# Notes on Hecke Operators

Simon Burton

March 8, 2018

## 1 Introduction

Group representation theory concerns itself with homomorphisms from a group $G$ to the general linear group over a field $k$ :

$$G \to \mathrm{GL}(n, k).$$

The definition of the general linear group makes sense not just for a field $k$, but also for a ring, or even a semi-ring (a ring without additive inverses.) In particular, we consider the semi-ring of truth-values:

$$\mathbb{F}_1 = \{\mathrm{false}, \mathrm{true}\}$$

with addition as disjunction and multiplication as conjunction. The notation $\mathbb{F}_1$ refers to the "field with one element", which however is not a field and doesn't have one element [5]. We have the following

**Theorem 1.** The group $\mathrm{GL}(n, \mathbb{F}_1)$ consists of $n \times n$ permutation matrices, and is therefore isomorphic to the permutation group $S_n$.

**Proof:** Adapted from [6]. ∎

And so we find that group representation theory over the semi-ring of truth values is the theory of groups acting on sets.

## 2 Groups acting on sets

We say a group $G$ *acts* on a set $X$ when there is a group homomorphism:

$$G \to \mathrm{Aut}(X).$$

We choose not to name this homomorphism, and instead confuse the elements of $G$ with their image in $\mathrm{Aut}(X)$. In this way we understood expressions such as $gx$ for $g \in G, x \in X$. This is similar to how a field acts on a vector space: we don't usually write the homomorphism, and instead just let elements of the field act on the vectors (on the left.)

We also call this setup a $G$-set $X$.

A map of $G$-sets $X \to Y$ is a set function $f : X \to Y$ that commutes with the group action. That is, for every $g \in G$ we have the commuting square:

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Y \end{array}$$

Thinking of $G$ as a one object category, a group action is then a set-valued functor and we see that a map of $G$-sets is the same as a natural transformation of functors. This gives the category of $G$-sets which we denote **GSet**.

For $x \in X$ the *stabilizer* is defined

$$\mathrm{Stab}(x) := \{g \in G \mid gx = x\}.$$

This is clearly a subgroup of $G$. Dually, the *fixed point set* of $g \in G$ is the subset of $X$ given by

$$\mathrm{Fix}(g) := \{x \in X \mid gx = x\}.$$

{In what sense are these really dual?}

The *orbit* of $x \in X$ is the set

$$\mathrm{Orbit}(x) := \{gx \mid g \in G\}.$$

A simple calculation shows that the stabilizers of points in the same orbit are related by conjugation: given a $G$-set $X$, with $x \in X, g \in G$, we have

$$\mathrm{Stab}(gx) = g\mathrm{Stab}(x)g^{-1}.$$

Given an arbitrary group $G$, there are many ways to cook up a set that $G$ acts on. The most important such recipe is the following. Let $H$ be any subgroup of $G$, and let $X = \{gH\}_{g\in G}$ be the set of left cosets of $H$. Then $G$ acts on $X$ by left-multiplication, and this action is transitive. We get the *left regular action* when $H = \{1\}$ and we get the trivial action when $H = G$. But this recipe has a converse: any transitive $G$-set $X$ is isomorphic to a $G$-set $\{gH\}_{g\in G}$ for some subgroup $H \in G$. The subgroup can be chosen to be the stabilizer of any point $x \in X$.

We now state the *orbit-stabilizer theorem*.

**Theorem 2.** Given a $G$-set $X$ and a point $x \in X$ there is a bijection of sets:

$$\mathrm{Orbit}(x) \times \mathrm{Stab}(x) \cong G.$$

**Proof:** Let $H$ be the subgroup

$$H = \mathrm{Stab}(x).$$

Then $G$ is partitioned into cosets $\{gH \mid g \in G\}$. We claim that this set of cosets is in bijection with the orbit of $x$, with bijection given by the relation

$$\begin{aligned} \mathrm{Orbit}(x) &\to \{gH \mid g \in G\} \\ gx &\mapsto gH. \end{aligned}$$

