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Algebraic geometry

algebra of dual numbers

Coalgebraic geometry

Cartesian closed world of coalgebras

$$\text{Ring} \xrightarrow{\text{Spectrum}} \Lambda$$

co-exponential

exponential

$\varepsilon^2=0$

the line

$$L^{\varepsilon} \cong TL$$

$$x, dx$$

$$\operatorname{Spec}(K[X]) = \text{Line}$$

$$\operatorname{Spec}\left(K[\varepsilon]/\varepsilon^2\right) = \text{walking tangent vector}$$

Commutative Hopf algebra

comultiplication is from group structure, multiplication geometry

algebra of functions

Cocommutative Hopf algebra

multiplication is from convolution group structure, comultiplication geometry

coalgebra of measures

both commutative:

is a Fourier dual pair of Abelian groups

See also: Lawvere extensive & intensive

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co exponential

cobase: $K[X]$ the theory of a quantity X.

coexponent: $K[T]/T^2 = \varepsilon$ the theory of a quantity T with $T^2 = 0$.

coexponential: the theory of: a ~~[illegible]~~ model of the cobase in the universe provided by the coexponent.

$$\boxed{\begin{aligned} \text{aside} \quad \varepsilon^{\varepsilon} &= \text{endomorphisms of } K[T]/T^2 \\ &= \text{solutions of } T^2=0 \text{ in } K[T]/T^2 \\ &\quad \text{models} \end{aligned}}$$

coexponent $\to$ B

cobase $\to$ A

A^B : the theory of an A model in the universe provided by B.

model = spectrum

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Example $A = K[x,y] / x^2+y^2=1$

$B = K[Z]$

$$A^B = \begin{array}{l} x \mapsto x_0, x_1, x_2 \dots \\ y \mapsto y_0, y_1, y_2 \dots \end{array}$$

s.t. $x_0^2 + y_0^2 = 1$

$x_1 y_0 + x_0 y_1 = 0$

$x_2 y_0 + x_1 y_1 + x_0 y_2 = 0$

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Example

$$A = K[X,Y]/\cancel{\ }XY=0$$

$$B = K[T]/T^2$$

$$A^B = K[X_0,X_1,Y_0,Y_1]/(X_0+X_1T)(Y_0+Y_1T)=0$$

$$X_0Y_0 = 0$$

$$X_1Y_0 + X_0Y_1 = 0$$

$$Y_0X_1 + X_0Y_1 = 0$$

singularities give rise to tangent vectors that "don't go anywhere" these are formal sums of tangent vectors that "do go somewhere". Walking tangent vector & also goes nowhere.

Zariski tangent space: linear hull of the tangent cone of a curve at a point.

Example

$$A = K[T]/T^2$$

$$B = K[T]/T^2$$

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More co-exponential examples

cobase: $A = K[x,y] / x^2+y^2=1 \xrightarrow{\text{spec}}$ the circle

coexponent: $B = K[T] / T^2$

coexponent: the theory of "$x^2+y^2=1$" in B.

$$x \mapsto x_0 + x_1 T$$
$$y \mapsto y_0 + y_1 T$$
$$1 \mapsto 1$$

$$(x_0 + x_1 T)^2 + (y_0 + y_1 T)^2 = 1$$

$$x_0^2 + 2x_0x_1T + y_0^2 + 2y_0y_1T = 1$$

$$K[x_0, x_1, y_0, y_1] \Big/ \left\{ x_0^2 + y_0^2 = 1,\ 2x_0x_1 + 2y_0y_1 = 0 \right\}$$

$$(x_0, y_0)\cdot(x_1, y_1) = 0$$

$$K[x, dx, y, dy] / x^2 + y^2 = 1$$

$$\cancel{2}x\,dx + \cancel{2}y\,dy = 0$$

tangent bundle projects onto

$$K[x, y] / x^2 + y^2 = 1$$