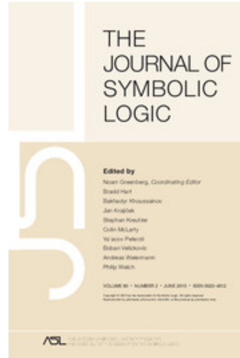

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INEQUIVALENT REPRESENTATIONS OF GEOMETRIC RELATION ALGEBRAS

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Abstract. It is shown that the automorphism group of a relation algebra $\mathfrak{B}_P$ constructed from a projective geometry P is isomorphic to the collineation group of P. Also, the base automorphism group of a representation of $\mathfrak{B}_P$ over an affine geometry D is isomorphic to the quotient of the collineation group of D by the dilatation subgroup. Consequently, the total number of inequivalent representations of $\mathfrak{B}_P$, for finite geometries P, is the sum of the numbers

$$\frac{|\mathrm{Col}(P)|}{|\mathrm{Col}(D)|\ /\ |\mathrm{Dil}(D)|},$$

where D ranges over a list of the non-isomorphic affine geometries having P as their geometry at infinity. This formula is used to compute the number of inequivalent representations of relation algebras constructed over projective lines of order at most 10. For instance, the relation algebra constructed over the projective line of order 9 has 56,700 mutually inequivalent representations.

The calculus of relations arose in the middle of the 19th century as the study of laws expressing properties of certain natural operations on binary relations. In modern terminology, it was the study of laws holding in full *algebras of relations*: algebras of the form

$$\mathfrak{R}(D) = \langle A, \cup, \sim, |, ^{-1}, I_D \rangle,$$

with a universe A consisting of all binary relations on a *base set* D, with the operations of forming the union $R \cup S$ of two relations R and S, the complement $\sim R$ of a relation R (with respect to the universal relation $D \times D$), the composition $R \mid S$ of two relations R and S,

$$R \mid S = \{(x, y) : \exists z[(x, z) \in R \text{ and } (z, y) \in S]\},$$

and the inverse (or converse) R^{-1} of a relation R,

$$R^{-1} = \{(y, x) : (x, y) \in R\},$$

and with I_D, the identity relation on D, as a distinguished constant. (The adjective "full" means the universe consists of *all* binary relations on the set D).

In 1941, Tarski [21] proposed an abstract approach to the subject, based on a finite number of postulates that eventually took the form of equations. The models of these postulates came to be called (abstract) *relation algebras*. Of course every

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concrete algebra of binary relations is a model of the postulates, and Tarski posed the fundamental problem: *Is every abstract relation algebra isomorphic to some* (not necessarily full) *algebra of binary relations*? A positive answer would imply that the set of postulates is semantically complete: every law true of all algebras of relations would be derivable from the postulates. Lyndon [14] showed, however, that the answer to the representation problem is negative: there are finite relation algebras that are not representable, that is, they are not isomorphic to any algebras of relations.

A number of years later, Jónsson [9] constructed examples of non-representable relation algebras in a step-by-step fashion from non-Desarguesian projective planes. Seizing upon this idea, Lyndon showed that by modifying Jónsson's construction somewhat, non-representable relation algebras could be obtained directly from non-Desarguesian planes without using a step-by-step process. In fact, starting with *any* projective geometry P, Lyndon's version of the construction yields a relation algebra $\mathfrak{B}_P$. Lyndon characterized when this algebra has a representation — an isomorphism onto an algebra of binary relations (in the infinite case, a complete representation — a representation in which a supremum of an infinite set of elements is always mapped to the union of the relations that form the images of the elements), and also the form that (complete) representations have to take.[1] A corollary of his general theorems is that $\mathfrak{B}_P$ has, essentially, a unique (complete) representation when the geometry P has dimension at least three or when it is a Desarguesian plane. It follows from Jónsson's argument that there is no representation when P is a non-Desarguesian plane. Some important results concerning representability and non-representability in the case when P is a projective line were established by Lyndon, but the general problem of determining the number of (complete) representations in this case was left unsettled.[2]

Historically, the algebras constructed over projective lines have been the most important of Lyndon's algebras. In fact, when people speak of "Lyndon algebras", they are usually referring to this special case. Monk [16] used them to show that the class RRA of representable relation algebras is not finitely axiomatizable. (Tarski [22] had shown earlier that this class is axiomatizable by a set of equations). Jónsson [10] used them to show that RRA cannot be axiomatized by any set of equations that involves only finitely many variables. More recently, classes of these algebras have been the objects of study in investigations regarding decision problems (see Andréka-Givant-Németi [1] and Stebletsova-Venema [19]), axiomatizability (see Givant [6] and Stebletsova [18]), and inequivalent (non-base-isomorphic) representations (see Hirsch-Hodkinson [8]).

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[1]The assumption that the representation is complete is needed when the projective geometry P is infinite, and is therefore inserted parenthetically.

[2]The typescript of a preliminary version of [15] contains the following statement as part of Theorem 1.

> "If $d = 1$ [if the dimension is one], $A(G)$ [in our notation, $\mathfrak{B}_P$] has a representation if and only if there exists a projective plane of the same order as G, and the equivalence classes of representations are in one-to-one correspondence with the isomorphism classes of projective planes of this order."

Evidently, Lyndon believed at one point that, when P is a projective line of order n, one could count the number of inequivalent representations of $\mathfrak{B}_P$ by counting the number of non-isomorphic projective planes of order n. He discovered the error and corrected it in [15], but the corrected version does not give a usable algorithm.

The present paper derives a simple and general formula for determining the representations of Lyndon's algebras constructed over any finite projective geometry. Of course, every representation can be replicated arbitrarily often by using a bijection to replace the elements of the base set with new elements. Such representations should be treated as being essentially the same — in technical jargon, they are *equivalent*. The real problem is to count the number of inequivalent representations, and this is what the formula does. The formula is used to compute the number of inequivalent representations of each algebra $\mathfrak{B}_P$ over a projective line P of order at most ten. This is the largest order for which the requisite information about affine planes is currently known. The calculation of the number of representations of $\mathfrak{B}_P$ for the projective line P of order nine illustrates the intricacy of the geometric questions that must be answered in order to carry out the computation.

It is also our intention to give a somewhat different perspective on some of Lyndon's original results. For that reason, we present proofs of a few of his theorems in a way that will, hopefully, provide insight into his original motivations.

There is a close connection between relation algebras and first-order logic. It may be of some interest to the reader to get a sense of the logical content of the results presented in this paper. Let P be a finite projective geometry of order at least three. The algebra $\mathfrak{B}_P$ associated with P is an algebraic version of a specific theory T_P of binary relations constructed from the geometry P in an appropriate first-order language with equality. The non-logical symbols of T_P are binary predicates R_p, one for each point p in P. The axioms of T_P say that the predicates denote symmetric, irreflexive relations, and every pair of distinct elements is in exactly one of these relations. Finally, for any two distinct points p and q of the geometry, the following sentences are also axioms (where Γ is the set of points collinear with, but different from, p and q):

$$\forall x \forall y (\exists z [R_p(x,z) \wedge R_q(z,y)] \longleftrightarrow \bigvee_{r \in \Gamma} R_r(x,y)),$$
$$\forall x \forall y (\exists z [R_p(x,z) \wedge R_p(z,y)] \longleftrightarrow R_p(x,y) \vee x = y).$$

The formula for computing the number of inequivalent representations of the algebra $\mathfrak{B}_P$ is really a formula for computing the number of non-isomorphic models of the theory T_P.

The paper is organized as follows. Section 1 surveys the general geometric notions and results that are need throughout the paper. Section 2 studies algebras $\mathfrak{A}_D$ of binary relations constructed from affine geometries D — so-called *affine algebras* — that are the basis of Lyndon's *affine* representations of $\mathfrak{B}_P$. Lyndon proved that every (complete) representation of $\mathfrak{B}_P$ is equivalent to an affine representation. Thus, to count the number of inequivalent (complete) representations of a geometric relation algebra, up to equivalence, it suffices to count the number of inequivalent affine representations. Lyndon also proved that $\mathfrak{B}_P$ has an affine representation over D — that is, $\mathfrak{B}_P$ is isomorphic to $\mathfrak{A}_D$ — just in case P is isomorphic to the geometry at infinity of D. Assume, therefore, that P is the geometry at infinity of D, and that φ is an isomorphism from $\mathfrak{B}_P$ to $\mathfrak{A}_D$.

In Section 2.1 it is shown that the automorphism group of $\mathfrak{A}_D$ is isomorphic to the collineation group of P, in symbols,

$$\text{(1)} \qquad \operatorname{Aut}(\mathfrak{A}_D) \cong \operatorname{Col}(P).$$

(An abstract version of this theorem replaces $\mathfrak{A}_D$ with $\mathfrak{B}_P$ — see Section 3.) In Section 2.2, it is proved that the subgroup of automorphisms of $\mathfrak{A}_D$ induced by bijections of the base set D — so-called *base automorphisms* — is isomorphic to the quotient of the group of collineations of D by the subgroup of dilatations of D, in symbols,

$$\mathrm{Aut}_{\mathrm{b}}(\mathfrak{A}_D) \cong \mathrm{Col}(D) \,/\, \mathrm{Dil}(D). \tag{2}$$

Lyndon's definition of the abstract relation algebra $\mathfrak{B}_P$ constructed from the projective geometry P is given in Section 3. It is proved in Section 3.1 that the affine representations of $\mathfrak{B}_P$ over D are precisely the mappings $\sigma \circ \varphi$, where σ ranges over $\mathrm{Aut}(\mathfrak{A}_D)$, and that two such mappings, $\sigma \circ \varphi$ and $\pi \circ \varphi$, are equivalent just in case $\pi \circ \sigma^{-1}$ is in $\mathrm{Aut}_{\mathrm{b}}(\mathfrak{A}_D)$. Thus, the number of inequivalent affine representations of $\mathfrak{B}_P$ over D is just the number of cosets of $\mathrm{Aut}_{\mathrm{b}}(\mathfrak{A}_D)$ in $\mathrm{Aut}(\mathfrak{A}_D)$:

$$|\mathrm{Aut}(\mathfrak{A}_D) : \mathrm{Aut}_{\mathrm{b}}(\mathfrak{A}_D)|. \tag{3}$$

Together, (1)–(3) yield the formula

$$\frac{|\mathrm{Col}(P)|}{|\mathrm{Col}(D)| \,/\, |\mathrm{Dil}(D)|} \tag{4}$$

for computing the number of inequivalent affine representations of $\mathfrak{B}_P$ over D, for finite P. To obtain the total number of inequivalent representations of $\mathfrak{B}_P$, term (4) is summed over the (mutually non-isomorphic) affine geometries D having P as their geometry at infinity.

This formula is used to compute, in Table 2, the number of inequivalent representations of $\mathfrak{B}_P$ for the projective lines P of order at most ten. The calculations when the order is different from nine are not difficult: there is at most one affine plane for each of these orders — the Desarguesian plane — and the requisite geometric information about its group of collineations and subgroup of dilatations is well known from the literature. (This information is summarized at the beginning of Section 4 for the convenience of the reader.) For the projective line of order nine, there are seven affine planes derived from the following four well-known projective planes of order nine: the Desarguesian plane over the field of order nine, the plane over the unique non-commutative near-field of order nine, the dual of the preceding plane, and the Hughes plane of order nine. In the case of the last three planes, it takes a fair amount of work to extract the requisite information about collineations and dilatations of the derived affine planes from the results in the literature about collineations of the projective planes. This is done for the three planes in Sections 4.2, 4.3, and 4.4 respectively.

As regards incomplete representations of infinite geometric relation algebras, it is shown in Section 3.1 that the existence of an incomplete representation implies the existence of a complete one. In Section 3.2 it is proved that the existence of a representation implies the existence of as many inequivalent incomplete representations as there are sets.

§1. Geometry. A *geometry* is usually conceived as a two-sorted structure consisting of points, lines, and a relation of incidence between points and lines. In dimensions higher than two, there may be a third sort consisting of planes. A set of points is *collinear* if the points all lie on one line, and *coplanar* if they all lie in one plane. Two lines are said to be *parallel* if they are coplanar and are either equal or have no point of intersection. A *four-point* (or a *quadrangle*) is a set of four points, no three of which are collinear.

An *affine geometry* is a geometry satisfying the following postulates:

(1) Through any two (distinct) points there passes one and only one line.

(2) Through a given point p not on a line ℓ, there passes one and only one line that is parallel to ℓ and coplanar with it.

(3) Every line has at least two points.

(4) Through three non-collinear points there passes one and only one plane.

(5) Two lines parallel to a third line are parallel to each other.

The unique line through two points p and q is usually denoted by pq. All lines have the same number of points, and this number is called the *order* of the geometry. The relation of parallelism between lines is an equivalence relation, and the equivalence classes are called *parallel classes* (of lines). Parallel classes are an abstract way of talking about *direction*. In an affine plane of order n, there are $n + 1$ parallel classes, each parallel class has n lines, and altogether there are n^2 points. Also, associated with each affine geometry is a uniquely determined number called its *dimension*.

A *projective geometry* is a geometry that satisfies the following postulates:

(1) Through any two points there passes one and only one line.

(2) For distinct points p, q, s and t, if lines pq and st intersect in a point, then so do lines ps and qt.

(3) Every line has at least three points.

Essentially, axiom (2) says that two coplanar lines always intersect. The notions of order and dimension are also defined for projective geometries. For instance, in a projective geometry of order n, every line has (by definition) $n + 1$ points, and every point is incident with $n + 1$ lines. In a projective plane of order n, there are $n^2 + n + 1$ points and the same number of lines.

Every affine geometry D can be extended to a projective geometry D^* of the same order and dimension (see [20], pp. 158–159). For each parallel class Φ of lines in D, we add a new "point at infinity" that lies on all the lines in Φ, and for each parallel class of planes, we add a new "line at infinity" that lies on each of the parallel planes. We shall refer to D^* as the *projective extension of* D. It is uniquely determined up to isomorphisms that leave D fixed. The set of points and lines at infinity constitute, in themselves, a projective geometry — the "geometry at infinity". It does not matter how the points at infinity are chosen (as long as they are not points of D). For the sake of concreteness, and to simplify notation, we shall assume that the parallel class Φ is itself chosen as the point at infinity on each of the lines in Φ.

