

Sheet diagrams for bimonoidal categories

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Abstract

Bimonoidal categories (also known as rig categories) are categories with two monoidal structures, one of which distributes over the other. We formally define sheet diagrams, a graphical calculus for bimonoidal categories that was informally introduced by Staton. Sheet diagrams are string diagrams drawn on a branching surface, which is itself an extruded string diagram. Our main result is a soundness and completeness theorem of the usual form for graphical calculi: we show that sheet diagrams form the free bimonoidal category on a signature.

1 Introduction

String diagrams for monoidal categories are graphical representations of morphisms in monoidal categories [Hotz, 1965, Joyal and Street, 1991]. The equational theory for monoidal categories corresponds to certain permissible topological transformations from one diagram to another, so that one diagram can be deformed into another if and only if they are equal up to the axioms of monoidal categories. Thus one can reason about monoidal categories graphically, by deforming the corresponding string diagrams.

Monoidal categories are common in applied settings, usually as a minimal theory of processes, where the categorical composition and monoidal product of morphisms are respectively thought of as sequential and parallel composition of processes. String diagrams have the simultaneous advantages that they are so intuitive that beginners can pick them up without even having knowledge of category theory, while also being entirely formal thanks to the soundness and completeness theorems which relate deformations to axiomatic reasoning with algebraic expressions.¹ String diagrams have been defined for several more refined notions of monoidal categories, a survey of which can be found in Selinger [2010].

However convenient this syntax might be, it is limited by the expressivity of the monoidal structure. It is common to work in settings where two monoidal

¹Such results are sometimes called *coherence theorems*, but we reserve this term to results about commutation of all diagrams of a certain sort.

structures are used; however, monoidal string diagrams are by definition a syntax for one particular monoidal structure. And a priori, there is no reason why one might expect that more than one monoidal structures interact well with each other. However, there are certain instances when there is some sort of useful interaction between both monoidal structures.

For the purpose of this paper we focus our attention to a particular type of distributivity between two monoidal structures on a category. Bimonoidal categories (also known as rig categories) are categories with a monoidal structure \otimes and a symmetric monoidal structure \oplus with a natural isomorphisms

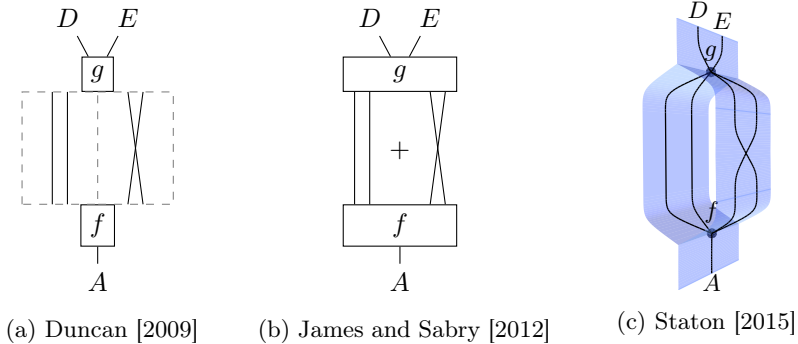
$$\delta_{A,B,C} : A \otimes (B \oplus C) \rightarrow (A \otimes B) \oplus (A \otimes C)$$

$$\delta_{A,B,C}^{\#} : (A \oplus B) \otimes C \rightarrow (A \otimes B) \oplus (A \otimes C)$$

distributing \otimes over \oplus from the left and the right, satisfying certain coherence laws. Many well-known categories have such a structure, for example **Set** with disjoint unions and cartesian products, or **Vect** with direct sums and tensor products. Some informal attempts have been made at drawing string diagrams for such categories. Consider for instance the following linear map, a morphism in **Vect**:

$$A \xrightarrow{f} (B \otimes C) \oplus (B \otimes C) \xrightarrow{1_{B \otimes B} \oplus \gamma_{B,C}} (B \otimes C) \oplus (C \otimes B) \xrightarrow{g} D \otimes E$$

where $\gamma_{B,C} : B \otimes C \rightarrow C \otimes B$ is the symmetry for \otimes . Authors have used various informal conventions to represent such a morphism as a diagram:



These conventions all communicate the structure of a morphism to readers, but do not a priori enjoy a soundness and completeness theorem. In this paper we develop the formal theory of diagrams for bimonoidal categories (also known as rig categories, semiring categories). We provide a definition of the class of diagrams (that we call *sheet diagrams*), their deformations and a soundness and completeness theorem for them. Our sheet diagrams follow the three-dimensional style used by Staton [2015] and Delpeuch [2019], retrospectively justifying their use as formal reasoning tools.

Bimonoidal categories have found applications in a variety of fields: probability theory [Fritz and Perrone, 2018], quantum information [Staton, 2015], dataflow computations [Delpeuch, 2019] and reversible computation [James and Sabry, 2012]. They are also studied in K-theory [Guillou, 2009, Gomez, 2009].

Sheet diagrams could potentially be used in each of these fields, but one important obstacle for using these diagrams is the difficulty of typesetting and manipulating them. We introduce a web-based tool called SheetShow,² which renders sheet diagrams as vector graphics based on a purely combinatorial description of their topology. We give an overview of the data structures of this tool in Appendix A, which are based on work by Delpuch and Vicary [2018].

2 Acknowledgements

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3 Bimonoidal categories

Definition 1. A bimonoidal category, or rig category, is a category \mathcal{C} with a monoidal structure (\mathcal{C}, \cdot, I) and a symmetric monoidal structure (\mathcal{C}, \oplus, O) with natural isomorphisms called the left and right distributors:

$$\begin{aligned}\delta_{A,B,C} &: A(B \oplus C) \rightarrow AB \oplus AC \\ \delta_{A,B,C}^\# &: (A \oplus B)C \rightarrow AC \oplus BC\end{aligned}$$

and isomorphisms called the left and right annihilator:

$$\begin{aligned}\lambda_A^* &: OA \rightarrow O \\ \rho_A^* &: AO \rightarrow O\end{aligned}$$

satisfying the coherence conditions given in Appendix B.

For instance, **Set** is bimonoidal when equipped with the monoidal structures $(\mathbf{Set}, \times, \{\star\})$ and $(\mathbf{Set}, \sqcup, \emptyset)$, where \times is the cartesian product and \sqcup is the disjoint union. Similarly, \mathbf{Vect}_k (the category of vector spaces on a field k) is bimonoidal for $(\mathbf{Vect}, \otimes, k)$ and $(\mathbf{Vect}, \oplus, O)$.

One important question when defining categorical structures such as this one is whether they satisfy coherence. By coherence, we mean that any two parallel morphisms generated by the structural isomorphisms $\delta, \delta^\#, \lambda^*, \rho^*$ and those of the monoidal structures are equal. For bimonoidal categories, this unfortunately does not hold, since the symmetry of (\mathcal{C}, \oplus, O) makes this impossible: $\gamma_{A,A} : A \oplus A \rightarrow A \oplus A \neq 1_A \oplus 1_A$.

However, a restricted form of coherence for bimonoidal categories was proved by Laplaza [1972]. It applies when the domain (or equivalently codomain) of the parallel pair of morphisms is *regular*, which essentially means that no two occurrences of the same object generator can be swapped by a symmetry. We will make this precise in Section 5 as Theorem 4. We finish this section by a semi-strictification theorem for bimonoidal categories.

²Available at <https://wetneb.github.io/sheetshow/> (web app) and <https://github.com/wetneb/sheetshow> (source code)

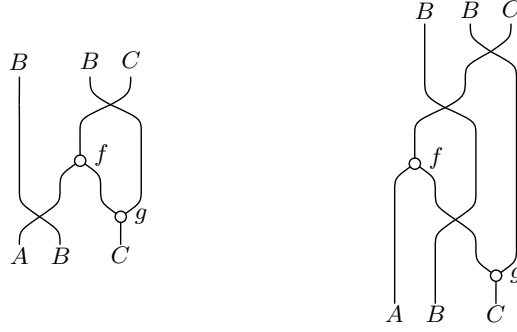


Figure 2: Two isomorphic symmetric monoidal string diagrams

Definition 2. A bimonoidal category is left-semistrict (resp. right-semistrict) if both monoidal structures are strict and $\delta_{A,B,C}^\#$ (resp. $\delta_{A,B,C}$) is an identity for all A, B, C . It is semistrict if it is either left- or right-semistrict.

Theorem 1. [Guillou, 2009] Any bimonoidal category is equivalent to a semistrict one.

4 Symmetric monoidal string diagrams

In this section we briefly recall results on string diagrams for symmetric monoidal categories. For instance, the diagram in Figure 2 represents the morphism $(1_B \otimes \gamma_{C,B}) \circ (1_B \otimes f \otimes 1_B) \circ (\gamma_{A,B} \otimes g)$, where $f : A \otimes A \rightarrow C$, $g : C \rightarrow A \otimes B$, where $\gamma_{A,B} : A \otimes B \rightarrow B \otimes A$ is the symmetry.

