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Algebraic geometry

algebra of dual numbers

Coalgebraic geometry

Cartesian closed
world of coalgebras

Ring $\xrightarrow{\text{Spectrum}}$ Λ

co-exponential
 $\varepsilon^2=0$
the line $\rightarrow L \xrightarrow{\varepsilon} \varepsilon$
 $L \cong TL$

exponential

X, dX

$\text{Spec}(K[X]) = \text{Line}$

$\text{Spec}\left(\frac{K[\varepsilon]}{\varepsilon^2}\right) = \text{Walking tangent vector}$

Commutative Hopf
algebra

multiplication is
from group structure,
multiplication
geometry

algebra
of
functions

Co-commutative Hopf
algebra

multiplication is from
co-group structure

comultiplication
geometry

coalgebra
of
measures

both commutative:
is a Fourier
dual
pair of Abelian
groups

See also:
Lawvere
extensive
& intensive

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coexponential

cobase: $K[X]$

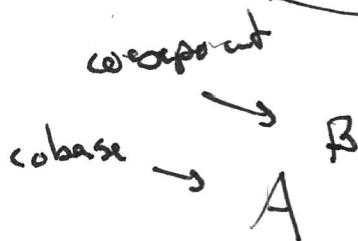
the theory of a quantity X .

coexponent: $K[T]_{T^2} = \varepsilon$ the theory of a quantity T with $T^2=0$.

coexponential:

the theory of: a ^{model} ~~universe~~ of the cobase is
the universe provided by the
coexponent.

aside $\varepsilon^{\varepsilon}$ = endomorphisms of $K[T]_{T^2}$
= solutions of $T^2=0$ in $K(r)_{T^2}$.
models



: the theory of an A model is
the universe provided by B .

model = spectrum

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3/4/19

Example

$$A = K[x, y] / x^2 + y^2 = 1$$

$$B = K[\Sigma]$$

$$A^B = X \mapsto x_0, x_1, x_2, \dots$$

$$Y \mapsto y_0, y_1, y_2, \dots$$

$$\text{s.t. } x_0^2 + y_0^2 = 1$$

$$x_1 y_0 + x_0 y_1 = 0$$

$$x_2 y_0 + x_1 y_1 + x_0 y_2 = 0$$

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3/4/19

Example

$$A = K[x, y] / \cancel{XY} XY = 0$$

$$B = K[\tau] / \tau^2$$

$$A^B = K[x_0, x_1, y_0, y_1] / (x_0 + x_1, \tau)(y_0 + y_1, \tau) = 0$$

$$x_0 y_0 = 0$$

$$x_1 y_0 + x_0 y_1 = 0$$

$$y_0 x_1 + x_0 y_1 = 0$$

singularities give
rise to tangent
vectors that
"don't go anywhere"
these are formal
sums of tangent
vectors that
"do go somewhere".
Walking tangent
vector ϵ also goes
nowhere.

Zariski tangent

space: linear
hull of
the tangent cone
of a curve at a
point.

Example

$$A = K[\tau] / \tau^2$$

$$B = K[\tau] / \tau^2$$

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3/4/19

More co-exponential examples

co-base: $A = K[x, y] / x^2 + y^2 = 1$ $\xrightarrow{\text{spec}}$ the circle

co-fiber: $B = K[t] / t^2$

co-fiber: the theory of " $x^2 + y^2 = 1$ " in B .

$$x \mapsto x_0 + x_1 t$$

$$y \mapsto y_0 + y_1 t$$

$$1 \mapsto 1$$

~~III~~

$$(x_0 + x_1 t)^2 + (y_0 + y_1 t)^2 = 1$$

$$x_0^2 + 2x_0 x_1 t + y_0^2 + 2y_0 y_1 t = 1$$

$$K[x_0, x_1, y_0, y_1] / \{x_0^2 + y_0^2 = 1,$$

$$2x_0 x_1 + 2y_0 y_1 = 0\}$$

$$(x_0, y_0) \cdot (x_1, y_1) = 0$$

$$K[x_0, y_0, x_1, y_1] / x^2 + y^2 = 1$$

$$x dx + y dy = 0$$

tangent
bundle
projects
onto

$$K[x, y] / x^2 + y^2 = 1$$