An *isomorphism*, or *collineation*, between two geometries is a bijection of points to points and lines to lines that preserves the relation of incidence. Equivalently, it is a bijection of points to points that preserves the (ternary) relation of collinearity. Collineations of a geometry to itself are sometimes called *automorphisms* of the geometry. Various types of collineations play an important role in this paper. For instance, a *dilatation* is a collineation (or automorphism) of an affine geometry that maps each line to a parallel line.

The following extension theorem will be used several times in this paper. (See, for example, [20], p. 103, Theorem 3.3.1 for a statement and proof of the planar case.)

THEOREM 1. *Every collineation between affine geometries extends in a unique way to a collineation between their projective extensions.*

PROOF. Let σ be a collineation between two affine geometries D and D'. Then σ must map parallel lines to parallel lines (since it preserves coplanarity, intersection, and non-intersection). So σ maps the lines of a parallel class Φ of D bijectively to the lines of a parallel class Ψ of D'. Define $\hat{\sigma}$ to be the extension of σ determined by setting

$$\hat{\sigma}(\Phi) = \Psi = \{\sigma(\ell) : \ell \in \Phi\}.$$

It is a simple matter to check that this extension maps the set of points at infinity bijectively to itself. It remains to check that the extension preserves collinearity.

First of all, suppose Φ is collinear with the distinct affine points p and q. In other words, suppose Φ is the point at infinity on the affine line pq. Then the line pq is in Φ. If σ maps the lines of Φ to the lines of Ψ, then the line $\sigma(pq)$ is in Ψ. Hence, Ψ will be the point at infinity on the line $\sigma(pq)$. Thus, the points $\hat{\sigma}(\Phi)$, $\sigma(p)$, and $\sigma(q)$ will be collinear, by definition of $\hat{\sigma}$. (Conversely, if points $\hat{\sigma}(\Phi)$, $\sigma(p)$, and $\sigma(q)$ are distinct and collinear, then so are points Φ, p, and q.)

Suppose now that Δ, Φ, and Ψ are three collinear points at infinity. Then there is an affine plane in D whose line at infinity contains all three points. Let k, ℓ, and m be three lines in this plane that are in the parallel classes Δ, Φ, and Ψ respectively. Because these lines are coplanar (and collineations preserve coplanarity), the images $\sigma(k)$, $\sigma(\ell)$, and $\sigma(m)$ will also be coplanar — say, they all lie on a plane π of D'. Then the points at infinity on these image lines — the parallel classes that contain them — will be collinear: they will all lie on the line at infinity of the plane π. By definition, these parallel classes must be $\hat{\sigma}(\Delta)$, $\hat{\sigma}(\Phi)$, and $\hat{\sigma}(\Psi)$. Thus, $\hat{\sigma}$ preserves collinearity on lines at infinity as well.

Finally, suppose τ is a collineation of the projective extension of D that agrees with σ on D. Since the parallel class Φ is the point at infinity of each affine line in Φ, its image $\tau(\Phi)$ must be the point at infinity on the image of each affine line in Φ. In other words, its image must be the parallel class $\{\tau(\ell) : \ell \in \Phi\}$. But then

$$\tau(\Phi) = \{\tau(\ell) : \ell \in \Phi\} = \{\sigma(\ell) : \ell \in \Phi\} = \hat{\sigma}(\Phi),$$

since τ and σ agree on D. ⊣

We will usually use the same symbol to denote an affine collineation and its unique projective extension. Notice that the projective extension of an affine collineation (when restricted to the geometry at infinity) is a collineation between the geometries at infinity. Thus, every affine collineation has a natural extension that yields a collineation of the geometries at infinity.

The last sections of the paper contain a presentation of the other, more specialized facts about affine and projective collineations that will be needed in the examples of this paper.

§2. Affine relation algebras. Let D be an affine geometry. In terms of D, define a complete, atomic subalgebra of the full algebra of relations over D as follows. The atoms of the algebra are the identity relation I_D and relations defined in terms of parallel classes (of lines). For each parallel class Φ, set

$$R_\Phi = \{(p, q) : p, q \text{ are distinct points of } D \text{ and } pq \in \Phi\}$$

(where pq denotes the unique line through the two points p and q). For distinct points p and q in D, there is one and only one parallel class that contains the line pq. Thus, the relations R_Φ are pairwise disjoint, and disjoint from I_D, and their union, together with I_D, is the universal relation $D \times D$. It follows that they automatically form the atoms of a complete, atomic Boolean set algebra whose universe A_D consists of arbitrary unions of the atomic relations. Notice that

$$R_\Phi^{-1} = R_\Phi,$$

since a line pq is in Φ just in case qp (the same line) is in Φ. The operation of forming inverses is therefore the identity operation on A_D. The operation of relational composition between two atomic relations (different from the identity relation) is determined by the following theorem.

THEOREM 2. *Suppose Φ and Ψ are distinct parallel classes.*

(i) $R_\Phi \mid R_\Phi = I_D \cup R_\Phi$ *when D has order different from* 2.
(ii) $R_\Phi \mid R_\Phi = I_D$ *when D has order* 2.
(iii) $R_\Phi \mid R_\Psi = \bigcup\{R_\Delta : \Delta$ *is collinear with and distinct from* $\Phi, \Psi\}$.

PROOF. Let Φ and Ψ be parallel classes, and p and q two points in D. By the parallel postulate, there is a unique line ℓ in Φ through p. Similarly, there is a unique line in Ψ through q. The pair (p, q) is in $R_\Phi \mid R_\Psi$ just in case the lines ℓ and m intersect in a point different from p and q. Indeed, the existence a point s different from p and q such that ps is in Φ and sq is in Ψ is equivalent (by the uniqueness of the lines ℓ and m) to the existence of a point s different from p and q such that ps coincides with ℓ, and sq with m.

Suppose the parallel classes (or, equivalently, the points at infinity) Φ and Ψ are equal. Then the lines ℓ and m are parallel. Hence, they intersect just in case they are equal, that is, just in case either $p = q$ or else $p \neq q$ and $pq = \ell = m$. In the first case, the line ℓ always has a point s different from p (since every line has at least two points), so the (equal) lines ℓ and m do intersect in a point different from p and q. In the second case, there is such a point s just in case lines have at least three points, that is, just in case the order of the geometry D is at least 3. This establishes parts (i) and (ii) of the theorem.

Assume, now, that the parallel classes (points at infinity) Φ and Ψ are distinct. Let k be the unique line at infinity passing through these two points. The affine lines ℓ and m certainly are not parallel (they belong to different parallel classes), so they intersect if and only if they are coplanar (and in fact coplanar with the line k). Treat first the case when the pair (p, q) is in the composition $R_\Phi \mid R_\Psi$. Then there is a point s on the intersection of the two lines ℓ and m that is distinct from p and q. The points s, p, and q cannot be collinear (otherwise, we would have $\ell = sp = sq = m$ and hence $\Phi = \Psi$). So they determine a projective plane containing the three lines pq, $sp = \ell$, and $sq = m$. The plane contains the points at infinity on the lines ℓ and m, namely Φ and Ψ, and therefore also the line through these points, namely k. Since pq, ℓ, and m are not affinely parallel, the projective line pq intersects the line at infinity k in a point Δ that is different from Φ and Ψ. By definition, the line pq is in the parallel class Δ, so the pair (p, q) is in R_Δ. This establishes the inclusion from left to right in (iii).

For the reverse inclusion, suppose the pair (p, q) is in the relation R_Δ for some parallel class Δ collinear with, but different from, Φ and Ψ. Then the points p and q are distinct, by definition of the relation R_Δ, and the lines ℓ, m, and pq are coplanar and pairwise not affinely parallel. (In more detail, the line k — through the points at infinity Φ, Ψ,and Δ — and the affine point p lie in a projective plane. Since this plane contains the points Δ and p, it must contain the collinear point q. Since it contains p and Φ, it must contain all the points of ℓ. Since it contains q and Ψ, it must contain all the points of m. The lines are not affinely parallel, because their points at infinity are different). In particular, the lines ℓ and m, being not affinely parallel, must intersect in a unique (affine) point s. This point of intersection cannot equal p or q, for this would force the line pq to coincide either with m or ℓ. Consequently, the pair (p, q) is in the composition $R_\Phi \mid R_\Psi$. ⊣

It is worth pointing out that, in case D is an affine *plane*, the formula in (iii) takes the form

$$R_\Phi \mid R_\Psi = D \times D \sim (R_\Phi \cup R_\Psi \cup I_D).$$

It follows from the theorem (and some trivial remarks regarding the identity relation) that the relational composition of two atoms of A_D is a relation in A_D. Relational composition distributes across arbitrary unions, and the set A_D is closed under arbitrary unions. Therefore, A_D must be closed under arbitrary relational compositions. In particular, it is a (complete) subuniverse of the full algebra of relations over the set D. The corresponding algebra $\mathfrak{A}_D$ is therefore a complete, atomic algebra of relations — a complete subalgebra of $\mathfrak{R}(D)$. We shall call it the *affine relation algebra over* D. The description in Theorem 2 of the relational composition operation involves the (projective) geometry at infinity, but the definition of $\mathfrak{A}_D$ depends only on D, and not on the geometry at infinity. The formulas in Theorem 2 show that A_D is closely related to the modular lattice of subspaces of the geometry at infinity (of D).

It should be pointed out that the definition of the algebra $\mathfrak{A}_D$ and the statement of Theorem 2 are both implicit in [15].

2.1. Isomorphisms between affine relation algebras. We now approach the main task of this paper: a description of the isomorphisms between affine relation algebras. Fix two affine geometries D and D'.

THEOREM 3. *Every collineation τ between the geometries at infinity of D and D' induces a unique isomorphism σ between the algebras $\mathfrak{A}_D$ and $\mathfrak{A}_{D'}$ via the condition*

(i) $$\sigma(R_\Phi) = R_\Psi \qquad \textit{if and only if} \qquad \tau(\Phi) = \Psi.$$

PROOF. Suppose τ is a collineation between the geometries at infinity of D and D'. Certainly, the geometries at infinity — and hence also D and D' — must have the same order. (The order is one less than the number of points on a line, and a collineation preserves the number of points on a line.) By convention, the points at infinity are the parallel classes. Thus, τ is a bijection of the parallel classes of D to the parallel classes of D'. Condition (i) therefore makes sense, and, together with the condition

$$\sigma(I_D) = I_{D'},$$

defines a bijection σ from the atoms of $\mathfrak{A}_D$ to the atoms of $\mathfrak{A}_{D'}$. The elements of affine relation algebras are the (disjoint) unions of the atomic relations. Therefore, if σ is extended to the entire universe of $\mathfrak{A}_D$ by requiring it to preserve arbitrary unions — for any collection X of atoms of $\mathfrak{A}_D$,

$$\sigma(\bigcup X) = \bigcup\{\sigma(S) : S \in X\},$$

then this extension takes the elements of $\mathfrak{A}_D$ bijectively to the elements of $\mathfrak{A}_{D'}$ and preserves union and complementation. In other words, it is a Boolean isomorphism. By definition, it preserves the identity relation. The operation of inversion in an affine relation algebra is the identity operation, so it is certainly preserved by σ.

It remains to show that σ preserves relational composition. Because σ preserves arbitrary unions, it suffices to show that the relational composition of two atomic relations is preserved. This verification is trivial when one of the atomic relations is the identity. *For the other cases, suppose Φ and Θ are distinct parallel classes of D*, and

(1) $$\tau(\Phi) = \Psi, \qquad \tau(\Theta) = \Omega.$$

Then, by Theorem 2,

(2) $$R_\Phi \mid R_\Theta = \bigcup\{R_\Delta : \Delta \text{ is collinear with and distinct from } \Phi, \Theta\},$$

and, since Ψ and Ω are also distinct,

(3) $$R_\Psi \mid R_\Omega = \bigcup\{R_\Xi : \Xi \text{ is collinear with and distinct from } \Psi, \Omega\}.$$

From (2) and the definition of σ it follows that

(4) $$\sigma(R_\Phi \mid R_\Theta) = \bigcup\{\sigma(R_\Delta) : \Delta \text{ is collinear with and distinct from } \Phi, \Theta\}.$$

The image of R_Δ under σ is the relation R_Ξ, where $\tau(\Delta) = \Xi$. Because τ is a collineation of the geometry at infinity, it maps points Δ collinear with and distinct from Φ and Θ to points Ξ collinear with and distinct from Ψ and Ω (see (1)). Thus, by (4),

(5) $$\sigma(R_\Phi \mid R_\Theta) = \bigcup\{R_\Xi : \Xi \text{ is collinear with and distinct from } \Psi, \Omega\}.$$

The right-hand sides of equations (3) and (5) are equal, so

$$\sigma(R_\Phi \mid R_\Theta) = R_\Psi \mid R_\Omega = \sigma(R_\Phi) \mid \sigma(R_\Theta).$$

(The second equality uses (1) and the definition of σ in (i).)

It must also be verified that σ preserves the relational composition of an atom (different from the identity relation) with itself. If D has order $d > 2$, then so does D', and

$$R_\Phi \mid R_\Phi = \bigcup\{R_\Phi, I_D\}, \qquad R_\Psi \mid R_\Psi = \bigcup\{R_\Psi, I_{D'}\},$$

by Theorem 2. Therefore,

$$\begin{aligned}\sigma(R_\Phi \mid R_\Phi) &= \bigcup\{\sigma(R_\Phi), \sigma(I_D)\}\\ &= \bigcup\{R_\Psi, I_{D'}\}\\ &= R_\Psi \mid R_\Psi\\ &= \sigma(R_\Phi) \mid \sigma(R_\Phi),\end{aligned}$$

by the definition of σ. The case when D has order 2 is even simpler, and is left to the reader.

