Moduli and Cohomology

by

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Submitted in Partial Fulfillment

of the

Requirements for the Degree

Doctor of Philosophy

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2008

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Acknowledgments

I owe Mark Behrens, Mike Hopkins, Tyler Lawson, Jack Morava, Paul Goerss, Ethan Pribble, Peter May, and Bob Bruner each for their support, very useful conversations, and willingness to answer all my tedious questions. From when I was an undergraduate, I owe John Erdman and Lawrence Wheeler each for having more faith in me than I had in myself. More than anyone else, I owe Doug Ravenel for going through great pains over the course of five years to give me the opportunity to have a career as a researcher in mathematics. Anything of value that I ever contribute to mathematics should be credited as least as much to Doug as to myself, as I would never have been able to become useful to mathematics without his help.

Abstract

We provide an algebro-geometric framework for Hopf algebroids in which they become equivalent to flat algebraic stacks, modulo many technical details, which we describe and handle. We produce isomorphisms in flat cohomology between objects on either side of this equivalence, and we produce resolutions and spectral sequences for performing basic cohomological tasks in the relevant categories. We apply these results to the moduli stacks and classifying Hopf algebroids of formal A-modules, with A a number ring, obtaining some splitting results about these moduli objects. We lay the groundwork for further methods, currently in preparation, generalizing chromatic theory from stable homotopy theory for general use in algebraic geometry as well as advanced applications in computational stable homotopy.

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0.1 Preface.

We do not recommend that this document be used in any attempt to learn the material contained within. The author is in the process of rewriting each section, for conciseness and clarity; the total lack of either quality in this thesis makes it a poor choice for anyone hoping to learn something.

This document is a doctoral thesis, but the material in it describes the first parts of a work (currently in a long process of being written down) which really should be taken as a whole. The goal is to compute portions of $\pi_*(S)$, the stable homotopy of spheres, including the heights 1,2,3, and 4 layers in the chromatic spectral sequence, at all primes; we use a generalized, highly functorial version of the chromatic theory of [Ravenel, 1986], aided by number-theoretic and algebro-geometric tools, to accomplish this. These computations also result in some very suggestive, very conjectural (but supported by empirical evidence, from the computations) formulas relating the orders of the groups in the height n layer of the chromatic spectral sequence to certain factors in special values of Artin L-functions of n-dimensional representations of the Galois groups of some extensions of $\mathbb{Q}$. We consider this a highly worthwhile goal.

In this thesis we lay down the groundwork for our exposition, which will have to come later, of the generalized chromatic theory we use to accomplish the new computations. In particular, in this thesis we demonstrate a cohomology-preserving equivalence between certain pro-rigidified fpqc stacks and certain "sheafy" flat Hopf algebroids. Our approach includes stacks with non-affine diagonal, which is a departure from previous approaches to stacks by stable homotopy theorists, such as [Pribble, 2004] and [Goerss, 2008] and [Hopkins, 1999]; we feel that this is important because it allows *all* of the flat algebraic stacks considered by geometers to be attacked by the methods of computational stable homotopy theory, and not merely those over an affine base scheme and with affine diagonal. Unfortunately this violates the saying, which is already on its way to becoming a slogan, that "rigidified flat stacks are equivalent to flat Hopf algebroids"; but this saying wasn't even true without re-defining algebraic stacks to only include those with affine base scheme and affine diagonal, which is certainly not what algebraic geometers (e.g. [Laumon and Moret-Bailly, 2000]) mean by an algebraic stack. The restriction to an affine base scheme is, in some ways, benign, but refusing to consider stacks with non-affine diagonal is a more serious problem. The Hilbert stack of a non-quasiseparated scheme X, for instance, does not in general have affine diagonal; see [Rydh, 2008].

Along the way in our dealing with stacks and Hopf algebroids we give a treatment of some of the general machinery of sheaves on a site taking values in an abelian category, or in multiple abelian categories, a formalism which we need for our applications. We also demonstrate an equivalence of flat algebraic spaces with a class of algebraic objects which are essentially Hopf algebroids which are missing their coproduct maps.

After developing and proving our main results about stacks and Hopf algebroids we apply them to moduli stacks of formal modules, which are the crucial objects in our computational approach to $\pi_*(S)$, as the E_2 term of the Adams-Novikov spectral sequence converging to $\pi_*(S)$ is the flat cohomology $H^*_{\mathrm{fpqc}}(\mathcal{M}_{fg}; \omega^{*\otimes})$ of the moduli stack of formal groups, and our new computations are made possible by study of the restriction map induced in cohomology by the stack map $\mathcal{M}_{fm_A} \to \mathcal{M}_{fg}$ classifying the underlying formal group of the universal formal A-module, with A a finite extension of $\mathcal{Z}$.

Some things are missing from this account, which we hope to include later. The most important is the question of gradings; the Hopf algebroids arising in stable homotopy are equipped with a $\mathbb{Z}$-grading. This grading becomes a $\mathbb{G}_m$-action on the associated pro-rigidified fpqc stack and there is some bookkeeping to be done to come to grips with this $\mathbb{G}_m$-action. This is the main oversight we hope to rectify before putting this written material out for general distribution. We also hope to clarify and streamline the exposition, and to come up with reasonable alternatives to the unreasonable names we have given some objects, left in capitals in the text, to indicate that they are only temporary names.

Most importantly we hope to continue this project, typing up our results and making more of the technology available for making new computations of the Adams-Novikov E_2-term.

The following chart is useful for keeping track of some of the objects which we prove are equivalent to one another.

Sets -objects	commutative R-algebras	affine S-schemes	S-schemes
Setoids-objects	commutative R-algebroids	Ravenel S-spaces in the canonical topology	rigidified S-spaces in the canonical topology
SmCat-objects	commutative R-bialgebroids	**SmCat**-affine S-schemes	**SmCat**-schemes over S
Monoids-objects	commutative R-bialgebras	**Monoids**-affine S-schemes	**Monoids**-schemes over S
Groups-objects	commutative Hopf algebras over R	affine group S-schemes	group S-schemes
Groupoids-objects	commutative Hopf algebroids over R	Ravenel S-stacks in the canonical topology	rigidified S-stacks, of schematic descent class, in the canonical topology

1 How to use this document.

1.1 What do you do with a flat stack?

A certain motto has become folklore among topologists in the past ten years:

Motto 1.1.1. There exists a cohomology-preserving anti-equivalence between flat Hopf algebroids and rigidified flat stacks.

Unfortunately, this motto has never been made substantial by a published account, including proofs, of this anti-equivalence. Several unpublished accounts do exist: the course notes [Hopkins, 1999] of Mike Hopkins and the Ph.D. thesis [Pribble, 2004] of Ethan Pribble, for example. These unpublished accounts are informative and they have served topologists well so far. Unfortunately, I do not think we have yet found the really "right" way to state Motto 1.1.1. As far as I am aware, there are four problems with the version of Motto 1.1.1 stated above. I will list them in order, from "most immediately troublesome" to "least immediately troublesome." The first of the four is the only one which has been addressed in the existing (unpublished) literature on Hopf algebroid/algebraic stack equivalences.

1. **Finite-dimensionality.** The Hopf algebroids arising in homotopy theory give rise to stacks which are, using the dimension conventions in force in alge-

braic geometry (see [Behrend, 2003], for example), of "negative countably infinite dimension." The complex cobordism Hopf algebroid $(\pi_*(MU), \pi_*(MU \wedge MU))$, for instance, which is an object of critical importance for stable homotopy theory, has the property that $\pi_*(MU \wedge MU)$ is infinite-dimensional as an $\pi_*(MU)$-algebra, and this property is shared by some of the other most important Hopf algebroids from computational stable homotopy, such as the BP-homology Hopf algebroid $(\pi_*(BP), \pi_*(BP \wedge BP))$ and the mod p homology Hopf algebroid $(\mathbb{F}_p, \mathcal{A})$ (i.e., $(\pi_*(H\mathbb{F}_p), \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p))$. Clearly, any translation between Hopf algebroids and rigidified stacks which is useful to homotopy theorists will have to be able to handle these Hopf algebroids.

Unfortunately, when we form the groupoid scheme

$$\operatorname{Spec} \pi_* MU \rightleftarrows \operatorname{Spec} \pi_*(MU \wedge MU)$$

and stackify the $\operatorname{Spec} \mathbb{Z}$-prestack it represents, we have an epimorphism

$$\operatorname{Spec} \pi_* MU \to \mathcal{X}$$

of $\operatorname{Spec} \mathbb{Z}$-stacks in the fpqc topology, but it is *not* a presentation, *for dimensional reasons*: the morphism fails to be smooth, but it is "ho-pro-smooth" (a filtered homotopy limit of smooth morphisms). By this we mean the following: we have that $\pi_*(MU \wedge MU) \cong \pi_*(MU)[w_1, w_2, \dots]$ and the stackification $\operatorname{Spec} \pi_*(MU) \to \mathcal{X}_i$ of the prestack represented by the groupoid scheme

$$\operatorname{Spec} \pi_* MU \rightleftarrows \operatorname{Spec} (\pi_*(MU)[w_1, w_2, \dots, w_i])$$

is a (fppf) presentation, hence a rigidified fppf $\operatorname{Spec} \mathbb{Z}$-stack; and the fpqc epimorphism $\operatorname{Spec} \pi_* MU \to \mathcal{X}$ is the homotopy limit of the morphisms $\operatorname{Spec} \pi_* MU \to \mathcal{X}_i$. In particular, $\mathcal{X} \simeq \operatorname{holim}_i \mathcal{X}_i$. So we are "ho-pro-smooth" but not quite smooth. Note that we have had to switch between the fppf and fpqc topologies, because while each $\operatorname{Spec} \pi_* MU \to \mathcal{X}_i$ is faithfully flat and finitely presented, the morphism $\operatorname{Spec} \pi_* MU \to \mathcal{X}$ is faithfully flat but not finitely presented; however, it is affine, hence quasicompact, and is a cover in the fpqc topology but not the fppf topology. These (related) problems are common to many of the important Hopf algebroids encountered in algebraic topology.

There are two workarounds for this, both of which are (to my knowledge) inventions of Paul Goerss.

(a) **Weaken the smoothness requirement in the definition of an algebraic stack.** One can simply re-define "algebraic stack" to mean a stack admitting a *quasismooth*, rather than smooth, cover; Goerss mentions this in [Goerss, 2004], while in another paper, he throws out the smoothness condition entirely ([Goerss, 2008]). Goerss also re-defines stacks so that they will have affine diagonal. I worry that re-defining basic terms is an impediment to communication between algebraic topologists and algebraic geometers, and whatever benefits such re-definitions afford us in the near future will be outweighed by the confusions and miscommunications we may suffer later, so I use the term "algebraic stack" in its most common sense, e.g. [Laumon and Moret-Bailly, 2000], which excludes those stacks (e.g. the stack associated to $(\pi_* MU, \pi_*(MU \wedge MU))$) whose only covers by schemes are quasismooth and not smooth. So one could invent a term ("quasistack"?) to describe stacks admitting a cover which is quasismooth, not necessarily smooth, and satisfies all other axioms for being an algebraic stack; such "quasistacks" would be very appropriate objects for an antiequivalence between flat Hopf algebroids and fpqc stacks.

(b) **Consider homotopy limits of algebraic stacks.** The other solution, and the one we adopt in this thesis, is to construct our anti-2-equivalence between a category of ho-pro-rigidified (homotopy pro-rigidified) fpqc stacks, and a suitable category of flat smooth Hopf algebroids. In particular, let $X \to \mathcal{X}_i$ be an fppf rigidified stack for each i in a small filtered index category; then by a "ho-pro-rigidified fpqc stack" we will mean $X \to \operatorname{holim}_i \mathcal{X}_i$ as long as $\mathcal{X}_i$ is an fpqc stack (not necessarily algebraic). A topological example would be precisely the family of rigidified fppf stacks associated to the family of "truncated complex cobordisms" above, in 1; the Hopf algebroids which constitute the usual suspects in homotopy theory all stackify to ho-pro-rigidified fpqc stacks.

We hope that there is some analogue of Neron-Popescu desingularization in the world of flat stacks, which would take the form of an equivalence between our "quasi-stacks" and our "ho-pro-rigidified stacks," possibly with additional hypotheses; see [Swan, 1998] for this result for affine schemes, and [Maltenfort, 2002] for useful discussion of the various smoothness criteria (for schemes, not stacks).

2. **Non-affine base schemes.** The usual procedure ([Hopkins, 1999],[Pribble, 2004]) for producing a Hopf algebroid from an fpqc S-stack $\mathcal{X}$ equipped with an fpqc cover by an affine X is to form the groupoid S-scheme with objects X and $X \times_{\mathcal{X}} X$, and the associated structure maps, and to apply the global sections functor throughout. When the diagonal map $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is affine, this produces a Hopf algebroid over the commutative ring $\Gamma(S)$, and when S is an affine scheme, we can reverse the process.

However, if we are given an fpqc stack $\mathcal{X}$ over a base scheme S which is not affine—perhaps one which lives naturally over a projective base, for instance, a moduli object of branched covers of $\mathbb{P}^n$ for some n—then we may lose something in the process. Suppose we are given a cover $X \xrightarrow{P} \mathcal{X}$ which is an affine morphism and with X an affine scheme. Then when we form the cogroupoid object $(\Gamma(X), \Gamma(X \times_{\mathcal{X}} X))$, it is a cogroupoid object in commutative $\Gamma(S)$-algebras. If S is, for instance, projective space over some base scheme, its global sections carry very little of its geometric data, and in fact we cannot recover $\mathcal{X} \to S$ from $(\Gamma(X), \Gamma(X \times_{\mathcal{X}} X))$, even up to 2-categorical equivalence; instead we are only able to recover $\mathcal{X} \to \operatorname{Spec} \Gamma(S)$, again up to 2-categorical equivalence. So Motto 1.1.1 has hitherto been only true over affine schemes.

The way to remedy this problem is to re-define Hopf algebroids $(A, \mathbb{\Gamma})$: instead of letting A and $\mathbb{\Gamma}$ be commutative R-algebras with R some commutative ring, we instead let A and $\mathbb{\Gamma}$ be quasicoherent $\mathcal{O}_S$-algebras, where $\mathcal{O}_S$ is the structure ring sheaf of the local Zariski site on S. When the stack $\mathcal{X}$ has affine diagonal, then every fpqc cover $X \to \mathcal{X}$ with X an affine S-scheme is in fact an affine morphism, and as a result when we push them forward along their S-scheme structure maps, we get quasicoherent commutative $\mathcal{O}_S$-algebras, and from these algebras, the original affine S-schemes (and the structure maps between them) can recovered, by the classical bijection between affine S-schemes and quasicoherent $\mathcal{O}_S$-algebras.

So we have the task of rephrasing the usual methods for handling Hopf algebroids so that they work with $\mathcal{O}_S$-algebras. This is not difficult and we accomplish the most basic parts of this task in this thesis.

When $\mathcal{X}$ does not have affine diagonal, then $X \times_{\mathcal{X}} X$ is not affine over X, and this method does not work, as a pushforward of a scheme's structure sheaf along a non-affine morphism does not contain enough data to reconstruct the original scheme.

3. **Stacks with non-affine diagonal.** When $\mathcal{X}$ is an fpqc stack with non-affine diagonal then even the above method for handling stacks over non-affine base schemes is insufficient. The algebraic S-space $X \times_{\mathcal{X}} X$ is not necessarily even a scheme, much less an affine scheme; standard definitions

in algebraic geometry (e.g. [Laumon and Moret-Bailly, 2000]) require only that $X \times_{\mathcal{X}} X$ be a quasicompact, separated algebraic S-space. In this case the cohomology of the naive Hopf algebroid $(\Gamma(X), \Gamma(X \times_{\mathcal{X}} X))$ is the zero-line of the E_2-term of the Cech-to-derived-functor-cohomology spectral sequence converging to $H^*_{\mathrm{fpqc}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, i.e., the spectral sequence we get by taking the derived functors of global sections on the Cech nerve

$$\begin{tikzcd} X & X \times_{\mathcal{X}} X \arrow[l, shift left] \arrow[l, shift right] & X \times_{\mathcal{X}} X \times_{\mathcal{X}} X \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & \cdots . \end{tikzcd}$$

In this case we do not have enough information from the above Zariski $\mathcal{O}_S$-algebra definition of a Hopf algebroid to produce objects out of such stacks which the original stacks can be reconstructed from. In this case, what suffices is to extend the definition of a Hopf algebroid (A, Γ) once more, so that A and Γ are fpqc $\mathcal{O}_S$-algebras, i.e., algebras over the structure ring sheaf $\mathcal{O}_S$ of the local fpqc site on S. In this case taking the pushforward of the structure sheaf of $X \times_{\mathcal{X}} X$ along its map down to S does not lose any information, since $X \times_{\mathcal{X}} X$ is flat over the scheme X affine (by assumption) over S. One can then reconstruct the original S-stack from this kind of Hopf algebroid: a cogroupoid object in algebras over the structure ring sheaf of the local fpqc site on S. We have not laid out the details for this approach in this thesis, but we will lay out the details when this material is prepared for publication.

We note that it is not farfetched for a stack to have non-affine diagonal; indeed, such stacks solve some naturally-occuring moduli problems and the restriction (as in [Goerss, 2008],[Hopkins, 1999],[Pribble, 2004]) to stacks with affine diagonal is a significant restriction. Hilbert stacks of non-quasiseparated schemes do not typically have affine diagonal (see [Rydh, 2008]) and Vakil, in [Vakil, 2006], has produced a kind of "Murphy's Law" indicating that any kind of singularity, including those which spoil such properties as having affine diagonal, appear in a whole range of very natural moduli problems.

4. **Non-quasicoherent sheaves on a stack.** Part of Motto 1.1.1 is the idea that comodules over a Hopf algebroid are equivalent to quasicoherent modules over the structure sheaf of its associated stack. After suitable hypothesis are made, this is indeed true; but one sometimes needs to handle non-quasicoherent sheaves on a stack, e.g. the pushforward of $\mathcal{O}_{X\times_{\mathcal{X}}X}$ to $\mathcal{X}$ when $\mathcal{X}$ fails to have affine diagonal. For this, one needs to consider non-quasicoherent algebras on the structure sheaf $\mathcal{O}_X$ of the local fpqc site on X, equipped with descent data, i.e., comodule structure maps satisfying a certain cocycle condition (a stack-theoretic generalization of descent by Beck's theorem, as in [Bénabou and Roubaud, 1970]).

5. **Abelian sheaves on a stack.** To construct ℓ-adic cohomology it is not sufficient to consider modules and algebras over the structure sheaf of a scheme or stack, no matter how large and data-rich of a site we choose; working over $\operatorname{Spec} \mathbb{F}_p$, that structure sheaf will always have rings of characteristic p for its sections. One must consider sheaves of abelian groups on the local fpqc site of a stack $\mathcal{X}$ in order to have constructible torsion sheaves of order prime to p, e.g. the constant sheaves $\mathbb{Z}/\ell^n\mathbb{Z}$ on $\mathcal{X}$ with ℓ a prime, $\ell \neq p$, which gives us ℓ-adic stack cohomology

$$H^*(\mathcal{X}, \mathbb{Q}_\ell) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \lim_{n\to\infty} H^*(\mathcal{X}_{\mathrm{fpqc}}; \mathbb{Z}/\ell^n\mathbb{Z}).$$

   Such a construction is important, as we can see by e.g. Behrend's success in [Behrend, 2003] in proving a Lefschetz fixed-point formula for (the related case of) Deligne-Mumford stacks; ℓ-adic stack cohomology makes possible some potentially powerful point-counting methods and a link to ζ-functions.

   At this point I think that Motto 1.1.1 breaks entirely; there does not seem to be any way to define the constant sheaves $\mathbb{Z}/\ell^n\mathbb{Z}$ as comodules, even

> "sheafy" comodules, on a Hopf algebroid related to $\mathcal{X}$. I think this is a sign that, while we may translate much of the language of algebraic geometry into the language of stable homotopy and chromatic theory, ultimately the theoretical framework of algebraic geometry is a richer world than that of Hopf algebroids, and the Grothendieckified chromatic computational apparatus will connect most fruitfully to the rest of mathematics as well as answer its own questions if it is developed in the language of algebraic geometry.

So, what do you do with a flat stack? The answer is that you compute its cohomology. The chromatic theory used to compute the Adams-Novikov E_2-term, i.e., the flat cohomology of the moduli stack of formal groups, admits a broad and powerful generalization, whose primary aim is to reduce the computation of the cohomology of sheaves on an fpqc stack to the computation of the cohomology of an associated G_p-module for sufficiently many geometric points p of the stack, where G_p is the automorphism group of p, and then we get some immediate tools for computing the cohomology of that G_p-module. In other words, chromatic theory reduces the computation of flat stack cohomology to the computation of group cohomology. This process is fruitfully described using (suitable "sheafy" generalizations of) Hopf algebroids, but the language which is most suited for the future of the subject is the language of algebraic geometry; so one may translate the stack into Hopf-algebroid-theoretic terms and apply chromatic methods to compute its fpqc cohomology, or one may use our imported, generalized chromatic theory in the world of stacks, without translating into Hopf algebroids.

2 Moduli problems taking values in groupoids.

2.1 Commutative Hopf algebroids.

This section is intended as a supplement to, but not a replacement for, Ravenel's Appendix 1. Much of the material in Ravenel's Appendix 1 is not reproduced here. We will frequently use the following notation, which is standard in this subject: given a commutative ring R and two R-bimodules M, N, i.e., abelian groups M, N each equipped with both a left and a right action of R, we will write $M \otimes_R N$ for the tensor product of M *as a right R-module* with N *as a left R-module*. This means that $M \otimes_R N$ is not necessarily isomorphic to $N \otimes_R M$.

We recall that, "classically," given a commutative ring R, a *commutative Hopf algebroid* over R is a cogroupoid object in the category of commutative R-algebras. associate to X two functors $X_{\mathrm{ob}}, X_{\mathrm{mor}} : \mathbf{CAlg}(R) \rightarrow \mathbf{Sets}$, the first given by associating to an algebra B the set of objects of $X(B)$, and the second given by associating to an algebra B the set of morphisms of $X(B)$. There are certain morphisms between X_{ob} and X_{mor}, to which we will give the rather loaded names

$\epsilon^*, \eta_L^*, \eta_R^*, \Delta^*$, and χ^*; namely,

$$\begin{array}{rcll} X_{\mathrm{ob}} & \xrightarrow{\epsilon^*} X_{\mathrm{mor}} & \text{sends an object of } X(B) \text{ to its identity morphism,} \\ X_{\mathrm{mor}} & \xrightarrow{\eta_L^*} X_{\mathrm{ob}} & \text{sends a morphism of } X(B) \text{ to its source object,} \\ X_{\mathrm{mor}} & \xrightarrow{\eta_R^*} X_{\mathrm{ob}} & \text{sends a morphism of } X(B) \text{ to its target object,} \\ X_{\mathrm{mor}} \times_{X_{\mathrm{ob}}} X_{\mathrm{mor}} & \xrightarrow{\Delta^*} X_{\mathrm{mor}} & \text{sends composable pair of morphisms in } X(B) \text{ to their composite,} \\ X_{\mathrm{mor}} & \xrightarrow{\chi^*} X_{\mathrm{mor}} & \text{sends a morphism in } X(B) \text{ to its inverse.)} \end{array}$$

These structure maps satisfy certain axioms, such as $\chi \circ \chi = 1_{X_{\mathrm{mor}}}$. We say that X is *co-represented* by a pair of commutative R-algebras (A, Γ) if $\hom_{\mathbf{CAlg}(R)}(A, -)$ is naturally isomorphic to the functor X_{ob}, $\hom_{\mathbf{CAlg}(R)}(\Gamma, -)$ is naturally isomorphic to the functor X_{mor}, and there are R-algebra morphisms

$$\begin{array}{ccc} \Gamma & \xrightarrow{\epsilon} & A, \\ A & \xrightarrow{\eta_L} & \Gamma, \\ A & \xrightarrow{\eta_R} & \Gamma, \\ \Gamma & \xrightarrow{\Delta} & \Gamma \otimes_A \Gamma, \\ \Gamma & \xrightarrow{\chi} & \Gamma \end{array}$$

inducing the maps $\epsilon^*, \eta_L^*, \eta_R^*, \Delta^*$, and χ^* between the functors $\hom_{\mathbf{CAlg}(R)}(A, -)$ and $\hom_{\mathbf{CAlg}(R)}(\Gamma, -)$. So a commutative Hopf algebroid over R will be (a certain algebro-geometric generalization of) a pair of commutative R-algebras (A, Γ), together with structure maps $\epsilon, \eta_L, \eta_R, \Delta, \chi$, which, taken all together, co-represents some functor X from commutative R-algebras to groupoids. Familiar examples of commutative Hopf algebroids over R include:

1. commutative Hopf algebras over R, which are the *cogroup* objects in commutative R-algebras, i.e., pairs (A, Γ) which represent a groupoid-valued functor X on commutative R-algebras, such that $X(B)$ always has only one object; since this means that $\hom_{\mathbf{CAlg}(R)}(A, B)$ always has only one object,

it follows that A must be the initial object in commutative R-algebras, i.e., R itself. It also necessarily follows that $\eta_L = \eta_R$, since every morphism in a group (viewed as a category) has the same source and target.

2. any commutative R-algebra A can be made into a commutative Hopf algebroid by making all its structure maps the identity map.

3. pairs $(\pi_0(E), \pi_0(E \wedge E))$, where E is an even periodic ring spectrum: we get ϵ by applying π_0 to the ring spectrum multiplication map $E \wedge E \to E$; we get η_L and η_R by applying π_0 to the maps $E \simeq E \wedge S \xrightarrow{1_E \wedge \eta} E \wedge E$ and $E \simeq S \wedge E \xrightarrow{\eta \wedge 1_E} E \wedge E$, respectively, where η is the ring spectrum unit map $S \xrightarrow{\eta} E$; we get Δ by applying π_0 to the composite $E \wedge_S E \simeq E \wedge S \wedge E \to E \wedge E \wedge E \simeq (E \wedge E) \wedge_E (E \wedge E)$; and we get χ by applying π_0 to the factor swap map $E \wedge E \to E \wedge E$.

4. some algebraic stacks will give rise to commutative Hopf algebroids, as we will see in Prop. **??**.

The example of the commutative Hopf algebroid associated to an even periodic ring spectrum will be an important one for the purposes of this paper; in particular, we will generally be dealing with *graded* commutative Hopf algebroids, like those arising from ring spectra.

Remark 2.1.1. So far we have discussed commutative Hopf algebroids as they have usually been used by stable homotopy theorists, as a diagram of commutative R-algebras for some commutative ring R. However, we are going to show that commutative Hopf algebroids over R are equivalent to certain kinds of stacks over $\operatorname{Spec} R$. Stacks over non-affine base schemes are sometimes studied by algebraists and as such we want to extend the old definition of a Hopf algebroid, so that we can consider Hopf algebroids over an arbitrary base scheme S, and we want to recover the classical case of a commutative Hopf algebroid over a commutative ring R as (in our new language) a commutative Hopf algebroid over $\operatorname{Spec} R$. This is quite easy to accomplish, using Prop. B.1.13. The reader who is uncomfortable with commutative Hopf algebroids over arbitrary base schemes can simply always take S to be an affine scheme, in which case all our commutative Hopf algebroids are (in the language of e.g. [Ravenel, 1986]) Hopf algebroids over $\Gamma(\mathcal{O}_S)$; one can then leave out the word "quasicoherent" from most definitions and theorems, as non-quasicoherent modules are a purely geometric phenomenon.

Now suppose we are given a base scheme S and a pair of commutative $\mathcal{O}_S$-algebras (A, Γ) and maps $\epsilon, \eta_L, \eta_R, \Delta, \chi$ between them. We want to decide whether the pair (A, Γ) corepresents a functor from commutative $\mathcal{O}_S$-algebras to groupoids.

Proposition 2.1.2. Structure maps of a co-SmCat object. *Let S be a scheme. Let (A, Γ) be a pair of quasicoherent commutative $\mathcal{O}_S$-algebras and let $\epsilon, \eta_L, \eta_R, \Delta$ be $\mathcal{O}_S$-algebra maps with source and target as follows:*

$$\begin{aligned} \Gamma &\xrightarrow{\epsilon} A \\ A &\xrightarrow{\eta_L} \Gamma \\ A &\xrightarrow{\eta_R} \Gamma \\ \Gamma &\xrightarrow{\Delta} \Gamma \otimes_A \Gamma \end{aligned}$$

Then (A, Γ) corepresents a functor from commutative $\mathcal{O}_S$-algebras to small categories *if and only if all of the following conditions are met:*

1. *Δ is an A-bimodule morphism (This ensures that, on the level of the category that our functor associates to any $\mathcal{O}_S$-algebra, the composite $g \circ f$ has the same source as f and the same target as g). (We recall that Γ is a left A-module (resp. graded left A-module) via η_L and a right A-module (resp. graded right A-module) via η_R, hence an A-bimodule, and there is a canonical A-bimodule structure on $\Gamma \otimes_A \Gamma$, so we can ask whether $\Gamma \xrightarrow{\Delta} \Gamma \otimes_A \Gamma$ is an A-bimodule morphism (resp. graded A-bimodule morphism).) This is*

equivalent to requiring that $\Delta \circ \eta_L = \iota_1 \circ \eta_L$ *and* $\Delta \circ \eta_R = \iota_2 \circ \eta_R$*, where* $\iota_1 : \gamma \mapsto \gamma \otimes 1$ *and* $\iota_2 : \gamma \mapsto 1 \otimes \gamma$ *are the canonical (pushout) morphisms* $\Gamma \to \Gamma \otimes_A \Gamma$*.*

2. ϵ *is an A-bimodule morphism (equivalently,* $\epsilon \circ \eta_L = \epsilon \circ \eta_R = \mathrm{id}_A$*) (the identity morphism on an object has that object for its source and target).*

3. $(\mathrm{id}_\Gamma \otimes_A \epsilon) \circ \Delta = (\epsilon \otimes_A \mathrm{id}_\Gamma) \circ \Delta = \mathrm{id}_\Gamma$ *(composing a morphism f with an identity morphism on either the left or the right yields f again),*

4. $(\mathrm{id}_\Gamma \otimes_A \Delta) \circ \Delta = (\Delta \otimes_A \mathrm{id}_\Gamma) \circ \Delta$ *(composition of morphisms is associative).*

Proof. Given (A, Γ) and the maps $\epsilon, \eta_L, \eta_R, \Delta$ satisfying 2, 3, and 4, for any commutative $\mathcal{O}_S$-algebra B we want to construct a small category. We do this as follows: for any "objects" $f, g \in \hom_{\mathbf{CAlg}(\mathcal{O}_S)}(A, B)$, we take the intersection

$$(\eta_L^*)^{-1}(f) \cap (\eta_R^*)^{-1}(g)$$

$$= \{f\} \times_{\hom_{\mathbf{CAlg}(\mathcal{O}_S)}(A,B)} \hom_{\mathbf{CAlg}(\mathcal{O}_S)}(\Gamma, B) \times_{\hom_{\mathbf{CAlg}(\mathcal{O}_S)}(A,B)} \{g\}$$

$$\subseteq \hom_{\mathbf{CAlg}(\mathcal{O}_S)}(\Gamma, B)$$

to get the set of "morphisms" from f to g. Any "morphism" in $\hom_{\mathbf{CAlg}(\mathcal{O}_S)}(\Gamma, B)$ has a unique source and target, since η_L^* and η_R^* are well-defined maps of sets, so our morphism sets are disjoint. Given objects $f, g, h \in \hom_{\mathbf{CAlg}(\mathcal{O}_S)}(A, B)$ and morphisms $\psi \in (\eta_L^*)^{-1}(f) \cap (\eta_R^*)^{-1}(g), \phi \in (\eta_L^*)^{-1}(g) \cap (\eta_R^*)^{-1}(h)$, knowing that $\eta_R^*(\psi) = \eta_L^*(\phi)$ gives us a unique lift of

$$(\phi, \psi) \in \hom_{\mathbf{CAlg}(\mathcal{O}_S)}(\Gamma, B) \times_{\hom_{\mathbf{CAlg}(\mathcal{O}_S)}} (\Gamma, B)$$

to

$$\begin{aligned} (\phi, \psi) &\in \hom_{\mathbf{CAlg}(\mathcal{O}_S)}(\Gamma, B) \times_{\hom_{\mathbf{CAlg}(\mathcal{O}_S)}(A,B)} \hom_{\mathbf{CAlg}(\mathcal{O}_S)}(\Gamma, B) \\ &\cong \hom_{\mathbf{CAlg}(\mathcal{O}_S)}(\Gamma \otimes_A \Gamma, B). \end{aligned}$$

We define the law of composition on our small category by letting $\phi \circ \psi$ be $\Delta^*(\phi, \psi) \in \hom_{\mathbf{CAlg}(\mathcal{O}_S)}(\Gamma, B)$. Axiom 1 guarantees that the source of $\phi \circ \psi$ will be the source of ψ and the target of $\phi \circ \psi$ will be the target of ϕ, and axiom 4 guarantees that this composition law is associative. Axioms 2 and 3 give the existence of an identity morphism for each object. We conclude that the set of objects, the set of morphisms, and the composition law given in this way is actually a small category.

Now let X be a functor $\mathbf{CAlg}(\mathcal{O}_S) \to \mathbf{SmCat}$ which is corepresented by (A, Γ) and the maps $\epsilon, \eta_L, \eta_R, \Delta$. Then it is immediate that (A, Γ) satisfies the four conditions, since $X(B)$ is a small category for any commutative $\mathcal{O}_S$-algebra B. □

We will have need to refer to these objects, "co-small-category objects in commutative $\mathcal{O}_S$-algebras," so we should name them (an already existing name in the non-graded case would be "internal category in schemes affine over S" (see [Leinster, 2004]), but we prefer something more concise and also more accessible to those not versed in category theory). Since a bialgebra is a name for "a Hopf algebra which is missing a conjugation map," and our objects which need a name will be "Hopf algebroids which are missing conjugation maps," a name suggests itself:

Definition 2.1.3. *A pair of commutative (resp. graded commutative) quasicoherent Zariski $\mathcal{O}_S$-algebras (A, Γ) and $\mathcal{O}_S$-algebra morphisms (resp. graded $\mathcal{O}_S$-algebra morphisms) $\epsilon, \eta_L, \eta_R, \Delta, \chi$ between them satisfying any of the equivalent conditions of Prop. 2.1.2 is called a* commutative bialgebroid *over S (resp.* graded commutative bialgebroid *over S). When η_L and η_R are both flat morphisms we say that (A, Γ) is a* flat commutative bialgebroid.

(By a Zariski $\mathcal{O}_S$-algebra we mean a sheaf of algebras over the structure sheaf $\mathcal{O}_S$ of the local Zariski *site on S, i.e., a $\mathcal{O}_S$ in the sense of (INSERT REF. TO*

HARTSHORNE'S ALG GEOM).)

We are careful to say *commutative* bialgebroid and *commutative* Hopf algebroid, although it is the custom in algebraic topology to leave off the word "commutative," because there is a such thing as a non-commutative bialgebroid and a non-commutative Hopf algebroid outside of algebraic topology, e.g. in the study of quantum groups.

Proposition 2.1.4. Structure maps of a co-Groupoids object. *Let* (A, Γ) *be a pair of commutative (resp. graded commutative)* $\mathcal{O}_S$*-algebras and let* $\epsilon, \eta_L, \eta_R, \Delta, \chi$ *be* $\mathcal{O}_S$*-algebra morphisms (resp. graded* $\mathcal{O}_S$*-algebra morphisms) with source and target as specified above. Then the following conditions are equivalent:*

1. (A, Γ) *corepresents a functor from commutative (resp. graded commutative)* $\mathcal{O}_S$*-algebras to groupoids.*

2. *Each of the following three conditions holds:*

   (a) (A, Γ) *and the maps* $\epsilon, \eta_L, \eta_R, \Delta$ *represent a functor from commutative (resp. graded commutative)* $\mathcal{O}_S$*-algebras to small categories, as in Prop. 2.1.2. (That is,* (A, Γ) *is a commutative bialgebroid (resp. graded commutative bialgebroid) over* S*.)*

   (b) $\chi \circ \chi = 1_\Gamma$ *(the inverse of the inverse of a function is the function itself).*

   (c) **(Ravenel's extension criterion.)** *The maps* $\nabla \circ (\mathrm{id}_\Gamma \otimes_{\mathcal{O}_S} \chi), \nabla \circ (\chi \otimes_{\mathcal{O}_S} \mathrm{id}_\Gamma) : \Gamma \otimes_{\mathcal{O}_S} \Gamma \to \Gamma$ *have extensions* $\overline{\nabla \circ (\mathrm{id}_\Gamma \otimes_{\mathcal{O}_S} \chi)}$ *and* $\overline{\nabla \circ (\chi \otimes_{\mathcal{O}_S} \mathrm{id}_\Gamma)}$ *to* $\Gamma \otimes_A \Gamma$*, making the following diagram commute:*

$$\begin{tikzcd}
 & \Gamma \otimes_R \Gamma \arrow[dl, "\nabla\circ(\chi\otimes_{\mathcal{O}_S}\mathrm{id}_\Gamma)"'] \arrow[d] \arrow[dr, "\nabla\circ(\mathrm{id}_\Gamma\otimes_{\mathcal{O}_S}\chi)"] & \\
\Gamma & \Gamma \otimes_A \Gamma \arrow[l, "\overline{\nabla\circ(\chi\otimes_{\mathcal{O}_S}\mathrm{id}_\Gamma)}"'] \arrow[r, "\overline{\nabla\circ(\mathrm{id}_\Gamma\otimes_{\mathcal{O}_S}\chi)}"] & \Gamma \\
A \arrow[u, "\eta_R"] & \Gamma \arrow[l, "\epsilon"'] \arrow[r, "\epsilon"] \arrow[u, "\Delta"'] & A \arrow[u, "\eta_L"']
\end{tikzcd}$$

(any function is composable, on either side, with its inverse; the composition $f \circ f^{-1}$ yields the identity map on f's target; the composition $f^{-1} \circ f$ yields the identity map on f's source).

Proof. That the second two statements are equivalent follows immediately from the argument noted in **??** $(\overline{\Delta \circ (\mathrm{id}_{\Gamma} \otimes_{\mathcal{O}_S} \chi)}(\gamma_1 \eta_L(r) \otimes \gamma_2) = \overline{\Delta \circ (\mathrm{id}_{\Gamma} \otimes_{\mathcal{O}_S} \chi)}(\gamma_1 \otimes \eta_R(r)\gamma_2))$. That the first statement is equivalent to either of the other two follows immediately from the definition of a groupoid and from the fact that a morphism of groupoids is simply a functor of the underlying small categories. □

Definition 2.1.5. *A pair of commutative (resp. graded commutative) quasicoherent $\mathcal{O}_S$-algebras (A, Γ) with structure maps $\epsilon, \eta_L, \eta_R, \Delta, \chi$ satisfying any of these equivalent conditions is called a* commutative Hopf algebroid *over S (resp.* graded commutative Hopf algebroid *over S). When η_L is flat (equivalently, when η_R is flat) we say that (A, Γ) is a* flat commutative Hopf algebroid. *If $\mathcal{D}$ is a descent class in S-schemes and (A, Γ) is a commutative Hopf algebroid over S such that* $\mathrm{Spec}_S\, \eta_L$ *and* $\mathrm{Spec}_S\, \eta_R$ *are both in $\mathcal{D}$, we say that (A, Γ) is a* commutative Hopf algebroid of descent class $\mathcal{D}$. *(Note that, since A, Γ are quasicoherent, every commutative Hopf algebroid is automatically of affine descent class. Some commutative Hopf algebroids may be of even stronger descent class, however. This is in contrast to the cases of algebraic stacks or algebraic spaces, where there are important examples failing to be of affine descent class. When one says "commutative Hopf algebroid" without specifying the descent class, it is to be assumed that the affine descent class is meant.)*

Given a commutative ring R, the Hopf algebroids over R of [Ravenel, 1986] are, in our language, commutative Hopf algebroids of affine descent class over $\mathrm{Spec}\, R$.

One characterization of a graded commutative bialgebroid or Hopf algebroid is as a co-small-category object or cogroupoid object in graded commutative quasicoherent algebras over the structure sheaf of a scheme. We must be careful, though, because in practice our commutative bialgebroids and commutative Hopf algebroids will be graded, *but* we will sometimes want to consider them as functors on the *ungraded* category of commutative algebras. For example, the Lazard algebras (L, LB) have a certain grading on them, but it is in the *ungraded* category that they corepresent the functor assigning to any commutative ring R the set of formal group laws over R (the grading does carry algebraically significant information in this case, though, which we will explore later). For this reason we will always need to specify whether we are considering our graded commutative bialgebroids and graded commutative Hopf algebroids as functors on the graded or the ungraded commutative $\mathcal{O}_S$-algebras.

Definition 2.1.6. *Let (A, Γ) be a commutative bialgebroid over a scheme S. A* left Γ-comodule *(resp.* right Γ-comodule*) is a A-module B together with a morphism of A-modules $B \xrightarrow{\psi} \Gamma \otimes_A B$ (resp. $B \xrightarrow{\psi} B \otimes_A \Gamma$) such that the following diagrams commute:*

(unit condition)
$$\begin{tikzcd} B \arrow[r, "\psi"] \arrow[dr, "\mathrm{id}_B"'] & \Gamma \otimes_A B \arrow[d, "\epsilon \otimes_A \mathrm{id}_B"] \\ & A \otimes_A B \end{tikzcd} \quad (2.1.1) \qquad \left(\textit{resp.} \begin{tikzcd} B \arrow[r, "\psi"] \arrow[dr, "\mathrm{id}_B"'] & B \otimes_A \Gamma \arrow[d, "\mathrm{id}_B \otimes_A \epsilon"] \\ & B \otimes_A A \end{tikzcd} \right) \quad (2.1.2)$$

(associativity)
$$\begin{tikzcd} B \arrow[r, "\psi"] \arrow[d, "\psi"] & \Gamma \otimes_A B \arrow[d, "\Delta \otimes_A \mathrm{id}_B"] \\ \Gamma \otimes_A B \arrow[r, "\mathrm{id}_\Gamma \otimes_A \psi"] & \Gamma \otimes_A \Gamma \otimes_A B \end{tikzcd} \quad (2.1.3) \qquad \left(\textit{resp.} \begin{tikzcd} B \arrow[r, "\psi"] \arrow[d, "\psi"] & B \otimes_A \Gamma \arrow[d, "\mathrm{id}_B \otimes_A \Delta"] \\ B \otimes_A \Gamma \arrow[r, "\psi \otimes_A \mathrm{id}_\Gamma"] & B \otimes_A \Gamma \otimes_A \Gamma \end{tikzcd} \right) \quad (2.1.4)$$

If (A, Γ) is a graded bialgebroid, B is a graded commutative A-algebra, and ψ

is a morphism of graded commutative A-morphisms such that the above diagrams commute, we call B a graded left $\mathbb{\Gamma}$-comodule *(resp.* graded right $\mathbb{\Gamma}$-comodule*). When B is a commutative A-algebra such that ψ is a morphism of commutative A-algebras, we call B a* left $\mathbb{\Gamma}$-comodule algebra *(resp.* right $\mathbb{\Gamma}$-comodule algebra*). We say that B is* quasicoherent *as a comodule when it is quasicoherent as an A-algebra.*

Remark 2.1.7. When S is a scheme and $(A, \mathbb{\Gamma})$ is a commutative Hopf algebroid over S, a $\mathbb{\Gamma}$-comodule is *almost* just a sheaf of $\mathbb{\Gamma}$-comodules, in the sense of Ravenel, on S; but it is not quite the same, since to two Zariski opens U, V in S, we do not associate two comodules from the same category of comodules: to U we associate a $(\mathcal{O}_U \otimes_{\mathcal{O}_S} A, \mathcal{O}_U \otimes_{\mathcal{O}_S} \mathbb{\Gamma})$-comodule and to V we associate a $(\mathcal{O}_V \otimes_{\mathcal{O}_S} A, \mathcal{O}_V \otimes_{\mathcal{O}_S} \mathbb{\Gamma})$-comodule. The proper way to view a $\mathbb{\Gamma}$-comodule as a sheaf of Ravenel's comodules is using the formalism of $(\mathcal{C}, \mathfrak{A}, \tau)$-modules, as in Def. A.2.8; the category of left $\mathbb{\Gamma}$-comodules is the category of $(\mathcal{C}, \mathfrak{A}, \tau)$-modules, where $(\mathcal{C}, \tau)$ is the local Zariski site S_{Zar}, and $\mathfrak{A}(U)$ is the category of left $\mathcal{O}_U \otimes_{\mathcal{O}_S} \mathbb{\Gamma}$-comodules, i.e., left comodules over the commutative Hopf algebroid $(\mathcal{O}_U \otimes_{\mathcal{O}_S} \mathbb{\Gamma}, \mathcal{O}_U \otimes_{\mathcal{O}_S} \mathbb{\Gamma})$ over U.

2.1.1 Algebraic stacks.

A flat stack over a base scheme S will be, roughly, a sheaf of groupoids on the big fpqc site $(\mathbf{Aff}/S)_{\mathrm{fpqc}}$. We make this precise in the definitions below, but for the sake of generality—and in particular, the change-of-site theorems we will wish to use—we will allow other sites in our definitions.

Definition 2.1.8. *Let S be a scheme and let τ be a Grothendieck topology on $\mathbf{Aff}/S$.*

1. *An* S-groupoid *consists of a category $\mathcal{X}$ and a functor $a : \mathcal{X} \to \mathbf{Aff}/S$ satifying the following conditions:*

(a) for any morphism $V \xrightarrow{\phi} U$ *in* $\mathbf{Aff}/S$ *and any object* $x \in \mathrm{ob}\,\mathcal{X}$ *such that* $a(x) = U$*, there exists at least one morphism* $y \xrightarrow{f} x$ *in* $\mathcal{X}$ *such that* $a(f) = \phi$*; and*

(b) for any diagram

$$\begin{tikzcd} z \arrow[dr, "h"] & \\ & x \\ y \arrow[ur, "f"'] & \end{tikzcd}$$

in $\mathcal{X}$ *with image*

$$\begin{tikzcd} W \arrow[dr, "\chi"] & \\ & U \\ V \arrow[ur, "\phi"'] & \end{tikzcd}$$

in $\mathbf{Aff}/S$*, and any morphism* $W \xrightarrow{\psi} V$ *in* $\mathbf{Aff}/S$ *such that* $\chi = \phi \circ \psi$*, there exists exactly one morphism* $z \xrightarrow{g} y$ *such that* $h = f \circ g$ *and with* $a(g) = \psi$*.*

For any $U \in \mathrm{ob}\,\mathbf{Aff}/S$ *we will write* $\mathcal{X}_U$ *for the fiber category in* $\mathcal{X}$ *over* U*, i.e., the objects of* $\mathcal{X}$ *which are mapped to* U *by* a*, and the morphisms of* $\mathcal{X}$ *which are mapped to* id_U *by* a*; so* $\mathcal{X}_U$ *is a groupoid.*

2. *Let* $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ *and* $\mathcal{X}' \xrightarrow{f'} \mathcal{Y}$ *be morphisms of* S*-groupoids. Then the* 2-categorical pullback $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}'$ *is the* S*-groupoid with objects*

$$\mathrm{ob}\,(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}')_U = \{(x, x', g) : x \in \mathrm{ob}\,\mathcal{X}_U, x' \in \mathrm{ob}\,\mathcal{X}'_U, g \in \hom_{\mathcal{Y}_U}(f(x), f'(x'))\}$$

and, for $(x_1, x'_1, g_1), (x_2, x'_2, g_2) \in \mathrm{ob}\,\mathcal{X}_U$*, morphisms given by*

$$\hom_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}')_U}((x_1, x'_1, g_1), (x_2, x'_2, g_2)) = \{(x_1 \xrightarrow{h} x_2, x'_1 \xrightarrow{h'} x'_2) : g_2 \circ f(h) = f'(h') \circ g_1\}$$

with composition defined in the obvious way. In general when we write a pullback of S*-groupoids we will mean this (2-categorical) pullback.*

3. *A morphism of S-groupoids $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ is said to be* representable (in the τ topology, of descent class $\mathcal{D}$) *(resp.* schematic, affine*) if, for any affine scheme $U \to \mathcal{Y}$, the pullback $U \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic S-space in the τ topology of descent class $\mathcal{D}$ (resp. an S-scheme, an affine S-scheme).*

4. *If P is a stable class of morphisms in $\mathbf{Sch}/S$, then we say that a schematic morphism of S-groupoids $\mathcal{X} \to \mathcal{Y}$* is in P *if, for any affine S-scheme $U \to \mathcal{Y}$ over $\mathcal{Y}$, the pullback morphism $U \times_{\mathcal{Y}} \mathcal{X} \to U$ is in P. For instance, at times we will let P be "separated," "quasiseparated," and "quasicompact," among other examples.*

5. *A* prestack over S in the τ-topology, *or* S-prestack in the τ topology, *is an S-groupoid satisfying the additional condition:*

   (a) *for any $U \in \mathrm{ob}\ \mathbf{Aff}/S$ and any $x, y \in \mathrm{ob}\ \mathcal{X}_U$, the presheaf*

$$\begin{aligned} \mathrm{Isom}(x, y) : \mathbf{Aff}/U &\to \mathbf{Sets} \\ (V \to U) &\mapsto \hom_{\mathcal{X}_V}(x_V, y_V) \end{aligned}$$

   is a sheaf on the site $\tau(\mathbf{Aff}/S)$.

6. *A* stack over S in the τ-topology, *or* S-stack in the τ-topology, *is an S-prestack in the τ-topology satisfying the additional condition:*

   (a) *for any covering family $(V_i \xrightarrow{\phi_i} U)_{i \in I}$ in τ, "every descent datum is effective," i.e., given any* descent datum *for $(V_i \xrightarrow{\phi_i} U)_{i \in I}$, which is a collection of objects $x_i \in \mathrm{ob}\ \mathcal{X}_{V_i}$ and isomorphisms $f_{ji} : (x_i | V_j \times_U V_i) \longrightarrow (x_j | V_j \times_U V_i)$ satisfying the "cocycle condition"*

$$(f_{ki} | V_k \times_U V_j \times_U V_i) = (f_{kj} | V_k \times_U V_j \times_U V_i) \circ (f_{ji} | V_k \times_U V_j \times_U V_i)$$

in $\mathcal{X}_{V_k \times_U V_j \times_U V_i}$ *for all* $i, j, k \in I$*, the descent datum is "effective," meaning that there exists an object* $x \in \mathrm{ob}\ \mathcal{X}_U$ *and isomorphisms* $f_i : (x|V_i) \longrightarrow x_i$ *in* $\mathcal{X}_{V_i}$ *for all* $i \in I$*, such that*

$$(f_j|V_j \times_U V_i) = f_{ji} \circ (f_i|V_j \times_U V_i)$$

for all $i, j \in I$*.*

Proposition 2.1.9. *1. Let* S *be a base scheme and let* $(\mathbf{Aff}\,/S)^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathbf{SmCat}$ *be a small-category-valued presheaf on affines over* S*. Then the following construction yields a category* $\underline{\mathcal{F}}$ *with a projection map* $\underline{\mathcal{F}}^{\mathrm{op}} \xrightarrow{a} \mathbf{Aff}\,/S$*: we let* $\mathrm{ob}\ \underline{\mathcal{F}}$ *be the disjoint union*

$$\mathrm{ob}\ \underline{\mathcal{F}} = \coprod_{U \in \mathrm{ob}\ \mathbf{Aff}\,/S} \mathrm{ob}\ \mathcal{F}(U)$$

and, given $x \in \mathcal{F}(U), x' \in \mathcal{F}(U')$ *for some* $U, U' \in \mathrm{ob}\ \mathbf{Aff}\,/S$*, we define the morphism set by*

$$\hom_{\underline{\mathcal{F}}}(x, x') = \left\{ (f_1, f_2) : U' \xrightarrow{f_1} U, f_2 \in \hom_{\mathcal{F}(U')}((\mathcal{F}(f_1))(x), x') \right\},$$

and finally, given $(f_1, f_2) \in \hom_{\underline{\mathcal{F}}}(x, x')$ *and* $(g_1, g_2) \in \hom_{\underline{\mathcal{F}}}(x', x'')$*, we define composition in* $\underline{\mathcal{F}}$ *by*

$$(g_1, g_2) \circ (f_1, f_2) = (f_1 \circ g_1, g_2 \circ (\mathcal{F}(g_1))(f_2)).$$

The projection map a *is given by sending* $x \in \mathcal{F}(U)$ *to* $a(x) = U$ *and by sending a morphism* $x \xrightarrow{(f_1,f_2)} x'$ *to* $a((f_1, f_2)) = f_1$*.*

2. *If* $\mathcal{F}$ *takes values in* **Groupoids** $\subseteq$ **SmCat***, then* $\underline{\mathcal{F}}^{\mathrm{op}} \xrightarrow{a} \mathbf{Aff}\,/S$ *is an* S*-groupoid.*

3. *Let* S *be a base scheme and let* $(\mathbf{Aff}\,/S)^{\mathrm{op}} \xrightarrow{\mathcal{F}}$ **Groupoids** *be a groupoid-valued presheaf on affines over* S*. We form the simplicial-set-valued presheaf* $(\mathbf{Aff}\,/S)^{\mathrm{op}} \xrightarrow{\mathcal{N}_\cdot \circ \mathcal{F}}$ **SSet** *and then project to the first and second sets to get*

two set-valued presheaves $(\mathbf{Aff}/S)^{\mathrm{op}} \xrightarrow{\mathcal{N}_1\circ\mathcal{F}} \mathbf{Sets}$ *and* $(\mathbf{Aff}/S)^{\mathrm{op}} \xrightarrow{\mathcal{N}_2\circ\mathcal{F}} \mathbf{Sets}$ *on affines over* S*. If* $\mathcal{N}_1\circ\mathcal{F}$ *and* $\mathcal{N}_2\circ\mathcal{F}$ *are sheaves in the topology* τ*, then* $\underline{\mathcal{F}}$ *is an* S*-prestack in the topology* τ*.*

Proof. 1. This is a tedious verification which we leave out.

2. Let $V \xrightarrow{\phi} U$ be a morphism in $\mathbf{Aff}/S$ and let $x \in \underline{\mathcal{F}}$ such that $a(x) = U$; then let $y = (\mathcal{F}(\phi))(x)$, and we have the morphism $y \xrightarrow{(\phi,\mathrm{id}_y)} x$.

   Now let $z \xrightarrow{h} x$ and $y \xrightarrow{f} x$ be morphisms in $\underline{\mathcal{F}}$ and let $a(z) \xrightarrow{\psi} a(y)$ be a morphism in $\mathbf{Aff}/S$ such that $a(h) = a(f)\circ\psi$; then $f = (f_1, f_2)$ and $h = (h_1, h_2)$ for some S-scheme morphisms f_1, h_1 and some morphisms f_2, h_2 in $\underline{\mathcal{F}}_{a(y)}$ and $\underline{\mathcal{F}}_{a(z)}$, respectively. Then the only morphism $z \xrightarrow{g} y$ satisfying $a(g) = \psi$ and $h = f\circ g$ is $g = (\psi, h_2\circ((\mathcal{F})(\psi))(f_2)^{-1})$.

3. Given $x, y \in \mathcal{F}(U)$, the isometry sheaf is given by its value on $V \xrightarrow{f} U$:

$$
\begin{aligned}
& \mathrm{Isom}_{\mathcal{F}(V)}(\mathcal{F}(f)(x), \mathcal{F}(f)(y)) \\
= \; & \{(\mathcal{N}_0\circ\mathcal{F})(f)(x)\} \times_{\mathcal{N}_1\circ\mathcal{F}(V)} \mathcal{N}_2\circ\mathcal{F}(V) \times_{\mathcal{N}_1\circ\mathcal{F}(V)} \{(\mathcal{N}_0\circ\mathcal{F})(f)(y)\} \\
= \; & s_0^{-1}(\mathcal{F}(f)(x)) \cap s_1^{-1}(\mathcal{F}(f)(y)) \\
\subseteq \; & \mathcal{N}_2\circ\mathcal{F}(V),
\end{aligned}
$$

   a pullback of sheaves in the τ topology, so it is itself a sheaf in the τ topology.

□

Proposition 2.1.10. *Let* S *be a scheme and let* X *be a scheme affine over* S*, and let* $\mathcal{F}$ *be the presheaf of sets on* $\mathbf{Aff}/S$ *represented by* X*. Then* $\underline{\mathcal{F}}$ *is an* S*-stack in the fpqc topology on* $\mathbf{Aff}/S$*.*

Proof. Since there are no non-identity morphisms in $\underline{\mathcal{F}}_U$ for any $U \in \mathrm{ob}\,(\mathbf{Aff}/S)$, a descent datum for $\mathcal{F}$ and a morphism $U \xrightarrow{f} V$ of affines over S consists of a map $x \in \hom_{\mathbf{Aff}/S}(U, X)$ such that $x\circ\pi_1 = x\circ\pi_2$ as maps $U\times_V U \to X$, where

π_1 and π_2 are the two projection maps $U \times_V U \to U$. We show that x descends to $\mathcal{F}(V)$ by covering S by affines, showing that x is effective on each, and then gluing. Let T be an affine open subscheme of S. Since $x \circ \pi_1 = x \circ \pi_2$ we have the following commutative diagram of commutative rings:

$$\begin{tikzcd}
\Gamma(T, \mathcal{A}(U \times_V U)) & \Gamma(T, \mathcal{A}(U)) \arrow[l, "\Gamma(T,\pi_2)"'] & \\
\Gamma(T, \mathcal{A}(U)) \arrow[u, "\Gamma(T,\pi_1)"'] & \Gamma(T, \mathcal{A}(V)) \arrow[l, "\Gamma(T,\mathcal{A}(f))"'] \arrow[u, "\Gamma(T,\mathcal{A}(f))"'] & \\
& & \Gamma(T, \mathcal{A}(X)) \arrow[ull, "\Gamma(T,\mathcal{A}(x))"] \arrow[uul, "\Gamma(T,\mathcal{A}(x))"']
\end{tikzcd}$$

and we want to produce a map $\Gamma(T, \mathcal{A}(X)) \to \Gamma(T, \mathcal{A}(V))$ making the diagram commute. We have an exact sequence of $\Gamma(T, \mathcal{A}(V))$-modules

$$\begin{tikzcd}
\Gamma(T, \mathcal{A}(V)) \otimes_{\Gamma(T,\mathcal{A}(V))} \Gamma(T, \mathcal{A}(U)) \arrow[r, "\Gamma(T,\mathcal{A}(f))\otimes_{\Gamma(T,\mathcal{A}(V))}\Gamma(T,\mathcal{A}(U))"] & \Gamma(T, \mathcal{A}(U)) \otimes_{\Gamma(T,\mathcal{A}(V))} \Gamma(T, \mathcal{A}(U)) \arrow[dl, "\delta"] \\
\operatorname{coker} \Gamma(T, \mathcal{A}(f)) \otimes_{\Gamma(T,\mathcal{A}(V))} \Gamma(T, \mathcal{A}(U)) \arrow[r] & 0
\end{tikzcd}$$

and $u \otimes 1 = 1 \otimes u$ in $\Gamma(T, \mathcal{A}(U)) \otimes_{\Gamma(T,\mathcal{A}(V))} \Gamma(T, \mathcal{A}(U)) \cong \Gamma(T, \mathcal{A}(U \times_V U))$ only if $\delta(u \otimes 1 - 1 \otimes u) = \delta(u \otimes 1) = 0$ in $\operatorname{coker} \Gamma(T, f) \otimes_{\Gamma(T,\mathcal{A}(V))} \Gamma(T, \mathcal{A}(U))$, i.e., only if u is in V. So the map $\Gamma(T, \mathcal{A}(X)) \overset{\Gamma(T,\mathcal{A}(x))}{\longrightarrow} \Gamma(T, \mathcal{A}(U))$ factors through $\Gamma(T, \mathcal{A}(V))$, and the descent datum given by x is effective.