To show that this relation is actually a function, let $gx = hx$ for $g, h \in G$. Then $h^{-1}gx = x$ and so $h^{-1}g \in H$, and the cosets $gH$ and $hH$ are identical. {finish} ∎

**Example 3.** The rank-nullity theorem says that given a linear map on finite-dimensional vector spaces $A : V \to V$,

$$\mathrm{Dim}(\mathrm{Im}(A)) + \mathrm{Dim}(\mathrm{Ker}(A)) = \mathrm{Dim}(V).$$

Such a map gives a group action: it is the additive group of $V$ acting on the set $V$ by addition. That is, any $v \in V$ acts on $x \in V$ as $v : x \mapsto x + Av$. Now we see that given any $x \in V$ the stabilizer subgroup $\mathrm{Stab}(x)$ of this action is precisely the kernel of $A$. The orbit of $x$ is $x$ plus the image of $A$.

Working with a vector space over a finite field, we can take the cardinality of these sets as in the formula $|\mathrm{Orbit}(x)||\mathrm{Stab}(x)| = |G|$ and take the logarithm of this where the base is the size of the field and we get exactly the rank-nullity equation.

Over an infinite field this doesn't work and we need to think more along the lines of a categorified orbit-stabilizer theorem. {Does this really work?} In this case, for each $x \in V$ we can find a bijection:

$$\mathrm{Orbit}(x) \cong G/\mathrm{Stab}(x)$$

and this bijection gives us the First Isomorphism Theorem:

$$\mathrm{Im}(A) \cong V/\mathrm{Ker}(A).$$

{Question: Can we bootstrap this one more time in order to say something about the size of the homology groups in a chain complex? Perhaps thinking of a (length 2) chain complex as a 2-category?} ∎

For a $G$-set $X$, the *orbit space* is defined as the set of orbits:

$$G\backslash X := \{\mathrm{Orbit}(x)\}_{x \in X}.$$

The following is known as "Burnside's lemma".

**Theorem 4.** Given a $G$-set $X$,

$$|G\backslash X| = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|.$$

**Proof:** {todo} ∎

An action is *faithful* when for each $g \in G$ with $g \neq 1$ we have $\mathrm{Fix}(g) \neq X$. An action is *free* when for each $g \in G$ with $g \neq 1$ we have $\mathrm{Fix}(g) = \phi$.

{define a torsor}

A $G$-set with only one orbit is called *simple* (or *transitive*, or *indecomposable*).

Given two $G$-sets $X$ and $Y$, we define the sum $X + Y$ {...} and the product $X \times Y$ {...}

The following theorem shows that $G$-sets are semi-simple.

**Theorem 5.** For a $G$-set $X$, we have

$$X = \sum_{i=1}^{n} X_i$$

with $X_i$ simple, and the summation is unique up to reordering.

**Proof:** Easy. ∎

The next theorem is a kind of "Schur's lemma for $G$-sets".

**Theorem 6.** Given $G$-sets $X$ and $Y$:

(a) For $f : X \to Y$, we have $f(X)$ is a $G$-set, $f(X) \subseteq Y$.
(b) For $f : X \to Y$, and $X \neq \phi$, if $Y$ is simple then $f$ is surjective.
(c) For $f : X \to X$ with $X$ simple, $f$ is an automorpism.

**Proof:** Easy. ▮

## 2.1 The category of canonical orbits

Taken from [1, 4].

## 2.2 Double cosets

Given a map of sets $f : X \to Y$, that is $G$-*invariant* ie., $f(x) = f(gx)$ for all $g \in G, x \in X$, we can *lift* to a unique map $\tilde{f} : G\backslash X \to Y$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \\ & G\backslash X \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

where $p$ is the projection map $p : X \to G\backslash X$. We use this observation in the proof of the following.

**Proposition 7.** Given a group $G$ with subgroups $H$ and $K$, the orbit-space of the $G$-set $G/H \times G/K$ is in bijection with the $H, K$ double cosets:

$$G\backslash(G/H \times G/K) \cong \{HgK\}_{g\in G}.$$

**Proof:** We will show that the following relation

$$f : G/H \times G/K \to \{HgK\}_{g\in G}$$
$$f(aH, bK) := Ha^{-1}bK,$$

defines a $G$-invariant function that lifts to the required bijection on the orbits.

