad f''(1) = 26$

$f'''(x) = 12 \quad f'''(1) = 12$

Taylor's Series

$$f(x) = f(1) + \frac{(x-1)}{1!} f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1)$$

$$= 4 + 21(x-1) + \frac{26}{2}(x-1)^2 + \frac{12}{6}(x-1)^3$$

$$f(x) = 4 + 21(x-1) + 13(x-1)^2 + 2(x-1)^3$$

$$(1-x^2)y_1^2 = 4y$$

$$(1-x^2)\,2y_1y_2 - 2xy_1^2 = 4y_1$$

$$(1-x^2)y_2 - xy_1 = 2$$

Differentiating n times using Leibniz

$$\left[(1-x^2)y_{n+2} + nc_1(-2x)y_{n+1} + nc_2(-2)y_n\right]$$

$$-\left\{xy_{n+1} + nc_1 \cdot 1 \cdot y_n\right\} = 0$$

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - xy_{n+1} - \frac{n(n-1)}{\cancel{2}}\cancel{2}y_n - ny_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$$

c) Find the n^{th} derivative of

$$\frac{1}{5(x-1)} + \frac{-3/2}{-5/2\,(2x+3)}$$

$$\frac{1}{5}(x-1)^{-1} + \frac{3}{5}(2x+3)^{-1}$$

$$D^n\left[\frac{1}{5}(x-1)^{-1} + \frac{3}{5}(2x+3)^{-1}\right]$$

$$= \frac{1}{5}\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{3}{5}\frac{(-1)^n n!\,2^n}{(2x+3)^{n+1}}$$

3. b) Using Maclaurin's series expand $\tan x$ upto the term containing x^5.

Let $y = \tan x$

$y(0) = 0 \quad y_1(x) = \sec^2 x \quad y_1(0) = 1$

$y_2 = 2\sec x(\sec x \tan x) \quad y_2(0) = 0$

$y_2 = 2\sec^2 x \tan x$

$y_3 = 2\left\{2\sec x(\sec x \tan x)\tan x + \sec^2 x \sec^2 x\right\}$

$y_3(0) = 2\{2(1)(0) + 1\} = 2$

$y_3 = 4\sec^2 x \tan^2 x + 2\sec^4 x$

$y_4 = 4\left(2\sec x \sec x \tan x \tan^2 x + \sec^2 x (2\tan x \sec^2 x)\right)$

$+ 2 \cdot 4 \sec^3 x \sec x \tan x$

$= 8\sec^2 x \tan^3 x + 8\sec^4 x \tan x + 8\sec^4 x \tan x$

$= 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$

$y_4(0) = 8(1)(0) + 16(1)(0) = 0$

~~$y_5 = 8\{3\sec^2 x(\sec x \tan x)\tan^3 x + 3\tan^2 x \sec^2 x \sec^3 x\}$~~

$$y_4 = 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$$

$$y_5 = 8\left(2\sec x \sec x \tan x \tan^3 x + 3\tan^2 x \sec^2 x \sec^2 x\right) + 16\left(4\sec^3 x \sec x \tan x \tan x + \sec^4 x \sec^2 x\right)$$

$$y_5(0) = 8(0) + 16(0+1) = 16$$

Maclaurin Series

$$y(x) = y_0 + \frac{x}{1!}y_1 + \frac{x^2}{2!}y_2 + \frac{x^3}{3!}y_3 + \frac{x^4}{4!}y_4 + \frac{x^5}{5!}y_5 + \cdots$$

$$\tan x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(16)$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

4.a) $u = x\log xy$ where $x^3+y^3+3xy=1$

Find $\frac{du}{dx}$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$u = x \log xy$$

$$\frac{\partial u}{\partial x} = 1 . \log xy + x . \frac{1}{xy}(y)$$

$$x^3 + y^3 + 3xy = 1$$

differentiating w.r.t x

$$3x^2 + 3y^2\frac{dy}{dx} + 3\left(x\frac{dy}{dx} + y\right) = 0$$

$$(x^2 + y) + (y^2 + x)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(x^2+y)}{(y^2+x)}$$