Definition 3. A monoidal signature Σ is given by a set of object symbols $\text{Ob}(\Sigma)$, a set of morphism symbols $\text{Mor}(\Sigma)$ and domain and codomain functions $\text{dom}, \text{cod} : \text{Mor}(\Sigma) \rightarrow \text{Ob}(\Sigma)^*$, where $\text{Ob}(\Sigma)^*$ is the set of finite words on $\text{Ob}(\Sigma)$.

For instance, we can define a monoidal signature Σ as:

$$\begin{aligned} \text{Ob}(\Sigma) &= \{A, B, C\} \\ \text{Mor}(\Sigma) &= \{f, g\} \\ \text{dom}(f) &= [A, A] \\ \text{cod}(f) &= [C] \\ \text{dom}(g) &= [C] \\ \text{cod}(g) &= [A, B] \end{aligned}$$

Definition 4. A symmetric monoidal string diagram on a symmetric monoidal signature Σ is an anchored progressive polarised diagram of Σ in the sense of Joyal and Street [1991].

The complete definition from Joyal and Street [1991] In the example signature above, one can draw the string diagrams of Figure 2.

Definition 5. An isomorphism of symmetric monoidal string diagrams $\phi : D \rightarrow D'$ is an anchored isomorphism of progressive plane diagrams in the sense of Joyal and Street [1991].

The two example diagrams in Figure 2 are isomorphic. Given a monoidal signature Σ , one can form a category $\mathbb{F}_s(\Sigma)$ whose objects are $\text{Ob}(\Sigma)^*$ and morphisms are isomorphism classes of symmetric monoidal string diagrams on Σ .

Theorem 2. *[Joyal and Street, 1991, Selinger, 2010] A well-formed equation between morphisms in the language of symmetric monoidal categories follows from the axioms of symmetric monoidal categories if and only if it holds, up to isomorphism of diagrams, in the graphical language. In other words, $\mathbb{F}_s(\Sigma)$ is the free symmetric monoidal category on Σ .*

Notice that the objects of the free symmetric monoidal category, $\text{Ob}(\Sigma)^*$, correspond to elements of the free monoid on the object symbols of the signature. (The fact that the set of finite words is the free monoid can be seen as a 1-dimensional analogue of the previous theorem.)

The aim of this article is to provide analogous results for bimonoidal categories: we characterize the free bimonoidal category on a signature as a certain category of diagrams quotiented by certain topological equivalences.

5 Bimonoidal signatures

Given a set X , let $R(X)$ denote the set of expressions generated by the two binary operators \cdot and \oplus , the two unary symbols I and O , and elements of X as unary symbols. The set $R(X)$ is the set of bimonoidal object expressions on generators X .

Definition 6. *A bimonoidal signature Σ consists of a set $\text{Ob}(\Sigma)$ of object symbols, a set $\text{Mor}(\Sigma)$ of morphism symbols, together with domain and codomain functions $\text{dom}, \text{cod} : \text{Mor}(\Sigma) \rightarrow R(\text{Ob}(\Sigma))$. We write $f : \text{dom}(f) \rightarrow \text{cod}(f)$.*

One can then give a tautological definition of the free bimonoidal category such a signature generates.

Definition 7. *Given a bimonoidal signature Σ , one can define the free bimonoidal category $\bar{\Sigma}$ that it generates: $\text{Ob}(\bar{\Sigma}) = R(\text{Ob}(\Sigma))$, and morphisms of $\bar{\Sigma}$ are equivalence classes of morphism expressions built out of $\text{Mor}(\Sigma)$, structural isomorphisms from Definition 1 and identities quotiented by the axioms of bimonoidal categories.*

This definition is not particularly easy to work with, since morphism expressions can easily get cluttered with structural isomorphisms due to the interplay between the three binary operations involved: \circ , \cdot and \oplus . By defining sheet diagrams for bimonoidal categories, we will provide a more practical description of $\bar{\Sigma}$, where the bimonoidal axioms are interpreted as topological deformations.

Definition 6 is the most general form of a bimonoidal signature, but again not the most convenient to work with. Just like monoidal signatures normalize the domains and codomains of their morphism symbols by forgetting the bracketing of their products, we will introduce a similar normalization for bimonoidal signatures. First, by Theorem 1 we can assume up to equivalence that both the additive and multiplicative monoidal structures are strict. We therefore omit the associators and unitors in what follows.

Definition 8. An expression $\phi \in R(X)$ is normalized when it is a sum of products of generators. Each summand can have no factors (in which case the summand is simply I) and the sum can have no summands, in which case $\phi = O$. A bimonoidal signature is normalized when all its domains and codomains of morphism symbols are normalized.

For any bimonoidal signature, we want to define a normalized version of it. This requires a bit more care than in the monoidal case because there can be multiple ways to normalize an object expression. For instance, the expression $(A \oplus B)(C \oplus D)$ is equivalent to two normalized forms: $AC \oplus AD \oplus BC \oplus BD$ and $AC \oplus BC \oplus AD \oplus BD$.

Definition 9. For any expression $A \in R(X)$ we define its normal form $N(A) = \bigoplus_i A_i$ where A_i are products, by induction:

- $N(O) = O$
- $N(I) = I$
- $N(A \oplus B) = N(A) \oplus N(B)$
- $N(A \cdot B) = \bigoplus_i \bigoplus_j A_i B_j$, where $N(A) = \bigoplus_i A_i$ and $N(B) = \bigoplus_j B_j$ with A_i, B_j products of generators.

Definition 10. Let A_1, \dots, A_p and B_1, \dots, B_q be object expressions. We define an isomorphism $\Delta_{p,q} : (\bigoplus_i A_i)(\bigoplus_j B_j) \rightarrow \bigoplus_i \bigoplus_j A_i B_j$ by induction of p :

- For $p = 0$, $\Delta_{0,q} = \lambda_{\bigoplus_j B_j}^*$;
- For $p = 1$, $\Delta_{1,q}$ is obtained by repeated applications of the distributor δ .
- For $p > 1$, $(\bigoplus_i A_i)(\bigoplus_j B_j) \rightarrow_\delta (\bigoplus_{i=1}^{p-1} A_i)(\bigoplus_j B_j) \oplus A_p(\bigoplus_j B_j)$. We can apply $\Delta_{p-1,q}$ on the left-hand side and an iteration of δ on the right-hand side.

Definition 11. For any object expression $A \in R(X)$ we define its normalization morphism $n_A : A \rightarrow N(A)$ by induction:

- $n_O = 1_O$
- $n_I = 1_I$
- $n_{A \oplus B} = n_A \oplus n_B$
- $n_{A \cdot B} = \Delta_{p,q} \circ (n_A \cdot n_B)$ where $N(A) = \bigoplus_{i=1}^p A_i$ and $N(B) = \bigoplus_{j=1}^q B_j$

Definition 12. Given any bimonoidal signature Σ , we can build a normalized signature $N(\Sigma)$ where all domains and codomains are normalized by N :

$$\begin{aligned} \text{Ob } N(\Sigma) &= \text{Ob } \Sigma \\ \text{Mor } N(\Sigma) &= \{f' : N(\text{dom } f) \rightarrow N(\text{cod } f) \mid f \in \text{Mor } \Sigma\} \end{aligned}$$

Theorem 3. For any bimonoidal signature Σ , the categories $\overline{\Sigma}$ and $\overline{N(\Sigma)}$ are isomorphic.

Proof. Consider a morphism expression ϕ in $\overline{\Sigma}$. We can translate it to a morphism expression $U(\phi)$ in $\overline{N(\Sigma)}$ by replacing each occurrence of a morphism generator $f \in \text{Mor } \Sigma$ by $n_{\text{cod } f}^{-1} \circ f' \circ n_{\text{dom } f}$. This is a valid expression in $\overline{N(\Sigma)}$ since $n_{\text{dom } f}$ and $n_{\text{cod } f}^{-1}$ are only built out of structural isomorphisms, without any morphism generators in them. We can similarly define a translation of any expression ψ in $\overline{N(\Sigma)}$ to $V(\psi)$ in $\overline{\Sigma}$ by replacing any morphism generator $f' \in \text{Mor } N(\Sigma)$ by $n_{\text{cod } f} \circ f \circ n_{\text{dom } f}^{-1}$.

Let us show that both translations respect the axioms of bimonoidal categories. If f_1, f_2 are equivalent morphism expressions in $\overline{\Sigma}$, then $U(f_1)$ and $U(f_2)$ are equivalent in $\overline{N(\Sigma)}$ by instantiation of its generators. Conversely, if $U(f_1)$ and $U(f_2)$ are equivalent in $\overline{N(\Sigma)}$, then $V(U(f_1))$ and $V(U(f_2))$ are equivalent by the same argument. As $V(U(f_1)) \sim f_1$ and similarly for f_2 , $f_1 \sim f_2$. So U and V define functors, which are the identity on objects and are mutually inverse. \square

As a consequence, we will only consider normalized signatures in what follows, as this will not restrict the generality of our results.

Let us turn to coherence. As mentioned earlier, coherence does not hold for bimonoidal categories in general, but only when some conditions on their domain (or equivalently codomain) are met.