To prove uniqueness, suppose σ' is any isomorphism between the affine relation algebras that satisfies condition (i). Then σ and σ' agree on the atoms of $\mathfrak{A}_D$, so they must agree everywhere. ⊣

The converse of the previous theorem is also true: every isomorphism between two affine relation algebras induces a collineation between the geometries at infinity.

THEOREM 4. *Every isomorphism σ between the relation algebras $\mathfrak{A}_D$ and $\mathfrak{A}_{D'}$ induces a collineation τ between the geometries at infinity of D and D' via the condition*

$$\text{(i)} \qquad \tau(\Phi) = \Psi \quad \textit{if and only if} \quad \sigma(R_\Phi) = R_\Psi.$$

PROOF. Let σ be any isomorphism between the affine relation algebras $\mathfrak{A}_D$ and $\mathfrak{A}_{D'}$. Then σ certainly maps the atoms of the first algebra bijectively to the atoms of second, and it maps the identity relation on D (which is an atom of the first algebra) to the identity relation on D' (an atom of the second algebra). In particular, for every parallel class Φ of D (that is, for every point Φ at infinity) there is a unique parallel class Ψ of D' such that

$$\sigma(R_\Phi) = R_\Psi.$$

Thus, the correspondence τ determined by (i) is well defined. Because σ maps the non-identity atoms of $\mathfrak{A}_D$ bijectively to the non-identity atoms of $\mathfrak{A}_{D'}$, the mapping τ is a bijection between the points at infinity of D and D'.

It remains to check that τ preserves collinearity. Suppose Φ, Θ, and Δ are distinct parallel classes — points at infinity of D — and

$$\text{(1)} \qquad \sigma(R_\Phi) = R_\Psi, \qquad \sigma(R_\Theta) = R_\Omega, \qquad \sigma(R_\Delta) = R_\Xi.$$

Then

$$\text{(2)} \qquad \tau(\Phi) = \Psi, \qquad \tau(\Theta) = \Omega, \qquad \tau(\Delta) = \Xi,$$

by definition of τ, and these three parallel classes are also distinct. The point Δ, being distinct from Φ and Θ, is collinear with them just in case

$$\text{(3)} \qquad R_\Delta \subseteq R_\Phi \mid R_\Theta,$$

by Theorem 2 (using also the fact that distinct atomic relations are pairwise disjoint). Since σ is a relation algebraic isomorphism, equation (3) holds just in case

$$\sigma(R_\Delta) \subseteq \sigma(R_\Phi) \mid \sigma(R_\Theta).$$

This inclusion may be rewritten as

$$\text{(4)} \qquad R_\Xi \subseteq R_\Psi \mid R_\Omega,$$

by (1). Inclusion (4) is equivalent to the point Ξ being collinear with Ψ and Ω, by Theorem 2. In view of (2), this means that (4) is equivalent to the point $\tau(\Delta)$ being collinear with $\tau(\Phi)$ and $\tau(\Theta)$. Putting all of these equivalences together: the collinearity of the three points Φ, Θ, and Δ is equivalent to the collinearity of the points $\tau(\Phi)$, $\tau(\Theta)$, and $\tau(\Delta)$, as was to be proved. ⊣

A collineation τ between the geometries at infinity of D and D' induces an isomorphism σ_τ between the affine relation algebras $\mathfrak{A}_D$ and $\mathfrak{A}_{D'}$, by Theorem 3. Conversely, an isomorphism σ between the affine relation algebras induces a collineation τ_σ between the geometries at infinity, by Theorem 4. Conditions (i) in the statements of the theorems make clear that if we start with a collineation τ between the geometries at infinity, pass to the induced isomorphism σ_τ between the affine relation algebras, and then pass to the induced collineation τ_{σ_τ} between the geometries at infinity, we end up where we started:

$$\tau_{\sigma_\tau} = \tau.$$

Similarly, if we start with an isomorphism σ between the affine relation algebras, pass to the induced collineation τ_σ between the geometries at infinity, and then pass to the induced isomorphism σ_{τ_σ} between the affine relation algebras, we end up where we started:

$$\sigma_{\tau_\sigma} = \sigma.$$

COROLLARY 5. *The isomorphisms between affine relation algebras are precisely the mappings induced by (projective) collineations between the corresponding geometries at infinity.*

The next theorem applies these various observations to automorphisms of affine relation algebras.

THEOREM 6. *The correspondence taking each collineation τ of the geometry at infinity of D to the induced automorphism σ_τ of the affine relation algebra $\mathfrak{A}_D$ is an isomorphism from the group of collineations of the geometry at infinity to the group of automorphisms of $\mathfrak{A}_D$. Its inverse is the correspondence taking each automorphism σ of $\mathfrak{A}_D$ to the induced collineation τ_σ of the geometry at infinity.*

PROOF. In view of the remarks preceding Corollary 5, the correspondence taking the collineation τ to the induced automorphism σ_τ is a bijection from the group of collineations of the geometry at infinity of D to the group of automorphisms of the affine relation algebra $\mathfrak{A}_D$; its inverse is the correspondence taking an automorphism σ of $\mathfrak{A}_D$ to the induced collineation τ_σ of the geometry at infinity. It remains to prove that this correspondence preserves the operation of composition between collineations. Suppose τ and γ are collineations of the geometry at infinity. Then

$$\sigma_{\tau\circ\gamma}(R_\Phi) = R_{\tau\circ\gamma(\Phi)} = R_{\tau(\gamma(\Phi))} = \sigma_\tau(R_{\gamma(\Phi)}) = \sigma_\tau(\sigma_\gamma(R_\Phi)),$$

by condition (i) of Theorem 3. This shows that the isomorphisms $\sigma_{\tau\circ\gamma}$ and $\sigma_\tau\circ\sigma_\gamma$ agree on atoms. Consequently, they agree everywhere. ⊣

Here are some examples that illustrate the last theorem. Suppose D is an affine plane of order n. Then its geometry at infinity is a projective line with $n+1$ points — the $n+1$ parallel classes of D. The collineations of the geometry at infinity are arbitrary permutations of the $n+1$ parallel classes. (Any such permutation automatically preserves collinearity: there is only one line at infinity.) Therefore, the group of automorphisms of the affine relation algebra $\mathfrak{A}_D$ is canonically isomorphic to the group of all permutations of the $n+1$ parallel classes. In particular, it has $(n+1)!$ elements.

For a second example, suppose D is a 3-dimensional affine geometry of finite order n. Then D satisfies the Desarguesian property. Consequently, its geometry at infinity is a projective plane P that satisfies the Desarguesian property. In particular, P can be coordinatized by a finite field F of prime power cardinality n. If n is the kth power of a prime, then the group of collineations of the plane P — and hence also the group of automorphisms of the affine relation algebra $\mathfrak{A}_D$ — has cardinality

$$k \cdot n^3 \cdot (n^3 - 1) \cdot (n^2 - 1)$$

(see Theorem 21). A similar argument leads to an analysis of the size of the group of automorphisms of the affine relation algebra over any affine geometry of finite order and finite dimension at least three. (See, for example, [5], p. 31, item 11, for an analysis of the size of the collineation group of an arbitrary Desarguesian projective geometry.)

2.2. Base isomorphisms. Suppose δ is a bijection between sets D and D'. Then δ induces a natural bijection σ between the binary relations on D and those on D':

$$\sigma(S) = \{(\delta(p), \delta(q)) : (p, q) \in S\}$$

for every binary relation S on D. This induced bijection of relations preserves all set-theoretically defined operations on relations. In particular, it preserves arbitrary unions, complementation, relational composition, and inversion, and it preserves the identity relation. If $\mathfrak{B}$ and $\mathfrak{B}'$ are algebras of relations on the base sets D and D' respectively, then σ will be an isomorphism between the two algebras just in case it maps the universe of $\mathfrak{B}$ onto the universe of $\mathfrak{B}'$. If, in addition, $\mathfrak{B}$ and $\mathfrak{B}'$ are complete and atomic, then it suffices to show that σ maps the sets of atoms of $\mathfrak{B}$ onto the set of atoms of $\mathfrak{B}'$.

Under what conditions does a bijection between the sets of points underlying two affine geometries induce a isomorphism between the corresponding affine relation algebras?

THEOREM 7. *A bijection between the sets of points of two affine geometries induces an isomorphism between the corresponding affine relation algebras just in case the bijection is an (affine) collineation.*

PROOF. Let δ be a bijection from the set of points of an affine geometry D to the set of points of an affine geometry D', and σ the bijection of relations induced by δ:

$$\sigma(S) = \{(\delta(p), \delta(q)) : (p, q) \in S\} \tag{1}$$

for each binary relation S on D. We will show that σ maps the non-identity atoms of $\mathfrak{A}_D$ bijectively to the non-identity atoms of $\mathfrak{A}_{D'}$ (and hence the universe of the first algebra bijectively to the universe of the second algebra) just in case δ is a collineation. For each parallel class Φ of D, the atomic relation R_Φ is defined by the formula

$$R_\Phi = \{(p, q) : p, q \text{ are distinct points of } D \text{ and } pq \in \Phi\}.$$

By (1), the image of this relation under σ has the form

$$\sigma(R_\Phi) = \{(\delta(p), \delta(q)) : p, q \text{ are distinct points of } D \text{ and } pq \in \Phi\} \tag{2}$$

Suppose δ is a collineation. Then δ extends in a unique way to a collineation between the projective extensions of D and D', by Theorem 1. In particular, the restriction of (the extension of) δ to the geometries at infinity is a (projective) collineation between these geometries, say

$$\Psi = \delta(\Phi) = \{\delta(\ell) : \ell \in \Phi\} \tag{3}$$

for a given parallel class Φ of D. A line pq is in the parallel class Φ just in case the three points p, q, and Φ are collinear (since Φ is the point at infinity on the lines it contains). These three points are collinear just in case the points $\delta(p)$, $\delta(q)$, and $\delta(\Phi)$ are collinear (because δ is a collineation), and this happens just in case the line $\delta(p)\delta(q)$ is in the set $\delta(\Phi)$. Combine (2), (3), and the previous observations to write

$$\begin{aligned}\sigma(R_\Phi) &= \{(\delta(p),\delta(q)) : \delta(p),\delta(q) \text{ are distinct points of } D' \text{ and } \delta(p)\delta(q) \in \delta(\Phi)\}\\ &= \{(s,t) : s,t \text{ are distinct points of } D' \text{ and } st \in \Psi\}\\ &= R_\Psi.\end{aligned}$$

Since δ is a bijection of the parallel classes (the points at infinity), it follows that σ maps the atomic non-identity relations of $\mathfrak{A}_D$ bijectively to the atomic non-identity relations of $\mathfrak{A}_{D'}$. Of course, σ also maps the remaining atoms — the identity relations — to one another. Consequently, it is an isomorphism between the affine relation algebras, by the remarks preceding the statement of the theorem. (Notice that σ is just the isomorphism induced by the restriction of the extended δ to the geometries at infinity, as in Theorem 3.)

Assume, now, that σ is a bijection of the atomic, non-identity relations, and suppose

$$\sigma(R_\Phi) = R_\Psi. \tag{4}$$

Then

$$R_\Psi = \{(\delta(p),\delta(q)) : p,q \text{ are distinct points of } D \text{ and } pq \in \Phi\}, \tag{5}$$

by (2) and (4). But

$$R_\Psi = \{(s,t) : s,t \text{ are distinct points of } D' \text{ and } st \in \Psi\}, \tag{6}$$

by definition. It follows from these two equations that δ must map the lines pq in Φ to the lines st in Ψ. In other words, δ is a collineation. In more detail, suppose p, q, and r are three distinct collinear points, with pq in Φ. Put

$$\delta(p) = s, \qquad \delta(q) = t, \qquad \delta(r) = u.$$

Then the pairs

$$(s,t), \qquad (s,u), \qquad (t,u)$$

are in R_Ψ, by (5). Therefore, the lines st, su, and tu are in Ψ, by (6). Parallel lines with a point in common are equal, so the points s, t, and u must be collinear. ⊣

Isomorphisms between algebras of relations that are induced by bijections between their base sets are usually called *base isomorphisms*. Using this terminology, the preceding theorem may be rephrased: *base isomorphisms between affine relation algebras are precisely the mappings induced by collineations between the underlying affine geometries.* It may happen, however, that two distinct collineations induce the same base isomorphism.

THEOREM 8. *Two collineations δ and ρ between affine geometries induce the same base isomorphism between the corresponding affine relation algebras just in case the composition $\rho^{-1} \circ \delta$ is a dilatation.*

PROOF. Suppose δ and ρ are both collineations between affine geometries D and D'. The induced isomorphisms between the affine relation algebras $\mathfrak{A}_D$ and $\mathfrak{A}_{D'}$ are completely determined by the actions of (the extensions of) δ and ρ on the points at infinity. (See the parenthetical remark in the previous proof, and Theorems 3 and 4.) Thus, the induced isomorphisms coincide just in case δ and ρ agree on points at infinity. To say that δ and ρ agree on points at infinity is to say that the composition $\rho^{-1} \circ \delta$ is the identity function on the points at infinity of D. In other words, it maps each parallel class of D to itself. In view of the definition of the extension of an affine isomorphism (see Theorem 1), this means that $\rho^{-1} \circ \delta$ maps each line in a parallel class to another line in the same parallel class. In other words, it is a dilatation. ⊣

The specialization of the previous two theorems to the case when the geometries D and D' coincide is of particular interest. In this case, the set of collineations of D forms a group, as does the set of base automorphisms of the relation algebra $\mathfrak{A}_D$.