So we have morphisms $\Gamma(T, \mathcal{A}(X)) \to \Gamma(T, \mathcal{A}(V))$ for each affine open subscheme T of S; it is easy to check that their intersections agree and so they glue properly. $\square$

Corollary 2.1.11. *Let S be a scheme and let $X \to S$ be affine over S, and let $\mathcal{F}$ be the presheaf of sets on* $\mathbf{Aff}/S$ *represented by X. Then $\underline{\mathcal{F}}$ is an S-stack in any topology τ on* $\mathbf{Aff}/S$ *coarser than the fpqc topology.*

Definition 2.1.12. *Let S be a scheme and let τ be a Grothendieck topology on* $\mathbf{Aff}/S$.

1. *A morphism* $\mathcal{X} \to \mathcal{Y}$ *of* S*-stacks in the* τ *topology is an* epimorphism *if, for any scheme* $U \to \mathcal{Y}$ *over* $\mathcal{Y}$ *which is affine over* S*, there exists a covering* $\{U' \to U\}$ *of* U *in* τ *which fits into a 2-commutative square*

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U . \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

2. *An* algebraic S-stack in the τ-topology of descent class $\mathcal{D}$ *is an* S*-stack in the* τ*-topology satisfying the following additional conditions:*

   (a) *the diagonal morphism* $\mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times_S \mathcal{X}$ *is schematic and in* $\mathcal{D}$*,*

   (b) *there exists an algebraic space* X *and a (*1*-)morphism of* S*-stacks in the* τ*-topology*

   $$X \xrightarrow{P} \mathcal{X},$$

   which is an epimorphism and smooth. The map P *is called a* presentation *of* $\mathcal{X}$*.*

3. *A* Deligne-Mumford S-stack in the τ-topology (of descent class $\mathcal{D}$ *is an* S*-stack in the* τ*-topology (of descent class* $\mathcal{D}$*) which admits a presentation* P *which is an étale morphism.*

4. *A* rigidified S-stack in the τ-topology (of descent class $\mathcal{D}$) *is an algebraic* S*-stack in the* τ*-topology (of descent class* $\mathcal{D}$*) together with a particular choice of presentation* $X \xrightarrow{P} \mathcal{X}$*.*

5. *A* Ravenel S-stack in the τ-topology *is an algebraic* S*-stack* $\mathcal{X}$ *in the* τ*-topology of affine descent class together with a scheme* X *affine over* S *and an epimorphism* $X \xrightarrow{P} \mathcal{X}$ *in the* τ*-topology, such that the diagonal map* $\mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times_S \mathcal{X}$ *is schematic and of affine descent class, and there exists an inverse system* $\{X \xrightarrow{P_i} \mathcal{X}_i\}$ *in algebraic* S*-stacks in the* τ*-topology of affine*

descent class together with a surjection $\mathcal{X} \to \mathcal{X}_i$ for each i, compatible with the maps P and P_i, such that when we pass to the homotopy limit, we recover $X \xrightarrow{P} \mathcal{X}$ up to homotopy equivalence, i.e., we have a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "P"] & X \arrow[d, "\operatorname{holim}_i P_i"] \\ \mathcal{X} \arrow[r, "\cong"] & \operatorname{holim}_i \mathcal{X}_i. \end{tikzcd}$$

6. *A* rigidified Ravenel S-stack in the τ-topology *is a Ravenel S-stack in the τ-topology together with a particular choice of epimorphism P satisfying the conditions in the definition of a Ravenel stack.*

The reason we define Ravenel stacks, instead of being content with algebraic stacks, is that the stacks we care most about as algebraic topologists will *not* be algebraic, rather, they will be a homotopy limit of algebraic stacks, and furthermore, they will come equipped with a (pro-)presentation by an affine scheme (not just an algebraic space). This is because the Hopf algebroids (A, Γ) associated to important cohomology theories in topology, such as complex cobordism, have the property that Γ is of infinite Krull dimension as an A-algebra; so the usual version of smoothness (incorporated into the definition of an algebraic stack) will not hold, because Γ is not finite type over A. But Γ is the union of sub-Hopf-algebroids whose associated stacks *are* smooth; hence our definition of a Ravenel stack. (Another way to resolve this issue is due to Hopkins, in [Hopkins, 1999], where he weakens his requirements for a stack to be algebraic, by only demanding that the presentation be *quasi-smooth*, rather than smooth. We believe that this is worth investigating, but also more technically demanding, so we leave it off for now.)

Lemma 2.1.13. *Let S be a scheme, τ a Grothendieck topology on $\mathbf{Sch}/S$, and $\mathcal{D}$ an effective descent class that LIFTS OVER DIAGONALS. Let $X \xrightarrow{P} \mathcal{X}$ be* *a rigidified S-stack in the τ-topology of descent class $\mathcal{D}$ and suppose that the S-scheme structure map $X \to S$ of X belongs to $\mathcal{D}$. Then the fiber product $X \times_{\mathcal{X}} X$ is an algebraic S-space in the τ-topology of descent class $\mathcal{D}$ whose projection maps $X \times_{\mathcal{X}} X \to X$, diagonal map $X \to X \times_{\mathcal{X}} X$, and structure map $X \times_{\mathcal{X}} X \to S$ all belong to $\mathcal{D}$.*

Proof. We have the diagrams in $\mathbf{Stacks}/S$

$$
\begin{aligned}
Z_1 &= \left(\begin{tikzcd} & X \arrow[d, "P"] \\ \mathcal{X} \arrow[r, "\mathrm{id}_{\mathcal{X}}"] & \mathcal{X} \end{tikzcd} \right) \\
Z_2 &= \left(\begin{tikzcd} & S \arrow[d, "\mathrm{id}_S"] \\ \mathcal{X} \arrow[r] & S \end{tikzcd} \right)
\end{aligned}
$$

and an evident map $Z_1 \to Z_2$. Now

$$
\begin{aligned}
\lim \left(\begin{tikzcd} X \times_S X \arrow[dr] & \\ & \mathcal{X} \times_S \mathcal{X} \\ \mathcal{X} \arrow[ur] & \end{tikzcd} \right)
&\cong \lim \left(\lim \left(\begin{tikzcd} Z_1 \arrow[dr] & \\ & Z_2 \\ Z_1 \arrow[ur] & \end{tikzcd} \right) \right) \\
&\cong \lim \left(\begin{tikzcd} \lim Z_1 \arrow[dr] & \\ & \lim Z_2 \\ \lim Z_1 & \end{tikzcd} \right) \\
&\cong \lim \left(\begin{tikzcd} X \arrow[dr] & \\ & \mathcal{X} \\ X \arrow[ur] & \end{tikzcd} \right) \\
&\cong X \times_{\mathcal{X}} X.
\end{aligned}
$$

As a result we have the 2-categorical pullback square

$$\begin{tikzcd} X \times_{\mathcal{X}} X \arrow[r] \arrow[d] & X \times_S X \arrow[d, "P \times_S P"] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

and since the bottom map is in $\mathcal{D}$, the top map also is; its composites with the projections $X \times_S X \to X$ and structure map $X \times_S X \to S$ are also in $\mathcal{D}$ as the projections and structure map are in $\mathcal{D}$, since $X \to S$ is. That the diagonal $X \to X \times_{\mathcal{X}} X$ is in $\mathcal{D}$ follows from the fact that $\mathcal{D}$ LIFTS OVER DIAGONALS. By Cor.**??**, X is an algebraic S-space in the τ topology of descent class $\mathcal{D}$, as given a presentation of X we get an induced (by pullback) presentation of $X \times_{\mathcal{X}} X$ and the structure maps of the associated THINGY stay in $\mathcal{D}$. □

Definition 2.1.14. *Let $X \xrightarrow{P} \mathcal{X}$ be a rigidified S-stack in the τ-topology of descent class $\mathcal{D}$. Then we form the* bar construction of $X \xrightarrow{P} \mathcal{X}$, *also called the* nerve of $X \xrightarrow{P} \mathcal{X}$, *by*

$$B_\bullet(X \xrightarrow{P} \mathcal{X}) = \left(\begin{tikzcd} X \arrow[r, shift left, "\Delta"] & X \times_{\mathcal{X}} X \arrow[l, shift left] \arrow[l, shift left=2] \arrow[r, shift left, "\Delta \times_{\mathcal{X}} \mathrm{id}_X"] \arrow[r, shift right, "\mathrm{id}_X \times_{\mathcal{X}} \Delta"'] & \cdots \end{tikzcd} \right),$$

a simplicial object in algebraic S-stacks in the τ-topology of descent class $\mathcal{D}$ whose structure maps down to S are in $\mathcal{D}$ (by Lemma 2.1.13), where the morphisms going to the left are appropriate projection maps.

Note that

$$B_\bullet(X \xrightarrow{P} \mathcal{X}) =$$
$$\operatorname{cosk}_3^\infty \left(\begin{tikzcd} X \arrow[r, shift left, "\Delta"] & X \times_{\mathcal{X}} X \arrow[l, shift left] \arrow[l, shift left=2] \arrow[r, shift left, "\Delta \times_{\mathcal{X}} \mathrm{id}_X"] \arrow[r, shift right, "\mathrm{id}_X \times_{\mathcal{X}} \Delta"'] & X \times_{\mathcal{X}} X \times_{\mathcal{X}} X \times_{\mathcal{X}} X \end{tikzcd} \right),$$

i.e., we can recover the entire bar construction of $X \xrightarrow{P} \mathcal{X}$ from its first three stages; this is because the simplicial-set-valued functor on S-schemes represented by $B_\bullet(X \xrightarrow{P} \mathcal{X})$ takes an S-scheme S' to the nerve of the groupoid $\mathcal{X}(S')$, and nerves of groupoids have a distinct structure among all simplicial sets. When

$X \xrightarrow{P} \mathcal{X}$ is Ravenel we will often think of $B_\bullet(X \xrightarrow{P} \mathcal{X})$ as a groupoid object in schemes affine over S, or a groupoid affine S-scheme for short; when P is not Ravenel, but is at least a schematic morphism, then the bar construction instead yields a more general object, a groupoid S-scheme; and in the case that P fails to be even be schematic, the bar construction yields only a groupoid algebraic S-space.

The bar construction gives us a way to pass from a Ravenel S-stack to a groupoid affine S-scheme. Once we have a groupoid affine S-scheme, we can apply the $\mathcal{A}(-)$ functor (see Prop.**??**) to get a commutative Hopf algebroid over S:

Process 2.1.15. From stacks to Hopf algebroids. Let S be a base scheme. If $\mathcal{X}$ is an S-stack in the fpqc topology of descent class $\mathcal{D}$, X is a scheme affine over S, and $X \xrightarrow{P} \mathcal{X}$ is an affine fpqc epimorphism (e.g. if $\mathcal{X}$ has affine diagonal), then we truncate $B_\bullet(X \xrightarrow{P} \mathcal{X})$ after the third stage to get a groupoid object in schemes affine over S; if $X \xrightarrow{P} \mathcal{X}$ is an fpqc presentation of an S-stack in the fpqc topology, of descent class $\mathcal{D}$ containing all affine morphisms of S schemes, such that $\mathcal{X} \cong \operatorname{holim}_i \mathcal{X}_i$ is a filtered homotopy limit with each $X \longrightarrow \mathcal{X}_i$ a rigidified fpqc stack, then the 3-truncated bar construction of $X \xrightarrow{P} \mathcal{X}$ is a groupoid object in schemes affine over S, and we then apply $\mathcal{A}(-)$ to get a faithfully flat smooth commutative Hopf algebroid $(\mathcal{A}(X), \mathcal{A}(X \times_{\mathcal{X}} X))$ over S of descent class $\mathcal{D}$.

As a special case we have a faithfully flat smooth Hopf algebroid over $\operatorname{Spec} R$, in the sense of [Ravenel, 1986], associated to any S-stack in the fpqc topology of affine descent class equipped with a fpqc cover satisfying the above homotopy limit property. If the cover is a presentation, i.e., if we are given a rigidified S-stack in the fpqc topology of affine descent class (making the homotopy limit property redundant), then the associated Hopf algebroid is faithfully flat and smooth with Γ finitely generated as an A-algebra.

We may also go the opposite way, taking a groupoid affine S-scheme, considering the S-groupoid it represents, and stackifying it.

Proposition 2.1.16. *Let S be a base scheme and let τ be a Grothendieck topology on $\mathbf{Aff}/S$. Let*

$$\mathbf{Stacks}_\tau/S \xrightarrow{I} \mathbf{Prestacks}_\tau/S$$

denote the forgetful functor from stacks to prestacks. The functor I has a weak left adjoint J, which we call stackification, *in the sense that there is a natural equivalence of categories*

$$\hom_{\mathbf{Stacks}_\tau}(J(\mathcal{X}), \mathcal{Y}) \cong \hom_{\mathbf{Prestacks}_\tau}(\mathcal{X}, I(\mathcal{Y}))$$

for all stacks $\mathcal{X}$ and prestacks $\mathcal{Y}$; we construct J explicitly as follows:

1. *let $\mathcal{X}$ be an S-prestack in the τ topology; then for each scheme U affine over S, we define the objects of the groupoid $J(\mathcal{X})_U$ by*

   $$\mathrm{ob}\ J(\mathcal{X})_U = \left\{(U' \xrightarrow{f} U, x, f') : \{f\} \in \mathrm{Cov}\,\tau \text{ and } (x, f') \text{ is a descent datum for } \{f\}\right\}$$

2. *and given two such objects $(U_1' \xrightarrow{f_1} U, x_1, f_1')$ and $(U_2' \xrightarrow{f_2}, x_2, f_2')$, their morphism set is given by*

   $$\hom_{J(x)}((U_1' \xrightarrow{f_1} U, x_1, f_1'), (U_2' \xrightarrow{f_2}, x_2, f_2')) = \{x_1|U_1' \times_U U_2' \xrightarrow{g} x_2|U_1' \times_U U_2' : g \text{ is compatible w}$$

   We also have that $J \circ I$ is the identity functor on $\mathbf{Stacks}_\tau/S$, i.e., stackifying a prestack which is already a stack does not change it.

Proposition 2.1.17. *Let S be a base scheme. Let $X.$ be a faithfully flat and of descent class $\mathcal{D}$ (i.e., $X_1 \to X_0$ is faithfully flat and in $\mathcal{D}$) groupoid scheme over S, and we write $\mathcal{F}$ for the groupoid-valued presheaf on $\mathbf{Aff}/S$ represented by $X.$. Then the map $X_0 \to X.$ of groupoid schemes over S, where by X_0 we mean the groupoid scheme with all objects given by X_0 and all morphisms the identity morphism, stackifies to an epimorphism $X_0 \to J(\underline{\mathcal{F}})$, and the diagonal map $J(\underline{\mathcal{F}}) \to J(\underline{\mathcal{F}}) \times_S J(\underline{\mathcal{F}})$ is schematic and in $\mathcal{D}$.*

Proof. If $U \to J(\underline{\mathcal{F}})$ is a morphism of stacks with U a scheme affine over S, then $X_0 \times_{J(\underline{\mathcal{F}})} U \to U$ is in $\mathcal{D}$; in fact, $X_0 \times_{J(\underline{\mathcal{F}})} U \to U \cong X_0 \times_{X_0} X_1 \times_{X_0} U$. We see immediately that it is a stack associated to a presheaf of sets, because from the definition of the 2-categorical pullback, there is no room for non-identity morphisms in $(X_0 \times_{J(\underline{\mathcal{F}})} U)_V$ for any V affine over S. By inspection we see that $(X_0 \times_{J(\underline{\mathcal{F}})} U)_V$ as a set is identical to $\hom_{\mathbf{Aff}/S}(V, X_0 \times_{J(\underline{\mathcal{F}})} U)$. As a consequence $X_0 \to J(\underline{\mathcal{F}})$ is representable and epimorphic in the fpqc topology and the diagonal $J(\underline{\mathcal{F}}) \to J(\underline{\mathcal{F}}) \times_S J(\underline{\mathcal{F}})$ is in $\mathcal{D}$. □

Lemma 2.1.18.

1. *Let S be a scheme and let (A, Γ) be a commutative Hopf algebroid over S such that $A \xrightarrow{\eta_L} \Gamma$ and $A \xrightarrow{\eta_R} \Gamma$ are faithfully flat morphisms; let $\mathcal{X}$ denote the stackification of the S-prestack represented by $(\mathrm{Spec}_S A, \mathrm{Spec}_S \Gamma)$. Then the associated presentation $\mathrm{Spec}_S A \to \mathcal{X}$ (from Prop. 2.1.17) is an affine morphism.*

2. *Let S be a scheme, let $\mathcal{D}$ be an effective descent class of morphisms in $\mathbf{Sch}/S$ in the fpqc topology, and let (A, Γ) be a commutative Hopf algebroid over S such that $A \xrightarrow{\eta_L} \Gamma$ and $A \xrightarrow{\eta_R} \Gamma$ are faithfully flat morphisms with the morphisms $\mathrm{Spec}_S(\eta_L)$ and $\mathrm{Spec}_S(\eta_R)$ and $\mathrm{Spec}_S \Gamma \to \mathrm{Spec}_S A \times_S \mathrm{Spec}_S A$ all in $\mathcal{D}$; let $\mathcal{X}$ denote the stackification of the S-prestack represented by $(\mathrm{Spec}_S A, \mathrm{Spec}_S \Gamma)$. Then the associated presentation $\mathrm{Spec}_S A \to \mathcal{X}$ (from Prop. 2.1.17) is in $\mathcal{D}$.*

Proof.

1. Choose a scheme Y affine over S with a morphism $Y \to \mathcal{X}$. We need to show that $Y \times_{\mathcal{X}} X$ is a scheme affine over S. We choose an fpqc cover $W \to Y$, with W a scheme affine over S equipped with a map to $\mathrm{Spec}_S A$ making the following diagram 2-commute (this is a 2-categorical version of

the argument of Prop. I.5.9 of [Knutson, 1971]):

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec}_S A \arrow[r] & \mathcal{X}. \end{tikzcd}$$

We have the 2-commutative diagram

$$\begin{tikzcd} W \times_{\mathcal{X}} \operatorname{Spec}_S A \arrow[r] \arrow[d] & Y \times_{\mathcal{X}} \operatorname{Spec}_S A \arrow[d] \\ \operatorname{Spec}_S \mathbb{\Gamma} \arrow[r] & \operatorname{Spec}_S A \times_S \operatorname{Spec}_S A. \end{tikzcd} \tag{2.1.5}$$

This is a homotopy pullback diagram, as we have the diagrams

$$Z_0 \cong \left(\begin{tikzcd} Y \arrow[dr] & \\ & \mathcal{X} \\ \operatorname{Spec}_S A \arrow[ur] & \end{tikzcd} \right), Z_1 \cong \left(\begin{tikzcd} W \arrow[dr] & \\ & S \\ \operatorname{Spec}_S A \arrow[ur] & \end{tikzcd} \right),$$

$$Z_2 \cong \left(\begin{tikzcd} Y \arrow[dr] & \\ & S \\ \operatorname{Spec}_S A \arrow[ur] & \end{tikzcd} \right)$$

with evident maps between them giving a 2-commuting diagram

$$\begin{tikzcd} Z_0 \arrow[dr] & \\ & Z_1 . \\ Z_2 \arrow[ur] & \end{tikzcd}$$

Now we have

$$
\begin{aligned}
\operatorname{holim}\left(\begin{tikzcd} \operatorname{holim} Z_0 \arrow[dr] & \\ & \operatorname{holim} Z_1 \\ \operatorname{holim} Z_2 \arrow[ur] & \end{tikzcd}\right) &\simeq \operatorname{holim}\left(\begin{tikzcd} \operatorname{holim} Z_0 \arrow[dr] & \\ & \operatorname{holim} Z_1 \\ \operatorname{holim} Z_2 \arrow[ur] & \end{tikzcd}\right) \\
&\simeq \operatorname{holim}\left(\begin{tikzcd} W \arrow[dr] & \\ & \mathcal{X} \\ \operatorname{Spec}_S A \arrow[ur] & \end{tikzcd}\right) \\
&\simeq W \times_{\mathcal{X}} \operatorname{Spec}_S A,
\end{aligned}
$$

so diagram 2.1.5 is a homotopy pullback diagram. Now since the map $\operatorname{Spec}_S \mathbb{\Gamma} \to \operatorname{Spec}_S A \times_S \operatorname{Spec}_S A$ is affine and affineness is preserved under (HOMOTOPY BASE CHANGE! ADD THIS POINT) base change (Prop A.3.4), the morphism $W \times_{\mathcal{X}} \operatorname{Spec}_S A \to W \times_S \operatorname{Spec}_S A$ is also affine; finally, we have another homotopy pullback square

$$
\begin{tikzcd}
W \times_{\mathcal{X}} \operatorname{Spec}_S A \arrow[r] \arrow[d] & W \times_S \operatorname{Spec}_S A \arrow[d] \\
Y \times_{\mathcal{X}} \operatorname{Spec}_S A \arrow[r] & Y \times_S \operatorname{Spec}_S A
\end{tikzcd}
$$

and since $W \times_S \operatorname{Spec}_S A \to Y \times_S \operatorname{Spec}_S A$ is an fpqc cover, so is $W \times_{\mathcal{X}} \operatorname{Spec}_S A \to Y \times_{\mathcal{X}} \operatorname{Spec}_S A$, and by fpqc descent (by Prop. A.3.4, affine morphisms form an effective descent class in the fpqc topology), since $W \times_{\mathcal{X}} \operatorname{Spec}_S A \to W \times_S \operatorname{Spec}_S A$ is affine, so is $Y \times_{\mathcal{X}} \operatorname{Spec}_S A \to Y \times_S \operatorname{Spec}_S A$ and hence also the composite $Y \times_{\mathcal{X}} \operatorname{Spec}_S A \to Y \times_S \operatorname{Spec}_S A \to \operatorname{Spec}_S A \to S$. Hence $Y \times_{\mathcal{X}} \operatorname{Spec}_S A$ is affine over S.

2. This follows the same method of proof as that given above, with morphism of the effective descent class $\mathcal{D}$ in place of affine morphisms.

$\square$

Process 2.1.19. From Hopf algebroids to stacks. Let S be a base scheme. If (A, Γ) is a faithfully flat commutative Hopf algebroid over S of descent class $\mathcal{D}$, then $(\mathrm{Spec}_S A, \mathrm{Spec}_S \Gamma)$ is a groupoid affine S-scheme and hence represents an S-prestack $J(\mathcal{F}_{(A,\Gamma)})$, equipped with an affine fpqc epimorphism $\mathrm{Spec}_S A \rightarrow J(\mathcal{F}_{(A,\Gamma)})$. By Prop. 2.1.17 this stackifies to an S-stack $\underline{J(\mathcal{F}_{(A,\Gamma)})}$ equipped with an fpqc epimorphism $\mathrm{Spec}_S A \rightarrow \underline{J(\mathcal{F}_{(A,\Gamma)})}$ such that the diagonal $\underline{J(\mathcal{F}_{(A,\Gamma)})} \rightarrow \underline{J(\mathcal{F}_{(A,\Gamma)})} \times_S \underline{J(\mathcal{F}_{(A,\Gamma)})}$ is affine and of descent class $\mathcal{D}$. When (A, Γ) is of affine descent class and is a filtered colimit of smooth sub-Hopf-algebroids (A, Γ_i) (i.e., Γ_i is smooth (Lemma 2.1.18 gives that the rigidified fpqc stack associated to a smooth Hopf algebroid is also smooth) over A for each i, and $\Gamma = \bigcup_i \Gamma_i$) then $\mathrm{Spec}_S A \rightarrow \underline{J(\mathcal{F}_{(A,\Gamma_i)})}$ is a rigidified S-stack in the fpqc topology of affine descent class, and $\mathrm{Spec}_S A \rightarrow \mathrm{holim}_i \underline{J(\mathcal{F}_{(A,\Gamma_i)})}$ is a Ravenel S-stack (in the fpqc topology). When (A, Γ) is itself a smooth faithfully flat commutative Hopf algebroid over S of affine descent class, then $\underline{J(\mathcal{F}_{(A,\Gamma)})}$ is an algebraic S-stack in the fpqc topology of affine descent class, and $\mathrm{Spec}_S A \rightarrow \underline{J(\mathcal{F}_{(A,\Gamma)})}$ is a presentation of $\underline{J(\mathcal{F}_{(A,\Gamma)})}$.

Definition 2.1.20. *1. Let S be a scheme and let $\mathcal{X}$ be an S-stack in the τ topology. Then we define* the local τ-site on $\mathcal{X}$ *by*

$$\mathrm{ob}\ \tau(\mathcal{X}) =$$

$$\{U \xrightarrow{f} \mathcal{X} : U \textit{ is an } S\textit{-scheme, } f \textit{ is affine, and } f \in \mathrm{Cov}\,\tau\}$$

$$\hom_{\tau(\mathcal{X})}(U \xrightarrow{f} \mathcal{X}, U' \xrightarrow{f'} \mathcal{X}) = \{(U \xrightarrow{g} U', \alpha \in \hom_{\mathcal{X}_U}(f, f' \circ g))\}$$

$$\mathrm{Cov}(\tau(\mathcal{X})) =$$

$$\{\{U_i \xrightarrow{f_i} U\} : \textit{each } U_i \textit{ is an } S\textit{-scheme and } \{U_i \xrightarrow{f_i} U\} \in \mathrm{Cov}\,\tau\}.$$

When we write $f \in \mathrm{Cov}\,\tau$ when f is an affine map $U \xrightarrow{f} \mathcal{X}$ of stacks, we mean that, for any affine scheme V over $\mathcal{X}$, the pullback $U \times_{\mathcal{X}} V \rightarrow V$ is *in a covering in the τ topology. In principle this allows us to extend the Grothendieck topology on the local τ-site of a stack $\mathcal{X}$ to a Grothendieck topology on the category of all S-stacks equipped with affine morphisms to $\mathcal{X}$; however, all we need for our purposes in these notes is the local τ-site on $\mathcal{X}$ as we have already defined it.*

2. *We define a τ $\mathcal{O}_S$-*algebra sheaf on $\mathcal{X}$ *as a sheaf of $\mathcal{O}_S$-algebras $\mathcal{F}$ on $\tau(\mathcal{X})$ such that, for each morphism of affines $U \xrightarrow{f} U'$ in $\operatorname{Cov}\tau$, the map*

$$\operatorname{Spec}_S \mathcal{F}(U) \xrightarrow{\operatorname{Spec}_S \mathcal{F}(f)} \operatorname{Spec}_S \mathcal{F}(U')$$

is in $\operatorname{Cov}\tau$. (Compare 3.18 of [Pribble, 2004].)

3. *A pair consisting of an algebraic S-stack in the τ-topology and a τ $\mathcal{O}_S$-algebra sheaf on $\mathcal{X}$ is called a* ringed algebraic S-stack in the τ-topology.

4. *Let $(\mathcal{X}, \mathcal{A})$ be a ringed algebraic S-stack in the τ-topology. Then an $\mathcal{A}$-*module *is a sheaf of $\mathcal{O}_S$-modules on $\tau(\mathcal{X})$ together with an $\mathcal{A}(U)$-module structure on $\mathcal{F}(U)$ for each affine U over $\mathcal{X}$, and such that, for any morphism $U \xrightarrow{f} U'$ of affines over $\mathcal{X}$, the map $\mathcal{F}(U') \xrightarrow{\mathcal{F}(f)} \mathcal{F}(U)$ is a morphism of $\mathcal{A}(U')$-modules, or equivalently, the adjoint map $\mathcal{A}(U) \otimes_{\mathcal{A}(U')} \mathcal{F}(U') \xrightarrow{\mathcal{F}(f)^\natural} \mathcal{F}(U)$ is an $\mathcal{A}(U)$-module morphism. We sometimes write $\mathbf{Mod}(\mathcal{A})$ for the category of $\mathcal{A}$-modules.*

5. *An $\mathcal{A}$-module is called* Cartesian *if, for each f, the map $\mathcal{F}(f)^\natural$ is an isomorphism. We sometimes write $\mathbf{Mod}(\mathcal{A})_{\mathrm{Cart}}$ for the category of Cartesian $\mathcal{A}$-modules, i.e., the full subcategory of $\mathbf{Mod}(\mathcal{A})$ whose objects are the Cartesian $\mathcal{A}$-modules.*

Lemma 2.1.21. *Let S be a base scheme and let $\mathcal{X}$ be an S-stack in the τ-topology. Then the sheaf on $\tau(\mathcal{X})$ which sends each affine $U \xrightarrow{f} S$ to its associated quasicoherent $\mathcal{O}_S$-module $f_*\mathcal{O}_U$ is a τ $\mathcal{O}_S$-algebra sheaf, which we call* the structure sheaf of $\mathcal{X}$, *and we write $\mathcal{O}_{\mathcal{X}}$ for it.*

Proof. Immediate from the definition. □

Proposition 2.1.22. *Let S be a scheme and let $\mathcal{X}$ be an S-stack in the fpqc topology, and let $X \xrightarrow{P} \mathcal{X}$ be an fpqc affine epimorphism, with X a scheme affine over S. To any Cartesian $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$ we can associate the quasicoherent $\mathcal{O}_X$-module $P^*\mathcal{F}$ and also a map*

$$\mathcal{F}(X) \xrightarrow{\mathcal{F}(P \times_{\mathrm{id}_{\mathcal{X}}} \mathrm{id}_X)} \mathcal{F}(X \times_{\mathcal{X}} X) \xrightarrow{\cong} \mathcal{O}_{X \times_{\mathcal{X}} X} \otimes_{\mathcal{O}_X} \mathcal{F}(X),$$

of quasicoherent $\mathcal{O}_X$-modules, where the isomorphism is by the definition of Cartesianness for $\mathcal{F}$ (Def. 2.1.20). We will write ψ for the composite map above. This map makes the following diagrams commute:

$$\text{(counit)} \quad \begin{tikzcd} \mathcal{F}(X) \arrow[r, "\psi"] \arrow[dr, "\mathrm{id}_{\mathcal{F}(X)}"'] & \mathcal{O}_{X \times_{\mathcal{X}} X} \otimes_{\mathcal{O}_X} \mathcal{F}(X) \arrow[d, "\mathcal{O}_\Delta \otimes_{\mathcal{O}_X} \mathrm{id}_{\mathcal{F}(X)}"] \\ & \mathcal{F}(X) \end{tikzcd}$$

$$\text{(coassociativity)} \quad \begin{tikzcd} \mathcal{F}(X) \arrow[r, "\psi"] \arrow[d, "\psi"] & \mathcal{O}_{X \times_{\mathcal{X}} X} \otimes_{\mathcal{O}_X} \mathcal{F}(X) \arrow[d, "\mathrm{id}_{\mathcal{O}_{X \times_{\mathcal{X}} X}} \otimes_{\mathcal{O}_X} \psi"] \\ \mathcal{O}_{X \times_{\mathcal{X}} X} \otimes_{\mathcal{O}_X} \mathcal{F}(X) \arrow[r, "\Delta \otimes_{\mathcal{O}_X} \mathrm{id}_{\mathcal{F}(X)}"] & \mathcal{O}_{X \times_{\mathcal{X}} X} \otimes_{\mathcal{O}_X} \mathcal{O}_{X \times_{\mathcal{X}} X} \otimes_{\mathcal{O}_X} \mathcal{F}(X) \end{tikzcd} .$$

We will refer to a Cartesian $\mathcal{O}_X$-module $\mathcal{G}$ together with a map

$$\mathcal{G} \xrightarrow{\phi} \mathcal{O}_{X \times_{\mathcal{X}} X} \otimes_{\mathcal{O}_X} \mathcal{G},$$

satisfying the above counit and coassociativity conditions, as a Cartesian $\mathcal{O}_X$-module with descent datum relative to P. *Such objects form a category, which we write* $\mathbf{Mod}_{\mathrm{Cart}}(\mathcal{O}_X) + \mathbf{Desc}(P)$.

The functor $\mathbf{Mod}_{\mathrm{Cart}}(\mathcal{O}_{\mathcal{X}}) \to \mathbf{Mod}_{\mathrm{qcoh}}(\mathcal{O}_X) + \mathbf{Desc}(P)$ *is an equivalence of categories.*

Proof. The counit property follows from the fact that the diagram

$$\begin{tikzcd} X \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}_X"] & \\ X \times_{\mathcal{X}} X \arrow[r, "P \times_{\mathcal{X}} \mathrm{id}_X"] & X \end{tikzcd}$$

commutes, from the universal property of the (2-categorical) pullback. When we apply $\mathcal{F}$ we get

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "\psi"] \arrow[dr, "\mathrm{id}_{\mathcal{F}(X)}"'] & \mathcal{O}_{X\times_{\mathcal{X}}X} \otimes_{\mathcal{O}_X} \mathcal{F}(X) \arrow[r, "\cong"] \arrow[d, "\mathcal{O}_\Delta \otimes_{\mathcal{O}_X} \mathrm{id}_{\mathcal{F}(X)}"] & \mathcal{F}(X\times_{\mathcal{X}} X) \arrow[dl, "\mathcal{F}(\Delta)"] \\ & \mathcal{F}(X) & \end{tikzcd}.$$

The coassociativity property follows from the fact that the diagram

$$\begin{tikzcd} X\times_{\mathcal{X}} X \times_{\mathcal{X}} X \arrow[r, "\mathrm{id}_X \times_{\mathcal{X}} P \times_{\mathcal{X}} \mathrm{id}_X"] \arrow[d, "P\times_{\mathcal{X}} \mathrm{id}_X \times_{\mathcal{X}} \mathrm{id}_X"] & X\times_{\mathcal{X}} X \arrow[d, "P\times_{\mathcal{X}}\mathrm{id}_X"] \\ X\times_{\mathcal{X}} X \arrow[r, "P\times_{\mathcal{X}}\mathrm{id}_X"] & X \end{tikzcd}$$

(2-)commutes, so when we apply $\mathcal{F}$ we get

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "\psi"] \arrow[d, "\psi"] & \mathcal{F}(X\times_{\mathcal{X}} X) \arrow[r, "\cong"] \arrow[d, "\mathcal{F}(P\times_{\mathcal{X}}\mathrm{id}_X \times_{\mathcal{X}} \mathrm{id}_X)"] & \mathcal{O}_{X\times_{\mathcal{X}}X} \otimes_{\mathcal{O}_X} \mathcal{F}(X) \arrow[dd, "\Delta\otimes_{\mathcal{O}_X}\mathrm{id}_{\mathcal{F}(X)}"] \\ \mathcal{F}(X\times_{\mathcal{X}} X) \arrow[r, "\mathcal{F}(\mathrm{id}_X \times_{\mathcal{X}} P \times_{\mathcal{X}} \mathrm{id}_X)"] \arrow[d, "\cong"] & \mathcal{F}(X\times_{\mathcal{X}} X \times_{\mathcal{X}} X) \arrow[dr, "\cong"] & \\ \mathcal{O}_{X\times_{\mathcal{X}}X} \otimes_{\mathcal{O}_X} \mathcal{F}(X) \arrow[rr, "\mathrm{id}_{\mathcal{O}_{X\times_{\mathcal{X}}X}} \otimes_{\mathcal{O}_X} \psi"] & & \mathcal{O}_{X\times_{\mathcal{X}}X} \otimes_{\mathcal{O}_X} \mathcal{O}_{X\times_{\mathcal{X}}X} \otimes_{\mathcal{O}_X} \mathcal{F}(X). \end{tikzcd}$$

Now given a quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ with descent datum $\mathcal{F}(X) \xrightarrow{\psi} \mathcal{O}_{X\times_{\mathcal{X}}X} \otimes_{\mathcal{O}_X} \mathcal{F}(X)$ relative to P, we need to construct a Cartesian $\mathcal{O}_{\mathcal{X}}$-module. We compose the map ϕ with the Cartesianness isomorphism $\mathcal{O}_{X\times_{\mathcal{X}}X} \otimes_{\mathcal{O}_X} \mathcal{F}(X) \xrightarrow{\cong} \mathcal{F}(X\times_{\mathcal{X}} X)$, tensor up over $\mathcal{O}_X$ with $\mathcal{O}_{U\times_{\mathcal{X}}X}$, and compose with further Cartesianness isomorphisms, to get the isomorphic equalizer sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}(U) \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{F}(U)}"] & \mathcal{F}(U\times_{\mathcal{X}} X) \oplus \mathcal{F}(U\times_{\mathcal{X}} X) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & \mathcal{F}(U\times_{\mathcal{X}} X\times_{\mathcal{X}} X) \arrow[d, "\cong"] \\ 0 \arrow[r] & \mathcal{F}(U) \arrow[r] & \mathcal{O}_{U\times_{\mathcal{X}}X} \otimes_{\mathcal{O}_X} \mathcal{F}(X) \oplus \mathcal{O}_{U\times_{\mathcal{X}}X} \otimes_{\mathcal{O}_X} \mathcal{F}(X) \arrow[r, shift left] \arrow[r, shift right] & \mathcal{O}_{U\times_{\mathcal{X}}X} \otimes_{\mathcal{O}_X} \mathcal{F}(X\otimes_{\mathcal{X}} X) \end{tikzcd},$$

where we get the vertical isomorphisms by the Cartesianness of $\mathcal{F}$ and we know that the top sequence is an equalizer sequence since $\mathcal{F}$ is a sheaf. Now, since we know $\mathcal{F}(X)$ and $\mathcal{F}(X\times_{\mathcal{X}} X)$ and the morphism between them, the equalizer sequences above uniquely determine $\mathcal{F}(U)$, giving us a Cartesian sheaf over $\mathcal{X}$. $\square$

(It would be interesting to study the forgetful map $\mathbf{Mod}_{\mathrm{qcoh}}(\mathcal{O}_X)+\mathbf{Desc}(P) \rightarrow \mathbf{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{X}))$ from the perspective of fiber bundles of categories or toposes. Does this perspective yield any new information?)

2.2 Cohomology.

Definition 2.2.1. 1. *Let S be a scheme and $(A, \mathbb{\Gamma})$ a commutative Hopf algebroid over S, and let M be a right $\mathbb{\Gamma}$-comodule and N a left $\mathbb{\Gamma}$-comodule, with comodule structure maps $M \xrightarrow{\psi_M} M \otimes_A \mathbb{\Gamma}$ and $N \xrightarrow{\psi_N} \mathbb{\Gamma} \otimes_A N$, respectively. Then the* cotensor product sheaf *$M \boxast_{\mathbb{\Gamma}} N$ is the $\mathcal{O}_S$-module given as the equalizer*

$$0 \longrightarrow M \boxast_{\mathbb{\Gamma}} N \longrightarrow M \otimes_A N \underset{\mathrm{id}_M \otimes_A \psi_N}{\overset{\psi_M \otimes_A \mathrm{id}_N}{\rightrightarrows}} M \otimes_A \mathbb{\Gamma} \otimes_A N.$$

The cotensor product is functorial (covariant) in M and N, i.e., $(M, N) \rightarrow M \boxast_{\mathbb{\Gamma}} N$ is a bifunctor from the product category (right $\mathbb{\Gamma}$-comodules) $\times$ (left $\mathbb{\Gamma}$-comodules) to the category of $\mathcal{O}_S$-modules, and it is commutative by construction, i.e., there is a natural isomorphism $M \boxast_{\mathbb{\Gamma}} N \cong N \boxast_{\mathbb{\Gamma}} M$, where in the second cotensor product we have exchanged the right $\mathbb{\Gamma}$-comodule structure on M for the equivalent left $\mathbb{\Gamma}$-comodule structure on M, and vice versa for N.

2. *Let S be a scheme and $(A, \mathbb{\Gamma})$ a commutative Hopf algebroid over S, and let M be a right $\mathbb{\Gamma}$-comodule and N a left $\mathbb{\Gamma}$-comodule, with comodule structure maps $M \xrightarrow{\psi_M} M \otimes_A \mathbb{\Gamma}$ and $N \xrightarrow{\psi_N} \mathbb{\Gamma} \otimes_A N$, respectively. Then the* cotensor product *$M \square`_{\mathbb{\Gamma}} N$ is the $\Gamma(\mathcal{O}_S)$-module given as the equalizer*

$$0 \longrightarrow M \square_{\mathbb{\Gamma}} N \longrightarrow \Gamma(M \otimes_A N) \underset{\Gamma(\mathrm{id}_M \otimes_A \psi_N)}{\overset{(\psi_M \otimes_A \mathrm{id}_N)}{\rightrightarrows}} \Gamma(M \otimes_A \mathbb{\Gamma} \otimes_A N),$$

i.e., $M \square_{\mathbb{\Gamma}} N \cong \Gamma(M \boxast_{\mathbb{\Gamma}} N)$. The cotensor product is functorial (covariant) in M and N, i.e., $(M, N) \rightarrow M \square_{\mathbb{\Gamma}} N$ is a bifunctor from the product category *(right $\mathbb{\Gamma}$-comodules) $\times$ (left $\mathbb{\Gamma}$-comodules) to the category of $\Gamma(\mathcal{O}_S)$-modules, and it is commutative by construction, i.e., there is a natural isomorphism $M\Box_{\mathbb{\Gamma}}N \cong N\Box_{\mathbb{\Gamma}}M$, where in the second cotensor product we have exchanged the right $\mathbb{\Gamma}$-comodule structure on M for the equivalent left $\mathbb{\Gamma}$-comodule structure on M, and vice versa for N.*

Proposition 2.2.2. *Let S be a scheme.*

1. *Let $(A, \mathbb{\Gamma})$ be a flat (i.e., the maps η_L and η_R are flat) commutative Hopf algebroid over S. Then the category $\mathbf{Comod}_{\mathbb{\Gamma}}$ of left (resp. right) $\mathbb{\Gamma}$-comodules is an abelian category. (When η_L or η_R fails to be flat, we are not guaranteed the existence of kernels in $\mathbf{Comod}_{\mathbb{\Gamma}}$.)*

2. *Let $(A, \mathbb{\Gamma})$ be a flat (i.e., the maps η_L and η_R are flat) commutative Hopf algebroid over S. Then the category $(\mathbf{Comod}_{\mathbb{\Gamma}})_{\mathrm{qcoh}}$ of quasicoherent $\mathbb{\Gamma}$-comodules is an abelian subcategory of $\mathbf{Comod}_{\mathbb{\Gamma}}$, i.e., the product and coproduct of quasicoherent comodules is quasicoherent, the kernel and cokernel of a morphism of quasicoherent comodules is quasicoherent, and the zero object in comodules is quasicoherent.*

3. *Let $\mathcal{X}$ be an algebraic S-stack in the fpqc topology. Then the category $\mathbf{Mod}(\mathcal{O}_{\mathcal{X}})$ of $\mathcal{O}_{\mathcal{X}}$-modules is an abelian category.*

4. *Let $\mathcal{X} \simeq \mathrm{holim}_i\, \mathcal{X}_i$ be a ho-pro-algebraic S-stack in the fpqc topology, i.e., it is an fpqc S-stack which is a filtered homotopy limit of algebraic S-stacks in the fpqc topology. Then the category $\mathbf{Mod}(\mathcal{O}_{\mathcal{X}})$ of $\mathcal{O}_{\mathcal{X}}$-modules is an abelian category.*

Proof. 1. The zero object in right $\mathbb{\Gamma}$-comodules is given by the A-module A together with comodule structure map $A \xrightarrow{\eta_L} \mathbb{\Gamma}$. The zero object in left $\mathbb{\Gamma}$ is A with comodule structure map given by η_R.

The product and coproduct (the two are naturally isomorphic in any abelian category) of right Γ-comodules M, N with structure maps ϕ_M, ϕ_N is given by the structure map which is the composite

$$M \oplus N \xrightarrow{\phi_M \oplus \phi_N} (M \otimes_A \Gamma) \oplus (N \otimes_A \Gamma) \xrightarrow{\cong} (M \oplus N) \otimes_A \Gamma.$$

The reader may verify that this structure map satisfies coassociativity and counitality and also that this comodule satisfies the universal properties of a product and coproduct.

Given a map $M \xrightarrow{f} N$ of right Γ-comodules with structure maps ϕ_M, ϕ_N, we define a kernel comodule as having the underlying A-module $\ker f$ and structure map $\phi_{\ker f}$ given by the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \ker f \arrow[r] \arrow[d, "\phi_{\ker f}"] & M \arrow[r, "f"] \arrow[d, "\phi_M"] & N \arrow[r] \arrow[d, "\phi_N"] & 0 \arrow[d] \\
0 \arrow[r] & \ker f \otimes_A \Gamma \arrow[r] & M \otimes_A \Gamma \arrow[r, "f \otimes_A \mathrm{id}_\Gamma"] & N \otimes_A \Gamma \arrow[r] & 0
\end{tikzcd}$$

(if η_L, η_R were not flat, then the bottom row could fail to be exact and we would not be able to define a kernel of a map of comodules in this way). We can define cokernels of comodule maps in an analogous way (which does not demand that η_L, η_R be flat, since tensoring up with Γ is right exact already) and the task of proving that these satisfy the kernel and cokernel axioms in the definition of an abelian category is left to the reader.

2. That products, coproducts, kernels, cokernels, and the zero in $\mathbf{Comod}_\Gamma$ are quasicoherent if all involved comodules are quasicoherent; this follows immediately from the explicit constructions in the proof of the previous part of this proposition.

3. This follows from Lemma **??**.

4. This follows from Lemma **??**.

5. This follows from Lemma **??**.

□

Proposition 2.2.3. *Let S be a scheme and let $(A, \mathbb{\Gamma})$ be a commutative flat (i.e., η_L, η_R are flat) Hopf algebroid over S, and let M be a right Γ-comodule. Then:*

1. *the functor*

$$\begin{array}{ccc} \mathbf{Comod}_{\mathbb{\Gamma}} & \xrightarrow{M \boxast_{\mathbb{\Gamma}} -} & \mathbf{Mod}_{\mathcal{O}_S} \\ N & \mapsto & M \boxast_{\mathbb{\Gamma}} N \end{array}$$

   is additive and left exact.

2. *the functor*

$$\begin{array}{ccc} \mathbf{Comod}_{\mathbb{\Gamma}} & \xrightarrow{M \square_{\mathbb{\Gamma}} -} & \mathbf{Mod}_{\Gamma(\mathcal{O}_S)} \\ N & \mapsto & M \square_{\mathbb{\Gamma}} N \end{array}$$

   is additive and left exact.

Proof. Checking additivity is routine. Left exactness comes, in both cases, from the cotensor product being defined as a kernel, i.e., a limit; so it commutes with other limits, such as kernels, and is hence left exact. □

Definition 2.2.4. *We will write $\mathit{Cotor}^*_{\mathbb{\Gamma}}(M, -)$ for the right derived functors of $M \boxast_{\mathbb{\Gamma}} -$ and we will write $\mathrm{Cotor}^*_{\mathbb{\Gamma}}(M, -)$ for the right derived functors of $M \square_{\mathbb{\Gamma}} -$.*

Proposition 2.2.5. Ext in the category of comodules is Cotor.

Let S be a scheme and $(A, \mathbb{\Gamma})$ a commutative Hopf algebroid over S, and let M be a right $\mathbb{\Gamma}$-comodule and N a left $\mathbb{\Gamma}$-comodule, with comodule structure maps $M \xrightarrow{\psi_M} M \otimes_A \mathbb{\Gamma}$ and $N \xrightarrow{\psi_N} \mathbb{\Gamma} \otimes_A N$, respectively.

1. *Suppose M is locally free as an A-module, i.e., M is a locally free $\mathcal{O}_{\mathrm{Spec}_S A}$-module. Then we have an isomorphism* $\hom_{\mathbf{Mod}_A}(M, A) \otimes_{\Gamma(A)} \Gamma(\mathbb{F}) \cong \hom_{\mathbf{Mod}_A}(M, \mathbb{F})$ *(Lemma B.1.12) and a right* $(\Gamma(A), \Gamma(\mathbb{F}))$*-comodule structure on* $\hom_{\mathbf{Mod}_A}(M, A)$ *given by the composite*

$$\hom_{\mathbf{Mod}_A}(M, A) \xrightarrow{\hom_{\mathbf{Mod}_A}(\mathrm{id}_M, \eta_L)} \hom_{\mathbf{Mod}_A}(M, \mathbb{F}) \xrightarrow{\cong} \hom_{\mathbf{Mod}_A}(M, A) \otimes_{\Gamma(A)} \Gamma(\mathbb{F}).$$

2. *If M is an A-algebra and it is locally free as an A-module, then the right* $\Gamma(\mathbb{F})$*-comodule structure on* $\hom_{\mathbf{Mod}_A}(M, A)$ *preserves multiplication, i.e., it is a right* $\Gamma(\mathbb{F})$*-comodule algebra structure map.*

3. *If M is locally free as an A-module then we have the isomorphism*

$$\hom_{\mathbf{Mod}_A}(M, A) \Box_{\mathbb{F}} N \cong \hom_{\mathbf{Comod}_{\mathbb{F}}}(M, N).$$

   As the special case $M \cong A$ we get the isomorphism

$$A \Box_{\mathbb{F}} N \cong \hom_{\mathbf{Comod}_{\mathbb{F}}}(A, N)$$

   which is natural in N but not natural *in A! Taking derived functors of the isomorphic functors* $A \Box_{\mathbb{F}} - \cong \hom_{\mathbf{Comod}_{\mathbb{F}}}(A, -)$ *we get the isomorphism*

$$\mathrm{Cotor}^*_{\mathbb{F}}(A, N) \cong \mathrm{Ext}^*_{\mathbf{Comod}_{\mathbb{F}}}(A, N)$$

   which is again natural in N but not natural *in A! This has been a point of some confusion in the past.*

4. *Suppose M is locally free as an A-module, i.e., M is a locally free $\mathcal{O}_{\mathrm{Spec}_S A}$-module. Then we have an isomorphism of A-modules* $\mathcal{hom}_{\mathbf{Mod}_A}(M, A) \otimes_A \mathbb{F} \cong \mathcal{hom}_{\mathbf{Mod}_A}(M, \mathbb{F})$ *(Lemma B.1.12) and a right $\mathbb{F}$-comodule structure on* $\mathcal{hom}_{\mathbf{Mod}_A}(M, A)$ *given by the composite*

$$\mathcal{hom}_{\mathbf{Mod}_A}(M, A) \xrightarrow{\mathcal{hom}_{\mathbf{Mod}_A}(\mathrm{id}_M, \eta_L)} \mathcal{hom}_{\mathbf{Mod}_A}(M, \mathbb{F}) \xrightarrow{\cong} \mathcal{hom}_{\mathbf{Mod}_A}(M, A) \otimes_A \mathbb{F}.$$

5. *If M is an A-algebra and it is locally free as an A-module, then the right $\mathbb{\Gamma}$-comodule structure on* $\hom_{\mathbf{Mod}_A}(M, A)$ *preserves multiplication, i.e., it is a right $\mathbb{\Gamma}$-comodule algebra structure map.*

6. *If M is locally free as an A-module then we have the isomorphism*

$$\mathcal{hom}_{\mathbf{Mod}_A}(M, A) \boxast_{\mathbb{\Gamma}} N \cong \mathcal{hom}_{\mathbf{Comod}_{\mathbb{\Gamma}}}(M, N).$$

As the special case $M \cong A$ we get the isomorphism

$$A \boxast_{\mathbb{\Gamma}} N \cong \mathcal{hom}_{\mathbf{Comod}_{\mathbb{\Gamma}}}(A, N)$$

which is natural in N but not natural in *A! Taking derived functors of the isomorphic functors $A \boxast_{\mathbb{\Gamma}} - \cong \mathcal{hom}_{\mathbf{Comod}_{\mathbb{\Gamma}}}(A, -)$ we get the isomorphism*

$$\mathcal{Cotor}^*_{\mathbb{\Gamma}}(A, N) \cong \mathcal{Ext}^*_{\mathbf{Comod}_{\mathbb{\Gamma}}}(A, N)$$

which is again natural in N but not natural in *A!*

Proof. 1. Counitality and coassociativity follow from the counitality and coassociativity of A as a right $\mathbb{\Gamma}$-comodule.

2. Immediate.

3. Immediate corollary from the previous parts of this lemma.

4. Counitality and coassociativity follow from the counitality and coassociativity of A as a right $\mathbb{\Gamma}$-comodule.

5. Immediate.

6. Immediate corollary from the previous parts of this lemma.

□

Lemma 2.2.6. *Let S be a scheme and $(A, \mathbb{\Gamma})$ a flat commutative Hopf algebroid over S.*

1. *If M is a right $\mathbb{\Gamma}$-comodule then $M \boxast_{\mathbb{\Gamma}} -$ is exact on injectives, and takes injectives to injectives.*

2. *The category $\mathbf{Comod}_{\mathbb{\Gamma}}$ has enough injectives.*

3. *The functor $\mathbf{Comod}(\mathbb{\Gamma}) \xrightarrow{\Gamma} \mathbf{Comod}(\Gamma(\mathbb{\Gamma}))$ takes injectives to acyclics (by "acyclic" here we mean acyclic with respect to* Cotor*, i.e., $\mathscr{F} \mapsto \operatorname{Cotor}^i_{\mathbb{\Gamma}}(A, \Gamma(\mathscr{F}))$ is zero when $i > 0$).*

Proof. 1. Let $\mathscr{F}$ be an injective A-comodule and let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of A-comodules. Then we have the commutative diagram with exact rows and columns

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & M' \boxast_{\mathbb{\Gamma}} \mathscr{F} \arrow[r] \arrow[d] & M' \otimes_A \mathscr{F} \arrow[r] \arrow[d] & M' \otimes_A \mathbb{\Gamma} \otimes_A \mathscr{F} \arrow[d] \\
0 \arrow[r] & M \boxast_{\mathbb{\Gamma}} \mathscr{F} \arrow[r] \arrow[d] & M \otimes_A \mathscr{F} \arrow[r] \arrow[d] & M \otimes_A \mathbb{\Gamma} \otimes_A \mathscr{F} \arrow[d] \\
0 \arrow[r] & M'' \boxast_{\mathbb{\Gamma}} \mathscr{F} \arrow[r] & M'' \otimes_A \mathscr{F} \arrow[r] \arrow[d] & M'' \otimes_A \mathbb{\Gamma} \otimes_A \mathscr{F} \arrow[d] \\
& & 0 & 0
\end{tikzcd}$$

and by a diagram chase we have that $M \boxast_{\mathbb{\Gamma}} \mathscr{F} \to M'' \boxast_{\mathbb{\Gamma}} \mathscr{F}$ is surjective.

2. The method of constructing acyclic resolutions described in Process A.2.22 for $(\mathcal{C}, \mathfrak{A}, \tau)$-modules works in this case and in fact produces injective resolutions. In this case $(\mathcal{C}, \tau)$ is the local Zariski site on S and $\mathfrak{A}(U)$ is the category of $\mathscr{A}(U) \otimes_{\mathcal{O}_S} \mathbb{\Gamma}$-comodules, where we write $\mathscr{A}(U)$ for the sheaf of $\mathcal{O}_S$-algebras associated to the S-scheme U.

3. Let M be an A-comodule and let $\mathcal{F}$ be an injective A-comodule. Then $\mathcal{F} \circledast_{\mathbb{\Gamma}} M$ is injective so $\Gamma(\mathcal{F} \circledast_{\mathbb{\Gamma}} -) \cong \Gamma(\mathcal{F})\square_{\Gamma(\mathbb{\Gamma})}\Gamma(-)$ is exact.

$\square$

Corollary 2.2.7. Local-to-global spectral sequence for Cotor**.** *Let S be a scheme and $(A, \mathbb{\Gamma})$ a flat commutative Hopf algebroid over S, and let M be a right $\mathbb{\Gamma}$-comodule and N a left $\mathbb{\Gamma}$-comodule. We have the composable pair of left exact additive functors*

$$\mathbf{Comod}(\mathbb{\Gamma}) \xrightarrow{\Gamma} \mathbf{Comod}(\Gamma(\mathbb{\Gamma})) \xrightarrow{-\square_{\mathbb{\Gamma}} A} \mathbf{Mod}(\mathcal{O}_S)$$

and, since Γ sends injectives to acyclics, by Lemma **??**, *we have a Grothendieck spectral sequence*

$$E_2^{*,*} \cong H^*(S, \mathcal{C}otor^*_{\mathbb{\Gamma}}(A, \mathcal{F}))$$

converging to

$$E_\infty^{*,*} \cong \mathrm{Cotor}^*_{\mathbb{\Gamma}}(A, \mathcal{F}).$$

We recall that the bar construction of a rigidified stack $X \xrightarrow{P} \mathcal{X}$ (Def. 2.1.14) is a simplicial algebraic S-space whose ith object is the i-fold fiber product $X \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} X$, and whose face maps are given by

$$\begin{array}{ccc} \bigtimes_{\mathcal{X}}^{i+1} X & \xrightarrow{d_j} & \bigtimes_{\mathcal{X}}^{i} X \\ (x_0, x_1, \ldots, x_i) & \mapsto & (x_0, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_i) \end{array}$$

(we will not here use the degeneracy maps).