Definition 13. [Laplaza, 1972] *An object $A \in \text{Ob } \Sigma$ is regular when all the summands in $N(A)$ are distinct (they are all different lists of generators) and for each summand of $N(A)$, its factors are all different (it is a product of distinct generators).*

For instance, $AB \oplus BA$ is regular, but $AB \oplus AB$ and $AA \oplus AB$ are not. Note that the second condition (each summand being a product of distinct generators) was required by Laplaza because they assumed the multiplicative monoidal structure to be symmetric. We only assume the additive structure to be symmetric, so this condition should be superfluous in our case. We keep it for the sake of accurate citation as we will be able to accommodate this artificial restriction later on, when using the following theorem:

Theorem 4. [Laplaza, 1972] *Let A, B be regular objects of $\overline{\Sigma}$. For all morphisms $f, g : A \rightarrow B$ generated by structural isomorphisms of $\overline{\Sigma}$, $f = g$.*

We finish this section with a few lemmas on the normalization function N . Let $i : \text{Ob } \overline{\Sigma} \rightarrow \text{Ob } \overline{\Sigma}$ such that $i(I \cdot I) = I$ and $i(x) = x$ for any other $x \in \text{Ob } \overline{\Sigma}$. In what follows, we assume all sums and product left-associated.

Lemma 1. *For all objects $A, B, C \in \text{Ob } \overline{\Sigma}$,*

$$N(A \cdot N(B \cdot C)) = N(N(A \cdot B) \cdot C)$$

Proof. Let $N(A) = \bigoplus_i A_i$, $N(B) = \bigoplus_j B_j$, $N(C) = \bigoplus_k C_k$. Then

$$\begin{aligned}
N(A \cdot N(B \cdot C)) &= N(A \cdot (B \cdot C)) \\
&= \bigoplus_i \bigoplus_{j,k} A_i (B_j C_k) \\
&= \bigoplus_{i,j} \bigoplus_k (A_i B_j) C_k \\
&= N(N(A \cdot B) \cdot C) \\
&= N((A \cdot B) \cdot C) \quad \square
\end{aligned}$$

6 Sheet diagrams

In this section, we assume a fixed normalized bimonoidal signature Σ . Because bimonoidal categories are symmetric monoidal categories for their additive structure, we can already use string diagrams for symmetric monoidal categories to reason about bimonoidal categories. Such a diagrammatic language treats the multiplicative structure as opaque, but we will see in this section how to extend the language to take the second monoidal structure into account as well.

The first step consists in defining a monoidal signature which we will use for our symmetric monoidal diagrams.

Definition 14. *The monoidal signature Γ consists of*

$$\begin{aligned}
\text{Ob } \Gamma &= \text{Ob } \Sigma^* \\
\text{Mor } \Gamma &= (\text{Mor } \Sigma \cup \{1_A | A \in \text{Ob } \Sigma\})^* \\
\text{dom } \Pi_{i=1}^n f_i &= N(\Pi_{i=1}^n \text{dom } f_i) \\
\text{cod } \Pi_{i=1}^n f_i &= N(\Pi_{i=1}^n \text{cod } f_i)
\end{aligned}$$

The signature Γ generates \mathcal{C} , a symmetric monoidal category for the additive structure. Object generators are multiplicative products of bimonoidal generators, therefore objects in \mathcal{C} are sums of products of bimonoidal generators.

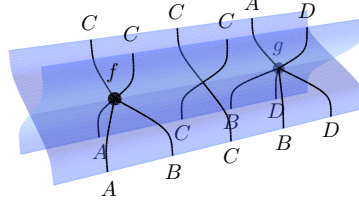
Similarly, morphism generators in Γ are multiplicative products of morphism generators in Σ , except that we also allow for identities to be part of the product. The domain of a morphism generator is obtained by normalizing the product of the domains of its components.

For instance, consider object generators A, B, C, D and bimonoidal morphism generators $f : A \oplus AB \rightarrow C$ and $g : BD \rightarrow A \oplus D$. We can form $f \cdot 1_C \cdot g \in \text{Mor } \Gamma$. Its domain is $N((A \oplus AB) \cdot C \cdot BD) = ACBD \oplus ABCBD$ and its codomain is $N(C \cdot C \cdot (A \oplus D)) = CCA \oplus CCD$. When drawn as a string diagram generator, it looks as follows:

$$\begin{array}{ccc}
CCA & & CCD \\
& \searrow & \swarrow \\
& \text{---} f \cdot 1_C \cdot g \text{---} & \\
& \swarrow & \searrow \\
ACBD & & ABCBD
\end{array}$$

This is not very informative, since only the additive monoidal structure is reflected by the diagram. The multiplicative structure is left unanalyzed because it is internal to the object and morphism generators.

To fix this issue, we extrude our monoidal diagram into a sheet diagram. Edges of our monoidal diagram become sheets, and vertices become seams. On the sheets, we can draw wires whose connectivity reflects which factor of the morphism generator they are coming from:



Informally, each factor of the morphism generator corresponds to a node on the seam, in the same order. These nodes are represented here by small black spheres, except for the middle one which corresponds to an identity, which we therefore leave unmarked. This idea can be generalized to arbitrary diagrams. This section defines these diagrams and the associated class of topological transformations.

Definition 15. Let $\Pi_{i=1}^n f_i \in \text{Mor } \Gamma$ and let $\text{dom } \Pi_{i=1}^n f_i = \bigotimes_{j=1}^p \Pi_{k=1}^q A_{jk}$ be its domain. For each j, k we define the origin of A_{jk} in $\Pi_{i=1}^n f_i$ as the index $1 \leq i \leq n$ such that A_{jk} occurs in $\text{dom } f_i$. If A_{jk} occurs in the domain of more than one morphism factor, this can be made unambiguous by adding indices to the object symbols involved in the domains before normalization. Similarly, the origin of an object symbol in the codomain of a morphism generator is defined.

Definition 16. Given a symmetric monoidal string diagram S on Γ , a sheet diagram D for S is a collection of manifolds in \mathbb{R}^3 :

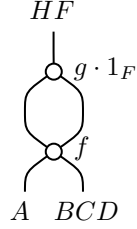
- for each vertex $v \in \mathbb{R}^2$ of S , there is a seam $v \times [0, 1] \in \mathbb{R}^2$. Let $\Pi_{i=1}^n f_i$ be the morphism generators associated with v . For each i we pick an $x_{vi} \in (0, 1)$ such that $x_{v1} < \dots < x_{vn}$ and add a node $v \times x_{vi}$, which is included in the vertex's seam;
- for each edge $e \subset \mathbb{R}^2$ of S , there is a sheet $e \times [0, 1] \subset \mathbb{R}^3$. Let $p_e : [0, 1] \rightarrow \mathbb{R}^2$ be a parametrization of e from source to target (bottom up). Let $\Pi_{i=1}^n A_{ei}$ be the type associated to e in S . For each i we pick a parametrized segment $\gamma_{ei} : [0, 1] \rightarrow [0, 1]$ which gives rise to a wire $w_{ei} : t \in [0, 1] \rightarrow p_e(t) \times \gamma_{ei}(t)$ included in the sheet for edge e . We require that for all $t \in (0, 1)$ and $i < j$, $\gamma_{ei}(t) < \gamma_{ej}(t)$.

Furthermore, we require the following conditions:

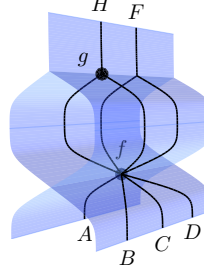
1. In S , if an edge e connects to a vertex v from below (into its domain), then for all wires w_{ei} on the sheet corresponding to e in D , $\gamma_{ei}(1) = x_{vj}$ where j is the origin of A_{ei} in the domain of v ;
2. In S , if an edge e connects to a vertex v from above (out of its codomain), then for all wires w_{ei} on the sheet corresponding to e in D , $\gamma_{ei}(0) = x_{vj}$ where j is the origin of A_{ei} in the codomain of v .

The skeleton of a sheet diagram D is denoted by $S(D)$. The set of sheet diagrams of Γ is denoted by $D(\Gamma)$.

Let us consider an example of a sheet diagram.



(a) A string diagram S on Γ



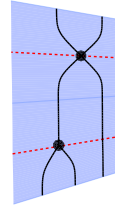
(b) A sheet diagram based on S

Obtaining a sheet diagram from a string diagram only requires picking node positions on each seam, and trajectories of the wires on each sheet, such that the two boundary conditions are satisfied. The geometry of the seams and sheets is directly inherited from that of the string diagram itself. The boundary conditions ensure proper typing of the diagram.