THEOREM 9. *The correspondence taking each collineation δ of an affine geometry D to the induced base automorphism σ_δ of the relation algebra $\mathfrak{A}_D$ is a homomorphism from the group of collineations of D onto the group of base automorphisms of $\mathfrak{A}_D$. Its kernel is the subgroup of dilatations of D.*

PROOF. The correspondence certainly maps the set of collineations of D onto the set of base automorphisms of $\mathfrak{A}_D$, by Theorem 7. It is easy to verify (as in the proof of Theorem 6) that the correspondence preserves composition, and is therefore a group homomorphism. Its kernel — the set of collineations mapped to the identity base isomorphism — is the group of dilatations of D, by Theorem 8. ⊣

This theorem can be written in a succinct symbolic fashion. Suppose

$$\mathrm{Col}(D), \qquad \mathrm{Dil}(D), \qquad \mathrm{Aut}_{\mathfrak{b}}(\mathfrak{A}_D)$$

denote the group of collineations of the affine geometry D, the normal subgroup of dilatations of D, and the group of base automorphisms of the relation algebra $\mathfrak{A}_D$ respectively. Then the last theorem can be rendered symbolically as

$$\mathrm{Aut}_{\mathfrak{b}}(\mathfrak{A}_D) \cong \mathrm{Col}(D) \,/\, \mathrm{Dil}(D).$$

Some examples will serve to illustrate the theorem. Suppose D is a Desarguesian plane of finite order n. Then n is the kth power of a prime, for some positive integer k. The plane D has

$$k \cdot n^3 \cdot (n^2 - 1) \cdot (n - 1)$$

collineations and $n^2 \cdot (n - 1)$ dilatations (see Theorems 23 and 24). Consequently, there are

$$\begin{aligned}[k \cdot n^3 \cdot (n^2 - 1) \cdot (n - 1)] \,/\, [n^2 \cdot (n - 1)] &= k \cdot n \cdot (n^2 - 1) \\ &= k \cdot (n + 1) \cdot n \cdot (n - 1)\end{aligned}$$

distinct base automorphisms of the affine relation algebra $\mathfrak{A}_D$, by the previous theorem.

As the example makes clear, to determine the number of base automorphisms of an affine algebra, non-trivial information about the number of collineations and dilatations of the geometry must be known. This point becomes even clearer if one tries to determine the number of base automorphisms of the affine algebras constructed over affine planes of order nine. The needed geometric information about these planes may be deduced from known results in the literature, but this information is certainly not "well known" or "trivial". Some details may be found in the last sections of the paper.

There are four projective planes of order nine: the plane Δ constructed from the field of order nine, a plane Ω constructed from the unique non-commutative near-field of order nine, its dual Ω^d, and the "Hughes plane" Ψ of order nine. There are seven affine planes of order nine: the plane Δ_0 obtained from Δ by deleting an arbitrary line, the planes Ω_0 and Ω_1 obtained from Ω by deleting the unique translation line and any other line, respectively, the planes Ω_0^d and Ω_1^d obtained from Ω^d by deleting a line through the translation point and any other line, respectively, and the planes Ψ_0 and Ψ_1 obtained from Ψ by deleting a real line and a complex line, respectively. The number of collineations and dilatations of each of these planes is given in the following table, along with the quotient of these two numbers. (See Corollaries 25, 38, 39, 42, 43, 57, and 58). This quotient is just the number of distinct base automorphisms of the corresponding affine relation algebra.

Affine planes	Δ_0	Ω_0	Ω_1	Ω_0^d	Ω_1^d	Ψ_0	Ψ_1
Collineations	933,120	311,040	3456	31,104	3840	2592	432
Dilatations	648	162	8	72	2	18	1
Quotient	1440	1920	432	432	1920	144	432

TABLE 1. The number of collineations and dilatations of the affine planes of order nine.

For an affine plane D, the number of automorphisms of $\mathfrak{A}_D$ (see the first example after Theorem 6) is, in general, substantially higher (exponentially higher) than the number of base automorphisms. It is therefore perhaps surprising that, for affine geometries of higher dimension, this is no longer the case. The reason is that, in higher dimensions, the geometry at infinity has enough structure to limit severely the number of collineations, which in turn limits the number of relation algebraic automorphisms.

THEOREM 10. *If D and D' are affine geometries of dimension at least three, then every isomorphism between $\mathfrak{A}_D$ and $\mathfrak{A}_{D'}$ is a base isomorphism.*

PROOF. Let σ be an isomorphism between the affine relation algebras $\mathfrak{A}_D$ and $\mathfrak{A}_{D'}$. There is a collineation τ between the geometries at infinity of D and D' that induces σ, by Corollary 5. Since the dimension of D is at least three, any isomorphism between the geometries at infinity extends to an isomorphism between the projective extensions of D and D', and in particular to an isomorphism between D and D' (see Theorem 61). Let $\hat{\tau}$ denote such an extension of τ. The base isomorphism between $\mathfrak{A}_D$ and $\mathfrak{A}_{D'}$ induced by $\hat{\tau}$ is completely determined by the action of $\hat{\tau}$ on the points at infinity. But on the points at infinity, $\hat{\tau}$ agrees with τ. Therefore, the *base isomorphism* induced by $\hat{\tau}$ must agree with the *isomorphism* induced by τ. In other words, it must be σ. ⊣

§3. Geometric relation algebras. The structure of the algebra of relations defined in terms of an affine geometry D (see, for instance, Theorem 2), as well as the structure of its automorphism group (see Corollary 5), is determined by the (projective) geometry at infinity of D. Lyndon [15] defined a class of abstract relation algebras in terms of arbitrary projective geometries.

Let P be any projective geometry (of any dimension ≥ 1) and i an arbitrary element not included in P. Take B to be the collection of all subsets of $P \cup \{i\}$. Identify each singleton $\{x\}$ with the element x itself. The power set B is the universe of a complete, atomic Boolean set algebra. Define a unary "conversion" operation $\smile$ on B by requiring it to be the identity function. Define a binary "relative multiplication" operation ; between atoms of B as follows. If x and y are distinct elements of the geometry P, then

$$x\,;x = \{i, x\}$$

when P has order different from 2,

$$x\,;x = i$$

when P has order 2, and

$$x\,;y = \{z \in P : z \text{ is collinear with and distinct from } x, y\}$$

(see Theorem 2). Relative multiplication with the element i is defined so that i is the identity element of the algebra:

$$i\,;x = x\,;i = x \qquad \text{and} \qquad i\,;i = i.$$

The operation of relative multiplication is extended to all of B by requiring it to distribute across arbitrary unions:

$$X\,;Y = \bigcup\{x\,;y : x \in X \text{ and } y \in Y\}$$

for subsets X and Y of $P \cup \{i\}$. Lyndon [15] proved that the algebra

$$\mathfrak{B}_P = \langle B, \cup, \sim, ;, \smile, i\rangle$$

is always an (abstract) relation algebra in the sense that it satisfies Tarski's ten postulates defining the class of relation algebras.[3,4] This algebra is usually called the *Lyndon algebra* or the *geometric relation algebra* over P. (The latter terminology is perhaps preferable, because it is more descriptive.)

[3]The exact nature of these postulates will not play a role in our discussion. However, the interested reader may find them, for example, on p. 235 of [23].

[4]Lyndon's construction is a modified version of an earlier construction of Jónsson [9]. Jónsson defined the relative product $x\,;y$ of two points x and y in the geometry to *include* the set of all points z collinear with and different from x and y, and he defined the relative product $x\,;x$ of a single point x with itself to *include* the identity element i (but he said nothing about the point x being included in this product). He then showed that, starting with this partially defined operation, one can construct an ascending chain of partial algebras so that, in the union of the chain, relative multiplication is a fully defined operation and satisfies Tarski's axioms — the principal one being the associative law for relative multiplication.

The theorems of Section 2.1 — but not those of Section 2.2 — have abstract versions that apply to the algebras $\mathfrak{B}_P$. For instance, the abstract versions of Theorems 3 and 4 can be combined into the following statement. *The restriction, to the set of atoms, of an isomorphism between relation algebras $\mathfrak{B}_P$ and $\mathfrak{B}_{P'}$ is a collineation between the projective geometries P and P'. Conversely, every collineation between these geometries has a unique extension to an isomorphism between the relation algebras.* The abstract version of Theorem 6 says

$$\mathrm{Aut}(\mathfrak{B}_P) \cong \mathrm{Col}(P).$$

In fact the correspondence taking each automorphism of $\mathfrak{B}_P$ to its restriction to the set of atoms is the desired group isomorphism. The proofs of the abstract versions are similar to, but somewhat simpler than, the proofs of the set-theoretic versions. There is no need to refer to a geometry at infinity. The abstract versions will not play a role in the subsequent discussion.

A *representation* of a (simple) relation algebra $\mathfrak{B}$ (over a set D) is an embedding ϑ of $\mathfrak{B}$ into the algebra of all binary relations on D.[5] The term "representation" is sometimes also used to refer to the image algebra — the algebra $\mathfrak{A}$ of binary relations that is the image of $\mathfrak{B}$ under the embedding ϑ. In this situation, we shall occasionally say that ϑ is a representation of $\mathfrak{B}$ *over* $\mathfrak{A}$ (an isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$). The representation ϑ is *complete* if, whenever a is the supremum of an infinite set X in $\mathfrak{B}$, then

$$\vartheta(a) = \bigcup\{\vartheta(x) : x \in X\}.^6$$

Two representations φ and ϑ of $\mathfrak{B}$ are *equivalent* if there is a base isomorphism δ between the image algebras such that $\varphi = \delta \circ \vartheta$. Essentially, this means that one representation can be obtained from the other simply by "renaming" the elements of the base set.

Suppose a projective geometry P is isomorphic to the geometry at infinity of an affine geometry D, say via a mapping π. Define a mapping ϑ on the atoms of the algebra $\mathfrak{B}_P$ by specifying

$$\vartheta(x) = R_\Phi \qquad \text{when} \qquad \pi(x) = \Phi,$$

[5]This is a restricted definition of representation that is appropriate to *simple* relation algebras — non-trivial relation algebras with precisely two congruence relations. In the general definition, a representation is an isomorphism onto an arbitrary algebra of relations. The difference between the two definitions is that, in the general definition, the Boolean unit of the representing algebra may not be the universal relation on the base set, but only some equivalence relation. It is shown in Theorem 4.28 of [11] that every representable relation algebra which is simple has a representation in which the Boolean unit *is* the universal relation.

There are two further reasons for adopting the restricted definition of representation in this paper. First, although it is not clear from the wording of [15] (second paragraph), it is the definition adopted by Lyndon. Indeed, his Theorem 1 — Theorem 12 below — only holds when the Boolean unit is the universal relation: this property is needed to verify the projective axiom that any two points lie on a line. Secondly, the base sets of general representations of a relation algebra $\mathfrak{B}$ can be blown up to arbitrarily large sizes by embedding $\mathfrak{B}$ into direct powers of the representing algebras, and this procedure preserves the completeness of representations. Therefore, there can be no bound on the number of inequivalent (complete) *general* representations that a simple relation algebra — even a finite one — may have.

[6]There is a slight chance that Lyndon intended his representations in [15] to always be complete, for he defines a representation as a "complete isomorphism". The qualifier "complete" is never explained, and it does not occur at all in an earlier draft of the paper. The difficulty with this hypothesis is that many of the theorems in [15] explicitly require the representation to be "completely additive".

for each point x in P, and

$$\vartheta(i) = I_D.$$

Extend ϑ to all of $\mathfrak{B}_P$ by requiring it to preserve arbitrary unions:

$$\vartheta(X) = \bigcup\{\vartheta(x) : x \in X\}.$$

It is immediate from the definitions of the operations of $\mathfrak{B}_P$ and from the description of the operations of the affine relation algebra $\mathfrak{A}_D$ (given, in part, in Theorem 2), that ϑ is an isomorphism between $\mathfrak{B}_P$ and $\mathfrak{A}_D$. In particular, it is a complete representation of $\mathfrak{B}_P$. Lyndon called such representations *affine representations*. If it is necessary to be more specific, we shall refer to ϑ as an *affine representation* (of $\mathfrak{B}_P$) *over* (the geometry) D, or as a *representation* (of $\mathfrak{B}_P$) *over* $\mathfrak{A}_D$. In fact, whenever a reference is made to a representation over an (affine) *geometry* D — as opposed to a *set* D — the intention is that the representation is affine. Using this terminology, we may formulate Lyndon's first theorem as follows.

THEOREM 11 (Lyndon [15]). *An isomorphism between a projective geometry P and the geometry at infinity of an affine geometry D induces an affine representation of the relation algebra $\mathfrak{B}_P$ over D.*

Lyndon proved that every complete representation of a geometric relation algebra is equivalent to an affine representation. Indeed, suppose ϑ is a complete representation of the algebra $\mathfrak{B}_P$ over a set D. By passing to an equivalent representation, it may be assumed that the sets D and P are disjoint. The representation ϑ determines the structure of an affine geometry on the set D. The points of the geometry are just the elements of D. The lines of the geometry are the lines through points of D in directions determined by the points of P. More precisely, the line through a point p in D, in the direction of a point x in P, is defined to be the set of points

$$\{q : (p,q) \in \vartheta(x)\} \cup \{p\}.$$

The proof that the set D endowed with this geometric structure is actually an affine geometry (or equivalently, that the geometry obtained from D by adjoining the points and lines of P as points and lines at infinity is a projective geometry) is carried out on pp. 24–25 of [15]. One can then verify that ϑ is just the representation of $\mathfrak{B}_P$ determined by the formula

$$\vartheta(x) = R_\Phi,$$

where Φ is the parallel class of lines with x as their point at infinity. Thus, ϑ is an affine representation of $\mathfrak{B}_P$ over the geometry D — an isomorphism from $\mathfrak{B}_P$ onto the affine relation algebra $\mathfrak{A}_D$.

THEOREM 12 (Lyndon [15]). *If P is a projective geometry, then every complete representation of the relation algebra $\mathfrak{B}_P$ is equivalent to an affine representation (over an affine geometry for which P is the geometry at infinity).*

3.1. Counting complete representations of geometric relation algebras. How many complete representations does a geometric relation algebra possess? By simply changing the "shape" of the elements in the base set of a given representation, it is possible to construct an arbitrary number of "new" representations. But these new representations look exactly like the original one in the sense that they are equivalent to it. The real question, then, is to determine the number of complete representations, *up to equivalence*. (Notice that, in the important case when the relation algebra is finite, the word "complete" is superfluous. The question then asks for the total number of all representations, up to equivalence.) Every complete representation of a geometric relation algebra is equivalent to an affine representation, by Theorem 12. So what must be computed is the number of affine representations of such an algebra, up to equivalence. The first observation to be made is a general result about arbitrary relation algebras.