(i) That $f$ defines a function: for any $h \in H, k \in K$, $f(aH, bK) = Ha^{-1}bK = H(h^{-1}a^{-1})(bk)K = f(ahH, bkK)$.

(ii) That $f$ is $G$-invariant: $f(gaH, gbK) = Ha^{-1}g^{-1}gbK = f(aH, bK)$.

(iii) $f$ is surjective, and so the lift $\tilde{f}$,

$$\begin{tikzcd} G/H \times G/K \arrow[rr, "f"] \arrow[drr, "p"'] & & Y \\ & & G\backslash(G/H \times G/K) \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

is also surjective.

(iv) That $\tilde{f}$ is injective,

$$\begin{aligned} f(aH, bK) = f(cH, dK) \;\; &\text{iff} \\ Ha^{-1}bK = Hc^{-1}dK \;\; &\text{iff} \\ \exists h, k \in G,\; a^{-1}b = hc^{-1}dk \;\; &\text{iff} \\ b = (ahc^{-1})dk \;\; &\text{iff} \\ bK = gdK,\; aH = gcH, \;\; &\text{where} \;\; g = ahc^{-1} \end{aligned}$$

And so $(aH, bK)$ and $(cH, dK)$ are in the same orbit. ∎

**Proposition 8.** {Not sure how this goes exactly...} Let $G$ be a group, with subgroups $H$ and $K$. Considered as an equivelance relation on the elements of $G$, the set of double cosets $\{HgK\}_{g\in G}$ is the finest mutual coarsening of the equivelance relations given by the right and left cosets, $\{Hg\}_{g\in G}$ and $\{gK\}_{g\in G}$. In other words, as equivelance relations we have

$$\{HgK\}_{g\in G} = \{Hg\}_{g\in G} \vee \{gK\}_{g\in G}.$$

**Proof:** {todo} ∎

### 2.3 Hecke operators

Let $\mathbb{C}[X]$ denote the complex vector space with basis $X$. Evidently, when $X$ is a $G$-set, we get a $\mathbb{C}$-linear representation of $G$ on $\mathbb{C}[X]$. A $\mathbb{C}$-linear representation of $G$ obtained in this way we call a *permutation representation.*

Given a point $x \in X$ we denote the corresponding basis vector in $\mathbb{C}[X]$ as $|x\rangle$, and corresponding dual vector as $\langle x|$. We also denote generic vectors in $\mathbb{C}[X]$ by $|v\rangle$, $|u\rangle$, etc.

Given $G$-sets $X$ and $Y$, and points $x \in X, y \in Y$ we define the *Hecke operator* as the linear operator

$$\mathbb{C}[X] \xrightarrow{r_{x,y}} \mathbb{C}[Y]$$

given by

$$r_{x,y} := \frac{1}{|\mathrm{Stab}(x)||\mathrm{Stab}(y)|} \sum_{g\in G} |gy\rangle\langle gx|.$$

**Lemma 9.** The Hecke operators are $G$-rep homomorphisms:

$$r_{x,y} \in \mathrm{Hom}_{\mathbf{GRep}}(\mathbb{C}[X], \mathbb{C}[Y]).$$

**Proof:** We need to show that $gr_{x,y}|v\rangle = r_{x,y}g|v\rangle$ for $|v\rangle \in \mathbb{C}[X], g \in G$. By linearity we need only consider this equation on basis vectors: $gr_{x,y}|x'\rangle = r_{x,y}g|x'\rangle$ for $x' \in X, g \in G$. Computing:

$$\begin{aligned}
\mathrm{LHS} &= g\sum_{h\in H} |hy\rangle \\
&= \sum_{h\in H} gh|y\rangle \\
\mathrm{RHS} &= \sum_{h\in G} |hy\rangle\langle hx|gx\rangle \\
&= \sum_{h\in G, hx=gx} |hy\rangle \\
&= \sum_{h\in H} gh|y\rangle.
\end{aligned}$$

