James Dolan ① 27/5/19

Def:
adjunct in a 2 category

unit
$$\begin{tikzcd} a \arrow[r, bend left, "L"] \arrow[rr, bend right, "1_a"'] & b \arrow[r, bend left, "R"] \arrow[d, phantom, "\Uparrow"] & a \\ & {} & \end{tikzcd}$$

counit
$$\begin{tikzcd} b \arrow[r, bend left, "R"] \arrow[rr, bend right, "1_b"'] & a \arrow[r, bend left, "L"] \arrow[d, phantom, "\Downarrow"] & b \\ & {} & \end{tikzcd}$$

s.t. triangle identities

In $\mathbb{C}Vec$:

$$1_* = \mathbb{C}$$

unit $\mathbb{C} \longrightarrow \overset{R}{U} \otimes \overset{L}{V}$

counit $\underset{L}{V} \otimes \underset{R}{U} \to \mathbb{C}$

Categorified linear algebra

* adjoint functors
* adjoint 1-cell in a 2-category

Category of complex vector spaces $\mathbb{C}\hat{V}ec$

Can we find an adjoint pair of objects?

From space V get a functor $V \otimes \_$.

$\mathbb{C}\tilde{V}ec$ is monoidal category: a single object 2-category

$$\mathbb{C}\hat{V}ec = Hom(*, *)$$

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$V^* = \mathrm{Hom}(V, \mathbb{C})$

eval (co-unit):

$$\underset{L}{V^*} \otimes \underset{R}{V} \to \mathbb{C}$$

coevaluation (unit):

$$\mathbb{C} \to \underset{R}{V} \otimes \underset{L}{V^*}$$

Need $V^{**} \cong V$ so finite dim.
($V \to V^{**}$ in general.)
Compact closed category
(compact linear operator)

$$\frac{[V, \mathbb{C}] \otimes V \to [V, V]}{[\mathbb{C}, V] \to [[V, \mathbb{C}], [V, V]]}$$

Enriched contravariant functoriality of $[-, V]$.
~~Contravariant~~ Covariant F:

$$[X, Y] \to [F(X), F(Y)]$$

Contravariant G

$$[X, Y] \to [G(Y), G(X)]$$

Take $G = [-, V]$
then $[X, Y] \to [(Y, V], [X, V]]$

triangle identities

$$\begin{tikzcd} a \arrow[r, "L"] \arrow[dr, "1_a"'] & b \arrow[d, "R"] \arrow[dr, "1_b"] & \\ & a \arrow[r, "L"'] & b \end{tikzcd}$$

$$\begin{tikzcd} b \arrow[d, "R"'] \arrow[dr, "1_b"] & \\ a \arrow[r, "L"] \arrow[dr, "1_a"'] & b \arrow[d, "R"] \\ & a \end{tikzcd}$$

Thm two right adjoints R, R' have a unique isomorphism

$$R \to R'$$

of adjunctions

$[\mathbb{C}, V] \to [[V, V], [\mathbb{C}, V]]$
or
$[V, \mathbb{C}] \to [[\mathbb{C}, V], [V, V]]$

$$V^* \otimes V \to [V, V]$$

compact operators (finite sums of 1D image ones) — all operators

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Categorify ~~Vec~~ modules over a com ring
locally presentable category
(is cocomplete category)
categorify adjoint linear operators
left adjoints preserve colimits

Adjoint functor theorem:

left adjoint functor $\Leftrightarrow$ colimit preserving functor

2-category:

objects: locally pres cats

morphism: left adjoint functors

decat'ed	cat'ed
free modules	presheaf category
inclusion of generators	Yoneda embedding

Thm a presheaf cat on $\check{C}$ is locally presentable iff $\check{C}$ is essentially small.

Question When is $\check{C}$ a presheaf?

Karubi envelope
splitting idempotents

$V^* \otimes W \to [V, W]$

isomorphism if V is finite di.

Categorify:

vector space

$\to$ presheaf category

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Co-unit $V^* \otimes V \to \mathbb{C}$

always exists

unit $\mathbb{C} \to V^* \otimes V$

only exists in FinVec

$U \to V$ $\quad U^* \otimes V \to [U, V]$

gives "compact" operators only

the site category is a "basis", is rigid (like Vec over $\mathbb{N}$)

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Example of a non-free module.

