Notes on Hecke Operators

Simon Burton

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1 Introduction

Group representation theory concerns itself with homomorphisms from a group G to the general linear group over a field k :

$$G \to \mathrm{GL}(n, k).$$

The definition of the general linear group makes sense not just for a field k, but also for a ring, or even a semi-ring (a ring without additive inverses.) In particular, we consider the semi-ring of truth-values:

$$\mathbb{F}_1 = \{\text{false}, \text{true}\}$$

with addition as disjunction and multiplication as conjunction. The notation $\mathbb{F}_1$ refers to the "field with one element", which however is not a field and doesn't have one element [5]. We have the following

Theorem 1. The group $\mathrm{GL}(n, \mathbb{F}_1)$ consists of $n \times n$ permutation matrices, and is therefore isomorphic to the permutation group S_n.

Proof: Adapted from [6]. ∎

And so we find that group representation theory over the semi-ring of truth values is the theory of groups acting on sets.

2 Groups acting on sets

We say a group G *acts* on a set X when there is a group homomorphism:

$$G \to \mathrm{Aut}(X).$$

We choose not to name this homomorphism, and instead confuse the elements of G with their image in $\mathrm{Aut}(X)$. In this way we understood expressions such as gx for $g \in G, x \in X$. This is similar to how a field acts on a vector space: we don't usually write the homomorphism, and instead just let elements of the field act on the vectors (on the left.)

We also call this setup a G-set X.

A map of G-sets $X \to Y$ is a set function $f : X \to Y$ that commutes with the group action. That is, for every $g \in G$ we have the commuting square:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

Thinking of G as a one object category, a group action is then a set-valued functor and we see that a map of G-sets is the same as a natural transformation of functors. This gives the category of G-sets which we denote **GSet**.

For $x \in X$ the *stabilizer* is defined

$$\mathrm{Stab}(x) := \{g \in G \mid gx = x\}.$$

This is clearly a subgroup of G. Dually, the *fixed point set* of $g \in G$ is the subset of X given by

$$\mathrm{Fix}(g) := \{x \in X \mid gx = x\}.$$

{In what sense are these really dual?}

The *orbit* of $x \in X$ is the set

$$\mathrm{Orbit}(x) := \{gx \mid g \in G\}.$$

A simple calculation shows that the stabilizers of points in the same orbit are related by conjugation: given a G-set X, with $x \in X, g \in G$, we have

$$\mathrm{Stab}(gx) = g\mathrm{Stab}(x)g^{-1}.$$

Given an arbitrary group G, there are many ways to cook up a set that G acts on. The most important such recipe is the following. Let H be any subgroup of G, and let $X = \{gH\}_{g \in G}$ be the set of left cosets of H. Then G acts on X by left-multiplication, and this action is transitive. We get the *left regular action* when $H = \{1\}$ and we get the trivial action when $H = G$. But this recipe has a converse: any transitive G-set X is isomorphic to a G-set $\{gH\}_{g \in G}$ for some subgroup $H \in G$. The subgroup can be chosen to be the stabilizer of any point $x \in X$.

We now state the *orbit-stabilizer theorem*.

Theorem 2. Given a G-set X and a point $x \in X$ there is a bijection of sets:

$$\mathrm{Orbit}(x) \times \mathrm{Stab}(x) \cong G.$$

Proof: Let H be the subgroup

$$H = \mathrm{Stab}(x).$$

Then G is partitioned into cosets $\{gH \mid g \in G\}$. We claim that this set of cosets is in bijection with the orbit of x, with bijection given by the relation

$$\begin{aligned} \mathrm{Orbit}(x) &\to \{gH \mid g \in G\} \\ gx &\mapsto gH. \end{aligned}$$

To show that this relation is actually a function, let $gx = hx$ for $g, h \in G$. Then $h^{-1}gx = x$ and so $h^{-1}g \in H$, and the cosets gH and hH are identical. {finish} ∎

Example 3. The rank-nullity theorem says that given a linear map on finite-dimensional vector spaces $A : V \to V$,

$$\mathrm{Dim}(\mathrm{Im}(A)) + \mathrm{Dim}(\mathrm{Ker}(A)) = \mathrm{Dim}(V).$$

Such a map gives a group action: it is the additive group of V acting on the set V by addition. That is, any $v \in V$ acts on $x \in V$ as $v : x \mapsto x + Av$. Now we see that given any $x \in V$ the stabilizer subgroup $\mathrm{Stab}(x)$ of this action is precisely the kernel of A. The orbit of x is x plus the image of A.