THEOREM 13. *Suppose a relation algebra $\mathfrak{B}$ is representable as an algebra $\mathfrak{A}$ of binary relations. Then the number of inequivalent representations of $\mathfrak{B}$ over $\mathfrak{A}$ is just the number of right cosets of* $\mathrm{Aut}_{\mathrm{b}}(\mathfrak{A})$ *in* $\mathrm{Aut}(\mathfrak{A})$.

PROOF. Let φ be a representation of $\mathfrak{B}$ over $\mathfrak{A}$ — an isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$. For every automorphism σ of $\mathfrak{A}$, the composition $\sigma \circ \varphi$ is again a representation of $\mathfrak{B}$ over $\mathfrak{A}$. Two such compositions $\sigma \circ \varphi$ and $\pi \circ \varphi$ are equal just in case the automorphisms σ and π are equal. Furthermore, every representation ϑ of $\mathfrak{B}$ over $\mathfrak{A}$ can be written in the form $\sigma \circ \varphi$ for some automorphism σ: the composition

$$\sigma = \vartheta \circ \varphi^{-1}$$

is an automorphism of $\mathfrak{A}$, and

$$\vartheta = \sigma \circ \varphi.$$

Thus, the representations of $\mathfrak{B}$ over $\mathfrak{A}$ are precisely the compositions, with φ, of the automorphisms of $\mathfrak{A}$.

Two representations $\sigma \circ \varphi$ and $\pi \circ \varphi$ are defined to be equivalent just in case there is a base automorphism μ of $\mathfrak{A}$ such that

$$\pi \circ \varphi = \mu \circ \sigma \circ \varphi,$$

or, what amounts to the same thing, such that

$$\pi = \mu \circ \sigma.$$

In other words, the two representations are equivalent just in case $\pi \circ \sigma^{-1}$ is a base automorphism of $\mathfrak{A}$. This shows that the representations over $\mathfrak{A}$ that are equivalent to a given representation $\sigma \circ \varphi$ are just the compositions $\mu \circ \sigma \circ \varphi$, where μ ranges over the base automorphisms of $\mathfrak{A}$. In other words they are the composition with φ of the elements of the right coset of $\mathrm{Aut}_{\mathrm{b}}(\mathfrak{A})$ by σ. ⊣

The next observation is a side remark about using the formula in the preceding theorem to count complete representations. The observation is really about Boolean algebras and their expansions, but to avoid unnecessary distractions, it is formulated for relation algebras. For an algebra $\mathfrak{A}$ of binary relations (or, more generally, a Boolean set algebra with additional operations) let's say that *suprema are unions* if, whenever an infinite subset X of the universe has a supremum (least upper bound) in $\mathfrak{A}$, then $\bigcup X$ is this supremum, or, equivalently, then $\bigcup X$ is in $\mathfrak{A}$.

LEMMA 14. *For algebras $\mathfrak{A}$ of relations, the following conditions are equivalent.*

(i) *In $\mathfrak{A}$, suprema are unions.*
(ii) *There is a representation (of a relation algebra) over $\mathfrak{A}$ that is complete.*
(iii) *Every representation (of every relation algebra) over $\mathfrak{A}$ is complete.*

PROOF. Suppose (i) holds, with the intention of deriving (iii). Let ϑ be a representation of a relation algebra $\mathfrak{B}$ over $\mathfrak{A}$. To check that ϑ is complete, suppose X is an infinite set in $\mathfrak{B}$ with a supremum a. Then $\vartheta(a)$ must be the supremum of $\{\vartheta(x) : x \in X\}$ in $\mathfrak{A}$. Apply (i) to obtain

$$\vartheta(a) = \bigcup\{\vartheta(x) : x \in X\}. \tag{1}$$

It is obvious that (iii) implies (ii): the identity automorphism of $\mathfrak{A}$ is a representation over $\mathfrak{A}$, and it must be complete, by (iii). For the implication from (ii) to (i), let ϑ be a complete representation of a relation algebra $\mathfrak{B}$ over $\mathfrak{A}$. To check that in $\mathfrak{A}$ suprema are unions, suppose Y is an infinite set in $\mathfrak{A}$ with a supremum b. Take a to be the pre-image of b, and X the set of pre-images of elements of Y, in $\mathfrak{B}$, under ϑ. Then a is the supremum of X in $\mathfrak{B}$, so (1) holds, by the assumption that ϑ is complete. In other words, $b = \bigcup Y$. ⊣

In particular, whenever $\mathfrak{A}$ is a complete algebra of relations, every representation over $\mathfrak{A}$ is complete. Thus, counting the number of (inequivalent) complete representations of a relation algebra $\mathfrak{B}$ over $\mathfrak{A}$ is the same as counting the number of (inequivalent) representations. In particular, the formula in Theorem 13 applies.

In the following corollary, the notation $|S|$ denotes the cardinality of the set S.

COROLLARY 15. *If $\mathfrak{B}_P$ is finite and representable over an affine geometry D, then it has exactly*

$$\frac{|\mathrm{Col}(P)|}{|\mathrm{Col}(D)|\,/\,|\mathrm{Dil}(D)|} \tag{i}$$

inequivalent affine representations over D.

PROOF. The automorphism group of $\mathfrak{A}_D$ is isomorphic to $\mathrm{Col}(P)$, by Theorem 6. The base automorphism group of $\mathfrak{A}_D$ is isomorphic to the quotient $\mathrm{Col}(D)/\mathrm{Dil}(D)$, by Theorem 9. Therefore, the number right cosets of $\mathrm{Aut_b}(\mathfrak{A}_D)$ in $\mathrm{Aut}(\mathfrak{A}_D)$ is precisely the quotient (i), by Lagrange's Theorem from basic group theory. The desired result now follows from Theorem 13. ⊣

The finiteness requirement in the preceding corollary is needed in order for the numerical quotient (i) to make sense.

Consider, now, affine representations ϑ and ψ of $\mathfrak{B}_P$ over different affine geometries, say D and D'. By definition, the two representations are equivalent just in case there is a base isomorphism σ from $\mathfrak{A}_D$ to $\mathfrak{A}_{D'}$ with $\sigma \circ \vartheta = \psi$. This amounts to saying that the composition $\psi \circ \vartheta^{-1}$ is a base isomorphism between the representing algebras. Every base isomorphism between affine relation algebras arises from an isomorphism between the underlying affine geometries, and conversely, by Theorem 7. If the geometries D and D' are not isomorphic, there can be no such base isomorphism, and consequently the representations ϑ and ψ cannot be equivalent. On the other hand, if D and D' *are* isomorphic, then every affine representation (of $\mathfrak{B}_P$) over D' is equivalent to an affine representation over D. Therefore, the representations of $\mathfrak{B}_P$ over D' are already accounted for (up to equivalence) when the inequivalent affine representations of $\mathfrak{B}_P$ over D are counted. In more detail, suppose ψ is an affine representation over D', and let σ be the base isomorphism between $\mathfrak{A}_D$ and $\mathfrak{A}_{D'}$ induced by an isomorphism between the underlying affine geometries. Then the composition $\chi = \sigma^{-1} \circ \psi$ is an affine representation of $\mathfrak{B}_P$ over D that is equivalent to ψ, since $\sigma \circ \chi = \psi$. Summarizing: *The number of* *inequivalent affine representations of any (finite or infinite) algebra $\mathfrak{B}_P$ is just the sum of the numbers of inequivalent affine representations of the algebra over the mutually non-isomorphic affine geometries that have P as their geometry at infinity.*

THEOREM 16. *Suppose P is a finite projective geometry, and $\langle D_\xi : \xi < \kappa \rangle$ a complete list, up to isomorphism, of the affine geometries that have P as their geometry at infinity. Then the number of inequivalent representations of $\mathfrak{B}_P$ is the sum (over ξ) of*

$$\frac{|\mathrm{Col}(P)|}{|\mathrm{Col}(D_\xi)| \,/\, |\mathrm{Dil}(D_\xi)|}.$$

Here are some examples that illustrate this theorem. Suppose P is a projective line of finite order $n \geq 2$. Then P is just a set with $n + 1$ points. Consequently, $\mathrm{Col}(P)$ consists of all possible permutations of $n + 1$ elements. In other words,

$$|\mathrm{Col}(P)| = (n + 1)!$$

The line P is isomorphic to the line at infinity of (the projective extension of) any affine plane of order n. For the orders $n = 2, 3, 4, 5, 7$, and 8, there is a unique affine plane of order n: the Desarguesian plane over the field of cardinality n. (There is no affine plane of order 6, by the Bruck-Ryser Theorem [3], and no affine plane of order 10, by the computer-aided analysis of Lam, Thiel, and Swiercz [13].) As we saw in the previous section, if D is the Desarguesian plane of order n, then

$$|\mathrm{Col}(D)| \,/\, |\mathrm{Dil}(D)| = k \cdot (n + 1) \cdot n \cdot (n - 1),$$

where n is the kth power of a prime. Therefore, there are

$$(n + 1)! \,/\, [k \cdot (n + 1) \cdot n \cdot (n - 1)] = (n - 2)! \,/\, k$$

inequivalent representations of $\mathfrak{B}_P$ over D. Table 2 below lists the number of inequivalent representations for the smallest projective lines.

The projective line P of order nine is of special interest. Its collineation group has cardinality

$$10! = 3{,}628{,}800.$$

As was mentioned earlier, there are seven affine planes of order nine, and P is isomorphic to the line at infinity of each of them. The number of inequivalent representations of $\mathfrak{B}_P$ over one of these planes D is the quotient of $10!$ by another quotient: the number of collineations of D divided by the number of dilatations of D (by Corollary 15). This second quotient is computed in the third line of Table 1. Therefore, the number of inequivalent representations of $\mathfrak{B}_P$ is the sum of the quotients

$$10!\,/\,1440, \quad 10!\,/\,1920, \quad 10!\,/\,432, \quad 10!\,/\,432,$$
$$10!\,/\,1920, \quad 10!\,/\,144, \quad 10!\,/\,432.$$

In other words, it is the sum

$$2{,}520 + 1{,}890 + 8{,}400 + 8{,}400 + 1{,}890 + 25{,}200 + 8{,}400 = 56{,}700.$$

It may be worth emphasizing that only seven algebras of relations are used as images in all of these many inequivalent representations. For instance, the affine algebra

$\mathfrak{A}_{\Psi_0}$ is the image algebra for 25,200 of the inequivalent representations of $\mathfrak{B}_P$ (where Ψ_0 is the affine plane obtained from the Hughes plane of order nine by deleting a real line). The relatively large number of representations involving this algebra is caused by the fact that the plane Ψ_0, which is relatively poor in structure, has a small ratio of collineations to dilatations. This leads to a small number of base automorphisms of the corresponding affine algebra, and hence to small equivalence classes of representations.

Order of P	Number of inequivalent representations of $\mathfrak{B}_P$
2	1
3	1
4	1
5	6
6	0
7	120
8	240
9	56,700
10	0

TABLE 2. The projective lines P of order ≤ 10, and the number of inequivalent representations of $\mathfrak{B}_P$.

The computation of the number of inequivalent representations of the geometric relation algebra over the projective line of order n has been reduced to three questions about affine geometry: (1) Up to isomorphism, how may affine planes of order n are there? (2) What is the size of the collineation group of each of these planes? (3) What is the size of the dilatation group of each of these planes? Unfortunately, the answer, even to the first question, is unknown for every $n \geq 11$ that is not excluded by the Bruck-Ryser Theorem. In particular, it is not known whether there is just one affine plane of order n when n is a prime number ≥ 11. Thus, the geometric information needed to compute the number of representations of the geometric relation algebra of order n is unknown for all $n \geq 11$ (except those n excluded by the Bruck-Ryser Theorem).

Nevertheless, the formula in Theorem 16 still gives a useful lower bound on the number of inequivalent representations. For instance, there is at least one affine plane of order 29: the Desarguesian plane over the field of cardinality 29. Therefore, there are at least

$$30! \,/\, (30 \cdot 29 \cdot 28) = 27! \approx 1.0888869 \times 10^{28}$$

inequivalent representations of the geometric relation algebra over the projective line of order 29. Hirsch-Hodkinson [8], p. 253, Proposition 18, showed that there are at least two inequivalent representations of this algebra.

For projective geometries of dimension higher than one, the computation of the number of inequivalent complete representations of the geometric relation algebras simplifies enormously.

THEOREM 17 (Lyndon [15]). *Suppose P is a projective geometry of dimension at least three or else a Desarguesian plane. Then $\mathfrak{B}_P$ has exactly one complete representation, up to equivalence.*

PROOF. Every Desarguesian projective geometry P — in particular, every projective geometry of dimension at least three — is isomorphic to the geometry at infinity of an affine geometry D of one higher dimension. There is, then, at least one affine representation of $\mathfrak{B}_P$, by Theorem 11. The geometry D is unique, up to isomorphism (see Theorem 61). Two affine representations φ and ϑ of the algebra $\mathfrak{B}_P$ over D are always equivalent: the composition $\vartheta \circ \varphi^{-1}$ is an automorphism of the affine algebra $\mathfrak{A}_D$, and any automorphism of $\mathfrak{A}_D$ is a base automorphism, by Theorem 10. ⊣

When P is a non-Desarguesian plane, the algebra $\mathfrak{B}_P$ has no complete representations, as was pointed out in [15].[7] In fact, one can show — using the proof Jónsson [9] gave for the non-representability of his relation algebras — that geometric relation algebras over non-Desarguesian planes have no representations whatsoever. For the sake of completeness (and to clear up an apparently ambiguous situation in the literature — see the previous footnote), we give this argument.

