James Dolan ① 5/4/19

Dictionaries allow to talk about the deep thing without actually going there.

pseudo coexponential:

1) cobase: a commRing

2) pseudo-coexponent

an affine algebraic commRing (or CommAlgebra)

a commutative k-algebra object in the category of affine k-algebraic varieties.

algebraic object

geometric world

If $k = \mathbb{Z}$ then a k-algebra is a ring.

A bicomm Hopf ring is an abelian group object in the category of affine $\mathbb{Z}$-algebraic varieties.

That is:

Abelian cogroup object in the category of commutative rings

$A \to A \otimes A$

Theorem coproduct in CommRing (& CommAlg) is tensor product of modules

$$\frac{K[a, b, \ldots]}{\begin{matrix} p() \\ q() \end{matrix}} \otimes \frac{K[a', b' \ldots]}{\begin{matrix} p'() \\ q'() \end{matrix}} = \frac{K[a \otimes a', a \otimes b', \ldots]}{\begin{matrix} p(), \\ q_p() \ldots \end{matrix}}$$

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A morphism of affine k-algebra varieties — Geometry

the opposite of

A morphism of comm algebras — Algebra

Example

the line $\quad K[x]$

the line is $Spec(K[x])$

endomorphisms: $K[x] \to K[x]$

$1 \mapsto 1$

$x \mapsto p(x)$

for some polynomial p.

In general $f : \underbrace{K[x, y, \dots]}_{R} \to \underbrace{K[x', y', \dots]}_{R'}$

$x \mapsto p(x', y', \dots)$

$y \mapsto p(\quad)$

$\vdots$

looking for a solution of R in the world R'.

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$k[x] \to k[x,y] / x^2+y^2=1$

$x \mapsto p(x)$

"Gauge fix" a basis, or normal form (Grobner basis)

$\{1, x, xy, x^2, x^2y, x^3, x^3y, \dots\}$

Geometry

circle $\to$ line

real circle $\to$ complex line

periodic motion

Glitch: Non-compact

Affine varieties have missing points at infinity

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Example $K[x] \to K[x,y]$

$x \mapsto p(x,y)$

Geometrically: $(x,y) \mapsto p(x,y)$

plane $\to$ line

Example

$K[x,y] \to K[x]$

$x \mapsto p(x)$

$y \mapsto q(x)$

Geometrically: $x \mapsto (p(x), q(x))$

line $\to$ plane

Example

$K[x,y]/_{x^2+y^2=1} \to K[x]$

$x \mapsto p(x)$

$y \mapsto q(x)$

s.t. $p(x)^2 + q(x)^2 = 1$

Geometrically: Line $\to$ Circle

only constant functions

(GAGA fails because line is non-compact)

Example $K[x] \to K[x,y]/_{x^2+y^2=1}$

$x \mapsto p(x)$

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Example

Circle × Circle $\rightsquigarrow$ $K[x,y]/_{x^2+y^2=1} \otimes K[x,y]/_{x^2+y^2=1}$

"$K[\cos, \sin]$" $\otimes$ "$K[\cos, \sin]$"

$$= K[x_1, y_1, x_2, y_2] \Big/ {\scriptstyle x_1^2+y_1^2=1,\ x_2^2+y_2^2=1}$$

$$K[x,y]/_{x^2+y^2=1} \longrightarrow K[x_1, y_1, x_2, y_2] \Big/ {\scriptstyle x_1^2+y_1^2=1,\ x_2^2+y_2^2=1}$$

$$x \mapsto p(x_1, y_1, x_2, y_2)$$

$$y \mapsto q(x_1, y_1, x_2, y_2)$$

s.t. $p(\ )^2 + q(\ )^2 = 1$

$\sin(\theta_1+\theta_2)^2 + \cos(\theta_1+\theta_2)^2 = 1$

(Geometrically: group $S_1 \subset \mathbb{C}^\times$)

Same kind of Hopf algebras as affine algebraic groups.