Note that nothing interesting happens on sheets themselves: wires flow vertically, without being able to cross, from the bottom seam to the top seam of the sheet (or the diagram boundaries). However, for seams which have only one input sheet and one output sheet, we will use the convention of not drawing the seam between the two sheets, informally treating them as the same sheet. This makes it possible to draw monoidal string diagrams for the multiplicative product “on the sheets”. In the following example we mark the seams with dashed red lines, which will be omitted in the future:



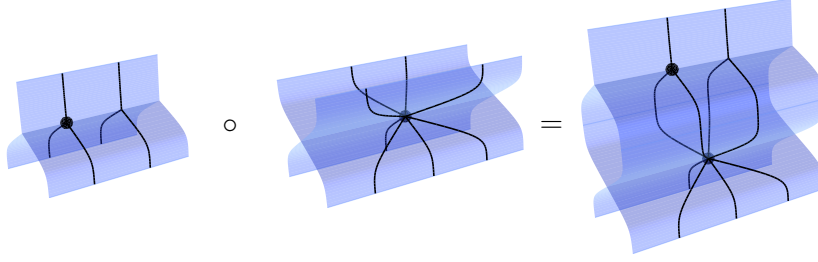
(a) Skeleton



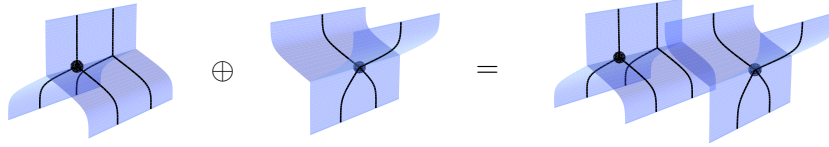
(b) Sheet diagram

Definition 17. Let D be a sheet diagram. Its domain $\text{dom } D$ is the domain of its skeleton, and similarly for its codomain $\text{cod } D$.

Definition 18. Let D_1 and D_2 be sheet diagrams such that $\text{cod } D_1 = \text{dom } D_2$. The sheet diagram $D_2 \circ D_1$ can be obtained by stacking D_2 on top of D_1 , binding sheets and wires at the boundary with linear interpolation.



Definition 19. Let D_1 and D_2 be sheet diagrams. One can form the sheet diagram $D_1 \oplus D_2$ by adjoining D_2 to the right of D_1 . We have $S(D_1 \oplus D_2) = S(D_1) \oplus S(D_2)$

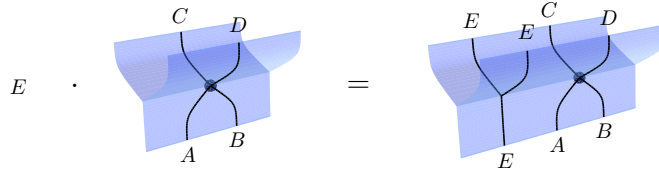


To obtain a bimonoidal category of sheet diagrams, the last binary operation we need to define is the tensor product. This is more intricate since it is not naturally represented by the structure of the skeleton. We start by defining the whiskering of a diagram with an object, in other words a tensor product where one of the factors is an identity.

Definition 20. Let $A \in \text{Ob } \Sigma$ and D be a sheet diagram. The left whiskering of D by A , denoted by $A \cdot D$, is obtained from D by:

- adding a node n_v on each seam corresponding to vertex v in $S(D)$, before all other nodes on the seam;
- adding a wire w_e on each sheet corresponding to edge e in $S(D)$, before all other wires on the seam;
- replacing all symmetries $\gamma_{U,V}$ in $S(D)$ by the whiskered symmetry $\gamma_{AU,AV}$;

Such that if sheet e connects to seam v , wire w_e connects to node n_v . We have $A \cdot D : N(A \cdot \text{dom } D) \rightarrow N(A \cdot \text{cod } D)$. The right whiskering is defined similarly, placing the new nodes and wires after the existing ones instead.



We now turn to the general definition of the tensor product. We first need to define a family of isomorphisms to reorder summands.

Definition 21. Let $p, q \in \mathbb{N}$ and $X_1, \dots, X_p, Y_1, \dots, Y_q \in \text{Ob } \Gamma$. We define an isomorphism

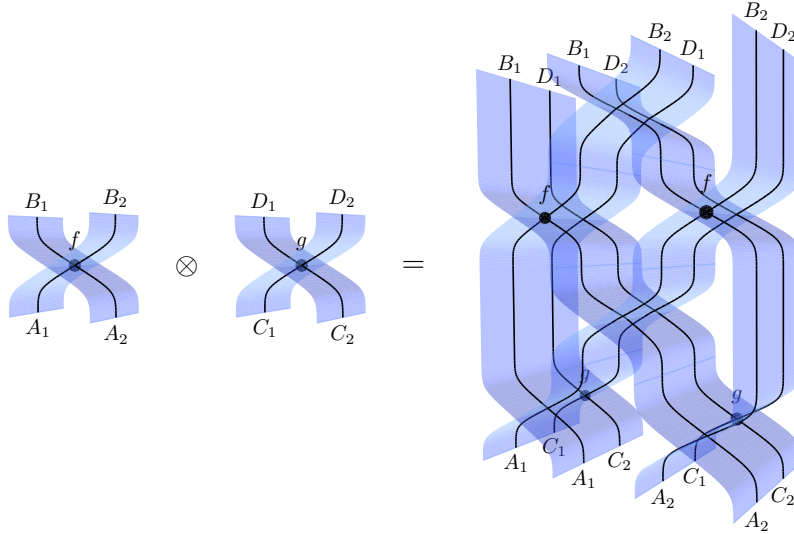
$$E_{XY} : \bigoplus_{i=1}^p \bigoplus_{j=1}^q X_i Y_j \rightarrow \bigoplus_{j=1}^q \bigoplus_{i=1}^p X_i Y_j$$

which reorders the summands as indicated by the commutation of sums, by repeated application of the symmetry for \oplus .

The definition of the tensor product of arbitrary diagrams exploits the exchange law to express it as a composition of whiskered diagrams.

Definition 22. Let $f : \bigoplus_i A_i \rightarrow \bigoplus_j B_j$ and $g : \bigoplus_k C_k \rightarrow \bigoplus_l D_l$ be sheet diagrams, where $A_i, B_j, C_k, D_l \in \text{Ob } \Gamma$. We can define the tensor product of D_1 and D_2 as:

$$\begin{aligned} f \otimes g &:= E_{BD}^{-1} \circ \left(\bigoplus_l f D_l \right) \circ E_{AD} \circ \left(\bigoplus_i A_i g \right) \\ &: \bigoplus_i \bigoplus_k A_i C_k \rightarrow \bigoplus_l \bigoplus_j B_j D_l \end{aligned}$$



Note that we made a choice here: $f \otimes g$ could have equivalently been defined as $(\bigoplus_j B_j g) \circ E_{BC}^{-1} \circ (\bigoplus_k f C_k) \circ E_{AC}^{-1}$. In the next section, we will define a class of isotopies which will let us relate the two expressions (Lemma 10). Before that, we introduce one last product, that of morphism generators for which we do not need to resort to whiskering.

Definition 23. Let $f : \bigoplus_k A_k \rightarrow \bigoplus_l B_l$ and $g : \bigoplus_p C_p \rightarrow \bigoplus_q D_q \in \text{Mor } \Gamma$. They can be decomposed as products of morphism generators in Σ or identities: $f = \Pi_i f_i$ and $g = \Pi_j g_j$. We define their product as

$$fg := \Pi_i f_i \Pi_j g_j$$

In other words we concatenate the lists of generators and identities that compose them.

Lemma 2. *For all $f, g \in \text{Mor } \Gamma$, $\text{dom } fg = N((\text{dom } f)(\text{dom } g))$ and $\text{cod } fg = N((\text{cod } f)(\text{cod } g))$.*

Proof. This is a direct consequence of the associativity of N (Lemma 1) and the definition of domains and codomaing in Γ (Definition 14). \square

7 Isomorphisms of sheet diagrams

We first begin with a lemma on isomorphisms of monoidal string diagrams.

Lemma 3. *Given an anchored isomorphism of progressive plane diagrams between diagrams S_1 and S_2 , there is an open graph isomorphism ϕ between the open graphs defined by S_1 and S_2 . In other words there are bijections between their sets of nodes, their sets of edges, and those respect the adjacency relations.*

Definition 24. *Given sheet diagrams D_1, D_2 with $\text{dom } D_1 = \text{dom } D_2$ and $\text{cod } D_1 = \text{cod } D_2$, a regular isomorphism of sheet diagrams from D_1 to D_2 is given by an anchored isomorphism of progressive plane diagrams $\alpha : S(D_1) \rightarrow S(D_2)$, as well as:*

- for each node n_{vi} on a seam v in D_1 , a continuous map $x_{vi}^* : [0, 1] \rightarrow [0, 1]$ such that $x_{vi}^*(0) = x_{vi}$ and $x_{vi}^*(1) = x_{\phi(v)i}$, where $n_{\phi(v)i} \in D_2$;
- for each wire w_{ei} on a sheet e in D_1 , a continuous isotopy $\gamma_{ei}^* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ such that $\gamma_{ei}^*(0, t) = \gamma_{ei}(t)$ and $\gamma_{ei}^*(1, t) = \gamma_{\phi(e)i}(t)$ for all $t \in [0, 1]$, where $w_{\phi(e)i} \in D_2$

Finally, at each time $t \in [0, 1]$, the sheet diagram made of the skeleton $\alpha(t)$, the node positions $x_{vi}^*(t)$ and the wire paths $\gamma_{ei}^*(t)$ is required to be a valid sheet diagram.