$$\frac{du}{dx} = (\log xy + 1) + x . \frac{1}{xy} \cancel{x} \frac{-(x^2+y)}{(y^2+x)}$$

$$= 1 + \log xy + \frac{x}{y} - \left(\frac{x^2+y}{y^2+x}\right)$$

$$= \log e + \log xy - \frac{x}{y}\left(\frac{x^2+y}{y^2+x}\right)$$

$$= \log exy - \frac{x}{y}\left(\frac{x^2+y}{y^2+x}\right)$$

4. b) $z = f(x, y)$

$x = e^{u} + e^{-v} \quad y = e^{-u} - e^{v}$

P.T $z_u - z_v = x z_x - y z_y$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x} e^{u} + \frac{\partial z}{\partial y}(-e^{-u}) \quad - (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v}) \quad - (2)$$

(1) − (2)

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^{u} + e^{-v}) + \frac{\partial z}{\partial y}(-e^{-u} + e^{v})$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

4 c) If $u = x + 2y^2 - z^3 \quad v = 4x^2 y z$

$w = 2z^2 - xy$ find $J\left(\begin{matrix} u & v & w \\ x & y & z \end{matrix}\right)$

at $(1, -1, 0)$

$$J\begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

$$= 1\left(16x^2z^2 + 4x^3y\right) - 6y\left(32xyz^2 + 4x^2y^2\right) - 3z^2\left(-8x^2yz + 4x^2yz\right)$$

$$J \text{ at } (1\ -1\ 0) = -4 + 6(4) = 20$$

5. a) Obtain the reduction formula for $\int \sin^n x \, dx$

Let $I_n = \int \sin^n x \, dx$

$$= \int \sin^{n-1} x \; \sin x \, dx$$

$$= \sin^{n-1} x \cos x - \int (n-1) \sin^{n-2} x \cos x \cos x \, dx$$

$$= \sin^{n-1} x \cos x - (n-1) \int \sin^{n-2} x \cos^2 x \, dx$$

$$= \sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x\,(1-\sin^2 x)\,dx$$

$$= \sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x\,(1-\sin^2 x)\,dx$$

$$= \sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

$$I_n = \sin^{n-1}x \cos x + (n-1)\,I_{n-2} - (n-1)\,I_n$$

$$I_n(1+n-1) = \sin^{n-1}x \cos x + (n-1)\,I_{n-2}$$

$$I_n = \frac{1}{n}\left\{\sin^{n-1}x \cos x + (n-1)\,I_{n-2}\right\}$$

b) Evaluate $\displaystyle\int_0^a \frac{x^7}{\sqrt{a^2-x^2}}\,dx$

put $x = a\sin\theta$ $\quad dx = a\cos\theta\,d\theta$

$x=0 \quad \theta = 0 \;;\quad x = a \quad \sin\theta = 1 \quad \theta = \pi/2$

$$I = \int_0^{\pi/2} \frac{a^7\sin^7\theta}{\sqrt{a^2-a^2\sin^2\theta}}\,(a\cos\theta)\,d\theta$$

$$= a^7\int_0^{\pi/2} \sin^7\theta\,d\theta = a^7\,I_7$$

$$= a^7\,\frac{7-1}{7}\,\frac{7-3}{7-2}\,\frac{7-5}{7-4}$$

$$= a^7\,\frac{16}{35}$$

c) Evaluate $\int_1^2 \int_3^4 (xy + e^y)\, dy\, dx$

$$\int_{x=1}^{2} \left\{ x\left(\frac{y^2}{2}\right)_3^4 + \left(e^y\right)_3^4 \right\} dx$$

$$= \int_{x=1}^{2} \left\{ \frac{16-9}{2}(x) + e^4 - e^3 \right\} dx$$

$$= \frac{7}{2}\int_{x=1}^{2} x\, dx + \left(e^4 - e^3\right)\int_{x=1}^{2} dx$$

$$= \frac{7}{2}\left(\frac{x^2}{2}\right)_1^2 + \left(e^4 - e^3\right)(x)_1^2$$

$$= \frac{21}{4} + \left(e^4 - e^3\right)$$

6.a) Evaluate $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\, dx\, dy\, dz$

