Notes on Hecke Operators

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1 Introduction

Group representation theory concerns itself with homomorphisms from a group G to the general linear group over a field k :

$$G \to \mathrm{GL}(n, k).$$

The definition of the general linear group makes sense not just for a field k, but also for a ring, or even a semi-ring (a ring without additive inverses.) In particular, we consider the semi-ring of truth-values:

$$\mathbb{F}_1 = \{\mathrm{false}, \mathrm{true}\}$$

with addition as disjunction and multiplication as conjunction. The notation $\mathbb{F}_1$ refers to the "field with one element", which however is not a field and doesn't have one element [5]. We have the following

Theorem 1. The group $\mathrm{GL}(n, \mathbb{F}_1)$ consists of $n \times n$ permutation matrices, and is therefore isomorphic to the permutation group S_n.

Proof: Adapted from [6]. ∎

And so we find that group representation theory over the semi-ring of truth values is the theory of groups acting on sets.

2 Groups acting on sets

We say a group G *acts* on a set X when there is a group homomorphism:

$$G \to \mathrm{Aut}(X).$$

We choose not to name this homomorphism, and instead confuse the elements of G with their image in $\mathrm{Aut}(X)$. In this way we understood expressions such as gx for $g \in G, x \in X$. This is similar to how a field acts on a vector space: we don't usually write the homomorphism, and instead just let elements of the field act on the vectors (on the left.)

We also call this setup a G-set X.

A map of G-sets $X \to Y$ is a set function $f : X \to Y$ that commutes with the group action. That is, for every $g \in G$ we have the commuting square:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

Thinking of G as a one object category, a group action is then a set-valued functor and we see that a map of G-sets is the same as a natural transformation of functors. This gives the category of G-sets which we denote **GSet**.

For $x \in X$ the *stabilizer* is defined

$$\mathrm{Stab}(x) := \{g \in G \mid gx = x\}.$$

This is clearly a subgroup of G. Dually, the *fixed point set* of $g \in G$ is the subset of X given by

$$\mathrm{Fix}(g) := \{x \in X \mid gx = x\}.$$

{In what sense are these really dual?}

The *orbit* of $x \in X$ is the set

$$\mathrm{Orbit}(x) := \{gx \mid g \in G\}.$$

A simple calculation shows that the stabilizers of points in the same orbit are related by conjugation: given a G-set X, with $x \in X, g \in G$, we have

$$\mathrm{Stab}(gx) = g\mathrm{Stab}(x)g^{-1}.$$

Given an arbitrary group G, there are many ways to cook up a set that G acts on. The most important such recipe is the following. Let H be any subgroup of G, and let $X = \{gH\}_{g \in G}$ be the set of left cosets of H. Then G acts on X by left-multiplication, and this action is transitive. We get the *left regular action* when $H = \{1\}$ and we get the trivial action when $H = G$. But this recipe has a converse: any transitive G-set X is isomorphic to a G-set $\{gH\}_{g \in G}$ for some subgroup $H \in G$. The subgroup can be chosen to be the stabilizer of any point $x \in X$.

We now state the *orbit-stabilizer theorem*.

Theorem 2. Given a G-set X and a point $x \in X$ there is a bijection of sets:

$$\mathrm{Orbit}(x) \times \mathrm{Stab}(x) \cong G.$$

Proof: Let H be the subgroup

$$H = \mathrm{Stab}(x).$$

Then G is partitioned into cosets $\{gH \mid g \in G\}$. We claim that this set of cosets is in bijection with the orbit of x, with bijection given by the relation

$$\begin{aligned} \mathrm{Orbit}(x) &\to \{gH \mid g \in G\} \\ gx &\mapsto gH. \end{aligned}$$

To show that this relation is actually a function, let $gx = hx$ for $g, h \in G$. Then $h^{-1}gx = x$ and so $h^{-1}g \in H$, and the cosets gH and hH are identical. {finish} ∎

Example 3. The rank-nullity theorem says that given a linear map on finite-dimensional vector spaces $A : V \to V$,

$$\mathrm{Dim}(\mathrm{Im}(A)) + \mathrm{Dim}(\mathrm{Ker}(A)) = \mathrm{Dim}(V).$$

Such a map gives a group action: it is the additive group of V acting on the set V by addition. That is, any $v \in V$ acts on $x \in V$ as $v : x \mapsto x + Av$. Now we see that given any $x \in V$ the stabilizer subgroup $\mathrm{Stab}(x)$ of this action is precisely the kernel of A. The orbit of x is x plus the image of A.