Definition 2.2.8. *Let S be a scheme, let τ be a Grothendieck topology on* $\mathbf{Sch}/S$, *let $X \xrightarrow{P} \mathcal{X}$ be a rigidified S-stack in the τ topology, and let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{X}$. Then we define the* Cech complex of P with coefficients in $\mathcal{F}$, *with notation $\check{C}^\bullet(X/\mathcal{X}; \mathcal{F})$ or $\check{C}^\bullet(P; \mathcal{F})$, as the sequence of abelian groups*

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{d^0} \mathcal{F}(X \times_{\mathcal{X}} X) \xrightarrow{d^1} \mathcal{F}(X \times_{\mathcal{X}} X \times_{\mathcal{X}} X) \xrightarrow{d^2} \cdots$$

where the ith term in the sequence is $\mathcal{F}(\times_{\mathcal{X}}^{i} X)$ and the ith differential d^i is given by

$$\begin{array}{rcl} \mathcal{F}(\overset{i}{\underset{\mathcal{X}}{\times}} X) & \xrightarrow{d^i} & \mathcal{F}(\overset{i+1}{\underset{\mathcal{X}}{\times}} X) \\ d^i(x) & = & \sum_{j=0}^{i}(-1)^j \mathcal{F}(d_j)(x), \end{array}$$

where the d_j are the face maps from the bar construction on P.

We define the Cech cohomology of $\mathcal{F}$ with respect to the covering P, *written $\check{H}^*(X/\mathcal{X};\mathcal{F})$ or $\check{H}^*(P;\mathcal{F})$, as the cohomology groups of the Cech complex $\check{C}^\bullet(X/\mathcal{X};\mathcal{F})$.*

We now know how to turn a faithfully flat commutative Hopf algebroid (A,Γ) over S into an S-stack in the fpqc topology with an affine fpqc epimorphism $\operatorname{Spec}_S A \xrightarrow{P} \mathcal{X}(A,\Gamma)$. The stack $\mathcal{X}(A,\Gamma)$ has affine diagonal (Process 2.1.19). When (A,Γ) is a filtered colimit of smooth sub-Hopf-algebroids (A,Γ_i) (i.e., Γ_i is smooth over A for each i, and $\bigcup_i \Gamma_i = \Gamma$) then $\operatorname{holim}_i(\operatorname{Spec}_S A \xrightarrow{P_i} \mathcal{X}(A,\Gamma_i))$ is the Ravenel S-stack $(\operatorname{Spec}_S A \xrightarrow{P} \mathcal{X}(A,\Gamma))$ in the fpqc topology.

Furthermore, given a Ravenel Hopf algebroid (A,Γ), we have an equivalence of categories between quasicoherent left Γ-comodules and Cartesian $\mathcal{O}_{\mathcal{X}(A,\Gamma)}$-modules (INSERT REF TO BEHREND); so we have isomorphisms in cohomology

$$\begin{array}{rcl} \operatorname{Cotor}_\Gamma(A,M) & \cong & \operatorname{Ext}_{\mathbf{Comod}(\Gamma)_{\mathrm{qcoh}}}(A,M) \\ & \cong & H^*_{\mathrm{fpqc}}(\mathcal{X}(A,\Gamma),\widetilde{M}) \end{array}$$

functorial in M.

Proposition 2.2.9. *Let S be a scheme and let $\mathcal{X}$ be an S-stack in the fpqc topology, and let $X \xrightarrow{P} \mathcal{X}$ be an affine fpqc epimorphism, with X a scheme affine over S. To each Cartesian $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$ we associate a quasicoherent comodule $\bar{\mathcal{F}}$ over* *the faithfully flat commutative Hopf algebroid* $(\mathcal{A}(X), \mathcal{A}(X \times_{\mathcal{X}} X))$ *over* S*; the comodule* $\bar{\mathcal{F}}$ *is the* $\mathcal{O}_X$*-module given by Prop. 2.1.22 with comodule structure map*

$$\bar{\mathcal{F}} \to \mathcal{O}_{X \times_{\mathcal{X}} X} \otimes_{\mathcal{O}_X} \bar{\mathcal{F}}$$

given by the descent datum

$$\mathcal{F}(X) \xrightarrow{\psi} \mathcal{O}_{X \times_{\mathcal{X}} X} \otimes_{\mathcal{O}_X} \mathcal{F}(X).$$

We have a covariant functor

$$\begin{array}{rcl} \mathbf{Mod}(\mathcal{O}_{\mathcal{X}})_{\mathrm{Cart}} & \to & \mathbf{Comod}((\mathcal{A}(X), \mathcal{A}(X \times_{\mathcal{X}} X)))_{\mathrm{qcoh}} \\ \mathcal{F} & \mapsto & \bar{\mathcal{F}}. \end{array}$$

Given a faithfully flat smooth commutative Hopf algebroid $(A, \Gamma) \cong (A, \cup_i \Gamma_i)$ *over* S *with* i *ranging over a filtered index category and each* (A, Γ_i) *a faithfully flat smooth commutative Hopf algebroid with* Γ_i *finitely generated as an* A*-algebra, to each quasicoherent* Γ*-comodule* M *we associate a Cartesian module* $\widetilde{M}$ *over the associated ho-pro-rigidified fpqc stack* $\mathrm{Spec}_S A \to \mathcal{X}(A, \Gamma)$ *over* S*; it is the* $\mathcal{O}_{\mathcal{X}(A,\Gamma)}$*-module associated (by Prop. 2.1.22) to the* A*-module* M *with descent datum given by the comodule structure map of* M*. We have a covariant functor*

$$\begin{array}{rcl} \mathbf{Comod}((\mathcal{A}(X), \mathcal{A}(X \times_{\mathcal{X}} X)))_{\mathrm{qcoh}} & \to & \mathbf{Mod}(\mathcal{O}_{\mathcal{X}})_{\mathrm{Cart}} \\ M & \mapsto & \widetilde{M}. \end{array}$$

We have natural isomorphisms $M \cong \bar{\widetilde{M}}$ *and* $\mathcal{F} \cong \widetilde{\bar{\mathcal{F}}}$*.*

For any quasicoherent M *and Cartesian* $\mathcal{F}$ *as above, we can consider* $\Gamma(\mathcal{F}, \mathcal{X} \times_S -)$ *as a functor taking values on the Zariski opens of* S*, and we will write* $\mathcal{H}^0(\mathcal{X}; \mathcal{F})$ *for the resulting Zariski* $\mathcal{O}_S$*-module. Then we have isomorphisms of* $\Gamma(\mathcal{O}_S)$*-modules*

$$\begin{array}{rcl} \mathcal{H}^0(\mathcal{X}; \mathcal{F}) & \cong & \mathcal{A}(X) \boxast_{\mathcal{A}(X \times_{\mathcal{X}} X)} \bar{\mathcal{F}}, \\ A \boxast_{\Gamma} M & \cong & \mathcal{H}^0(\mathcal{X}; \widetilde{M}), \\ \Gamma(\mathcal{F}) & \cong & \mathcal{A}(X) \square_{\mathcal{A}(X \times_{\mathcal{X}} X)} \bar{\mathcal{F}}, \\ A \square_{\Gamma} M & \cong & \Gamma(\widetilde{M}) \end{array}$$

natural in $\mathcal{F}$ and M (we remind the reader that our notation $\Gamma(\mathcal{F})$ means global sections of $\mathcal{F}$ as a $\Gamma(\mathcal{O}_X)$-module, i.e., $\Gamma(\mathcal{F}) \cong \mathcal{F}(X)$), arising from the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H}^0(\mathcal{X}, \mathcal{F}) \arrow[r] \arrow[d] & \mathcal{H}^0(X, \mathcal{F}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & \mathcal{H}^0(X \times_{\mathcal{X}} X; \mathcal{F}) \arrow[d, "\cong"] \\
0 \arrow[r] & \bar{\mathcal{F}} \boxast_{\mathcal{A}(X \times_{\mathcal{X}} X)} \mathcal{A}(X) \arrow[r] & \bar{\mathcal{F}} \arrow[r, shift left] \arrow[r, shift right] & \mathcal{A}(X \times_{\mathcal{X}} X) \otimes_{\mathcal{A}(X)} \bar{\mathcal{F}}
\end{tikzcd}$$

with rows equalizer sequences, forcing the leftmost vertical map to be an isomorphism (the rightmost vertical map is a quasicoherence isomorphism).

As a result we have that the category of quasicoherent comodules over a faithfully flat commutative Hopf algebroid satisfying the above colimit property is equivalent to the category of Cartesian modules over its associated ho-pro-rigidified fpqc algebraic stack, and vice versa; and we get isomorphisms in flat cohomology (a priori these are isomorphisms in quasicoherent/Cartesian flat cohomology, i.e., the derived functors of global sections/cotensor product on the category of Cartesian sheaves/quasicoherent comodules on the flat site; but from Cor. A.2.25 we know that this is isomorphic to flat cohomology)

$$\begin{aligned}
H^*(\mathcal{X}_{\mathrm{fl}}; \mathcal{F}) &\cong \mathrm{Cotor}^*_{\mathcal{A}(X \times_{\mathcal{X}} X)}(\mathcal{A}(X), \bar{\mathcal{F}}), \\
H^*(\mathcal{X}(A,\Gamma)_{\mathrm{fl}}; \widetilde{M}) &\cong \mathrm{Cotor}^*_{\Gamma}(A, M), \\
\mathcal{H}^*(\mathcal{X}_{fl}; \mathcal{F}) &\cong \mathcal{C}otor^*_{\mathcal{A}(X \times_{\mathcal{X}} X)}(\mathcal{A}(X), \bar{\mathcal{F}}), \\
\mathcal{H}^*(\mathcal{X}(A,\Gamma)_{\mathrm{fl}}; \widetilde{M}) &\cong \mathcal{C}otor^*_{\Gamma}(A, M).
\end{aligned}$$

Now by the composable pair of left exact additive functors

$$\mathrm{bb} - \mathbf{Mod}(O_{\mathcal{X}}) \xrightarrow{\mathcal{H}^0} \mathbf{Mod}(\mathcal{O}_S) \xrightarrow{\Gamma} \mathbf{Mod}(\Gamma(\mathcal{O}_S)$$

we have a Grothendieck spectral sequence

$$E_2^{*,*} \cong H^*(S; \mathcal{H}^*(\mathcal{X}; \mathcal{F})) \Rightarrow E_\infty^{*,*} \cong H^*(\mathcal{X}, \mathcal{F})$$

which is naturally isomorphic, by our equivalence of modules over a stack and comodules over a Hopf algebroid, to the local-to-global Cotor spectral sequence of Prop. 2.2.7.

Since X is affine over S and $\mathcal{F}$ is quasicoherent (over X) we have that the Zariski cohomology $\mathcal{O}_S$-module $\mathcal{H}^i_{\mathrm{Zar}}(X;\mathcal{F}) \cong 0$ when $i > 0$. The change-of-topologies spectral sequence $\mathcal{H}^*_{\mathrm{Zar}}(X; R^*f_*W(\mathcal{F})) \Rightarrow \mathcal{H}^*_{\mathrm{fl}}(X; W(\mathcal{F}))$ collapses in this case since f_*, the map forgetting the data of a sheaf on the non-Zariski fpqc opens, is exact (Prop. III.3.7 of [Milne, 1980]). So $\mathcal{H}^i_{\mathrm{fl}}(X;\mathcal{F}) \cong 0$ for $i > 0$.

Since the stack $\mathcal{X}$ is of affine descent class, the cover $X \to \mathcal{X}$ is an affine morphism (this is the argument from Prop. I.5.9 of [Knutson, 1971]) and as a result so are all the products $X \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} X$ appearing in the bar construction of $X/\mathcal{X}$. As a result, the $\mathcal{O}_S$-modules $\mathcal{H}^i(\check{C}^j(X/\mathcal{X};\mathcal{F}))$ are trivial for $i > 0$ and the Cech complex of sheaves *computes flat cohomology:*

$$\check{\mathcal{H}}^*(X/\mathcal{X};\mathcal{F}) \cong \mathcal{H}^*(\mathcal{X}_{\mathrm{fl}};\mathcal{F}).$$

2.3 Tensoring up.

We need to develop a theory of "tensoring up" for commutative Hopf algebroids and, eventually, for graded commutative Hopf algebroids; or equivalently, a theory of base change for stacks. For moduli problems which are solvable with ordinary commutative rings, or graded commutative rings, tensoring up is a straightforward affair; see Prop. B.1.10. Now, we can define "tensoring up" for commutative Hopf algebroids in a way which is analogous to this (very classical—[Grothendieck, 1960]) kind, in which we begin with a commutative Hopf algebroid defined over S and then want to get one defined over S', *or* we can define the tensor product of a commutative Hopf algebroid (A, Γ) with an A-algebra B, giving us an algebraic object $(B, B \otimes_A \Gamma)$ which is still defined over commutative R-algebras, which requires certain additional structure on B, *or* we can "tensor up on both sides" and form a commutative Hopf algebroid $(B, B \otimes_A \Gamma \otimes_A B)$, which requires no additional structure on B. In the second case we must be careful of some technical difficulties which have become manifest among topologists as a general unease about tensoring up commutative Hopf algebroids. This unease is at least partly unfounded, however, because once the three kinds of "tensoring up" have been described, and basic properties have been proven about them, we will have no great difficulty using them. We will give all three constructions in the following propositions.

Proposition 2.3.1. *Let (A, Γ) be a bialgebroid (resp. Hopf algebroid, graded bialgebroid, graded Hopf algebroid) over a scheme S, and let $S \xrightarrow{f} S'$ be a morphism of commutative rings. Then $(f_*A, f_*\Gamma)$ is a bialgebroid (resp. Hopf algebroid, graded bialgebroid, graded Hopf algebroid) over S', and there is an isomorphism of small categories (resp. groupoids, small categories, groupoids)*

$$\hom_{\mathbf{GrAlg}(\mathcal{O}_{S'})}((f_*A, f_*\Gamma), X) \cong \hom_{\mathbf{GrAlg}(\mathcal{O}_S)}((A, \Gamma), f^*X)$$

which is natural in X.

Proof. From Prop. B.1.5 we know that $\hom_{\mathbf{GrAlg}(\mathcal{O}_{S'})}(f_*A, X) \cong \hom_{\mathbf{GrAlg}(\mathcal{O}_S)}(A, f^*X)$ and $\hom_{\mathbf{GrAlg}(\mathcal{O}_{S'})}(f_*\Gamma, X) \cong \hom_{\mathbf{GrAlg}(\mathcal{O}_S)}(\Gamma, f^*X)$, i.e., the small category (resp. groupoid, small category, groupoid) of graded $\mathcal{O}_{S'}$-algebra maps from $(f_*A, f_*\Gamma)$ to X has object set and morphism set both isomorphic to those of the small category (resp. groupoid, small category, groupoid) of graded $\mathcal{O}_S$-algebra maps from (A, Γ) to f^*X, and the naturality of the isomorphism $\hom_{\mathbf{GrAlg}(\mathcal{O}_{S'})}(f_*A, X) \cong \hom_{\mathbf{GrAlg}(\mathcal{O}_S)}(A, f^*X)$ in A gives that the isomorphisms of object and morphism sets commute with the rest of the structure maps of the two small categories (resp. groupoids, small categories, groupoids), so the two small categories (resp. groupoids, small categories, groupoids) are isomorphic; finally, the naturality in

X of the isomorphism from Prop. B.1.5 gives that the isomorphism of small categories (resp. groupoids, small categories, groupoids) is natural in X. $\square$

Remark 2.3.2. Given a scheme-morphism $S \xrightarrow{f} S'$ and a flat commutative Hopf algebroid (A, Γ) over S which is a filtered colimit of flat smooth commutative Hopf algebroids over S, we have the associated Ravenel stack $\mathrm{Spec}_S\, A \xrightarrow{P} \mathcal{X}_{(A,\Gamma)}$ and we may ask what its relation to the associated Ravenel stack of $(f_*A, f_*\Gamma)$; the answer is that the associated Ravenel stack of $(f_*A, f_*\Gamma)$ is $(\mathrm{Spec}_S\, A) \times_S S' \xrightarrow{P \times_S S'} \mathcal{X}_{(A,\Gamma)} \times_S S'$. We see this immediately from, for instance, considering what groupoid-valued functor is represented by the associated Ravenel stack of $(f_*A, f_*\Gamma)$.

If R is a commutative ring, (A, Γ) a bialgebroid or a Hopf algebroid over R, and $A \xrightarrow{f} B$ is a morphism of commutative R-algebras, then we cannot immediately make a bialgebroid $(B, B \otimes_A \Gamma)$ or $(B, \Gamma \otimes_A B)$; in the first case (tensoring up on the left), we fail to get a well-defined right unit map η_R, and in the second case (tensoring up on the right), we fail to get a well-defined η_L. We provide an illustrative example: for our Hopf algebroid we consider $(\mathbb{F}_p[x], \mathbb{F}_p[y, z])$, with $\eta_L(x) = y$ and $\eta_R(x) = z$ (these are the only structure maps we will use right now, but we can fill in the rest of the structure of the Hopf algebroid with maps $\epsilon(y) = \epsilon(z) = x$ and $\chi(y) = z$, $\chi(z) = y$, and $\Delta(y) = y \otimes 1 + 1 \otimes y$, $\Delta(z) = z \otimes 1 + 1 \otimes z$) as a Hopf algebroid over $\mathbb{F}_p$, and we will try to tensor up on the left over the morphism $\mathbb{F}_p[x] \to \mathbb{F}_p[x]/(x^p)$. We get a pair of commutative $\mathbb{F}_p$-algebras $(\mathbb{F}_p[x]/(x^p), \mathbb{F}_p[x]/(x^p) \otimes_{\mathbb{F}_p[x]} \mathbb{F}_p[y, z]) \cong (\mathbb{F}_p[x]/(x^p), \mathbb{F}_p[y, z]/(y^p))$ and a left unit map $\eta_L(x) = y$ which is well-defined, but the right unit map which one would like to define by $\eta_R(x) = z$ fails to be well-defined, as $0 = \eta_R(0) = \eta_R(x^p) \neq \eta_R(x)^p = z^p$. In general, when one has a bialgebroid (A, Γ) over R and a commutative R-algebra map $A \to B$, one uses the left unit on (A, Γ) to define the left unit $B \to B \otimes_A \Gamma$, making the following diagram 2.3.6 commute, but one does not get a commutative R-algebra map $B \to B \otimes_A \Gamma$ making the following diagram 2.3.7 commute:

$$\begin{tikzcd} A \arrow[r, "\eta_L"] \arrow[d, "f"] & \mathbb{\Gamma} \arrow[d] \\ B \arrow[r] & B \otimes_A \mathbb{\Gamma} \end{tikzcd} \tag{2.3.6}$$

$$\begin{tikzcd} A \arrow[r, "\eta_R"] \arrow[d, "f"] & \mathbb{\Gamma} \arrow[d] \\ B \arrow[r] & B \otimes_A \mathbb{\Gamma} \end{tikzcd} \tag{2.3.7}$$

We want to get around this problem and define a left tensoring-up of this kind, so we can ask: what additional structure can we require on B which will give us a suitable right unit map on $(B, B \otimes_A \mathbb{\Gamma})$? To answer this question, as well as for other important purposes, we return to comodules.

Remark 2.3.3. What is a comodule algebra in more categorical language? Let $(A, \mathbb{\Gamma})$ be a bialgebra over a commutative ring R, and let B be a $\mathbb{\Gamma}$-comodule. We adopt the language that, given some commutative R-algebra S, the objects in the set $\hom_{\mathbf{CAlg}}(A, S)$ are "objects classified by A," or "A-objects" for short, because in practice our bialgebras will generally be solutions to moduli problems, so they really are "classifying" something. The map $A \xrightarrow{f} B$ gives us a map $\hom_{\mathbf{CAlg}(R)}(B, -) \xrightarrow{f^*} \hom_{\mathbf{CAlg}(R)}(A, -)$ of functors; the comodule algebra structure map $B \xrightarrow{\phi} B \otimes_A \mathbb{\Gamma}$ tells us that, given an object X classified by B, and a morphism (classified by $\mathbb{\Gamma}$) from $f^*(X)$ to an object Y classified by A, we have a unique object classified by B. If the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\eta_R"] \arrow[d, "f"] & \mathbb{\Gamma} \arrow[d, "f \otimes_A \mathrm{id}_{\mathbb{\Gamma}}"] \\ B \arrow[r, "\psi"] & B \otimes_A \mathbb{\Gamma} \end{tikzcd}$$

then the object classified by B is in $(f^*)^{-1}(Y)$. In other words, a comodule algebra map ψ making the above diagram commute is a kind of descent datum, since, given an A-object X and an A-morphism to it from the "underlying" A-object of some B-object, we can "descend" X to a B-object. This is dual to the usual descent procedure; in general this is a process which is best performed in a setting where everything is of the opposite variance, i.e., a geometric setting.

On the level of the ho-pro-rigidified stack $\operatorname{Spec}_S A \to \mathcal{X}(A, \Gamma)$ associated to (A, Γ), specifying a comodule algebra structure map $B \to B \otimes_A \Gamma$ satisfying property 3 is equivalent to specifying a map filling in the dotted line in the 2-commutative diagram below:

$$\begin{tikzcd}
 & \operatorname{Spec}_S B \arrow[dr] & \\
\operatorname{Spec}_S B \times_{\operatorname{Spec}_S A} \operatorname{Spec}_S \Gamma \arrow[ur, dashed] \arrow[d] \arrow[dr] & & \operatorname{Spec}_S A \arrow[d] \\
\operatorname{Spec}_S B \arrow[dr] & \operatorname{Spec}_S \Gamma \arrow[ur] \arrow[d] & \mathcal{X} \\
 & \operatorname{Spec}_S A \arrow[ur] &
\end{tikzcd}$$

One notices that the above diagram is a cube missing a vertex. Given a comodule algebra structure map filling in the above dotted line, we may use (a homotopical or 2-categorical) descent to produce a stack $\mathcal{Y}$ filling in the missing vertex, equipped with an fpqc presentation by $\operatorname{Spec}_S B$; when B is faithfully flat as an A-algebra then $\mathcal{Y}$ is faithfully flat and affine over $\mathcal{X}$. However, many of the most interesting cases are those in which B is not flat as an A-algebra, and in those circumstances we are performing descent over a weaker (for instance, affine) descent class than the fpqc descent class; and we will get that $\mathcal{Y} \to \mathcal{X}$ is affine but not necessarily flat. See Cor. 3.1.33, for example.

Lemma 2.3.4. *If S is a scheme and the Hopf algebroid (A, Γ) is a commutative Hopf algebra over S, i.e., $A \cong \mathcal{O}_S$, then all comodule algebras over (A, Γ) satisfy the descent condition.*

Proof. The diagram defining the descent condition becomes trivial when $\mathcal{O}_S \cong A$. □

Definition 2.3.5. *Let S be a scheme and let (A, Γ) be a graded bialgebroid over S. Then we let* $\mathbf{GrMod}_A$ *be the category of graded right A-modules, we let*

$\mathbf{GrComod}_{\mathbb{\Gamma}}$ *be the category of graded right* $\mathbb{\Gamma}$*-comodules, and let* $\mathbf{GrComodAlg}_{\mathbb{\Gamma}}$ *be the category of graded right* $\mathbb{\Gamma}$*-comodule algebras. We will let* `forget` *denote the forgetful functor from* $\mathbf{GrComod}_{\mathbb{\Gamma}}$ *to* $\mathbf{GrMod}_A$*. We will use the same notation for the forgetful functor from* $\mathbf{GrComodAlg}_{\mathbb{\Gamma}}$ *to* $\mathbf{GrAlg}(A)$*. We define a functor* $\texttt{extend} : \mathbf{GrMod}_A \to \mathbf{GrComod}_{\mathbb{\Gamma}}$ *by letting* $\texttt{extend}(M) = M \otimes_A \mathbb{\Gamma}$ *with* $\mathbb{\Gamma}$*-comodule structure map*

$$M \otimes_A \mathbb{\Gamma} \xrightarrow{\mathrm{id}_M \otimes_A \Delta} M \otimes_A \mathbb{\Gamma} \otimes_A \mathbb{\Gamma}.$$

We note that if we apply `extend` *to a graded commutative* A*-algebra, then we get a graded right* $\mathbb{\Gamma}$*-comodule algebra, i.e., we also have a functor (which we use the same notation for)* $\texttt{extend} : \mathbf{GrAlg}(A) \to \mathbf{GrComodAlg}_{\mathbb{\Gamma}}$*. (A priori, we do not know that* `extend` *actually takes values in* $\mathbf{GrComod}_{\mathbb{\Gamma}}$ *or* $\mathbf{GrComodAlg}_{\mathbb{\Gamma}}$*, but we prove this below, in Prop. 1 and Prop. 2.)*

The right $\mathbb{\Gamma}$*-comodule (resp. right* $\mathbb{\Gamma}$*-comodule algebra)* $\texttt{extend}(M)$ *will be referred to as the "extended right* $\mathbb{\Gamma}$*-comodule on* M*" (resp. "extended right* $\mathbb{\Gamma}$*-comodule algebra on* M*").*

All of these constructions can equally well be made for left modules, left comodules, left algebras, and left comodule algebras, in a completely parallel way.

Proposition 2.3.6. 1. *Given a right* A*-module* M*, the object* $\texttt{extend}(M)$ *is in fact a right* $\mathbb{\Gamma}$*-comodule.*

2. *Given a commutative* A*-algebra* M*, the object* $\texttt{extend}(M)$ *is in fact a right* $\mathbb{\Gamma}$*-comodule algebra.*

3. *The functor* $\texttt{extend} : \mathbf{GrMod}_A \to \mathbf{GrComod}_{\mathbb{\Gamma}}$ *is right adjoint to the functor* $\texttt{forget} : \mathbf{GrComod}_{\mathbb{\Gamma}} \to \mathbf{GrMod}_A$*.*

4. *The functor* $\texttt{extend} : \mathbf{GrAlg}(A) \to \mathbf{GComodAlg}_{\mathbb{\Gamma}}$ *is right adjoint to the functor* `forget` : $\mathbf{GrComodAlg}_{\mathbb{\Gamma}}$ *to* $\mathbf{GrAlg}(A)$*.*

Proof. 1. We need to show that properties 1 and 1 hold. Inspection of the appropriate diagrams shows us that this follows immediately from (A, Γ) satisfying properties 3 and 4.

2. We must simply show that the map $\Gamma \otimes_A M \to \Gamma \otimes_A \Gamma \otimes_A M$ is an A-algebra morphism; this follows from its definition as a tensor product of A-algebra morphisms.

3. See Prop. 2.3.18.

4. See Prop. 2.3.18.

$\square$

As it is standard to call a functor "free" if it is left adjoint to a forgetful functor, we will sometimes refer to `extend`(M) as a "cofree" comodule or "cofree" comodule algebra.

Definition 2.3.7. *Let S be a scheme and let $(A, \Gamma) \xrightarrow{(f_1, f_2)} (B, \Sigma)$ be a morphism of bialgebroid over S. Then we define the right Σ-comodule $\Gamma \otimes_A B$ by the structure map*

$$\Gamma \otimes_A B \xrightarrow{\Delta} \Gamma \otimes_A \Gamma \xrightarrow{\mathrm{id}_\Gamma \otimes_A f_2} \Gamma \otimes_A \Sigma \xrightarrow{\cong} (\Gamma \otimes_A B) \otimes_B \Sigma.$$

A completely parallel construction makes $B \otimes_A \Gamma$ into a left Σ-comodule. A priori, we do not know that these are actually comodules, but we prove this below. (These comodules will play a very significant role in the change-of-rings spectral sequence, and its applications.)

Lemma 2.3.8. *$\Gamma \otimes_A B$, with structure map as above, is a right Σ-comodule.*

Proof. On inspection of the diagram for property 1, it follows from property **??** of (A, Γ), and property 1 follows from property 4 of (A, Γ) and (B, Σ) and the fact that f_2 commutes with coproducts. $\square$

Proposition 2.3.9. *Let (A, Γ) be a bialgebroid over a commutative ring R, and let B be a right (resp. left) Γ-comodule algebra, such that the following diagram commutes:*

$$\begin{tikzcd} A \arrow[r, "\eta_R"] \arrow[d, "f"] & \Gamma \arrow[d, "f\otimes_A \mathrm{id}_\Gamma"] \\ B \arrow[r, "\psi"] & B \otimes_A \Gamma \end{tikzcd} \tag{2.3.8}$$

$$\left(\text{resp. } \begin{tikzcd} A \arrow[r, "\eta_L"] \arrow[d, "f"] & \Gamma \arrow[d, "\mathrm{id}_\Gamma \otimes_A f"] \\ B \arrow[r, "\psi"] & \Gamma \otimes_A B \end{tikzcd} \right), \tag{2.3.9}$$

where f is the A-algebra structure map $A \xrightarrow{f} B$. Then the algebraic object given by the pair $(B, B \otimes_A \Gamma)$ (resp. $(B, \Gamma \otimes_A B)$), with its right unit $B \to B \otimes_A \Gamma$ (resp. left unit $B \to \Gamma \otimes_A B$)) equal to the comodule structure map on B, is a bialgebroid over R. If (A, Γ) is a Hopf algebroid, then so is $(B, B \otimes_A \Gamma)$ (resp. $(B, \Gamma \otimes_A B)$); if (A, Γ) is a graded bialgebroid and B is a graded right (resp. left) Γ-comodule, then so is $(B, B \otimes_A \Gamma)$ (resp. $(B, \Gamma \otimes_A B)$); if (A, Γ) is a graded Hopf algebroid and B is a graded right (resp. left) Γ-comodule, then so is $(B, B \otimes_A \Gamma)$ (resp. $(B, \Gamma \otimes_A B)$).

As a functor from commutative $\mathcal{O}_S$-algebras to groupoids, the value taken by the bialgebroid $(B, B \otimes_A \Gamma)$ on a commutative $\mathcal{O}_S$-algebra T has object set $\hom_{\mathbf{Alg}(\mathcal{O}_S)}(B, T)$ (the "T-objects") and the morphisms in this groupoid are the A-morphisms from the underlying A-object of a B-object to the B-object which we get by "descending" the target of the A-morphism to a B-object over the A-morphism, using the structure map of the comodule algebra B.

Proof. Note that the condition on the comodule algebra B guarantees that ψ "extends" η_R (resp. η_L) in a way that allows us to define a coproduct on $(B, B \otimes_A$

Γ) (resp. $(B, \Gamma \otimes_A B)$), as we need an isomorphism $B \otimes_A \Gamma \otimes_A \Gamma \cong (B \otimes_A \Gamma) \otimes_B (B \otimes_A \Gamma)$ (resp. $\Gamma \otimes_A \Gamma \otimes_A B \cong (\Gamma \otimes_A B) \otimes_B (\Gamma \otimes_A B)$).

First, we make explicit the structure maps on $(B, B \otimes_A \Gamma)$ (we note that, throughout this proof, we will consistently use the symbols η_L, η_R, ϵ, and Δ (and χ if (A, Γ) is a Hopf algebroid) to denote the structure maps on (A, Γ), and $\psi : B \to B \otimes_A \Gamma$ to denote the comodule structure map of B):

$$\text{augmentation: } B \otimes_A \Gamma \xrightarrow{\mathrm{id}_B \otimes_A \epsilon} B$$

$$\text{left unit: } B \xrightarrow{\mathrm{id}_B \otimes_A \eta_L} B \otimes_A \Gamma$$

$$\text{right unit: } B \xrightarrow{\psi} B \otimes_A \Gamma$$

$$\text{coproduct: } B \otimes_A \Gamma \xrightarrow{\mathrm{id}_B \otimes_A \Delta} B \otimes_A \Gamma \otimes_A \Gamma \xrightarrow{\cong} (B \otimes_A \Gamma) \otimes_B (B \otimes_A \Gamma)$$

$$\text{conjugation, when } (A, \Gamma) \text{ is a Hopf algebroid: } B \otimes_A \Gamma \xrightarrow{\mathrm{id}_B \otimes_A \chi} B \otimes_A \Gamma.$$

We now show that these structure maps satisfy the axioms from Prop.2.1.2. First we show that the coproduct on $(B, B \otimes_A \Gamma)$ is a left B-module morphism, i.e., that this diagram commutes:

$$\begin{tikzcd}
B \arrow[r, "\cong"] \arrow[d, "\mathrm{id}_B \otimes_A \eta_L"'] & B \otimes_A A \otimes_A A \arrow[rr, "\mathrm{id}_B \otimes_A \eta_L \otimes_A \eta_L"] & & B \otimes_A \Gamma \otimes_A \Gamma \arrow[d, "\cong"] \\
B \otimes_A \Gamma \arrow[r, "\mathrm{id}_B \otimes_A \Delta"] & B \otimes_A \Gamma \otimes_A \Gamma \arrow[rr, "\cong"] & & (B \otimes_A \Gamma) \otimes_B (B \otimes_A \Gamma),
\end{tikzcd}$$

whose commutativity follows from Δ being a left A-module morphism.

We now check that the coproduct on $(B, B \otimes_A \Gamma)$ is also a right B-module morphism:

$$\begin{tikzcd}
B \arrow[rr, "\cong"] \arrow[d, "\psi"'] & & B \otimes_B B \arrow[d, "\psi \otimes_B \psi"] \\
B \otimes_A \Gamma \arrow[r, "\mathrm{id}_B \otimes_A \Delta"] & B \otimes_A \Gamma \otimes_A \Gamma \arrow[r, "\cong"] & (B \otimes_A \Gamma) \otimes_B (B \otimes_A \Gamma),
\end{tikzcd}$$

whose commutativity follows from $(\mathrm{id}_B \otimes_A \Delta) \circ \psi = (\psi \otimes_A \mathrm{id}_\Gamma) \otimes_A \psi$, one of the axioms for B being a Γ-comodule.

We now check that the augmentation on $(B, B \otimes_A \Gamma)$ is a left B-module morphism, i.e., $\mathrm{id}_B = (\mathrm{id}_B \otimes_A \epsilon) \circ (\mathrm{id}_B \otimes_A \eta_L)$, which follows immediately from $\mathrm{id}_A = \epsilon \circ \eta_L$; and we check that the augmentation on $(B, B \otimes_A \Gamma)$ is a right B-module morphism, i.e., $(\mathrm{id}_B \otimes_A \epsilon) \circ \psi = \mathrm{id}_B$, which is precisely the other axiom for B being a Γ-comodule.

We now check property 3 of Prop. 2.1.2, i.e., the commutativity of the following diagram:

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}_B \otimes_A \Delta"] \arrow[d, "\mathrm{id}_B \otimes_A \Delta"] & B \otimes_A \Gamma \otimes_A \Gamma \arrow[d, "\mathrm{id}_B \otimes_A \mathrm{id}_\Gamma \otimes_A \epsilon"] \\ B \otimes_A \Gamma \otimes_A \Gamma \arrow[r, "\mathrm{id}_B \otimes_A \epsilon \otimes_A \mathrm{id}_\Gamma"] & B \otimes_A \Gamma, \end{tikzcd}$$

which follows from property 3 being satisfied by (A, Γ).

The last property we need to verify is the commutativity of the diagram:

$$\begin{tikzcd} B \otimes_A \Gamma \arrow[r, "\mathrm{id}_B \otimes_A \Delta"] \arrow[d, "\mathrm{id}_B \otimes_A \Delta"] & B \otimes_A \Gamma \otimes_A \Gamma \arrow[d, "\mathrm{id}_B \otimes_A \Delta \otimes_A \mathrm{id}_\Gamma"] \\ B \otimes_A \Gamma \otimes_A \Gamma \arrow[r, "\mathrm{id}_B \otimes_A \mathrm{id}_\Gamma \otimes_A \Delta"] & B \otimes_A \Gamma \otimes_A \Gamma \otimes_A \Gamma, \end{tikzcd}$$

which again follows immediately from the analogous property for (A, Γ).

In the graded cases, it is very easy to check by inspection of the above structure maps and diagrams that, since ψ is graded and all structure maps of (A, Γ) are graded maps, $(B, B \otimes_A \Gamma)$ and its structure maps are graded.

This proof has been put in terms of a right Γ-comodule algebra and $(B, B \otimes_A \Gamma)$ but the same methods work with obvious minor changes to give the stated result in terms of a left Γ-comodule algebra and $(B, \Gamma \otimes_A B)$. $\square$

Proposition 2.3.10. *Let (A, Γ) be a commutative Hopf algebroid over a scheme S, and let B be a commutative A-algebra. Suppose we have a commutative Hopf algebroid $(B, \Gamma \otimes_A B)$ (resp. $(B, B \otimes_A \Gamma)$); then the A-algebra structure map $A \to B$ on B, together with the map $\Gamma \otimes_A A \to \Gamma \otimes_A B$ that we get by tensoring it up over A with Γ, gives us a morphism of commutative Hopf algebroids $(A, \Gamma) \to (B, \Gamma \otimes_A B)$*

(resp. $(A, \Gamma) \to (B, B \otimes_A \Gamma)$) if and only if B is a left (resp. right) Γ-comodule algebra satisfying property 3 (resp. property 3).

Proof. Suppose that B is a left Γ-comodule algebra; then 3 guarantees, by Prop.2.3.9, that $(B, \Gamma \otimes_A B)$ is a commutative Hopf algebroid; checking that the above map is a map of commutative Hopf algebroids is trivial except for when we check that the left units commute with the map $(A, \Gamma) \to (B, \Gamma \otimes_A B)$, and this is equivalent to property 3.

Now suppose that $(A, \Gamma) \to (B, \Gamma \otimes_A B)$, given as above, is a morphism of commutative Hopf algebroids; then it is easy to check that the left unit on $(B, \Gamma \otimes_A B)$ is a left Γ-comodule algebra structure map on B, satisfying property 3. $\square$

Remark 2.3.11. The associated pro-rigidified fpqc stack of $(B, B \otimes_A \Gamma)$ is the $\mathrm{Spec}_S B \to \mathcal{Y}$ of Remark 2.3.3; among other things this implies that $\mathrm{Spec}_S B \times_{\mathcal{Y}} \mathrm{Spec}_S B \simeq \mathrm{Spec}_S(B \otimes_A \Gamma)$.

For commutative $\mathcal{O}_S$-algebra morphisms $A \to B$ which are surjective, we can be even more explicit about when we can tensor up on one side to form a Hopf algebroid $(B, B \otimes_A \Gamma)$ or $(B, \Gamma \otimes_A B)$:

Remark 2.3.12. Thinking of a Hopf algebroid (A, Γ) over a commutative ring R as solving some moduli problem, i.e., classifying "A-objects" and "A-isomorphisms" between them, if I is an $\mathcal{O}_S$-ideal sheaf of the $\mathcal{O}_S$-algebra A and we can form the Hopf algebroid $(A/I, \Gamma \otimes_A (A/I))$, what does $(A/I, \Gamma \otimes_A (A/I))$ classify? Let X be some commutative $\mathcal{O}_S$-algebra, and recall from Lemma **??** that A/I-objects over R form a subset of the A-objects over S; on examination of diagram 3 we see that A/I can be made into a left Γ-comodule algebra extending η_L if and only if any A-object which is A-isomorphic to an A/I-object is itself an A/I-object; in other words, when such a comodule algebra structure exists on A/I, the R-algebras $\Gamma \otimes_A (A/I)$ and $(A/I) \otimes_A \Gamma \otimes_A (A/I)$ corepresent the same set-valued functor on commutative

$\mathcal{O}_S$-algebras, so they are isomorphic as commutative $\mathcal{O}_S$-algebras; and we have isomorphisms $(A/I, (A/I) \otimes_A \Gamma) \cong (A/I, \Gamma \otimes_A (A/I)) \cong (A/I, (A/I) \otimes_A \Gamma \otimes_A (A/I))$ of Hopf algebroids over S.

In other words, an ideal sheaf I of A, regarded as a right (resp. left) Γ-comodule by the map η_L (resp. η_R), satisfies property 3 (resp. property 3) exactly when it is an invariant ideal of (A, Γ)—which we prove in the next proposition.

Proposition 2.3.13. *Let (A, Γ) be a commutative Hopf algebroid over a scheme S, and let I be an $\mathcal{O}_S$-ideal sheaf of A. Then the following conditions are equivalent:*

1. *I is an invariant ideal sheaf of A.*
2. *$(A/I, \Gamma/(\eta_L(I)))$ is a commutative Hopf algebroid over S and the obvious quotient maps $A \to A/I$ and $\Gamma \to \Gamma/(\eta_L(I))$ form a morphism of commutative Hopf algebroids $(A, \Gamma) \to (A/I, \Gamma/(\eta_L(I)))$.*
3. *$(A/I, \Gamma/(\eta_R(I)))$ is a commutative Hopf algebroid over S and the obvious quotient maps $A \to A/I$ and $\Gamma \to \Gamma/(\eta_R(I))$ form a morphism of commutative Hopf algebroids $(A, \Gamma) \to (A/I, \Gamma/(\eta_R(I)))$.*
4. *A/I admits a left Γ-comodule algebra structure map ψ extending η_L, as in 3.*
5. *A/I admits a right Γ-comodule algebra structure map ψ extending η_R, as in 3.*

Proof. 1. **Condition 4 implies condition 1.** If A/I is a left Γ-comodule algebra extending η_L, then by the Hopf algebroid isomorphisms in Remark 2.3.12 we have $\Gamma/(\eta_L(I)) = \Gamma/(\eta_R(I)) = (\Gamma/(\eta_R(I)))/(\eta_L(I)) = (\Gamma/(\eta_L(I)))/(\eta_R(I))$, i.e., $(\eta_L(I)) = (\eta_R(I))$, so by Prop. **??**, I is invariant.

2. **Conditions 2, 3, 4, and 5 are equivalent.** Clear from Prop. 2.3.9 and Prop. 2.3.10.

3. **Condition 1 implies condition 4.** We have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\eta_L"] \arrow[d] & \Gamma \arrow[d] \\ A/I \arrow[r, "\psi"] & \Gamma/(\eta_L(I)) \end{tikzcd}$$

since $\Gamma \otimes_A (A/I) \cong \Gamma/(\eta_R(I)) \cong \Gamma/(\eta_L(I))$, so ϕ extends η_L. That this choice of ψ fulfills 1 and 1 is left to the reader.

$\square$

Finally, we consider the tensor product on both sides:

Proposition 2.3.14. *Let (A, Γ) be a commutative bialgebroid over a scheme S, and let $A \xrightarrow{f} B$ be a morphism of commutative $\mathcal{O}_S$-algebras. Then the pair of R-algebras $(B, B \otimes_A \Gamma \otimes_A B)$ with structure maps*

$$\textit{augmentation: } B \otimes_A \Gamma \otimes_A B \xrightarrow{\mathrm{id}_B \otimes_A \epsilon \otimes_A \mathrm{id}_B} B \otimes_A B \xrightarrow{\nabla} B$$

$$\textit{left unit: } B \xrightarrow{\cong} B \otimes_A A \otimes_{\mathcal{O}_S} \mathcal{O}_S \xrightarrow{\mathrm{id}_B \otimes_A \eta_L \otimes_{s_A} \mathrm{id}_{s_B}} B \otimes_A \Gamma \otimes_A B$$

$$\textit{right unit: } B \xrightarrow{\cong} \mathcal{O}_S \otimes_{\mathcal{O}_S} A \otimes_A B \xrightarrow{\mathrm{id}_{s_B} \otimes_{s_A} \eta_R \otimes_A \mathrm{id}_B} B \otimes_A \Gamma \otimes_A B$$

$$\begin{aligned} \textit{coproduct: } \quad & B \otimes_A \Gamma \otimes_A B \xrightarrow{\mathrm{id}_B \otimes_A \Delta \otimes_A \mathrm{id}_B} B \otimes_A \Gamma \otimes_A \Gamma \otimes_A B \\ & \xrightarrow{\cong} B \otimes_A \Gamma \otimes_A A \otimes_A \Gamma \otimes_A B \xrightarrow{\mathrm{id}_B \otimes_A \mathrm{id}_\Gamma \otimes_A f \otimes_A \mathrm{id}_\Gamma \otimes_A \mathrm{id}_B} B \otimes_A \Gamma \otimes_B B \otimes_A \Gamma \otimes_A B \\ & \xrightarrow{\cong} (B \otimes_A \Gamma \otimes_A B) \otimes_B (B \otimes_A \Gamma \otimes_A B) \end{aligned}$$

defines a commutative bialgebroid $(B, B \otimes_A \Gamma \otimes_A B)$ over S. If the structure maps for the left or right unit are unclear to the reader, we suggest drawing out the involved tensor products as colimits of diagrams, and the maps between them *which give us the structure maps when we pass to the colimits. We have used* s_A, s_B *to denote the* $\mathcal{O}_S$*-algebra structure maps* $\mathcal{O}_S \to A$, $\mathcal{O}_S \to B$, *respectively.*

When (A, Γ) *is a commutative Hopf algebroid over* S, *the following map defined sectionwise gives us a conjugation on* $B \otimes_A \Gamma \otimes_A B$ *turning* $(B, B \otimes_A \Gamma \otimes_A B)$ *into a commutative Hopf algebroid over* S:

$$\begin{aligned} B \otimes_A \Gamma \otimes_A B &\to B \otimes_A \Gamma \otimes_A B \\ b_1 \otimes \gamma \otimes b_2 &\mapsto b_2 \otimes \chi(\gamma) \otimes b_1. \end{aligned}$$

As a functor from commutative $\mathcal{O}_S$*-algebras to groupoids, the value taken by the bialgebroid* $(B, B \otimes_A \Gamma \otimes_A B)$ *on a commutative* R*-algebra* S *has object set* $\hom_{\mathbf{CAlg}(\mathcal{O}_S)}(B, \mathcal{O}_S)$ *(the "*B*-objects") and the morphisms in this groupoid are the* A*-morphisms from the underlying* A*-object of a* B*-object to the underlying* A*-object of another* B*-object.*

Proof. Checking the conditions in Prop. 2.1.2 and Prop. 2.1.4 amounts to a sequence of easy diagram chases. The details are left to the reader. $\square$

In general we have maps of bialgebroids or Hopf algebroids $(A, \Gamma) \to (B, B \otimes_A \Gamma)$ and $(A, \Gamma) \to (B, B \otimes_A \Gamma \otimes_A B)$ but we do not generally have a map of bialgebroids or Hopf algebroids $(B, B \otimes_A \Gamma) \to (B, B \otimes_A \Gamma \otimes_A B)$.

Remark 2.3.15. We have the 2-commutative diagram

$$\begin{tikzcd}
 & & \operatorname{Spec}_S B \arrow[dr] & \\
\operatorname{Spec}_S B \times_{\operatorname{Spec}_S A} \operatorname{Spec}_S \Gamma \times_{\operatorname{Spec}_S A} \operatorname{Spec}_S B \arrow[urr] \arrow[d] \arrow[drr] & & & \operatorname{Spec}_S A \arrow[d] \\
\operatorname{Spec}_S B \arrow[drr] & & \operatorname{Spec}_S \Gamma \arrow[ur] \arrow[d] & \mathcal{X} \\
 & & \operatorname{Spec}_S A \arrow[ur] &
\end{tikzcd}$$

One notices that the above diagram is a cube missing a vertex, as in Remark 2.3.3. Given a comodule algebra structure map filling in the above dotted line, we may use (a homotopical or 2-categorical) descent to produce a stack $\mathcal{Y}$ filling in the missing vertex, equipped with an fpqc presentation by $\mathrm{Spec}_S B$; when B is faithfully flat as an A-algebra then $\mathcal{Y}$ is faithfully flat and affine over $\mathcal{X}$, and under those conditions one has not only a cohomology isomorphism but also a kind of Morita equivalence between the $\mathcal{O}_{\mathcal{X}}$-modules and the $\mathcal{O}_{\mathcal{Y}}$-modules, which implies the cohomology isomorphism; see [Hopkins, 1995].

Proposition 2.3.16. *Let S be a scheme and let $(A, \Gamma) \to (B, \Sigma)$ be a map of commutative bialgebroids over S.*

1. *If M is a left (resp. right) B-module and $M \xrightarrow{\psi} \Gamma \otimes_A M$ (resp. $M \xrightarrow{\psi} M \otimes_A \Gamma$) is a left (resp. right) Γ-comodule structure map, then the composite*
$$M \xrightarrow{\psi} \Gamma \otimes_A M \longrightarrow \Sigma \otimes_B M \left(\text{resp. } M \xrightarrow{\psi} M \otimes_A \Gamma \longrightarrow M \otimes_B \Sigma\right)$$
is a left (resp. right) Σ-comodule structure map on M.

2. *If M a left (resp. right) B-module and also a left (resp. right) Γ-comodule algebra, then the left (resp. right) Σ-comodule structure map defined as above is also a left (resp. right) Σ-comodule algebra structure map.*

3. *If M is a left (resp. right) B-module and also a left (resp. right) Γ-comodule algebra satisfying condition (resp. condition), then the left (resp. right) Σ-comodule algebra structure map defined as above also satisfies condition (resp. condition).*

Proof. We provide proofs for the left comodule case. The case of a right comodule is completely analogous.

1. We first verify that the given map $M \to \Sigma \otimes_B M$ satisfies the unit and associativity conditions and . First, the commutativity of

$$\begin{tikzcd}
M \arrow[r, "\psi"] \arrow[dr, "\mathrm{id}_M"'] & \Gamma \otimes_A M \arrow[r] \arrow[d, "\epsilon \otimes_A \mathrm{id}_M"] & \Sigma \otimes_B M \arrow[d, "\epsilon \otimes_B M"] \\
& A \otimes_A M \arrow[r] & B \otimes_B M
\end{tikzcd}$$

follows from the commutativity of the following diagram of diagrams in $\mathbf{CAlg}(R)$, whose colimit is the square appearing in the above diagram:

$$\begin{tikzcd}
& & \Gamma \arrow[rrr, bend left] \arrow[ddd, "\epsilon"] & & & \Sigma \arrow[ddd, "\epsilon"] \\
A \arrow[urr] \arrow[rrr, bend left] \arrow[drr] \arrow[ddd, "\cong"] & & & B \arrow[urr] \arrow[drr] \arrow[ddd, "\cong"] & & \\
& & M \arrow[rrr, bend left, "\cong"] \arrow[ddd, "\cong"] & & & M \arrow[ddd, "\cong"] \\
& & A \arrow[rrr, bend left] & & & B \\
A \arrow[urr] \arrow[rrr, bend left] \arrow[drr] & & & B \arrow[urr] \arrow[drr] & & \\
& & M \arrow[rrr, bend left, "\cong"] & & & M
\end{tikzcd} .$$

The associativity condition, i.e., commutativity of the diagram

$$\begin{tikzcd}
& \Gamma \otimes_A M \arrow[r] \arrow[dr, "\Delta \otimes_A \mathrm{id}_M"] & \Sigma \otimes_B M \arrow[dr, "\Delta \otimes_B \mathrm{id}_M"] & \\
M \arrow[ur, "\psi"] \arrow[dr, "\psi"'] & & \Gamma \otimes_A \Gamma \otimes_A M \arrow[r] & \Sigma \otimes_B \Sigma \otimes_B M \\
& \Gamma \otimes_A M \arrow[r] \arrow[ur, "\mathrm{id}_\Gamma \otimes_A \psi"] & \Sigma \otimes_B M \arrow[ur, "\mathrm{id}_\Sigma \otimes_B \psi"] &
\end{tikzcd}$$

follows from the commutativity of the following two diagrams of diagrams in $\mathbf{CAlg}(R)$, whose colimits are the squares on the upper right and lower right, respectively, in the above diagram:

$$
\begin{tikzcd}
 & \Gamma \arrow[rr] \arrow[ddd, "\Delta"] & & \Sigma \arrow[ddd, "\Delta"] \\
A \arrow[ur] \arrow[rr] \arrow[dr] \arrow[ddd, "\cong"] & & B \arrow[ur] \arrow[dr] \arrow[ddd, "\cong"] & \\
 & M \arrow[rr, "\cong"] \arrow[ddd, "\cong"] & & M \arrow[ddd, "\cong"] \\
 & \Gamma \otimes_A \Gamma \arrow[rr] & & \Sigma \otimes_B \Sigma \\
A \arrow[ur] \arrow[rr] \arrow[dr] & & B \arrow[ur] \arrow[dr] & \\
 & M \arrow[rr, "\cong"] & & M
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & \Gamma \arrow[rr] & & \Sigma \quad . \\
A \arrow[ur] \arrow[rr] \arrow[dr, "\cong"] & & B \arrow[ur] \arrow[dr, "\cong"] & \\
 & \Gamma \otimes_A M \arrow[rr, "\cong"] & & \Sigma \otimes_B M \\
 & \Gamma \arrow[rr] \arrow[uuu] & & \Sigma \arrow[uuu] \\
A \arrow[ur, "\psi"] \arrow[rr] \arrow[dr] \arrow[uuu, "\cong"] & & B \arrow[ur, "\psi"] \arrow[dr] \arrow[uuu, "\cong"] & \\
 & M \arrow[rr, "\cong"] \arrow[uuu] & & M \arrow[uuu]
\end{tikzcd}
$$

2. That the Σ-comodule structure map on M is a comodule algebra structure map follows from the maps $M \xrightarrow{\psi} \Gamma \otimes_A M$ and $\Gamma \otimes_A M \to \Sigma \otimes_B M$ both being R-algebra morphisms.

3. Condition for M follows from the commutativity of the following diagram:

$$\begin{tikzcd}
 & B \arrow[rrr, bend left=10] \arrow[dr, "\eta_L"] & & & M \arrow[dr] & \\
A \arrow[ur] \arrow[dr, "\eta_L"'] \arrow[rrr, bend left=10] & & \Sigma \arrow[r] & M \arrow[ur, "\cong"] \arrow[rr, bend left=10, "\psi"] \arrow[dr] & & \Sigma \otimes_B M \\
 & \Gamma \arrow[ur] \arrow[rrr, bend left=10, "\psi"] & & & \Gamma \otimes_A M \arrow[ur] &
\end{tikzcd} .$$

The five faces of the cube apart from

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "\eta_L"] & M \arrow[d] \\
\Sigma \arrow[r, "\psi"] & \Sigma \otimes_B M
\end{tikzcd}$$

are commutative, so this sixth face is also commutative, which gives us condition for $M \xrightarrow{\psi} \Sigma \otimes_B M$.

$\square$

Lemma 2.3.17. *Let S be a scheme, let (A, Γ) be a commutative Hopf algebroid over S, let M be a left Γ-comodule with comodule structure map $M \xrightarrow{\phi_M} \Gamma \otimes_A M$, and let B be a flat A-module. Then we can form the left Γ-comodule $M \otimes_A B$ with structure map $\phi_{M \otimes_A B}$ equal to*

$$M \otimes_A B \xrightarrow{\phi_M \otimes_A \mathrm{id}_B} \Gamma \otimes_A M \otimes_A B$$

and for any right Γ-comodule B we have the isomorphisms

$$N \boxast_\Gamma (M \otimes_A B) \cong (N \boxast_\Gamma M) \otimes_A B$$

and, as a consequence (by taking global sections),

$$N \square_\Gamma (M \otimes_A B) \cong (N \square_\Gamma M) \otimes_A B,$$

with both isomorphisms natural in M,N, and B.

Proof. We may tensor up the equalizer sequence defining $N \circledast_\Gamma M$ over A with B, and since B is flat over A, what we get is also an equalizer sequence; then we have the following commutative diagram with rows equalizer sequences:

$$\begin{tikzcd}
0 \arrow[r] & (N \circledast_\Gamma M) \otimes_A B \arrow[r] \arrow[d] & (N \otimes_A M) \otimes_A B \arrow[r, shift left, "(\phi_N \otimes_A \mathrm{id}_M) \otimes_A \mathrm{id}_B"] \arrow[r, shift right, "(\mathrm{id}_N \otimes_A \phi_M) \otimes_A \mathrm{id}_B"'] \arrow[d, "\cong"] & (N \otimes_A \Gamma \otimes_A M) \otimes_A B \arrow[d, "\cong"] \\
0 \arrow[r] & N \circledast_\Gamma (M \otimes_A B) \arrow[r] & N \otimes_A (M \otimes_A B) \arrow[r, shift left, "\phi_N \otimes_A \mathrm{id}_{M \otimes_A B}"] \arrow[r, shift right, "\mathrm{id}_N \otimes_A \phi_{M \otimes_A B}"'] & N \otimes_A \Gamma \otimes_A (M \otimes_A B),
\end{tikzcd}$$

giving us the desired isomorphism. □

Proposition 2.3.18. *Let S be a scheme and let $(A, \Gamma) \xrightarrow{f} (B, \Sigma)$ be a map of commutative bialgebroids over S. If N is a left (B, Σ)-comodule then we have a left Γ-comodule structure map on $(\Gamma \otimes_A B)\Box_\Sigma N$ given by the composite*

$$(\Gamma \otimes_A B)\Box_\Sigma N \xrightarrow{(\Delta \otimes_A \mathrm{id}_B)\Box_\Sigma \mathrm{id}_N} (\Gamma \otimes_A \Gamma \otimes_A B)\Box_\Sigma N \xrightarrow{\cong} \Gamma \otimes_A ((\Gamma \otimes_A B)\Box_\Sigma N),$$

with the isomorphism given by Lemma 2.3.17. We will write f^*N for this Γ-comodule.

With S and f as before, let M be a left Γ-comodule with structure map $M \xrightarrow{\phi_M} \Gamma \otimes_A M$. We will write $A \xrightarrow{f'} B$ and $\Gamma \xrightarrow{f''} \Sigma$ for the component maps of f. Then we define a left Σ-comodule structure on $M \otimes_A B$ by the composite

$$M \otimes_A B \xrightarrow{\phi_M \otimes_A \mathrm{id}_B} \Gamma \otimes_A M \otimes_A B \xrightarrow{f'' \otimes_{f'} \mathrm{id}_{M \otimes_A B}} \Sigma \otimes_B (M \otimes_A B).$$

We will write f_*M for this Σ-comodule.

Now the functor $\mathbf{Comod}(\Gamma) \xrightarrow{f_*} \mathbf{Comod}(\Sigma)$ is left adjoint to the functor $\mathbf{Comod}(\Sigma) \xrightarrow{f^*} \mathbf{Comod}(\Gamma)$, i.e., we have an isomorphism

$$\hom_{\mathbf{Comod}(\Sigma)}(f_*M, N) \cong \hom_{\mathbf{Comod}(\Gamma)}(M, f^*N)$$

natural in M and N.