$$\int_{z=0}^{1} \int_{y=0}^{1} \left(e^x\right)_0^1 e^{y+z}\, dy\, dz$$

$$\int_{z=0}^{1} \int_{y=0}^{1} \left(e^1 - e^0\right) e^y e^z\, dy\, dz$$

$$= \left(\int_{z=0}^{1} e^z\, dz \int_{y=0}^{1} e^y\, dy\right)(e-1)$$

$$(e-1)(e-1)(e-1) = (e-1)^3$$

b) Find the value of $\Gamma\frac{1}{2}$

We k.T $\Gamma n = 2\int_0^{\infty} x^{2n-1} e^{-x^2} dx$

$$\Gamma\tfrac{1}{2} = 2\int_0^{\infty} e^{-x^2} dx \quad (1)$$

$$\Gamma\tfrac{1}{2} = 2\int_0^{\infty} e^{-y^2} dy \quad (2)$$

(1) × (2)

$$\Gamma\tfrac{1}{2}\,\Gamma\tfrac{1}{2} = \left(2\int_0^{\infty} e^{-x^2} dx\right)\left(2\int_0^{\infty} e^{-y^2} dy\right)$$

$$= 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dx\, dy$$

$x = r\cos\theta \quad y = r\sin\theta \quad dx\,dy = r\,dr\,d\theta$

$r \longrightarrow 0$ to $\infty \quad \theta \longrightarrow 0$ to $\pi/2$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-(r^2\cos^2\theta + r^2\sin^2\theta)} r\,dr\,d\theta$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} r\,dr\,d\theta$$

put $t = r^2$ $\quad dt = 2r\,dr$

$$r\,dr = \frac{1}{2}\,dt$$

$$\left(\Gamma\tfrac{1}{2}\right)^2 = 4\int_{\theta=0}^{\pi/2}\left(\int_{t=0}^{\infty} e^{-t}\,\frac{1}{2}\,dt\right)d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} -\left(e^{-t}\right)_0^{\infty}\,d\theta$$

$$= 2\left(\theta\right)_0^{\pi/2} = \pi$$

$$\left(\Gamma\tfrac{1}{2}\right)^2 = \pi \qquad \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

6c) P.T $\quad B(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$

We K.T $\quad \Gamma m = 2\int_0^{\infty} e^{-x^2}x^{2n-1}dx$

$$\Gamma n = 2\int_0^{\infty} e^{-y^2}y^{2m-1}dy$$

$$\Gamma m\,\Gamma n = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}x^{2n-1}y^{2m-1}\,dx\,dy$$

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$dx\,dy = r\,dr\,d\theta$$

$$\Gamma m \ \Gamma n = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2(\cos^2\theta+\sin^2\theta)} \cancel{r\,dr\,d\theta} \, (r\cos\theta)^{2n-1}(r\sin\theta)^{2m-1} r\,dr\,d\theta$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} r^{2m+2n-1} \sin^{2m-1}\theta \cos^{2n-1}\theta \, dr\, d\theta$$

$$= \left(2\int_{r=0}^{\infty} e^{-r^2} r^{2m+2n-1} dr\right)\left(2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta\right)$$

$$= \Gamma m+n \ B(m,n)$$

$$B(m,n) = \frac{\Gamma m \ \Gamma n}{\Gamma m+n}$$

7.a) Solve $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$

$$\frac{dy}{dx} = e^{-2y} e^{3x} + e^{-2y} x^2$$

$$dy = e^{-2y}(e^{3x} + x^2)\, dx$$

$$e^{2y}\, dy = (e^{3x} + x^2)\, dx$$

Intg $\frac{1}{2} e^{2y} = \frac{e^{3x}}{3} + \frac{x^3}{3} + c$

b) Solve $\frac{dy}{dx} = \frac{x^2-y^2}{xy}$

put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x^2 - v^2x^2}{x^2v}$$

$$v + x\frac{dv}{dx} = \frac{1-v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1-v^2}{v} - v = \frac{1-2v^2}{v}$$