Working with a vector space over a finite field, we can take the cardinality of these sets as in the formula $|\mathrm{Orbit}(x)||\mathrm{Stab}(x)| = |G|$ and take the logarithm of this where the base is the size of the field and we get exactly the rank-nullity equation.

Over an infinite field this doesn't work and we need to think more along the lines of a categorified orbit-stabilizer theorem. {Does this really work?} In this case, for each $x \in V$ we can find a bijection:

$$\mathrm{Orbit}(x) \cong G/\mathrm{Stab}(x)$$

and this bijection gives us the First Isomorphism Theorem:

$$\mathrm{Im}(A) \cong V/\mathrm{Ker}(A).$$

{Question: Can we bootstrap this one more time in order to say something about the size of the homology groups in a chain complex? Perhaps thinking of a (length 2) chain complex as a 2-category?} ∎

For a G-set X, the *orbit space* is defined as the set of orbits:

$$G\backslash X := \{\mathrm{Orbit}(x)\}_{x\in X}.$$

The following is known as "Burnside's lemma".

Theorem 4. Given a G-set X,

$$|G\backslash X| = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|.$$

Proof: {todo} ∎

An action is *faithful* when for each $g \in G$ with $g \neq 1$ we have $\mathrm{Fix}(g) \neq X$. An action is *free* when for each $g \in G$ with $g \neq 1$ we have $\mathrm{Fix}(g) = \phi$.

{define a torsor}

A G-set with only one orbit is called *simple* (or *transitive*, or *indecomposable*).

Given two G-sets X and Y, we define the sum $X + Y$ {...} and the product $X \times Y$ {...}

The following theorem shows that G-sets are semi-simple.

Theorem 5. For a G-set X, we have

$$X = \sum_{i=1}^{n} X_i$$

with X_i simple, and the summation is unique up to reordering.

Proof: Easy. ∎

The next theorem is a kind of "Schur's lemma for G-sets".

Theorem 6. Given G-sets X and Y:

(a) For $f : X \to Y$, we have $f(X)$ is a G-set, $f(X) \subseteq Y$.
(b) For $f : X \to Y$, and $X \neq \phi$, if Y is simple then f is surjective.
(c) For $f : X \to X$ with X simple, f is an automorpism.

Proof: Easy. ▮

2.1 The category of canonical orbits

Taken from [1, 4].

2.2 Double cosets

Given a map of sets $f : X \to Y$, that is G-*invariant* ie., $f(x) = f(gx)$ for all $g \in G, x \in X$, we can *lift* to a unique map $\tilde{f} : G\backslash X \to Y$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \\ & G\backslash X \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

where p is the projection map $p : X \to G\backslash X$. We use this observation in the proof of the following.

Proposition 7. Given a group G with subgroups H and K, the orbit-space of the G-set $G/H \times G/K$ is in bijection with the H, K double cosets:

$$G\backslash(G/H \times G/K) \cong \{HgK\}_{g\in G}.$$

Proof: We will show that the following relation

$$f : G/H \times G/K \to \{HgK\}_{g\in G}$$
$$f(aH, bK) := Ha^{-1}bK,$$

defines a G-invariant function that lifts to the required bijection on the orbits.

(i) That f defines a function: for any $h \in H, k \in K$, $f(aH, bK) = Ha^{-1}bK = H(h^{-1}a^{-1})(bk)K = f(ahH, bkK)$.

