Notes on Hecke Operators

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0.1 Group acting on a set

We say a group G *acts* on a set X when there is a group homomorphism:

$$G \to \mathrm{Aut}(X).$$

We choose not to name this homomorphism, and instead confuse the elements of G with their image in $\mathrm{Aut}(X)$. In this way we understood expressions such as gx for $g \in G, x \in X$. This is similar to how a field acts on a vector space: we don't usually write the homomorphism, and instead just let elements of the field act on the vectors (on the left.)

We also call this setup a G-set X.

A map of G-sets $X \to Y$ is a set function $f : X \to Y$ that commutes with the group action. That is, for every $g \in G$ we have the commuting square:

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Y \end{array}$$

Thinking of G as a one object category, a group action is then a set-valued functor and we see that a map of G-sets is the same as a natural transformation of functors. This gives the category of G-sets which we denote **GSet**.

For $x \in X$ the *stabilizer* is defined

$$\mathrm{Stab}(x) := \{g \in G \mid gx = x\}.$$

This is clearly a subgroup of G. Dually, the *fixed point set* of $g \in G$ is the subset of X given by

$$\mathrm{Fix}(g) := \{x \in X \mid gx = x\}.$$

{In what sense are these really dual?}

The *orbit* of $x \in X$ is the set

$$\mathrm{Orbit}(x) := \{gx \mid g \in G\}.$$

The following lemma shows that the stabilizers of points in the same orbit are related by conjugation.

Lemma 1. Given a G-set X, with $x \in X, g \in G$, we have

$$\mathrm{Stab}(gx) = g\mathrm{Stab}(x)g^{-1}.$$

▮

Theorem (orbit-stabilizer). Given a G-set X and a point $x \in X$ there is a **Set** bijection:

$$\mathrm{Orbit}(x) \times \mathrm{Stab}(x) \cong G.$$

Proof: Let H be the subgroup

$$H = \mathrm{Stab}(x).$$

Then G is partitioned into cosets $\{gH \mid g \in G\}$. We claim that this set of cosets is in bijection with the orbit of x. The bijection is given by

$$\begin{aligned} \mathrm{Orbit}(x) &\to \{gH \mid g \in G\} \\ gx &\mapsto gH. \end{aligned}$$

To show that this is well defined, let $gx = hx$ for $g, h \in G$. Then $h^{-1}gx = x$ and so $h^{-1}g \in H$, and the cosets gH and hH are identical. {finish} ∎

0.2 Hecke operators

Let $\mathbb{C}[X]$ denote the complex vector space with basis X. Evidently, when X is a G-set, we get a $\mathbb{C}$-linear representation of G on $\mathbb{C}[X]$. A $\mathbb{C}$-linear representation of G obtained in this way we call a *permutation representation.*

Given a point $x \in X$ we denote the corresponding basis vector in $\mathbb{C}[X]$ as $|x\rangle$, and corresponding dual vector as $\langle x|$. We also denote generic vectors in $\mathbb{C}[X]$ by $|v\rangle$, $|u\rangle$, etc.

Given G-sets X and Y, and points $x \in X, y \in Y$ we define the *Hecke operator* as the linear operator

$$\mathbb{C}[X] \xrightarrow{r_{x,y}} \mathbb{C}[Y]$$

given by

$$r_{x,y} := \frac{1}{|\mathrm{Stab}(x)||\mathrm{Stab}(y)|} \sum_{g \in G} |gy\rangle\langle gx|.$$

Lemma 2. The Hecke operators are G-rep homomorphisms:

$$r_{x,y} \in \mathrm{Hom}_{\mathbf{GRep}}(\mathbb{C}[X], \mathbb{C}[Y]).$$

Proof: We need to show that $gr_{x,y}|v\rangle = r_{x,y}g|v\rangle$ for $|v\rangle \in \mathbb{C}[X], g \in G$. By linearity we need only consider this equation on basis vectors: $gr_{x,y}|x'\rangle = r_{x,y}g|x'\rangle$ for $x' \in X, g \in G$. Computing:

$$\begin{aligned} \mathrm{LHS} &= g \sum_{h \in H} |hy\rangle \\ &= \sum_{h \in H} gh|y\rangle \\ \mathrm{RHS} &= \sum_{h \in G} |hy\rangle\langle hx|gx\rangle \\ &= \sum_{h \in G, hx = gx} |hy\rangle \\ &= \sum_{h \in H} gh|y\rangle. \end{aligned}$$

{what is H?} ∎

Theorem 3. Given two permutation representation $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ of a group G, the Hecke operators

$$\{r_{x,y} \mid x \in X, y \in Y\}$$

form a basis for the linear space

$$\mathrm{Hom}_{\mathbf{GRep}}(\mathbb{C}[X], \mathbb{C}[Y]).$$

Proof: Let $f \in \mathrm{Hom}_{\mathbf{GRep}}(\mathbb{C}[X], \mathbb{C}[Y])$. Then for any $g \in G, x \in X, y \in Y$ we have:

$$\begin{aligned} gf|x\rangle &= fg|x\rangle \\ f|x\rangle &= g^{-1}fg|x\rangle \\ \langle y|f|x\rangle &= \langle y|g^{-1}fg|x\rangle \\ &= \langle gy|f|gx\rangle. \end{aligned}$$

ie., the matrix for f is constant on the orbits of $X \times Y$ and so f is a sum of Hecke operators. ∎

Lemma 4. Given a doubly transitive action $G \to \mathrm{Aut}(X)$ the permutation representation $\mathbb{C}[X]$ breaks into exactly two irreducible representations.

Proof: There are two Hecke operators corresponding to the diagonal matrix, and the off-diagonal matrix. The result follows by the previous theorem and Schur's lemma. ∎

0.3 Bibliographic notes

See [1]

References

[1] A. Dress. *Notes on the theory of representations of finite groups, Part I: The Burnside ring of a finite group and some AGN-applications.* 1971.