Proof. An element of $\hom_{\mathbf{Comod}(\Gamma)}(M, f^*N)$ gives a commuting square

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\phi_M"] & (\Gamma \otimes_A B) \boxtimes_\Sigma N \arrow[d, "\phi_{f^*N}"] \\ \Gamma \otimes_A M \arrow[r] & \Gamma \otimes_A ((\Gamma \otimes_A B) \boxtimes_\Sigma N) \end{tikzcd}$$

and, by the definition of the cotensor product sheaf as an equalizer, this square is equivalent to a commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\phi_M"] & \Gamma \otimes_A B \otimes_B N \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\Delta \otimes_A \mathrm{id}_B \otimes_B N"] & \Gamma \otimes_A B \otimes_B \Sigma \otimes_B N \arrow[d, "\Delta \otimes_A \mathrm{id}_{B \otimes_B \Sigma \otimes_B N}"] \\ \Gamma \otimes_A M \arrow[r] & \Gamma \otimes_A \Gamma \otimes_A B \otimes_B N \arrow[r, shift left] \arrow[r, shift right] & \Gamma \otimes_A \Gamma \otimes_A B \otimes_B \Sigma \otimes_B N \end{tikzcd}$$

with horizontal maps from the definition of $(\Gamma \otimes_A B) \boxtimes_\Sigma N$ and the left Γ-comodule structure on it. By a diagram chase one verifies that a choice of map $M \to \Gamma \otimes_A B \otimes_B N$ fitting into the above diagram is equivalent to a choice of map $M \otimes_A B \xrightarrow{\lambda} N$ with

$$\begin{tikzcd} M \otimes_A B \arrow[r, "\lambda"] \arrow[d, "\psi_M \otimes_A \mathrm{id}_B"] & N \arrow[dd, "\psi_N"] \\ \Gamma \otimes_A M \otimes_A B \arrow[d, "f'' \otimes_{f'} \mathrm{id}_{M \otimes_A B}"] & \\ \Sigma \otimes_B (M \otimes_A B) \arrow[r, "\mathrm{id}_\Sigma \otimes_B \lambda"] & \Sigma \otimes_B N \end{tikzcd}$$

and such a map defines a unique element of $\hom_{\mathbf{Comod}(\Sigma)}(f_*M, N)$. $\square$

Remark 2.3.19. With S, f, M, N as above, we have a morphism of fpqc stacks

$$\mathcal{X}(B, \Sigma) \xrightarrow{f} \mathcal{X}(A, \Gamma)$$

induced by f, and sheaves $\widetilde{M}$ over $\mathcal{X}(A, \Gamma)$ and $\widetilde{N}$ over $\mathcal{X}(B, \Sigma)$. Now we will write $f_*\widetilde{N}$ for the $\mathcal{O}_{\mathcal{X}(A,\Gamma)}$-module $\widetilde{f^*N}$ associated to the Γ-comodule f^*N, and we call this *the direct image sheaf* of $\widetilde{N}$. We will write $f^*\widetilde{M}$ for the $\mathcal{O}_{\mathcal{X}(B,\Sigma)}$-module $\widetilde{f_*M}$ associated to the Σ-comodule f_*M, and we call this *the inverse image sheaf* of $\widetilde{N}$.

Now the adjunction of Prop. 2.3.18 gives us that the functor $\mathbf{Mod}(\mathcal{O}_{\mathcal{X}(B,\Sigma)}) \xrightarrow{f_*} \mathbf{Mod}(\mathcal{O}_{\mathcal{X}(A,\Gamma)})$ is right adjoint to the functor $\mathbf{Mod}(\mathcal{O}_{\mathcal{X}(A,\Gamma)}) \xrightarrow{f^*} \mathbf{Mod}(\mathcal{O}_{\mathcal{X}(B,\Sigma)})$, i.e., we have an isomorphism

$$\hom_{\mathbf{Mod}(\mathcal{O}_{\mathcal{X}(A,\Gamma)})}(\widetilde{M}, f_*\widetilde{N}) \cong \hom_{\mathbf{Mod}(\mathcal{O}_{\mathcal{X}(B,\Sigma)})}(f^*\widetilde{M}, \widetilde{N})$$

natural in M and N.

Once a suitable (pro-)presentation $X \to \mathcal{X}$ of an fpqc stack has been chosen, $\mathcal{O}_{\mathcal{X}}$-modules are equivalent to $\mathcal{A}(X \times_{\mathcal{X}} X)$-comodules, so we can take the above as our definitions of direct image and inverse image sheaves for arbitrary sheaves on pro-rigidified fpqc stacks.

Proposition 2.3.20. *Let (A, Γ) be a Hopf algebroid over a scheme S, and let Γ_L be Γ considered as an A-module via $A \xrightarrow{\eta_L} \Gamma$, and let Γ_R be Γ considered as an A-module via $A \xrightarrow{\eta_R} \Gamma$.*

1. *$\Gamma_L \cong \Gamma_R$ as A-modules.*

2. *Let $(A, \Gamma^{\mathrm{op}})$ denote the Hopf algebroid whose augmentation and conjugation are identical to those of (A, Γ), whose left unit is identical to the right unit on (A, Γ), whose right unit is identical to the left unit on (A, Γ), and whose coproduct is $(\chi \otimes_A \chi) \circ \Delta \circ \chi$, where Δ is the coproduct on (A, Γ). Then $(A, \Gamma) \cong (A, \Gamma^{\mathrm{op}})$ as Hopf algebroids. If (A, Γ) is a graded Hopf algebroid then the isomorphism is an isomorphism of graded Hopf algebroids.*

3. *If $A \xrightarrow{f} B$ is a map of commutative $\mathcal{O}_S$-algebras then there is an isomorphism of Hopf algebroids $(B, \Gamma \otimes_A B) \cong (B, B \otimes_A \Gamma)$.*

Proof. 1. Working sectionwise, we recall that the commutativity of the following diagram is one of the conditions for (A, Γ) to be a Hopf algebroid:

$$\begin{tikzcd} A \arrow[r, "\eta_L"] \arrow[dr, "\eta_R"'] & \Gamma \arrow[d, "\chi"] \\ & \Gamma \end{tikzcd},$$

i.e.,

$$\chi(r \cdot x) = \chi(\eta_L(r)x) = \eta_R(r)\chi(x) = r \cdot \chi(x),$$

so χ is an A-linear map $\Gamma_L \longrightarrow \Gamma_R$, and it is bijective, so it is an isomorphism of A-modules.

2. The map $(A, \Gamma) \xrightarrow{(1_A, \chi)} (A, \Gamma^{\mathrm{op}})$ is the desired isomorphism. We have $\eta_R = \chi \circ \eta_L$ and $\eta_L = \chi \circ \eta_R$ by the axioms for (A, Γ) to be a Hopf algebroid, so $(1, \chi)$ commutes with the left and right units; obviously $(1, \chi)$ commutes with conjugation; the coproduct on $A, \Gamma^{\mathrm{op}})$ is constructed so that $(1, \chi)$ commutes with it ($\Delta \circ \chi = (\chi \otimes \chi) \circ \Delta$); and as for commuting with the augmentations, we have

$$\begin{aligned} \chi \circ \epsilon &= \epsilon \circ \eta_R \circ \chi \circ \epsilon \\ &= \epsilon \circ \eta_L \circ \epsilon \\ &= \epsilon. \end{aligned}$$

3. When we tensor up the isomorphism $(A, \Gamma) \cong (A, \Gamma^{\mathrm{op}})$ with B, we get an isomorphism of Hopf algebroids

$$(B, \Gamma \otimes_A B) \cong (B, \Gamma^{\mathrm{op}} \otimes_A B) \cong (B, B \otimes_A \Gamma).$$

Note that this isomorphism relies on (A, Γ) being a Hopf algebroid, and not an arbitrary bialgebroid.

$\square$

We recall a special case of the cotensor product:

Proposition 2.3.21. *For a commutative Hopf algebroid (A, Γ) over an affine scheme S with A, Γ both quasicoherent as $\mathcal{O}_S$-algebras, and a right Γ-comodule M, the cotensor product $M \Box_\Gamma A$ is the sub-$\Gamma(\mathcal{O}_S)$-module of M consisting of elements m such that $\psi_M(m) = m \otimes 1 \in \Gamma(S, M \otimes_A \Gamma)$. The natural right Γ-comodule* *structure on $M\Box_\Gamma A$ is the trivial one; this is true for arbitrary base scheme S and without quasicoherence assumptions on A and Γ.*

Proof. This follows immediately from the equalizer sequence which defines $M\Box_\Gamma A$:

$$0 \longrightarrow M\Box_\Gamma A \longrightarrow M \otimes_A A \xrightarrow{g} M \otimes_A \Gamma \otimes_A A,$$

where $g = \psi_M \otimes A - M \otimes \psi_A$. Since $\psi_A = \eta_R$, the right unit map on (A, Γ), an element $m \otimes r \in M \otimes_A A$ is in $M\Box_\Gamma A$ if and only if $g(m \otimes r) = 0$, i.e., iff

$$\psi_M(m) \otimes r = m \otimes \psi_A(r),$$

which again is true iff

$$\psi_M(mr) \otimes 1 = mr \otimes \psi_A(1) = mr \otimes 1.$$

$\square$

Proposition 2.3.22. *Let S be a scheme and let $(A, \Gamma) \xrightarrow{\gamma} (B, \Sigma)$ be a morphism of commutative Hopf algebroids over S, and let M be a left Σ-comodule. Then the map*

$$(\Gamma \otimes_A B)\Box_\Sigma M \to M$$

is a monomorphism of left Σ-comodules, with equality when $\Gamma \xrightarrow{\gamma} \Sigma$ is a monomorphism of $\mathcal{O}_S$-algebras.

Proof. We first show that the map is as claimed as a morphism of B-modules, and then we show that the comodule structure on $(\Gamma\Box_\Gamma A) \otimes_A B$ is trivial.

We will write $A \xrightarrow{f'} B$ and $\Gamma \xrightarrow{f''} \Sigma$ for the component morphisms of the Hopf algebroid morphism f, and we will write $M \xrightarrow{\psi_M} \Sigma \otimes_B M$ for the Σ-comodule structure map of M. Consider the commutative diagram of A-modules

$$\begin{tikzcd}
 & \Gamma \otimes_A M \arrow[r, shift left, "((\mathrm{id}_\Gamma \otimes_A f'')\circ\Delta)\otimes_{f'}\mathrm{id}_M"] \arrow[r, shift right, "\mathrm{id}_\Gamma \otimes_A \psi_M"'] \arrow[d, "\epsilon\otimes_A \mathrm{id}_M"] & \Gamma \otimes_A \mathrm{id}_\Sigma \otimes_B \mathrm{id}_M \\
0 \arrow[r] & M. &
\end{tikzcd}$$

The kernel of $\epsilon \otimes_A \mathrm{id}_M$ is $(\ker \epsilon) \otimes_A M$, since A is injective over A and so $\mathcal{Tor}_1^A(A, M) \cong 0$. Now suppose T is an A-module mapping to $\mathbb{\Gamma} \otimes_A M$ and commuting with the maps in the upper row in diagram 2.3. Let $T \xrightarrow{g} \mathbb{\Gamma} \otimes_A M$ be the map. Then, by the hypothesis on T and by the axioms defining (INSERT REF TO DEF OF A HOPF ALGEBROID!) a Hopf algebroid, we have

$$\begin{aligned} g &= (((\mathrm{id}_{\mathbb{\Gamma}} \otimes_A \epsilon) \circ \Delta) \otimes_A \mathrm{id}_M) \circ g \\ &= ((\mathrm{id}_{\mathbb{\Gamma}} \otimes_A \epsilon \otimes_A \mathrm{id}_M) \circ (\Delta \otimes_A \mathrm{id}_M)) \circ g \\ &= ((\mathrm{id}_{\mathbb{\Gamma}} \otimes_A \epsilon \otimes_A \mathrm{id}_M) \circ (\mathrm{id}_{\mathbb{\Gamma}} \otimes_A \phi_M)) \circ g \\ &= ((\epsilon \otimes_A \mathrm{id}_{\mathbb{\Gamma}} \otimes_A \mathrm{id}_M) \circ (\mathrm{id}_{\mathbb{\Gamma}} \otimes_A \phi_M)) \circ g \end{aligned}$$

which is the zero morphism (by evaluation of the leftmost tensor summands in the composite) if g factors through $(\ker \epsilon) \otimes_A M$. Hence the limit of the diagram 2.3 is zero, and so $\ker((\mathbb{\Gamma} \otimes_A B)\Box_{\mathbb{\Sigma}} M \mapsto M) = 0$.

$\square$

Remark 2.3.23. On the level of the associated $\mathcal{O}_X$-sheaves, this is the map $f_* f^* \widetilde{F} \to \widetilde{F}$. It is closely related to the trace map used in the theory of Grothendieck duality (INSERT REF TO HARTSHORNE "RES AND DUALITY") and we will use it in constructing the stack-theoretic Cousin complex.

2.3.1 Leray spectral sequences.

Proposition 2.3.24. The change-of-rings spectral sequence. *Let S be a scheme and let $(A, \mathbb{\Gamma}) \xrightarrow{f} (B, \mathbb{\Sigma})$ be a morphism of commutative Hopf algebroids over S and let M be a left $\mathbb{\Gamma}$-comodule which is flat as an A-module, and let N a right $\mathbb{\Sigma}$-comodule. Then the double complex*

$$\mathcal{C}_{\mathbb{\Gamma}}^*(M, \mathcal{C}_{\mathbb{\Sigma}}^*(\mathbb{\Gamma} \otimes_A B, N))$$

gives rise to a spectral sequence of $\mathcal{O}_S$-modules

$$E_2^{*,*} \cong \mathcal{Cotor}_{\mathbb{\Gamma}}^*(M, \mathcal{Cotor}_{\mathbb{\Sigma}}^*(\mathbb{\Gamma} \otimes_A B, N)) \Rightarrow E_\infty^{*,*} \cong \mathcal{Cotor}_{\mathbb{\Sigma}}^*(\Gamma \otimes_A B, N).$$

Proof. We may filter the double complex by the first degree to get a spectral sequence with $E_1^{*,*} \cong \mathcal{C}_{\Gamma}^*(M, \mathcal{C}otor_{\Sigma}^*(\Gamma \otimes_A B, N))$ and hence the desired E_2 term. Filtering the double complex by the second degree yields a spectral sequence with

$$\begin{aligned} E_1^{*,*} &\cong \mathcal{C}otor_{\Gamma}^*(M, \mathcal{C}_{\Sigma}^*(\Gamma \otimes_A B, N)) \\ &\cong \mathcal{C}otor_{\Gamma}^*(M, \Gamma \otimes_A \Sigma^{\otimes *} \otimes_B N)) \\ &\cong M \otimes_A \Sigma^{\otimes *} \otimes_B N \\ &\cong \mathcal{C}_{\Sigma}^*(M \otimes_A B, N) \end{aligned}$$

so $E_2 \cong E_\infty \cong \mathcal{C}otor_{\Sigma}^*(M \otimes_A B, N)$. This proof is from A1.3.11(b) of [Ravenel, 1986] and the general theory of the spectral sequences of a double complex is covered in [Cartan and Eilenberg, 1999] □

Corollary 2.3.25. The Leray spectral sequence. *Let S be a scheme and let $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ be a morphism of pro-algebraic fpqc stacks and let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_{\mathcal{X}}$-module. Then we have a Leray spectral sequence of $\mathcal{O}_S$-modules*

$$E_2^{*,*} \cong \mathcal{H}_{\mathrm{fpqc}}^*(\mathcal{Y}, R^* f_* \mathcal{F})) \Rightarrow E_\infty^{*,*} \cong \mathcal{H}_{\mathrm{fpqc}}^*(\mathcal{X}, \mathcal{F}).$$

Proposition 2.3.26. Tensoring up, in cohomology. *Let $(A, \Gamma) \xrightarrow{f} (B, \Sigma)$ be a split map of graded Hopf algebroids, i.e., B is a Γ-comodule algebra and $\Sigma \cong \Gamma \otimes_A B$ or, equivalently by Prop. 2.3.20, $\Sigma \cong B \otimes_A \Gamma$. Let N be a left Σ-comodule. Then there is an isomorphism*

$$\mathrm{Cotor}_{\Gamma}^{*,*}(A, N) \cong \mathrm{Cotor}_{\Sigma}^{*,*}(B, N).$$

Proof. The $E_2^{*,*,*}$ of the change-of-rings spectral sequence for f^* is

$$\mathrm{Cotor}_{\Gamma}^{r,t}(A, \mathrm{Cotor}_{\Sigma}^{s,t}(\Gamma \otimes_A B, N)) \cong \mathrm{Cotor}_{\Gamma}^{r,t}(A, N),$$

since $\Gamma \otimes_A B \cong \Sigma$ is free, hence relatively injective, over Σ. This E_2 is concentrated on the $s = 0$ line (or the $s = 0$ plane, if you like, since it is trigraded) and so there is no room for differentials. Hence $E_2 \cong E_\infty \cong \mathrm{Cotor}_{\Sigma}^{*,*}(B, N)$. □

Proposition 2.3.27. *Let* $(A, \Gamma) \xrightarrow{f} (B, \Sigma)$ *be a map of Hopf algebroids which is split onto a direct summand of* (B, Σ)*, i.e.,* $\Sigma \cong \Gamma \otimes_A (B \oplus T)$ *(or, equivalently by Prop. 2.3.20,* $\Sigma \cong (B \oplus T) \otimes_A \Gamma$*) for some commutative* A*-algebra* T*. If* $A \longrightarrow B$ *is injective and* Γ *is flat over* A*, then*

$$\mathrm{Cotor}^{*,*}_{\Sigma}(B, B) \cong \mathrm{Cotor}^{*,*}_{\Gamma}(A, (\Gamma \Box_{\Gamma} A) \otimes_A B).$$

Proof. We again use the change-of-rings spectral sequence. The right Σ-comodule $\Gamma \otimes_A B$ is a direct summand of Σ, so it is relatively injective, so the E_2-term of this spectral sequence is concentrated on the $s = 0$ line and it collapses at E_2. Using Prop. 2.3.22 and the fact that

$$\Gamma \cong \Gamma \otimes_A A \longrightarrow \Gamma \otimes_A B \longrightarrow \Gamma \otimes_A (B \oplus T) \cong \Sigma$$

is injective, $(\Gamma \otimes_A B) \Box_{\Sigma} B \cong (\Gamma \Box_{\Gamma} A) \otimes_A B$, and we have the stated isomorphism. $\Box$

2.3.2 Invariant ideals and some commutative algebra.

Lemma 2.3.28. *Let* (A, Γ) *be a Hopf algebroid and let* $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ *be a collection of invariant ideals of* (A, Γ)*, indexed by some set* $\mathcal{A}$*. Then* $I = \bigcup_{\alpha \in \mathcal{A}} I_\alpha$ *is invariant.*

Proof. Choose $x \in I$ and an $\alpha \in \mathcal{A}$ such that $x \in I_\alpha$. Then $\eta_L(x) \in \eta_R(I_\alpha)\Gamma \subseteq \eta_R(I)\Gamma$. $\Box$

Proposition 2.3.29. *Let* (A, Γ) *be a Hopf algebroid, let* $I \subset A$ *be an invariant ideal, and let* $j \in A$ *be such that* $\eta_L(j) \equiv \eta_R(j) \mod \eta_R(I)$ *(or, equivalently,* $\mod \eta_L(I)$*). Then* $I + (j)$ *is an invariant ideal.*

Proof. Choose an element $x \in I + (j)$; we want to show that $\eta_L(x) \in \eta_R(I + (j))\Gamma$. Choose decompositions $x = i + jr$ and $\eta_L(j) = \eta_R(j) + \eta_R(i')$ and $\eta_L(i) = \eta_R(i'')\gamma$, where $i, i', i'' \in I$ and $r \in A$ and $\gamma \in \Gamma$. Now we have

$$\begin{aligned} \eta_L(x) &= \eta_L(i) + \eta_L(j)\eta_L(r) \\ &= \eta_R(i'')\gamma + (\eta_R(j) + \eta_R(i'))\eta_L(r) \\ &\in \eta_R(I + (j))\Gamma. \end{aligned}$$

□

Corollary 2.3.30. *Let (A, Γ) be a Hopf algebroid and let $I = (a_1, a_2, \dots)$ be an ideal in A with $\eta_L(a_1) = \eta_R(a_1)$ and*

$$\eta_L(a_i) \equiv \eta_R(a_i) \mod (\eta_L(a_1), \dots, \eta_L(a_{i-1}))$$

for all $i > 1$. Then I is invariant.

Proof. Clearly $(a_1) \subseteq A$ is invariant. We use Prop. 3.2.7 for the inductive step, to get that $(a_1, \dots a_i) \subseteq A$ is invariant for all positive integers i; and, using Lemma 3.2.6, the union of all of these invariant ideals, I, is also invariant. □

2.4 Preliminaries on commutative algebra.

2.4.1 Some combinatorial commutative algebra.

Definition 2.4.1. *We consider sequences $I = (i_1, \dots, i_m)$ of positive integers; if $J = (j_1, \dots j_n)$ is another such sequence then we define $IJ = (i_1, \dots, i_m, j_1, \dots j_n)$, and this concatenation puts a monoid structure on the set of all such sequences. We define several operations and operators on these sequences:*

$$\begin{aligned} I'' &= (i_1 + i_2, i_3, i_4, \dots, i_m), \\ I''' &= (i_2, i_3, i_4, \dots, i_m), \\ |I| &= m, \\ ||I|| &= \sum_{i=1}^{m} i_m, \end{aligned}$$

and, given a choice of p-adic number ring A, we define the integer-valued function Π_A *on all such sequences by*

$$\begin{aligned} \Pi_A(\varnothing) &= 1, \\ \Pi_A(h) &= \pi_A - \pi_A^{q^h} \text{ for } h \text{ an integer}, \\ \Pi_A(I) &= \Pi_A(||I||)\Pi_A((i_1, i_2, \ldots, i_{m-1})). \end{aligned}$$

Lemma 2.4.2. *Let E/K have ramification degree e. Then for any sequence I of positive integers we have*

$$|\Pi_A(I)|_p = |\Pi_B(I)|_p^e.$$

Proof. For any positive integer h we have

$$\begin{aligned} |\Pi_A(h)|_p &= |\pi_A(1 - \pi_A^{q^h - 1})|_p \\ &= |\pi_A|_p \\ &= |\pi_B|_p^e \\ &= |\Pi_B(h)|_p^E. \end{aligned}$$

Now we proceed by induction on the length of the sequence I. Suppose we have proven the lemma for all I with $|I| < n$; then, for any I with $|I| = n$, we have

$$\begin{aligned} |\Pi_A(I)|_p &= |\Pi_A(||I||)|_p |\Pi_A((i_1, \ldots, i_{n-1}))|_p \\ &= |\Pi_B(||I||)|_p^e |\Pi_B((i_1, \ldots, i_{n-1}))|_p^e. \end{aligned}$$

□

Proposition 2.4.3. (Generalized Witt polynomials.) *For each sequence I and number ring A as above and any choice of positive integer m there is a symmetric polynomial $w_I^A = w_I^A(x_1, x_2, ..., x_m)$, in m variables, of degree $q^{||I||}$, and with coefficients in A, where $w_\varnothing^A = \sum_{t=1}^m x_t$ and*

$$\sum_{t=1}^m x_t^{q^{||K||}} = \sum_{IJ=K} \frac{\Pi_A(K)}{\Pi_A(I)} (w_J^A)^{q^{||I||}} \text{ and} \tag{2.4.10}$$

$$w_I^A \equiv (w_{|I|}^A)^{q^{||I||-|I|}} \mod \pi_A. \tag{2.4.11}$$

Proof. This will be an inductive proof. Assume that the proposition is true for all sequences I with $|I| < m$. Then let $K = (k_1, \ldots, k_m)$. Obviously $|K''| = |K'''|$ and $||K'''|| + k_1 = ||K''|| = ||K||$ and we have

$$\begin{aligned} w^A_{K''} &\equiv_{\pi_A} (w^A_{|K''|})^{q^{||K''||-|K''|}} \\ &= (w^A_{|K'''|})^{q^{k_1+||K'''||-|K'''|}} \\ &= \left((w^A_{|K'''|})^{q^{||K'''||-|K'''|}}\right)^{q^{k_1}} \\ &\equiv_{\pi_A} (w^A_{K'''})^{q^{k_1}}, \text{ so} \\ w^A_{K''} &\equiv_{\pi_A} (w^A_{K'''})^{q^{k_1}}. \end{aligned}$$

We need a simple-minded lemma about the function Π_A and the double-prime operator:

$$\begin{aligned} \Pi_A(K) &= \prod_{i=1}^{|K|} \Pi_A(||(k_1, \ldots, k_i)||) \\ &= \Pi_A(||(k_1)||) \prod_{i=2}^{|K|} \Pi_A(||(k_1 + k_2, k_3, \ldots, k_i)||) \\ &= \Pi_A(k_1)\Pi_A(K''). \end{aligned}$$

Now

$$\begin{aligned} \sum_{IJ=K''} \frac{\Pi_A(K'')}{\Pi_A(I)} (w^A_J)^{q^{||I||}} &= \sum_{t=1}^{m} x_t^{q^{||K''||}} \\ &= \sum_{t=1}^{m} x_t^{q^{||K||}} \\ &= \sum_{IJ=K} \frac{\Pi_A(K)}{\Pi_A(I)} (w^A_J)^{q^{||I||}} \end{aligned}$$

and, partially expanding each side, from the identity $\Pi_A(K) = \Pi_A(K'')\Pi_A(k_1)$ we get

$$
\begin{aligned}
\Pi_q(K)w_K^A + \frac{\Pi_A(K)}{\Pi_A(k_1)}(w_{K'''}^A)^{q^{k_1}} + \\
\sum_{IJ=K,|I|\geqslant 2} \frac{\Pi_A(K)}{\Pi_A(I)}(w_J^A)^{q^{||I||}} &= \sum_{IJ=K} \frac{\Pi_A(K)}{\Pi_A(I)}(w_J^A)^{q^{||I||}} \\
&= \sum_{IJ=K''} \frac{\Pi_A(K'')}{\Pi_A(I)}(w_J^A)^{q^{||I||}} \\
&= \Pi_A(K'')w_{K''}^A + \sum_{IJ=K'',|I|\geqslant 1} \frac{\Pi_A(K'')}{\Pi_A(I)}(w_J^A)^{q^{||I||}}.
\end{aligned}
$$

We compare the sums $\sum_{IJ=K,|I|\geqslant 2} \frac{\Pi_q(K)}{\Pi_q(I)}(w_J^A)^{q^{||I||}}$ and $\sum_{IJ=K'',|I|\geqslant 1} \frac{\Pi_q(K'')}{\Pi_q(I)}(w_J^A)^{q^{||I||}}$: in the first, we sum over the sequences

$$\{I_2 = (k_1, k_2), I_3 = (k_1, k_2, k_3), I_4 = (k_1, k_2, k_3, k_4), \ldots, I_m = (k_1, \ldots, k_m)\}$$

and in the second, we sum over the sequences

$$\{I_2' = (k_1+k_2), I_3' = (k_1+k_2, k_3), I_4' = (k_1+k_2, k_3, k_4), \ldots, I_m' = (k_1+k_2, k_3, \ldots, k_m)\}.$$

Now for each i let J_i, J_i' be the sequences such that $I_iJ_i = K$ and $I_i'J_i' = K''$ (so $J_i = J_i'$ for $i > 1$), and we have

$$
\begin{aligned}
\frac{\Pi_A(K)}{\Pi_A(I_i)}(w_{J_i}^A)^{q^{||I_i||}} &= \frac{\Pi_A(K)}{\Pi_A(k_1)\prod_{j=2}^i \Pi_A(||I_j||)}(w_{J_i}^A)^{q^{||I_i||}} \\
&= \frac{\Pi_A(K)}{\Pi_A(k_1)\Pi_A(k_1+k_2)\prod_{j=3}^i \Pi_A(||I_j'||)}(w_{J_i}^A)^{q^{||I_i||}} \\
&= \frac{\Pi_A(K'')}{\Pi_A(k_1+k_2)\prod_{j=3}^i \Pi_A(||I_j'||)}(w_{J_i'}^A)^{q^{||I_i'||}} \\
&= \frac{\Pi_A(K'')}{\Pi_A(I_i')}(w_{J_i'}^A)^{q^{||I_i'||}},
\end{aligned}
$$

giving us

$$\Pi_A(K)\left(w_K^A + \frac{(w_{K'''}^A)^{q^{k_1}}}{\Pi_A(k_1)}\right) = \Pi_A(K'')w_{K''}^A,$$

that is,

$$w_K^A = \frac{w_{K''}^A - (w_{K'''}^A)^{q^{k_1}}}{\Pi_A(k_1)}.$$

Now, by equation 2.4.12, $w^A_{K''} - (w^A_{K'''})^{q^{k_1}}$ is divisible by π_A, and by the definition of Π_A, we have $\Pi_A(k_1) \equiv \pi_A \mod \pi_A^2$. Hence w^A_K has coefficients in A, and by construction, it satisfies equations 2.4.10 and 2.4.11. □

Lemma 2.4.4. *Let A be a p-adic number ring, let R be a commutative A-algebra, and let $S, R_1, R_2, \ldots$ be ideals in R. Let $x, y \in R$. Then*

$$\begin{aligned} x &\equiv y \mod \pi S + \sum_{i>0} R_i \textit{ implies} \\ x^{q^h} &\equiv y^{q^h} \mod \pi^{h+1} S + \sum_{j=0}^{h} \pi^j \sum_{i>0} R_i^{q^{h-j}}. \end{aligned}$$

Proof. We begin with the case $h = 1$. Suppose $x = y + \pi s + \sum_{i>0} r_i$, with $s \in R$ and $r_i \in R_i$ (obviously almost all of the r_i must be zero). Then

$$x^q = y^q + \sum_{0<j<q} \binom{q}{j} y^{q-j} \left(\pi s + \sum_{i>0} r_i\right)^j + (\pi s + \sum_{i>0} r_i)^q,$$

and $\binom{q}{j} y^{q-j} \left(\pi s + \sum_{i>0} r_i\right)^j \in \pi^2 S + \pi \sum_{i>0} A_i$ for $0 < j < q$, and $(\pi s + \sum_{i>0} r_i)^q \in \pi^2 S + \pi \sum_{i>0} A_i + \sum_{i>0} A_i^q$.

Now we take care of the inductive step: suppose we know that

$$(x')^{q^h} \equiv (y')^{q^h} \mod \pi^{h+1} S' + \sum_{j=0}^{h} \pi^j \sum_{i>0} (R_i')^{q^{h-j}},$$

for some h. Then let $S = \pi S'$ and let $R_1, R_2, \ldots$ be the sequence of ideals $R_1^q, R_2^q, \ldots, \pi R_1, \pi R_2, \ldots$, and apply the $h = 1$ case, with $x = (x')^{q^h}$, $y = (y')^{q^h}$, to get the $h + 1$ case of the stated proposition. □

3 Moduli of formal modules.

3.1 Splittings of moduli of formal modules.

We will making many constructions which are functorial on finite field extensions, and studying the maps induced in these functors by such a field extension. Our general notation for a field extension will be E/K and we will use A, B to denote the rings of integers in K, E, respectively, and we will use e, f, n to denote the ramification degree, residue degree, and total degree, respectively, of E/K. Uniformizers of K, E will be written π_K, π_E respectively, unless it is clear from context what the uniformizer has to be (e.g. E/K is unramified) and then we will sometimes write simply π.

Let K be a p-adic number field. We consider the full Lazard ring

$$L^A \cong A[S_2^A, S_3^A, S_4^A, \dots]$$

and the A-typical Lazard ring

$$V^A \cong A[v_1^A, v_2^A, \dots] \cong A[V_1^A, V_2^A, \dots]$$

with $\{v_i^A\}$ the Araki generators, i.e., if the universal formal A-module law on V^A has fgl-logarithm

$$\lim_{h\to\infty} p^{-h}[p^h](x) = \log(x) = \sum_{i\geqslant 0} \ell_i^A x^{p^i},$$

then the log coefficients ℓ_i^A satisfy

$$\pi \ell_h^A = \sum_{i=0}^{h} \ell_i^A (v_{h-i}^A)^{q^i}, \tag{3.1.1}$$

and $\{V_i^A\}$ the Hazewinkel generators, which satisfy

$$\pi \ell_h^A = \sum_{i=0}^{h-1} \ell_i^A (V_{h-i}^A)^{q^i}, \tag{3.1.2}$$

The Araki v_i^A agrees mod π with the Hazewinkel V_i^A.

Definition 3.1.1. *These rings are actually objects in* $\mathbf{GrCAlg}(A)$*. The gradings are given by*

$$\begin{aligned} |S_i^A| &= 2(i-1) \\ |v_i^A| &= 2(q^i-1) \\ |V_i^A| &= 2(q^i-1) \\ |t_i^A| &= 2(q^i-1) \end{aligned}$$

where q is the cardinality of the residue field of K. These algebras are connected, i.e., the direct summand consisting of elements of degree 0 *is precisely the image of the unit map, so we let the augmentation be projection to this direct summand.*

The fact that all generators are in even degrees conveniently allows us to ignore the graded-commutativity convention.

Definition 3.1.2. *In order to use the combinatorics of Prop. 2.4.3, given a sequence $(i_1, i_2, \ldots, i_m)$ of positive integers, we define the polynomial $v_I^A \in V^A$ by*

$$\begin{aligned} v_\varnothing^A &= 1 \\ v_{(i_1,i_2,\ldots,i_m)}^A &= v_{i_1}^A (v_{(i_2,i_3,\ldots,i_m)}^A)^{q^{i_1}}. \end{aligned}$$

Similarly we let $t_I^A \in V^A T$ be

$$\begin{aligned} t_\varnothing^A &= 1 \\ t_{(i_1,i_2,\ldots,i_m)}^A &= t_{i_1}^A (t_{(i_2,i_3,\ldots,i_m)}^A)^{q^{i_1}}. \end{aligned}$$

The degrees of v_I^A and t_I^A are both $2(q^{||I||}-1)$.

Hazewinkel has a convenient formula for ℓ_i^A in terms of his generators; we wish to derive a similar formula in terms of the Araki generators, and then to rewrite the formula using some of the combinatorics we have developed.

Proposition 3.1.3.

$$\begin{aligned}
(\pi - \pi^{q^n})\ell_h^A &= \sum_{i_1+\ldots+i_r=h} \left(v_{i_1}^A \prod_{j=2}^{r} \frac{(v_{i_j}^A)^{q^{\sum_{k=1}^{j-1} i_k}}}{\pi - \pi^{q^{\sum_{k=1}^{j-1} i_k}}} \right) \quad \textit{and} \\
\ell_h^A &= \sum_{||I||=h} \frac{v_I^A}{\Pi_A(I)},
\end{aligned}$$

where all i_j are positive integers, and the I are sequences of positive integers with v_I^A and Π_A as in Prop. 2.4.3.

Proof. First, we note that the formula gives us $(\pi - \pi^q)\ell_1^A = v_1^A$, which is the correct answer, by equation (3.1.1).

We proceed by induction. Suppose the proposition holds for all ℓ_i^A with $1 \leqslant i < h$. Then

$$\begin{aligned}
(\pi - \pi^{q^h})\ell_h^A &= \sum_{i=0}^{h-1} \ell_i^A (v_{h-i}^A)^{q^i} \\
&= \sum_{i=0}^{h-1} \frac{(v_{h-i}^A)^{q^i}}{\pi - \pi^{q^i}} \sum_{i_1+\ldots+i_r=i} \left(v_{i_1}^A \prod_{j=2}^{r} \frac{(v_{i_j}^A)^{q^{\sum_{k=1}^{j-1} i_k}}}{\pi - \pi^{q^{\sum_{k=1}^{j-1} i_k}}} \right) \\
&= \sum_{i_1+\ldots+i_r=h} \left(v_{i_1}^A \prod_{j=2}^{r} \frac{(v_{i_j}^A)^{q^{\sum_{k=1}^{j-1} i_k}}}{\pi - \pi^{q^{\sum_{k=1}^{j-1} i_k}}} \right).
\end{aligned}$$

Now we note that there is an obvious bijection between sequences of integers $i_1, \ldots, i_r$ with $\sum_{j=1}^r i_j = h$ and sequences I with $||I|| = h$, and

$$\frac{v_{i_1}^A}{\pi - \pi^{q^h}} \prod_{j=2}^{r} \frac{(v_{i_j}^A)^{q^{\sum_{k=1}^{j-1} i_k}}}{\pi - \pi^{q^{\sum_{k=1}^{j-1} i_k}}} = \frac{v_{(i_1,\ldots,i_r)}^A}{\Pi_A((i_1, \ldots, i_r))},$$

giving us the second equality. $\square$

Corollary 3.1.4. *Given a sequence of elements $x_1, ..., x_m \in V^A$ we have*

$$\sum_{t=1}^{m}{}^{F} x_t = \sum_{I}{}^{F} v_I^A w_I^A(x_1, x_2, ..., x_m),$$

where F is the universal formal A-module law on V^A, and the right-hand sum is taken over all finite sequences I of positive integers.

Proof. From Prop.3.1.3 we have

$$\log_F(X) = \sum_I \frac{v_I^A X^{q^{||I||}}}{\Pi_A(I)},$$

and now we have

$$\begin{aligned}
\log_F\left(\sum_J{}^F v_J^A w_J^A\right) &= \sum_J \log_F(v_J^A w_J^A) \\
&= \sum_{I,J} \frac{v_{IJ}^A (w_J^A)^{q^{||I||}}}{\Pi_A(I)} \\
&= \sum_{I,J} \frac{v_{IJ}^A}{\Pi_A(IJ)} \frac{\Pi_A(IJ)}{\Pi_A(I)} (w_J^A)^{q^{||I||}} \\
&= \sum_{t,K} \frac{v_K^A}{\Pi_A(K)} x_t^{q^{||K||}} \\
&= \sum_t \log_F(x_t) \\
&= \log_F \sum_{t=1}^{m}{}^F x_t.
\end{aligned}$$

□

The following lemma will be useful later, when handling logarithms.

Lemma 3.1.5. *1. Let R be a commutative A-algebra such that the algebra structure map $A \to R$ is injective, let $V^A \xrightarrow{\gamma} R \otimes_A K$ be a morphism of commutative A-algebras, and let h, j be positive integers. Suppose $\gamma(\ell_i^A) = 0$ for all $i < j$ with $h \nmid i$. Then $\gamma(V_i^A) = 0$ for all $i < j$ with $h \nmid i$.*

2. *Let R be a commutative A-algebra such that the algebra structure map $A \to R$ is injective, let $V^A \xrightarrow{\gamma} R \otimes_A K$ be a morphism of commutative A-algebras, and let h be a positive integer. Suppose $\gamma(\ell_i^A) = 0$ for all i with $h \nmid i$. Then $\gamma(V_i^A) = 0$ for all i with $h \nmid i$.*

Proof. 1. First, if $h = 1$, the proposition is vacuous. If $h > 1$, then $\gamma(\ell_1^A) = \pi_A^{-1} V_1^A = 0$ implies $\gamma(V_1^A) = 0$.

Now we proceed by induction. Let j' be a positive integer, $j' < j$. Suppose $\gamma(V_i^A) = 0$ for all $i < j'$ with $h \nmid i$. We show that $\gamma(V_{j'}^A) = 0$ if $h \nmid i$: assume $h \nmid j'$. Then

$$\begin{aligned} 0 &= \gamma(\ell_{j'}^A) \\ &= \pi_A^{-1} \sum_{a=0}^{j-1} \gamma(\ell_a^A) \gamma(V_{j-a}^A)^{q^a} \\ &= 0, \end{aligned}$$

since either $h \nmid a$ or $h \nmid (j - a)$ for each a in the sum.

2. This follows from the first part of the lemma, by induction on j.

□

Hazewinkel's generators are easier to compute with in some contexts but later we will derive a formula for $\eta_R : V^A \to V^A T$ which is very well-behaved in terms of the Araki generators, and since we will eventually want to consider a number of Bockstein spectral sequences in which the differentials are determined by η_R, we will have many occasions in which we will use the Araki generators. For the rest of this chapter, however, we will be using the Hazewinkel generators of V^A.

There are also the algebras classifying strict isomorphisms of formal A-module laws: the full algebra is $L^A B$, which we will not have much occasion to use, but the A-typical algebra is

$$V^A T \cong V^A[t_1^A, t_2^A, t_3^A, \dots]$$

and the Hopf algebroid $(V^A, V^A T)$ is one of the main objects of study in these notes. There is a map $L^A \longrightarrow V^A$ classifying the universal A-typical formal A-module law; it is given by

$$S_i^A \mapsto \begin{cases} V_j^A & \text{if } i = q^j \\ 0 & \text{if } i \text{ is not a power of } q \end{cases}.$$

(see [Hazewinkel, 1978].)

The ring V^A has prime ideals

$$I_h^A = (\pi_A, V_1^A, \ldots, V_{h-1}^A),$$

for all positive integers h (we adopt the convention, which generalizes the standard conventions of BP-theory, that $I_1^A = (\pi)$ and $V_0^A = \pi$). These ideals are of critical importance to almost everything that follows in these notes.

The universal A-typical formal A-module law on V^A is a formal A-module law, so it is classified by a map $L^A \xrightarrow{\Theta} V^A$; in addition, any formal A-module law is canonically isomorphic to an A-typical one (see [Hazewinkel, 1978]), so we get a map $V^A \longrightarrow L^A$ which we will call *Cartier typicalization*. Since the typicalization of a formal A-module law that is already A-typical is identical to itself, the map $L^A \longrightarrow V^A$ is a retraction of the Cartier typicalization map.

Proposition 3.1.6. *Let K be a p-adic number field. Then the composite*

$$V^A \xrightarrow{CT} L^A \xrightarrow{\Theta} V^A$$

is the identity map on V^A. Hence V^A is a retract of L^A.

Proof. This composite classifies the Cartier A-typicalization of the universal A-typical formal A-module law, but since the A-typicalization of a formal A-module law which is already A-typical is identical to itself (see [Hazewinkel, 1978]), this map $V^A \longrightarrow V^A$ classifies the universal A-typical formal A-module law, i.e., it is the identity map. $\square$

Given a particular choice of embedding of fields $K \xrightarrow{\sigma} E$, there is an induced map of Lazard rings

$$V^A \xrightarrow{\gamma_\sigma} V^B,$$

given by classifying the A-typical formal A-module law underlying the universal B-typical formal B-module law on V^B. We will eventually compute this map, modulo certain ideals, for a large set of extensions E/K, but for now we will compute it just for E/K unramified.

Proposition 3.1.7. *Let E/K be unramified of degree f, both fields having uniformizer π, and let q be the cardinality of the residue field of K. Using the Hazewinkel generators for V^A and V^B, the map $V^A \xrightarrow{\gamma_\sigma} V^B$ is determined by*

$$\gamma_\sigma(V_i^A) = \begin{cases} V_{i/f}^B & \text{if } f \mid i \\ 0 & \text{if } f \nmid i \end{cases}.$$

Proof. First, since $\gamma_\sigma(\ell_i^A) = 0$ if $f \nmid i$, we have that $\gamma_\sigma(V_i^A) = 0$ for $i < f$, and

$$\begin{aligned} \frac{1}{\pi} V_1^B &= \ell_1^B \\ &= \gamma_\sigma(\ell_f^A) \\ &= \gamma_\sigma\left(\frac{1}{\pi}\sum_{i=0}^{f-1} \ell_i^A (V_{f-i}^A)^{q^i}\right) \\ &= \frac{1}{\pi}\gamma_\sigma(V_f^A), \end{aligned}$$

so $\gamma_\sigma(V_f^A) = V_1^B$.

We proceed by induction. Suppose that $\gamma_\sigma(V_i^A) = 0$ if $f \nmid i$ and $i < hf$ and

$\gamma_\sigma(V_i^A) = V_{i/f}^B$ if $f \mid i$ and $i < hf$. Then

$$\begin{aligned}
\gamma_\sigma(\ell_{hf}^A) &= \gamma_\sigma\left(\sum_{i=0}^{hf-1} \ell_i^A (V_{hf-i}^A)^{q^i}\right) \\
&= \sum_{i=0}^{h-1} \ell_i^B \gamma_\sigma(V_{(h-i)f}^A)^{q^{if}} \\
&= \gamma_\sigma(V_{hf}^A) + \sum_{i=1}^{h-1} \ell_i^B (V_{h-i}^B)^{q^{if}} \\
&= \ell_h^B = V_h^B + \sum_{i=1}^{h-1} \ell_i^B (V_{h-i}^B)^{q^{if}},
\end{aligned}$$

so $\gamma_\sigma(V_{hf}^A) = V_h^B$.

Now suppose that $0 < j < f$ and $\gamma_\sigma(V_i^A) = 0$ if $f \nmid i$ and $i < hf + j$ and $\gamma_\sigma(V_i^A) = V_{i/f}^B$ if $f \mid i$ and $i < hf$. We want to show that $\gamma_\sigma(V_{hf+j}^A) = 0$. We see that

$$\begin{aligned}
0 &= \gamma_\sigma(\ell_{hf+j}^A) \\
&= \gamma_\sigma\left(\sum_{i=0}^{hf+j-1} \ell_i^A (V_{hf+j-i}^A)^{q^i}\right) \\
&= \sum_{i=0}^{h} \ell_i^B \gamma_\sigma(V_{(h-i)f+j}^A)^{q^{if}} \\
&= \gamma_\sigma(V_{hf+j}^A).
\end{aligned}$$

This concludes the induction. □

Corollary 3.1.8. *Let E/K be unramified of degree f. Then*

$$V^B \cong (V^A \otimes_A B)/\left(\{V_i^A : f \nmid i\}\right).$$

When E/K is ramified, the map γ_σ is much more complicated. We will return to it later, but for now we will at least show its injectivity and rational surjectivity.

Proposition 3.1.9. *Let E/K be totally ramified. Then*

$$V^A \otimes_A B \otimes_B E \xrightarrow{\gamma'_\sigma \otimes_B E} V^B \otimes_B E$$

is surjective.

Proof. We will use Hazewinkel generators here. Given any V_i^B, we want to produce some element of $V^A \otimes_A B$ which maps to it. We begin with $i = 1$. Since $\ell_1^A = \pi_K^{-1} V_1^A$ and $\ell_1^B = \pi_E^{-1} V_1^B$ we have $\gamma'_\sigma(\frac{\pi_E}{\pi_K} V_1^A) = V_1^B$.

Now we proceed by induction. Suppose that we have shown that there is an element in $(V_1^A, \ldots, V_{j-1}^A) \subseteq V^A \otimes_A B$ which maps via γ'_σ to V_{j-1}^B. Then

$$\gamma'_\sigma(\pi_K^{-1} \sum_{i=0}^{j-1} \ell_i^A (V_{j-i}^A)^{q^i}) = \pi_E^{-1} \sum_{i=0}^{j-1} \ell_i^B (V_{j-i}^B)^{q^i}$$

and hence

$$\begin{aligned} \gamma'_\sigma(\pi_K^{-1} V_j^A) &= \pi_E^{-1} \sum_{i=0}^{j-1} \ell_i^B (V_{j-i}^B)^{q^i} - \gamma'_\sigma(\pi_K^{-1} \sum_{i=1}^{j-1} \ell_i^A (V_{j-i}^A)^{q^i}) \\ &\equiv \pi_E^{-1} V_j^B \mod (V_1^B, V_2^B, \ldots, V_{j-1}^B). \end{aligned} \tag{3.1.3}$$

So $\gamma'_\sigma(\frac{\pi_E}{\pi_K} V_j^A) \equiv V_j^B$ modulo terms hit by elements in the ideal generated by Hazewinkel generators of lower degree. This completes the induction. $\square$

Corollary 3.1.10. *Let E/K be a totally ramified, finite extension of p-adic number fields, and choose an $x \in V^B$. Then there exists some integer a such that*

$$\pi_E^a x \in \mathrm{im}\,(V^A \otimes_A B \xrightarrow{\gamma'_\sigma} V^B).$$

Corollary 3.1.11. *Let E/K be a totally ramified, finite extension of p-adic number fields, and let F be an A-typical formal A-module over a commutative B-algebra R. If F admits an extension to a B-typical formal B-module (i.e., a factorization of the structure map $A \xrightarrow{\rho} \mathrm{End}(F)$ through B) then that extension is unique.*

Proof. We need to show that, given maps

$$\begin{tikzcd} V^A \otimes_A B \arrow[r, "\gamma'_\sigma"] \arrow[dr, "\theta"] & V^B \\ & R \end{tikzcd}$$

where θ is the classifying map of F, if there exists a map $V^B \xrightarrow{g} R$ making the diagram commute, then that map g is unique.

The map g is determined by its values on the generators V_i^B of V^B. For any choice of i, let a be an integer such that $\pi_E^a V_i^B \in \operatorname{im} \gamma'_\sigma$ (the existence of such an a is guaranteed by Cor. 3.1.10). Now let $\overline{\pi_E^a V_i^B}$ be a lift of $\pi_E^a V_i^B$ to $V^A \otimes_A B$, and in order for the diagram to commute, $g(\pi_E^a V_i^B)$ must be equal to $f(\overline{\pi_E^a V_i^B})$, so

$$g(V_i^B) = \pi_E^{-a} f(\overline{\pi_E^a V_i^B}),$$

completely determining g. Hence g is unique. $\square$

Corollary 3.1.12. *Let E/K be a finite extension of p-adic number fields. Let F be an A-typical formal A-module law over a commutative B-algebra R. Then, if F admits an extension to an B-typical formal B-module law, that extension is unique.*

Proof. Let E_{nr} be the maximal subextension of E which is unramified over K. Then $V^A \longrightarrow V^{O_{E_{\mathrm{nr}}}}$ is surjective, so any map $V^A \otimes_A B \longrightarrow R$ admitting a factorization through $V^A \otimes_A B \longrightarrow V^{O_{E_{\mathrm{nr}}}} \otimes_{O_{E_{\mathrm{nr}}}} B$ admits only one such factorization, i.e., if there is an $O_{E_{\mathrm{nr}}}$-typical formal $O_{E_{\mathrm{nr}}}$-module law extending F, it is unique.

Now we use Cor. 3.1.11 to see that if there is an extension of this $O_{E_{\mathrm{nr}}}$-typical formal $O_{E_{\mathrm{nr}}}$-module law to a B-typical formal B-module law, then that B-typical formal B-module law is unique. $\square$

We will see that, when E/K is unramified, there may exist multiple extensions of the structure map $A \xrightarrow{\rho} \mathrm{End}(F)$ of a A-typical formal A-module law to the structure map $B \longrightarrow \mathrm{End}(F)$ of a formal B-module law, but as a result of the previous proposition, only one of these extensions yields an B-typical formal B-module law.

For the proof of the next proposition we will use a monomial ordering on V^A. This may also come in handy later, e.g. for Grobner basis purposes.

Definition 3.1.13. *We put the following ordering on the Hazewinkel generators of V^A:*

$$V_i^A \leqslant V_j^A \quad \textit{iff} \quad i \leqslant j$$

and we put the lexicographic order on the monomials of V^A.

Since this total ordering on the generators of V^A, V^B is preserved by $V^A \xrightarrow{\gamma_\sigma} V^B$ when E/K is totally ramified (equation 3.1.3), the ordering on the monomials of V^A, V^B is also preserved by γ_σ. The ordering on the monomials in V^A is a total ordering (see e.g. [Cox *et al.*, 2005]).

Proposition 3.1.14. *Let E/K be a totally ramified, finite extension of p-adic number fields. Then $V^A \xrightarrow{\gamma_\sigma} V^B$ is injective.*

Proof. Suppose $x \in \ker \gamma_\sigma$. Let x_0 be the sum of the monomial terms of x of highest order in the lexicographic ordering. Then, since γ_σ preserves the ordering of monomials, $\gamma_\sigma(x_0) = 0$. However, since the lexicographic ordering on V^A is a total ordering, x_0 is a monomial; let $x_0 = \prod_{i \in I}(V_i^A)^{\epsilon_i}$ for some set I of positive integers and positive integers $\{\epsilon_i\}_{i \in I}$. Then the terms of $\gamma_\sigma(x_0)$ of highest order in the lexicographic ordering on V^B consist of just the monomial $\prod_{i \in I}(V_i^B)^{\epsilon_i}$ (equation 3.1.3). Since $\gamma_\sigma(x_0) = 0$ this implies that $\prod_{i \in I}(V_i^B)^{\epsilon_i} = 0$, i.e., $x_0 = 0$ and finally $x = 0$. $\square$

At first glance this suggests that V^A and V^B are not very different when E/K is totally ramified: they only differ by a cokernel which consists entirely of π_E-torsion. When we begin to localize and complete at invariant primes, however, we will see that this π_E-torsion is very important.

We recall that, for any formal A-module law F over a commutative A-algebra A such that $A \hookrightarrow A \otimes_A K$ is injective, there exists a power series $\log_F(X) \in \mathbb{Q} \otimes_{\mathbb{Z}} A[X]$ with the properties that $F'(0) = 1$ and

$$F(X, Y) = \log_F^{-1}(\log_F(X) + \log_F(Y)),$$

i.e., $\log_F$ is the logarithm of the underlying formal group law of F. We recall that ρ is the standard notation for the structure map $A \xrightarrow{\rho} \mathrm{End}(F)$, and $\log_F$ has the additional property that $\rho(\alpha) = \log_F^{-1}(\alpha \log_F(X))$ (see [Hazewinkel, 1978]).

Definition 3.1.15. *Let R be a commutative ring and A a commutative R-algebra, and let $PS(A)$ denote the group of power series $F(X) \in A[[X]]$ in one variable such that $F(0) = 0$ and $F'(0) \in A^\times$, with the group operation given by composition of power series. Let $PS^0(A) \subseteq PS(A)$ be the subgroup consisting of all power series F such that $F'(0) = 1$.*

Proposition 3.1.16. *These groups fit into a short exact sequence*

$$1 \longrightarrow PS^0(A) \longrightarrow PS(A) \longrightarrow A^\times \longrightarrow 1,$$

which is split.

Proof. Let $F, G \in PS(A)$ with $F'(0) = \alpha_F$, $G'(0) = \alpha_G$. Then $(F \circ G)'(0) = \alpha_F \alpha_G$, so the map $PS(A) \longrightarrow A^\times$ given by $F \mapsto F'(0)$ is a group homomorphism. That it is surjective is obvious (for any $\alpha \in A^\times$, the power series $\alpha X \in PS(A)$ maps to it), and its kernel is clearly the power series F such that $F'(0) = 1$, i.e., $PS^0(A)$. The splitting $A^\times \longrightarrow PS(A)$ is given by $\alpha \mapsto \alpha X$. □

The functors $A \mapsto PS(A)$ and $A \mapsto PS^0(A)$ are representable group-valued functors on R-algebras; their representing Hopf algebras are of the form

$$R[PS] \cong R[l_1^{\pm 1}, l_2, l_3, ...],$$

$$R[PS^0] \cong R[l_2, l_3, l_4, ...],$$

with l_i representing the ith coefficient in a power series. There is an obvious surjection $R[PS] \longrightarrow R[PS^0]$ sending l_1 to $1 \in R$. The coalgebra structure on these Hopf algebras is what carries the interesting information (about how power series compose). Since $PS(\mathbb{Z})$ is the automorphism group of the universal formal group law on $L^{\mathbb{Z}}$ there is an isomorphism $H^*(\mathbb{Z}[PS]) \cong \mathrm{Ext}^*_{MU_*MU}(MU_*, MU_*)$, and the coproduct on $\mathbb{Z}[PS]$ can be derived from that of (MU_*, MU_*MU) (see [Ravenel, 1986]).

We recall that a formal A-module law is called *A-typical* if it is classified by a map from the formal A-module law Lazard ring V^A. Over a commutative A-algebra A such that $A \mapsto A \otimes_A K$ is injective, the A-typicality of F is equivalent to $\log_F(X)$ being of the form $\sum_{i \geqslant 0} l_i X^{q^i}$, with q the cardinality of the residue field of A, for some collection $\{l_i \in A \otimes_A K\}$ (see 21.5.10 of [Hazewinkel, 1978]). The term p-typical is in wider circulation, and in the cases of interest to us here, it is equivalent to $\mathbb{Z}_p$-typical.

Proposition 3.1.17. *Let F be a p-typical formal A-module with a logarithm and let p^f be the cardinality of the residue field of K. Then the following conditions are equivalent:*

1. $[\zeta]_F(X) = \zeta X$ *for some primitive* $(p^f - 1)$*th root of unity* $\zeta \in A$.

2. F *is A-typical.*

3. $[\zeta]_F(X) = \zeta X$ *for all* $(p^f - 1)$*th roots of unity* $\zeta \in A$.

4. *The following diagram is commutative:*

$$\begin{tikzcd} \mu_{(p)}(A) \arrow[r, "\rho|_{\mu_{(p)}(A)}"] \arrow[dr, "s"'] & \mathrm{Aut}(F) \arrow[d] \\ & PS(A) \end{tikzcd},$$

where $\mu_{(p)}(A)$ is the group of roots of unity in A of order prime to p.

Proof. **Condition 1 implies condition 2** Let R be the ring over which F is defined and let $\{u_i\}$ be the log coefficients of F. Since $[\zeta]_F$ is by definition

$\log_F^{-1}(\zeta \log_F(X))$, assuming $\zeta X = [\zeta]_F(X)$ gives us

$$\begin{aligned}
\sum_{i\geqslant 0} \zeta^{p^i} u_i X^{p^i} &= \sum_{i\geqslant 0} u_i(\zeta X)^{p^i} \\
&= \log_F(\zeta X) \\
&= \log_F([\zeta]_F(X)) \\
&= \zeta \log_F(X) \\
&= \sum_{i\geqslant 0} \zeta u_i X^i,
\end{aligned}$$

so $u_i = 0$ for all i such that $\zeta^{p^i} \neq \zeta$, i.e., all i such that $(p^f - 1) \nmid (p^i - 1)$. We have factorizations $p^f - 1 = \prod_{d|f} \Phi_d(p)$ and $p^i - 1 = \prod_{d|i} \Phi_d(p)$, where $\Phi_d(X)$ is the dth cyclotomic polynomial, which is irreducible. These factorizations are unique factorizations since $\mathbb{Z}[X]$ is a UFD, so when $f \nmid i$, we see that $\Phi_{\frac{f}{\mathrm{lcd}(f,i)}}(p)$ appears in the factorization of $p^f - 1$ but not in the factorization of $p^i - 1$. Hence $(p^f - 1)|(p^i - 1)$ only if $f \mid i$, and $u_i = 0$ if $f \nmid i$. This tells us that $\log_F(X) = \sum_{i\geqslant 0} u_{fi} X^{p^{fi}}$, so F is A-typical.

Condition 2 implies condition 3 We know that F's logarithm is of the form $\log_F(X) = \sum_{i\geqslant 0} u_i X^{p^{fi}}$ for some $\{u_i\}$. Choose a $p^f - 1$th root of unity $\zeta \in A$. Now

$$\begin{aligned}
[\zeta]_F(X) &= \log_F^{-1}(\zeta \log_F(X)) \\
&= \log_F^{-1}(\zeta \sum_{i\geqslant 0} u_i X^{p^{fi}}) \\
&= \log_F^{-1}(\sum_{i\geqslant 0} u_i \zeta^{p^{fi}} X^{p^{fi}}) \\
&= \log_F^{-1}(\log_F(\zeta X)) = \zeta X.
\end{aligned}$$

Condition 3 implies condition 1 This is immediate.