Lemma 4. *Regular isomorphism of sheet diagrams preserves their interpretation as bimonoidal morphisms.*

Proof. As the interpretation of a sheet diagram D is the interpretation of its skeleton $S(D)$, this is a simple consequence of Theorem 2. \square

However, regular isomorphism of sheet diagrams does not capture the entire equational theory of bimonoidal categories: the exchange law for the multiplicative monoidal structure is missing. For instance, the following diagrams have the same interpretation, but they are not regularly isomorphic:



Their interpretations are equal by the exchange rule for the multiplicative product:

$$\begin{aligned} ((g \cdot 1_B) \oplus (g \cdot 1_C)) \circ (1_A \cdot f) &= (g \cdot (1_{B \oplus C})) \circ (1_A \cdot f) \\ &= (1_C \cdot f) \circ (g \cdot 1_D) \end{aligned}$$

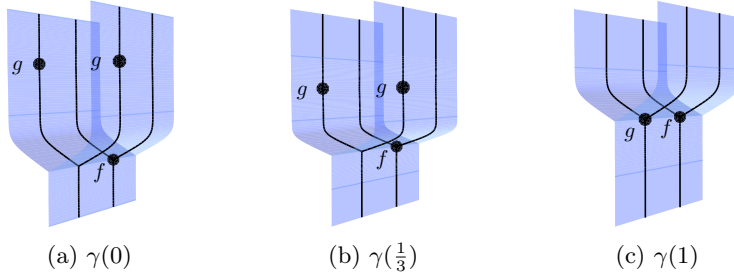
Therefore we need to broaden our class of isomorphisms to capture multiplicative exchange too.

Definition 25. Let $f : \bigoplus_i A_i \rightarrow \bigoplus_j B_j$ and $g : \bigoplus_k C_k \rightarrow \bigoplus_l D_l$ be morphism generators in Γ (diagrams with a single seam), where $A_i, B_j, C_k, D_l \in \text{Ob } \Gamma$. A tensor merge from $\alpha = E_{BD}^{-1} \circ (\bigoplus_l f D_l) \circ E_{AD} \circ (\bigoplus_i A_i g)$ to $\beta = fg$ is a function $\gamma : [0, 1] \rightarrow D(\Gamma)$ (where $D(\Gamma)$ is the set of sheet diagrams on Γ) such that:

- $\gamma(0) = \alpha$ and $\gamma(1) = \beta$;
- the restriction of γ on $[0, t)$ for all $0 < t < 1$ is a regular isomorphism of sheet diagrams
- for each seam $s \in \alpha$, $\lim_{t \rightarrow 1} s(t)$ is the unique seam of β ;
- for each sheet $s \in \alpha$ with one end on the boundary of the diagram, $\lim_{t \rightarrow 1} s(t)$ is the unique sheet in β connected to the same boundary at the same ordinal position;
- for each sheet $s \in \alpha$ not connected to the boundary, $\lim_{t \rightarrow 1} s(t)$ is the unique seam of β ;
- for each node $n \in \alpha$ on a seam s , $\lim_{t \rightarrow 1} n(t)$ is a node in the unique seam of β , with the same ordinal position;
- for each wire w on a sheet $s \in \alpha$ that connects to the boundary, $\lim_{t \rightarrow 1} w(t)$ is a wire on $\lim_{t \rightarrow 1} s(t)$ with the same ordinal position.

Similarly one can define a tensor merge from $(\bigoplus_j B_j g) \circ E_{BC}^{-1} \circ (\bigoplus_k f C_k) \circ E_{AC}^{-1}$ to fg . Finally, a tensor explosion $\gamma : \alpha \rightarrow \beta$ is a tensor merge in reverse, i.e. when $t \mapsto \gamma(1 - t)$ is a tensor merge.

For example, the following are steps of a tensor merge:



We can extend the notions of tensor merges and tensor explosions to wider contexts, where the seams to merge or explode are parts of a larger diagram.

Definition 26. Let $\gamma : \alpha \rightarrow \beta$ be a tensor merge and $C(x)$ be a sheet diagram with a hole, such that $C(\alpha)$ is a valid sheet diagram. Since α and β have the same domain and codomain, $C(\beta)$ is also a valid sheet diagram. The function $C(\gamma) : [0, 1] \rightarrow C(\Sigma)$ defined by $C(\gamma) : t \mapsto C(\gamma(t))$ is called a tensor merge in context. Similarly, we define tensor explosions in context.

Lemma 5. For all tensor merge or explosion in context $C(\gamma) : C(\alpha) \rightarrow C(\beta)$, $C(\alpha)$ and $C(\beta)$ are equal as bimonoidal morphisms.

Proof. By Definition 25, the start and end diagrams of tensor merges or explosions are equal by multiplicative exchange. By composition, this extends to contexts. \square

We can now define our most general notion of isotopy for sheet diagrams.

Definition 27. A bimonoidal isotopy between sheet diagrams D_1, D_2 is a function $\gamma : [0, 1] \rightarrow D(\Sigma)$ such that $\gamma(0) = D_1$, $\gamma(1) = D_2$ and for all $t \in [0, 1]$, there exists $\epsilon > 0$ such that on $[t - \epsilon, t]$, γ is either a regular isomorphism or a tensor merge, and on $[t, t + \epsilon]$, γ is either a regular isomorphism or a tensor explosion.

Lemma 6. Bimonoidal isotopy preserves the interpretation of diagrams.

Proof. Since $[0, 1]$ is connected, it is enough to show that the interpretation of $\gamma(t)$ is locally constant for all $t \in [0, 1]$. By Lemma 4, the interpretation is constant during regular isomorphisms and by Lemma 5, tensor merges and explosions in context also preserve interpretation. \square

Lemma 7. Composition, sum and tensor of sheet diagrams all respect bimonoidal isotopy.

Proof. Given two sheet diagrams α, β and bimonoidal isotopies $\gamma : \alpha \rightarrow \alpha'$, $\gamma' : \beta \rightarrow \beta'$, we obtain a bimonoidal isotopy from $\alpha \oplus \beta$ to $\alpha' \oplus \beta'$ by first running γ while β stays still, then running γ' while α' stays still. Note that we do not run both transformations in parallel because our definition of bimonoidal isotopy only allows for one tensor merge or explosion at a time. The case for the composition of diagrams is similar. By Definition 22, the diagram $\alpha \otimes \beta$ contains in general multiple copies of α and β : we obtain an isotopy by running γ on each copy of α in sequence, and then γ' on copies of β . \square

Definition 28. The category of sheet diagrams $D(\Sigma)$ has sums of products of object symbols from Σ as objects, and equivalence classes of sheet diagrams under sheet diagram isotopy as morphisms. Domains and codomains are given by Definition 17, composition by Definition 18. It has a symmetric monoidal structure \oplus given by Definition 19.

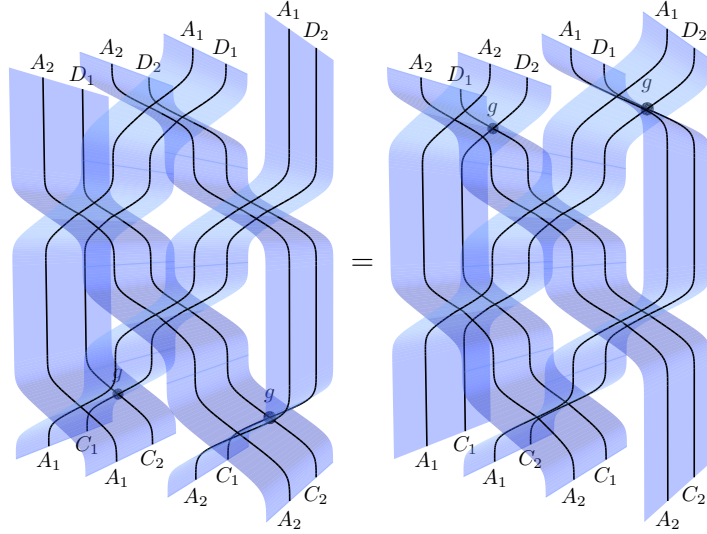
To equip $D(\Sigma)$ with a multiplicative monoidal structure, we need to show that our tensor product (Definition 22) satisfies the exchange law. Tensor merges and explosions are only defined for morphism generators in Γ (single seams), not arbitrary diagrams, so we cannot just use one tensor merge followed by one tensor explosion in general.

