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Synthetic differential geometry

Categorification of infinitesimal numbers.

Category of spaces

add the walking tangent vector as an idealised infinitesimal space: $\mathcal{E}$

Cartesian closed (currying)

$\hom(A,B) \cong B^A$

tangent bundle TX

see also a short Lawvere paper

$TX = X^{\mathcal{E}}$ is how the fairy-tale space $\mathcal{E}$ becomes an actual space.

vector fields are a section

$$\begin{tikzcd} & TX \arrow[dl] \\ X \arrow[r] & X \arrow[u, "v"'] \end{tikzcd}$$

Question: how to show this extra syntax "$\mathcal{E}$" plays nicely? I'm so lazy...

Algebraic geometry history: nullstellensatz

"zero-place theorem"

An algebraic variety ~~can be~~ is equivalent to the space of zero places of polynomial functions that are zero on the algebraic variety.

Ie. Ideals in polynomial rings....

Flea in the ointment:

$x^2 = 0$

non-radical ideal (not closed under taking radicals)

So the bijection between ideals & varieties fails.

Sol'n: exclude nilpotent elements

$$\frac{\text{generators}}{\text{relations}} : \frac{\text{polynomials}}{\text{ideal}} \cong \frac{\text{cartesian product space}}{\text{variety}}$$

points of the ideal is all the functions that are zero on the variety

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Category of commutative rings $\overset{\text{tug of war}}{\underset{\ldots}{\overset{???}{\simeq}}}$ Category of algebraic Varieties

First problem: not cartesian closed

take L: line

then X^L: space of paths in X
is infinite dimensional
..... FAIL
(dream of functional analysis)

~~...~~ but $X^{\varepsilon} = TX$ exists

Idea make a category of infinitesimal spaces: it is cartesian closed, and acts on the category of (algebraic varieties) spaces.

(other ideas: denotational semantics, topos theory, subobject classifier.)

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Syntactic	Semantic
Commutative rings	Algebraic varieties

Define

A category of infinitesimal spaces, cartesian closed & locally presentable (colimits)

(cartesian product distributes over colimits

therefore

$a \times \_$ has a right adjoint which is exp)

called the category of cocommutative coalgebras

(looks like a discrete space with infinitesimals on each point...)

When

Is it an algebra & when is it a ring?

Algebras look like rings with a special sub-ring.

Defn Cartesian closed: has

i) terminal object

ii) product $X \times Y$

iii) exponential X^Y

s.t.

$\mathrm{Hom}(X \times Z, Y) \cong \mathrm{Hom}(X, Y^Z)$

naturally in X & Y.

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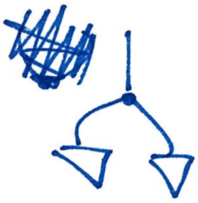
Take a cocommutative coalgebra over a commutative base ring K (e.g. integers, or a field)

↑ time

algebra: $V \otimes V \to V$



co-algebra: $V^* \to V^* \otimes V^*$

Examples: ring	Space
ring of polynomials in one variable (looks like an infinite dim vector space)	the line
the Study numbers:	walking tangent vector
$K[x]/x^2$ "dual" numbers (is a finite dimensional vector space so this works) $\xrightarrow{\text{dualize}}$	coalgebra

Questions: what is ε^ε?

punchline/conclusion:

Categorify this story