Condition 3 is equivalent to condition 4 Every element in $\mu_{(p)}(A)$ is a $(p^f - 1)$th root of unity (see 2.4.3 Prop. 2 of [Robert, 2000]). Let $\zeta \in \mu_{(p)}(A)$, and now $s(\zeta) = \zeta(X)$ while ζ's image under the composite map

$$\mu_{(p)}(A) \hookrightarrow \mathrm{Aut}(F) \hookrightarrow PS(A)$$

is the power series $[\zeta]_F(X)$.

$\square$

Proposition 3.1.18. *Let E/K be a totally ramified extension of p-adic number fields and let A be a commutative B-algebra. If F/A is a A-typical formal B-module law with a logarithm, then it is B-typical.*

Proof. Since E/K is totally ramified, $\log_F(X) = \sum_{i \geqslant 0} l_i X^{q^{fi}}$, where q^f is the cardinality of the residue fields of both A and B; so it is immediate that F is B-typical if it is A-typical.

Alternatively, since the triangle in the following diagram commutes:

$$\begin{tikzcd}
\mu_{(p)}(B) \arrow[r, "\cong"] & \mu_{(p)}(A) \arrow[r, "\rho|_{\mu_{(p)}(B)}"] \arrow[dr, "s"] & \mathrm{Aut}(F) \arrow[d] \\
 & & PS(A)
\end{tikzcd},$$

composing with the isomorphism $\mu_{(p)}(B) \cong \mu_{(p)}(A)$ gives us that
$\mu_{(p)}(B) \xrightarrow{s} PS(B)$ is equal to $\mu_{(p)}(B) \hookrightarrow \mathrm{Aut}(F) \hookrightarrow PS(B)$. $\square$

Now we remove the hypothesis that the formal module has a logarithm.

Proposition 3.1.19. *Let E/K be a totally ramified extension of p-adic number fields and let R be a commutative B-algebra. If F is a A-typical formal B-module law over R, then it is B-typical.*

Proof. Let S be an B-algebra which surjects on to A and such that $S \longrightarrow S \otimes_B E$ is injective. Such an algebra always exists, e.g. $B[Z_a : a \in R]$ (this strategy of proof is adapted from 21.7.18 of [Hazewinkel, 1978]). Let $S \xrightarrow{w} R$ be a surjection. We choose a lift $\bar{F}$ of F to S, and since L^B and $V^A \otimes_A B$ are free B-algebras, we can choose lifts $L^B \longrightarrow S$, $V^A \otimes_A B \longrightarrow S$ of the classifying maps of F as a formal B-module law and as an A-typical formal A-module law, respectively. Now by construction $\bar{F}$ is a A-typical formal B-module law with a logarithm, so it is also B-typical and there exists a map $V^B \longrightarrow S$ classifying it; composing this map with w gives us F as an B-typical formal B-module law. We include a diagram showing all these maps:

$$\begin{tikzcd}
L^A \otimes_A B \arrow[r] \arrow[d] & V^A \otimes_A B \arrow[d] \arrow[ddr] & & \\
L^B \arrow[r] \arrow[drr] & V^B \arrow[dr] & & \\
& & S \arrow[r, "w"] & R
\end{tikzcd}$$

□

Corollary 3.1.20. *Let E/K be a totally ramified extension of p-adic number fields. Then $V^B \cong L^B \otimes_{L^A} V^A$.*

Proof. Given a commutative B-algebra A with an A-typical formal B-module law on it, i.e., maps making the following diagram commute:

$$\begin{tikzcd}
L^A \otimes_A B \arrow[r] \arrow[d] & V^A \otimes_A B \arrow[d] \arrow[ddr] & \\
L^B \arrow[r] \arrow[drr] & V^B & \\
& & A
\end{tikzcd},$$

F is B-typical, i.e., there exists a unique map $V^B \longrightarrow A$ making the above diagram commute. So the square in the above diagram is a pushout square in commutative B-algebras, i.e., $V^B \cong L^B \otimes_{L^A \otimes_A B} (V^A \otimes_A B)$.

Now we use Prop. **??** to get the desired result. □

The situation is very different for E/K unramified.

Proposition 3.1.21. *Let E/K be unramified of degree f. Then $V^A \otimes_{L^A} L^B$ surjects on to V^B, but this map is not an isomorphism unless $f = 1$.*

Proof. We consider the following diagrams in the category of graded commutative B-algebras:

$$\begin{tikzcd} & V^A \otimes_A B \\ L^A \otimes_A B \arrow[ur, "\Theta"] \arrow[dr] & \\ & L^B \end{tikzcd} \tag{3.1.4}$$

$$\begin{tikzcd} & B \\ B \arrow[ur] \arrow[dr] & \\ & V^B \end{tikzcd} \tag{3.1.5}$$

We refer to diagram (3.1.4) as X_1 and diagram (3.1.5) as X_2. There is a map $X_1 \longrightarrow X_2$ given by Θ and the augmentations and a map $X_2 \longrightarrow X_1$ given by Cartier typicalization and the unit maps, and it is trivial to check that the composite $X_2 \longrightarrow X_1 \longrightarrow X_2$ is the identity on X_2 and that the map $\operatorname{colim} X_1 \longrightarrow \operatorname{colim} X_2$ is the map produced via the universal property of the pushout of X_1 by mapping L^B and $V^A \otimes_A B$ to V^B. Hence $\operatorname{colim} X_2 \cong V^B$ is a retract of $\operatorname{colim} X_1 \cong (V^A \otimes_A B) \otimes_{L^A \otimes_A B} L^B$, and the map $(V^A \otimes_A B) \otimes_{L^A \otimes_A B} L^B \longrightarrow V^B$ is a surjection.

However, when $f > 1$ the map g is not an isomorphism: we let γ_σ be the map $V^A \otimes_A B \xrightarrow{\gamma_\sigma} V^B$, and then it is certainly true that $\gamma_\sigma(V_1^A) = 0$, by Prop. 3.1.7, so $g(1 \otimes V_1^A) = 0$. However, we show that $1 \otimes V_1^A \neq 0 \in (V^A \otimes_A B) \otimes_{L^A \otimes_A B} L^B$. Considering $(V^A \otimes_A B) \otimes_{L^A \otimes_A B} L^B$ as a quotient of $(V^A \otimes_A B) \otimes_B L^B \cong B[S_2^B, S_3^B, \dots][V_1^A, V_2^A, \dots]$, all the $S_i^B \otimes 1$ are identified with zero for $q \nmid i$, but $\pi S_q^B \otimes 1 \sim 1 \otimes V_1^A$ since S_q^A maps to both in the appropriate pushout diagram, and since all involved maps are graded, there are no other elements in $L^A \otimes_A B$ which can map to $V_1^A \in V^A \otimes_A B$. So $1 \otimes V_1^A$ is nonzero in $L^B \otimes_{L^A} V^A$, although it is divisible by π there, which is a surprising twist. $\square$

We could now ask how far this map is from being an isomorphism. We retain the notation X_1 for diagram (3.1.4) and X_2 for diagram (3.1.5). We let $\bar{CT}$ denote the given map $X_2 \xrightarrow{\bar{CT}} X_1$, and then we have an exact sequence of diagrams of graded commutative B-algebras

$$0 \longrightarrow X_2 \xrightarrow{\bar{CT}} X_1 \longrightarrow \operatorname{coker} \bar{CT} \longrightarrow 0$$

and, applying colim, a diagram with exact top row

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{colim} X_2 \arrow[r, "\operatorname{colim} \bar{CT}"] \arrow[d, "\cong"] & \operatorname{colim} X_1 \arrow[r] \arrow[d, "\cong"] & \operatorname{coker} \bar{CT} \arrow[r] \arrow[d, "\cong"] & 0 . \\
 & V^B & L^B \otimes_{L^A} V^A & V^A \otimes_{L^A} \left(L^B/V^B\right) &
\end{tikzcd}$$

We can think of the B-algebra $\left(L^B \otimes_{L^A} V^A\right)/V^B \cong V^A \otimes_{L^A} \left(L^B/V^B\right)$ as measuring the failure of extensions of A-typical formal A-module laws to formal-B module laws to be B-typical, i.e., in some way it measures the existence of certain unusual formal power series that we describe below.

Corollary 3.1.22. *When E/K is unramified and nontrivial there exists at least one A-typical formal B-module which is not B-typical. In particular, the universal A-typical formal B-module law on $L^B \otimes_{L^A} V^A$ is not B-typical.*

Proof. If an B-typical formal B-module law were equivalent to a A-typical formal B-module law then the following diagram would be a pushout square (as in the totally ramified case):

$$\begin{tikzcd}
L^A \otimes_A B \arrow[r] \arrow[d] & V^A \otimes_A B . \arrow[d] \\
L^B \arrow[r] & V^B
\end{tikzcd}$$

But from the preceding proposition we know that this is not a pushout square. $\square$

Corollary 3.1.23. *Let E/K be unramified of degree f and let ζ be a primitive (q^f-1)th root of unity. Then there exists at least one power series $G(X) \in B[[X]]$ such that $G^{\frac{q^f-1}{q-1}}(X) = \zeta^{\frac{q^f-1}{q-1}}X$ (where by G^a we mean G composed with itself a times, not G multiplied with itself a times!) and $G(X) = \alpha_1 X + \alpha_2 X^2 + \alpha_3 X^3 + \dots$ with infinitely many of the α_i nonzero.*

In particular, G generates a cyclic group of order $q^f - 1$ in $PS(B)$, and while $G^{\frac{q^f-1}{q-1}} \in \operatorname{im} s$, we have that G itself is not in $\operatorname{im} s$. We note that G is conjugate to an element in $\operatorname{im} s$, however, since $G(X) = \log_F^{-1}((\zeta \log_F)(X))$.

Proof. We choose an A-typical formal B-module law F over B which is not B-typical.

(One way to do this is to choose a nonzero $x \in B$ and then consider the formal B-module law classified by the map $L^B \longrightarrow B$ sending S_q^B to x and all other S_i^B to zero (there are, of course, other, less trivial means of producing suitable formal modules of this kind). Then we have a map $V^A \otimes_A B \longrightarrow B$ sending V_1^A to πx and all other V_i^A to zero making the following diagram commutative:

$$\begin{tikzcd}
L^A \otimes_A B \arrow[r] \arrow[d] & V^A \otimes_A B \arrow[ddr] & \\
L^B \arrow[drr] & & \\
& & B
\end{tikzcd},$$

so we get an induced map from the pushout $L^B \otimes_{L^A} V^A$ to B but it is impossible for it to factor through the classifying map $L^B \otimes_{L^A} V^A \longrightarrow V^B$ because the only element of $L^B \otimes_{L^A} V^A$ that is nonzero in B is zero in V^B.)

Now our desired power series G is $[\zeta]_F(X)$. Since $G^{\frac{q^f-1}{q-1}}$ is the image of $\rho(\zeta^{\frac{q^f-1}{q-1}}) \in \operatorname{End}(F)$ and F is A-typical, $G^{\frac{q^f-1}{q-1}}(X) = \zeta^{\frac{q^f-1}{q-1}}X$.

It remains to be shown that, if $G(X) = \sum_{i\geqslant 1} \alpha_i X^i$, then infinitely many α_i are nonzero. Suppose this is not so, and let j be the largest integer such that α_j

is nonzero. Then

$$\begin{aligned} G^{\frac{q^f-1}{q-1}}(X) &\equiv \alpha_j^{1+j+j^2+\cdots+j^{\frac{q^f-1}{q-1}-1}} X^{j^{\frac{q^f-1}{q-1}}} \\ &\equiv \alpha_j^{\frac{j^{\frac{q^f-1}{q-1}}-1}{q-1}} X^{j^{\frac{q^f-1}{q-1}}} \mod X, X^2, X^3, \dots, X^{j^{\frac{q^f-1}{q-1}}-1}, \end{aligned}$$

which is nonzero, so $G^{\frac{q^f-1}{q-1}}(X)$ cannot be $\zeta^{\frac{q^f-1}{q-1}}X$. □

Any power series $G \in B[[X]]$ as in the above corollary (the ζ-series of an A-typical formal B-module law which is not B-typical) will converge in $\mathbf{m}_E - \{0\}$, the complement of zero in the maximal ideal of B, and we can easily bound the number of fixed points in that region. It would be interesting, at some point, to study the analytic properties of such power series as functions on that maximal ideal. This construction is associating, to each power series of a specific type, a free action of a cyclic group on $\mathbf{m}_E - \{0\}$, which is analytic in an appropriate sense; there are various things one could do with this. One would be to consider this as a family of representations of C_{q^f-1} in the analytic automorphism group of $\mathbf{m}_E - \{0\}$–an algebraic analogue of the study of group representations in the diffeomorphisms of a sphere.

We recall that the structure of L^A is given by

$$L^A \cong A[S_2^A, S_3^A, S_4^A, \dots],$$

where the universal formal A-module law on L^A has logarithm satisfying the functional equation

$$\log_{F_A}(X) = X + \sum_{i\geqslant 2} S_i^A X^i - \sum_{j\geqslant 1} S_{q^j}^A X^{q^j} + \sum_{j\geqslant 1} \pi_K^{-1} S_{q^j}^A \sigma_*^j \left(\log_{F_A}(X^{q^j})\right),$$

where σ is the Frobenius (qth power) map on the generators S_i^A of L^A, and for any power series $g \in A[[X]]$ we let $(\sigma_*^j)(g)$ be σ^j applied to the coefficients of g (see 21.4.8 of [Hazewinkel, 1978]). We want a formula of some kind for the log coefficients of this formal A-module law.

Proposition 3.1.24. *Let* $\log_{F_A}(X) = \sum_{i \geqslant 1} u_i^A X^i$. *Fix an integer* $i > 1$ *and let* I_i *denote the set of all ordered tuples* $(j_1, \ldots, j_r)$, $r \geqslant 1$, *of integers* $\geqslant 2$ *such that* $\prod_{h=1}^r j_h = i$, *i.e., "ordered factorizations of* i*." We define a nonnegative-integer-valued function* μ_q *on* I_i *by letting* $\mu_q(j_1, \ldots, j_r)$ *be the number of integers in the sequence* $(j_1, \ldots, j_r)$ *which are powers of* q. *Then*

$$u_i^A = \sum_{(j_1,\ldots,j_r) \in I_i} \pi_K^{-\mu_q(j_1,\ldots,j_r)} \prod_{h=1}^r \left(S_{j_h}^A\right)^{\prod_{a=1}^{h-1} j_a}.$$

Proof. Define $\tilde{S}_i^A$ by

$$\tilde{S}_i^A = \begin{cases} S_i^A & \text{if i is not a power of } q \\ \pi_K^{-1} S_i^A & \text{if i is a power of } q \end{cases}.$$

Then, from the functional equation for $\log_{F_A}$ we obtain u_i^A as the sum of all the compositions

$$\tilde{S}_{j_1}^A \sigma_*^{j_1} \left(\tilde{S}_{j_2}^A \sigma_*^{j_2} \left(\tilde{S}_{j_3}^A \sigma_*^{j_3} \left(\ldots \sigma_*^{j_{r-1}} \left(\tilde{S}_{j_r}^A \right) \right) \ldots \right) \right.$$

where the sum ranges across all $(j_1, \ldots, j_r) \in I_i$. This expression for u_i^A simplifies to

$$\begin{aligned} u_i^A &= \sum_{(j_1,\ldots,j_r) \in I_i} \pi_K^{-\mu_q(j_1,\ldots,j_r)} (S_{j_1}^A)(S_{j_2}^A)^{j_1}(S_{j_3}^A)^{j_1 j_2} \ldots (S_{j_r}^A)^{j_1 j_2 \cdots j_{r-1}} \\ &= \sum_{(j_1,\ldots,j_r) \in I_i} \pi_K^{-\mu_q(j_1,\ldots,j_r)} \prod_{h=1}^r \left(S_{j_h}^A\right)^{\prod_{a=1}^{h-1} j_a}. \end{aligned}$$

□

The following are easy special cases of the above formula, but we also supply independent proofs.

Proposition 3.1.25. *Let* $\log_{F_A}(X) = \sum_{i \geqslant 1} u_i^A X^i$. *The* u_i *are polynomials in* $S_2^A, S_3^A, \ldots$ *and we will write* $u_i(S_2^A, S_3^A, \ldots)$ *when we want to emphasize this. We describe* u_i *for various values of* i*:*

1. *If* $q \nmid i$ *then* $u_i^A = S_i^A$.

2. *If $i = q^j$ for a positive integer j then $u_i^A(S_2^A, S_3^A, \dots) = \ell_j^A(V_1^A, V_2^A, ...)$, where ℓ_j^A is the jth log coefficient (with Hazewinkel generators) of the universal A-typical formal A-module law, as a polynomial in $V_1^A, V_2^A, \dots$, and the variable V_h^A in the second polynomial corresponds to the variable $S_{q^h}^A$ in the first polynomial. The variables S_h^A with h not a power of q do not appear in the polynomial $u_{q^j}(S_2^A, S_3^A, \dots)$.*

Proof. 1. If $q \nmid i$ then it is clear from the functional equation for $\log_{F_A}$ that $u_i^A = S_i^A$.

2. We see from the functional equation that S_h^A appears in u_i^A only if $h|i$. In the case we consider here, that $i = q^j$, this implies that S_h^A appears in u_i^A only if h is a power of q. Since no S_b^A where b is a power of q appears in $\sum_{h\geqslant 2} S_h^A(X^{q^a h}) - \sum_{h\geqslant 1} S_{q^h}^A(X^{q^a h})$ for any integer $a \geqslant 0$, that part of the functional equation contributes nothing to $u_{q^j}^A$, and we see that the power series F obtained from the functional equation

$$F(X) = X + \sum_{h\geqslant 1} \pi_K^{-1} S_{q^h}^A \sigma_*^h F(X^{q^h})$$

has q^jth coefficient identical to $u_{q^j}^A$. But the functional equation which defines the logarithm of the universal A-typical formal A-module law is

$$\log_{F_{VA}}(X) = X + \sum_{h\geqslant 1} \pi_K^{-1} V_h^A \sigma_*^h \log_{F_{VA}}(X^{q^h}),$$

which, replacing V_h^A with $S_{q^h}^A$, is identical to the functional equation just above.

□

Definition 3.1.26. (Extending a formal module structure via an isomorphism.) *Given an isomorphism of formal A-modules $F \xrightarrow{\phi} G$ and a formal B-module structure on F, there is a natural formal B-module structure on G given by*

$$B \xrightarrow{\rho_G} \mathrm{End}(G)$$

$$\alpha \mapsto \phi \circ \rho_F(\alpha) \circ \phi^{-1}.$$

Since the data of a formal B-module law F over R, a formal A-module law G over R, and a strict isomorphism $F \to G$ of formal A-modules over R is equivalent to a map from the diagram

$$\begin{tikzcd} L^A \arrow[r, "\gamma", "\sigma"'] \arrow[d, "\eta_L"] & L^B \\ L^A T & \\ L^A \arrow[u, "\eta_R"] & \end{tikzcd}$$

to R, the above definition gives us a map from L^B to $L^B \otimes_{L^A} L^A T$ classifying the formal B-module law obtained by the definition. We will call this map $L^B \xrightarrow{\psi} L^B \otimes_{L^A} L^A T$. When F is B-typical and G is A-typical, we have a priori only a map $L^B \to V^B \otimes_{V^A} V^A T$, but when E/K is totally ramified then we know that the extension of G to a formal B-module law is B-typical (Prop. 3.1.18), so when E/K is totally ramified we have a map $V^B \xrightarrow{\psi'} V^B \otimes_{V^A} V^A T$.

Proposition 3.1.27.

1. *The map ψ is a right $L^A T$-comodule algebra structure map on L^B, and it satisfies condition 3.*
2. *The map ψ' (only defined when E/K is totally ramified) is a right $V^A T$-comodule structure map on V^B, and it satisfies condition 3.*

Proof. To prove both cases, we refer to Definition 2.1.6. The unit condition is equivalent, on inspection of the diagram 1, to the extension of a formal B-module law F over the isomorphism id_R being again F itself, which is clearly true; and the associativity condition is equivalent, on inspection of the diagram 1, to the following: given a formal B-module law F_1 and formal A-module laws F_2, F_3, all over R, and strict isomorphisms of formal A-module laws $F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} F_3$, extending F_2 to a formal B-module law over f_1 and then extending F_3 to a formal B-module law over f_2 gives us the same formal B-module law as extending F_3 to a formal B-module law over $f_2 \circ f_1$. In the first case, the relevant structure map is

$$\begin{aligned} B &\xrightarrow{\rho} \operatorname{End}(F_3) \\ \alpha &\mapsto f_2 \circ (f_1 \circ \rho_{F_1}(\alpha) \circ f_1^{-1}) \circ f_2^{-1}, \end{aligned}$$

and in the second case, the relevant structure map is

$$\begin{aligned} B &\xrightarrow{\rho'} \operatorname{End} F_3 \\ \alpha &\mapsto (f_2 \circ f_1) \circ \rho_{F_1}(\alpha) \circ (f_2 \circ f_1)^{-1}. \end{aligned}$$

Now $\rho = \rho'$, giving us the associativity condition.

For the first case, the comodule algebra L^B, we note that condition 3 is equivalent to the following statement: given a formal B-module law F, a formal A-module law G, and a strict isomorphism of formal A-module laws $F \xrightarrow{f} G$, if we extend G to a formal B-module law via f, its underlying formal A-module law is G itself, and this is clearly true. For the second case, the comodule algebra V^B, we note that condition 3 is equivalent to the following statement: given a B-typical formal B-module law F, an A-typical formal A-module law G, and a strict isomorphism of A-typical formal A-module laws $F \xrightarrow{f} G$, if we extend G to a B-typical formal B-module law via f, its underlying A-typical formal A-module law is G itself, and this is again clearly true. □

Proposition 3.1.28.

1. *Let $F \xrightarrow{\phi} G$ be a strict isomorphism of formal A-modules. Let F have a formal B-module structure compatible with its underlying formal A-module structure. Then, with the formal B-structure on G induced by ϕ, ϕ is a strict isomorphism of formal B-modules.*

   In other words, the map of Hopf algebroids $(L^A, L^AT) \to (L^B, L^BT)$ factors through $(L^B, L^B \otimes_{L^A} L^AT)$.

2. *Let E/K be totally ramified and let $F \xrightarrow{\phi} G$ be a strict isomorphism of A-typical formal A-modules. Let F have a B-typical formal B-module structure*

compatible with its underlying formal A-module structure. Then, with the formal B-structure on G induced by ϕ, ϕ is a strict isomorphism of B-typical formal B-modules.

In other words, when E/K is totally ramified, the map of Hopf algebroids $(V^A, V^A T) \to (V^B, V^B T)$ factors through $(V^B, V^B \otimes_{V^A} V^A T)$.

Proof. In both cases, the map ϕ is a strict isomorphism of formal B-modules if $\phi(\rho_F(\alpha))(X) = \rho_G(\alpha)(\phi(X))$. But due to the way that ρ_G is defined, we have $\rho_G(\alpha)(\phi(X)) = \phi(\rho_F(\alpha)(\phi^{-1}(\phi(X))) = \phi(\rho_F(\alpha))(X)$. □

We recall that a morphism of formal A-module laws over a commutative A-algebra R is just a power series $R[[X]]$ satisfying appropriate axioms. With this in mind, it is not a priori impossible for the same power series to be a morphism $F_1 \longrightarrow G_1$ and also a morphism $F_2 \longrightarrow G_2$ of formal A-module laws with $F_1 \neq F_2$ or $G_1 \neq G_2$. We want to rule out at least a certain case of this:

Lemma 3.1.29. *Let R be a commutative A-algebra and let $\phi \in R[[X]]$ be a power series such that it is an isomorphism $F \xrightarrow{\phi} G_1$ and an isomorphism $F \xrightarrow{\phi} G_2$ of formal A-module laws. Then $G_1 = G_2$.*

Proof. The identity map $(\phi \circ \phi^{-1})(X) = X$ is a morphism $G_1 \longrightarrow G_2$, so

$$(\phi \circ \phi^{-1}) G_1(X, Y) = G_2((\phi \circ \phi^{-1})(X), (\phi \circ \phi^{-1})(Y)),$$

but $(\phi \circ \phi^{-1})(X) = X$, so $G_1(X, Y) = G_2(X, Y)$; similarly, $(\phi \circ \phi^{-1}) \circ (\rho_{G_1}(\alpha)) = (\rho_{G_2}(\alpha)) \circ (\phi \circ \phi^{-1})$, so $\rho_{G_1} = \rho_{G_2}$. □

Corollary 3.1.30. *Let E/K be a finite extension of p-adic number fields and let $F \xrightarrow{\phi} G$ be a strict isomorphism of formal B-module laws; let $\tilde{G}$ denote the formal A-module law underlying G and let $\tilde{\phi}$ denote the strict isomorphism of formal A-module laws underlying ϕ; finally, let G' be the formal B-module law induced by $F \xrightarrow{\tilde{\phi}} \tilde{G}$, using Def. 3.1.26. Then $G = G'$.*

Proof. The map ϕ is strict isomorphism of formal B-module laws $F \longrightarrow G$ and also a strict isomorphism of formal B-module laws $F \longrightarrow G'$, by Prop. 3.1.28; so by Lemma 3.1.29, $G = G'$. $\square$

Proposition 3.1.31. *1. The map $L^B \otimes_{L^A} L^A T \to L^B T$ from Prop. 3.1.28 is an isomorphism of graded commutative B-algebras, and the left unit maps $L^B \xrightarrow{\eta_L} L^B T$, $L^B \xrightarrow{\eta_L} L^B \otimes_{L^A} L^A T$ commute with this isomorphism.*

2. When E/K is totally ramified, the map $V^B \otimes_{V^A} V^A T \to V^B T$ from Prop. 3.1.28 is an isomorphism of graded commutative B-algebras, and the left unit maps $V^B \xrightarrow{\eta_L} V^B T$, $V^B \xrightarrow{\eta_L} V^B \otimes_{V^A} V^A T$ commute with this isomorphism.

Proof. 1. We first name some diagrams.

$$\left(\begin{tikzcd} L^A \arrow[r] \arrow[d, "\eta_L"] & L^B \\ L^A T & \\ L^A \arrow[r] \arrow[u, "\eta_R"] & L^A \end{tikzcd}\right) \xrightarrow{\Xi_1} \left(\begin{tikzcd} L^A \otimes_A B \arrow[r] \arrow[d, "\eta_L"] & L^B \\ L^A T \otimes_A B & \\ L^A \otimes_A B \arrow[r] \arrow[u, "\eta_R"] & L^A \otimes_A B \end{tikzcd}\right) \xrightarrow{\Xi_2} \left(\begin{tikzcd} L^B \arrow[r] \arrow[d, "\eta_L"] & L^B \\ L^B T & \\ L^B \arrow[r] \arrow[u, "\eta_R"] & L^B \end{tikzcd}\right)$$

We will refer to the above three diagrams as X_1, X_2, and X_3, from left to right, and there are maps $X_1 \xrightarrow{\Xi_1} X_2 \xrightarrow{\Xi_2} X_3$ as indicated, which we have constructed in the previous propositions. The map Ξ_1 is a morphism of diagrams in $\mathbf{GCAlg}(A)$, while Ξ_2 is a morphism of diagrams in $\mathbf{GCAlg}(B)$.

From Def. 3.1.26 and Cor. 3.1.30, we know that to specify a strict isomorphism of formal B-module laws is the same thing as to specify a source formal B-module law, a target formal A-module law, and a strict isomorphism of formal A-module laws, i.e., morphisms from X_2 to commutative B-algebras are in bijection with morphisms from X_3 to commutative B-algebras. So by the Yoneda Lemma, Ξ_2 induces an isomorphism in $\mathbf{GCAlg}(B)$ on passing to colimits, and by Prop. **??**, Ξ_1 does too. Since these isomorphisms were obtained by taking the colimits of the morphisms of the above diagrams in which the maps η_L appear, the isomorphisms commute with the left unit maps η_L.

2. We repeat the same argument as above, with the following diagrams:

$$\left(\begin{tikzcd} V^A \arrow[r] \arrow[d, "\eta_L"] & V^B \\ V^AT & \\ V^A \arrow[u, "\eta_R"] \arrow[r] & V^A \end{tikzcd}\right) \xrightarrow{\Xi_1} \left(\begin{tikzcd} V^A \otimes_A B \arrow[r] \arrow[d, "\eta_L"] & V^B \\ V^AT \otimes_A B & \\ V^A \otimes_A B \arrow[u, "\eta_R"] \arrow[r] & V^A \otimes_A B \end{tikzcd}\right) \xrightarrow{\Xi_2} \left(\begin{tikzcd} V^B \arrow[r] \arrow[d, "\eta_L"] & V^B \\ V^BT & \\ V^B \arrow[u, "\eta_R"] \arrow[r] & V^B. \end{tikzcd}\right)$$

□

Bringing most of the results of this section together, we have:

Corollary 3.1.32. *1. For any finite E/K the map of Hopf algebroids $(L^A, L^AT) \longrightarrow (L^B, L^BT)$ classifying the underlying formal A-module structures on (L^B, L^BT) is split, i.e., $(L^B, L^BT) \cong (L^B, L^B \otimes_{L^A} L^AT)$ as Hopf algebroids over B, and*

$$\mathrm{Cotor}^{*,*}_{L^BT}(L^B, L^B) \cong \mathrm{Cotor}^{*,*}_{L^AT}(L^A, L^B).$$

2. For any totally ramified E/K, the map of Hopf algebroids $(V^A, V^AT) \longrightarrow (V^B, V^BT)$ classifying the underlying formal A-module structures on (V^B, V^BT) is split, i.e., $(V^B, V^BT) \cong (V^B, V^B \otimes_{V^A} V^AT)$ as Hopf algebroids over B, and

$$\mathrm{Cotor}^{*,*}_{V^BT}(V^B, V^B) \cong \mathrm{Cotor}^{*,*}_{V^AT}(V^A, V^B).$$

Proof. The splittings follow immediately from Prop. 1 and Prop. 2. The isomorphisms in cohomology follow immediately from Prop. 2.3.26. □

Corollary 3.1.33. *1. For any finite E/K the Leray spectral sequence for the map of moduli stacks $\mathcal{M}_{FM_B} \to \mathcal{M}_{FM_A}$ classifying the underlying formal A-module of the universal formal B-module collapses at E_2 and we have the isomorphism in flat cohomology $H^*_{\mathrm{fpqc}}(\mathcal{M}_{FM_B}, \mathcal{F}) \cong H^*_{\mathrm{fpqc}}(\mathcal{M}_{FM_A}, f_*\mathcal{F})$ for any $\mathcal{O}_{\mathcal{M}_{FM_B}}$-module $\mathcal{F}$.*

2. *For any totally ramified E/K the Leray spectral sequence for the map of moduli stacks $\mathcal{M}_{B\ typ-FM_B} \rightarrow \mathcal{M}_{A\ typ-FM_A}$ classifying the underlying A-typical formal A-module of the universal B-typical formal B-module collapses at E_2 and we have the isomorphism in flat cohomology $H^*_{\mathrm{fpqc}}(\mathcal{M}_{B\ typ-FM_B}, \mathcal{F}) \cong H^*_{\mathrm{fpqc}}(\mathcal{M}_{A\ typ-FM_A}, f_*\mathcal{F})$ for any quasicoherent $\mathcal{O}_{\mathcal{M}_{B\ typ-FM_B}}$-module $\mathcal{F}$.*

3.2 The right unit map and invariant primes.

This section consists mostly of generalizations of the methods and results of A2 of [Ravenel, 1986] to the case of formal A-modules.

Proposition 3.2.1. *Let $F \xrightarrow{f} G$ be an isomorphism of formal A-modules with logarithms over a commutative A-algebra. Then*

$$\log_G(X) = \log_F(f^{-1}(X)).$$

Proof. Since f^{-1} is also a well-defined morphism of formal A-module laws, we have

$$\begin{aligned} f^{-1}(G(X,Y)) &= F(f^{-1}(X), f^{-1}(Y)) \\ &= \log_F^{-1}(\log_F(f^{-1}(X)) + \log_F(f^{-1}(Y))). \end{aligned}$$

We apply f to both sides:

$$\begin{aligned} G(X,Y) &= f(\log_F^{-1}(\log_F(f^{-1}(X)) + \log_F(f^{-1}(Y)))) \\ &= (\log_F \circ f^{-1})^{-1}((\log_F \circ f^{-1})(X) + (\log_F \circ f^{-1})(Y)). \end{aligned}$$

So $(\log_F \circ f^{-1})$ is a logarithm for G. By the uniqueness of the logarithm, we have $\log_G(X) = (\log_F \circ f^{-1})(Y)$. □

Proposition 3.2.2. *Let $F \xrightarrow{f} G$ be an isomorphism of formal A-module laws with logarithms, and let F be A-typical. Then G is A-typical if and only if*

$$f^{-1}(X) = \sum_{i \geqslant 0}{}^F t_i X^{q^i}$$

for some collection $\{t_i \in A\}$ *with* $t_0 = 1$.

Proof. Assume that f^{-1} is of the above form.

$$\begin{aligned}\log_G(X) &= \log_F(f^{-1}(X)) \\ &= \sum_{i\geqslant 0} \log_F(t_i X^{q^i}) \\ &= \sum_{i\geqslant 0}\sum_{j\geqslant 0} m_j (t_i X^{q^i})^{q^j}\end{aligned}$$

for some $\{m_j \in A \otimes_A K\}$, since F is A-typical. Reindexing,

$$= \sum_{i\geqslant 0} \left(\sum_{j=0}^{i} m_j t_{i-j}^{q^j} \right) X^{q^i},$$

so G is A-typical.

Now assume that F, G are both A-typical with logarithms $\log_F(X) = \sum_{i\geqslant 0} f_1(\ell_i^A) X^{q^i}$ and $\log_G(X) = \sum_{i\geqslant 0} f_2(\ell_i^A) X^{q^i}$. Now if we let

$$t_i = f_2(\ell_i^A) - \sum_{j=0}^{i-1} f_1(\ell_{i-j}^A) t_j^{q^{i-j}},$$

then

$$\begin{aligned}\log_G(X) &= \sum_{i\geqslant 0} \left(\sum_{j=0}^{i} f_1(\ell_i^A) t_{i-j}^{q^j} \right) X^{q^i} \\ &= \sum_{i\geqslant 0}\sum_{j\geqslant 0} f_1(\ell_j^A)(t_i X^{q^i})^{q^j} \\ &= \sum_{i\geqslant 0} \log_F(t_i X^{q^i}) \\ &= \log_F(f^{-1}(X)).\end{aligned}$$

Hence

$$\begin{aligned}f^{-1}(X) &= \log_F^{-1}\left(\sum_{i\geqslant 0} \log_F(t_i X^{q^i}) \right) \\ &= \sum_{i\geqslant 0}{}^F t_i X^{q^i}.\end{aligned}$$

$\square$

We recall that the way that the Hopf algebroid (V^A, V^AT) works is that, given two A-typical formal A-module laws F, G over a commutative A-algebra A which are classified by maps $f, g : V^A \longrightarrow A$ and a strict isomorphism $\gamma : F \longrightarrow G$, there exists a map $\Gamma : V^AT \longrightarrow A$ classifying γ, i.e., $\Gamma \circ \eta_L = f_1$ and $\Gamma \circ \eta_R = f_2$. Now, if F and G have logarithms, $\log_F(X) = \sum_{i\geqslant 0} f_1(\ell_1^A)X^{q^i}$ and

$$\log_G(X) = \sum_{i\geqslant 0} f_2(\ell_i^A)X^{q^i} = \sum_{i\geqslant 0}(\Gamma \circ \eta_R)(\ell_i^A)X^{q^i}.$$

The left unit is, with either the Araki or Hazewinkel generators, the obvious isomorphism of V^A on to the degree zero part of V^AT. The right unit is much more subtle and will figure prominently in much of what follows.

Proposition 3.2.3. *The right unit* $V^A \xrightarrow{\eta_R} V^AT$ *satisfies*

$$\eta_R(\ell_i^A) = \sum_{j=0}^{i} \eta_L(\ell_j^A)\left(t_{i-j}^A\right)^{q^j},$$

where $t_0^A = \ell_0^A = 1$.

Proof. The identity map on V^AT classifies the universal strict isomorphism of A-typical formal A-module laws, i.e., the strict isomorphism between the A-typical formal module law on V^AT induced by η_L and the one induced by η_R. For the duration of this proof we will name them F and G, respectively, and we will let f be the strict isomorphism between them that is induced by the identity on V^AT. Both F and G have logarithms, so by Prop. 3.2.2, $f^{-1}(x) = \sum_{i\geqslant 0}{}^F t_i x^{q^i}$. Now

$$\begin{aligned}
\log_G(x) &= \sum_{i\geqslant 0} \eta_R(\ell_i^A)x^{q^i} \\
&= \log_F(f^{-1}(x)) \\
&= \sum_{i\geqslant 0} \log_F(t_i x^{q^i}) \\
&= \sum_{i,j\geqslant 0} \eta_L(\ell_j^A)(t_i x^{q^i})^{q^j} \\
&= \sum_{i,j\geqslant 0} \eta_L(\ell_j^A)t_i^{q^j} x^{q^{i+j}}.
\end{aligned}$$

We match the coefficients of the x^{q^i}-term to get $\eta_R(\ell_i^A) = \sum_{j=0}^{i} \eta_L(\ell_j^A) t_{i-j}^{q^j}$. □

Proposition 3.2.4. (The right unit formula.) *The right unit η_R satisfies*

$$\sum_{i,j \geqslant 0}{}^F t_i^A \eta_R(v_j^A)^{q^i} = \sum_{i,j \geqslant 0}{}^F \eta_L(v_i^A)(t_j^A)^{q^i}$$

where F is the formal A-module law on $V^A T$ induced by the left unit map η_L.

Proof. We apply η_R to equation 3.1.1:

$$\eta_R(\pi \ell_h^A) = \sum_{i=0}^{h} \eta_R(\ell_i^A) \eta_R(v_{h-i}^A)^{q^i}$$

and now we use Prop. 3.2.3:

$$\begin{aligned} \eta_R(\pi \ell_h^A) &= \pi \sum_{i=0}^{h} \eta_L(\ell_i^A)(t_{h-i}^A)^{q^i} \\ &= \sum_{i=0}^{h} \sum_{j=0}^{i} \eta_L(\ell_j^A)(t_{i-j}^A)^{q^j} \eta_R(v_{h-i}^A)^{q^i} \end{aligned}$$

Now we sum over all nonnegative h (in an ungraded situation this would be a heuristic move at best, but with the gradings on V^A and $V^A T$, there is only a finite number of elements in each degree in this sum, so this sum makes sense):

$$\sum_{h,i \geqslant 0} \pi \eta_L(\ell_i^A)(t_{h-i}^A)^{q^i} = \sum_{h,i,j \geqslant 0} \eta_L(\ell_j^A)(t_{i-j}^{q^j}) \eta_R(v_{h-i})^{q^i},$$

and now we reindex, substitute for $\pi \ell_i^A$ using formula 3.1.1, and then reindex again to get

$$\begin{aligned} \sum_{i,j,k \geqslant 0} \eta_L(\ell_i^A)(t_j^A)^{q^i} \eta_R(v_k^A)^{q^{i+j}} &= \sum_{i,j \geqslant 0} \eta_L(\pi \ell_i^A)(t_j^A)^{q^i} \\ &= \sum_{i,j,k \geqslant 0} \eta_L(\ell_k^A (v_{i-j}^A)^{q^k})(t_j^A)^{q^i} \\ &= \sum_{i,j,k \geqslant 0} \eta_L(\ell_i^A) \eta_L(v_j^A)^{q^i} (t_k^A)^{q^{i+j}}. \end{aligned}$$

Now $\log_F(x) = \sum_{i\geqslant 0} \eta_L(\ell_i^A)x^{q^i}$, so:

$$\begin{aligned}
\sum_{j,k\geqslant 0} \log_F\left(t_j^A \eta_R(v_k^A)^{q^j}\right) &= \sum_{i,j,k\geqslant 0} \eta_L(\ell_i^A)(t_j^A)^{q^i}\eta_R(v_k^A)^{q^{i+j}} \\
&= \sum_{i,j,k\geqslant 0} \eta_L(\ell_i^A)\eta_L(v_j^A)^{q^i}(t_k^A)^{q^{i+j}} \\
&= \sum_{j,k\geqslant 0} \log_F\left(\eta_L(v_j^A)(t_k^A)^{q^j}\right).
\end{aligned}$$

We apply $\log_F^{-1}$ to get the theorem as stated. □

We recall the following basic property of invariant primes.

Lemma 3.2.5. *Let (R, Γ) be a Hopf algebroid with Γ flat over R and let I be an ideal in R. The following conditions are equivalent:*

1. *I is a sub-right-Γ-comodule of R, i.e., the following diagram commutes:*

$$\begin{tikzcd}
I \arrow[r] \arrow[d, hook] & \Gamma \otimes_R I \arrow[d, hook] \\
R \arrow[r] & \Gamma \otimes_R R.
\end{tikzcd}$$

2. *$\eta_R(I) \subseteq \eta_L(I)\Gamma$*

3. *$\eta_L(I) \subseteq \eta_R(I)\Gamma$*

Lemma 3.2.6. *Let (R, Γ) be a Hopf algebroid and let $\{I_\alpha\}_{\alpha\in A}$ be a collection of invariant ideals of (R, Γ), indexed by some set A. Then $I = \bigcup_{\alpha\in A} I_\alpha$ is invariant.*

Proof. Choose $x \in I$ and an $\alpha \in A$ such that $x \in I_\alpha$. Then $\eta_L(x) \in \eta_R(I_\alpha)\Gamma \subseteq \eta_R(I)\Gamma$. □

Proposition 3.2.7. *Let (R, Γ) be a Hopf algebroid, let $I \subset R$ be an invariant ideal, and let $j \in R$ be such that $\eta_L(j) \equiv \eta_R(j) \mod \eta_R(I)$ (or, equivalently, $\mod \eta_L(I)$). Then $I + (j)$ is an invariant ideal.*

Proof. Choose an element $x \in I + (j)$; we want to show that $\eta_L(x) \in \eta_R(I + (j))\Gamma$. Choose decompositions $x = i + jr$ and $\eta_L(j) = \eta_R(j) + \eta_R(i')$ and $\eta_L(i) = \eta_R(i'')\gamma$, where $i, i', i'' \in I$ and $r \in R$ and $\gamma \in \Gamma$. Now we have

$$\begin{aligned} \eta_L(x) &= \eta_L(i) + \eta_L(j)\eta_L(r) \\ &= \eta_R(i'')\gamma + (\eta_R(j) + \eta_R(i'))\eta_L(r) \\ &\in \eta_R(I + (j))\Gamma. \end{aligned}$$

□

Corollary 3.2.8. *Let (R, Γ) be a Hopf algebroid and let $I = (a_1, a_2, \dots)$ be an ideal in R with $\eta_L(a_1) = \eta_R(a_1)$ and*

$$\eta_L(a_i) \equiv \eta_R(a_i) \mod (\eta_L(a_1), \dots, \eta_L(a_{i-1}))$$

for all $i > 1$. Then I is invariant.

Proof. Clearly $(a_1) \subseteq R$ is invariant. We use Prop. 3.2.7 for the inductive step, to get that $(a_1, \dots a_i) \subseteq R$ is invariant for all positive integers i; and, using Lemma 3.2.6, the union of all of these invariant ideals, I, is also invariant. □

We recall that $I_h^A = (\pi, v_1^A, \dots, v_{h-1}^A) \subseteq V^A$ for $h > 0$, and $I_0^A = (0)$.

Proposition 3.2.9. *I_h^A is invariant, and $\eta_L(v_h^A) \equiv \eta_R(v_h^A) \mod I_h^A$.*

Proof. Let $j < h$; we need to show that $\eta_R(v_j^A) \equiv 0 \mod \eta_L(I_h^A)$. Suppose

$\eta_R(v_i^A) \equiv 0 \mod \eta_L(I_h^A)$ for all $i < j$. Then

$$\begin{aligned}
\eta_R(v_j^A) &\equiv \eta_R(v_j^A) + \sum_{i=1}^{j} \eta_R(\ell_i(v_{j-i}^A)^{q^i}) \\
&\equiv \eta_R((\pi_A - \pi_A^{q^j})^{-1}\ell_j^A) \text{ (using formula 3.1.1)} \\
&\equiv (\pi_A - \pi_A^{q^j})^{-1} \sum_{i=0}^{j} \eta_L(\ell_i^A) \left(t_{j-i}^A\right)^{q^i} \text{ (using Prop. 3.2.3)} \\
&\equiv (\pi_A - \pi_A^{q^j})^{-1} \eta_L(\ell_j^A) \\
&\equiv \eta_L\left(\sum_{i=0}^{j} \ell_i^A \left(v_{j-i}^A\right)^{q^i}\right) \\
&\equiv 0 \mod I_h^A.
\end{aligned}$$

When $h > 1$ and $j = 1$ we have

$$\eta_R(v_1^A) = \eta_L(v_1^A) + (\pi_A - \pi_A^q)t_1^A \equiv 0 \mod I_2^A,$$

which starts the induction.

For the second part of the proposition, we have

$$\begin{aligned}
\eta_R(v_h^A) &\equiv \eta_R(v_h^A) + \sum_{i=1}^{h} \eta_R(\ell_i(v_{h-i}^A)^{q^i}) \\
&\equiv \eta_R((\pi_A + \pi_A^{q^h})\ell_h^A) \\
&\equiv (\pi_A + \pi_A^{q^h}) \sum_{j=0}^{h} \eta_L(\ell_j^A) \left(t_{h-j}^A\right)^{q^j} \\
&\equiv (\pi_A + \pi_A^{q^h})\eta_L(\ell_h^A) \\
&\equiv \eta_L(v_h^A) \mod I_h^A.
\end{aligned}$$

$\square$

Proposition 3.2.10. *Let h be a positive integer and let k be the residue field of A. Then*

$$\mathrm{Cotor}^0_{V^AT}(V^A, V^A/I_h^A) \cong k[v_h^A] \text{ and}$$

$$\mathrm{Cotor}^0_{V^AT}(V^A, V^A) \cong A.$$

Proof. We recall that $v_0^A = \pi_A$ and $I_0^A = (0)$ by convention.

$$\begin{aligned}
\mathrm{Cotor}^0_{V^AT}(V^A, V^A/I_h^A) &\cong V^A\square_{V^AT}(V^A/I_h^A) \\
&= \{x \in V^A/I_h^A : \psi(x) = 1 \otimes x \in V^AT \otimes_{V^A} (V^A/I_h^A)\} \\
&= \{x \in V^A/I_h^A : \eta_R(x) \equiv \eta_L(x) \mod I_h^A\},
\end{aligned}$$

following x through the commutative diagram

$$\begin{tikzcd}
V^A/I_h^A \arrow[r, "\eta_R"] \arrow[dr] & V^AT/(\eta_R(I_h^A)) \arrow[d, "\cong"] \\
 & V^AT \otimes_{V^A} (V^A/I_h^A).
\end{tikzcd}$$

Now by Prop. 3.2.9, $\eta_R(v_h^A)^i \equiv \eta_L(v_h^A)^i \mod I_h^A$ for all positive integers i, so $A \subseteq V^A\square_{V^AT}V^A$, and $k[v_h^A] \subseteq V^A\square_{V^AT}(V^A/I_h^A)$ when $h > 0$.

We must show that this exhausts $V^A\square_{V^AT}(V^A/I_h^A)$. We reduce the right unit formula (Prop. 3.2.4) modulo $I_h^A + (t_1^A, t_2^A, \ldots, t_{j-1}^A)$ and consider terms of equal grading to get

$$\begin{aligned}
t_j^A\eta_R(v_h^A)^{q^j} +_F \eta_R(v_{h+j}^A) &\equiv (\eta_L(v_h^A)t_j^A)^{q^h}) +_F \eta_L(v_{h+j}^A) \mod I_h^A + (t_1^A, \ldots, t_{j-1}^A) \\
\sum_{i\geqslant 0} t_i^A\left((t_j^A\eta_R(v_h^A)^{q^j})^{q^i} + \eta_R(v_{h+j}^A)^{q^i}\right) &\equiv \log(t_j^A\eta_R(v_h^A)^{q^j}) + \log(\eta_R(v_{h+j}^A)) \\
&\equiv \log(\eta_L(v_h^A)t_j^A)^{q^h})) + \log \eta_L(v_{h+j}^A) \\
&\equiv \sum_{i\geqslant 0} t_i^A\left((\eta_L(v_h^A)(t_j^A)^{q^h})^{q^i} + \eta_L(v_{h+j}^A)^{q^i}\right) \\
&\qquad \mod I_h^A + (t_1^A, \ldots, t_{j-1}^A).
\end{aligned}$$

In each of these two sums there is only one term in each grading, so we match gradings to get

$$t_i^A\left((t_j^A)^{q^i}\eta_R(v_h^A)^{q^{i+j}} + \eta_R(v_{h+j}^A)^{q^i}\right) \equiv t_i^A\left(\eta_L(v_h^A)^{q^i}(t_j^A)^{q^{h+i}} + \eta_L(v_{h+j}^A)^{q^i}\right) \mod I_h^A+(t_1^A, \ldots, t_{j-1}^A).$$

Now the ideal $(\eta_L(I_h^A, t_1^A, \ldots, t_{j-1}^A))$ is prime in V^AT, so $V^AT/(\eta_L(I_h^A, t_1^A, \ldots, t_{j-1}^A)) \cong V^AT/(\eta_R(I_h^A, t_1^A, \ldots, t_{j-1}^A))$ has no zero divisors and so for $i \geqslant j$ we have

$$\begin{aligned}
(t_j^A \eta_R(v_h^A)^{q^j} + \eta_R(v_{h+j}^A))^{q^i} &\equiv (t_j^A)^{q^i} \eta_R(v_h^A)^{q^{i+j}} + \eta_R(v_{h+j}^A)^{q^i} \\
&\equiv \eta_L(v_h^A)^{q^i} (t_j^A)^{q^{h+i}} + \eta_L(v_{h+j}^A)^{q^i} \\
&\equiv (\eta_L(v_h^A)(t_j^A)^{q^h} + \eta_L(v_{h+j}^A))^{q^i} \mod I_h^A + (t_1^A, \ldots, t_{j-1}^A), \text{ and finally} \\
t_j^A \eta_R(v_h^A)^{q^j} + \eta_R(v_{h+j}^A) &\equiv \eta_L(v_h^A)(t_j^A)^{q^h} + \eta_L(v_{h+j}^A) \mod I_h^A + (t_1^A, \ldots, t_{j-1}^A).
\end{aligned}$$

So the set $\{\eta_L(v_{h+j}^A), \eta_R(v_{h+j}^A) : j > 0\} \cup \{v_h^A\}$ is algebraically independent in V^AT/I_h^A (i.e. it generates a free k-algebra in V^AT/I_h^A) and if $\eta_R(x) = \eta_L(x)$ then x is a k-polynomial in v_h^A. Hence $A = V^A \square_{V^AT} V^A$ and $k[v_h^A] = V^A \square_{V^AT} (V^A/I_h^A)$ for $h > 0$. $\square$

Corollary 3.2.11.

$$0 \longrightarrow \Sigma^{2(q^h-1)} V^A/I_h^A \xrightarrow{v_h^A} V^A/I_h^A \longrightarrow V^A/I_{h+1}^A \longrightarrow 0$$

is a short exact sequence of V^AT-comodules.

Proof. Since $V^A \square_{V^AT} (V^A/I_h^A) \cong \hom_{V^AT-\text{comod}}(V^A, V^A/I_h^A)$ (see e.g. Ravenel A1), multiplication by v_h^A is a left V^AT-comodule map on V^A/I_h^A. It is clear that the cokernel is V^A/I_h^A as a V^A-module and its left V^AT-comodule structure is the same as that induced on V^A/I_h^A as a quotient of V^A. $\square$

Proposition 3.2.12. *Let $J \subseteq V^A$ be an invariant prime. Then $J = I_h^A$ for some $0 \leqslant h \leqslant \infty$, where $I_\infty^A = \bigcup_h I_h^A$.*

Proof. Suppose $J \subseteq V^A$ is an invariant prime such that $I_i^A \neq J$ for all finite $i \geqslant 0$. Let h be a nonnegative integer such that $I_h^A \subseteq J$ (this is at least true for $h = 0$). Let J' be the direct summand of J (since J is a graded A-module) consisting of the elements in the smallest dimension of J which has elements not contained in I_h^A. We choose an A-basis S for J and we put the lexicographic ordering on S

(see Def. 3.1.13). Since this is a total ordering on any A-basis for S there will be a minimal element $x \in S$, and $\eta_R(x)$ will not contain terms of higher lexicographic order than x, so $\eta_R(x) = \eta_L(x)$, i.e., $x \in \mathrm{Cotor}^0_{V^AT}(V^A, V^A/I^A_h)$, so x is a power of v^A_h. Since J is prime we have $v^A_h \in J$ and $I^A_{h+1} \subseteq J$. By induction, $I^A_h \subseteq J$ for all finite $h \geqslant 0$, so $I^A_\infty \subseteq J$; but I^A_∞ is maximal, so $I^A_\infty = J$. □

Recall that a sequence $(x_0, x_1, \ldots, x_{h-1})$ of elements in a polynomial algebra R over A is called regular if x_i is not a zero divisor in $R/(x_0, \ldots, x_{i-1})$ for all $i < h$, i.e.,

$$0 \longrightarrow R/(x_0, \ldots, x_{i-1}) \xrightarrow{x_i} R/(x_0, \ldots, x_{i-1}) \longrightarrow R/(x_0, \ldots, x_i) \longrightarrow 0$$

is exact for all $i < h$, and we know from Prop. 3.2.12 and Cor. 3.2.11 that, for $R = V^A$, the ideal $I = (x_0, \ldots, x_{i-1})$ is invariant if the above sequence is exact.

Corollary 3.2.13. *All invariant primes in V^A are regular.*

Now we want to produce a certain set of invariant regular ideals in V^A, which are not necessarily prime, and which by no means exhaust all invariant regular ideals in V^A, but will be useful to refer back to.

Proposition 3.2.14. *Let A be a p-adic number ring with residue field $\mathbb{F}_q$, and let $i_1, i_2, \ldots, i_h$ be a sequence of positive integers such that, for each m with $0 < m < h$, the integer i_{m+1} is divisible by the smallest power of q not less than i_m, and let $j \geqslant 0$. Then the regular ideal $(\pi^{j+1}, (v^A_1)^{i_1 q^j}, (v^A_2)^{i_2 q^{2j}}, \ldots (v^A_h)^{i_h q^{hj}})$ is invariant.*

Proof. We will show that

$$\eta_L((v^A_h)^{i_h q^{hj}}) \equiv \eta_R((v^A_h)^{i_h q^{hj}}) \mod (\eta_L(\pi^{j+1}), \eta_L(v^A_1)^{i_1 q^j}, \ldots, \eta_L(v^A_{h-1})^{i_{h-1} q^{(h-1)j}}),$$

and then Cor.3.2.8 together with the obvious induction on the sequence

$$\pi^{j+1}, (v^A_1)^{i_1 q^j}, (v^A_2)^{i_2 q^{2j}}, \ldots, (v^A_h)^{i_h q^{hj}}$$

gives us the proposition as stated. But this congruence is true if we can prove the stronger congruence

$$\eta_L((v_h^A)^{i_h q^{hj}}) \equiv \eta_R((v_h^A)^{i_h q^{hj}}) \mod (\eta_L(\pi^{j+1}), \eta_L(v_1^A)^{q^{c_1+j}}, \ldots, \eta_L(v_{h-1}^A)^{q^{c_{h-1}+(h-1)j}}),$$

where c_m is the least integer such that $q^{c_m} \geqslant i_m$. Now the condition on i_h is that it is divisible by $q^{c_{h-1}}$; we show that

$$\eta_L(v_h^A)^{q^{hj+c_{h-1}}} \equiv \eta_R(v_h^A)^{q^{hj+c_{h-1}}} \mod (\eta_L(\pi^{j+1}), \eta_L(v_1^A)^{q^{c_1+j}}, \ldots, \eta_L(v_{h-1}^A)^{q^{c_{h-1}+(h-1)j}}),$$

i.e., the $i_h = q^{c_{h-1}}$ case, and the other cases follow immediately from it. We apply Lemma 2.4.4 with $S = (1)$ and

$$R_i = \begin{cases} (\eta_L(v_i^A)) & \text{if } 0 \leqslant i < h \\ (0) & \text{if } i \geqslant h, \end{cases}$$

and we get

$$\eta_L(v_h^A)^{q^{hj+c_{h-1}}} \equiv \eta_R(v_h^A)^{q^{hj+c_{h-1}}} \mod (\pi^{hj+c_{h-1}+1}) + \sum_{m=0}^{hj+c_{h-1}} (\pi^m) \sum_{i=0}^{h-1} (\eta_L(v_i^A)^{q^{hj+c_{h-1}-m}}),$$

and

$$(\pi^{hj+c_{h-1}+1}) + \sum_{m=0}^{hj+c_{h-1}} (\pi^m) \sum_{i=0}^{h-1} (\eta_L(v_i^A)^{q^{hj+c_{h-1}-m}}) \subseteq (\eta_L(\pi^{j+1}), \eta_L(v_1^A)^{q^{c_1+j}}, \ldots, \eta_L(v_{h-1}^A)^{q^{c_{h-1}+(h-1)j}}),$$

as desired. $\square$

3.3 The coproduct map and the b^A elements.

We recall the way that the coproduct map works on (V^A, V^AT). Maps from the diagram

$$X \cong \left(\begin{tikzcd}[column sep=small] V^A \arrow[dr, "\eta_L"] & \\ & V^AT \\ V^A \arrow[ur, "\eta_R"] \arrow[dr, "\eta_L"] & \\ & V^AT \\ V^A \arrow[ur, "\eta_R"] & \end{tikzcd}\right)$$

to a commutative A-algebra R correspond to pairs of strict isomorphisms $F \longrightarrow G \longrightarrow H$ of A-typical formal A-module laws over R, and the map $V^AT \xrightarrow{\Delta} V^AT \otimes_{V^A} V^AT \cong \operatorname{colim} X$ classifies the composite of the two strict isomorphisms.