Lemma 8. *Any diagram $f \in D(\Sigma)$ can be written in general position, such that no two seams or additive symmetries are at the same height, up to bimonoidal isotopy. It can therefore be expressed as a sequential composition of slices, which are sums of at most one seam or additive symmetry and a finite number of identities.*

Proof. This is a straightforward generalization of the same result for symmetric monoidal string diagrams, which can be found in Joyal and Street [1991]. \square

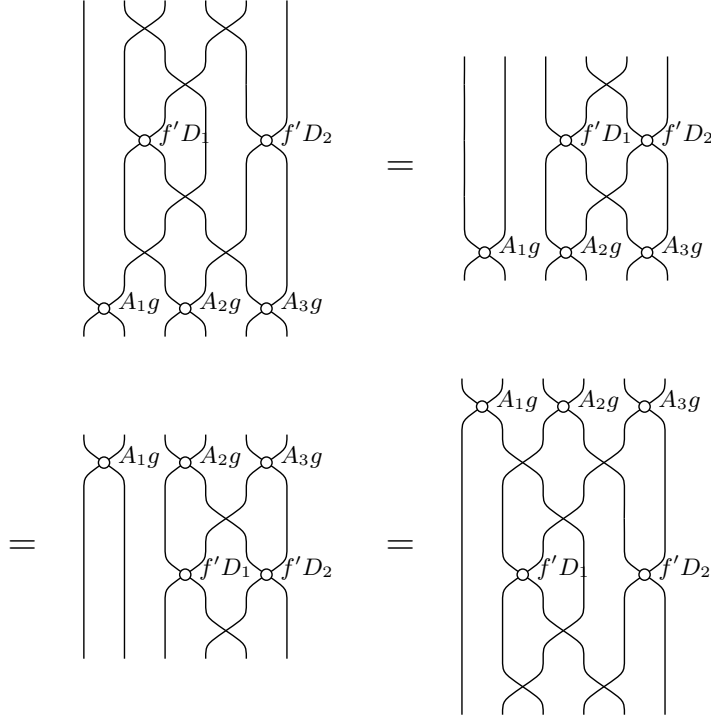
Lemma 9. *Let $f : \bigoplus_i A_i \rightarrow \bigoplus_j B_j$ and $g : \bigoplus_k C_k \rightarrow \bigoplus_l D_l$ be slices, where $A_i, B_j, C_k, D_l \in \text{Ob } \Gamma$. Then there is a bimonoidal isotopy between $E_{BD}^{-1} \circ (\bigoplus_l f D_l) \circ E_{AD} \circ (\bigoplus_i A_i g)$ and $(\bigoplus_j B_j g) \circ E_{BC}^{-1} \circ (\bigoplus_k f C_k) \circ E_{AC}^{-1}$.*

Proof. We proceed by induction on the sum of numbers of identities in the summands of f and g . When there are no identities in f or g , there are three cases. If both f and g are seams, then the two expressions are isotopic via a tensor merge and explosion by construction. If both f and g are symmetries, then the two expressions only consist of symmetries and identities which induce the same permutation of the summands of their domain, so they are isotopic. Finally if one of f and g is a seam and the other is a symmetry, let us assume by symmetry that $f = \gamma_{A_1, A_2}$ and g is a seam. The isotopy holds by pull through moves:



Notice that in this transformation nothing interesting is happening on the third dimension: in the future, we will resort to classical two dimensional string diagrams for such isotopies.

Now for the inductive case, assume there is an identity in $f = 1_{A_1} \oplus f'$. The isotopy holds as follows:



The second equality uses the induction hypothesis on f' and g , other steps are regular isotopies. \square

Lemma 10. *Let $f : \bigoplus_i A_i \rightarrow \bigoplus_j B_j$ and $g : \bigoplus_k C_k \rightarrow \bigoplus_l D_l$ be sheet diagrams, where $A_i, B_j, C_k, D_l \in \text{Ob } \Gamma$. Then there is a bimonoidal isotopy between $E_{BD}^{-1} \circ (\bigoplus_l f D_l) \circ E_{AD} \circ (\bigoplus_i A_i g)$ and $(\bigoplus_j B_j g) \circ E_{BC}^{-1} \circ (\bigoplus_k f C_k) \circ E_{AC}^{-1}$.*

Proof. Up to a regular isotopy, we can assume that f and g are in general position and therefore expressed as a sequential composition of slices. We can then apply Lemma 9 repeatedly, exchanging neighbouring slices of f and g until all slices of f are below all slices of g . \square

Lemma 11. *$D(\Sigma)$ can be equipped with a monoidal structure $(D(\Sigma), \otimes, I)$, given on objects by $A \otimes B = N(A \cdot B)$ and on morphisms by Definition 22*

Proof. The product on objects is unital ($N(A \cdot I) = N(I \cdot A) = N(A)$) and associative by Lemma 1. By Lemma 10, the exchange law for \otimes is satisfied, hence the result. \square

Lemma 12. *$D(\Sigma)$ is bimonoidal for (\oplus, O) and (\otimes, I) .*

Proof. The monoidal structure $(D(\Sigma), \oplus, O)$ is given by Theorem 2, and the monoidal structure $(D(\Sigma), \otimes, I)$ is given by Lemma 11.

Since $N((A \oplus B)C) = N(AC) \oplus N(BC)$, we can define:

$$\delta_{A,B,C}^\# : (A \oplus B) \otimes C \rightarrow A \otimes B \oplus B \otimes C = 1_{N(AC) \oplus N(BC)}$$

For $\delta_{A,B,C} : A \otimes (B \oplus C) \rightarrow A \otimes B \oplus A \otimes C$, decompose $N(A) = \bigoplus_i A_i$ with $A_i \in \text{Ob } \Gamma$. We have

$$N(A(B \oplus C)) = \bigoplus_i N(A_i(B \oplus C)) = \bigoplus_i N(A_i B) \oplus N(A_i C)$$

$$N(AC) \oplus N(BC) = \bigoplus_i N(A_i B) \oplus \bigoplus_i N(A_i C)$$

Therefore we define $\delta_{A,B,C}$ as the reordering map from $\bigoplus_i N(A_i B) \oplus N(A_i C)$ to $\bigoplus_i N(A_i B) \oplus \bigoplus_i N(A_i C)$.

Since $N(OA) = N(AO) = O$ for all A , we define $\lambda^* : O \otimes A \rightarrow O$ and $\rho_A^* : AO \rightarrow O$ as 1_O (the empty sheet diagram).

We now need to check the coherence axioms of Appendix B. Let us consider the first axiom:

$$\begin{array}{ccc} A(B \oplus C) & \xrightarrow{\delta_{A,B,C}} & AB \oplus AC \\ \downarrow 1_A \gamma'_{B,C} & \text{(I)} & \downarrow \gamma'_{AB,AC} \\ A(C \oplus B) & \xrightarrow{\delta_{A,C,B}} & AC \oplus AB \end{array}$$

In $D(\Sigma)$, all sides of this square are composites of the additive symmetry γ . Therefore it is sufficient to check that both paths induce the same permutation, by coherence for symmetric monoidal categories. Equivalently, one can also use coherence for regular objects of bimonoidal categories (Theorem 4) by choosing A, B and C as sums of generators all distinct. One can then obtain commutation for the general case by instantiation (substituting the generators by the actual summands). The other axioms can be treated in similar ways:

- (I), (V), (VI), (VIII), (IX) hold by bimonoidal coherence for regular objects;
- (III) holds since $\delta^\# = 1$ and $\gamma_{A,B} 1_C = \gamma'_{AC,BC}$ by definition;
- (IV) simplifies thanks to $\delta^\# = 1$ and holds by monoidal coherence for (\oplus, O)
- (VII) simplifies thanks to $\delta^\# = 1$ and holds by monoidal coherence for (\otimes, I)
- (X), (XII), (XIII), (XIV), (XVI), (XVII), (XVIII) hold as all sides equal 1_O
- (XIX), (XX), (XXI), (XXII) hold as all sides are identities
- (XXIII) and (XXIV) hold as $\delta_{I,A,B} = 1_{A \oplus B} = \delta_{A,B,I}^\#$

□

Theorem 5. $D(\Sigma)$ and $\bar{\Sigma}$ are bimonoidally equivalent, i.e. $D(\Sigma)$ is the free bimonoidal category on Σ .

Proof. The interpretation of diagrams is a well-defined function $[\cdot] : D(\Sigma) \rightarrow \bar{\Sigma}$ by Lemma 6 and is a bimonoidal functor by construction. For the reverse direction, by freeness of $\bar{\Sigma}$ there is a unique bimonoidal functor $F : \bar{\Sigma} \rightarrow D(\Sigma)$ mapping each generator in Σ to its representation in $D(\Sigma)$.

$$\begin{array}{ccc} & F & \\ \bar{\Sigma} & \xrightarrow{\quad} & D(\Sigma) \\ & [\cdot] & \end{array}$$

Let us show that these form an equivalence. First, $F \circ [\cdot]$ is the identity on objects, on morphism generators and therefore by induction on all morphisms. Second, $[\cdot] \circ F$ is not the identity but $n_A : A \rightarrow N(A)$ is a natural isomorphism from the identity to $[\cdot] \circ F$. Its naturality can be shown by induction on f :

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ n_A \downarrow & & \downarrow n_B \\ N(A) & \xrightarrow{F(f)} & N(B) \end{array}$$

For f generator, the vertical sides are identities (by assumption that domains and codomains of generators are normalized, from Section 5). For f structural isomorphism, the square commutes by regular coherence. By induction, it holds for all morphisms. \square

8 Baez's conjecture

Recently, a conjecture attributed to Baez was confirmed by Elgueta [2020], who showed that the groupoid of finite sets and bijections is biinitial in the 2-category of bimonoidal categories. The category of finite sets has indeed a bimonoidal structure, where disjoint union of sets is the monoidal addition and cartesian product is the monoidal multiplication.

This result can also be obtained via string diagrams. Indeed, the free bimonoidal category on an empty signature, $\bar{\emptyset}$, is biinitial. This is a direct consequence of the universal property: any bimonoidal functor from $\bar{\emptyset}$ to a bimonoidal category \mathcal{C} is determined (up to equivalence) by the image of the generators of $\bar{\emptyset}$, but there are no such generators.