(ii) That f is G-invariant: $f(gaH, gbK) = Ha^{-1}g^{-1}gbK = f(aH, bK)$.

(iii) f is surjective, and so the lift $\tilde{f}$,

$$\begin{tikzcd} G/H \times G/K \arrow[r, "f"] \arrow[dr, "p"'] & Y \\ & G\backslash(G/H \times G/K) \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

is also surjective.

(iv) That $\tilde{f}$ is injective,

$$\begin{aligned} f(aH, bK) = f(cH, dK) &\ \text{ iff} \\ Ha^{-1}bK = Hc^{-1}dK &\ \text{ iff} \\ \exists h, k \in G,\ a^{-1}b = hc^{-1}dk &\ \text{ iff} \\ b = (ahc^{-1})dk &\ \text{ iff} \\ bK = gdK,\ aH = gcH, &\ \text{ where } g = ahc^{-1} \end{aligned}$$

And so (aH, bK) and (cH, dK) are in the same orbit. ▮

Proposition 8. {Not sure how this goes exactly...} Let G be a group, with subgroups H and K. Considered as an equivelance relation on the elements of G, the set of double cosets $\{HgK\}_{g\in G}$ is the finest mutual coarsening of the equivelance relations given by the right and left cosets, $\{Hg\}_{g\in G}$ and $\{gK\}_{g\in G}$. In other words, as equivelance relations we have

$$\{HgK\}_{g\in G} = \{Hg\}_{g\in G} \vee \{gK\}_{g\in G}.$$

Proof: {todo} ▮

2.3 Hecke operators

Let $\mathbb{C}[X]$ denote the complex vector space with basis X. Evidently, when X is a G-set, we get a $\mathbb{C}$-linear representation of G on $\mathbb{C}[X]$. A $\mathbb{C}$-linear representation of G obtained in this way we call a *permutation representation.*

Given a point $x \in X$ we denote the corresponding basis vector in $\mathbb{C}[X]$ as $|x\rangle$, and corresponding dual vector as $\langle x|$. We also denote generic vectors in $\mathbb{C}[X]$ by $|v\rangle$, $|u\rangle$, etc.

Given G-sets X and Y, and points $x \in X, y \in Y$ we define the *Hecke operator* as the linear operator

$$\mathbb{C}[X] \xrightarrow{r_{x,y}} \mathbb{C}[Y]$$

given by

$$r_{x,y} := \frac{1}{|\mathrm{Stab}(x)||\mathrm{Stab}(y)|} \sum_{g\in G} |gy\rangle\langle gx|.$$

Lemma 9. The Hecke operators are G-rep homomorphisms:

$$r_{x,y} \in \mathrm{Hom}_{\mathbf{GRep}}(\mathbb{C}[X], \mathbb{C}[Y]).$$

Proof: We need to show that $gr_{x,y}|v\rangle = r_{x,y}g|v\rangle$ for $|v\rangle \in \mathbb{C}[X], g \in G$. By linearity we need only consider this equation on basis vectors: $gr_{x,y}|x'\rangle = r_{x,y}g|x'\rangle$ for $x' \in X, g \in G$. Computing:

$$\begin{aligned}
\mathrm{LHS} &= g \sum_{h\in H} |hy\rangle \\
&= \sum_{h\in H} gh|y\rangle \\
\mathrm{RHS} &= \sum_{h\in G} |hy\rangle\langle hx|gx\rangle \\
&= \sum_{h\in G, hx=gx} |hy\rangle \\
&= \sum_{h\in H} gh|y\rangle.
\end{aligned}$$

{what is H?} ▮

Theorem 10. Given two permutation representation $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ of a group G, the Hecke operators

$$\{r_{x,y} \mid x \in X, y \in Y\}$$

form a basis for the linear space

$$\mathrm{Hom}_{\mathbf{GRep}}(\mathbb{C}[X], \mathbb{C}[Y]).$$

Proof: Let $f \in \mathrm{Hom}_{\mathbf{GRep}}(\mathbb{C}[X], \mathbb{C}[Y])$. Then for any $g \in G, x \in X, y \in Y$ we have:

$$\begin{aligned} gf|x\rangle &= fg|x\rangle \\ f|x\rangle &= g^{-1}fg|x\rangle \\ \langle y|f|x\rangle &= \langle y|g^{-1}fg|x\rangle \\ &= \langle gy|f|gx\rangle. \end{aligned}$$