{what is $H$?} ∎

**Theorem 10.** Given two permutation representation $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ of a group $G$, the Hecke operators

$$\{r_{x,y} \mid x \in X, y \in Y\}$$

form a basis for the linear space

$$\mathrm{Hom}_{\mathbf{GRep}}(\mathbb{C}[X], \mathbb{C}[Y]).$$

**Proof:** Let $f \in \mathrm{Hom}_{\mathbf{GRep}}(\mathbb{C}[X], \mathbb{C}[Y])$. Then for any $g \in G, x \in X, y \in Y$ we have:

$$\begin{aligned} gf|x\rangle &= fg|x\rangle \\ f|x\rangle &= g^{-1}fg|x\rangle \\ \langle y|f|x\rangle &= \langle y|g^{-1}fg|x\rangle \\ &= \langle gy|f|gx\rangle. \end{aligned}$$

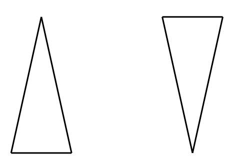
ie., the matrix for $f$ is constant on the orbits of $X \times Y$ and so $f$ is a sum of Hecke operators. $\blacksquare$

**Corollary 11.** Given a doubly transitive action $G \to \mathrm{Aut}(X)$ the permutation representation $\mathbb{C}[X]$ breaks into exactly two irreducible representations.

**Proof:** There are two Hecke operators corresponding to the diagonal matrix, and the off-diagonal matrix. The result follows by the previous theorem and Schur's lemma. $\blacksquare$

# 3 Kleinien geometry

Given a homogeneous "space" $X$ the idea is to refer to, or define, geometric figures (a "line" or "point", etc.) by the subgroup that fixes that figure.



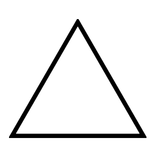
structure symmetry

## 3.1 Example: $S_3$

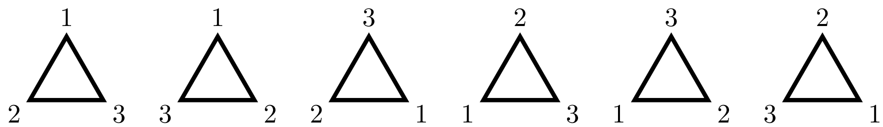
We consider the group $G = S_3$ the permutation group on three things. Enumerating these three things, we write the elements of $S_3$ according to their action:

$$S_3 = \big\{(1)(2)(3), (1)(23), (13)(2), (12)(3), (123), (132)\big\}.$$

This group acts on the triangle:

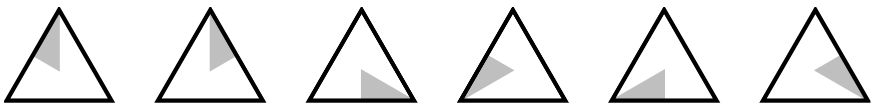


but we can only see this action if we add some *structure*. For example, we can number the points (vertices) of the triangle:

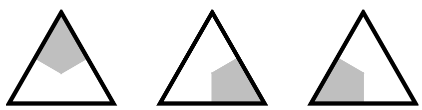



We call this kind of structure a *frame* because it allows us to see everything that is going on.

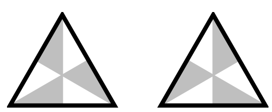
Here we do the same, but instead of numbering the points we shade a portion of the interior. This is a more geometric kind of frame.



By shading a different portion we find the structure corresponding to a *point*:



There is one other structure that we haven't found yet:



This structure we call an *orientation*.