Therefore, to prove Baez' conjecture it is enough to characterize the free bimonoidal category on an empty signature. By Theorem 5, $\bar{\emptyset}$ is bimonoidally equivalent to $D(\emptyset)$. The objects of $D(\emptyset)$ are finite sums of the multiplicative monoidal unit. The morphisms of $D(\emptyset)$ are string diagrams on the empty signature. We can analyze the geometry of such string diagrams. All the sheets in such diagrams are empty: they cannot have any wires on them, since those wires would need to be annotated by an object from the signature. Similarly, the diagrams do not contain any seams, since each seam contains at least one node which is annotated by a morphism generator. Therefore, the diagrams are only made of identities of empty sheets and additive symmetries.

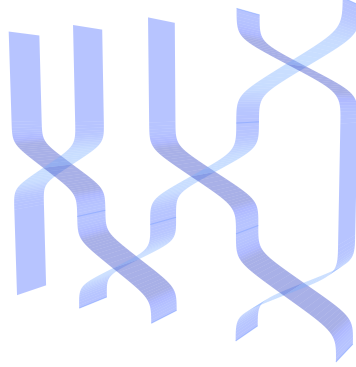


Figure 7: A sheet diagram on the empty signature

Hence, a morphism in $D(\emptyset)$ induces a permutation of its domain, which is equal to its codomain. Furthermore, morphisms in $D(\emptyset)$ are equivalence classes of string diagrams up to bimonoidal equivalence. Two string diagrams induce the same permutation of their common domain when their skeletons are isomorphic as symmetric monoidal diagrams, and therefore when the diagrams are in bimonoidal equivalence. Therefore, the category $D(\emptyset)$ is equivalent to the groupoid of finite sets, hence the result.

References

- Antonin Delpeuch. A complete language for faceted dataflow programs. In *arXiv:1906.05937 [Cs, Math]*, To Appear in Proceedings of the Applied Category Theory 2019, Oxford, June 2019.
- Antonin Delpeuch and Jamie Vicary. Normalization for planar string diagrams and a quadratic equivalence algorithm. *arXiv:1804.07832 [cs]*, April 2018.
- Ross Duncan. Generalized Proof-Nets for Compact Categories with Biproducts. In Simon Gay and Ian Mackie, editors, *Semantic Techniques in Quantum Computation*, pages 70–134. Cambridge University Press, Cambridge, 2009. ISBN 978-1-139-19331-3. doi: 10.1017/CBO9781139193313.004.
- Josep Elgueta. The groupoid of finite sets is biinitial in the 2-category of rig categories. *arXiv:2004.08684 [math]*, April 2020.
- Tobias Fritz and Paolo Perrone. Bimonoidal Structure of Probability Monads. *Electronic Notes in Theoretical Computer Science*, 341:121–149, December 2018. ISSN 15710661. doi: 10.1016/j.entcs.2018.11.007.
- Jose Manuel Gomez. From fibered symmetric bimonoidal categories to symmetric spectra. *arXiv:0905.3156 [math]*, May 2009.
- Bertrand Guillou. Strictification of categories weakly enriched in symmetric monoidal categories. *arXiv:0909.5270 [math]*, September 2009.
- Guenter Hotz. Eine Algebraisierung des Syntheseproblems von Schaltkreisen I. *Elektronische Informationsverarbeitung und Kybernetik*, 1(3):185–205, 1965.

- Roshan P. James and Amr Sabry. Information effects. *ACM SIGPLAN Notices*, 47(1):73–84, January 2012. ISSN 0362-1340. doi: 10.1145/2103621.2103667.
- André Joyal and Ross Street. The geometry of tensor calculus, I. *Advances in Mathematics*, 88(1):55–112, 1991. ISSN 0001-8708. doi: 10.1016/0001-8708(91)90003-p.
- Miguel L. Laplaza. Coherence for distributivity. In G. M. Kelly, M. Laplaza, G. Lewis, and Saunders Mac Lane, editors, *Coherence in Categories*, volume 281, pages 29–65. Springer Berlin Heidelberg, Berlin, Heidelberg, 1972. ISBN 978-3-540-05963-9 978-3-540-37958-4. doi: 10.1007/BFb0059555.
- P. Selinger. A Survey of Graphical Languages for Monoidal Categories. In Bob Coecke, editor, *New Structures for Physics*, number 813 in Lecture Notes in Physics, pages 289–355. Springer Berlin Heidelberg, 2010. ISBN 978-3-642-12820-2 978-3-642-12821-9. doi: 10.1007/978-3-642-12821-9_4.
- Sam Staton. Algebraic Effects, Linearity, and Quantum Programming Languages. In *Proceedings of the 42nd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages - POPL '15*, pages 395–406, Mumbai, India, 2015. ACM Press. ISBN 978-1-4503-3300-9. doi: 10.1145/2676726.2676999.

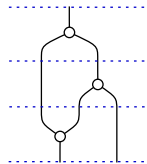
A Data structures for sheet diagrams

In this section we give a short primer on the declarative format used to represent diagrams in SheetShow³, the tool we used to render all sheet diagrams in this article.

The first step to understand the format is to explain the data structure we use to represent monoidal string diagrams. These are simpler than bimonoidal diagrams as they can be drawn in the plane.

A.1 Data structure for monoidal string diagrams

A string diagrams in a monoidal category is in *general position* when no two nodes share the same height. Any string diagram can be put in general position without changing its meaning. A diagram in general position can be decomposed as a sequence of horizontal *slices*, each of which contains exactly one node.



One can therefore encode such a diagram as follows:

³Available at <https://wetneb.github.io/sheetshow/> and <https://github.com/wetneb/sheetshow> (source)

- The number of wires crossing the input boundary (or a list of the objects annotating them, in a typed context);
- The list of slices, each of which can be described by the following data:
 - The number of wires passing to the left of the node in the slice. We call this the *offset*;
 - The number of input wires consumed by the node;
 - The number of output wires produced by the node (or again, their list of types).

For the sample diagram above, this gives us the following encoding (with inputs at the bottom of the diagram):

```
inputs: 2
slices:
- offset: 0
  inputs: 1
  outputs: 2
- offset: 1
  inputs: 2
  outputs: 1
- offset: 0
  inputs: 2
  outputs: 1
```

Each slice can be augmented to store details about the morphism in that slice (such as a label or types, for instance). This data structure is well suited to reason about string diagrams and there are efficient algorithms to determine if two diagrams are equivalent up to exchanges. For more details about this data structure, see Delpuch and Vicary [2018].

A.2 Bimonoidal diagrams

Sheet diagrams in bimonoidal categories are obtained by extruding symmetric monoidal string diagrams for the additive monoidal structure (\mathcal{C}, \oplus, O) . Therefore our data structure for bimonoidal diagrams is based on that for monoidal diagrams.

A bimonoidal diagram is described by:

- The number of input sheets, and the number of input wires on each of these input sheets;
- The slices of the bimonoidal diagram, which are seams between sheets. They are each described by:
 - The number of sheets passing to the left of the seam. We call this, again, the *offset*;
 - The number of input sheets joined by the seam;
 - The number of output sheets produced by the seam;
 - The nodes present on the seam.

Each seam can have multiple nodes on it. Each of these can connect to some wires on each input sheet (not necessarily the same number of wires for each input sheet) and similarly for output sheets. We describe them with the following data:

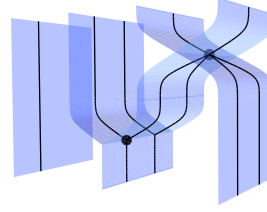
- The number of wires passing through the seam without touching a node, to the left of the node being described. We call this the *offset* of the node;
- For each input sheet, the number of wires connected to the node;
- For each output sheet, the number of wires connected to the node.

For instance:

```

inputs: [ 1, 2, 2 ]
slices:
- offset: 1
  inputs: 1
  outputs: 2
  nodes:
  - offset: 0
    inputs: [ 1 ]
    outputs: [ 1, 1 ]
  - offset: 2
    inputs: 2
    outputs: 2
  nodes:
  - offset: 0
    inputs: [ 2, 2 ]
    outputs: [ 1, 1 ]

```



Again, additional metadata can be added on the geometry to annotate it with labels, types, and represent symmetries for the additive and multiplicative structures. For more details about these, consult SheetShow's documentation: <https://sheetshow.readthedocs.org/en/latest/>.