THEOREM 18 (Jónsson [9], Lyndon [15]). *If P is a non-Desarguesian projective plane, then $\mathfrak{B}_P$ is not representable at all.*

PROOF. Let φ be a representation of the algebra $\mathfrak{B}_P$ over a set D, with the goal of showing that the geometry P must be Desarguesian. Consider two triangles pqr and $p'q'r'$ in P that are perspective from a point s, that is, the lines pp', qq', and rr' are concurrent in s (see Figure 1). In the algebra $\mathfrak{B}_P$, it is possible to express algebraically (the non-trivial case of) this geometric assumption:

$$(1)\qquad s = (p\,;p')\cdot(q\,;q') = (p\,;p')\cdot(r\,;r') = (q\,;q')\cdot(r\,;r').$$

Denote the points of intersection of the corresponding pairs of opposite sides by e, f, and g (see Figure 1). The fact that they are the points of intersection can be expressed algebraically in $\mathfrak{B}_P$ via the equations

$$(2)\qquad e = (p\,;q)\cdot(p'\,;q'), \qquad f = (p\,;r)\cdot(p'\,;r'), \qquad g = (q\,;r)\cdot(q'\,;r').$$

Let (x, y) be an arbitrary pair of elements in $\varphi(e)$, the relation representing the point e. Since φ preserves relative and Boolean multiplication, it follows from the

[7]This result is stated without proof and in a somewhat misleading way in [15], as a corollary to Theorem 1. Lyndon asserts that, when P is non-Desarguesian, the algebra $\mathfrak{B}_P$ has no representation. The intended argument presumably proceeds by contraposition as follows. Suppose $\mathfrak{B}_P$ is [completely] representable. Then it must have an affine representation, by Theorem 12, so the plane P is isomorphic to the plane at infinity of an affine geometry of dimension three. But an affine geometry of dimension three is always Desarguesian (and hence so is the plane at infinity). Clearly, this argument requires either the assumption that P is finite or the assumption that the representation is complete, neither of which is explicitly made by Lyndon. Probably, this was an oversight on his part. In many of the other theorems and corollaries of his paper, Lyndon explicitly makes the assumption that the representations must be completely additive for the results to hold. However, assumptions concerning complete additivity do not occur at all in an earlier draft of [15]. They were added later, when Lyndon realized they were needed. In all likelihood, he overlooked replacing "representation" by "complete representation" in the conclusion of the corollary.

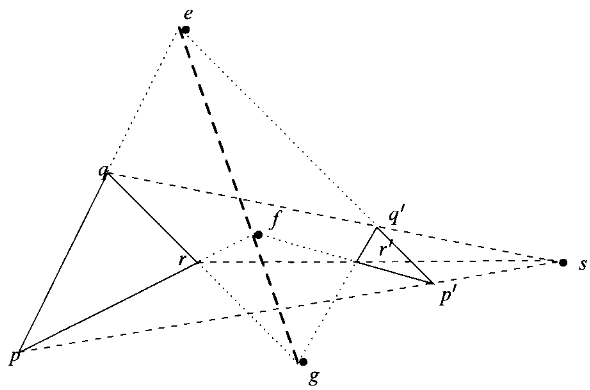

FIGURE 1. Two triangles perspective from a point.

first equation in (2) that there must be elements z and w in D such that

$$(3)\quad (x, z) \in \varphi(p), \qquad (z, y) \in \varphi(q) \quad \text{and} \quad (x, w) \in \varphi(p'), \qquad (w, y) \in \varphi(q').$$

Consequently, the pair (z, w) is in $\varphi((p ; p') \cdot (q ; q'))$, by (3) and the isomorphism properties of φ. (Recall, in this connection, that the converse operation is the identity operation in $\mathfrak{B}_P$). The product $(p ; p') \cdot (q ; q')$ coincides with the point s, and the point s is included in the relative product $r ; r'$, by (1). Therefore, the pair (z, w) is in $\varphi(r ; r')$, by the isomorphism properties of φ. This implies that there is an element u such that

$$(4)\qquad (z, u) \in \varphi(r) \quad \text{and} \quad (u, w) \in \varphi(r').$$

Combine (3) with (4), and use the isomorphism properties of φ, to conclude that the pair (x, u) is in $\varphi((p ; r) \cdot (p' ; r'))$, and the pair (u, y) is in $\varphi((q ; r) \cdot (q' ; r'))$. Recall, in this connection, that relative multiplication is commutative in the algebra $\mathfrak{B}_P$, since conversion is the identity operation:

$$a ; b = (a ; b)^{\smile} = b^{\smile} ; a^{\smile} = b ; a.$$

The products

$$(p ; r) \cdot (p' ; r') \qquad \text{and} \qquad (q ; r) \cdot (q' ; r')$$

coincide with the points f and g, by (2). Therefore, the pair (x, u) is in $\varphi(f)$, and the pair (u, y) is in $\varphi(g)$. This means that the pair (x, y) is in $\varphi(f ; g)$.

We have shown that every pair in $\varphi(e)$ is in $\varphi(f ; g)$. It follows that

$$(5)\qquad e \leq f ; g$$

in $\mathfrak{B}_P$, since φ is an isomorphism. But equation (5) is equivalent to saying that the point e is collinear with f and g, by definition of relative multiplication in $\mathfrak{B}_P$. In other words, the points of intersection of the corresponding opposite sides of two perspective triangles are collinear. This is just what it means for the geometry P to be Desarguesian. ⊣

COROLLARY 19. *A geometric relation algebra is representable if and only if it is completely representable.*

PROOF. For the non-trivial direction, suppose a geometric relation algebra $\mathfrak{B}_P$ is representable . In the case when the dimension of P is at least two, P must be Desarguesian. (If the dimension of P is greater than two, it is automatically Desarguesian. If the dimension is two, then it must be Desarguesian, since in the opposite case $\mathfrak{B}_P$ could not be representable, by the previous theorem.) A Desarguesian geometry is always isomorphic to the geometry at infinity of an affine geometry of one higher dimension. Consequently, $\mathfrak{B}_P$ is completely representable, by Theorem 11. Suppose P has dimension one. If P is infinite, then it is isomorphic to the line at infinity of an affine plane. So $\mathfrak{B}_P$ is completely representable, by Theorem 11. If P is finite, then any representation is automatically complete. ⊣

The previous corollary says, in particular, that a representable geometric relation algebra is completely representable. It does not say that all representations are necessarily complete. In fact, the vast majority of the representations are necessarily incomplete.

3.2. Counting representations of relation algebras. In the previous section, we described a method for calculating the number of complete representations that a geometric relation algebra possesses, up to equivalence. For finite algebras, this number coincides with the number of representations, since a representation of a finite algebra is always complete. But in the case of an infinite algebra, the two numbers are different. How many *representations* (complete or not) does an infinite geometric relation algebra possess, up to equivalence? The answer is: there are as many inequivalent representations as there are sets.

THEOREM 20. *If a (simple) relation algebra of cardinality $\kappa \geq \omega$ has any representations at all, then it has representations over sets of every cardinality at least κ. Representations over sets of different cardinality are never equivalent.*

PROOF. Suppose φ is a representation of an infinite relation algebra $\mathfrak{A}$ over a set D. Since the algebra is infinite, the set D must also be infinite. For each element a of A, let R_a be the binary relation

$$R_a = \varphi(a),$$

and let $\mathfrak{D}$ be the relational structure

$$\mathfrak{D} = \langle D, R_a \rangle_{a \in A}.$$

Apply the upward or downward Löwenheim-Skolem-Tarski Theorem to obtain an elementary extension or an elementary substructure

$$\mathfrak{C} = \langle C, S_a \rangle_{a \in A}$$

of $\mathfrak{D}$ of any given cardinality greater than or equal to κ (the number of binary relations of the relational structure $\mathfrak{D}$). Then the mapping ϑ from $\mathfrak{A}$ into the algebra of all binary relations on the set C that is defined by the rule

$$\vartheta(a) = S_a$$

is a representation of $\mathfrak{A}$. For instance, to show that ϑ preserves relative multiplication, suppose $a \,;\, b = c$ in $\mathfrak{A}$. Since φ is a representation, we have

$$R_a \mid R_b = R_c.$$

Therefore, the sentence

$$\forall x \forall y [R_c(x, y) \leftrightarrow \exists z (R_a(x, z) \wedge R_b(z, y))]$$

holds in $\mathfrak{D}$. Because $\mathfrak{C}$ is an elementary extension or substructure of $\mathfrak{D}$, the same sentence (with "R" replaced by "S") must hold in $\mathfrak{C}$. Consequently,

$$S_a \mid S_b = S_c,$$

which means that

$$\vartheta(a \,;\, b) = \vartheta(a) \mid \vartheta(b).$$

The other required properties of ϑ (one-oneness, and the preservation of finite unions, complements, converses, and the identity element) follow in a similar fashion from the fact that these properties are definable using sets of first-order sentences. Therefore, they must be preserved under the passage to an elementary extension or substructure.

It is obvious that two representations over sets of different cardinality cannot be equivalent, since there can be no bijection between the two sets. ⊣

§4. Collineations of affine and projective planes. This section contains the facts about collineations of affine and projective planes that are needed in the paper. The majority of them concern collineations of the affine and projective planes of order nine. They are consequences of results known from the literature, but in most cases, some work is required to put them in the form needed for the applications here. Enough details are provided (with appropriate references to the literature) so that the reader not well acquainted with projective geometry can gain a sense of the arguments involved.

The first theorem is a classical result about projective planes constructed over finite fields (see, for example, [20], p. 135). It will be used to calculate the number of collineations of affine planes constructed over these fields (see Theorem 23).

THEOREM 21. *A projective plane over a field of cardinality n (the kth power of a prime) has $k \cdot n^3 \cdot (n^3 - 1) \cdot (n^2 - 1)$ collineations.*

PROOF. Here is a sketch of the proof. Let P be a projective plane constructed over a field of cardinality n, the kth power of a prime. By the Fundamental Theorem of Projective Geometry, every collineation of P is the composition of a collineation induced by an automorphism of field with a projectivity of the plane P. There are k automorphisms of the field (since its cardinality is the kth power of a prime), and each of them induces a distinct collineation of P. A projectivity of the plane P is uniquely determined by its action on any fixed four-point. Suppose p, q, r, and s are a four-point: four points, no three of which are collinear. The image p' of p may be an arbitrary point in the plane, so there are $n^2 + n + 1$ choices. The image q' of q may be any point different from q, so there are $n^2 + n$ choices. The image r' of r may be any point not collinear with p' and q', so there are n^2 choices (there are $n + 1$ points on the line $p'q'$ that may not be selected). Finally, the image s' of s may be any point not on one of the three lines through p', q', and r'. There are n^2 points not on the line $p'q'$. Of these, $2n - 1$ points are on one of the two lines $r'p'$ and $r'q'$ (one point occurs on both lines). Thus, there are $n^2 - 2n + 1$ choices for the point s'. Altogether, then, there are

$$(n^2+n+1)\cdot(n^2+n)\cdot n^2\cdot(n^2-2n+1)=n^3\cdot(n^3-1)\cdot(n^2-1)$$

possible choices for the image four-point, and hence there are this many projectivities of the plane P. ⊣

The *orbit* of a line or point under the group of all collineations is the set of images of the line or point under actions by collineations of the group, in symbols

$$\mathrm{Orb}(\ell)=\{\varphi(\ell):\varphi\in\mathrm{Col}(P)\}.$$

The following useful theorem was pointed out to the author by William Kantor: the number of collineations of an affine plane is equal to the number of collineations of its projective extension divided by the size of the orbit of the line at infinity.

THEOREM 22. *Let D be the affine plane obtained from a (finite) projective plane P by deleting a line ℓ. Then*

$$|\mathrm{Col}(D)|=|\mathrm{Col}(P)|\ /\ |\mathrm{Orb}(\ell)|.$$

PROOF. The collineations of D may be viewed as the (restrictions of the) collineations of P that leave the line ℓ fixed, by Theorem 1. The latter form a subgroup G of $\mathrm{Col}(P)$. The cardinality of a group equals the cardinality of any subgroup times the index of the subgroup — the number of its left cosets. In particular,

$$|G|\cdot m=|\mathrm{Col}(P)|, \tag{1}$$

where m is the number of left cosets of G. The collineations in a left coset of G all map ℓ to the same line, and collineations in different left cosets map ℓ to different lines. Indeed, collineations ϑ and φ agree on ℓ just in case $\vartheta^{-1}\circ\varphi$ fixes ℓ, which happens just in case $\vartheta^{-1}\circ\varphi$ is in G, that is, just in case ϑ and φ are in the same left coset of G. Thus, the index m coincides with the number of different images of ℓ under collineations of P. In other words,

$$m=|\mathrm{Orb}(\ell)|, \tag{2}$$

the size of the orbit of ℓ. Together, (1) and (2) give the desired result. ⊣

It is easy to compute the number of collineations of a Desarguesian affine plane using the two previous theorems. (A generalization of this theorem to higher dimensions can be found, for instance, in [5], p. 32, item 13.)

THEOREM 23. *An affine plane over a field of cardinality n (the kth power of a prime) has $k\cdot n^3\cdot(n^2-1)\cdot(n-1)$ collineations.*

PROOF. Let D be an affine plane over a field of cardinality n — the kth power of a prime — and P its projective extension. It is well known that every line of P can be mapped to every other line, since P is Desarguesian. Consequently, the orbit of the line at infinity has size n^2+n+1 (the number of lines in P). The size of $\mathrm{Col}(P)$ is

$$k\cdot n^3\cdot(n^3-1)\cdot(n^2-1)=k\cdot n^3\cdot(n^2+n+1)\cdot(n-1)\cdot(n^2-1),$$

by Theorem 21. Therefore, the size of $\mathrm{Col}(D)$ is

$$[k\cdot n^3\cdot(n^3-1)\cdot(n^2-1)]\ /\ [n^2+n+1]=k\cdot n^3\cdot(n^2-1)\cdot(n-1).$$

by Theorem 22. ⊣

Collineations of projective geometries are classified, in part, by the points and lines that they leave fixed. When a collineation leaves every line through a point fixed, that point is called a *center* of the collineation. Dually, when a collineation leaves every point on a line fixed, that line is called an *axis* of the collineation. It is a fundamental fact of projective geometry that a collineation has a center if and only if it has an axis (see [5], p. 30, item 9). Moreover, the center and axis are unique in the sense that a collineation with two centers or two axis can only be the identity (*op. cit.*, p. 30, item 8). Collineations with a center are called *central* collineations. They fall into two categories: *elations* are central collineations for which the center lies on the axis, and *homologies* are central collineations for which the center does not lie on the axis. Every central collineation of a projective plane is uniquely determined by its action on a single point not lying on the axis and different from the center (see [4], pp. 53–54, Theorems 6.22 and 6.23). For instance, if it leaves some such point fixed, then it must be the identity.