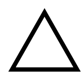
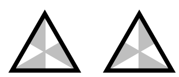
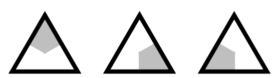
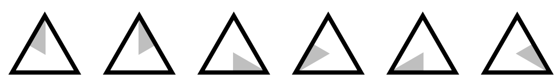
Using the Kleinien perspective, we associate each of these structures with the subgroup of $G$ that stabilizes the structure. And vice-versa: every subgroup of $G$ corresponds to some kind of structure.

| structure | stabilizer subgroup $H \leq G$ | cosets $\{gH\}_{g\in G}$ |
|---|---|---|
| nothing | $\big\{(),(23),(13),(12),(123),(132)\big\}$ | $\big\{(),(23),(13),(12),(123),(132)\big\}$ |
| orientation | $\big\{(),(123),(132)\big\}$ | $\big\{(),(123),(132)\big\},\ \big\{(23),(13),(12)\big\}$ |
| point | $\big\{(),(23)\big\}$ | $\big\{(),(23)\big\},\ \big\{(13),(123)\big\},\ \big\{(12),(132)\big\}$ |
| ” | $\big\{(),(13)\big\}$ | $\big\{(),(13)\big\},\ \big\{(23),(132)\big\},\ \big\{(12),(123)\big\}$ |
| ” | $\big\{(),(12)\big\}$ | $\big\{(),(12)\big\},\ \big\{(13),(132)\big\},\ \big\{(23),(123)\big\}$ |
| frame | $\big\{()\big\}$ | $\big\{()\big\},\big\{(23)\big\},\big\{(13)\big\},\big\{(12)\big\},\big\{(123)\big\},\big\{(132)\big\}$ |

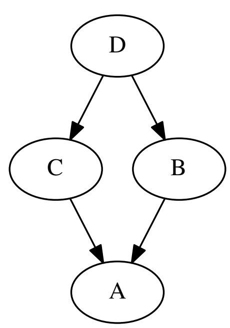
Each of the three subgroups that stabilize a point are conjugate to each other, and so the corresponding $G$-sets $G/H$ are isomorphic. So in calculations we just pick one of these subgroups to stand for the point structure. We label each of these subgroups (structures) with the letters A, B, C and D.

| structure | stabilizer subgroup | order | #cosets | #conjugates |
|---|---|---|---|---|
| nothing | $A = S_3$ | 6 | 1 | 1 |
| orientation | $B$ | 3 | 2 | 1 |
| point | $C$ | 2 | 3 | 3 |
| frame | $D = \{1\}$ | 1 | 6 | 1 |

The same table in picture form:

| structure | subgroup | orbit |
|---|---|---|
| nothing | $A$ |  |
| orientation | $B$ |  |
| point | $C$ |  |
| frame | $D$ |  |

The ordering of the table is misleading. It's really a partially ordered set:



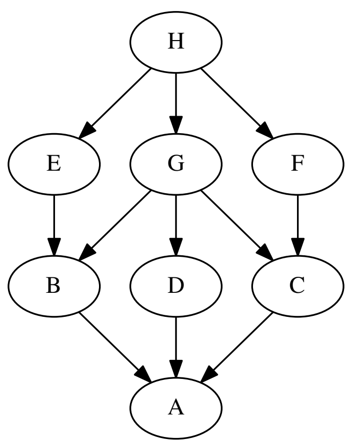

We next consider the conjunction (logical "and") of two structures.

| $\times$ | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| $A$ | $A$ | $B$ | $C$ | $D$ |
| $B$ | $B$ | $2B$ | $D$ | $2D$ |
| $C$ | $C$ | $D$ | $C+D$ | $3D$ |
| $D$ | $D$ | $2D$ | $3D$ | $6D$ |

### 3.2 Example: $D_8$

The dihedral group of order 8.