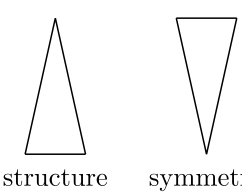
ie., the matrix for f is constant on the orbits of $X \times Y$ and so f is a sum of Hecke operators. ∎

Corollary 11. Given a doubly transitive action $G \to \mathrm{Aut}(X)$ the permutation representation $\mathbb{C}[X]$ breaks into exactly two irreducible representations.

Proof: There are two Hecke operators corresponding to the diagonal matrix, and the off-diagonal matrix. The result follows by the previous theorem and Schur's lemma. ∎

3 Kleinien geometry

Given a homogeneous "space" X the idea is to refer to, or define, geometric figures (a "line" or "point", etc.) by the subgroup that fixes that figure.



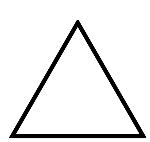
3.1 Example: the triangle

This is about the simplest example that has some substance to it. Because it is our first example we go into great detail.

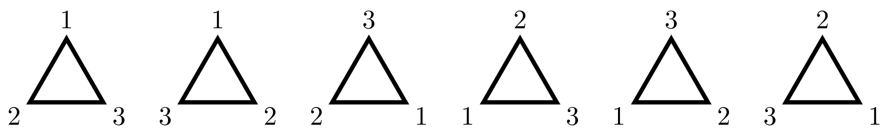
We consider the group $G = S_3$ acting as the permutation group on three things. Enumerating these three things, we write the elements of S_3 according to their action:

$$S_3 = \big\{(1)(2)(3), (1)(23), (13)(2), (12)(3), (123), (132)\big\}.$$

This group acts by isometries on the equi-angle triangle:

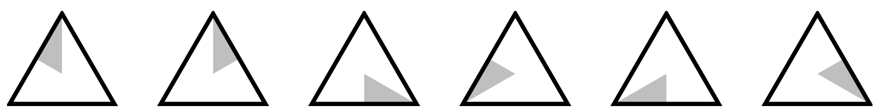


but we can only "see" this action if we add some *structure*. For example, we can number the points (vertices) of the triangle:

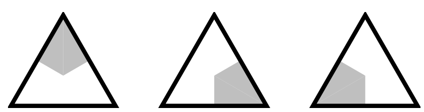


We call this kind of structure a *frame* because it allows us to see everything that is going on with the group action.

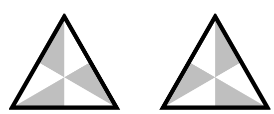
Here we do the same, but instead of numbering the points we shade a portion of the interior. This is a more graphical depiction of a frame.



By shading a different portion we find the structure corresponding to a *point*:



There is one other structure which we call an *orientation*:



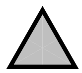
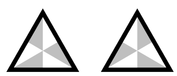
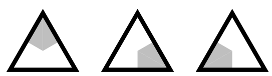
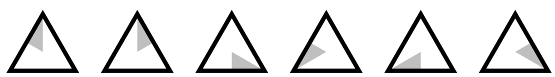
Using the Kleinien perspective, we associate each of these structures with the subgroup of G that stabilizes the structure. And vice-versa: every subgroup of G corresponds to some kind of structure.