$$\frac{v}{1-2v^2}dv = \frac{dx}{x}$$

put $t = 1-2v^2$

$dt = -4v\,dv$

$$-\frac{1}{4}\frac{dt}{t} = \frac{dx}{x}$$

$$\frac{dt}{t} = -4\frac{dx}{x}$$

Intg

$$\log t = -4\log x + \log c$$

$$t = cx^{-4}$$

$(1-2v^2) = cx^{-4}$ is the soln

where $v = \frac{y}{x}$

7.c) Solve $\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1)$

Given equation is a Linear Diffl eqn

$$y(\text{IF}) = \int Q(\text{IF})\,dx + c$$

$$\text{IF} = e^{\int P\,dx} = e^{-\int \frac{1}{x+1}\,dx}$$

$$= e^{-\log(x+1)}$$

$$= (x+1)^{-1}$$

soln is $y(x+1)^{-1} = \int e^{3x}(x+1)\frac{1}{(x+1)^{+1}}\,dx + c$

$$y(x+1)^{-1} = \frac{e^{3x}}{3} + c$$

$$y = \frac{1}{3}\frac{e^{3x}}{(x+1)^{-1}} + \frac{c}{(x+1)^{-1}}$$

genl soln $y = \frac{1}{3}(x+1)e^{3x} + c(x+1)$

8.a) Solve $y'' + 5y' + 6y = e^{x}$

$$(D^2+5D+6)y = e^{x}$$

AE $m^2+5m+6=0$ $(m+2)(m+3)=0$

$$m = -2, -3$$

CF $c_1 e^{-2x} + c_2 e^{-3x}$

$$PI = \frac{1}{D^2+5D+6} e^x = \frac{1}{12} e^x$$

soln $y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{12} e^x$

b) Solve $y'' - 3y' + 2y = \sin 2x$

soln $y = CF + PI$

AE $m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$m = 1, 2$

CF $c_1 e^x + c_2 e^{2x}$

$$PI = \frac{1}{D^2 - 3D + 2} \sin 2x$$

$$= \frac{1}{-4 - 3D + 2} \sin 2x$$

$$\frac{1}{-(3D+2)} \sin 2x$$

$$= \frac{-(3D-2)}{(3D+2)(3D-2)} \sin 2x$$

$$= -\frac{\{3D(\sin 2x) - 2\sin 2x\}}{9(-4) - 4}$$

$$= \frac{1}{40}(6\cos 2x - 2\sin 2x)$$

c) Solve $(D^2-1)y = x\sin 3x + \cos x$

soln $y = CF + PI$

CF $\quad$ AE $m^2 - 1 = 0 \quad m = \pm 1$

CF $\quad c_1 e^{x} + c_2 e^{-x}$

$$PI_1 = \frac{1}{D^2-1} x\sin 3x$$

$$= \frac{1}{D^2-1} x \text{ IP } e^{i3x}$$

$$\text{I.P } \frac{1}{D^2-1} x e^{i3x}$$

$$\text{I.P} = e^{i3x} \frac{1}{(D+3i)^2-1} x$$

$$= \text{I.P } e^{i3x} \frac{1}{D^2+6Di-9-1} x$$

$$= \text{I.P } e^{i3x} \frac{1}{D^2+6Di-10} x$$

$$= \text{I.P } e^{i3x} \left[D^2+6Di-10\right]^{-1} x$$

$$= \text{I.P } e^{i3x} \left[(-10)^{-1} \left\{1 - \frac{D^2+6Di}{10}\right\}^{-1} x\right]$$

$$= -\frac{1}{10} \text{ I.P } e^{i3x} \left\{1 + \frac{D^2+6Di}{10}\right\} x$$

$$= -\frac{1}{10} \text{ I.P } e^{i3x} \left\{x + \frac{6i}{10}\right\}$$

$$= -\frac{1}{10}$$

$$= -\frac{1}{10} \overset{I.P}{\left(x + \frac{3i}{5}\right)\left(\cos 3x + i\sin 3x\right)}$$

$$= -\frac{1}{10}\left\{\frac{3}{5}\cos 3x + x\sin 3x\right\}$$

$$PI_2 = \frac{1}{D^2-1}\cos x \quad \text{replace } D^2 \text{ by } -1$$

$$= -\frac{1}{2}\cos x$$

soln $y = CF + PI_1 + PI_2$

~~Solve~~