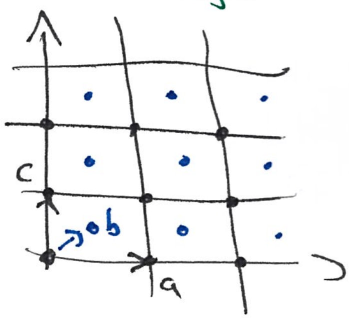
1) Integers over Nats:

$$\mathbb{N}[a,b] \big/ \{a+b=0\}.$$

Vector space:
n generators
m relators
n-m dims
k syzigies
n-m+k dims

2) $\mathbb{N}[a,b,c] \big/ \{a+c=2b\}$

b is average of a, c



pair of small cats $\check{A}$, $\check{B}$
left adjoint

$$L : \mathrm{presh}(\check{A}) \to \mathrm{presh}(\check{B})$$

are "matrices":
bi-presheaves:

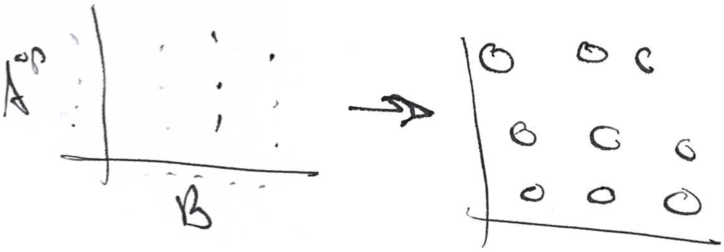
$$L \in \mathrm{presh}(\check{A}^{op} \times \check{B})$$

(or bimodules)

Example $\check{A}$, $\check{B}$ discrete cats
ie sets.

Presheave is family of indexed set.

$\mathrm{Psh}(A^{op} \times B)$:



looks like spans

Categorified presentation is a sketch

Example now add a relator:

m should be epimorphic:

def idempotent under pushout, ie.

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "m"'] & Y \arrow[d, "1_Y"] \\ Y \arrow[r, "1_Y"'] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout.

this is a colimit

Amazing trick:
we get the co-classical models of the theory.

Classical model of a colimit theory T is realized in category of sets.

Co-classical is "opposite of $\widetilde{Set}$.

Amazing fact

Syntactic category = co-classical models

Example
Doctrine: of colimits

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theory: "a pair of objects and a morphism between"

$$X \xrightarrow{m} Y$$

freely add colimits:

presheaves. ← the syntactic category of the theory

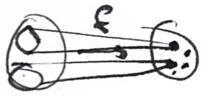
this is the Function category:

objects: $\cdot \xrightarrow{f} \cdot$ ← the construction

morphisms: $f \to g$

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

A function is also a set family: fibres



the representables are:

i) singleton, the walking element

ii) nothing, the walking set (empty...)

this holds for any colimit theory given by a small sketch!

Example

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theory of co-group G

co-mul: $G \longrightarrow G + G$

co-id: $G \longrightarrow 0$

co-classical models: groups.

Amazing trick: cat of groups is the syntactic category.

this is the universal model

left adjoint from the syntact category to any cocomplete category will pick out a model

in this left adj example: cat of groups to any cocomplete cat picks out a cogroup object by evaluating this left adj on free group on one generator (the generic group).

Example

presheaves on simplexes

$=$ simplicial sets

Example

theory: $X \overset{e}{\twoheadrightarrow} Y$

category of classical models: the category of surjections in Set

category of co classical models: injections in Set

within the syntactic category we have a generic epi:

claim: this morphism of injections is an epi.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & \{*\} \arrow[d, hook] \\ \{*\} \arrow[r] & \{*\} \end{tikzcd}$$

$X \twoheadrightarrow Y$

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Now take
the classical models of
the theory:

cogroup objects in
cat of sets
(there's only one cogroup!)
pretty useless

Analogous to
$$x^2+1=0$$
over $\mathbb{R}$ versus $\mathbb{C}$.

coclassical : $\mathbb{C}$ sees everything
classical : $\mathbb{R}$