structure	stabilizer subgroup $H \leq G$	cosets $\{gH\}_{g\in G}$
nothing	$\{(), (23), (13), (12), (123), (132)\}$	$\{(), (23), (13), (12), (123), (132)\}$
orientation	$\{(), (123), (132)\}$	$\{(), (123), (132)\},\ \{(23), (13), (12)\}$
point	$\{(), (23)\}$	$\{(), (23)\},\ \{(13), (123)\},\ \{(12), (132)\}$
”	$\{(), (13)\}$	$\{(), (13)\},\ \{(23), (132)\},\ \{(12), (123)\}$
”	$\{(), (12)\}$	$\{(), (12)\},\ \{(13), (132)\},\ \{(23), (123)\}$
frame	$\{()\}$	$\{()\}, \{(23)\}, \{(13)\}, \{(12)\}, \{(123)\}, \{(132)\}$

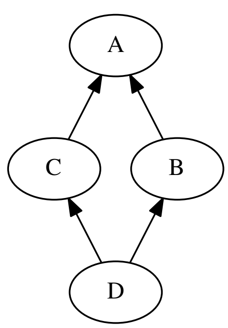
Each of the three subgroups that stabilize a point are conjugate to each other, and so the corresponding G-sets G/H are isomorphic. So in calculations we just pick one of these subgroups to stand for the point structure. We label each of these subgroups (structures) with the letters A, B, C and D.

structure	stabilizer subgroup	order	#cosets	#conjugates
nothing	$A = S_3$	6	1	1
orientation	B	3	2	1
point	C	2	3	3
frame	$D = \{1\}$	1	6	1

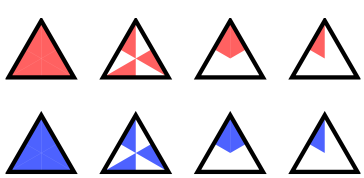
The same table in picture form:

structure	subgroup	orbit
nothing	A	
orientation	B	
point	C	
frame	D	

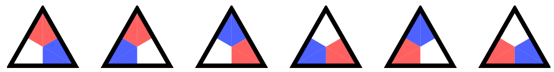
The ordering of the table is misleading. It's really a partially ordered set:



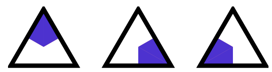
We next consider the conjunction (logical "and") of two structures. To keep track of the two structures when we combine them, we use two colours, red and blue.



When we have two points that are not the same this is what the orbit looks like:



This G-set is isomorphic to D. In other words, two distinct (ordered) points are sufficient structure to pick out a frame. When we have two points that are the same this is what the orbit looks like:

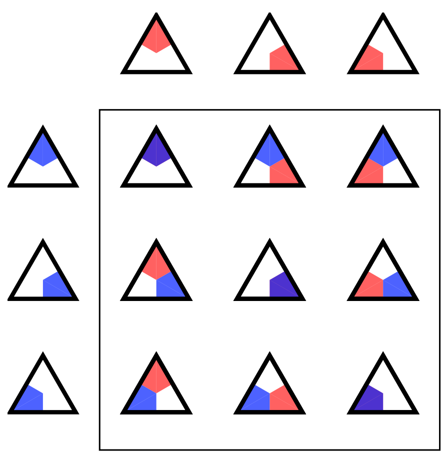


The two points overlap which we show as purple. The resulting G-set is isomorphic to the original G-set, C. And this makes sense: "two points that are identical" is really no more structure than just having one point.

We therefore have the isomorphism:

$$C \times C \cong C + D.$$

We can also examine the entire G-set $C \times C$ in a graphical way. Here we show a matrix, with rows labelled by the orbit of one kind of structure, and columns by the orbit of the other kind of structure. The entries of the matrix then show the elements of the G-set $C \times C$:

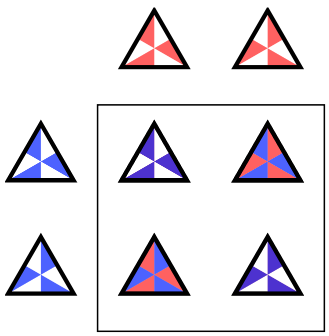


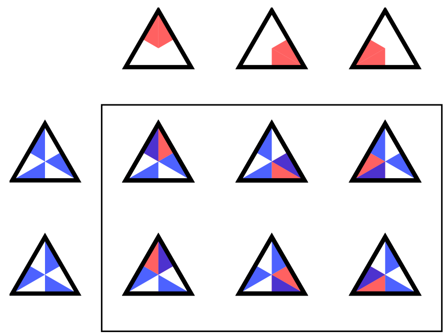
And we see the elements of this matrix break down into two orbits. One orbit is on the diagonal, and the other orbit is off the diagonal. Once again we see that $C \times C \cong C + D$. From this matrix we can also read off matrices for the two Hecke operators corresponding to these two orbits:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