Dilatations of an affine plane are really just central collineations of the projective extension, with axis the line at infinity: a collineation maps each line to a parallel line — a line with the same point at infinity — just in case it maps each point at infinity to itself. If the center lies on the line at infinity, then the dilatation is called a *translation* (of the affine plane) instead of an elation. A computation of the number of dilatations of a Desarguesian affine plane is straightforward.

THEOREM 24. *An affine plane over a field of cardinality n has $n^2 \cdot (n-1)$ dilatations.*

PROOF. For any two elements a, b in the field, there is a translation $\tau_{a,b}$ that maps each point (x, y) to the point $(x + a, y + b)$ (and maps points at infinity to themselves). These are the only translations, so there are n^2 of them. They form a normal subgroup of the group of all collineations of the plane.

For every non-zero element c in the field, there is a dilatation σ_c fixing $(0,0)$ that maps each point (x, y) to the point $(x \cdot c, y \cdot c)$ (and maps points at infinity to themselves). These are the only dilatations fixing $(0,0)$, so there are $n-1$ of them. Every other dilatation γ with a fixed point is the composition of a dilatation fixing $(0,0)$ and a translation (see [7], Proposition 7.8): if γ fixes (a, b), then

$$\gamma = \tau_{a,b} \circ \sigma_c \circ \tau_{a,b}^{-1} = \tau_{e,f} \circ \sigma_c$$

for some non-zero element c, and some pair of elements e, f (recall that the translations form a normal subgroup). Consequently, the dilatations are just the mappings $\tau_{e,f} \circ \sigma_c$, where e and f range over all elements of the field, and c over the non-zero elements of the field. These compositions are pairwise distinct, so there are $n^2 \cdot (n-1)$ of them. ⊣

We now consider, one at a time, the four projective planes of order nine.

4.1. The plane Δ. The plane Δ is constructed over the nine-element field F. The elements of F may be thought of as complex numbers $a + bi$, where a and b are integers modulo 3 (the numbers 0, 1, and -1). The field operations of addition and multiplication are defined just as in the case of the standard complex numbers, except that they are performed modulo 3. When the coefficient b is zero, the number $a + bi$ is said to be *real*, and otherwise it is *complex*. The *conjugate* of $a + bi$ is $a - bi$.

The points of the plane Δ are: (1) the pairs (x, y), with x and y elements of F; (2) the points at infinity: a point (m) for each element m in F, and a point (∞). The lines of Δ are: (1) for each pair of elements m and b from the field, the line consisting of the points (x, y) satisfying the equation $y = x \cdot m + b$, together with the point (m) at infinity; (2) for each element b of the field, the line consisting of the points (x, y) satisfying the equation $x = b$, together with the point (∞) at infinity; (3) the line ℓ consisting of the points at infinity.

An affine plane obtained from a projective plane P by deleting one of the lines (and all of the points on the line) will be called an *affine restriction* of P, or an affine plane *derived* from P. An example of such a restriction is the plane Δ_0 obtained from Δ by deleting the line ℓ at infinity. Theorems 23 and 24 immediately apply to Δ_0, and give the following information about its collineations and dilatations.

COROLLARY 25. *Δ_0 has* 933,120 *collineations and* 648 *dilatations.*

For any line k of Δ, there is a collineation that maps ℓ to k. (Pick a four-point, the first two points of which are on ℓ, and an image four-point, the first two points of which are on k. Invoke the theorem that any four-point can be mapped to any other four-point by a collineation. Compare [20], p. 141, Example 4.4.3(ii).) Such a collineation is, of course, an automorphism of Δ, and also an isomorphism between Δ_0 and the affine restriction of Δ obtained by deleting the line k.

COROLLARY 26. *All affine restrictions of Δ are isomorphic to Δ_0.*

4.2. The plane Ω. The plane Ω is constructed over a near-field Q of cardinality nine. (A finite near-field obeys almost the same axioms as a field. However multiplication is not required to be commutative or left-distributive over addition. It is associative and right-distributive over addition, there is a multiplicative identity element, and non-zero elements have two-sided multiplicative inverses.) This near-field is virtually identical to F, and the same terminology is used for both (real number, complex number, conjugate, etc.). The only difference between them is that the operation of multiplication is modified. Call a number $a + bi$ *even* if at least one of a and b is 0, and odd otherwise. Then the product of two numbers $a + bi$ and $c + di$ is defined to be

$$(a + bi) \odot (c + di) = \begin{cases} (a + bi) \cdot (c + di) & \text{when } c + di \text{ is even,} \\ (a - bi) \cdot (c + di) & \text{when } c + di \text{ is odd} \end{cases}$$

(see [17], p. 9). There will be no further occasion to refer to the multiplication of the field F, so we shall henceforth use the symbol " $\cdot$ " instead of " $\odot$ " to denote the multiplication of Q.

The points and lines of Ω are defined exactly as in the case of the geometry Δ (but notice that some of the lines are now different, because the operation of multiplication has been altered). The line ℓ of Δ continues to be a line of Ω, and it plays a critical role in the analysis of the collineations of Ω.

Here are some examples of collineations of the plane Ω (see [17], pp. 105–110, for details). Suppose a, b are elements of Q. Define $\tau_{a,b}$ to be the mapping that takes each point (x, y) to the point $(x + a, y + b)$, and takes each point on ℓ to itself. Then $\tau_{a,b}$ is an elation with axis ℓ. In other words, it is a *translation* of the affine plane obtained from Ω by deleting the line ℓ. The translations form a subgroup of the group of all collineations, and this group acts transitively on points not on ℓ.

In other words, for any two points (a,b) and (c,d) not on ℓ, there is a translation taking the point (a,b) to the point (c,d): it is the composition of $\tau_{c,d}$ with the inverse of $\tau_{a,b}$.

Next, suppose a and b are non-zero elements of Q. Define $\sigma_{a,b}$ to be the mapping that takes each point (x,y) to the point $(x \cdot a, y \cdot b)$, each point (m) on ℓ to the point $(a^{-1} \cdot m \cdot b)$, and (∞) to itself. Then $\sigma_{a,b}$ is a collineation of Ω. The mapping $\sigma_{a,1}$ is a homology with center (0) and axis the line $x = 0$. It maps the line $y = x$ to the line $y = x \cdot a^{-1}$. The mapping $\sigma_{1,b}$ is a homology with center (∞) and axis the line $y = 0$. It maps the line $y = x$ to the line $y = x \cdot b$. The only two mappings $\sigma_{a,b}$ that leave the line ℓ at infinity pointwise fixed are $\sigma_{1,1}$ (the identity mapping) and $\sigma_{-1,-1}$ (which maps (x,y) to $(-x,-y)$).

Another collineation of Ω is the mapping ν that takes each point (x,y) to the point $(x+y, x-y)$, interchanges (0) and (1), interchanges (-1) and (∞), and interchanges (m) and $(-m)$ for each complex number m. It maps the line $y = 0$ to the line $y = x$. Yet another one is the mapping ρ that takes each point (x,y) to the point (y,x), interchanges the points (0) and (∞), and interchanges the points (m) and (m^{-1}), for non-zero m in Q. It maps the line $y = 0$ to the line $x = 0$ and conversely.

The next theorem is needed to determine the sizes of collineation groups of affine restrictions of Ω, and the size of one dilatation group.

THEOREM 27. *There are* 162 *collineations of* Ω *with axis* ℓ. *There are*

$$162 \cdot 1920 = 311{,}040$$

collineations of Ω *in all, and each one maps the line* ℓ *to itself.*

PROOF. This theorem is stated and proved in [20]; see pp. 364–366, and in particular, Theorem 12.5.6, Theorem 12.5.9, Corollary 12.5.10, and Theorem 12.5.17. Here are the main ideas. The elations with axis ℓ are precisely the mappings $\tau_{a,b}$, and there are 81 of them. The only non-identity homology with center the origin $(0,0)$ and axis ℓ is $\sigma_{-1,-1}$. Every non-identity homology with axis ℓ may be obtained by composing translations with $\sigma_{-1,-1}$, as in the proof of Theorem 24. In fact, there are 81 non-identity homologies with axis ℓ, and they have the form

$$\tau_{a,b} \circ \sigma_{-1,-1} \circ \tau_{a,b}^{-1}$$

for a and b in Q. These 162 collineations with axis ℓ — the elations and the non-identity homologies — form a normal subgroup G of the group of all collineations of Ω.

The points on ℓ can be grouped into five pairs of *associates*: $\{(m), (-m)\}$, for non-zero m in Q, and $\{(0), (\infty)\}$. Any collineation of Ω must map pairs of associates to pairs of associates. For instance, if it takes (1) to $(-1+i)$, then it must take (-1) to $(1-i)$ (see [20], p. 364, Theorem 12.5.11). The line ℓ has 10 points, so there are 10 ways to assign a value to, say, (0) (and then the value assigned to its associate (∞) is determined), 8 remaining ways to assign a value to, say, (1) (and then the value assigned to its associate (-1) is determined), 6 remaining ways to assign a value to, say, (i), and 4 remaining ways to assign a value to, say, $(1+i)$. As soon as the values for four of the pairs have been determined, the value assigned to each element of the last pair is also determined (see [20], Theorem 12.5.12). This gives

$$10 \cdot 8 \cdot 6 \cdot 4 = 1920$$

different ways to assign values to the pairs of points on ℓ, and each of them is actually realized by some collineation (see [20], Theorems 12.5.13–12.5.16).

Collineations ϑ and φ assign the same values to the points on ℓ just in case $\vartheta^{-1} \circ \varphi$ fixes ℓ pointwise, and this happens just in case ϑ and φ are in the same coset of G. It follows that there must be 1920 cosets of G, and hence $1920 \cdot 162$ collineations in all. ⊣

The following corollary is stated in [20], p. 141, Example 4.4.3(iii). It computes the size of point orbits in Ω. This is important because points in Ω are lines in its dual. Thus, the corollary actually calculates the size of line orbits in the dual of Ω. Theorem 22 may then be used to compute the size of the collineation groups of derived affine planes. This will be done in Section 4.3.

COROLLARY 28. *Each point of ℓ can be mapped to every other point of ℓ — and to no others — by a collineation. Each point not on ℓ can be mapped to every other point not on ℓ — and to no others — by a collineation. Consequently, each point on ℓ has an orbit of size* 10, *and each point not on ℓ has an orbit of size* 81.

PROOF. The second part of the proof of the previous theorem shows that a point on ℓ can be mapped to any other point on ℓ. Since each collineation of Ω fixes ℓ, points on ℓ cannot be mapped to points not on ℓ, and conversely. Thus, the points on ℓ — of which there are 10 — all fall into the same orbit.

Any point not on ℓ can be mapped to any other point not on ℓ by a translation $\tau_{a,b}$. Thus, the points not on ℓ — of which there are 81 — must all fall into the same orbit. ⊣

The following corollary is formulated in Example 4.3.4 of [20], p. 136. More will be said about its purpose in a moment.

COROLLARY 29. *There is exactly one non-identity homology with axis ℓ and a given center.*

PROOF. The proof of Theorem 27 lists all the non-identity homologies with axis ℓ: there are 81 of them, and they have the form

$$\tau_{a,b} \circ \sigma_{-1,-1} \circ \tau_{a,b}^{-1}. \tag{1}$$

The center of the homology (1) is the point (a, b) (since the center of $\sigma_{-1,-1}$ is the origin). Thus, each homology in (1) has a different center. ⊣

The line ℓ plays a special role in the plane Ω, as Theorem 27 shows. The other lines of Ω all behave similarly to one another. (This result is stated in [20], p. 141, Example 4.4.3(iv).) One orbit contains them all, and they all lead to isomorphic derived affine planes.

THEOREM 30. *Any two lines of Ω different from ℓ can be mapped to one another by collineations.*

PROOF. Suppose k and m are distinct lines of Ω different from ℓ. Then they intersect at a point p. Consider three cases. If p is on ℓ, choose two other points: q on k, and r on m. Then there is a translation that takes q to r, and of course it leaves p fixed (since the axis of a translation is ℓ). The translation must map pq (the line k) to pr (the line m).

Next, suppose p is the point $(0, 0)$. The collineation $\sigma_{1,m}$ maps the line $y = x$ to the line $y = x \cdot m$, for non-zero m in Q. The collineation v maps the line $y = 0$ to the line $y = x$. The collineation ρ maps the line $y = 0$ to the line $x = 0$. In other words, there are collineations that map the line $y = 0$ to every other line through p. By composing these collineations and their inverses, we get collineations mapping any given line through p to any other given line through p. In particular, there is a collineation mapping k to m.

Finally, suppose p is an arbitrary point not on ℓ. There is a translation τ mapping the point p to the origin $(0, 0)$. The lines $\tau(k)$ and $\tau(m)$ must therefore intersect in the origin. By the results of the previous paragraph, there is a collineation γ mapping the line $\tau(k)$ to the line $\tau(m)$. Consequently, the composition $\tau^{-1} \circ \gamma \circ \tau$ maps the line k to the line m. ⊣

COROLLARY 31. *Affine restrictions of Ω obtained by deleting lines different from ℓ are isomorphic.*

What can one say about central collineations with an axis different from ℓ? The main result in this regard is Theorem 35 below, and it will be used to determine the dilatations of one of the affine restrictions of Ω. The next theorem gathers together some important facts concerning these collineations that are stated and proved in [17], pp. 110–112, as Theorems 4.3.7, 4.3.10, and 4.3.9. One of these facts uses the notion of the associate of a point on ℓ that was formulated in the proof of Theorem 27. Theorem 32 is needed to prove Corollaries 33 and 34, and all three results are used to prove Theorem 35. The two corollaries are also used to prove Theorem 37, which is really a result about the dual of Ω.