Proposition 3.3.1. *The coproduct on (V^A, V^AT) is determined by*

$$\sum_{i,m\geqslant 0} (\eta_L(\ell_m^A)\otimes 1)\Delta(t_i^A)^{q^m} = \sum_{i,j,m\geqslant 0} (\eta_L(\ell_m^A)(t_i^A)^{q^m}) \otimes (t_j^A)^{q^{i+m}}.$$

Proof. We have three A-typical formal A-module laws on $V^AT \otimes_{V^A} V^AT$, classified by the maps $\eta_L \otimes_{V^A} V^AT, V^AT \otimes_{V^A} \eta_R$, and $\eta_R \otimes_{V^A} V^AT = V^AT \otimes_{V^A} \eta_L$ which we will call F_1, F_2, F_3, respectively; and we have strict isomorphisms between them, classified by $V^AT \otimes_{V^A} V^A \xrightarrow{V^AT\otimes_{V^A}\eta_R} V^AT \otimes_{V^A} V^AT$ and $V^A \otimes_{V^A} V^AT \xrightarrow{\eta_L\otimes_{V^A}V^AT} V^AT \otimes_{V^A} V^AT$, and we will call them f_1, f_2, respectively. The coproduct map Δ is necessarily the classifying map of their composite. Using Prop. 3.2.2, we have

$$\begin{aligned} f_1^{-1}(X) &= \sum_{i\geqslant 0}{}^{F_1} (t_i^A \otimes 1)X^{q^i}, \\ f_2^{-1}(X) &= \sum_{j\geqslant 0}{}^{F_2} (1\otimes t_j^A)X^{q^j}, \text{ and} \\ (f_2\circ f_1)^{-1}(X) &= \sum_{i\geqslant 0}{}^{F_1} \Delta(t_i^A)X^{q^i}. \end{aligned}$$

Now we simply manipulate the above equations in the appropriate way:

$$\begin{aligned}
(f_2 \circ f_1)^{-1}(X) &= f_1^{-1}\left(\sum_{j\geqslant 0}{}^{F_2}(1\otimes t_j^A)X^{q^j}\right) \\
&= \sum_{j\geqslant 0}{}^{F_1} f_1^{-1}((1\otimes t_j^A)X^{q^j}) \\
&= \sum_{i,j\geqslant 0}{}^{F_1}(t_i^A\otimes 1)((1\otimes t_j^A)X^{q^j})^{q^i} \\
&= \sum_{i,j\geqslant 0}{}^{F_1}(t_i^A\otimes (t_j^A)^{q^i})X^{q^{i+j}} \\
&= \sum_{i\geqslant 0}{}^{F_1}\left(\sum_{0\leqslant j\leqslant i}{}^{F_1}(t_j^A)\otimes t_{i-j}^{q^j}\right)X^{q^i} \\
&= \sum_{i\geqslant 0}{}^{F_1}\Delta(t_i^A)X^{q^i}, \text{ i.e.,} \\
\sum_{i\geqslant 0}{}^{F_1}\Delta t_i^A &= \sum_{i,j\geqslant 0}{}^{F_1}(t_i^A)\otimes (t_j^A)^{q^i}, \text{ i.e.,} \\
\sum_{i,m\geqslant 0}(\eta_L(\ell_m^A)\otimes 1)\Delta(t_i^A)^{q^m} &= \sum_{i\geqslant 0}\log_{F_1}(\Delta(t_i^A)) \\
&= \sum_{i,j\geqslant 0}\log_{F_1}((t_i^A)\otimes (t_j^A)^{q^i}) \\
&= \sum_{i,j,m\geqslant 0}(\eta_L(\ell_m^A)\otimes 1)((t_i^A)\otimes (t_j^A)^{q^i})^{q^m} \\
&= \sum_{i,j,m\geqslant 0}(\eta_L(\ell_m^A)(t_i^A)^{q^m})\otimes (t_j^A)^{q^{i+m}}.
\end{aligned}$$

□

We want to use Cor. 3.1.4 to make coproduct computations in (V^A, V^AT), and for that sake it will be useful to have a way of simplifying expressions of the form $\sum_{h,i}{}^F a_{h,i}$, where all h are positive integers and $a_{h,i}$ has dimension $2(q^h-1)$ in some graded commutative V^A-algebra D. We will accomplish this using Prop. 3.3.1: given our set of elements $\{a_{h,i}\}\subseteq D$ we define subsets A_h, B_h of D as follows: $A_h = B_h = \varnothing$ for $h\leqslant 0$, while for $h>0$, we let $A_h = \{a_{h,i}\}$ and we define B_h recursively by

$$B_h = A_h \cup \bigcup_{|J|>0}\{v_J^A w_J^A(B_{h-||J||})\}.$$

Lemma 3.3.2. *With notation as above, $\sum_{h,i}^{F} a_{h,i} = \sum_{h>0}^{F} w_{\varnothing}(B_h)$, where F is the formal group law on D induced by the V^A-module structure map $V^A \longrightarrow D$.*

Proof. This is exactly as in Ravenel 4.3.11: we observe that

$$\sum{}^F a_{h,i} = \sum_{0<h<m} {}^F w_{\varnothing}^A(B_h) +_F \sum_{h<m, ||J||+h\geqslant m} {}^F v_J^A w_J^A(B_h) +_F \sum_{h\geqslant m} {}^F a_{h,i}$$

is true for $m = 1$, as it reads $\sum{}^F a_{h,i} = \sum_{h\geqslant 1}{}^F a_{h,i}$. We assume it is true for m and proceed by induction:

$$\begin{aligned}
\sum{}^F a_{h,i} &= \sum_{0<h<m} {}^F w_{\varnothing}^A(B_h) +_F \sum_{h<m, ||J||+h>m} {}^F v_J^A w_J^A(B_h) +_F \sum_{h>m} {}^F a_{h,i} +_F \sum{}^F B_m \\
&= \sum_{0<h<m} {}^F w_{\varnothing}^A(B_h) +_F \sum_{h<m, ||J||+h>m} {}^F v_J^A w_J^A(B_h) +_F \sum_{h>m} {}^F a_{h,i} +_F \sum_{J}{}^F v_J^A w_J^A(B_m) \\
&= \sum_{0<h\leqslant m} {}^F w_{\varnothing}^A(B_h) +_F \sum_{h\leqslant m, ||J||+h>m} {}^F v_J^A w_J^A(B_h) +_F \sum_{h>m} {}^F a_{h,i},
\end{aligned}$$

which is identical to the inductive hypothesis for $m + 1$. This completes the induction. □

Proposition 3.3.3. *We define M_h as the set $\{t_i^A \otimes (t_{h-i}^A)^{q^i} : 0 \leqslant i \leqslant h\} \subseteq V^A T \otimes_{V^A} V^A T$ and we define Δ_h as the subset*

$$\Delta_h = M_h \cup \bigcup_{|J|>0} \{(\eta_L(v_J^A) \otimes 1) w_J^A(\Delta_{h-||J||})\}$$

of $V^A T \otimes_{V^A} V^A T$ (so the V^A-action is given by $\eta_L \otimes V^A T$). Then we have a formula for the coproduct Δ on t_h^A:

$$\Delta(t_h^A) = w_{\varnothing}^A(\Delta_h).$$

Proof. Prop. 3.3.1 implies that $\sum_{i\geqslant 0} \log_{F_1}(\Delta(t_i^A)) = \sum_{i,j\geqslant 0} \log_{F_1}(t_i^A \otimes (t_j^A)^{q^i})$, where F_1 is the formal A-module law induced on $V^A T \otimes_{V^A} V^A T$ by $\eta_L \otimes_{V^A} V^A T$; i.e., $\sum_{i\geqslant 0}^{F_1} \Delta(t_i^A) = \sum_{i,j\geqslant 0}^{F_1} t_i^A \otimes (t_j^A)^{q^i}$, and, matching gradings, $\Delta(t_i^A) = \sum_{0\leqslant j\leqslant i}^{F_1} t_j^A \otimes (t_{i-j}^A)^{q^j}$. The stated proposition is now an immediate consequence of Lemma 3.3.2.

□

Definition 3.3.4. *For i a positive integer and j a nonnegative integer, we define the element $b^A_{i,j} \in V^A T$ by $b^A_{i,j} = w^A_{(j+1)}(\Delta_i)$.*

For instance, using the above formulas, we easily find that

$$b^A_{1,j} = -\frac{1}{\pi - \pi^{q^{j+1}}} \sum_{0<i<q^{j+1}} \binom{q^{j+1}}{i} (t^A_1)^i \otimes (t^A_1)^{q^{j+1}-i}.$$

Thinking of these elements as sitting inside the second stage of the cobar complex, they will be important in computations. In particular, we will find that the $b^A_{i,j}$ elements are responsible for periodic behavior in cohomology after localization at a geometric point.

We now need several easy lemmas.

Lemma 3.3.5. *1. Reducing the formula for $\Delta(t_k)$ (Prop. 3.3.3) modulo I^A_h when $0 < k \leqslant 2h$ yields the formula*

$$\Delta(t^A_k) = \sum_{i=0}^{k} t^A_i \otimes (t^A_{k-i})^{q^i} + \sum_{i=n}^{k-1} (\eta_L(v^A_i) \otimes 1) b^A_{k-i,i-1} \in V^A T \otimes_{V^A} V^A T / I^A_h.$$

2. For any formal group law F on $V^A T$ we have

$$\sum_{i,|I|\geqslant 0}{}^F [(-1)^{|I|}]_F (t^A_I (t^A_i)^{q^{||I||}}) = \sum_{i,|I|\geqslant 0}{}^F [(-1)^{|I|}]_F (t^A_i (t^A_I)^{q^i}) = 1.$$

3. Suppose $p > 2$ and let F be a p-typical formal group law, for instance, that underlying formal group law of an A-typical formal A-module law, where A is a p-adic number ring. Then $[-1]_F(X) = -X$.

Proof. 1. We have

$$\begin{aligned}
\Delta(t^A_k) &= w^A_{\varnothing}(\Delta_k) \\
&= w_{\varnothing}(\{t^A_i \otimes (t^A_{k-i})^{q^i}\} \cup \{(\eta_L(v^A_i) \otimes 1) w^A_i(\Delta_{k-i})\}) \\
&= \sum_{i=0}^{k} t^A_i \otimes (t^A_{k-i})^{q^i} + \sum_{i=n}^{k-1} (\eta_L(v^A_i) \otimes 1) b^A_{k-i,i-1}
\end{aligned}$$

2. This is exactly as in 4.3.16 of [Ravenel, 1986]: in the first expression, for each $I = (i_1, i_2, \ldots, i_h)$ with $h > 0$, the expression t_I^A appears twice, once as $t_I^A t_0^A$ and once as $t_{I'}^A (t_{i_h}^A)^{q^{||I'||}}$, where $I' = (i_1, i_2, \ldots, i_{h-1})$. The two terms have opposite formal sign. In the second expression, we have t_I^A appearing once as $t_0^A t_I^A$ and once as $(t_{(i_2,\ldots,i_h)}^A)^{q^{i_1}} t_{i_1}^A$, again with opposite formal sign.

3. First we assume that F has a logarithm. Let $\log_F(X) = \sum_{i\geqslant 0} \lambda_i X^{q^i}$ and let $[-1]_F(X) = \sum_{i\geqslant 1} a_i X^i$. Then

$$\sum_{i\geqslant 1} a_i X^i + \lambda_1 \left(\sum_{i\geqslant 1} a_i X^i \right)^p + \lambda_2 \left(\sum_{i\geqslant 1} a_i X^i \right)^{p^2} + \cdots = -\sum_{i\geqslant 0} \lambda_i X^{p^i},$$

which immediately forces $a_1 = -\lambda_0 = -1$ and $a_i = 0$ for $1 < i < p$. Suppose $a_i = 0$ for $1 < i < p^j$, j some positive integer; then $a_{p^j} + \lambda_j a_1^{p^j} = -\lambda_j$, i.e., $a_{p^j} = 0$ (this fails at $p = 2$; instead we would get $a_{p^j} = -p\lambda_j$) and immediately $a_i = 0$ for $1 < i < p^{j+1}$. By induction we have the proposition as stated, for formal group laws with logarithms; if F does not have a logarithm, we can find a ring R mapping to the underlying ring of F with an FGL $\bar{F}$ over R inducing F, and such that $\bar{F}$ has a logarithm (the universal example of R is BP_*); let g be this morphism and we have $g(\bar{F}(X, -X)) = g(0) = 0 = F(g(X), g(-X))$, so $[-1]_F(X) = g(-X) = -X$.

$\square$

Proposition 3.3.6. *Let h be a positive integer and let $N_h, R_h \subset V^A T$ be defined by*

$$\begin{aligned} N_h &= \bigcup_{||I||+i+j=n} \{(-1)^{|I|} t_I^A (v_i^A (t_j^A))^{q^{||I||}}\} \\ R_h &= N_h \cup \bigcup_{||J||=i, 0<i<h} \{\eta_R(v_J^A) w_J^A (R_{h-i})\}. \end{aligned}$$

Then $\eta_R(v_h^A) = w_\varnothing^A (R_h)$.

Proof. Let F be the formal group law on V^AT induced by η_L, and then from the previous lemma, we begin with $1 = \sum^F_{j,|K|\geqslant 0}[(-1)^{|K|}]_F t_j^A (t_K^A)^{q^j}$, and taking the logarithm we get

$$\begin{aligned}
\sum_{i\geqslant 0} \ell_i^A &= \sum_{j,|K|\geqslant 0} (-1)^{|K|} \log_F(t_j^A (t_K^A)^{q^j}) \\
&= \sum_{i,j,|K|\geqslant 0} \ell_i^A (-1)^{|K|} (t_j^A (t_K^A)^{q^j})^{q^i}
\end{aligned}$$

We substitute this expression for $\sum_{i\geqslant 0} \ell_i^A$ into $\sum_{i,j,k\geqslant 0} \ell_i^A (t_j^A)^{q^i} \eta_R(v_k^A)^{q^{i+j}} = \sum_{i,j\geqslant 0} \ell_i^A (v_j^A)^{q^i} (t_j^A)^{q^{i+j}}$, which we get from Prop.??, to get

$$\begin{aligned}
\sum_{i,j\geqslant 0} \eta_R(\ell_i^A)\eta_R(v_j^A)^{q^i} &= \sum_{i,j,k\geqslant 0} \ell_i^A (t_j^A)^{q^i} \eta_R(v_k^A)^{q^{i+j}} \\
&= \sum_{i,j\geqslant 0} \ell_i^A (v_j^A)^{q^i} (t_j^A)^{q^{i+j}} \\
&= \sum_{i,j,|K|,l,m\geqslant 0} (-1)^{|K|} \ell_i^A (t_j^A)^{q^i} (t_K^A)^{q^{i+j}} (v_l^A)^{q^{i+j+||K||}} (t_m^A)^{q^{i+j+||K||+l}} \\
&= \sum_{i,|J|,k,l\geqslant 0} (-1)^{|J|} \eta_R(\ell_i^A)(t_J^A)^{q^i} (v_K^A)^{q^{i+||J||}} (t_l^A)^{q^{i+||J||+k}} \\
&= \sum_{i,|J|,k,l\geqslant 0} (-1)^{|J|} \eta_R(\ell_i^A)(t_J^A (v_k^A (t_l^A)^{q^k})^{q^{||K||}})^{q^i}
\end{aligned}$$

and, letting F_2 denote the formal group law on V^AT induced by the right unit map (Ravenel uses $c(F)$ for this formal group law to emphasize that it is conjugate to the one induced by the left unit map), we apply $\log_{F_2}^{-1}$ to the expressions at the far left and right ends of this chain of equalities, and we get

$$\sum_{j\geqslant 0}{}^{F_2} \eta_R(v_j^A) = \sum_{|J|,k,l\geqslant 0}{}^{F_2} (-1)^{|J|} t_J^A (v_k^A (t_l^A)^{q^k})^{q^{||K||}},$$

and now applying Prop. 3.3.3 we get the proposition as stated. □

Definition 3.3.7. *Given a finite sequence J of positive integers and a positive integer i we define the element $c_{i,J} = w_J^A(R_i) \in V^AT$, and when $J = (j)$ we will simply write $c_{i,j}$ instead of $c_{i,J}$.*

3.4 The Morava kappa.

Definition 3.4.1. *Let A a p-adic number ring with residue field k, and let h be a positive integer. We put a V^A-module structure on $k[(v_h^A)^{\pm 1}]$ by the ring-morphism*

$$\begin{array}{ccl} V^A & \longrightarrow & k[(v_h^A)^{\pm 1}] \\ \alpha v_i^A & \mapsto & \begin{cases} 0 & \text{if } i \geqslant 1,\ i \neq h \\ \epsilon(\alpha) v_h^A & \text{if } i = h, \end{cases} \end{array}$$

where $\alpha \in A$ and $\epsilon(\alpha)$ is the reduction of α modulo the maximal ideal in A. We write $\kappa^A(h)$ for this V^A-module, and we refer to it as a Morava kappa*.*

Similarly we define V^A-modules $\bar{k}^A(h)_* \cong A[V_h^A]$ by a ring morphism identical to the one above, but without reducing the scalar in A. We can compose the structure map $V^A \to \bar{k}^A(h)_*$ with reduction modulo π_A on $\bar{k}^A(h)_*$ to get the V^A-module structure map of $\kappa^A(h)$, since V_h^A is congruent to v_h^A modulo π_A.

This module will play a crucial role in everything that follows in these notes. While this module is a generalization of the Morava K-theory ring $K(h)_*$, we have chosen the letter κ rather than K because when A is strictly larger than $\hat{\mathbb{Z}}_p$ it is not clear that this module is the coefficient ring of any kind of homology theory on spaces, much less one which deserves to be called a K-theory.

Proposition 3.4.2. *Given a positive integer h and an A-typical formal A-module law over R, the following conditions are equivalent:*

1. *For some choice of morphism $R' \xrightarrow{f} R$ of commutative A-algebras, such that the algebra structure map $A \to R'$ is injective, and some lift $\bar{F}$ of F to R', i.e., some $\bar{F}$ over R' such that $f(\bar{F}) = F$, the coefficients α_i of the logarithm*
$$\log_{\bar{F}}(X) = \sum_{i \geqslant 0} \alpha_i X^{q^i}$$

satisfy

$$\alpha_i = \begin{cases} 0 & \text{if } h \nmid i \\ \pi_A^{-j} (\pi_A \alpha_h)^{\frac{q^{hj}-1}{q^h-1}} & \text{if } i = hj \end{cases}.$$

2. *The classifying map $V^A \xrightarrow{\gamma_F} R$ of F factors through $\bar{k}^A(h)_*$.*

Proof. 1. **Condition 1 implies condition 2.** From what is given we know that the classifying map $V^A \to R$ factors through R'; we will show that the map $V^A \to R'$ factors through $\bar{k}^A(h)_*$.

Suppose that $\alpha_i = 0$ for all i with $h \nmid i$. Then, by Lemma 3.1.5, $\gamma_F(V_i^A) = 0$ for all i with $h \nmid i$. We have $\alpha_h = \pi_A^{-1}\gamma_F(V_h^A)$. Suppose that $\gamma_F(V_{hj}^A) = 0$ for all j with $1 < j < j'$, with j, j' integers, and suppose the expression for α_{hi} in the statement of the proposition holds. We show that $\gamma_F(V_{hj'}^A) = 0$:

$$\begin{aligned} \pi_A^{-j'} (\pi_A \alpha_h)^{\frac{q^{hj'}-1}{q^h-1}} &= \gamma_F(\ell_{hj'}^A) \\ &= \pi_A^{-1} \sum_{a=0}^{hj'-1} \gamma_F(\ell_a^A)\gamma_F(V_{hj'-a}^A)^{q^a} \\ &= \pi_A^{-1}(\gamma_F(V_{hj'}^A) + \gamma_F(\ell_{h(j'-1)}^A)\gamma_F(V_h^A)^{q^{h(j'-1)}} \\ &= \pi_A^{-1}(\gamma_F(V_{hj'}^A) + (\pi_A^{-j'+1} (\pi_A\alpha_h)^{\frac{q^{h(j'-1)}-1}{q^h-1}})(\pi_A\alpha_h)^{q^{h(j'-1)}}, \end{aligned}$$

so $\gamma_F(V_{hj'}^A) = 0$. Induction finishes the argument.

2. **Condition 2 implies condition 1.** We choose a commutative A-algebra R' and a morphism f; we want to construct a lift $\bar{F}$ of F to R' whose log coefficients are as described in condition 1. We will do this by factoring the map $\bar{k}^A(h)_* \xrightarrow{\gamma} R$ through R': since $\bar{k}^A(h)_*$ is a free A-algebra, we simply send V_h^A to any element in $f^{-1}(\gamma(V_h^A))$, and now $\bar{F}$ is the A-typical formal A-module law over R' classified by $V^A \to \bar{k}^A(h)_* \to R'$.

Now we consider the logarithm of $\bar{F}$. We recall that $\alpha_i = \gamma_{\bar{F}}(\ell_i^A)$. If $\gamma_{\bar{F}}$ factors through $K^A(h)_*$ then $\gamma_{\bar{F}}(\ell_i^A) = 0$ if the reduction of ℓ_i^A modulo

$(v_1^A, v_2^A, \ldots, v_{h-1}^A, v_{h+1}^A, \ldots)$ is zero. By Prop. 3.1.3 this is true if $h \nmid i$. Now $\alpha_h = \gamma_{\bar{F}}(\ell_h^A) = \pi_A^{-1}\gamma(V_h^A)$ and we have

$$\begin{aligned} \gamma_{\bar{F}}(\ell_{hi}^A) &= \pi_A^{-1}\sum_{a=0}^{hi-1} \gamma_{\bar{F}}(\ell_a^A)\gamma_{\bar{F}}(V_{hi-a}^A)^{q^a} \\ &= \pi_A^{-1}\gamma_{\bar{F}}(\ell_{h(i-1)}^A)\gamma_{\bar{F}}(\pi_A \ell_h^A)^{q^{h(i-1)}}. \end{aligned}$$

A quick induction gives the desired condition on α_{hi}.

□

Proposition 3.4.3.

$\kappa^A(h)\otimes_{V^A}V^AT\otimes_{V^A}\kappa^A(h) \cong \kappa^A(h)[t_1^A, t_2^A, \ldots]/(\{t_i^A(v_h^A)^{q^i} - v_h^A(t_i^A)^{q^h} : i \textit{ is a positive integer}\})$.

Proof. Tensoring V^AT on the left with $\kappa^A(h)$ produces $\kappa^A(h)\otimes_{V^A}V^AT \cong \kappa^A(h)[t_1^A, t_2^A, \ldots]$. The right unit formula (Prop. 3.2.4) gives us that $V^A \xrightarrow{\eta_R} \kappa^A(h) \otimes_{V^A} V^AT$ is determined by

$$\sum_{i\geqslant 0}{}^F t_i^A \eta_R(v_h^A)^{q^i} = \sum_{j\geqslant 0}{}^F v_h^A(t_j^A)^{q^h},$$

and in each of these formal sums there is only one element in each grading. Matching gradings, we get $t_i^A\eta_R(v_h^A)^{q^i} = v_h^A(t_i^A)^{q^h}$ in $\kappa^A(h) \otimes_{V^A} V^AT \otimes_{V^A} \kappa^A(h)$, giving us the relation in the statement of the theorem. □

Definition 3.4.4. *Let E/K be a finite extension of p-adic number fields and let h, j be positive integers. Then we define a map $\kappa^A(h) \xrightarrow{\gamma_\kappa} \kappa^B(j)$ by sending v_h^A to the image of v_h^A under the map $V^A \xrightarrow{\gamma} V^B \longrightarrow \kappa^B(j)$. Since this makes the square*

$$\begin{array}{ccc} V^A & \longrightarrow & \kappa^A(h) \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\kappa_\gamma} \\ V^B & \longrightarrow & \kappa^B(j) \end{array}$$

commute, we will call this the map induced in the Morava kappas by E/K. Obviously this map depends on a choice of h, j, and if we want something which is *functorial on the category of fields without a choice of h, j we can define the* total Morava kappa $\kappa(A) := \bigoplus_{h>0}\kappa^A(h)$*, and from the maps κ_γ we have a well-defined map $\kappa(A) \longrightarrow \kappa(B)$ induced by E/K.*

For any choice of h, the map κ_γ will be zero for most choices of j. We want to know when κ_γ is nonzero.

Proposition 3.4.5. *Let E/K be totally ramified of degree $n > 1$ and fix a positive integer j. The least h such that the map $\kappa^A(h) \xrightarrow{\gamma_\kappa} \kappa^B(j)$ is nonzero is $h = jn$. For $h = jn$ we have*

$$\gamma_\kappa(v^A_{jn}) = \epsilon\left(\frac{\pi_A - \pi_A^{q^{jn}}}{\prod_{m=1}^n(\pi_B - \pi_B^{q^{mj}})}\right)(v^B_j)^{\frac{q^{jn}-1}{q^j-1}},$$

where ϵ is reduction modulo the maximal ideal in B.

Proof. Assume $\kappa^A(h') \longrightarrow \kappa^B(j)$ is zero for all $h' < h$. Using Prop. 3.1.3 we have

$$\gamma(v^A_h) = \frac{\pi_A - \pi_A^{q^h}}{\pi_B - \pi_B^{q^h}}v^B_h +$$

$$\sum_{i_1+\cdots+i_r=h \text{ and } r>1}\left(\frac{\pi_A - \pi_A^{q^h}}{\pi_B - \pi_B^{q^h}}v^B_{i_1}\prod_{m=2}^r\frac{(v^B_{i_m})^{q^{\sum_{s=1}^{m-1} i_s}}}{\pi_B - \pi_B^{q^{\sum_{s=1}^{m-1} i_s}}} - \gamma(v^A_{i_1})\prod_{m=2}^r\frac{\gamma(v^A_{i_m})^{q^{\sum_{s=1}^{m-1} i_s}}}{\pi_A - \pi_A^{q^{\sum_{s=1}^{m-1} i_s}}}\right),$$

which is zero in $\kappa^A(j)$ unless $j \mid h$. Assume $j \mid h$; then we have

$$\begin{aligned}\gamma(v^A_h) &= \frac{\pi_A - \pi_A^{q^h}}{\pi_B - \pi_B^{q^h}}v^B_j\prod_{m=2}^{h/j}\frac{(v^B_j)^{q^{(m-1)j}}}{\pi_B - \pi_B^{q^{(m-1)j}}}\\ &= \frac{\pi_A - \pi_A^{q^h}}{\prod_{m=1}^{h/j}(\pi_B - \pi_B^{q^{mj}})}(v^B_j)^{\frac{q^h-1}{q^j-1}}\end{aligned}$$

and $\left|\frac{\pi_A - \pi_A^{q^h}}{\prod_{m=1}^{h/j}(\pi_B - \pi_B^{q^{mj}})}\right|_p \geqslant 1$ if and only if $\frac{h}{j} \geqslant n$; since, in this case, $r = \frac{h}{j} > 1$ and $n > 1$, we cannot have $\frac{h}{j} > n$, so the lowest h such that we get a nonzero map $\kappa^A(h) \longrightarrow \kappa^B(j)$ is $h = jn$. $\square$

We now know that the map $\kappa^A(jn)_* \xrightarrow{\gamma_\kappa} \kappa^B(j)_*$ is nontrivial.

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A Descent.

A.1 Sites, and sheaves on them.

The historical motivation for the development of Grothendieck topologies was a need to define sheaves on a variety which carry more data than sheaves on the category of (Zariski) open subsets of the variety; this was, at one point, a stumbling block for the development of the ℓ-adic cohomology theories which many geometers hoped to construct in order to approach ζ-functions of varieties, and the Weil conjectures, using cohomological machinery. Grothendieck's insight was that one should generalize the process of taking the category of open subsets of an algebraic variety X, or in other words, taking the category of schemes Y equipped with an open immersion $Y \to X$. One could instead take the category of schemes Y equipped with an étale open morphism $Y \to X$, and then consider **Sets**- or $\mathbf{Mod}_{\mathbb{Z}}$-valued sheaves on this category and the derived functors of their global sections $F \mapsto F(X)$ (this wound up providing, in many cases of interest to geometers, a suitable ℓ-adic cohomology theory). These sheaves are "richer" than the classical Zariski sheaves, in the sense that they take values on many more objects associated to X (not only all the Zariski open sets, but all the étale coverings of the Zariski open sets), and since they also must obey the sheaf axiom on those objects, there are also fewer (in a sense to be made precise) of these sheaves than there are Zariski sheaves. One can (and in this paper, we will) go further and consider sheaves which take values on (almost) all flat coverings of the Zariski open sets, getting even "richer" sheaves in the process. A Grothendieck topology on a category is additional data on that category which tells us which families of morphisms we consider "coverings" and as such it is the necessary data for defining sheaves on that category.

Definition A.1.1. Grothendieck topologies and sites. *Let $\mathcal{C}$ be a category. A* Grothendieck topology on $\mathcal{C}$ *consists of a set τ of families of maps $\{U_i \xrightarrow{\phi_i} U\}_{i\in I}$ in $\mathcal{C}$ called* coverings *satisfying the following axioms:*

1. *If ϕ is an isomorphism then $\phi \in \tau$. ("τ contains all isomorphisms.")*

2. *If $\{U_i \to U\} \in \tau$ and $\{V_{i,j} \to U_i\} \in \tau$ for every i then the family $\{V_{i,j} \to U\}$ obtained by composition is in τ. ("τ is closed under composition.")*

3. *If $\{U_i \to U\} \in \tau$ and $V \to U$ is a morphism in τ, then the categorical pullback $U_i \times_U V$ exists and $\{U_i \times_U V \to V\} \in \tau$ ("τ is stable under base change.")*

A site *is a pair $(\mathcal{C}, \tau)$ where $\mathcal{C}$ is a category and τ is a Grothendieck topology. The notations* $\mathrm{Cov}(\mathcal{C}, \tau)$ *for τ and* $\mathrm{Cat}(\mathcal{C}, \tau)$ *for $\mathcal{C}$ are standard in some sources and we will sometimes use them.*

A morphism $(\mathcal{C}, \tau) \xrightarrow{f} (\mathcal{C}', \tau')$ of sites is a functor of the underlying categories $\mathcal{C} \xrightarrow{f} \mathcal{C}'$ such that, if $\{U_i \xrightarrow{\phi_i} U\}_{i\in I} \in \tau$, then $\{f(U_i) \xrightarrow{f(\phi_i)} f(U)\}_{i\in I} \in \tau'$.

Remark A.1.2. This definition actually differs from that of [MR0, 1972], in which this notion of a Grothendieck topology is only considered a "pre-topology," the two notions of a site yield the same categories of sheaves, which is the important thing. This particular change in convention from SGA4 seems to be fairly standard in algebraic geometry; see for instance [Milne, 1980].

It is clear that a composite of site-morphisms is a site-morphism and hence there is a category (actually a 2-category, with the additional structure of natural transformations of site-morphisms) of sites whose underlying categories are small. We will not have occasion to work with this category in these notes, so we will not fix a name for it, but note that, while on the surface it seems too restrictive to only consider sites whose underlying categories are small, these are good "models" for many sites which occur in practice, since the nicest examples of sites (for instance, any site whose underlying category is a subcategory of the category of schemes locally of finite type over a fixed noetherian base scheme) have a small subcategory containing at least one object in each isomorphism class. This makes a naive sheafification of presheaves possible, when it otherwise would not be. We say more about this in Prop. A.1.10.

Definition A.1.3. Presheaves and sheaves on a site. *Let $\mathcal{X}$ be a site. A* presheaf on $\mathcal{X}$ with values in $\mathcal{C}$ *is a functor* $(\mathrm{Cat}\,\mathcal{X})^{\mathrm{op}} \xrightarrow{F} \mathcal{C}$. *A* sheaf on $\mathcal{X}$ with values in $\mathcal{C}$ *is a presheaf F on $\mathcal{X}$ with values in $\mathcal{C}$ such that, for any $\{U_i \to U\}_{i\in I} \in$* $\mathrm{Cov}\,\mathcal{X}$, *the following diagram (assembled from the maps given by the universal properties of the products involved) is an equalizer diagram:*

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j).$$

Lemma A.1.4. Continuous functors (on the target) take sheaves to sheaves. *Let $\mathcal{X}$ be a site, let $\mathcal{C}, \mathcal{D}$ be categories, let $\mathcal{F}$ be a sheaf* $(\mathrm{Cat}\,\mathcal{X})^{\mathrm{op}} \to \mathcal{C}$, *and let $\mathcal{C} \xrightarrow{G} \mathcal{D}$ be a continuous functor (i.e., a functor preserving all small limits). Then* $(\mathrm{Cat}\,\mathcal{X})^{\mathrm{op}} \xrightarrow{G\circ\mathcal{F}} \mathcal{D}$ *is a sheaf.*

Proof. The equalizer arising from applying $\mathcal{F}$ to a covering family in $\mathcal{X}$ is a small limit. □

Corollary A.1.5. *If $\mathcal{C}$ is a concrete category such that the underlying set of a product $\prod_i A_i$ in $\mathcal{C}$ is the same as the set-theoretic product of the underlying sets* *of the A_i, i.e., the structure functor $\mathcal{C} \to$ **Sets** of $\mathcal{C}$ as a concrete category is continuous, then a presheaf on $\mathcal{X}$ with values in $\mathcal{C}$ is a sheaf if and only if its underlying* **Sets***-valued presheaf on $\mathcal{X}$ is a sheaf.*

In practice, whatever Grothendieck topologies we work in, we will want all *representable* presheaves to be sheaves.

Definition A.1.6. UEEFs and the canonical topology.

1. *Let $\mathcal{C}$ be a category with arbitrary pullbacks. A family $\{U_i \to U\}_{i\in I}$ of maps in $\mathcal{C}$ is called a* universal effectively epimorphic family *(or* UEEF *for short) if, for any objects W, Z of $\mathcal{C}$ and any morphism $W \to U$ in $\mathcal{C}$, the following diagram in* **Sets** *is an equalizer diagram:*

   $$\hom_{\mathcal{C}}(W,Z) \longrightarrow \prod_{i\in I} \hom_{\mathcal{C}}(W \times_U U_i, Z) \rightrightarrows \prod_{i,j\in I} \hom_{\mathcal{C}}(W \times_U (U_i \times_U U_j), Z)\ .$$

   We call a map $V \xrightarrow{f} U$ in $\mathcal{C}$ a universal effective epimorphism *if $\{f\}$ is a UEEF.*

2. *Let $\mathcal{C}$ be a category with arbitrary pullbacks. Then there is a canonical Grothendieck topology $\tau_{\mathcal{C}}$ associated to $\mathcal{C}$: it is defined by letting $\operatorname{Cat}\tau_{\mathcal{C}}$ be $\mathcal{C}$ and by letting $\operatorname{Cov}\tau_{\mathcal{C}}$ be the UEEFs in $\mathcal{C}$.*

   (In the next lemma we will show that $\tau_{\mathcal{C}}$ is actually a Grothendieck topology.)

3. *Let $\mathcal{C}$ be a category with small pullbacks, and let τ be a Grothendieck topology on $\mathcal{C}$ such that $\operatorname{Cov}\tau \subseteq \operatorname{Cov}\tau_{\mathcal{C}}$, i.e., the identity map on $\mathcal{C}$ induces a morphism $\tau \to \tau_{\mathcal{C}}$ of Grothendieck topologies. Then we call τ a* subcanonical topology *on $\mathcal{C}$.*

Lemma A.1.7. A topology is subcanonical iff all representable presheaves are sheaves.

1. *Let $\mathcal{C}$ be a category with arbitrary pullbacks. Then $\tau_{\mathcal{C}}$ is actually a Grothendieck topology.*

2. *Let $(\mathcal{C}, \tau)$ be a site such that $\mathcal{C}$ is a category with arbitrary pullbacks. Then τ is subcanonical if and only if the* **Sets***-valued presheaf* $X \mapsto \hom_{\mathcal{C}}(X, Y)$ *is a sheaf for every object Y of $\mathcal{C}$.*

Proof. 1. Every isomorphism is clearly a UEE.

Let $\{U_i \xrightarrow{\phi_i} U\}$ be a UEEF and let $\{U_{i,j} \xrightarrow{\psi_{i,j}} U_i\}$ be a UEEF for each i; let W be an object over U in $\mathcal{C}$, and let V be an object in $\mathcal{C}$. Then we have the pair of morphisms

$$\prod_{i,j} \hom_{\mathcal{C}}(W \times_U U_{i,j}, V) \rightrightarrows \prod_{i,j,k,l} \hom_{\mathcal{C}}(W \times_U U_{i,j} \times_U U_{k,l}, V)$$

and we want to show that if x is an element in $\prod_{i,j} \hom_{\mathcal{C}}(W \times_U U_{i,j}, V)$ whose two restrictions to $\prod_{i,j,k,l} \hom_{\mathcal{C}}(W \times_U U_{i,j} \times_U U_{k,l}, V)$ are equal, then x is in the image of $\hom_{\mathcal{C}}(W, V)$ in $\prod_{i,j} \hom_{\mathcal{C}}(W \times_U U_{i,j}, V)$. We have the commutative diagram

$$\begin{tikzcd}
\hom_{\mathcal{C}}(W,V) \arrow[r] & \prod_i \hom_{\mathcal{C}}(W \times_U U_i, V) \arrow[r, shift left, "\rho'_1"] \arrow[r, shift right, "\rho'_2"'] \arrow[d] & \prod_{i,j} \hom_{\mathcal{C}}(W \times_U U_i \times_U U_j, V) \arrow[d] \\
& \prod_{i,j} \hom_{\mathcal{C}}(W \times_U U_{i,j}, V) \arrow[r, shift left, "\rho_1"] \arrow[r, shift right, "\rho_2"'] \arrow[d, shift right, "\rho''_2"'] \arrow[d, shift left, "\rho''_1"] & \prod_{i,j,k,l} \hom_{\mathcal{C}}(W \times_U U_{i,j} \times_U U_{k,l}, V) \\
& \prod_{i,j,k} \hom_{\mathcal{C}}(W \times_U U_{i,j} \times_{U_i} U_{i,k}, V) &
\end{tikzcd}$$

in which the top row and the middle column are both equalizer sequences, and, given $x \in \prod_{i,j} \hom_{\mathcal{C}}(W \times_U U_{i,j}, V)$ with $\rho_1(x) = \rho_2(x)$, it is certainly the case that $\rho''_1(x) = \rho''_2$, so x is in the image of $\prod_i \hom_{\mathcal{C}}(W \times_U U_i, V)$ in $\prod_{i,j} \hom_{\mathcal{C}}(W \times_U U_{i,j}, V)$, due to the middle column being an equalizer sequence. Choose an $x' \in \prod_i \hom_{\mathcal{C}}(W \times_U U_i, V)$ whose image in $\prod_{i,j} \hom_{\mathcal{C}}(W \times_U U_{i,j}, V)$ is x; then, due to the commutativity of the above diagram, $\rho'_1(x') = \rho'_2(x')$, and the fact that the top row is an equalizer sequence tells us that x' is in the image of $\hom_{\mathcal{C}}(W, V)$ in $\prod_i \hom_{\mathcal{C}}(W \times_U U_i, V)$ is x'. So the induced map in the above diagram from $\hom_{\mathcal{C}}(W, V)$ to the equalizer of ρ_1 and ρ_2 is a surjection. Now from the above commutative diagram, the injective map $\hom_{\mathcal{C}}(W,V) \hookrightarrow \prod_{i,j} \hom_{\mathcal{C}}(W \times_U U_{i,j}, V)$ factors through $\mathrm{eq}\{\rho_1,\rho_2\} \hookrightarrow \hom_{\mathcal{C}}(W \times_U U_{i,j}, V)$, also an injection, so the map $\hom_{\mathcal{C}}(W,V) \to \mathrm{eq}\{\rho_1,\rho_2\}$ is also an injection. We have already established that it is a surjection, so it is an isomorphism, and a composite of UEEFs is a UEEF.

That a pullback of a UEEF is also a UEEF is built into the definition of a UEEF.

2. If every covering in τ is a UEEF then, for any object Z in $\mathcal{C}$ and any covering $\{U_i \to U\}$ in τ, the presheaf represented by Z satisfies the sheaf axiom for the covering $\{U_i \to U\}$, by the definition of a UEEF; so the presheaf represented by Z is a sheaf. Conversely, if $\{U_i \to U\}$ is a covering in τ which is *not* a UEEF, this implies that there must exist some object Z of $\mathcal{C}$ such that

$$\hom_{\mathcal{C}}(U,Z) \longrightarrow \prod_{i\in I} \hom_{\mathcal{C}}(U_i,Z) \rightrightarrows \prod_{i,j\in I} \hom_{\mathcal{C}}(U_i \times_U U_j, Z)$$

is *not* an equalizer diagram, so the presheaf on τ represented by Z is not a sheaf on τ.

□

Since the purpose of defining sites is to consider sheaves on them, there is a set-theoretic difficulty with sites which do not have a small subcategory with at least one object in each isomorphism class; there can exist presheaves on such sites which do not admit a sheafification. One can either fix a universe, as in SGA (appendix of [MR0, 1972]), or one can use Waterhouse's method (from [Waterhouse, 1975]) of only defining sheafification on "basically bounded" presheaves. See the discussion following Prop. 2.2 of [Milne, 1980]. The fpqc topology is the most important topology for the purposes of this thesis, and the big fpqc site over a base scheme S does not admit a small subcategory having at least one object in each isomorphism class, so we will use Waterhouse's method. Note that there do exist presheaves on the big fpqc site which do not admit a sheafification; see the example following Thm. 5.4 in [Waterhouse, 1975]. These presheaves fail to satisfy Waterhouse's "basically bounded" condition.

Definition A.1.8. Notations for categories of (pre)sheaves, and basically bounded (pre)sheaves.

Let $\mathcal{C}, \mathcal{D}$ be categories. We write $\mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$ for the category of $\mathcal{D}$-valued presheaves on $\mathcal{C}$.

1. *Let $\mathcal{X}$ be a site, and let $\mathcal{D}$ be a category. We write $\mathcal{D}\,\mathrm{Sh}(\mathcal{X})$ for the category of $\mathcal{D}$-valued sheaves on $\mathcal{X}$, and when the collection of $\mathcal{D}$-valued sheaves on $\mathcal{X}$ satisfying some property P forms a category, we will write $P - \mathcal{D}\,\mathrm{Sh}(\mathcal{X})$ for that category.*

2. *Let $\mathcal{C}, \mathcal{D}$ be categories with $\mathcal{D}$ having equalizers and small colimits. Suppose $\mathcal{C}$ is a concrete category and suppose we choose a faithful functor $\mathcal{C} \xrightarrow{F} \mathbf{Sets}$ (that $\mathcal{C}$ is "concrete" simply means that at least one such faithful functor exists). Let m be a cardinal number and let $\mathcal{C}(m)$ be the full subcategory of $\mathcal{C}$ whose objects are those x such that $F(x)$ is a set of cardinality $\leqslant m$. We have the inclusion functor $\mathcal{C}(m) \xrightarrow{j(m)} \mathcal{C}$ and we have the induced functor on presheaves*

$$\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}} \xrightarrow{j(m)^*} \mathbf{Sets}^{\mathcal{C}(m)^{\mathrm{op}}}$$

given by $j(m)^*(\mathcal{F}) = \mathcal{F} \circ j(m)$. As $\mathcal{C}(m)$ is small and $\mathcal{D}$ has small colimits, any presheaf $\mathcal{F} \in \mathrm{ob}\,\mathbf{Sets}^{\mathcal{C}(m)}$ has a left Kan extension along $j(m)$; taking the left Kan extension along $j(m)$ of presheaves gives us a functor

$$\mathbf{Sets}^{\mathcal{C}(m)^{\mathrm{op}}} \xrightarrow{j(m)_*} \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$$

which is left adjoint to $j(m)^*$, by usual properties of left Kan extensions (see [Mac Lane, 1998]).

If a presheaf $\mathcal{F} \in \mathrm{ob}\ \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ *is in the image of* $j(m)_*$ *we will refer to* $\mathcal{F}$ *as* m-based. *If a presheaf* $\mathcal{F}$ *is* m*-based for some cardinal number* m*, then we refer to* $\mathcal{F}$ *as* basically bounded. *We write* $\bigcup_m \mathbf{Sets}^{\mathcal{C}(m)^{\mathrm{op}}}$ *or* $\mathrm{bb}-\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ *for the full subcategory of* $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ *consisting of basically bounded presheaves, and if* $\mathcal{X}$ *is a site with underlying category* $\mathcal{C}$*, we write* $\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{X})$ *for the intersection of* $\mathcal{D}\,\mathrm{Sh}(\mathcal{X})$ *with* $\mathrm{bb}-\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ *in* $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$.

3. *Recall the axioms L1,L2, and L3 of M. Artin (from [Artin, 1962]) for a category* $\mathcal{C}$*:*

   (a) **L1:** *Given a diagram*

$$\begin{tikzcd} & Y \\ X \arrow[ur] \arrow[dr] & \\ & Y' \end{tikzcd}$$

   in $\mathcal{C}$*, there exists a diagram*

$$\begin{tikzcd} Y \arrow[dr] & \\ & Z \\ Y' \arrow[ur] & \end{tikzcd}$$

   in $\mathcal{C}$ *such that the resulting square commutes.*

   (b) **L2:** *Given a diagram* $X \rightrightarrows Y$ *in* $\mathcal{C}$*, there exists an object* Z *and a map* $Y \to Z$ *such that the two composite maps* $X \to Z$ *are equal.*

   (c) **L3** $\mathcal{C}$ *is connected, i.e., for any objects* $X, Y \in \mathrm{ob}\ \mathcal{C}$ *there exists a (finite) set of objects* $Z_1, ..., Z_n$ *and a chain of morphisms*

$$\begin{tikzcd} X \arrow[dr] & & Z_2 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & Z_{n-1} \arrow[dl] \arrow[dr] & & Y \arrow[dl] \\ & Z_1 & & Z_3 & & Z_{n-2} & & Z_n & \end{tikzcd}$$

in $\mathcal{C}$.

*If $\mathcal{C}$ is a category equipped with a faithful functor $\mathcal{C} \longrightarrow$ **Sets** such that $\mathcal{C}(m)$ satisfies L1, L2, and L3 for any cardinal number m, then we say that $\mathcal{C}$ is* Artin-concrete.

Lemma A.1.9. *Let $\mathcal{C}$ be an Artin-concrete category and let $\mathcal{D}$ be an abelian category. Then for any cardinal number m, the left Kan extension*

$$\mathcal{D}^{\mathcal{C}(m)^{\mathrm{op}}} \xrightarrow{j(m)_*} \mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$$

is exact.

Proof. See I.2 of [Artin, 1962]. □

Proposition A.1.10. Sheafification. *Let $\mathcal{X}$ be a site with underlying category $\mathcal{C}$ equipped with a faithful functor $\mathcal{C} \to$ **Sets**, and let $\mathcal{D}$ be a category with equalizers and small colimits. The forgetful functor*

$$\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{X}) \xrightarrow{\texttt{forget}} \mathrm{bb}-\mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$$

has a left adjoint #, which we call "sheafification."

Proof. Let $\mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathcal{D}$ be a basically bounded presheaf, and let m be a cardinal number such that F is m-based. Now we will write $F(m)$ for F regarded as a sheaf on the subcategory $\mathcal{C}(m)$ of $\mathcal{C}$ which maps to sets of cardinality $\leqslant m$ via the faithful functor $\mathcal{C} \to$ **Sets**; F is obtained from $F(m)$ by left Kan extension along the inclusion of that subcategory into $\mathcal{C}$. We construct $\#(F(m))$ as follows: for any $U \in \mathrm{ob}\ \mathcal{C}(m)$ let $J_U(m)$ denote the category of τ-coverings $\{U_i \longrightarrow U\}$ of U with all U_i in $\mathcal{C}(m)$. Then F induces a functor $J_U(m)^{\mathrm{op}} \xrightarrow{F_U(m)} \mathcal{D}$ by

$$F_U(m)(\{U_a \longrightarrow U\}) = \mathrm{eq}\left\{ \textstyle\prod_a F(U_a) \rightrightarrows \prod_{a,b} F(U_a \times_U U_b) \right\}.$$

Let

$$F(m)^+(U) = \mathrm{colim}_{\{U_i \to U\} \in \mathrm{ob}\ J_U(m)}\, F_U(\{U_i \to U\}).$$

Functoriality in U, and that $F(m)^+$ is a separated presheaf, and that $F(m)^+$ is a sheaf if $F(m)$ is a separated presheaf, is proved as in section II.1 of [Artin, 1962]. Our sheafification $\#(F(m))$ is then $F(m)^{++}$. We take the left Kan extension of $F(m)^{++}$ along the inclusion of $\mathcal{C}(m)$ into $\mathcal{C}$ to get the sheaf $\#(F)$ defined on $\mathcal{C}$. $\square$

Lemma A.1.11. Inverse image and direct image (pre)sheaves.

1. *Let $\mathcal{C} \xrightarrow{f} \mathcal{C}'$ be a functor. Then we have a functor*

$$\begin{array}{ccc} \mathcal{D}^{(\mathcal{C}')^{\mathrm{op}}} & \xrightarrow{f^*} & \mathcal{D}^{\mathcal{C}^{\mathrm{op}}} \\ \mathcal{F} & \mapsto & \mathcal{F} \circ f. \end{array}$$

2. *Let $\mathcal{C}, \mathcal{C}'$ be concrete categories equipped with a fixed choice of faithful functor $\mathcal{C} \xrightarrow{G}$ **Sets**, $\mathcal{C}' \xrightarrow{G'}$ **Sets**. Let $(\mathcal{C}, \tau) \xrightarrow{f} (\mathcal{C}', \tau')$ be a morphism of sites such that G is naturally isomorphic to $G' \circ f$. Let $\mathcal{D}$ be a cocomplete category and let $\mathcal{F}$ be a basically bounded $\mathcal{D}$-valued sheaf on $(\mathcal{C}', \tau')$. Then $\mathcal{F} \circ f$ is a basically bounded $\mathcal{D}$-valued sheaf on $(\mathcal{C}, \tau)$, i.e.,*

$$\mathrm{bb}-\mathcal{D}^{(\mathcal{C}')^{\mathrm{op}}} \xrightarrow{f^*} \mathrm{bb}-\mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$$

restricts to a map

$$\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}'_{\tau'}) \xrightarrow{f^*} \mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}_{\tau}).$$

3. *The functor*

$$\mathrm{bb}-\mathcal{D}^{(\mathcal{C}')^{\mathrm{op}}} \xrightarrow{f^*} \mathrm{bb}-\mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$$

has a left adjoint

$$\mathrm{bb}-\mathcal{D}^{\mathcal{C}^{\mathrm{op}}} \xrightarrow{f_*} \mathrm{bb}-\mathcal{D}^{(\mathcal{C}')^{\mathrm{op}}}.$$

If $\mathcal{F}$ is a sheaf, it is not necessarily true that the presheaf $f_*\mathcal{F}$ is a sheaf; however, we can compose with sheafification to get that the functor

$$\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}'_{\tau'}) \xrightarrow{f^*} \mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}_{\tau})$$

has a left adjoint

$$\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}_\tau) \xrightarrow{\#\circ f_*} \mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}'_{\tau'}).$$

Proof. 1. Functoriality is immediate from the definition of the map f^*.

2. Let $\{U_i \xrightarrow{\phi_i} U\} \in \tau$; then $\mathcal{F}$ obeys the sheaf axiom on $\{f(U_i) \xrightarrow{f(\phi_i)} f(U)\} \in \tau'$, so $\mathcal{F} \circ f$ obeys the sheaf axiom on $\{U_i \xrightarrow{\phi_i} U\}$.

3. This sketch of a proof is from [Artin, 1962]. For each $Y \in \mathcal{C}'$ we construct the category I^f_Y where

$$\begin{aligned} \mathrm{ob}\ I^f_Y &= \{\text{pairs } (X,\psi) : X \in \mathrm{ob}\ \mathcal{C}, \psi \in \hom_{\mathcal{C}'}(Y, f(X))\}, \\ \hom_{I^f_Y}((X_1,\psi_1),(X_2,\psi_2)) &= \{\xi \in \hom_{\mathcal{C}}(X_1,X_2) : f(\xi)\circ\psi_1 = \psi_2\}. \end{aligned}$$

Any presheaf $\mathcal{F}$ on $\mathcal{C}$ "restricts" to a presheaf $\mathcal{F}_Y$ on I^f_Y by taking the composite $(I^f_Y)^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathcal{D}$, where the first functor is the projection $(X,\psi) \mapsto X$. This "restricted" presheaf is functorial, in the appropriate sense, in $\mathcal{F}$, as is I^f_Y in Y.

We now define f_* by:

$$(f_*\mathcal{F})(Y) = \mathrm{colim}\,\mathcal{F}_Y = \mathrm{colim}_{(X,\psi)\in I^f_Y}\,\mathcal{F}(X).$$

The functoriality of I^f_Y in Y makes $f_*\mathcal{F}$ a presheaf on $\mathcal{C}'$, and the functoriality of $\mathcal{F}_Y$ in $\mathcal{F}$ makes $\mathcal{F} \mapsto f_*\mathcal{F}$ a functor $\mathcal{D}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{D}^{(\mathcal{C}')^{\mathrm{op}}}$.

Now, for any $\xi \in \hom_{\mathcal{C}}(\mathcal{F}, f^*\mathcal{G})$, we have for each $(X,\psi) \in I^f_Y$ a map $F(X) \to G(f(X)) \xrightarrow{G(\psi)} G(Y)$ and so by the universal property of the colimit we have a unique map $\mathrm{colim}\,\mathcal{F}_Y \to \mathcal{G}(Y)$, which is functorial in Y and hence gives us a map of presheaves $f_*\mathcal{F} \to \mathcal{G}$ on $\mathcal{C}'$. Conversely, given a map $\mathrm{colim}\,\mathcal{F}_Y \to \mathcal{G}(Y)$ we get maps $\mathcal{F}(X) \to f_*\mathcal{F}(f(X)) \to \mathcal{G}(f(X))$, where the map on the left in the composite is induced by $(X, \mathrm{id}_{f(X)}) \in \mathrm{ob}\ I^f_{f(X)}$, and so a map $\mathcal{F} \to f^*\mathcal{G}$. These two maps are clearly inverses of each other, by their construction; hence f^* is right adjoint to f_*.

□

A.2 Abelian categories.

Sheaves are defined on sites, which we have already discussed; sheaves must take values in some category, and if we are to do homological algebra with those sheaves, they will need to take values in an abelian category. We develop some of the technology necessary for doing such homological algebra, especially for showing that the sheaf categories we care about have injective resolutions.

We recall the definitions of additive and abelian categories, and the extra conditions AB3, AB4, AB5, AB3*, AB4*, and AB5* for abelian categories which were introduced by Grothendieck in [Grothendieck, 1957]; English-language references for this material include [Bucur and Deleanu, 1968] and [Weibel, 1994]. The conditions AB3,...,AB5* are conditions on an abelian category which make it more amenable, in varying ways, to doing homological algebra.

Definition A.2.1. Abelian categories and the Grothendieck conditions AB3,...,AB5. *Let $\mathcal{C}$ be a category.*

1. *We say that $\mathcal{C}$ together with the structure of an abelian group on* $\hom_{\mathcal{C}}(A,B)$ *for each pair of objects $A, B \in$ ob $\mathcal{C}$ is an* additive category *if the following conditions are satisfied:*

   (a) *composition is bilinear, i.e., for every every triple of objects $A, B, C \in$ ob $\mathcal{C}$, we have*

   $$h \circ (f+g) = (h \circ f) + (h \circ g)$$

   for all $h \in \hom_{\mathcal{C}}(B,C)$ and $f, g \in \hom_{\mathcal{C}}(A,B)$, and we have

   $$(f+g) \circ h = (f \circ h) + (g \circ h)$$

for all $f, g \in \hom_{\mathcal{C}}(B, C)$ *and* $h \in \hom_{\mathcal{C}}(A, B)$*; and*

(b) *finite products exist in* $\mathcal{C}$*.*

2. *We say that* $\mathcal{C}$ *is an* abelian category *if* $\mathcal{C}$ *is an additive category satisfying the following conditions:*

   (a) *there is a zero object in* $\mathcal{C}$*;*

   (b) *every morphism in* $\mathcal{C}$ *has a kernel, i.e., every morphism* $A \xrightarrow{f} B$ *in* $\mathcal{C}$ *has an object* $\ker f \xrightarrow{i} A$ *over* A *such that* $f \circ i = 0$ *and* $\ker f$ *is terminal in the category of objects* $T \xrightarrow{g} A$ *over* A *in* $\mathcal{C}$ *such that* $f \circ g = 0$*;*

   (c) *every morphism in* $\mathcal{C}$ *has a cokernel, i.e., every morphism* $A \xrightarrow{f} B$ *in* $\mathcal{C}$ *has an object* $B \xrightarrow{P} \operatorname{coker} f$ *under* B *such that* $P \circ f = 0$ *and* $\operatorname{coker} f$ *is initial in the category of objects* $B \xrightarrow{g} T$ *under* B *in* $\mathcal{C}$ *such that* $g \circ f = 0$*;*

   (d) *for every morphism* f *in* $\mathcal{C}$*, if* $\ker f = 0$*, then* $\ker(\operatorname{coker} f) = f$*; and*

   (e) *for every morphism* f *in* $\mathcal{C}$*, if* $\operatorname{coker} f = 0$*, then* $\operatorname{coker}(\ker f) = f$*.*

   (Note that any abelian category automatically has equalizers, making it immediately a good place for sheaves to take values in; the equalizer of f, g *is* $\ker(f - g)$*.)*

3. *An abelian category* $\mathcal{C}$ *is said to satisfy* AB3 *if it is cocomplete, i.e., it contains all small colimits. (Note that this means we can sheafify basically bounded presheaves taking values in* $\mathcal{C}$*, by Prop. A.1.10.)*

4. *An abelian category* $\mathcal{C}$ *is said to satisfy* AB4 *if it satisfies AB3 and the coproduct of a small family of monomorphisms is a monomorphism.*

5. *An abelian category* $\mathcal{C}$ *is said to satisfy* AB5 *if it is cocomplete and filtered colimits of exact sequences are exact. (This implies that* $\mathcal{C}$ *satisfies AB4*

since we can take a small family of monomorphisms $\{0 \to A_i \to B_i\}_{i \in I}$ *and form the directed system of all finite coproducts of these monomorphisms; this is a filtered diagram with colimit the coproduct* $0 \to \bigoplus_{i\in I} A_i \to \bigoplus_{i \in I} B_i$*.)*

6. *An abelian category* $\mathcal{C}$ *is said to satisfy* AB3* *(resp.* AB4*, AB5**) if the dual abelian category* $\mathcal{C}^{\mathrm{op}}$ *satisfies* AB3 *(resp.* AB4, AB5*).*

Lemma A.2.2. ker -coker **exact sequence.** *Let* $A \xrightarrow{f} B \xrightarrow{g} C$ *be a composable pair of morphisms in an abelian category* $\mathcal{C}$*.*

1. *We have an exact sequence*

$$0 \to \ker f \to \ker(g \circ f) \to \ker g \to \operatorname{coker} f \to \operatorname{coker}(g \circ f) \to \operatorname{coker} g \to 0$$

   in $\mathcal{C}$*, with the maps given by the universal properties of* ker *and* coker*.*

2. *If* $\ker(g \circ f) \cong 0$ *then* $\ker f \cong 0$*.*

3. *If* $\operatorname{coker}(g \circ f) \cong 0$ *then* $\operatorname{coker} g \cong 0$*.*

4. *If the natural map* $\ker f \to \ker(g \circ f)$ *is an isomorphism and* $\operatorname{coker} f \cong 0$*, then* $\ker g \cong 0$

5. *If the natural map* $\operatorname{coker}(g \circ f) \to \operatorname{coker} g$ *is an isomorphism and* $\ker g \cong 0$ *then* $\operatorname{coker} f \cong 0$*.*

Proof. 1. We first describe the maps in the sequence

$$0 \to \ker f \to \ker(g \circ f) \to \ker g \to \operatorname{coker} f$$

before showing its exactness.