What we are doing here is (in great detail) examining how two structures can *geometrically relate* to each other. For two points, there are two such relationships: they can either be the same, or different.

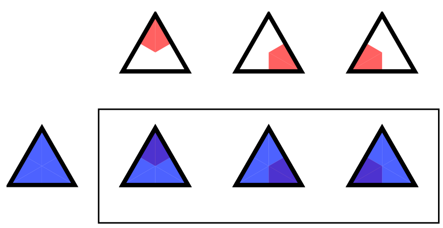
Here we show matrices for $B \times B \cong 2B$ and $B \times C \cong D$:



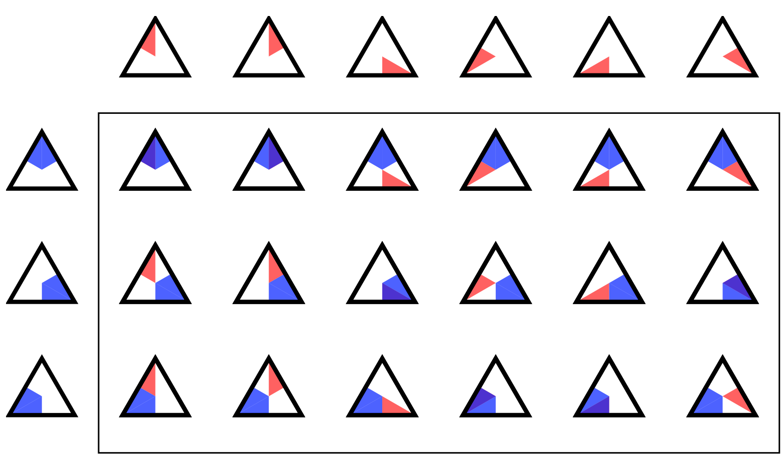


In terms of geometric relationships: there are two ways for an orientation to relate to another orientation (same or not), and there is only one way for a point to relate to an orientation.

The most boring structure of all is the nothing structure A. When we combine this structure with any other structure, we just get the other structure. For example, $A \times C \cong C$:



At the other end of the structures, when we combine a frame D with any other structure, we "see everything". What this means is for some structure (G-set) X, we find that $D \times X$ breaks into n copies of D, one for every element of X. For example, $D \times C \cong 3D$:



It is a little hard to see, but there are three different kinds of shapes in the above matrix, each of which is sufficient to serve as (isomorphic to) a frame. And we see that there are three ways for a point to relate to a frame. This makes sense: there are three points, and if we have a frame then we can see each of these three points as distinct.

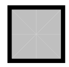
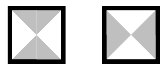
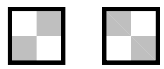
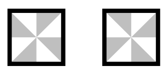
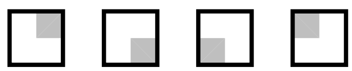
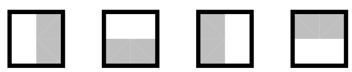
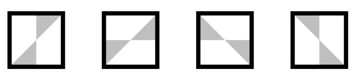
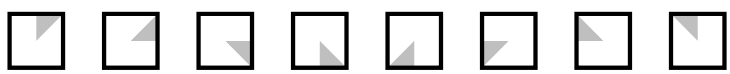
We summarize everything we know about multiplying these G-sets in the following table:

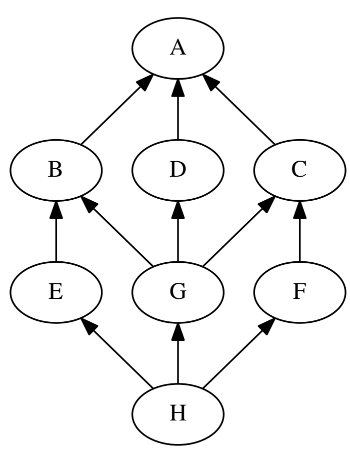
$\times$	A	B	C	D
A	A	B	C	D
B	B	$2B$	D	$2D$
C	C	D	$C+D$	$3D$
D	D	$2D$	$3D$	$6D$

3.2 Example: the square

The dihedral group of order 8.

structure	stabilizer subgroup	order	#cosets	#conjugates
nothing	$A = D_8$	8	1	1
short axis	B	4	2	1
long axis	C	4	2	1
orientation	D	4	2	1
point	E	2	4	2
line	F	2	4	2
	G	2	4	1
frame	$H = \{1\}$	1	8	1

structure	subgroup	orbit
nothing	A	
short axis	B	
long axis	C	
orientation	D	
point	E	
line	F	
s&l axis	G	
frame	H	



$\times$	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	$2B$	G	G	$2E$	H	$2G$	$2H$
C	C	G	$2C$	G	H	$2F$	$2G$	$2H$
D	D	G	G	$2D$	H	H	$2G$	$2H$
E	E	$2E$	H	H	$2E+H$	$2H$	$2H$	$4H$
F	F	H	$2F$	H	$2H$	$2F+H$	$2H$	$4H$
G	G	$2G$	$2G$	$2G$	$2H$	$2H$	$4G$	$4H$
H	H	$2H$	$2H$	$2H$	$4H$	$4H$	$4H$	$8H$

3.3 Example: S_4

$\times$	A	B	C	D	E	F	G	H	I	J	K
A	A	B	C	D	E	F	G	H	I	J	K
B	B	$2B$	F	H	J	$2F$	J	$2H$	K	$2J$	$2K$
C	C	F	$C+F$	I	$E+J$	$3F$	$G+J$	K	$I+K$	$3J$	$3K$
D	D	H	I	$D+I$	$2I$	K	K	$H+K$	$2I+K$	$2K$	$4K$
E	E	J	$E+J$	$2I$	$2E+K$	$3J$	$J+K$	$2K$	$2I+2K$	$2J+2K$	$6K$
F	F	$2F$	$3F$	K	$3J$	$6F$	$3J$	$2K$	$3K$	$6J$	$6K$
G	G	J	$G+J$	K	$J+K$	$3J$	$2G+K$	$2K$	$3K$	$2J+2K$	$6K$
H	H	$2H$	K	$H+K$	$2K$	$2K$	$2K$	$2H+2K$	$4K$	$4K$	$8K$
I	I	K	$I+K$	$2I+K$	$2I+2K$	$3K$	$3K$	$4K$	$2I+5K$	$6K$	$12K$
J	J	$2J$	$3J$	$2K$	$2J+2K$	$6J$	$2J+2K$	$4K$	$6K$	$4J+4K$	$12K$
K	K	$2K$	$3K$	$4K$	$6K$	$6K$	$6K$	$8K$	$12K$	$12K$	$24K$

3.4 Example: GL$(3, 2)$

4 Bibliographic notes

Group actions with applications to group theory: [2, 3].

See [4].

References

[1] G. E. Bredon. *Equivariant cohomology theories*, volume 34. Springer, 2006.

[2] K. Conrad. Group actions. `http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/gpaction.pdf`.

[3] K. Conrad. Transitive group actions. `http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/transitive.pdf`.

[4] A. Dress. *Notes on the theory of representations of finite groups, Part I: The Burnside ring of a finite group and some AGN-applications*. 1971.

[5] O. Lorscheid. F1 for everyone. *arXiv preprint arXiv:1801.05337*, 2018.

[6] D. Speyer. Invertible matrices of natural numbers are permutations. `https://mathoverflow.net/a/62126/39775`, 2011.