Working with a vector space over a finite field, we can take the cardinality of these sets as in the formula $|\mathrm{Orbit}(x)||\mathrm{Stab}(x)| = |G|$ and take the logarithm of this where the base is the size of the field and we get exactly the rank-nullity equation.

Over an infinite field this doesn't work and we need to think more along the lines of a categorified orbit-stabilizer theorem. {Does this really work?} In this case, for each $x \in V$ we can find a bijection:

$$\mathrm{Orbit}(x) \cong G/\mathrm{Stab}(x)$$

and this bijection gives us the First Isomorphism Theorem:

$$\mathrm{Im}(A) \cong V/\mathrm{Ker}(A).$$

{Question: Can we bootstrap this one more time in order to say something about the size of the homology groups in a chain complex? Perhaps thinking of a (length 2) chain complex as a 2-category?} ∎

For a G-set X, the *orbit space* is defined as the set of orbits:

$$G\backslash X := \{\mathrm{Orbit}(x)\}_{x \in X}.$$

The following is known as "Burnside's lemma".

Theorem 4. Given a G-set X,

$$|G\backslash X| = \frac{1}{|G|}\sum_{g\in G} |\mathrm{Fix}(g)|.$$

Proof: {todo} ∎

An action is *faithful* when for each $g \in G$ with $g \neq 1$ we have $\mathrm{Fix}(g) \neq X$. An action is *free* when for each $g \in G$ with $g \neq 1$ we have $\mathrm{Fix}(g) = \phi$.

{define a torsor}

A G-set X with only one orbit is called *simple* (or *transitive*, or *indecomposable*).

Given two G-sets X and Y, we define the sum $X + Y$ {...} and the product $X \times Y$ {...}

The following theorem shows that G-sets are semi-simple.

Theorem 5. For a G-set X, we have

$$X = \sum_{i=1}^{n} X_i$$

with X_i simple, and the summation is unique up to reordering.

Proof: Easy. ∎

The next theorem is a kind of "Schur's lemma for G-sets".

Theorem 6. Given G-sets X and Y:

(a) For $f : X \to Y$, we have $f(X)$ is a G-set, $f(X) \subseteq Y$.
(b) For $f : X \to Y$, and $X \neq \phi$, if Y is simple then f is surjective.
(c) For $f : X \to X$ with X simple, f is an automorpism.

Proof: Easy. ▮

2.1 The category of canonical orbits

Taken from [1, 4].

2.2 Double cosets

Given a map of sets $f : X \to Y$, that is G-*invariant* ie., $f(x) = f(gx)$ for all $g \in G, x \in X$, we can *lift* to a unique map $\tilde{f} : G\backslash X \to Y$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \\ & G\backslash X \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

where p is the projection map $p : X \to G\backslash X$. We use this observation in the proof of the following.

<u>**Proposition 7.**</u> Given a group G with subgroups H and K, the orbit-space of the G-set $G/H \times G/K$ is in bijection with the H, K double cosets:

$$G\backslash(G/H \times G/K) \cong \{HgK\}_{g \in G}.$$

Proof: We will show that the following relation

$$f : G/H \times G/K \to \{HgK\}_{g \in G}$$
$$f(aH, bK) := Ha^{-1}bK,$$

defines a G-invariant function that lifts to the required bijection on the orbits.

(i) That f defines a function: for any $h \in H, k \in K$, $f(aH, bK) = Ha^{-1}bK = H(h^{-1}a^{-1})(bk)K = f(ahH, bkK)$.

(ii) That f is G-invariant: $f(gaH, gbK) = Ha^{-1}g^{-1}gbK = f(aH, bK)$.

(iii) f is surjective, and so the lift $\tilde{f}$,

$$\begin{tikzcd} G/H \times G/K \arrow[r, "f"] \arrow[dr, "p"'] & Y \\ & G\backslash(G/H \times G/K) \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

is also surjective.