We have a map $\ker f \xrightarrow{j_1} \ker(g \circ f)$ because any object $T \xrightarrow{h} A$ satisfying $f \circ h = 0$ also satisfies $g \circ f \circ h = 0$.

We have a map $\ker(g \circ f) \xrightarrow{j_2} \ker g$ because the morphism $\ker(g \circ f) \xrightarrow{h} A$ satisfies $g \circ (f \circ h) = 0$.

We have a map $\ker g \xrightarrow{j_3} \operatorname{coker} f$ simply as the composite $\ker f \to B \to \operatorname{coker} f$.

Now we need to show exactness at all three joints. First, since $\ker f$ is itself final in the category of objects over $\ker f$ whose composite with g is zero, we know that j_1 is monomorphic. Now $\ker j_2$ is the final object in the category of objects $T \xrightarrow{h} \ker(g \circ f)$ such that $j_2 \circ h = 0$, i.e., the category of objects $T \xrightarrow{h} A$ such that $f \circ h = 0$, so the map $\ker j_2$ is precisely $\ker f$.

Now we want to consider $\operatorname{im} j_2 = \ker \operatorname{coker} j_2$ and $\ker j_3$. The object $\ker \operatorname{coker} j_2$ is the final object in the category of objects $T \xrightarrow{h} \ker g$ with $k \circ h = 0$ for any $\ker g \xrightarrow{k} T'$ with $k \circ j_2 = 0$, and as j_2 is given by composition with f, this describes exactly $\ker j_3$. Hence the sequence is exact.

We apply this proof also to the opposite sequence of $A \xrightarrow{f} B \xrightarrow{g} C$ in the opposite abelian category $\mathcal{C}^{\mathrm{op}}$ to get the exactness of the sequence

$$\ker g \to \operatorname{coker} f \to \operatorname{coker}(g \circ f) \to \operatorname{coker} g \to 0$$

in $\mathcal{C}$. Splicing the two exact sequences together, we get the exact sequence in the lemma.

2. The remaining parts of this lemma are immediate consequences of the first part.

□

Lemma A.2.3. The cokernel of a sheaf-morphism is a separated presheaf. *Let $\mathcal{X}$ be a site with underlying category $\mathcal{C}$ equipped with a faithful functor $\mathcal{C} \to$* **Sets***, and let $\mathcal{D}$ be an abelian category satisfying AB4. Let $\mathcal{F} \xrightarrow{f} \mathcal{G}$ be a morphism of basically bounded $\mathcal{D}$-valued sheaves on $\mathcal{X}$. Then the presheaf* $\operatorname{coker} f$ *defined by*

$$(\operatorname{coker} f)(U) = \operatorname{coker}\left(\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U)\right)$$

for all $U \in \operatorname{ob} \mathcal{C}$ is a separated presheaf.

Proof. Let $\{U_i \to U\}_{i \in I}$ be a cover in $\mathcal{X}$. We have the commutative diagram in $\mathcal{D}$ with rows equalizer sequences:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d, "f(U)"] & \prod_{i\in I} \mathcal{F}(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\prod_{i\in I} f(U_i)"] & \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j) \arrow[d] \\
\mathcal{G}(U) \arrow[r] \arrow[d] & \prod_{i\in I} \mathcal{G}(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[dd] & \prod_{i,j\in I} \mathcal{G}(U_i \times_U U_j) \arrow[dd] \\
(\operatorname{coker} f)(U) \arrow[d, "j"] & & \\
\operatorname{eq}\{\rho_1, \rho_2\} \arrow[r] & \prod_{i\in I} (\operatorname{coker} f)(U_i) \arrow[r, shift left, "\rho_1"] \arrow[r, shift right, "\rho_2"'] & \prod_{i,j\in I} (\operatorname{coker} f)(U_i \times_U U_j)
\end{tikzcd}$$

and we need to show that j (whose existence is given by the universal property of an equalizer) is an injection. Consider the composable sequence of morphisms

$$\mathcal{G}(U) \to (\operatorname{coker} f)(U) \xrightarrow{j} \operatorname{eq}\{\rho_1, \rho_2\}$$

appearing in the above commutative diagram; the left-hand morphism is surjective and has kernel $\mathcal{F}(U)$, isomorphic to the kernel of the composite $\mathcal{G}(U) \to \operatorname{eq}\{\rho_1, \rho_2\}$, since a kernel sheaf of a presheaf morphism is isomorphic to the kernel sheaf of the sheafification of the morphism; so by Lemma A.2.2, j is an injection. □

Lemma A.2.4. Basically bounded presheaves (with values in an abelian category) form an abelian category. *Let $\mathcal{C}$ be a category equipped with a faithful functor to* **Sets** *(this is so that we can consider basically bounded presheaves and sheaves on $\mathcal{C}$), and let $\mathcal{D}$ be an abelian category. Then the category* $\mathrm{bb}-\mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$ *(of basically bounded $\mathcal{D}$-valued presheaves on $\mathcal{C}$) is an abelian category. If $\mathcal{D}$ satisfies AB5 then so does* $\mathrm{bb}-\mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$.

Proof. A zero object, biproducts, kernels, and cokernels are all formed "sections-wise" in the obvious ways. Verifications are routine. If $\mathcal{D}$ satisfies AB5 then so does $\mathrm{bb}-\mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$ because colimits commute with colimits. □

Lemma A.2.5. Sheafification is exact. *Let $\mathcal{C}$ be a category equipped with a faithful functor to* **Sets** *(this is so that we can consider basically bounded presheaves and sheaves on $\mathcal{C}$), let τ be a Grothendieck topology on $\mathcal{C}$, and let $\mathcal{D}$ be an abelian category satisfying AB3 (this is so that we can sheafify presheaves taking values in $\mathcal{D}$). Then if*

$$0 \to \mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C} \to 0$$

is an exact sequence in the presheaf category $\mathrm{bb}-\mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$*, the sequence*

$$0 \to \#(\mathcal{A}) \xrightarrow{\#(f)} \#(\mathcal{B}) \xrightarrow{\#(g)} \#(\mathcal{C}) \to 0$$

in the sheaf category $\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}_\tau)$ *is exact.*

Proof. Right exactness follows because # is a left adjoint and hence preserves colimits. Left exactness is, for instance, covered by Thm 2.15.a in [Milne, 1980].

$\square$

Lemma A.2.6. Sheaves (with values in an abelian category) on a site form an abelian category. *Let $\mathcal{C}$ be a category equipped with a faithful functor to* **Sets** *(this is so that we can consider basically bounded presheaves and sheaves on $\mathcal{C}$), let τ be a Grothendieck topology on $\mathcal{C}$, and let $\mathcal{D}$ be an abelian category satisfying AB3 (this is so that we can sheafify presheaves taking values in $\mathcal{D}$). Then the category* $\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}_\tau)$ *(of basically bounded $\mathcal{D}$-valued sheaves on $\mathcal{C}$ in the τ topology) is an abelian category satisfying AB3.*

We describe the zero, biproducts, kernels, and cokernels in this category:

1. *The zero object is the constant presheaf (automatically a sheaf) given by $\mathcal{F}(U) = 0$, the zero object in $\mathcal{D}$, for each $U \in \mathrm{ob}\ \mathcal{C}$.*

2. *Let I be an index set and let $\{\mathcal{F}_i\}_{i\in I}$ be a set of basically bounded $\mathcal{D}$-valued presheaves on $\mathcal{C}_\tau$. Then the coproduct $\coprod_{i\in I} \mathcal{F}_i$ is the sheafification of the presheaf whose value on $U \in \mathrm{ob}\ \mathcal{C}$ is $\bigoplus_{i\in I}(\mathcal{F}_i(U))$. (When I is finite then this*

presheaf is already a sheaf, and is naturally isomorphic to the product sheaf as well, as it must be in order for $\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}_\tau)$ *to be an abelian category.)*

If $\mathcal{D}$ *satisfies AB4 then the coproduct presheaf is already a sheaf and sheafification is unnecessary.*

3. *The kernel of a morphism* $\mathcal{F} \xrightarrow{f} \mathcal{G}$ *in* $\mathrm{bb}-\mathrm{Sh}(\mathcal{C}_\tau)$ *is the kernel presheaf (automatically a sheaf) whose values are given by*

$$(\ker f)(U) = \ker\left(\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U)\right)$$

for each $U \in \mathcal{C}$*.*

4. *The cokernel of a morphism* $\mathcal{F} \xrightarrow{f} \mathcal{G}$ *in* $\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}_\tau)$ *is the sheafification of the cokernel presheaf whose values are given by*

$$(\operatorname{coker} f)(U) = \operatorname{coker}\left(\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U)\right)$$

for each $U \in \mathrm{ob}\ \mathcal{C}$*. (The cokernel presheaf is* not *automatically a sheaf.)*

If $\mathcal{D}$ *satisfies AB5 then so does* $\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}_\tau)$*.*

Proof. Equalizer sequences in $\mathcal{D}$ remain equalizer sequences after taking finite biproducts of them, so finite biproduct presheaves are also sheaves. This gives us that there exists a biproduct in $\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}_\tau)$.

Sheafification is left adjoint to the forgetful functor from sheaves to presheaves, and left adjoints preserve colimits, so sheafifying the coproduct presheaf yields the coproduct sheaf, for small coproducts.

If $\mathcal{F} \xrightarrow{f} \mathcal{G}$ is a morphism in $\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}_\tau)$ then for each τ-cover $\{U_i \to U\}_{i\in I}$ we have diagrams in $\mathcal{D}$

$$X_{\mathcal{F}} = \left(\prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)\right),$$

$$X_{\mathcal{G}} = \left(\prod_{i\in I} \mathcal{G}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{G}(U_i \times_U U_j)\right),$$

$$X_0 = \left(0 \rightrightarrows \prod_{i,j\in I} 0 \right)$$

and diagram-morphisms $X_{\mathcal{F}} \to X_{\mathcal{G}}$ and $X_0 \to X_{\mathcal{G}}$. We have that the equalizer diagram for the cover $\{U_I \to U\}_{i\in I}$ and the presheaf $\ker f$ is given by the limit diagram

$$\lim \left(\begin{tikzcd} X_{\mathcal{F}} \arrow[dr] & \\ & X_{\mathcal{G}} \\ X_0 \arrow[ur] & \end{tikzcd} \right)$$

and if the limit of that diagram, i.e., the equalizer, is $(\ker f)(U)$ then $\ker f$ satisfies effective descent for the cover $\{U_i \to U\}_{i\in I}$, and if it satisfies effective descent for every cover, then $\ker f$ is a sheaf. Now we have

$$\begin{aligned}
\lim \left(\lim \left(\begin{tikzcd} X_{\mathcal{F}} \arrow[dr] & \\ & X_{\mathcal{G}} \\ X_0 \arrow[ur] & \end{tikzcd} \right) \right) &\cong \lim \left(\begin{tikzcd} \lim X_{\mathcal{F}} \arrow[dr] & \\ & \lim X_{\mathcal{G}} \\ \lim X_0 \arrow[ur] & \end{tikzcd} \right) \\
&\cong \lim \left(\begin{tikzcd} \mathcal{F}(U) \arrow[dr] & \\ & \mathcal{G}(U) \\ 0 \arrow[ur] & \end{tikzcd} \right) \\
&\cong (\ker f)(U).
\end{aligned}$$

So the kernel presheaf $\ker f$ is a sheaf.

Sheafifying a cokernel presheaf yields a cokernel in the category of sheaves, since a cokernel is a colimit and left adjoints (such as sheafification) preserve colimits.

If $\mathcal{F} \xrightarrow{f} \mathcal{G}$ is a morphism in $\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}_\tau)$ with $\ker f = 0$ then we have the composable pair of morphisms

$$\mathcal{G} \xrightarrow{p} \operatorname{coker} f \xrightarrow{i} \#(\operatorname{coker} f)$$

in the presheaf category, $\mathrm{bb}-\mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$. We have the exact sequence of presheaves

$$0 \to \ker p \to \ker(i \circ p) \to \ker i \to 0 \tag{A.2.1}$$

from Lemma A.2.2, since $\operatorname{coker} p = 0$. Now since $\#(i)$ is an isomorphism, from Prop. A.1.10 and Lemma **??**, we have $\#(\ker i) = 0$ from Lemma A.2.5. As a result when we sheafify sequence A.2.1, we get the commutative diagram with exact (by Lemma A.2.5) rows:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \#(\ker p) \arrow[r] \arrow[d, "\cong"] & \#(\ker(i \circ p)) \arrow[r] \arrow[d, "\cong"] & \#(\ker i) \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d] \\
0 \arrow[r] & \ker \operatorname{coker} f \arrow[r] & \mathcal{F} \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

So we have an isomorphism of sheaves $\ker \operatorname{coker} f \cong \mathcal{F}$.

If $\mathcal{F} \xrightarrow{f} \mathcal{G}$ is a morphism in $\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}_\tau)$ with $\operatorname{coker} f = 0$ then the sequence

$$0 \to \ker f \to \mathcal{F} \xrightarrow{f} \mathcal{G} \to 0$$

is exact both as sheaves and as presheaves, so when we take the cokernel presheaf of the morphism $\ker f \to \mathcal{F}$, we get precisely $\mathcal{G}$; so $\operatorname{coker} \ker f = f$ if $\operatorname{coker} f = 0$.

If $\mathcal{D}$ satisfies AB5 and we have a diagram of exact sequences

$$0 \to \mathcal{F}_i' \to \mathcal{F}_i \to \mathcal{F}_i'' \to 0$$

in $\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}_\tau)$ with index $i \in I$ ranging over a filtered category I, then the colimit is the sheafification of the exact (by Prop. A.2.4) sequence of presheaves

$$0 \to \operatorname{colim}_i \mathcal{F}_i' \to \operatorname{colim}_i \mathcal{F}_i \to \operatorname{colim}_i \mathcal{F}_i'' \to 0,$$

and that sheafification is an exact sequence of sheaves in $\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}_\tau)$, by Lemma A.2.5.

□

Definition A.2.7. Ring sheaves. *Let $\mathcal{C}$ be a category equipped with a faithful functor to* **Sets** *(this is so that we can consider basically bounded presheaves and sheaves on $\mathcal{C}$), let τ be a Grothendieck topology on $\mathcal{C}$, and let $\mathfrak{A}$ be a basically bounded sheaf of commutative rings on the site $\mathcal{C}_\tau$. Then a* basically bounded $\mathfrak{A}$-module *consists of a basically bounded sheaf $\mathcal{F}$ of abelian groups on $\mathcal{C}_\tau$ together with an $\mathfrak{A}(U)$-module structure on $\mathcal{F}(U)$ for each $U \in \mathrm{ob}\ \mathcal{C}$, such that, for any $U \xrightarrow{f} V$ in $\mathcal{C}$, the morphism of abelian groups*

$$\mathcal{F}(V) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(U)$$

is a morphism of $\mathfrak{A}(V)$-modules, or equivalently, the adjoint morphism

$$\mathcal{F}(V) \otimes_{\mathfrak{A}(V)} \mathfrak{A}(U) \xrightarrow{\mathcal{F}(f)^\natural} \mathcal{F}(U)$$

is a morphism of $\mathfrak{A}(U)$-modules. We say that $\mathcal{F}$ is quasicoherent *when $\mathcal{F}(f)^\natural$ is an isomorphism of $\mathfrak{A}(U)$-modules for every $U \in \mathcal{C}$.*

A ring sheaf on a category is a way of "stitching together" categories of modules over commutative rings: if the category has only one object X, and we specify a ring sheaf $\mathfrak{A}$ on that category, then the category of $\mathfrak{A}$-modules is identical to the category of modules over the ring $\mathfrak{A}(X)$; if we have nontrivial morphisms in the category, and a ring sheaf $\mathfrak{A}$ on that category, then $\mathfrak{A}$-modules consist of particular diagrams of modules over varying commutative rings. The category of modules over a commutative ring is a fundamental example of an abelian category; we want to be able to stitch together other abelian categories in a similar way to how we produce modules over a ring sheaf on a category. To achieve this, we give the following definition.

Definition A.2.8. Locally abelianly categoried sites and their (pre)modules.

1. *A* locally abelianly categoried site *consists of a category $\mathcal{C}$ equipped with a (possibly trivial) Grothendieck topology τ and, for each object $U \in \mathrm{ob}\ \mathcal{C}$, an*

abelian category $\mathfrak{A}(U)$*, and, for each morphism* $U \xrightarrow{f} V$ *in* $\mathcal{C}$*, a pair of additive functors*

$$\mathfrak{A}(U) \xrightarrow{f_*} \mathfrak{A}(V)$$
$$\mathfrak{A}(V) \xrightarrow{f^*} \mathfrak{A}(U),$$

with f_* *left adjoint to* f^**, and such that* $(f \circ g)_* = f_* \circ g_*$ *(or equivalently,* $(f \circ g)^* = g^* \circ f^*$*) for all composable pairs* f, g *in* $\mathcal{C}$*, and* $(\mathrm{id}_U)_* = \mathrm{id}_{\mathfrak{A}(U)}$ *for all objects* U *in* $\mathcal{C}$

2. *If* $(\mathcal{C}, \mathfrak{A}, \tau)$ *is a locally abelianly categoried site, we define* a $\mathfrak{A}$-premodule $\mathcal{F}$ *as a choice of object* $\mathcal{F}(U) \in \mathrm{ob}\ \mathfrak{A}(U)$ *for each object* $U \in \mathrm{ob}\ \mathcal{C}$*, and for each* $U \xrightarrow{f} V$ *in* $\mathcal{C}$*, a morphism*

$$f_*\mathcal{F}(V) \xrightarrow{\mathcal{F}(f_*)} \mathcal{F}(U),$$

(which we will sometimes call a transition morphism*) or equivalently, a morphism*

$$\mathcal{F}(V) \xrightarrow{\mathcal{F}(f^*)} f^*\mathcal{F}(U),$$

such that $\mathcal{F}(f_*) \circ \mathcal{F}(g_*) = \mathcal{F}(g_* \circ f_*)$ *for any composable pair* f, g *of morphisms in* $\mathcal{C}$*, and such that* $\mathcal{F}(\mathrm{id}_U) = \mathrm{id}_{\mathcal{F}(U)}$ *for any object* U *in* $\mathcal{C}$*. We say that an* $\mathfrak{A}$*-premodule is* an $\mathfrak{A}$-module *if, for any cover* $\{U_i \xrightarrow{\phi_i} U\}_{i \in I}$ *in* $\mathcal{C}$*, we have the equalizer sequence in* $\mathfrak{A}(V)$*:*

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \bigoplus_{i \in I} \phi_i^* \mathcal{F}(U_i) \rightrightarrows \bigoplus_{i,j \in I} (\phi_i \circ \pi_1)^* \mathcal{F}(U_i \times_U U_j)$$

where π_1 *is the projection* $U_i \times_U U_j \to U_i$ *to the first coordinate (we could write* π_2 *for projection to the second coordinate instead, and then we have the isomorphism* $(\phi_i \circ \pi_1)^* \mathcal{F}(U_i \times_U U_j) \cong (\phi_j \circ \pi_2)^* \mathcal{F}(U_i \times_U U_j)$*, which is necessary to define the equalizer sequence in question).*

Given two $\mathfrak{A}$-premodules $\mathcal{F}, \mathcal{G}$, we define a morphism $\mathcal{F} \xrightarrow{\lambda} \mathcal{G}$ of $\mathfrak{A}$-premodules as, for each $U \in \mathcal{C}$, a morphism $\mathcal{F}(U) \xrightarrow{\lambda(U)} \mathcal{G}(U)$ in $\mathfrak{A}(U)$, such that, for any morphism $U \xrightarrow{f} V$ in $\mathcal{C}$, the following diagram in $\mathfrak{A}(U)$ commutes:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\lambda(U)"] \arrow[d, "\mathcal{F}(f^*)"] & \mathcal{G}(U) \arrow[d, "\mathcal{G}(f^*)"] \\
f^*\mathcal{F}(V) \arrow[r, "f^*\lambda(V)"] & f^*\mathcal{G}(V),
\end{tikzcd}$$

or equivalently, the adjoint diagram in $\mathfrak{A}(V)$ commutes:

$$\begin{tikzcd}
f_*\mathcal{F}(U) \arrow[r, "f_*\lambda(U)"] \arrow[d, "\mathcal{F}(f_*)"] & f_*\mathcal{G}(U) \arrow[d, "\mathcal{G}(f_*)"] \\
\mathcal{F}(V) \arrow[r, "\lambda(V)"] & \mathcal{G}(V).
\end{tikzcd}$$

A morphism of $\mathfrak{A}$-modules is a morphism of the underlying $\mathfrak{A}$-premodules.

We say that an $\mathfrak{A}$-premodule or $\mathfrak{A}$-module $\mathcal{F}$ is quasicoherent *if $\mathcal{F}(f_*)$ is an isomorphism for all morphisms f in $\mathcal{C}$.*

Given the above data and additionally a faithful functor $\mathcal{C} \to$ **Sets** *we can define $\mathcal{C}(m)$, basically bounded $\mathfrak{A}$-premodules, basically bounded $\mathfrak{A}$-modules, and so on in the vein of Def. A.1.8. The details are straightforward for the interested reader to work out. We will write* $\mathrm{bb}-\mathbf{Premod}(\mathfrak{A})$, $\mathrm{bb}-\mathbf{Premod}(\mathfrak{A})_{\mathrm{qcoh}}$, $\mathrm{bb}-\mathbf{Mod}(\mathfrak{A})$, *and* $\mathrm{bb}-\mathbf{Mod}(\mathfrak{A})_{\mathrm{qcoh}}$ *for the categories of basically bounded $\mathfrak{A}$-premodules, basically bounded quasicoherent $\mathfrak{A}$-premodules, basically bounded $\mathfrak{A}$-modules, and basically bounded quasicoherent $\mathfrak{A}$-modules, respectively.*

Remark A.2.9. Other ways of handling modules over a locally abelianly categoried site include the "twisted diagrams" of [R ondigs and H uttemann, 2008] and, an idea suggested to me by John Francis, simply insisting that $\mathfrak{A}(U)$ be a Grothendieck (AB5 and possessing a family of generators) category for each $U \in \mathcal{C}$ and taking the locally abelianly categoried site as a sheaf of categories on $\mathcal{C}$, giving one the adjoint functors in the definition of a locally abelianly categoried site automatically, by the adjoint functor theorem. Both of these approaches involve a set-theoretic difficulty which is bracketed by choosing a universe.

Example A.2.10. Examples of categories of $\mathfrak{A}$-modules over locally abelianly categoried sites. It is worth pointing out that, as a rule, the important object of study is generally the category of $\mathfrak{A}$-modules over a locally abelianly categoried site, and not the locally abelianly categoried site itself; for instance, we will sometimes want to know that a particular map between two locally abelianly categoried sites induces a map in cohomology with a certain property (such as being an isomorphism). The Miller-Ravenel and Morava change-of-rings isomorphisms, and their generalizations (which we hope to describe in the near future), are examples of this; see below where we discuss the example of locally abelianly categoried sites coming from comodules. In these cases the content of any theorem will really be about the involved categories of $\mathfrak{A}$-modules, but holding on to a locally abelianly categoried "underneath" the category of $\mathfrak{A}$-modules sometimes helps us prove the important theorems, and in any case it can be an aid to intuition. The situation resembles that of Morita theory where one wants to know that there is some kind of equivalence between the categories of modules over two different objects; one really cares about (often cohomological) properties of those modules and one uses Morita theory to "swap out" one underlying object for a more convenient object without changing the most important properties of its category of modules, and technically one could just work purely with the module categories and leave out the underlying objects, but this would make the statements of theorems, as well as the methods of proof, much more difficult to understand.

1. The most important example of a category of $\mathfrak{A}$-modules is the category of modules over a ring sheaf on a site. This example includes the module sheaves of algebraic geometry ala [Hartshorne, 1977], modules over the structure sheaf of an abelian variety (in the Zariski topology); it also includes the more modern examples that arose in the course of proving the

Weil conjectures, étale sheaves on the étale structure sheaf of a scheme (in this case our site $(\mathcal{C}, \tau)$ is typically set up as the small étale site of the scheme in question), and those that arose in studying abelian varieties over finite fields, sheaves on the fppf site of a scheme.

2. When $\mathcal{C}$ has only one object X, then the category of $\mathfrak{A}$-modules is just the abelian category $\mathfrak{A}(X)$; so any abelian category is also a locally abelianly categoried site.

3. Combining the previous two examples, when $\mathcal{C}$ has only one object X and $\mathfrak{A}(X)$ is a category of modules over a commutative ring, we get the categories of modules over a commutative ring as the very simplest examples of locally abelianly categoried sites.

4. When S is a scheme and we consider a rigidified stack $X \xrightarrow{P} \mathcal{X}$ over S in the fpqc topology, the category of modules over $\mathcal{X}$ is equivalent to the category of modules over the following locally abelianly categoried site: the site $(\mathcal{C}, \tau)$ is the (big) fpqc site over X, and for any scheme $Y \rightarrow X$ in the big fpqc site, the abelian category $\mathfrak{A}(Y)$ is the category of $\mathcal{O}_Y$-modules equipped with a particular choice of flat descent datum (it is not obvious that this is an abelian category, but we prove this in Prop. 2.1.22. We can weaker this further, and allow pro-rigidified fpqc stacks, i.e., homotopy limits of rigidified stacks; the presentation associated to a pro-rigidified stack need not be smooth but it is the homotopy limit of stacks with chosen covers which are necessarily smooth. It is probably the case that such a presentation is always quasi-smooth ([Hopkins, 1999] is relevant), and indeed that a choice of pro-rigidification (at least in the fpqc topology) is equivalent to a choice of quasismooth fpqc cover; but I know of nowhere where a result like this has been proven. It would be interesting to know if there are properties that the diagonal of the stack must have in order for such a result to hold.

5. When S is a scheme and we consider a commutative Hopf algebroid (A, Γ) over S, the category of Γ-comodules is the category of $(\mathcal{C}, \mathfrak{A}, \tau)$-modules, where $(\mathcal{C}, \tau)$ is the local Zariski site S_{Zar} and $\mathfrak{U}$ is the category of $(\mathcal{O}_U \otimes_{\mathcal{O}_S} A, \mathcal{O}_U \otimes_{\mathcal{O}_S} \Gamma)$-comodules for each Zariski open $U \subseteq S$.

Lemma A.2.11. Sheafification for (pre)modules on a locally abelianly categoried site. *Let $\mathcal{X}$ be a site with underlying category $\mathcal{C}$ equipped with a faithful functor $\mathcal{C} \to$ **Sets**, and let $(\mathcal{X}, \mathfrak{A})$ define a locally abelianly categoried site such that $\mathfrak{A}(U)$ satisfies AB3 for all $U \in \mathrm{ob}\ \mathcal{X}$. Then the forgetful functor*

$$\mathrm{bb}-\mathbf{Mod}(\mathfrak{A}) \stackrel{\texttt{forget}}{\longrightarrow} \mathrm{bb}-\mathbf{Premod}(\mathfrak{A})$$

has a left adjoint $\#$, which we call "sheafification" (since the only difference between an $\mathfrak{A}$-premodule and an $\mathfrak{A}$-module is a sheaf condition).

Proof. We proceed as in Prop. A.1.10: let $(\mathcal{C}, \mathfrak{A}, \tau)$ define a locally abelianly categoried site, let $\mathcal{C} \longrightarrow$ **Sets** be a faithful functor, let m be some cardinal number, and let $\mathcal{F}$ be an m-based $\mathfrak{A}$-premodule. For any $\mathcal{U} \in \mathrm{ob}\ \mathcal{C}(m)$ we let $J_U(m)$ be category of covers $\{U_i \xrightarrow{\phi_i} U\}_{i \in I}$ of U in the site $(\mathcal{C}(m), \tau\ |_{\mathcal{C}(m)})$. Then let

$$\mathcal{F}_U(\{U_i \xrightarrow{\phi_i} U\}_{i \in I}) = \mathrm{eq}\{\ \prod_{i \in I} \phi_i^* \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} (\phi_i \circ \pi_1)^* \mathcal{F}(U_i \times_U U_j)\ \}$$

with π_1 as in Def. A.2.8, and let

$$\mathcal{F}^+(m)(U) = \mathrm{colim}_{\{U_i \xrightarrow{\phi_i} U\}_{i \in I} \in \mathrm{ob}\ J_U(m)} \mathcal{F}_U(\{U_i \xrightarrow{\phi_i} U\}).$$

Then $\mathcal{F}^{++}(m)$ is an $(\mathcal{C}(m), \mathfrak{A}\ |_{\mathcal{C}(m)}, \tau\ |_{\mathcal{C}(m)})$-module and the left Kan extension of $\mathcal{F}^{++}(m)$ along the inclusion of $\mathcal{C}(m)$ into $\mathcal{C}$ is an $\mathfrak{A}$-module which we will call $\#\mathcal{F}$; the process is functorial in $\mathcal{F}$ and left adjoint to `forget`; see Prop. A.1.10. $\square$

Lemma A.2.12. *1.* **Premodules over a locally abelianly categoried site form an abelian category.** *Let $(\mathcal{C}, \mathfrak{A}, \tau)$ define a locally abelianly categoried site and let $\mathcal{C} \to$* **Sets** *be a faithful functor. Suppose $\mathfrak{A}(X)$ satisfies AB3 for all $X \in \mathrm{ob}\ \mathcal{C}$. Then $\mathrm{bb}-\mathbf{Premod}(\mathfrak{A})$ is an abelian category.*

We spell out the zero, kernels, and cokernels in this category.

The zero object in $\mathrm{bb}-\mathbf{Premod}(\mathfrak{A})$ *is the premodule whose value on any* $U \in \mathrm{ob}\ \mathcal{C}$ *is the zero object in* $\mathfrak{A}(U)$*.*

If $\mathcal{F} \xrightarrow{f} \mathcal{G}$ *is a morphism of* $\mathfrak{A}$*-premodules,* $\ker f$ *is the* $\mathfrak{A}$*-premodule whose value on* $U \in \mathrm{ob}\ \mathcal{C}$ *is* $\ker\left(\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U)\right)$*. If* $U \xrightarrow{\phi} V$ *is a morphism in* $\mathcal{C}$ *then we have the composite morphism*

$$(\ker f)(U) \longrightarrow \ker(\phi^* f(V)) \xrightarrow{\cong} \phi^* \ker f(V)$$

in $\hom_{\mathfrak{A}(U)}(\ker f(U), \phi^* \ker f(V))$*, and its adjoint morphism in* $\hom_{\mathfrak{A}(V)}(\phi_* \ker f(U), \ker f(V))$ *is the transition morphism* $(\ker f)(\phi)$*.*

If $\mathcal{F} \xrightarrow{f} \mathcal{G}$ *is a morphism of* $\mathfrak{A}$*-premodules,* $\mathrm{coker}\, f$ *is the* $\mathfrak{A}$*-premodule whose value on* $U \in \mathrm{ob}\ \mathcal{C}$ *is* $\mathrm{coker}\left(\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U)\right)$*. If* $U \xrightarrow{\phi} V$ *is a morphism in* $\mathcal{C}$ *then we have the composite morphism*

$$(\ker f)(U) \longrightarrow \ker(\phi^* f(V)) \xrightarrow{\cong} \phi^* \ker f(V)$$

in $\hom_{\mathfrak{A}(U)}(\ker f(U), \phi^* \ker f(V))$*, and its adjoint morphism in* $\hom_{\mathfrak{A}(V)}(\phi_* \ker f(U), \ker f(V))$ *is the transition morphism* $(\ker f)(\phi)$*.*

2. **Modules over a locally abelianly categoried site form an abelian category.** *Let* $(\mathcal{C}, \mathfrak{A}, \tau)$ *define a locally abelianly categoried site and let* $\mathcal{C} \to$ **Sets** *be a faithful functor. Suppose* $\mathfrak{A}(X)$ *satisfies AB3 for all* $X \in \mathrm{ob}\ \mathcal{C}$*. Then* $\mathrm{bb}-\mathbf{Mod}(\mathfrak{A})$ *is an abelian category.*

Proof. 1. As in Lemma A.2.4.

2. As in Lemma A.2.6; the cokernel module is, as before, only a premodule, and must be sheafified, using Prop. A.2.11.

□

Lemma A.2.13. *Let $(\mathcal{C}, \mathfrak{A}, \tau)$ define a locally abelianly categoried site and let $\mathcal{C} \to \mathbf{Sets}$ be a faithful functor.*

1. *If $\mathfrak{A}(X)$ satisfies AB5 for every $X \in \text{ob } \mathcal{X}$, then the $\mathfrak{A}$-premodule category* $\mathrm{bb} - \mathbf{Premod}(\mathfrak{A})$ *satisfies AB5.*

2. *The sheafification functor*

$$\mathrm{bb} - \mathbf{Premod}(\mathfrak{A}) \xrightarrow{\#} \mathrm{bb} - \mathbf{Mod}(\mathfrak{A})$$

   is additive and exact.

3. *If $\mathfrak{A}(X)$ satisfies AB5 for every $X \in \text{ob } \mathcal{X}$, then the $\mathfrak{A}$-module category* $\mathrm{bb} - \mathbf{Mod}(\mathfrak{A})$ *satisfies AB5.*

Proof.

1. Colimits of premodules are defined object by object in $\mathcal{C}$. As a result colim^1 is trivial for a filtered system of premodules if it is trivial at each $X \in \text{ob } \mathcal{C}$.

2. Same argument as Lemma A.2.5.

3. We may take any filtered system of short exact sequences in $\mathrm{bb} - \mathbf{Mod}(\mathfrak{A})$ and observe that the last four terms in the colimit sequence, $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$, are an exact sequence, and $\ker f$ is colim^1 of this filtered system; now, as the forgetful functor `forget` is left exact, we have that $\ker(\texttt{forget}\, f) \cong \texttt{forget}(\ker f)$, and since the premodule category $\mathrm{bb} - \mathbf{Premod}(\mathfrak{A})$ satisfies AB5, $\ker(\texttt{forget}\, f)$, which is colim^1 of the filtered system in the premodule category, must be zero. Now sheafification is exact and $\# \circ \texttt{forget} = \mathrm{id}_{\mathrm{bb} - \mathbf{Mod}(\mathfrak{A})}$ so $0 \cong \# \ker(\texttt{forget}\, f) \cong \ker f$ and colim^1 is trivial for all filtered systems in the module category $\mathrm{bb} - \mathbf{Mod}(\mathfrak{A})$.

$\square$

Corollary A.2.14. It's left exact to forget you have a sheaf. *When $\mathfrak{A}$ is an abelian category satisfying AB3, the inclusion functor*

$$\mathrm{bb}-\mathfrak{A}\,\mathrm{Sh}(\mathcal{X}) \xrightarrow{\texttt{forget}} \mathrm{bb}-\mathfrak{A}^{\mathcal{C}^{\mathrm{op}}}$$

is left exact.

Proof. The presheaf kernel of a map of sheaves is already a sheaf. Alternatively, since `forget` has a left adjoint, `forget` preserves limits, including kernels. □

Definition A.2.15. Global sections and cohomology.

1. *Let $\mathcal{X}$ be a site whose underlying category is concrete and has terminal object X. Let $\mathfrak{A}$ be an abelian category satisfying AB3. We define the* global sections functor

$$\begin{array}{ccc} \mathfrak{A}\,\mathrm{Sh}(\mathcal{X}) & \xrightarrow{\Gamma} & \mathfrak{A} \\ \mathcal{F} & \mapsto & \mathcal{F}(X). \end{array}$$

   The functor Γ is left exact (as we show in the next proposition) and we define the cohomology of X with coefficients in $\mathcal{F}$ *as the right derived functors of Γ:*

$$H^*(X, \mathcal{F}) \cong R^*\Gamma(\mathcal{F}).$$

   The cohomology functor $H^*(X, \mathcal{F})$ is functorial (and covariant) in $\mathcal{F}$ from the above definition.

2. *Let $(\mathcal{C}, \mathfrak{A}, \tau)$ be a locally abelianly categoried site such that $\mathfrak{A}(X)$ satisfies AB3 for all $X \in \mathcal{C}$. Suppose $\mathcal{C}$ has terminal object X. We define the* global sections functor

$$\begin{array}{ccc} \mathbf{Mod}((\mathcal{C}, \mathfrak{A}, \tau)) & \xrightarrow{\Gamma} & \mathfrak{A}(X) \\ \mathcal{F} & \mapsto & \mathcal{F}(X). \end{array}$$

The functor Γ *is left exact (as we show in the next proposition) and we define the* cohomology of X with coefficients in $\mathcal{F}$ *as the right derived functors of* Γ*:*

$$H^*(X, \mathcal{F}) \cong R^*\Gamma(\mathcal{F}).$$

We may also sometimes write $H^*_\tau(X, \mathcal{F})$ *(where* τ *is the Grothendieck topology of* $\mathcal{X}$*) or* $H^*(\mathcal{X}, \mathcal{F})$*; all of these notations are interchangeable. The cohomology functor* $H^*(X, \mathcal{F})$ *is functorial (and covariant) in* $\mathcal{F}$ *from the above definition.*

Lemma A.2.16. Γ *is left exact.*

Proof. Γ is the composite of the functor $\mathfrak{A}\,\mathrm{Sh}(\mathcal{X}) \xrightarrow{\mathtt{forget}} \mathfrak{A}^{\mathrm{Cat}\,\mathcal{X}^{\mathrm{op}}}$, which is left exact by Cor. A.2.14, with evaluation at X, and evaluation is exact on presheaves. A similar argument holds for Γ on $(\mathcal{C}, \mathfrak{A}, \tau)$-modules. $\square$

Generators. We recall the definition of category-theoretic generators. Some of the categories in which we would like to do homological algebra fail to have free objects, and this makes constructing resolutions more difficult in those categories; however, even when free objects do not exist, a set of generators for the category will give us at least some tools for constructing resolutions.

Definition A.2.17. Definitions needed for the Gabriel-Popescu theorem.

1. *Let* $\mathcal{C}$ *be a category and let* $\{U_i\}_{i\in I}$ *be a set of objects in* $\mathcal{C}$*. We say that* $\{U_i\}_{i\in I}$ *is a* family of generators for $\mathcal{C}$ *if, given a monomorphism* $A \xrightarrow{f} B$ *in* $\mathcal{C}$ *which is not an isomorphism, there exists some* $i \in I$ *and a morphism* $U_i \to B$ *which does not factor through* f*.*

   (It is actually important that the index set I *really* is *a set; one wants to have a set of generators and not a proper class of generators.)*

2. *A* Grothendieck category *is an abelian category satisfying AB5 and which has a family of generators.*

3. *Let $\mathcal{C}$ be an abelian category. A monomorphism $A \xrightarrow{f} B$ in $\mathcal{C}$ is called an* essential monomorphism *if, for any morphism $B \xrightarrow{g} B'$ such that $g \circ f$ is a monomorphism, g is also a monomorphism. (A standard example of such a morphism is the inclusion of a number ring A into its field of fractions $K(A)$; this is an essential monomorphism in the category of A-modules.)*

4. *If $\mathcal{C}$ is an abelian category such that every object in $\mathcal{C}$ admits an essential monomorphism into an injective object of $\mathcal{C}$, then we say that $\mathcal{C}$* has injective envelopes. *(This is stronger than saying that $\mathcal{C}$ has enough injectives; having injectives envelopes means that we can not only construct injective resolutions, but injective resolutions that are minimal in a certain sense).*

5. *Let $\mathcal{C}, \mathcal{C}'$ be abelian categories and let $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ be an exact additive functor with a fully faithful right adjoint. Then we say that $\mathcal{C}'$ is a* quotient category *of $\mathcal{C}$ (or, more specifically, that the pair consisting of F and its right adjoint defines $\mathcal{C}'$ as a quotient category of $\mathcal{C}$—this is somewhat like the distinction between saying that a group H is a quotient of a group G, and actually giving a surjective group morphism $G \to H$).*

Lemma A.2.18. *1. Let $\mathcal{C}, \mathcal{D}$ be categories and $\mathcal{C} \xrightarrow{T} \mathcal{D}$ a functor with right adjoint S. Recall that, since S, T are adjoint, we have natural transformations $\mathrm{id}_{\mathcal{C}} \xrightarrow{\Phi} (S \circ T)$ and $(T \circ S) \xrightarrow{\Psi} \mathrm{id}_{\mathcal{D}}$. The natural transformation Ψ is a natural equivalence if and only if S is full and faithful.*

2. *Let $\{U_i\}_{i \in I}$ be a set of objects in a strongly connected category $\mathcal{C}$ such that the coproduct $\coprod_{i \in I} U_i$ exists in $\mathcal{C}$ (we recall that a category $\mathcal{C}$ is said to be "strongly connected" if, for any pair of objects $A, B \in \mathrm{ob}\ \mathcal{C}$, there exists at least one morphism $A \to B$ in $\mathcal{C}$; this is a weaker condition on $\mathcal{C}$ than e.g. possessing a zero object). Then the following are equivalent:*

(a) $\{U_i\}_{i\in I}$ *is a family of generators of* $\mathcal{C}$.

(b) $\coprod_{i\in I} U_i$ *is a generator of* $\mathcal{C}$.

3. *If* $\mathcal{C}$ *is an abelian category with injective envelopes then every quotient category of* $\mathcal{C}$ *also has injective envelopes.*

   In particular, let $\mathcal{C} \xrightarrow{T} \mathcal{C}'$ *be an exact additive functor with a fully faithful right adjoint* S. *Suppose injective envelopes exist in* $\mathcal{C}$*; if* $M \in \operatorname{ob} \mathcal{C}$ *we will write* $M \xrightarrow{f_M} \widehat{M}$ *for the injective envelope of* M *in* $\mathcal{C}$. *Then we get the injective envelope* f_N *of* $N \in \operatorname{ob} \mathcal{C}'$ *as the composite*

$$N \xrightarrow{\cong} T(S(N)) \xrightarrow{T(f_{S(N)})} T(\widehat{S(N)}).$$

4. *Let* R *be a ring. Then the category* $\mathbf{Mod}(R)$ *of left (resp. right)* R*-modules has injective envelopes.*

Proof. 1. (We note that there is a typo in the statement of this theorem as it appears, as Prop. 1.13', in [Bucur and Deleanu, 1968].)

Let X, Y be two objects of $\mathcal{D}$. Then we have the commutative diagram

$$\begin{tikzcd}
\hom_{\mathcal{D}}(X,Y) \arrow[rr, "S(X{,}Y)"] \arrow[dr, "\hom_{\mathcal{D}}(\Psi(X){,}\mathrm{id}_Y)"'] & & \hom_{\mathcal{C}}(SX,SY) \arrow[dl, "\cong"] \\
& \hom_{\mathcal{D}}(TSX,Y) &
\end{tikzcd}$$

and $S(X,Y)$ is an isomorphism if and only if $\hom_{\mathcal{D}}(\Psi(X),Y)$ is; $S(X,Y)$ is an isomorphism if and only if S is full and faithful, and $\hom_{\mathcal{D}}(\Psi(X),Y)$ is an isomorphism, by the Yoneda Lemma, if and only if Ψ is a natural isomorphism.

2. Let $A \overset{f}{\hookrightarrow} B$ be a monomorphism in $\mathcal{C}$ which is not an isomorphism. If we have a morphism from U_j to B for some $j \in I$ which does not factor through A, then since $\mathcal{C}$ is strongly connected we may choose a morphism

$U_i \xrightarrow{f_i} B$ for each $i \in I$ with $i \neq j$, and then by the universal property of a coproduct, we have an induced morphism $\coprod_{i\in I} U_i \to B$ which does not factor through A since the component morphism $U_j \to B$ does not. Conversely, if $\coprod_{i\in I} U_i \xrightarrow{f} B$ is a morphism which does not factor through A, it induces (by the universal property of a coproduct) a morphism $U_i \xrightarrow{f_i} B$ for each $i \in I$, and if each f_i factors through A then so does f; so at least one f_i does not factor through A.

3. We need to show that $T(\widehat{S(N)})$ is injective and that f_N is an essential monomorphism. First, let $X \xrightarrow{f} Y$ be a monomorphism in $\mathcal{C}'$; we will work with the equivalent (since TS is naturally isomorphic to the identity functor) monomorphism $TSX \xrightarrow{TSf} TSY$. Let $TSX \xrightarrow{g} T(\widehat{S(N)})$ be a morphism in $\mathcal{C}'$. Now T is a full functor, as TS is naturally isomorphic to the identity functor, so every morphism in $\mathcal{D}$ must be in the image of T; and this tells us that $g = Th$ for some $SX \xrightarrow{h} \widehat{SN}$. Now we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & SX \arrow[r] \arrow[d, "h"] & SY \arrow[dl, dotted, "j"] \\ & \widehat{SN} & \end{tikzcd}$$

with the existence of j due to the injectivity of $\widehat{SN}$. Now we apply T to get the diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\cong"] & Y \arrow[d, "\cong"] \\ 0 \arrow[r] & TSX \arrow[r, "f'"] \arrow[d, "h"] & TSY \arrow[dl, dotted, "Tj"] \\ & T\widehat{SN} & \end{tikzcd}$$

giving us the injectivity of $T\widehat{SN}$.

Now we check that $N \xrightarrow{f_N} T\widehat{SN}$ is an essential monomorphism. Suppose we have a morphism $T\widehat{SN} \xrightarrow{g} X$ in $\mathcal{D}$ such that $g \circ f_N$ is monic. Now $X \cong TSX$

so we will write $T\widehat{SN} \xrightarrow{g'} TSX$ for that isomorphism composed with g, and we write $TSN \xrightarrow{f'_N} T\widehat{SN}$ for the isomorphism $TSN \cong N$ composed with f_N. Now as T is full we can choose a morphism $\widehat{SN} \xrightarrow{h} SX$ such that $Th = g'$. We have the morphisms

$$SN \xrightarrow{f_{SN}} \widehat{SN} \xrightarrow{h} SX$$

and we know that $T(h \circ f_{SN})$ is monic; choose $N \xrightarrow{j} X$ with $Sj = h \circ f_{SN}$ and now $TSj = j$ is monic, so $Sj = h \circ f_{SN}$ is monic, as S is a right adjoint and hence preserves small limits. Since $h \circ f_{SN}$ is monic and f_{SN} is an essential monomorphism, h is monic; and since T is exact, $Th = g'$ is monic. Hence f_N is an essential monomorphism.

4. Standard result; see [Cartan and Eilenberg, 1999] for example.

□

Proposition A.2.19. (Weak form of the) Gabriel-Popescu theorem. *Let $\mathcal{C}$ be an abelian category satisfying AB5 and let U be a generator of $\mathcal{C}$; we will write Λ for the ring $\hom_{\mathcal{C}}(U, U)$; let S be the functor $\mathcal{C} \to \mathbf{Mod}_\Lambda$ given by sending an object $V \in \operatorname{ob} \mathcal{C}$ to the Λ-module $\hom_{\mathcal{C}}(U, V)$. The functor S is fully faithful and has a left adjoint.*

In other words, every Grothendieck category is a quotient category of a category of modules over a ring.

Proof. See chapter 6 of [Bucur and Deleanu, 1968] for the full statement of the Gabriel-Popescu theorem, a proof of it and also the preceding lemma, and more treatment of related material. □

Corollary A.2.20. *Every Grothendieck category has injective envelopes.*

Lemma A.2.21. Categorical quotient maps preserve generators. *Let $\mathcal{C}, \mathcal{D}$ be categories and let $\mathcal{C} \xrightarrow{T} \mathcal{D}$ be a functor expressing $\mathcal{D}$ as a quotient category of $\mathcal{C}$. Then T takes generators to generators.*

Proof. Let U be a generator of $\mathcal{C}$. Let $A \overset{f}{\hookrightarrow} B$ be a monomorphism in $\mathcal{D}$ which is not an isomorphism. We want to know that $SA \xrightarrow{Sf} SB$ is not an isomorphism either. Suppose that $\operatorname{coker} Sf \cong 0$; then we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & TSA \arrow[r, "TSf"] \arrow[d, "\cong"] & TSB \arrow[r] \arrow[d, "\cong"] & T(\operatorname{coker} Sf) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r] & \operatorname{coker} f \arrow[r] & 0.
\end{tikzcd}$$

Now if $\operatorname{coker} Sf = 0$ then $0 \cong T(\operatorname{coker} Sf) \cong \operatorname{coker} f$ and f is an isomorphism, contradicting our assumption that f is not an isomorphism.

Then we have the commutative diagram

$$\begin{tikzcd}
\hom_{\mathcal{C}}(U, S(A)) \arrow[r, "\hom_{\mathcal{C}}(U,S(f))"] \arrow[d, "\cong"] & \hom_{\mathcal{C}}(U, S(B)) \arrow[d, "\cong"] \\
\hom_{\mathcal{D}}(T(U), A) \arrow[r, "\hom_{\mathcal{D}}(T(U),f)"] & \hom_{\mathcal{D}}(T(U), B)
\end{tikzcd}$$

whose top row is not an epimorphism, since S preserves limits and hence $S(f)$ is a monomorphism (but not an isomorphism) in $\mathcal{C}$, and U is a generator in $\mathcal{C}$. Hence the bottom row is not an epimorphism, and so there exists a map $T(U) \to B$ in $\mathcal{D}$ which does not factor through A. $\square$

We have this process as a corollary of the many lemmas we have proven in this section.

Process A.2.22. 1. Let $\mathcal{D}$ be an abelian category satisfying AB5 and which has a family of generators. Let $\mathcal{C}$ be an Artin-concrete category equipped with a Grothendieck topology τ and a terminal object. Then we can form acyclic resolutions in the category $\mathrm{bb}-\mathcal{D}\,\mathrm{Sh}(\mathcal{C}_\tau)$ of basically bounded $\mathcal{D}$-valued sheaves on $\mathcal{C}_\tau$, which are natural in the choice of sheaf. Specifically, let m be a cardinal number and let $\mathcal{F}$ be an m-based $\mathcal{D}$-valued sheaf on $\mathcal{C}_\tau$; then we take the injective envelope $\mathcal{F} \xrightarrow{d^0} \widehat{\mathcal{F}}$ in $\mathcal{C}(m)$, then map $\widehat{F}$ to the injective envelope $\operatorname{coker} d^0 \xrightarrow{d^1} \widehat{\operatorname{coker} d}^0$, etc., then apply $j(m)_*$ to get the acyclic resolution

$$0 \to j(m)_*\widehat{\mathcal{F}} \longrightarrow j(m)_*\widehat{\operatorname{coker} d}^0 \longrightarrow j(m)_*\widehat{\operatorname{coker} d}^1 \longrightarrow \dots.$$

This construction is natural in $\mathcal{F}$.

2. Let $(\mathcal{C}, \mathfrak{A}, \tau$ be a locally abelianly categoried site such that $\mathcal{C}$ is Artin-concrete and has a terminal object and such that, for any $X \in \operatorname{ob} \mathcal{C}$, the abelian category $\mathfrak{A}(X)$ satisfies AB5 and has a family of generators. Then we can form acyclic resolutions in the category $\mathrm{bb} - \mathbf{Mod}((\mathcal{C}, \mathfrak{A}, \tau))$ of basically bounded $(\mathcal{C}, \mathfrak{A}, \tau)$-modules, which are natural in the choice of $(\mathcal{C}, \mathfrak{A}, \tau)$-module. Specifically, let m be a cardinal number and let $\mathcal{F}$ be a $(\mathcal{C}(m), \mathfrak{A}|_{\mathcal{C}(m)}, \tau_{\mathcal{C}(m)})$-module; then we take the injective envelope $\mathcal{F} \xrightarrow{d^0} \widehat{\mathcal{F}}$ in $\mathbf{Mod}((\mathcal{C}(m), \mathfrak{A}|_{\mathcal{C}(m)}, \tau_{\mathcal{C}(m)}))$, then map $\widehat{F}$ to the injective envelope $\operatorname{coker} d^0 \xrightarrow{d^1} \widehat{\operatorname{coker} d}^0$, etc., then apply $j(m)_*$ to get the acyclic resolution

$$0 \to j(m)_*\widehat{\mathcal{F}} \longrightarrow j(m)_*\widehat{\operatorname{coker} d}^0 \longrightarrow j(m)_*\widehat{\operatorname{coker} d}^1 \longrightarrow \dots.$$

This construction is natural in $\mathcal{F}$.

Remark A.2.23. Let $\mathcal{D}$ be an abelian category with a generator U and let $\mathcal{C}$ be a category equipped with a faithful functor $\mathcal{C} \xrightarrow{F} \mathbf{Sets}$. Then the category $\mathrm{bb} - \mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$ of presheaves *does not necessarily have a family of generators*, and the same is true for the sheaf category $\mathrm{bb} - \mathcal{D}\,\mathrm{Sh}(\mathcal{C}_\tau)$ with τ Grothendieck topology on $\mathcal{C}$. For example, let $\mathcal{C}$ be the discrete (i.e., all morphisms are identity morphisms) category with objects in bijection with the cardinal numbers. In other words, $\mathcal{C}$ is the category obtained by throwing away all non-identity morphisms from a skeleton of the category of sets. We get a forgetful funtor $\mathcal{C} \to \mathbf{Sets}$ in the obvious way. Now suppose $\{X_i\}_{i \in I}$ is a family of generators for the category $\mathrm{bb} - \mathbf{Ab}^{\mathcal{C}^{\mathrm{op}}}$ of basically bounded abelian presheaves on $\mathcal{C}$. Then $\oplus_{i \in I} X_i$ is a generator for $\mathrm{bb} - \mathbf{Ab}^{\mathcal{C}^{\mathrm{op}}}$; since it is basically bounded we choose a cardinal number m such that

$\oplus_{i \in I} X_i$ is m-based. Now let $\mathcal{F}$ be the abelian presheaf on $\mathcal{C}(m+1)$ which takes the value 0 on all sets of cardinality $\leqslant m$ and the value $\mathbb{Z}$ on the set of cardinality m, and let $j(m+1)_* \mathcal{F}$ be the left Kan extension of $\mathcal{F}$ along the inclusion of categories $\mathcal{C}(m) \hookrightarrow \mathcal{C}$. Then there does not exist a morphism $\oplus_{i \in I} X_i \to j(m+1)_* \mathcal{F}$ which does not factor through the zero presheaf, contradicting the property of $\oplus_{i \in I} X_i$ being a generator. If we give $\mathcal{C}$ the Grothendieck topology in which the only covers are isomorphisms, then this counterexample also works in the sheaf category. As a result we aren't guaranteed to have injective envelopes in the entire basically bounded sheaf category, and the same holds for $(\mathcal{C}, \mathfrak{A}, \tau)$-(pre)modules.

Quasicoherent cohomology.

Proposition A.2.24. *Let $\mathcal{X}$ be a site with underlying category $\mathcal{C}$ equipped with a faithful functor $\mathcal{C} \to$ **Sets**, and let $(\mathcal{X}, \mathfrak{A})$ define a locally abelianly categoried site such that $\mathfrak{A}(U)$ satisfies AB3 for all $U \in \operatorname{ob} \mathcal{X}$. Then the inclusion of categories*

$$\mathrm{bb} - \mathbf{Mod}(\mathcal{C}, \mathfrak{A}, \tau)_{\mathrm{qcoh}} \xrightarrow{i} \mathrm{bb} - \mathbf{Mod}(\mathcal{C}, \mathfrak{A}, \tau)$$

is an exact additive functor.

Proof. We have the diagram of abelian categories and additive functors

$$\begin{tikzcd}
\mathrm{bb} - \mathbf{Premod}(\mathcal{C}, \mathfrak{A}, \tau)_{\mathrm{qcoh}} \arrow[r, shift left, "\#_{\mathrm{qcoh}}"] \arrow[d, "i_{Pre}"] & \mathrm{bb} - \mathbf{Mod}(\mathcal{C}, \mathfrak{A}, \tau)_{\mathrm{qcoh}} \arrow[l, shift left, "\texttt{forget}"] \arrow[d, "i"] \\
\mathbf{Premod}(\mathcal{C}, \mathfrak{A}, \tau) \arrow[r, "\#"] & \mathbf{Mod}(\mathcal{C}, \mathfrak{A}, \tau)
\end{tikzcd}$$

which is not commutative but in which most possible choices of composites of maps (for instance, any composites not including `forget`) between two fixed objects are equal; the reader can verify which composites are equal to which others. In particular we have $i = \# \circ \imath_{Pre} \circ \texttt{forget}$ and, as $\#$ and i_{Pre} are exact and `forget` is left exact, the functor i is left exact.

Now cokernels of morphisms of quasicoherent $(\mathcal{C}, \mathfrak{A}, \tau)$-modules are also quasicoherent, and in fact, for any morphism $U \xrightarrow{\phi} V$ in $\mathcal{C}$ and any morphism $\mathcal{F} \xrightarrow{f} \mathcal{G}$ of basically bounded $(\mathcal{C}, \mathfrak{A}, \tau)$-premodules, we have $\phi_*(\operatorname{coker} f)(U) = \operatorname{coker}(\phi_* f(U))$ as ϕ_* is a left adjoint and hence preserves cokernels; so if the cokernel of a morphism of basically bounded quasicoherent $(\mathcal{C}, \mathfrak{A}, \tau)$-modules is zero, then so is its cokernel in the category of basically bounded (not necessarily quasicoherent) $(\mathcal{C}, \mathfrak{A}, \tau)$-modules. So i is right exact. □

Corollary A.2.25. *Let $\mathcal{X}$ be a site with underlying category $\mathcal{C}$ possessing a terminal object X and equipped with a faithful functor $\mathcal{C} \to$ **Sets**, and let $(\mathcal{X}, \mathfrak{A})$ define a locally abelianly categoried site such that $\mathfrak{A}(U)$ satisfies AB3 for all $U \in \operatorname{ob} \mathcal{X}$. Then if i takes injectives (i.e., quasicoherent basically bounded modules which are injective in the category of quasicoherent basically bounded modules) to acyclics, we have an isomorphism in cohomology*

$$H^*_{\mathrm{qcoh}}(X; \mathcal{F}) \cong H^*(X; \mathcal{F})$$

for every quasicoherent basically bounded module $\mathcal{F}$ and natural in $\mathcal{F}$.