B Coherence axioms

The following coherence axioms are taken from Laplaza [1972]. Their axioms (II) and (XV) were removed as they only apply to symmetric bimonoidal categories. In the following, the monoidal structure (\mathcal{C}, \cdot, I) has coherence isomorphisms

$$\begin{aligned}
 \alpha_{A,B,C} &: A(BC) \rightarrow (AB)C \\
 \lambda_A &: IA \rightarrow A \\
 \rho_A &: AI \rightarrow A
 \end{aligned}$$

and the monoidal structure (\mathcal{C}, \oplus, O) has coherence isomorphisms

$$\begin{aligned}
 \alpha'_{A,B,C} &: A \oplus (B \oplus C) \rightarrow (A \oplus B) \oplus C \\
 \lambda'_A &: O \oplus A \rightarrow A \\
 \rho'_A &: A \oplus O \rightarrow A \\
 \gamma'_{A,B} &: A \oplus B \rightarrow B \oplus A
 \end{aligned}$$

$$\begin{array}{c}
\begin{array}{ccc}
A(B \oplus C) & \xrightarrow{\delta_{A,B,C}} & AB \oplus AC \\
\downarrow 1_A \gamma'_{B,C} & & \downarrow \gamma'_{AB,AC} \\
A(C \oplus B) & \xrightarrow{\delta_{A,C,B}} & AC \oplus AB
\end{array} \quad \text{(I)} \\
\\
\begin{array}{ccc}
(A \oplus B)C & \xrightarrow{\delta_{A,B,C}^\#} & AC \oplus BC \\
\downarrow \gamma'_{A,B} 1_C & & \downarrow \gamma'_{AC,BC} \\
(B \oplus A)C & \xrightarrow{\delta_{B,A,C}^\#} & BC \oplus AC
\end{array} \quad \text{(III)} \\
\\
\begin{array}{ccc}
(A \oplus (B \oplus C))D & \xrightarrow{\delta_{A,B \oplus C,D}^\#} & AD \oplus (B \oplus C)D \xrightarrow{1_{AD} \oplus \delta_{B,C,D}^\#} AD \oplus (BD \oplus CD) \\
\downarrow \alpha'_{A,B,C} 1_D & & \downarrow \alpha'_{AD,BD,CD} \\
((A \oplus B) \oplus C)D & \xrightarrow{\delta_{A \oplus B,C,D}^\#} & (A \oplus B)D \oplus CD \xrightarrow{\delta_{A,B,D}^\# \oplus 1_{CD}} (AD \oplus BD) \oplus CD \\
A(B \oplus (C \oplus D)) & \xrightarrow{\delta_{A,B,C \oplus D}} & AB \oplus A(C \oplus D) \xrightarrow{1_{AB} \oplus \delta_{A,C,D}} AB \oplus (AC \oplus AD) \\
\downarrow 1_A \alpha'_{B,C,D} & & \downarrow \alpha'_{AB,AC,AD} \\
A((B \oplus C) \oplus D) & \xrightarrow{\delta_{A,B \oplus C,D}} & A(B \oplus C) \oplus AD \xrightarrow{\delta_{A,B,C} \oplus 1_{AD}} (AB \oplus AC) \oplus AD \\
A(B(C \oplus D)) & \xrightarrow{1_A \delta_{B,C,D}} & A(BC \oplus BD) \xrightarrow{\delta_{A,BC,BD}} A(BC) \oplus A(BD) \\
\downarrow \alpha_{A,B,C \oplus D} & & \downarrow \alpha_{A,B,C} \oplus \alpha_{A,B,D} \\
(AB)(C \oplus D) & \xrightarrow{\delta_{AB,C,D}} & (AB)C \oplus (AB)D \\
(A \oplus B)(CD) & \xrightarrow{\delta_{A,B,CD}^\#} & A(CD) \oplus B(CD) \\
\downarrow \alpha_{A \oplus B,C,D} & & \downarrow \alpha_{A,C,D} \alpha_{B,C,D} \\
((A \oplus B)C)D & \xrightarrow{\delta_{A,B,C}^\# \oplus 1_D} & (AC \oplus BC)D \xrightarrow{\delta_{AC,BC,D}^\#} (AC)D \oplus (BC)D \\
A((B \oplus C)D) & \xrightarrow{1_A \delta_{B,C,D}^\#} & A(BD \oplus CD) \xrightarrow{\delta_{A,BD,CD}} A(BD) \oplus A(CD) \\
\downarrow \alpha_{A,B \oplus C,D} & & \downarrow \alpha_{A,B,D} \alpha_{A,C,D} \\
(A(B \oplus C))D & \xrightarrow{\delta_{A,B,C} 1_D} & (AB \oplus AC)D \xrightarrow{\delta_{AB,AC,D}^\#} (AB)D \oplus (AC)D
\end{array} \quad \text{(IV)} \\
\\
\text{(V)} \\
\\
\text{(VI)} \\
\\
\text{(VII)} \\
\\
\text{(VIII)}
\end{array}$$

$$\begin{array}{ccc}
A(C \oplus D) \oplus B(C \oplus D) & \xrightarrow{\delta_{A,C,D} \delta_{B,C,D}} & (AC \oplus AD) \oplus (BC \oplus BD) \\
\uparrow \delta_{A,B,C \oplus D}^\# & & \downarrow \alpha'_{AC \oplus AD, BC, BD} \\
(A \oplus B)(C \oplus D) & \text{(IX)} & ((AC \oplus AD) \oplus BC) \oplus BD \\
\downarrow \delta_{A \oplus B, C, D} & & \downarrow \alpha'^{-1}_{AC, AD, BC \oplus 1_{BD}} \\
(A \oplus B)C \oplus (A \oplus B)D & \xrightarrow{\delta_{A,B,C}^\# \delta_{A,B,D}^\#} & (AC \oplus (AD \oplus BC)) \oplus BD \\
& & \downarrow (1_{AC} \oplus \gamma'_{AD, BC}) \oplus 1_{BD} \\
& & (AC \oplus (BC \oplus AD)) \oplus BD \\
& & \uparrow \alpha'^{-1}_{AC, BC, AD \oplus 1_{BD}} \\
& & ((AC \oplus BC) \oplus AD) \oplus BD \\
& & \uparrow \alpha'_{AC \oplus BC, AD, BD}
\end{array}$$

$$\begin{array}{ccc}
O(A \oplus B) & \xrightarrow{\delta_{O,A,B}} & OA \oplus OB \\
\downarrow \lambda_{A \oplus B}^* & & \downarrow \lambda_A^* \oplus \lambda_B^* \\
O & \xleftarrow{\lambda'_O} & O \oplus O
\end{array} \quad \text{(XI)}$$

$$\begin{array}{ccc}
(A \oplus B)O & \xrightarrow{\delta_{A,B,O}^\#} & AO \oplus BO \\
\downarrow \rho_{A \oplus B}^* & & \downarrow \rho_A^* \oplus \rho_B^* \\
O & \xleftarrow{\lambda'_O} & O \oplus O
\end{array} \quad \text{(XII)}$$

$$\text{(XIII): } \rho_O = \lambda_O^* : OI \rightarrow O$$

$$\begin{array}{ccc}
O(AB) & \xrightarrow{\alpha_{O,A,B}} & (OA)B \\
\downarrow \lambda_{AB}^* & & \downarrow \lambda_A^* 1_B \\
O & \xleftarrow{\lambda_B^*} & OB
\end{array} \quad \text{(XVI)}$$

$$\text{(XIV): } \lambda_O = \rho_O^* : IO \rightarrow O$$

$$\begin{array}{ccc}
A(OB) & \xrightarrow{\alpha_{A,O,B}} & (AO)B \\
\downarrow 1_A \lambda_B^* & & \downarrow \rho_A^* 1_B \\
AO & & OB \\
\swarrow \rho_A^* & & \searrow \lambda_B^* \\
& O &
\end{array} \quad \text{(XVII)}$$

$$\begin{array}{ccc}
A(BO) & \xrightarrow{1_A \rho_B^*} & AO \\
\downarrow \alpha_{A,B,O} & & \downarrow \rho_A^* \\
(AB)O & \xrightarrow{\rho_{AB}^*} & O
\end{array} \quad \text{(XVIII)}$$

$$\begin{array}{ccc}
A(O \oplus B) \xrightarrow{\delta_{A,O,B}} AO \oplus AB & (O \oplus B)A \xrightarrow{\delta_{O,B,A}^\#} OA \oplus BA \\
\downarrow 1_A \lambda'_B & \downarrow \rho_A^* \oplus 1_{BA} \quad \lambda'_B 1_A & \downarrow \lambda_A^* \oplus 1_{BA} \\
AB \xleftarrow{\lambda'_{AB}} O \oplus AB & BA \xleftarrow{\lambda'_{BA}} O \oplus BA & \\
A(B \oplus O) \xrightarrow{\delta_{A,B,O}} AB \oplus AO & (A \oplus O)B \xrightarrow{\delta_{A,O,B}^\#} AB \oplus OB \\
\downarrow 1_A \rho'_B & \downarrow 1_{AB} \oplus \rho_A^* \quad \rho'_A 1_B & \downarrow 1_{AB} \oplus \lambda_B^* \\
AB \xleftarrow{\rho'_{AB}} AB \oplus O & AB \xleftarrow{\rho'_{AB}} AB \oplus O & \\
I(A \oplus B) \xrightarrow{\delta_{I,A,B}} IA \oplus IB & & \\
\swarrow \lambda_{A \oplus B} & \text{(XXIII)} & \searrow \lambda_A \oplus \lambda_B \\
& A \oplus B & \\
(A \oplus B)I \xrightarrow{\delta_{A,B,I}^\#} AI \oplus BI & & \\
\swarrow \rho_{A \oplus B} & \text{(XXIV)} & \searrow \rho_A \oplus \rho_B \\
& A \oplus B &
\end{array}$$

Figure 9: Coherence axioms