(iv) That $\tilde{f}$ is injective,

$$\begin{aligned} f(aH, bK) = f(cH, dK) &\text{ iff} \\ Ha^{-1}bK = Hc^{-1}dK &\text{ iff} \\ \exists h, k \in G,\ a^{-1}b = hc^{-1}dk &\text{ iff} \\ b = (ahc^{-1})dk &\text{ iff} \\ bK = gdK,\ aH = gcH, &\text{ where } g = ahc^{-1} \end{aligned}$$

And so (aH, bK) and (cH, dK) are in the same orbit. ▮

Proposition 8. {Not sure how this goes exactly...} Let G be a group, with subgroups H and K. Considered as an equivelance relation on the elements of G, the set of double cosets $\{HgK\}_{g\in G}$ is the finest mutual coarsening of the equivelance relations given by the right and left cosets, $\{Hg\}_{g\in G}$ and $\{gK\}_{g\in G}$. In other words, as equivelance relations we have

$$\{HgK\}_{g\in G} = \{Hg\}_{g\in G} \vee \{gK\}_{g\in G}.$$

Proof: {todo} ▮

2.3 Hecke operators

Let $\mathbb{C}[X]$ denote the complex vector space with basis X. Evidently, when X is a G-set, we get a $\mathbb{C}$-linear representation of G on $\mathbb{C}[X]$. A $\mathbb{C}$-linear representation of G obtained in this way we call a *permutation representation.*

Given a point $x \in X$ we denote the corresponding basis vector in $\mathbb{C}[X]$ as $|x\rangle$, and corresponding dual vector as $\langle x|$. We also denote generic vectors in $\mathbb{C}[X]$ by $|v\rangle$, $|u\rangle$, etc.

Given G-sets X and Y, and points $x \in X, y \in Y$ we define the *Hecke operator* as the linear operator

$$\mathbb{C}[X] \xrightarrow{r_{x,y}} \mathbb{C}[Y]$$

given by

$$r_{x,y} := \frac{1}{|\mathrm{Stab}(x)||\mathrm{Stab}(y)|} \sum_{g\in G} |gy\rangle\langle gx|.$$

Lemma 9. The Hecke operators are G-rep homomorphisms:

$$r_{x,y} \in \mathrm{Hom}_{\mathbf{GRep}}(\mathbb{C}[X], \mathbb{C}[Y]).$$

Proof: We need to show that $gr_{x,y}|v\rangle = r_{x,y}g|v\rangle$ for $|v\rangle \in \mathbb{C}[X], g \in G$. By linearity we need only consider this equation on basis vectors: $gr_{x,y}|x'\rangle = r_{x,y}g|x'\rangle$ for $x' \in X, g \in G$. Computing:

$$\begin{aligned}
\mathrm{LHS} &= g\sum_{h\in H} |hy\rangle \\
&= \sum_{h\in H} gh|y\rangle \\
\mathrm{RHS} &= \sum_{h\in G} |hy\rangle\langle hx|gx\rangle \\
&= \sum_{h\in G, hx=gx} |hy\rangle \\
&= \sum_{h\in H} gh|y\rangle.
\end{aligned}$$

{what is H?} ▮

Theorem 10. Given two permutation representation $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ of a group G, the Hecke operators

$$\{r_{x,y} \mid x \in X, y \in Y\}$$

form a basis for the linear space

$$\mathrm{Hom}_{\mathbf{GRep}}(\mathbb{C}[X], \mathbb{C}[Y]).$$

Proof: Let $f \in \mathrm{Hom}_{\mathbf{GRep}}(\mathbb{C}[X], \mathbb{C}[Y])$. Then for any $g \in G, x \in X, y \in Y$ we have:

$$\begin{aligned} gf|x\rangle &= fg|x\rangle \\ f|x\rangle &= g^{-1}fg|x\rangle \\ \langle y|f|x\rangle &= \langle y|g^{-1}fg|x\rangle \\ &= \langle gy|f|gx\rangle. \end{aligned}$$

ie., the matrix for f is constant on the orbits of $X \times Y$ and so f is a sum of Hecke operators. ∎

Corollary 11. Given a doubly transitive action $G \to \mathrm{Aut}(X)$ the permutation representation $\mathbb{C}[X]$ breaks into exactly two irreducible representations.

Proof: There are two Hecke operators corresponding to the diagonal matrix, and the off-diagonal matrix. The result follows by the previous theorem and Schur's lemma. ∎

3 Examples

4 Bibliographic notes

Group actions with applications to group theory: [2, 3].

See [4].

References

[1] G. E. Bredon. *Equivariant cohomology theories*, volume 34. Springer, 2006.

[2] K. Conrad. Group actions. `http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/gpaction.pdf`.

[3] K. Conrad. Transitive group actions. `http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/transitive.pdf`.

[4] A. Dress. *Notes on the theory of representations of finite groups, Part I: The Burnside ring of a finite group and some AGN-applications.* 1971.

[5] O. Lorscheid. F1 for everyone. *arXiv preprint arXiv:1801.05337*, 2018.

[6] D. Speyer. Invertible matrices of natural numbers are permutations. `https://mathoverflow.net/a/62126/39775`, 2011.