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6 23/7/19

Given a monad $T: C \to C$ construct the algebras for the monad

left adjoint .... the.

1 24/7/19

Example

theory of a cocategory object

syntactic category is category of small categories.

breaking the construction into stages:

primary operations:

domain, codomain, identity $M \xrightarrow{d} O$, $M \xrightarrow{c} O$, $O \xrightarrow{id} M$

secondary operations:

composition

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② 24/7/19

M_2: Monad on $\tilde{M}_1$-Alg

$$\begin{tikzcd} \tilde{M}_1\text{-Alg} \arrow[r, "\text{Free}"] & M_2\text{-Alg} \arrow[d, "\text{under}"] \\ & M_1\text{-Alg} \end{tikzcd}$$

Call this a <u>monad tower.</u>

We get a chain of adjunctions:

$$\begin{tikzcd} \tilde{C} \arrow[r, shift left, "L_1"] & M_1\text{-Alg} \arrow[l, shift left, "R_1"] \arrow[r, shift left, "L_2"] & M_2\text{-Alg} \arrow[l, shift left, "R_2"] & \cdots \end{tikzcd}$$

Question:

compose adjuncts to get composite monad:

$M_2 \circ M_1 := R_1 R_2 L_2 L_1$

Call this N.

right adjoint comparison functor

$$\tilde{M}_2\text{-Alg} \longrightarrow \tilde{N}\text{-Alg}$$

when this is an equivalence of categories, we call these monads composable

M_1: Monad on a

$\tilde{C}$: locally pres category

Eilenberg Moore category of algebras (Kleisli is subcategory) is also locally presentable ie. has a theory

$$\begin{tikzcd} \tilde{C} \arrow[r, "\text{Free}"] & \tilde{M}_1\text{-Alg} \arrow[d, "\text{Under}"] \\ & \tilde{C} \end{tikzcd}$$

the theory is: "take the sketch for $\tilde{C}$ and append co-M_1 algebra"

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Back to co-category example

Sketch

Object A

"arrows"

primary ops:

$A \xrightarrow{s} A$

$A \xrightarrow{t} A$

with

$s^2=s$, $t^2=t$,

$st=s$, $t=ts$

The theory of a cograph.

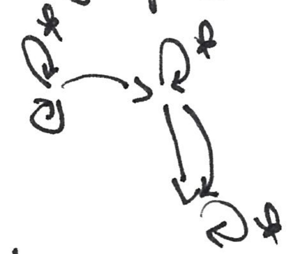
Syn Category

$\hat{C} = \hat{Set}$

The M_1 monad

Algebras for this monad are actions of the monoid $\{1, s, t\}$

ie. graphs



with vertices as degenerate special loop edges.

(simplicial sets with 0 & 1 di • objects)

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Sketch

$$\begin{tikzcd} \$A \arrow[r, "s"] \arrow[d, "t"'] & \$A \arrow[d] \\ \$A \arrow[r] & P \arrow[ul, phantom, "\ulcorner", very near start] & \\ & & *\$A \arrow[ul, "\text{composition}"'] \end{tikzcd}$$

\$ identity

\$ assoc

we have a morphism \$ into a colimit

Syntactic category

The M_2 monad on the category of graphs is the free category monad.

The M_2 algebras are the small categories.

$$\begin{tikzcd} \widehat{Set} \arrow[r, shift left, "L_1"] & \widehat{Graphs} \arrow[l, shift left, "R_1"] \arrow[r, shift left, "L_2"] & \widehat{SmallCats} \arrow[l, shift left, "R_2"] \end{tikzcd}$$

L_2L_1: free cat on free graph on a set of arrows.

$R_1R_2L_2L_1 : X \mapsto 3\cdot X$

this is the same as

$R_1L_1 : X \mapsto 3\cdot X$

So the comparison functor

$$M_2\text{-Alg} \longrightarrow N\text{-Alg}$$

fails to be an equivalence of categories.

Theorem you can't get this category as the algebras of any monad on $\widehat{Set}$.

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digression: What is a functor from a poset to a group?

Take

$$\begin{tikzcd} & \text{Small Groupoid} \arrow[d, "L"', "\text{freely invert morphisms}"] & \\ \text{Poset} \arrow[r, "\subseteq"] & \text{Small Cat} & \\ & \text{Small Groupoid} \arrow[u, "R"] & \end{tikzcd}$$

Daniel Kan references
Peter May references

Consider adjoint pairs

$$\text{Small Gpd} \xrightarrow[\subseteq]{R_G} \text{Sm Cat} \xrightarrow{L_G} \text{Sm Gpd}$$

$$\text{Poset} \xrightarrow[\subseteq]{R_P} \text{Small Cat} \xrightarrow{L_P} \text{Poset}$$

$$F = L_G R_P$$

Fundamental groupoid of a poset

F preserves connectedness of a small category.

Geometric realization of a POSET

the nerve of a poset is a simplicial sets defined as

For each composable n-chain of morphisms attach a n-simplex

eg 2

$$\begin{tikzcd} C & D \\ A \arrow[u] \arrow[ur] & B \arrow[u] \arrow[ul] \end{tikzcd}$$

as a groupoid:

$$\begin{tikzcd} C \arrow[r, shift left] \arrow[d, shift left] & D \arrow[l, shift left] \arrow[d, shift left] \\ A \arrow[u, shift left] \arrow[r, shift left] & B \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd}$$

just retain endomorphisms of A:

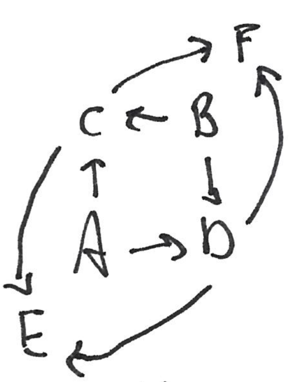
$$A^{D} \cong \mathbb{Z}$$

the homotopy group of the circle.

Homotopy theory

eg 3

$$\begin{tikzcd} E & F \\ C \arrow[u] \arrow[ur] & D \arrow[u] \arrow[ul] \\ A \arrow[u] \arrow[ur] & B \arrow[u] \arrow[ul] \end{tikzcd}$$



this is the suspension of eg3.
An octahedron (sphere).

eg.1

$$\begin{tikzcd} & A & \\ & B \arrow[u] & \\ C \arrow[ur] & & D \arrow[ul] \end{tikzcd}$$

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Example sketch with a tertiary operation

building on co-category sketch:

silly example:

(A) assign an arrow for each commuting square

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "*"'] & P \arrow[d] & \\
P \arrow[r] & S & \\
& & A \arrow[ul]
\end{tikzcd}
$$

relies on composition and so is a tertiary operation.

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7 24/7/19

Given a monad: $M : \check{C} \to \check{C}$

Keisli : Free M-algebras

$\downarrow$ inclusion

Eilenberg-Moore : all M-algebras

nPOV

The free M-algebra category is the 2-colimit of the monad.

The M-algebras ~~one~~ category is the 2-limit of the monad.

Kleisli category

freely introduces the Kleisli morphisms while preserving ~~coproduct~~ colimits but ~~does~~ may not itself be cocomplete, because of the new morphisms.

EM category

freely add colimits while preserving the old colimits.