Proof. When i takes injectives to acyclics, the composable pair of functors

$$\mathrm{bb} - \mathbf{Mod}(\mathcal{C}, \mathfrak{A}, \tau)_{\mathrm{qcoh}} \xrightarrow{i} \mathrm{bb} - \mathbf{Mod}(\mathcal{C}, \mathfrak{A}, \tau) \xrightarrow{\Gamma} \mathfrak{A}(X)$$

gives a Grothendieck spectral sequence

$$E_2^{*,*} \cong H^*(X; R^* i \mathcal{F}) \Rightarrow E_\infty^{*,*} \cong H^*_{\mathrm{qcoh}}(X; \mathcal{F})$$

and since i is exact, $R^t i \mathcal{F}$ vanishes for $t > 0$, and the spectral sequence collapses at E_2, giving us the desired isomorphism of cohomology with quasicoherent cohomology. □

Main examples. The following lemma is from Vistoli's section in [Fantechi *et al.*, 2005]; the idea is due to Kleiman.

Lemma A.2.26. fpqc maps. *Let $X \xrightarrow{f} Y$ be a surjective morphism of schemes. Then the following properties are equivalent:*

1. *Every quasicompact open subset of Y is the image of a quasicompact open subset of X.*

2. *There exists a cover $\{V_i\}$ of Y by open affine subschemes such that each V_i is the image of a quasicompact open subset of X.*

3. *Given a point $x \in X$, there exists a neighborhood U of x in X such that the image $f^{\rightarrow}(U)$ is open in Y and the restriction $U \xrightarrow{f|_U} f^{\rightarrow}(U)$ of f to U is quasicompact.*

4. *Given a point $x \in X$, there exists a quasicompact open neighborhood U of x in X such that the image $f^{\rightarrow}(U)$ is open and affine in Y.*

Proof. Prop. 2.33 of [Fantechi *et al.*, 2005]. □

Definition A.2.27. The fpqc topology. *A morphism of schemes is said to be* fpqc *if it is faithfully flat and satisfies the equivalent conditions of Lemma A.2.26. Given a scheme S, the* fpqc topology *on* $\mathbf{Sch}/S$ *is given by letting coverings be finite collections of morphisms $\{U_i \to U\}$ of S-schemes such that the induced map $\coprod_i U_i \to U$ is an fpqc morphism.*

Lemma A.2.28. Comparability of fpqc topology. *The fpqc topology really is a Grothendieck topology on* $\mathbf{Sch}/S$. *Any faithfully flat morphism locally of finite presentation is fpqc.*

Proof. Prop. 2.35 of [Fantechi *et al.*, 2005]. □

As a consequence of any faithfully flat morphism locally of finite presentation being fpqc, we have a site morphism $S_{\text{fppf}} \to S_{\text{fpqc}}$ for any scheme S. Since the fpqc topology admits comparison to the fppf topology, it also admits comparison to the étale and Zariski topologies (the fpqc topology is finer than all others listed); this is the motivation for the particular formulation of the fpqc topology that we have chosen. Some other formulations have the disadvantage of failing to admit comparison to the fppf topology; see the discussion in [Fantechi *et al.*, 2005].

A.3 Effective descent classes.

From now on we will sometimes write U to denote the functor on $\mathbf{Aff}/S$ or $\mathbf{Sch}/S$ represented by U, as well as the scheme U itself.

Definition A.3.1. Some descent class definitions. *Let $\mathcal{C}$ be a category with finite pullbacks and let τ be a subcanonical Grothendieck topology on $\mathcal{C}$. Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$.*

1. *We say that $\mathcal{D}$ is a* closed subcategory *of $\mathcal{C}$ if*

   (a) *$\mathcal{D}$ contains every isomorphism in $\mathcal{C}$, and*

   (b) *if*

$$\begin{array}{ccc} U & \longrightarrow & V \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \longrightarrow & Y \end{array}$$

   is a pullback square in $\mathcal{C}$ and f is in $\mathcal{D}$, then f' is in $\mathcal{D}$.

   (We note that being a closed subcategory of $\mathcal{C}$ depends only on the category $\mathcal{C}$, not on the Grothendieck topology τ.)

2. *We say that a class $\mathcal{D}$ of maps in $\mathcal{C}$ is* stable (under τ) *if $\mathcal{D}$ is a closed subcategory of $\mathcal{C}$ and for any $f : X \to Y$ in $\mathcal{C}$ and $\{Y_i \to Y\} \in \operatorname{Cov}\tau$ such that each $f_i : X \times_Y Y_i \to Y_i$ is in $\mathcal{D}$, the map f is also in $\mathcal{D}$.*

3. *If $\mathcal{D}$ is a stable class of maps in $\mathcal{C}$, we say that $\mathcal{D}$* satisfies effective τ-descent *(or, when the topology τ is obvious from context, we may say* satisfies

effective descent*) if, for every covering* $\{U_i \to U\} \in \mathrm{Cov}\,\tau$ *and every sheaf* $\mathcal{F}$ *of sets on* $\mathcal{C}$ *equipped with a map of sheaves* $\mathcal{F}(-) \to \hom_{\mathcal{C}}(-, U)$ *such that for each* i *the pullback sheaf* $U_i \times_U \mathcal{F}$ *is representable by some object* W_i *in* $\mathcal{C}$ *and such that* $W_i \to U_i$ *is in* $\mathcal{D}$ *for each* i*, then* $\mathcal{F}$ *is representable by some object in* $\mathcal{C}$*.*

4. *If* $\mathcal{D}$ *is an effective* τ*-descent class in* $\mathcal{C}$*, we say that* $\mathcal{D}$ *LIFTS OVER DIAGONALS if, for any objects* U, X *in* $\mathcal{C}$ *such that the diagonal map* $U \to U \times U$ *factors through* X *and both* $X \to U \times U$ *and* $U \to U \times U$ *are in* $\mathcal{D}$*, the factor map* $U \to X$ *is also in* $\mathcal{D}$*.*

5. *We say that* $\mathcal{D}$ *is* local on the source (in the τ topology) *if* $\mathcal{D}$ *is a subcategory of* $\mathcal{C}$ *such that, for any morphism* $X \xrightarrow{f} Y$ *in* $\mathcal{C}$ *and any cover* $\{X_i \xrightarrow{\phi_i} X\}$ *of* X *in* τ*,* $X \to Y$ *is in* $\mathcal{D}$ *if and only if every composite* $X_i \xrightarrow{f \circ \phi_i} Y$ *is in* $\mathcal{D}$*.*

6. *We say that* $\mathcal{D}$ *is* local on the target (in the τ topology) *if* $\mathcal{D}$ *is a subcategory of* $\mathcal{C}$ *such that, for any morphism* $X \xrightarrow{f} Y$ *in* $\mathcal{C}$ *and any cover* $\{Y_i \xrightarrow{\phi_i} Y\}$ *of* Y *in* τ*,* $X \to Y$ *is in* $\mathcal{D}$ *if and only if every pullback morphism* $X \times_Y Y_i \longrightarrow Y_i$ *is in* $\mathcal{D}$*.*

Definition A.3.2. APPROPRIATE topologies.

1. *If* $\mathcal{C}$ *is a category with finite pullbacks and* $\mathcal{B}$ *is a closed subcategory of* $\mathcal{C}$*, the* associated topology *to* $\mathcal{B}$*, sometimes written* $\tau_{\mathcal{B}}$*, is the Grothendieck topology on* $\mathcal{C}$ *where* $\{U_i \xrightarrow{\phi_i} U\} \in \mathrm{Cov}\,\tau_{\mathcal{B}}$ *iff each* ϕ_i *is in* $\mathcal{B}$ *and* $\{U_i \xrightarrow{\phi_i} U\}$ *is a UEEF.*

2. *If* $\mathcal{C}$ *is a category with finite pullbacks and* $\mathcal{B}$ *is a closed subcategory of* $\mathcal{C}$*, we say that* $\tau_{\mathcal{B}}$ *is a* APPROPRIATE *topology if:*

   (a) *for each set of maps* $\{X_i \xrightarrow{\phi_i} Y\}_{i \in I}$ *of* $\mathcal{C}$ *for which the coproduct* $X = \coprod_{i \in I} X_i$ *exists, the induced map* $\phi : X \to Y$ *is in* $\mathcal{B}$ *if and only if* ϕ_i *is in* $\mathcal{B}$ *for all* $i \in I$*; and* *(b) for each commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[dl, "g"] \\ Z & \end{tikzcd}$$

in $\mathcal{C}$ with h in $\mathcal{B}$,

i. if $\{f\} \in \operatorname{Cov} \tau_{\mathcal{B}}$ then g is in $\mathcal{B}$, and

ii. if g is in $\mathcal{B}$ then f is in $\mathcal{B}$.

Lemma A.3.3. *If $\mathcal{C}$ is a category with finite pullbacks and $\mathcal{B}$ is a closed subcategory of $\mathcal{C}$ then the associated topology $\tau_{\mathcal{B}}$ is a subcanonical Grothendieck topology on $\mathcal{C}$.*

Proof. Immediate from definitions. □

Proposition A.3.4. Examples of effective descent classes. *Let S be a scheme.*

1. *The class of affine morphisms is an effective descent class in* $\mathrm{fpqc}(\mathbf{Sch}/S)$.

2. *The class of quasicompact morphisms is an effective descent class in* $\mathrm{fpqc}(\mathbf{Sch}/S)$.

3. *The class of smooth morphisms is an effective descent class in* $\mathrm{fpqc}(\mathbf{Sch}/S)$.

Much more is true, many more classes of morphisms form effective descent classes in the fpqc *topology as well as in the* fppf *and étale topologies, but our needs are most immediately for the affine and quasicompact descent classes in the* fpqc *topology.*

Proof. These results are standard; [Fantechi *et al.*, 2005] is one place to look. □

B Moduli problems taking values in sets.

B.1 Preliminaries on commutative algebras and schemes.

The categories involved. Fix a commutative ring R. We denote by $\mathbf{CAlg}(R)$ the category of commutative unital R-algebras.

Fix a scheme S. We denote by $\mathbf{Aff}/S$ the category of schemes X equipped with an affine morphism to S. We denote by $\mathbf{Sch}/S$ the category of schemes X equipped with a morphism to S.

Base change.

Definition B.1.1. Base change for algebras. *Let R, S be commutative rings and let $R \xrightarrow{f} S$ be a ring morphism. We define functors f_*, f^* as follows:*

$$\begin{array}{rcl} f_* : \mathbf{CAlg}(R) & \to & \mathbf{CAlg}(S) \\ f_*(A) & = & A \otimes_R S \\ f^* : \mathbf{CAlg}(S) & \to & \mathbf{CAlg}(R) \\ f^*(A) & = & A \textit{ regarded as a } R\textit{-algebra via } f. \end{array}$$

Definition B.1.2. *Let R, S be schemes and let $R \xrightarrow{f} S$ be a morphism of schemes. We define functors f_*, f^* as follows:*

$$\begin{aligned} f^* : \mathbf{Sch}/S &\to \mathbf{Sch}/R \\ f^*(X) &= X \times_S R \\ f_* : \mathbf{Sch}/R &\to \mathbf{Sch}/S \\ f_*(A) &= A \text{ regarded as an } S\text{-scheme via } f. \end{aligned}$$

(The functor f_* will turn out to be a special case of Lemma A.1.11, where we embed $\mathbf{Sch}/S$ into $\mathbf{Sets}^{(\mathbf{Sch}/S)^{\mathrm{op}}}$ by sending X to the presheaf represented by S; that f_* is left adjoint to f^* is proved in Prop. B.1.5.)

Lemma B.1.3. Base change for schemes. *Let $R \xrightarrow{f} S$ be a morphism of schemes and let X be a scheme.*

1. *If X is affine over S then $f^*(X)$ is affine over R, i.e., we have a functor*

$$f^* : \mathbf{Aff}/S \to \mathbf{Aff}/R$$

   which agrees with the functor f^* defined on $\mathbf{Sch}/S$. If the morphism f is separated and X is affine over R then $f_*(X)$ is affine over S, i.e., if f is separated then we have a functor

$$f_* : \mathbf{Aff}/R \to \mathbf{Aff}/S$$

   which agrees with the functor f_* defined on $\mathbf{Sch}/R$.

2. *If f is separated then X is affine over R if and only if $f_*(X)$ is affine over S.*

3. *If X is quasicompact over S then $f^*(X)$ is quasicompact over R. If the morphism f is separated and X is quasicompact over R then $f_*(X)$ is quasicompact over S.*

4. *If f is separated then X is quasicompact over R if and only if $f_*(X)$ is quasicompact over S.*

5. *If f is quasicompact and $f_*(X)$ is quasicompact over S then X is quasicompact over R.*

Proof. 1. That affine morphisms are "stable"–preserved by base change–is 1.6.2.iii of [Grothendieck, 1961]. This directly implies that $f^*(X)$ is affine if X is affine.

For the second part, we have scheme-morphisms $X \xrightarrow{g} R \xrightarrow{f} S$ in which f is separated and g is affine, and we want to show that $f \circ g$ is affine. Let $\{U_i\}_{i \in I}$ be an open affine cover of S. Then $f^{\leftarrow}(U_i)$ is separated over the affine scheme U_i, so $(f \circ g)^{\leftarrow}(U_i)$ is affine over $f^{\leftarrow}(U_i)$ if and only if it is affine over $\operatorname{Spec}\mathbb{Z}$, and we are given that it is affine over $f^{\leftarrow}(U_i)$, so it is affine over $\operatorname{Spec}\mathbb{Z}$; however, U_i is affine, hence separated over $\operatorname{Spec}\mathbb{Z}$, so any scheme over U_i is affine over U_i if and only if it is affine over $\operatorname{Spec}\mathbb{Z}$, which $(f \circ g)^{\leftarrow}(U_i)$ is. So each $(f \circ g)^{\leftarrow}(U_i)$ is affine, and hence $f \circ g$ is an affine morphism (folio 5 of [Grothendieck, 1960], folio 1 of [Grothendieck, 1961]).

2. The "only if" follows from the previous part of this lemma. The "if" part is 5.5.12.v of [Grothendieck, 1960].

3. From 6.6.4 of [Grothendieck, 1960]. The argument is completely analogous to that of the previous part of this lemma.

4. The "only if" follows from the previous part of this lemma. The "if" part is 5.5.12.v of [Grothendieck, 1960].

5. Let $X \xrightarrow{g} R$ be the R-scheme structure map of X, and let $\{U_i\}$ be a covering of S by quasicompact open subschemes. Then $\{f^{\leftarrow}(U_i)\}$ is a covering of R by quasicompact open subschemes, and $g^{\leftarrow}(f^{\leftarrow}(U_i))$ is quasicompact for each i; so g is quasicompact.

□

Proposition B.1.4. Frobenius reciprocity for algebras. *The functor f_* is left adjoint to f^*, i.e., there exists an isomorphism of sets*

$$\hom_{\mathbf{CAlg}(S)}(f_*A, B) \cong \hom_{\mathbf{CAlg}(R)}(A, f^*B),$$

natural in the choices of $A \in \mathrm{ob}\ \mathbf{CAlg}(R)$ and $B \in \mathrm{ob}\ \mathbf{CAlg}(S)$.

Proof. Given a pair $A \in \mathrm{ob}\ \mathbf{CAlg}(R)$,$B \in \mathrm{ob}\ \mathbf{CAlg}(S)$ we specify a morphism of sets

$$\begin{aligned} \tau(A,B) : \hom_{\mathbf{CAlg}(S)}(f_*A, B) &\rightarrow \hom_{\mathbf{CAlg}(R)}(A, f^*B) \\ ((\tau(A,B))(\phi))(r) &= \phi(r \otimes 1), \end{aligned}$$

or in other words, to get $A \xrightarrow{\tau(A,B)(\phi)} f^*B$ we simply take the composite $A \rightarrow A \otimes_R S \rightarrow B$. It is routine to verify that $\tau(A,B)(\phi)$ is a morphism of R-algebras. Now, given $\phi \in \hom_{\mathbf{CAlg}(R)}(A, f^*B)$, we have a commutative square of ring-morphisms

$$\begin{tikzcd} R \arrow[r] \arrow[d, "f"'] & A \arrow[d, "\phi"] \\ S \arrow[r] & B \end{tikzcd}$$

and hence a ring-morphism from the pushout $A \otimes_R S$ to B. This gives us a map $\hom_{\mathbf{CAlg}(R)}(A, f^*B) \xrightarrow{\sigma(A,B)} \hom_{\mathbf{CAlg}(S)}(f_*A, B)$ inverse to $\tau(A,B)$; so $\tau(A,B)$ is an isomorphism of sets.

We must verify naturality. Given $A_1, A_2 \in \mathrm{ob}\ \mathbf{CAlg}(R)$ and $B_1, B_2 \in \mathrm{ob}\ \mathbf{CAlg}(S)$ together with morphisms $a : A_1 \rightarrow A_2$ and $b : B_1 \rightarrow B_2$, we have a diagram of maps:

$$\begin{tikzcd} \hom_{\mathbf{CAlg}(S)}(f_*A_2, B_1) \arrow[r, "\tau(A_2,B_1)"] \arrow[d, "\zeta"] & \hom_{\mathbf{CAlg}(R)}(A_2, f^*B_1) \arrow[d, "\theta"] \\ \hom_{\mathbf{CAlg}(S)}(f_*A_1, B_2) \arrow[r, "\tau(A_1,B_2)"] & \hom_{\mathbf{CAlg}(S)}(A_1, f^*B_2), \end{tikzcd}$$

where $\zeta(\phi) = b \circ \phi \circ (f_* a)$ and $\theta(\phi) = (f^* b) \circ \phi \circ a$. The commutativity of this diagram is equivalent to the composites $\tau(A_1, B_2) \circ \zeta$ and $\theta \circ \tau(A_2, B_1)$ being equal. We write out these composites: for any $\phi \in \hom_{\mathbf{CAlg}(S)}(f_* A_2, B_1)$ we have

$$\begin{aligned} (\tau(A_1, B_2) \circ \zeta)(\phi) &= b \circ \phi \circ (f_* a) \circ (1_{A_1} \otimes_R f) \text{ and} \\ (\theta \circ \tau(A_2, B_1))(\phi) &= b \circ \phi \circ (1_{A_2} \otimes_R f) \circ a, \end{aligned}$$

and finally our desired naturality is due to $(f_* a) \circ (1_{A_1} \otimes_R f) = (1_{A_2} \otimes_R f) \circ a$, which follows from the tensor product being a bifunctor. □

Proposition B.1.5. Frobenius reciprocity for schemes. *The functor f_* is left adjoint to f^*, i.e., there exists an isomorphism of sets*

$$\hom_{\mathbf{Sch}/S}(f_* A, B) \cong \hom_{\mathbf{Sch}/R}(A, f^* B),$$

natural in the choices of $A \in \operatorname{ob} \mathbf{Sch}/R$ and $B \in \operatorname{ob} \mathbf{Sch}/S$.

Proof. Given a pair $A \in \operatorname{ob} \mathbf{Sch}/R, B \in \operatorname{ob} \mathbf{Sch}/S$ we specify a morphism of sets

$$\tau(A, B) : \hom_{\mathbf{Sch}/R}(A, f^* B) \rightarrow \hom_{\mathbf{Sch}/S}(f_* A, B)$$

by letting $\tau(A, B)(\phi)$ be the composite $A \xrightarrow{\phi} B \times_S R \rightarrow B$, with the map $B \times_S R \rightarrow B$ being projection to the first factor. Now, given $\phi \in \hom_{\mathbf{Sch}/S}(f_* A, B)$, we have a commutative square of scheme-morphisms

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\ R \arrow[r, "f"] & S \end{tikzcd}$$

and hence a scheme-morphism from A to the pullback $B \times_S R$. This gives us a map $\hom_{\mathbf{Sch}/S}(f_* A, B) \xrightarrow{\sigma(A,B)} \hom_{\mathbf{Sch}/R}(A, f^* B)$ inverse to $\tau(A, B)$; so $\tau(A, B)$ is an isomorphism of sets.

We must verify naturality. Given $A_1, A_2 \in \text{ob } \mathbf{Sch}/R$ and $B_1, B_2 \in \text{ob } \mathbf{Sch}/S$ together with morphisms $a : A_1 \to A_2$ and $b : B_1 \to B_2$, we have a diagram of maps:

$$\begin{tikzcd}
\hom_{\mathbf{Sch}/R}(A_2, f^*B_1) \arrow[r, "\tau(A_2,B_1)"] \arrow[d, "\zeta"] & \hom_{\mathbf{Sch}/S}(f_*A_2, B_1) \arrow[d, "\theta"] \\
\hom_{\mathbf{Sch}/R}(A_1, f^*B_2) \arrow[r, "\tau(A_1,B_2)"] & \hom_{\mathbf{Sch}/S}(f_*A_1, B_2),
\end{tikzcd}$$

where $\zeta(\phi) = (f^*b) \circ \phi \circ a$ and $\theta(\phi) = b \circ \phi \circ (f_*a)$. The commutativity of this diagram is equivalent to the composites $\tau(A_1, B_2) \circ \zeta$ and $\theta \circ \tau(A_2, B_1)$ being equal. We write out these composites: for any $\phi \in \hom_{\mathbf{CAlg}(S)}(f_*A_2, B_1)$ we have

$$\begin{aligned}
(\tau(A_1, B_2) \circ \zeta)(\phi) &= \pi_{B_2} \circ (f^*b) \circ \phi \circ a \text{ and} \\
(\theta \circ \tau(A_2, B_1))(\phi) &= b \circ \pi_{B_1} \circ \phi \circ (f_*a),
\end{aligned}$$

where π_B is the projection $B \times_S R \to B$, and finally our desired naturality is due to $(\mathrm{id}_{B_2} \times_S f) \circ (b \times_S \mathrm{id}_R) = (b \times_S \mathrm{id}_S) \circ (\mathrm{id}_{B_1} \times_S f)$, which follows from the fiber product being a bifunctor. $\square$

Proposition B.1.6. Base change over a base-changed base, for algebras. *Given objects A, B, S, T and maps $S \longrightarrow A$ and $S \otimes_R T \longrightarrow B$ in $\mathbf{GCAlg}(R)$ there is an isomorphism $A \otimes_S B \cong (A \otimes_R T) \otimes_{S \otimes_R T} B$ of commutative R-algebras.*

Proof. We construct diagrams in $\mathbf{CAlg}(R)$ using the given data:

$$\begin{tikzcd}
 & A \\
R \arrow[ur] \arrow[dr] & \\
 & B
\end{tikzcd} \tag{B.1.1}$$

$$\begin{tikzcd}
 & A \otimes_R T \\
T \arrow[ur] \arrow[dr] & \\
 & B
\end{tikzcd} \tag{B.1.2}$$

$$\begin{tikzcd} & A \\ S \arrow[ur] \arrow[dr] & \\ & B \end{tikzcd} \tag{B.1.3}$$

$$\begin{tikzcd} & A \otimes_R T \\ S \otimes_R T \arrow[ur] \arrow[dr] & \\ & B \end{tikzcd} \tag{B.1.4}$$

We give the names X_1, X_2, X_3, X_4 to the diagrams (B.1.1),(B.1.2),(B.1.3),(B.1.4) respectively. These four diagrams fit into the following larger diagram:

$$\begin{tikzcd}
& A \arrow[rr] \arrow[ddd, bend left] & & A \otimes_R T \arrow[ddd, bend left] \\
R \arrow[ur] \arrow[rr] \arrow[dr] \arrow[ddd, bend left] & & T \arrow[ur] \arrow[dr] \arrow[ddd, bend left] & \\
& B \arrow[rr] \arrow[ddd, bend left] & & B \arrow[ddd, bend left] \\
& A \arrow[rr] & & A \otimes_R T \\
S \arrow[ur] \arrow[rr] \arrow[dr] & & S \otimes_R T \arrow[ur] \arrow[dr] & \\
& B \arrow[rr] & & B
\end{tikzcd}$$

in which every square is a pushout square, i.e.,

$$\operatorname{colim} \left(\begin{tikzcd} & X_2 \\ X_1 \arrow[ur] \arrow[dr] & \\ & X_3 \end{tikzcd} \right) \cong X_4$$

and, since we know from Cor. **??** that the map $\operatorname{colim} X_1 \longrightarrow \operatorname{colim} X_2$ is an isomorphism, we have

$$\begin{aligned}
(A \otimes_R T) \otimes_{S \otimes_R T} B &\cong \operatorname{colim} X_4 \\
&\cong \operatorname{colim} \operatorname{colim} \left(\begin{tikzcd}[row sep=small] & X_2 \\ X_1 \arrow[ur] \arrow[dr] & \\ & X_3 \end{tikzcd} \right) \\
&\cong \operatorname{colim} \left(\begin{tikzcd}[row sep=small] & \operatorname{colim} X_2 \\ \operatorname{colim} X_1 \arrow[ur] \arrow[dr] & \\ & \operatorname{colim} X_3 \end{tikzcd} \right) \\
&\cong \operatorname{colim} X_3 \\
&\cong A \otimes_S B,
\end{aligned}$$

as commutative R-algebras. □

Proposition B.1.7. Base change over a base-changed base, for schemes. *Given objects A, B, S, T and maps $A \longrightarrow S$ and $B \longrightarrow S \times_R T$ in* $\mathbf{Sch}/R$ *there is an isomorphism $A \times_S B \cong (A \times_R T) \times_{S \times_R T} B$ of R-schemes.*

Proof. We construct diagrams in $\mathbf{Sch}/R$ using the given data:

$$\begin{tikzcd} A \arrow[dr] & \\ & R \\ B \arrow[ur] & \end{tikzcd} \tag{B.1.5}$$

$$\begin{tikzcd}
A \times_R T \arrow[dr] & \\
& T \\
B \arrow[ur] &
\end{tikzcd} \tag{B.1.6}$$

$$\begin{tikzcd}
A \arrow[dr] & \\
& S \\
B \arrow[ur] &
\end{tikzcd} \tag{B.1.7}$$

$$\begin{tikzcd}
A \times_R T \arrow[dr] & \\
& S \times_R T \\
B \arrow[ur] &
\end{tikzcd} \tag{B.1.8}$$

We give the names X_1, X_2, X_3, X_4 to the diagrams (B.1.5),(B.1.6),(B.1.7),(B.1.8) respectively. These four diagrams fit into the following larger diagram:

$$\begin{tikzcd}
A \times_R T \arrow[rr] \arrow[dr] \arrow[ddd, bend left] & & A \arrow[dr] \arrow[ddd, bend left] & \\
& S \times_R T \arrow[rr] \arrow[ddd, bend left] & & S \arrow[ddd, bend left] \\
B \arrow[rr] \arrow[ur] \arrow[ddd, bend left] & & B \arrow[ur] \arrow[ddd, bend left] & \\
A \times_R T \arrow[rr] \arrow[dr] & & A \arrow[dr] & \\
& T \arrow[rr] & & R \\
B \arrow[rr] \arrow[ur] & & B \arrow[ur] &
\end{tikzcd}$$

in which every square is a pullback square, i.e.,

$$\lim \left(\begin{tikzcd} X_2 \arrow[dr] & \\ & X_1 \\ X_3 \arrow[ur] & \end{tikzcd} \right) \cong X_4$$

and, since we know from Cor. **??** that the map $\lim X_1 \longrightarrow \lim X_2$ is an isomorphism, we have

$$\begin{aligned} (A \times_R T) \times_{S \times_R T} B &\cong \lim X_4 \\ &\cong \lim \lim \left(\begin{tikzcd} X_2 \arrow[dr] & \\ & X_1 \\ X_3 \arrow[ur] & \end{tikzcd} \right) \\ &\cong \lim \left(\begin{tikzcd} \lim X_2 \arrow[dr] & \\ & \lim X_1 \\ \lim X_3 \arrow[ur] & \end{tikzcd} \right) \\ &\cong \lim X_3 \\ &\cong A \times_S B, \end{aligned}$$

as commutative R-algebras. □

Representability. A commutative R-algebra A determines a covariant functor $\hom_{\mathbf{CAlg}(R)}(A, -)$ from commutative R-algebras to sets. A covariant functor $\mathbf{CAlg}(R) \rightarrow \mathbf{Sets}$ which is isomorphic as a functor to $\hom_{\mathbf{CAlg}(R)}(A, -)$ is said to be *corepresented by A*. In practice this will occur when we consider moduli problems; for instance, given a commutative ring R one can consider the set of all commutative 1-dimensional formal group laws over R, and a map of rings gives a map (of the same variance) between sets of formal group laws over those rings, so we have a functor $\mathbf{CAlg}(\mathbb{Z}) \to \mathbf{Sets}$, and this functor is corepresented by the Lazard ring L (or, with the notation in force in this paper, $\mathbf{L}^{\mathbb{Z}}$). Some moduli problems are not solvable in $\mathbf{CAlg}(R)$ but are solvable in enlargements of the category of commutative R-algebras, such as the Hilbert scheme of a scheme $X/\operatorname{Spec} R$, which assigns to any commutative R-algebra A the set of closed subschemes of $\operatorname{Spec} A \times_{\operatorname{Spec} R} X$ (with mild hypothesis on X this moduli problem is solvable in algebraic $\operatorname{Spec} R$-spaces, a slight enlargement of the category of $\operatorname{Spec} R$-schemes), and some moduli problems which are not solvable in $\mathbf{CAlg}(R)$ or $\mathbf{GCAlg}(R)$ are solvable once we allow for more data in the moduli problem than simply a set (such as moduli of elliptic curves). We will return to these situations elsewhere in these notes. For now we adopt the following notation: we will refer to any element in the set $\hom_{\mathbf{CAlg}(R)}(A, B)$ as *an A-object (over B)*, taking inspiration from the theory of moduli; for instance, a formal group law is an $\mathbf{L}^{\mathbb{Z}}$-object.

The following result is elementary:

Lemma B.1.8. Ring-surjections represent injections. *Let R be a commutative ring and let $A \xrightarrow{s} B$ be a surjection of commutative R-algebras. Then, for any commutative R-algebra X, $\hom_{\mathbf{CAlg}(R)}(B, X) \to \hom_{\mathbf{CAlg}(R)}(A, X)$ is an injection, i.e., any A-object over X which "extends" to a B-object over X does so uniquely.*

If s has a nontrivial kernel, then over some R-algebra X, there exists an A-object which does not "extend" to a B-object over X.

Proof. Let $f, g : B \to X$ be such that $f \circ s = g \circ s$. Then for any $b \in B$, we choose $a \in s^{\leftarrow}(\{b\})$ and we have $(f \circ s)(a) = (g \circ s)(a) = f(b) = g(b)$. So $f = g$ and $\hom_{\mathbf{CAlg}(R)}(B, X) \to \hom_{\mathbf{CAlg}(R)}(A, X)$ is an injection. (Alternatively, we can say that this is true because every surjection of commutative R-algebras is right-cancellable—although the converse is not true.)

For the second part, let $i \in \ker s$ with $i \neq 0$, and choose some commutative R-algebra X with a morphism $A \xrightarrow{f} X$ of R-algebras such that $f(i) \neq 0$ (an example would be $X = A$ and $f = \mathrm{id}_A$). Then $f \in \hom_{\mathbf{CAlg}(R)}(A, X)$ but f cannot be in the image of $\hom_{\mathbf{CAlg}(R)}(B, X) \to \hom_{\mathbf{CAlg}(R)}(A, X)$. □

Lemma B.1.9. Closed immersions represent injections. *Let S be a scheme and let $B \xrightarrow{s} A$ be a closed immersion of S-schemes. Then, for any S-scheme X, $\hom_{\mathbf{Sch}/S}(X, B) \to \hom_{\mathbf{Sch}/S}(X, A)$ is an injection, i.e., any A-object over X which "lifts" to a B-object over X does so uniquely. If s is not onto, then over some S-scheme T, there exists an A-object which does not "lift" to a B-object over T.*

Proof. If $f, g : X \to B$ are two morphisms of S-schemes such that $f \circ s = g \circ s$, then for any affine subscheme $Y \subseteq A$, we restrict s to the domain $s^{\leftarrow}(Y)$ to get a closed immersion $s^{\leftarrow}(Y) \xrightarrow{s} Y$; since $s^{\leftarrow}(Y)$ is the complement of a Zariski open set in the affine scheme Y, it is itself affine and $s^{\leftarrow}(Y) \cong \operatorname{Spec} \Gamma(Y)/I$ for some ideal I in $\Gamma(Y)$. Now, by the affine case of this theorem handled in the left column, f and g agree on every $s^{\leftarrow}(Y)$, and B is covered by the $s^{\leftarrow}(Y)$, so f, g agree on all of B.

Now suppose that s is not onto, and choose a point $p \in A$ which is not in the image of A; let p inherit a structure sheaf from that of A, and now we have a scheme morphism $p \xrightarrow{f} A$ such that f is not equal to $s \circ g$ for any $p \xrightarrow{g} B$. □

Suppose we have a commutative ring R and a functor $\mathbf{CAlg}(R) \xrightarrow{X} \mathbf{Sets}$ which is corepresented by a commutative R-algebra A, and suppose that we have a morphism of commutative rings $R \xrightarrow{f} R'$. The morphism f puts an R-algebra structure on any R'-algebra, and gives us a functor $\mathbf{CAlg}(R') \xrightarrow{f^*} \mathbf{CAlg}(R)$; the composite $X \circ f^*$ is an extension of X to the category $\mathbf{CAlg}(R')$ in the sense that $X \circ f^*$ agrees with X on the R-algebra underlying any commutative R'-algebra.

Proposition B.1.10. *With R, X, A, f, R' as above, the functor $X \circ f^*$ is represented by the commutative S-algebra $A \otimes_R R'$.*

Proof. We have $\hom_{\mathbf{CAlg}(R')}(A \otimes_R R', B) \cong \hom_{\mathbf{CAlg}(R)}(A, f^* B) \cong (X \circ f^*)(B)$, from Prop. B.1.4, and this isomorphism is natural in B. $\square$

Proposition B.1.11. *With S, X, T, f, S' as above, the functor $X \circ f_*$ is represented by the S'-scheme $T \times_S S'$.*

Proof. We have $\hom_{\mathbf{Sch}/S'}(T \times_S S', Y) \cong \hom_{\mathbf{Sch}/S}(T, f_* Y) \cong (X \circ f_*)(Y)$, from Prop. B.1.5, and this isomorphism is natural in Y. $\square$

Lemma B.1.12. 1. *Let S be a scheme and let $\mathcal{L}$ be a locally free $\mathcal{O}_S$-module of finite rank and let $\mathcal{F}$ be an $\mathcal{O}_S$-module. Then the canonical map of sheaves*

$$\mathcal{hom}_{\mathbf{Mod}_{\mathcal{O}_S}}(\mathcal{L}, \mathcal{O}_S) \otimes_{\mathcal{O}_S} \mathcal{F} \to \mathcal{hom}_{\mathbf{Mod}_{\mathcal{O}_S}}(\mathcal{L}, \mathcal{F})$$

is an isomorphism.

2. *Let S be a scheme and let $\mathcal{L}$ be a locally free $\mathcal{O}_S$-module of finite rank and let $\mathcal{F}$ be an $\mathcal{O}_S$-module. Then the canonical map*

$$\hom_{\mathbf{Mod}_{\mathcal{O}_S}}(\mathcal{L}, \mathcal{O}_S) \otimes_{\Gamma(\mathcal{O}_S)} \Gamma(\mathcal{F}) \to \hom_{\mathbf{Mod}_{\mathcal{O}_S}}(\mathcal{L}, \mathcal{F})$$

is an isomorphism.

Proof. 1. 0, 5.4.2 of [Grothendieck, 1960].

2. Take global sections of the hom-sheaves in the previous part of this lemma.

$\square$

Proposition B.1.13. Equivalence of quasicoherent $\mathcal{O}_S$-algebras and affine S-schemes. *Let S be a scheme. We write $\mathbf{CAlg}(\mathcal{O}_S)_{\mathrm{qcoh}}$ for the category of* *quasicoherent commutative $\mathcal{O}_S$-algebras, and we write* $\mathbf{Aff}\,/S$ *for the category of schemes affine over* S*. We define the functors*

$$\begin{array}{rcl} \mathbf{Aff}\,/S & \xrightarrow{\mathcal{A}} & \mathbf{CAlg}(\mathcal{O}_S)_{\mathrm{qcoh}} \\ (X \xrightarrow{f} S) & \mapsto & f_*\mathcal{O}_X, \\ \mathbf{CAlg}(\mathcal{O}_S)_{\mathrm{qcoh}} & \xrightarrow{\mathrm{Spec}_S} & \mathbf{Aff}\,/S, \end{array}$$

where $\mathrm{Spec}_S\,\mathcal{F}$ *is given by taking an affine cover* $\{U_i \to S\}$ *of* S*, forming the affine schemes* $\mathrm{Spec}\,\mathcal{F}(U_i)$*, and gluing them along the subschemes* $\mathrm{Spec}\,\mathcal{F}(U_i \cap U_j)$*. Then* $\mathcal{A} \circ \mathrm{Spec}_S$ *is the identity functor on* $\mathbf{CAlg}(\mathcal{O}_S)_{\mathrm{qcoh}}$ *and* $\mathrm{Spec}_S \circ \mathcal{A}$ *is the identity functor on* $\mathbf{Aff}\,/S$*.*

Proof. That $\mathcal{A}(X)$ is a quasicoherent $\mathcal{O}_S$-algebra follows immediately from X being affine over S.

We reproduce 1.3.1 of [Grothendieck, 1961]: let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_S$-algebra. For each affine open $U \subseteq S$, let X_U be the scheme $\mathrm{Spec}(\Gamma(U, \mathcal{F}))$; as $\Gamma(U, \mathcal{F})$ is a $\Gamma(U, \mathcal{O}_S)$-algebra, X_U is a S-scheme. Since $\mathcal{F}$ is quasicoherent, the $\mathcal{O}_S$-algebra $\mathcal{A}(X_U)$ is identified canonically with $\mathcal{F}|_U$. Let V be another affine open of S, and let $X_{U,V}$ be the scheme induced X_U on $f_U^{-1}(U \cap V)$, and let f_U be the structure map $X_U \xrightarrow{f_U} S$; then $X_{U,V}$ and $X_{V,U}$ are affines over $U \cap V$, and by definition, $\mathcal{A}(X_{U,V})$ and $\mathcal{A}(X_{V,U})$ are canonically identified with $\mathcal{F}|_{U \cap V}$. There is then a canonical S-isomorphism $\theta_{U,V} : X_{V,U} \to X_{U,V}$; and then if W is another affine open in S, and if $\theta'_{U,V}, \theta'_{V,W}, \theta'_{U,W}$ are the restrictions of $\theta_{U,V}, \theta_{V,W}, \theta_{U,W}$ to the image of $U \cap V \cap W$ in X_V,X_W, and X_W, respectively, by the structure maps, we have $\theta'_{U,V} \circ \theta'_{V,W} = \theta'_{U,W}$. So there exists a scheme X and a covering $\{T_U\}$ of X by affine opens, and for each U an isomorphism $\phi_U : X_U \to T_U$, such that $\theta_{U,V} = \phi_U^{-1} \circ \phi_V$. The morphism $g_U = f_U \circ \phi_U^{-1}$ makes T_U into an S-scheme. It is clear, by definition, that X is affine over S and that $\mathcal{A}(T_U) = \mathcal{F}|_U$, hence $\mathcal{A}(X) = \mathcal{F}$. $\square$

B.2 Algebraic spaces.

We include here an important note: this section has been written with a definition slightly out of joint with the standard one, that of Knutson (INSERT REF TO KNUTSON). Knutson's algebraic spaces are like ours but satisfy the additional axiom that their cover by an affine scheme must express the algebraic space as the affine scheme modulo a τ-equivalence relation; in particular this limits how much "bigger" the covering scheme can be than the algebraic space. We are grateful to Sharon Hollander for pointing this out to us. Algebraic spaces in Knutson's sense are equivalent to *unicursal* **Setoids**-schemes; we plan to rewrite this section to be in accord with Knutson's definitions, and to include a discussion of unicursality, before the material in this section is published.

Definition B.2.1. Definitions of algebraic spaces. *Let S be a base scheme, τ a subcanonical Grothendieck topology on $\mathbf{Sch}/S$, and let $\mathcal{D}$ be an effective τ-descent class in $\mathbf{Sch}/S$.*

1. *An S*-space in the τ topology *is a sheaf $(\mathbf{Aff}/S)^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathbf{Sets}$ in the τ topology.*

2. *An* algebraic S-space in the τ topology of descent class $\mathcal{D}$ *is an S-space $\mathcal{F}$ in the τ topology such that there exists an S-scheme U, with $U \to S$ in $\mathcal{D}$, and a morphism of sheaves*

$$\hom_{\mathbf{Aff}/S}(-, U) \xrightarrow{P} \mathcal{F}(-)$$

such that, for any affine S-scheme V and sheaf morphism $\hom_{\mathbf{Aff}/S}(-, V) \to \mathcal{F}(-)$, the sheaf-theoretic pullback

$$\hom_{\mathbf{Aff}/S}(-, U) \times_{\mathcal{F}(-)} \hom_{\mathbf{Aff}/S}(-, V)$$

is representable by an S-scheme, and the pullback morphism

$$\hom_{\mathbf{Aff}/S}(-, U) \times_{\mathcal{F}(-)} \hom_{\mathbf{Aff}/S}(-, V) \to \hom_{\mathbf{Aff}/S}(-, V)$$

is induced by a morphism of schemes which is a cover in the τ topology; and the morphism of schemes inducing

$$\hom_{\mathbf{Aff}/S}(-,U)\times_{\mathcal{F}(-)}\hom_{\mathbf{Aff}/S}(-,U) \to \hom_{\mathbf{Aff}/S}(-,U)\times_{\hom_{\mathbf{Aff}/S}(-,S)}(\hom_{\mathbf{Aff}/S}(-,U)$$

belongs to the effective descent class $\mathcal{D}$. If U is affine over S then the map P is called a presentation of $\mathcal{F}$*.*

3. *If τ is a Grothendieck topology such that the class of affine morphisms forms an effective τ-descent class, then an algebraic S-space $\mathcal{F}$ in the τ-topology with presentation $U \xrightarrow{P} \mathcal{F}$ is called a* Ravenel S-space in the τ topology *if it is of affine descent class. The map P is called a* Ravenel presentation. *(We could speak of such objects even for a topology in which affine morphisms don't form an effective descent class, but this doesn't seem useful at the moment.)*

4. *A space equipped with a particular choice of presentation is said to be* rigidified. *A space equipped with a particular choice of Ravenel presentation is said to be* Ravenel-rigidified*.*

5. *A morphism of algebraic spaces is a morphism of the underlying sheaves.*

Proposition B.2.2. Structure maps of a Setoids-object, equivalences of algebraic spaces and Setoids-schemes. *Let S be a base scheme, τ a APPROPRIATE Grothendieck topology on* $\mathbf{Sch}/S$*, and let $\mathcal{D}$ be an effective τ-descent class in* $\mathbf{Sch}/S$ *which LIFTS OVER DIAGONALS.*

1. *Let $U \xrightarrow{P} \mathcal{F}$ be a rigidified S-space in the τ-topology of descent class $\mathcal{D}$. Then we take the pullback maps*

$$\begin{tikzcd} U \times_{\mathcal{F}} U \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & U \arrow[d, "P"] \\ U \arrow[r, "P"] & \mathcal{F}, \end{tikzcd}$$

the diagonal map $U \xrightarrow{\Delta_{U/\mathcal{F}}} U \times_{\mathcal{F}} U$*, and the factor swap map* $U \times_{\mathcal{F}} U \xrightarrow{T} U \times_{\mathcal{F}} U$*, and we observe that they obey the following identities:*

$$\begin{aligned} \pi_1 \circ \Delta_{U/\mathcal{F}} &= \pi_2 \circ \Delta_{U/\mathcal{F}} = \mathrm{id}_U \\ T \circ T &= \mathrm{id}_{U \times_{\mathcal{F}} U} \\ \pi_1 \circ T &= \pi_2 \\ \pi_2 \circ T &= \pi_1. \end{aligned}$$

Furthermore, the maps π_1, π_2, T*, and* $\Delta_{U/\mathcal{F}}$ *all belong to* $\mathcal{D}$*.*

2. *Let* X *be an* S*-scheme, let* U *be a scheme affine over* S*, and suppose we have morphisms of* S*-schemes*

$$\begin{aligned} X &\xrightarrow{\pi_1} U, \\ X &\xrightarrow{\pi_2} U, \\ X &\xrightarrow{T} X, \\ U &\xrightarrow{\Delta} X, \end{aligned}$$

with π_1 *and* π_2 *covers in the* τ *topology, and satisfying the axioms*

$$\begin{aligned} \pi_1 \circ \Delta &= \pi_2 \circ \Delta = \mathrm{id}_U \\ T \circ T &= \mathrm{id}_X \\ \pi_1 \circ T &= \pi_2 \\ \pi_2 \circ T &= \pi_1. \end{aligned}$$

Consider the map $X \to U \times_S U$ *coming from the universal property of the pullback, i.e., the unique map filling in this commutative diagram:*

$$\begin{tikzcd} X \arrow[dr] \arrow[drr, "\pi_1"] \arrow[ddr, "\pi_2"'] & & \\ & U \times_S U \arrow[r] \arrow[d] & U \arrow[d] \\ & U \arrow[r] & S. \end{tikzcd}$$

If the map $X \to U \times_S U$ *belongs to* $\mathcal{D}$*, then we call the object* $(X, U, \pi_1, \pi_2, T, \Delta)$ *a* **Setoids** *S-scheme, affine at stage* 1*, in the* τ *topology, of descent class* $\mathcal{D}$*. If we have an object* $(X, U, \pi_1, \pi_2, T, \Delta)$ *satisfying all of the same conditions except that the S-scheme U is not necessarily affine over S, then we refer to* $(X, U, \pi_1, \pi_2, T, \Delta)$ *as a* **Setoids** *S-scheme in the* τ *topology of descent class* $\mathcal{D}$*.*

If $(X, U, \pi_1, \pi_2, T, \Delta)$ *is a* **Setoids** *S-scheme, affine at stage* 1*, in the* τ *topology of descent class* $\mathcal{D}$*, then we define a presheaf* $(\mathbf{Aff}\,/S)^{\mathrm{op}} \xrightarrow{F} \mathbf{Sets}$ *as the coequalizer*

$$\hom_{\mathbf{Sch}\,/S}(-, X) \underset{\hom_{\mathbf{Sch}\,/S}(-,\pi_2)}{\overset{\hom_{\mathbf{Sch}\,/S}(-,\pi_1)}{\rightrightarrows}} \hom_{\mathbf{Sch}\,/S}(-, U) \longrightarrow F(-) \longrightarrow 0.$$

The sheafification $\mathcal{F}$ *of* F *in the* τ *topology is an algebraic S-space in the* τ *topology of descent class* $\mathcal{D}$*, and the composite*

$$\hom_{\mathbf{Aff}\,/S}(-, U) \to F \to \mathcal{F}$$

is a presentation of descent class $\mathcal{D}$*.*

3. **Characterization of maps of algebraic spaces.** *Let S be a base scheme,* τ *a APPROPRIATE Grothendieck topology on* $\mathbf{Sch}\,/S$*, and* $\mathcal{D}$ *an effective* τ*-descent class in* $\mathbf{Sch}\,/S$*. Let* A_1, A_2 *be algebraic S-spaces in the* τ*-topology of descent class* $\mathcal{D}$*, and let* $U_1 \xrightarrow{P_1} A_1$ *and* $U_2 \xrightarrow{P_2} A_2$ *be* τ*-coverings by S-schemes* U_1, U_2*.*

   (a) *If g, h are maps such that in the diagram*

$$\begin{tikzcd} U_1 \times_{A_1} U_1 \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, "g"] & U_1 \arrow[r, "P_1"] \arrow[d, "h"] & A_1 \\ U_2 \times_{A_2} U_2 \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & U_2 \arrow[r, "P_2"] & A_2 \end{tikzcd}$$

   we have $h \circ \pi_1 = \pi_1 \circ g$ *and* $h \circ \pi_2 = \pi_2 \circ g$*, then there exists a unique map* $A_1 \xrightarrow{f} A_2$ *with* $P_2 \circ h = f \circ P_1$*.*

(b) *Every map* $A_1 \xrightarrow{f} A_2$ *is induced in this way for some choice of* U_1, U_2, g, h.

4. *The category of* **Setoids** S*-schemes, affine at stage* 1*, in the* τ *topology of descent class* $\mathcal{D}$ *is equivalent to the category of rigidified* S*-spaces in the* τ *topology of descent class* $\mathcal{D}$.

   The equivalence works by sending a **Setoids** S*-scheme* $(X, U, \pi_1, \pi_2, T, \Delta)$ *to the rigidified* S*-space* $U \to \mathcal{F}$*, where* $\mathcal{F}$ *is the sheafification (in the* τ *topology) of the coequalizer presheaf of the sheaf morphisms represented by* $X \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} U$ *, while a rigidified* S*-space* $U \to \mathcal{F}$ *is sent to the* **Setoids** S*-scheme* $(U \times_{\mathcal{F}} U, U, \pi_1, \pi_2, T, \Delta)$.

5. *If* τ *is a Grothendieck topology such that the class of affine morphisms forms an effective* τ*-descent class, then if* $(X, U, \pi_1, \pi_2, T, \Delta)$ *is a* **Setoids** S*-scheme, affine at stage* 1*, in the* τ *topology of affine descent class, we call* $(X, U, \pi_1, \pi_2, T, \Delta)$ *a* Ravenel **Setoids** S-scheme in the τ topology*. The presentation* $U \to \mathcal{F}$ *associated to a Ravenel* **Setoids** S*-scheme is a Ravenel presentation, and the category of Ravenel* **Setoids** S*-schemes in the* τ *topology is equivalent to the category of Ravenel-rigidified* S*-spaces in the* τ *topology.*

6. *If* τ *is a Grothendieck topology such that the class of affine morphisms forms an effective* τ*-descent class, then to a Ravenel* **Setoids** S*-scheme in the* τ*-topology we can associate a co-***Setoids***-object in quasicoherent* $\mathcal{O}_S$*-algebras, with unit maps* η_L,η_R *(the maps coming from* π_1,π_2 *on the Ravenel* **Setoids** *scheme) such that* $\mathrm{Spec}_S\, \eta_L$ *and* $\mathrm{Spec}_S\, \eta_R$ *are covering maps in the* τ *topology. We call such an object an* S-algebroid in the τ topology *and the category of* S*-algebroids in the* τ *topology is antiequivalent to the category of Ravenel* **Setoids** S*-schemes in the* τ *topology.*

Proof. 1. Easy verification using universal properties. That the maps are in $\mathcal{D}$

is shown as follows: since $U \to S$ is in $\mathcal{D}$, both projection maps $U \times_S U \to U$ are also in $\mathcal{D}$, so the composite

$$U \times_{\mathcal{F}} U \to U \times_S U \to U,$$

where the right-hand map is projection to the ith coordinate ($i \in \{1, 2\}$), is the map π_i, and it is in $\mathcal{D}$ if $U \times_{\mathcal{F}} U \to U \times_S U$ is in $\mathcal{D}$, since $\mathcal{D}$ is a subcategory of **Sch**$/S$. The map T is in $\mathcal{D}$ since it is an isomorphism. Finally, Δ is in $\mathcal{D}$ by the hypothesis on $\mathcal{D}$ given at the beginning of the proposition.

2. First of all, we have the pullback diagram of sheaves

$$\begin{tikzcd} X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & U \arrow[d] \\ U \arrow[r] & \mathcal{F} \end{tikzcd}$$

We must first show that, for any $V \to \mathcal{F}$, the pullback $U \times_{\mathcal{F}} V \to V$ is represented by a morphism of S-schemes. That the sheaf $U \times_{\mathcal{F}} U$ is represented by X is a consequence of I.5.4 of [Knutson, 1971]. Suppose that $V \to \mathcal{F}$ factors through U; then $V \times_{\mathcal{F}} U \cong V \times_U (U \times_{\mathcal{F}} U) \cong V \times_U X$ is an S-scheme and the map $V \times_U X \to V \times_U U$ is a covering map in τ since π_1, π_2 are. The map $V \times_U X \to V \times_S U$ is $V \times_U (U \times_{\mathcal{F}} U) \to V \times_U (U \times_S U)$ so it is in the same effective descent class as $X \to U \times_S U$.

Now suppose that $V \to \mathcal{F}$ does not factor through U; then, again by I.5.4 of [Knutson, 1971], there exists an S-scheme $W \to V$ covering V in the τ-topology, and a map $W \to U$ making the diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & \mathcal{F} \end{tikzcd}$$

commute. We have the pullback square of sheaves on $\mathbf{Aff}/S$:

$$\begin{tikzcd} W \times_{\mathcal{F}} U \arrow[r] \arrow[d] & V \times_{\mathcal{F}} U \arrow[d] \\ W \times_S U \arrow[r] & V \times_S U \end{tikzcd}$$

and since the map $W \times_S U \to \mathcal{F}$ factors through U, the map $W \times_{\mathcal{F}} U \to W \times_S U$ is represented by a map of S-schemes; since the map $W \times_{\mathcal{F}} U \to V \times_{\mathcal{F}} U$ is a cover in the τ-topology, we get that $V \times_{\mathcal{F}} U \to V \times_S U$ is represented by a map in the same effective descent class as $U \times_{\mathcal{F}} U \to U \times_S U$, by descent; finally, since

$$\begin{tikzcd} W \times_{\mathcal{F}} U \arrow[r] \arrow[d] & V \times_{\mathcal{F}} U \arrow[d] \\ W \arrow[r] & V \end{tikzcd}$$

is a pullback square, $V \times_{\mathcal{F}} U \to V$ is represented by a covering map in the τ topology.

3. (a) From the argument given in the proof of the previous part of this proposition, A_i is the sheafification of the coequalizer presheaf of $\{\pi_1, \pi_2 : U_i \times_{A_i} U_i \to U_i\}$. Now the functoriality of coequalizers gives us the map $A_1 \to A_2$.

   (b) Given $f : A_1 \to A_2$, we have the composite map $f \circ \pi_1 \in \hom(U_1, A_2) = A_2(U_1)$; by the construction of the coequalizer sheaf A_2, $f \circ \pi_1 \in A_2(U_1)$ is given by a covering $W_1 \to U_1$ and a section $h \in U_2(W_1)$ such that the two images of h in $U_2 \times_{A_2} U_2(W_1 \times_{U_1} W_1)$ coincide. Now we have our map $h : W_1 \to U_2$ and the composite $W_1 \to U_1 \to A_1$ is a τ-covering of A_1, and the existence of $g : W_1 \times_{A_1} W_1 \to U_2 \times_{A_2} U_2$ follows from the universal property of a pullback.

   (c) Immediate from the preceding parts of this proposition.

   (d) Follows immediately from Lemma B.1.13.

$\square$

Corollary B.2.3. The diagram $\mathfrak{Setoids}$.

If τ is a Grothendieck topology such that the class of affine morphisms forms an effective τ-descent class, then the category of Ravenel S-spaces in the τ topology is antiequivalent to the category of functors

$$\mathfrak{Setoids}^{\mathrm{op}} \xrightarrow{F} \mathbf{CAlg}(\mathcal{O}_S)_{\mathrm{qcoh}}$$

sending π_1 and π_2 to (the maps of quasicoherent $\mathcal{O}_S$-algebras associated to) covers in the τ topology, where $\mathfrak{Setoids}$ is the category with two objects, X and U, and morphisms

$$\begin{aligned} X &\xrightarrow{\pi_1} U, \\ X &\xrightarrow{\pi_2} U, \\ X &\xrightarrow{T} X, \\ U &\xrightarrow{\Delta} X \end{aligned}$$

satisfying the axioms

$$\begin{aligned} \pi_1 \circ \Delta &= \pi_2 \circ \Delta = \mathrm{id}_U \\ T \circ T &= \mathrm{id}_X \\ \pi_1 \circ T &= \pi_2 \\ \pi_2 \circ T &= \pi_1. \end{aligned}$$

When S is affine, the image of F is a diagram of $\Gamma(S)$-algebras.

I am grateful to Paul Pearson for once mentioning multisets, which I had never heard of, casually in conversation, right at a moment when I was wondering what kind of functor a rigidified space would represent.

Remark B.2.4. What does a rigidified space represent? An algebraic S-space is—and represents—a set-valued functor on $\mathbf{Aff}\,/S$. However, making a choice of presentation $U \to A$, where A is an algebraic S-space and U is an S-scheme, gives us something more. Consider a **Setoids** S-scheme $(X, U, \pi_1, \pi_2, T, \Delta)$: the coequalizer presheaf F of π_1, π_2 has the property that $\hom_{\mathbf{Sch}\,/S}(X, U) \to F(X)$ is a surjection for any S-scheme X. Holding on to the choice of U means that we have effectively taken the functor represented by U, and put an equivalence relation on it: two elements $x, y \in \hom_{\mathbf{Sch}\,/S}(X, U)$ are equivalent if they have the same image in $F(X)$. This is where the setoids come from; a "setoid" (a construction which does not seem to be widely used outside of logic) is simply a set equipped with an equivalence relation. A **Setoids** S-scheme represents a functor from S-schemes to setoids.

Setoids are equivalent to another possibly-familiar object: we recall from combinatorics that a set in which elements can have multiple membership is called a *multiset*; specifying a multiset is equivalent to specifying a set and an equivalence relation on it (e.g. instead of writing $\{a, a, a, b, c, c\}$ we could have written $\{a, v, w, b, c, z\}$ and the equivalence relation generated by the relations $a \sim v, v \sim w, c \sim z$). A rigidified S-THINGY in any subcanonical topology then has an associated multiset object in S-schemes; we have chosen the notation $\mathfrak{Multi}$ above to indicate that it is the category describing a multiset-object.

Sheafifying F to get the algebraic S-space $\mathcal{F}$ means that $\hom_{\mathbf{Sch}\,/S}(X, U) \to \mathcal{F}(X)$ is no longer necessarily surjective for every S-scheme X, so a rigidified S-space is not directly a multiset object in S-schemes, but we can take the associated S-THINGY by Cor. **??**, which is a multiset object.