| structure | stabilizer subgroup | order | #cosets | #conjugates |
|---|---|---|---|---|
| nothing | $A = D_8$ | 8 | 1 | 1 |
| short axis | $B$ | 4 | 2 | 1 |
| long axis | $C$ | 4 | 2 | 1 |
| orientation | $D$ | 4 | 2 | 1 |
| point | $E$ | 2 | 4 | 2 |
| line | $F$ | 2 | 4 | 2 |
| | $G$ | 2 | 4 | 1 |
| frame | $H = \{1\}$ | 1 | 8 | 1 |




| $\times$ | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ |
|---|---|---|---|---|---|---|---|---|
| $A$ | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ |
| $B$ | $B$ | $2B$ | $G$ | $G$ | $2E$ | $H$ | $2G$ | $2H$ |
| $C$ | $C$ | $G$ | $2C$ | $G$ | $H$ | $2F$ | $2G$ | $2H$ |
| $D$ | $D$ | $G$ | $G$ | $2D$ | $H$ | $H$ | $2G$ | $2H$ |
| $E$ | $E$ | $2E$ | $H$ | $H$ | $2E+H$ | $2H$ | $2H$ | $4H$ |
| $F$ | $F$ | $H$ | $2F$ | $H$ | $2H$ | $2F+H$ | $2H$ | $4H$ |
| $G$ | $G$ | $2G$ | $2G$ | $2G$ | $2H$ | $2H$ | $4G$ | $4H$ |
| $H$ | $H$ | $2H$ | $2H$ | $2H$ | $4H$ | $4H$ | $4H$ | $8H$ |

### 3.3 Example: $S_4$

| $\times$ | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ | $I$ | $J$ | $K$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $A$ | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ | $I$ | $J$ | $K$ |
| $B$ | $B$ | $2B$ | $F$ | $H$ | $J$ | $2F$ | $J$ | $2H$ | $K$ | $2J$ | $2K$ |
| $C$ | $C$ | $F$ | $C{+}F$ | $I$ | $E{+}J$ | $3F$ | $G{+}J$ | $K$ | $I{+}K$ | $3J$ | $3K$ |
| $D$ | $D$ | $H$ | $I$ | $D{+}I$ | $2I$ | $K$ | $K$ | $H{+}K$ | $2I{+}K$ | $2K$ | $4K$ |
| $E$ | $E$ | $J$ | $E{+}J$ | $2I$ | $2E{+}K$ | $3J$ | $J{+}K$ | $2K$ | $2I{+}2K$ | $2J{+}2K$ | $6K$ |
| $F$ | $F$ | $2F$ | $3F$ | $K$ | $3J$ | $6F$ | $3J$ | $2K$ | $3K$ | $6J$ | $6K$ |
| $G$ | $G$ | $J$ | $G{+}J$ | $K$ | $J{+}K$ | $3J$ | $2G{+}K$ | $2K$ | $3K$ | $2J{+}2K$ | $6K$ |
| $H$ | $H$ | $2H$ | $K$ | $H{+}K$ | $2K$ | $2K$ | $2K$ | $2H{+}2K$ | $4K$ | $4K$ | $8K$ |
| $I$ | $I$ | $K$ | $I{+}K$ | $2I{+}K$ | $2I{+}2K$ | $3K$ | $3K$ | $4K$ | $2I{+}5K$ | $6K$ | $12K$ |
| $J$ | $J$ | $2J$ | $3J$ | $2K$ | $2J{+}2K$ | $6J$ | $2J{+}2K$ | $4K$ | $6K$ | $4J{+}4K$ | $12K$ |
| $K$ | $K$ | $2K$ | $3K$ | $4K$ | $6K$ | $6K$ | $6K$ | $8K$ | $12K$ | $12K$ | $24K$ |

### 3.4 Example: $\mathrm{GL}(3,2)$

## 4 Bibliographic notes

Group actions with applications to group theory: [2, 3].

See [4].

## References


[1] G. E. Bredon. *Equivariant cohomology theories*, volume 34. Springer, 2006.

[2] K. Conrad. Group actions. `http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/gpaction.pdf`.

[3] K. Conrad. Transitive group actions. `http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/transitive.pdf`.

[4] A. Dress. *Notes on the theory of representations of finite groups, Part I: The Burnside ring of a finite group and some AGN-applications*. 1971.

[5] O. Lorscheid. F1 for everyone. *arXiv preprint arXiv:1801.05337*, 2018.

[6] D. Speyer. Invertible matrices of natural numbers are permutations. `https://mathoverflow.net/a/62126/39775`, 2011.