James Dolan ① 12/7/19

Monad on a cocomplete category:
universal algebra
algebraic theory.
Underlying.
$R \cdot L$ = Free Group : $\check{Set} \to \check{Set}$
& unit : include generators
Exercise :
from any adjunction L, R
can build a monad.

$RL \otimes RL = RLRL$

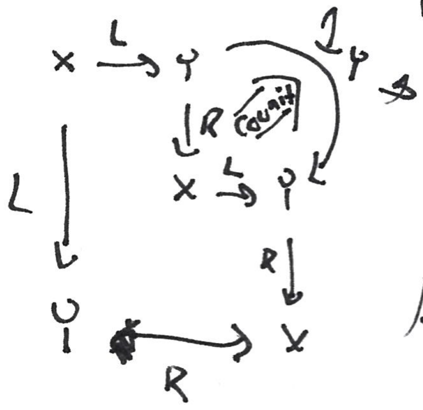
$\mu : RLRL \to RL$

unit $1_X \to RL$

counit $LR \to 1_Y$

Use the counit to construct μ.

$L : X \to Y$
$R : Y \to X$



Kliesli category:
category of free
Algebras on a monad

Eilenberg-Moore cat:
cat of all
algebras on a
monad.

Commutative monads on the tensor category of structure types.

1. Monads on Set

For a cat $\check{C}$

Monad on $\check{C}$ is a monoid object in the tensor category $End(\check{C})$ with composition as $\otimes$.

ie. $F : \check{C} \to \check{C}$

$\mu : F \otimes F \to F$ mul

$\eta : 1 \to F$ unit

Actions of a monoid.
monad
is an algebra of a monad
"riding"

Given a Monoidal category $\check{M}$ acting on a category $\check{C}$

A Monoid on $\check{M}$ gives action on objects of $\check{C}$.

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subject to cocompletion (preserving colimits)
Freely adjoin a morphism to the category of sets.

Eg. $p: 1 \to 0$

Sol'n take the believers, the objects that believe there is a map $p: 1 \to 0$.

A believer B is an object and a map

$$\hom(0, B) \xrightarrow{p_B} \hom(1, B).$$

So believers are the pointed sets.

(Not just a property of B, but extra structure on B.)

(Lemma the terminal object will believe anything. (is gullible).)

"Free algebra Functor"

$F: \hat{Set} \to \hat{Set+P} = \hat{Set}$.
that preserves colimits
(Left adjoints preserve colimits)

has same objects:

Kleisli morphism

~~Free$(S_1) \to$ Free(S_2)~~

~~ie $S_1 \to$ Free(~~

$S_1 \to S_2$

is

$S_1 \to M(S_2)$.

Composition:

$$S_1 \xrightarrow{a} S_2 \xrightarrow{b} S_3$$

ie.

$$S_1 \xrightarrow{a} M(S_2)$$

$$S_2 \xrightarrow{b} M(S_3)$$

$$S_1 \xrightarrow{a} M(S_2) \xrightarrow{M(b)} M(M(S_3)) \xrightarrow{\mu(M(S_3))} M(S_3)$$

$b \circ a$

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(To preserve the tensor structure we need a commutative monad.)

Example

In Cat freely adj

$a : 2 \to 2$

Soln

$p_B : B \times B \to B \times B$

a pair of binary operations.

$a : 2 \to 2$

$\Rightarrow a : 1 + 1 \to 2$

$\Rightarrow a_1 : 1 \to 2,\ a_2 : 1 \to 2.$

⇒ coloured trees.

Example

In ~~Cat^n~~ Set^n, freely adjoin

$a : 1 \to 2$

preserving colimits.

Soln:

$$\hom(2, B) \xrightarrow{a_B} \hom(1, B) = B$$

i.e. $B \times B \to B$

a binary operation on B.

$1 = \{g\}$

$1 \mapsto \mathrm{Free}(\{g\})$

$=$ 

$2 \mapsto \mathrm{Free}(\{g, h\})$

$= g,\ h,\ \bigwedge_{g\ g},\ \bigwedge_{g\ h},\ \bigwedge_{h\ g},\ \bigwedge_{h\ h}, \ldots$

then

$\mathrm{Free}(a) : \mathrm{Free}(1) \to \mathrm{Free}(2)$

$g \mapsto \bigwedge_{g\ h}$

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Example

freely adjoin $a : 1 \to 2$

s.t. cocomplete, ~~symmetric~~ tensor.

Sol'n

$B \times B \to B$

s.t.

$(a \times b) \times (c \times d)$

$= (a \times c) \times (b \times d)$

"Medial magma" gives a symmetric tensor cocomplete category.

Define structure types by freely adding object X.

Commutative monad on a cocomplete symmetric monoidal category.

$A \xrightarrow{C} B$

$D \xrightarrow{F} E$

$$\begin{array}{ccc} A \otimes D & \xrightarrow{C \otimes 1} & B \otimes D \\ {\scriptstyle 1_A \otimes F} \downarrow & & \downarrow {\scriptstyle 1_B \otimes F} \\ A \otimes E & \xrightarrow[C \otimes 1]{} & B \otimes E \end{array}$$

eg.

tensor functor

$(\times, \mathrm{Set}) \to (\otimes, \mathrm{Vec}^h)$

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Example

structure type:

$$X^2+2$$

freely add

$$X^2+2 \to 1$$

as an isomorphism
to create a categorified
complex numbers