THEOREM 32. (i) *If a non-identity elation of Ω has an axis different from ℓ, then its center cannot be the point of intersection of its axis with ℓ.*

(ii) *Every non-identity central collineation either has ℓ as its axis, or else its center is on ℓ.*

(iii) *If a non-identity collineation has its center on ℓ and if its axis is different from ℓ, then the point of intersection of the axis with ℓ must be the associate of the center.*

The next three corollaries and the next theorem are related to results stated in Example 4.3.4 of [20], pp. 136–137. First, it follows from parts (i) and (ii) of the previous theorem that no non-identity elation can have an axis different from ℓ.

COROLLARY 33. *Every elation of Ω has axis ℓ. Consequently, the elations of Ω are precisely the translations $\tau_{a,b}$.*

COROLLARY 34. *The collineations of Ω with a given axis $k \neq \ell$ are all homologies and have same center: the associate of the point of intersection of k with ℓ.*

PROOF. Certainly, no elation can have an axis k different from ℓ, by the previous corollary. Consider a homology with axis k. Its center must lie on line ℓ, by part (ii) of the previous theorem. The center cannot be the point of intersection of k with ℓ, since a homology is not an elation. Therefore, the center must be the associate of the point of intersection, by part (iii) of the previous theorem. ⊣

THEOREM 35. *There are eight collineations of Ω with a given axis $k \neq \ell$.*

PROOF. All collineations with axis k are homologies, and they all have the same center p (on ℓ), by the previous corollary. Because homologies are not elations, the center p does not lie on k. Homologies are completely determined by their action on a single point different from the center and not lying on the axis. Let q be such a point. The line pq intersects k in a point s different from p or q (see Figure 2). A homology with center p and axis k fixes the line pq (since this line passes through the center), and it also fixes the points p (the center) and s (a point on the axis). Therefore, it must map q to an image point on pq different from p and s. There are eight possibilities for such an image point (since the line pq has eight points different from p and s), so there are at most eight homologies with center p and axis k.

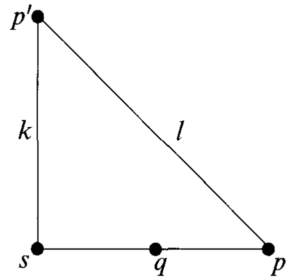

FIGURE 2.

For each of the eight choices for an image point r, there actually is a homology with center p and axis k mapping q to r. It suffices to show this for just one line k, since there are collineations (automorphisms) of Ω mapping any two lines different from ℓ to one another, by Theorem 30. Take k to be the line $x = 0$. It intersects ℓ in the point (∞), so, by Theorem 32(iii), the homologies with axis k have center (0), the associate of (∞). For every non-zero element a in Q, the collineation $\sigma_{a,1}$ (taking each point (x, y) to $(x \cdot a, y)$, each point (m) to $(a^{-1} \cdot m)$, and the point (∞) to itself) has axis $x = 0$ and center (0). The eight homologies $\sigma_{a,1}$ are clearly mutually distinct, since, for instance, they map $(1, 1)$ to different points. ⊣

Theorems 27 and 35 describe the collineations of Ω with a given axis. The next two results concern the collineations with a given center. This information is needed to determine the dilatations of affine restrictions of the dual of Ω, in the next section. Corollary 29 above was formulated in order to prove the next corollary.

COROLLARY 36. *There are two collineations of Ω with a given center not on ℓ.*

PROOF. If a point p is not on ℓ, then the axis of the collineations with center p must be ℓ, by Theorem 32(ii). Consequently, such collineations are homologies, and there are just two of them (including the identity collineation), by Corollary 29. ⊣

There are more collineations with a center on ℓ.

THEOREM 37. *There are 72 collineations of Ω with a given center on ℓ.*

PROOF. Let p be any point on ℓ, and p' its associate. There are, of course, nine lines through p' that are different from ℓ. For each such line k, there are seven non-identity collineations with axis k, by Theorem 35. Moreover, they all have center p, by Corollary 34. Thus, there are a total of $7 \cdot 9 = 63$ non-identity central collineations with axes different from ℓ and with center p. There are also eight non-identity elations with axis ℓ and center p, by Corollary 33. For instance, when p is the point (0), these elations are the translations $\tau_{a,0}$, which map the lines through (0) — the lines $y = b$ — to themselves. (In general, for each of the ten points p

on ℓ, there are eight non-identity translations with center p, yielding a total of 80 non-identity translations.) Finally, there is the identity collineation. This gives a total of $63 + 8 + 1 = 72$ collineations with center p. ⊣

We now turn to the affine restrictions of Ω. Let Ω_0 be the affine plane obtained from Ω by deleting the line ℓ, and Ω_1 the affine plane obtained from Ω by deleting some fixed line k different from ℓ.

COROLLARY 38. *The affine plane* Ω_0 *has* 311,040 *collineations and* 162 *dilatations.*

PROOF. This is an immediate consequence of Theorem 27. The affine collineations are just the (restrictions of the) collineations of the projective extension — in this case, Ω — that leave the line ℓ at infinity fixed. There are 311,040 of them, since every collineation of Ω leaves ℓ fixed. The dilatations are the central collineations that have the line at infinity as their axis, and there are 162 of them. ⊣

COROLLARY 39. *The affine plane* Ω_1 *has* 3456 *collineations and* 8 *dilatations.*

PROOF. The plane Ω has $9^2 + 9 + 1 = 91$ lines. Hence, k has an orbit of 90 lines, by Theorem 30. Divide 311,040 — the number of collineations of Ω — by 90 — the size of the orbit of k — to get 3456, the number of collineations of Ω_1, by Theorem 22. The dilatations are the central collineations of Ω with axis k, and there are eight of them, by Theorem 35. ⊣

COROLLARY 40. Ω *has two affine restrictions, up to isomorphism:* Ω_0 *and* Ω_1.

PROOF. Every affine restriction of Ω different from Ω_0 is isomorphic to Ω_1, by Corollary 31. The two restrictions Ω_0 and Ω_1 cannot be isomorphic, since their collineation groups have different cardinalities, by Corollaries 38 and 39. ⊣

4.3. The plane Ω^d. The plane Ω^d is dual to Ω. Its points are the lines of Ω, and its lines are the points of Ω. It has the same incidence relation as Ω, but with the role of points and lines reversed. Thus, it has precisely the same collineations as Ω. However, the axis of a collineation in Ω^d is the center of the collineation in Ω, and the center of a collineation in Ω^d is the axis of the collineation in Ω. With this in mind, it is easy to obtain from the previous theorems and corollaries the requisite information about the collineations and dilatations of the affine restrictions of Ω^d.

To refer to lines in Ω^d, we shall employ the letters that denoted points in Ω, and to refer to points in Ω^d, we shall employ the letters that denoted lines in Ω. The plane Ω^d has a special point ℓ that is fixed by all collineations: it is just the line ℓ of Ω.

COROLLARY 41. *Affine restrictions of* Ω^d *obtained by deleting lines through* ℓ *are isomorphic to one another. Similarly, affine restrictions of* Ω^d *obtained by deleting lines not passing through* ℓ *are isomorphic to each other.*

PROOF. The first assertion is just a restatement of the fact that there are collineations of Ω mapping any point on the line ℓ to any other point on ℓ. The second is a restatement of the fact that there are collineations mapping any point not on ℓ to any other point not on ℓ. See Corollary 28. ⊣

Let Ω_0^d denote the affine restriction of Ω^d obtained by deleting some fixed line through ℓ, and Ω_1^d the affine restriction of Ω^d obtained by deleting some fixed line not passing through ℓ. By the previous corollary, these two affine planes are uniquely determined up to isomorphism (that is, they do not depend on the particular choice of the line to be deleted).

COROLLARY 42. *The affine plane Ω_0^d has* 31,104 *collineations and* 72 *dilatations.*

PROOF. The number of collineations of Ω_0^d is equal to the number of collineations of Ω^d divided by the orbit of the line at infinity, by Theorem 22. The plane Ω^d has 311,040 collineations, by Theorem 27 (it has the same collineations as Ω), and the orbit of a line through ℓ has size 10, by Corollary 28 (the orbit of a point on line ℓ in Ω has size 10). The quotient of these two numbers is 31,104.

The number of dilatations of Ω_0^d is just the number of collineations of Ω^d that have the line at infinity — say, p — as their axis. This is the same as the number of collineations of Ω that have the point p as their center. Since p is incident with ℓ, this number is 72, by Theorem 37. ⊣

COROLLARY 43. *The affine plane Ω_1^d has* 3,840 *collineations and* 2 *dilatations.*

PROOF. The orbit of a line not through ℓ has size 81, by Corollary 28. Therefore, the plane Ω_1^d has

$$311{,}040 \ / \ 81 = 3{,}840$$

collineations, by Theorem 22. The dilatations of Ω_1^d are the collineations of Ω^d that have the line at infinity — say, q — as their axis. These are the collineations of Ω that have the point q as their center. Since q is not incident with ℓ, there are just two of them, by Corollary 36. ⊣

COROLLARY 44. *Ω^d has two affine restrictions up to isomorphism: Ω_0^d and Ω_1^d.*

PROOF. Every affine restriction of Ω^d is isomorphic to either Ω_0^d or Ω_1^d, by Corollary 41. The two restrictions Ω_0^d and Ω_1^d are not isomorphic since their collineation groups have different cardinalities, by Corollaries 42 and 43. ⊣

4.4. The plane Ψ. The "Hughes" plane Ψ of order nine was first investigated by Veblen-Wedderburn [24]. Just as Ω, it is constructed with the help of the near-field Q. Following [17], we use (*homogeneous coordinate*) *vectors* — ordered triples $x = (x_0, x_1, x_2)$ with coordinates in Q — to describe it. Define the *sum* $x + y$ of two vectors x and y to be the vector whose coordinates are the sums (in Q) of the corresponding coordinates of x and y. Define the *product* $x \cdot a$ of the vector x with a *scalar* (element) a of Q to be the vector whose coordinates are the products of the corresponding coordinates of x with a (the scalar a being multiplied on the right).

Two vectors are said to be equivalent if one is a non-zero scalar multiple of the other. The *points* of the projective plane Ψ are defined to be the equivalence classes of non-zero vectors. Thus, a point p consists of all non-zero scalar multiples of some non-zero vector (triple) of elements of Q. A point is *real* if some vector in it has only real coordinates (coordinates that are 0, 1, or -1), and otherwise it is *complex*. There are 13 real points in Ψ and 78 complex points. A point is often represented by a fixed vector (that belongs to the point). For instance, the phrase "the point $(1, 0, -1)$" refers to the unique point that contains the vector $(1, 0, -1)$.

The *lines* of Ψ are the sets of points

$$pq = \{p\} \cup \{p \cdot a + q : a \in Q\},$$

where p is a *real* point and q an arbitrary point distinct from p. This means that a real vector x in p and an arbitrary vector y in q are fixed, and the line pq is defined to consist of the ten points determined, in one case, by the vector x, and in the other nine cases by the vectors $x \cdot a + y$, where a ranges over the scalars in Q.

This definition does not depend on the choice of the vectors x and y (see [17], pp. 130–131). Notice that a is allowed to be zero, so the line pq contains both the points p and q. A line is *real* if it contains at least two real points, and *complex* otherwise.

The set of all real points of Ψ, together with the set of all real lines (restricted to the set of real points), form a subplane Δ_3 of Ψ that is isomorphic to the projective plane of order three (constructed over the three-element field). Every real line of Ψ corresponds to a unique real line of Δ_3, namely its restriction, and conversely. Thus, Ψ has 13 real lines (the number of lines of the projective plane of order three) and 78 complex lines. Every line of Δ_3 — and consequently every real line of Ψ — has 4 real points (the number of points on each line of the projective plane of order three). It follows that real lines of Ψ have 6 complex points. Every complex line pq has exactly one real point: it has at most one real point, by definition, and at least one real point, namely p. Two real lines, when restricted to Δ_3, intersect in a point of Δ_3. Therefore, any two real lines intersect in a real point. There is exactly one real line passing through any given complex point q: there is at most one such line, since two real lines have a real intersection; there is at least one such line, namely the line determined by q and its conjugate. (This line must contain both the sum and the difference of q with its conjugate, and both the sum and difference are real points). See Chapter 5 of [17] for details.

By the Fundamental Theorem of Projective Geometry (applied to the plane Δ_3), the collineations of Δ_3 are precisely the transformations induced by 3 by 3 non-singular (invertible) matrices M with real coordinates (coordinates 0, 1, and -1) via the operation of matrix multiplication (where, for this purpose, vectors x are thought of as 3 by 1 column vectors, and the product Mx as the product of a 3 by 3 matrix with a 3 by 1 column). If x is allowed to range over arbitrary non-zero vectors with entries from Q, then the matrix M induces a collineation of the projective plane Ψ. The basic reason is that the entries of M are the real numbers 0, 1, and -1, and multiplication by these numbers is commutative and right-distributive over addition in Q. Consequently, M acts linearly on vectors: the point represented by a vector $x \cdot a + y \cdot b$ (with x, y vectors, and a, b scalars in Q) is mapped by M to the point represented by the vector $(Mx) \cdot a + (My) \cdot b$. Hence, if the induced correspondence maps points p and q to p' and q', then it maps each point $p \cdot a + q$ on the line pq to the corresponding point $p' \cdot a + q'$ on the line $p'q'$, and hence must be a collineation. Because a collineation of Δ_3 maps, *per force*, real points to real points and real lines to real lines, the extended collineation induced by M on Ψ must also map real points to real points, and real line to real lines.

Homologies and elations of Δ_3 (which are matrix induced), extend to homologies and elations of Ψ with the same center and axis. In more detail, suppose a collineation of Δ_3 — induced, say, by the matrix M — is not the identity, and has a real axis ℓ and real center r. Let p and q be two real points on ℓ. Then the points of ℓ in Ψ can be written in the form $p \cdot a + q$, for a in Q. Each such point is mapped by the extended collineation to the point $p \cdot a + q$, by the above remarks (because the collineation fixes the real points p and q). In other words, ℓ remains the axis of the extended collineation of Ψ. Every non-identity collineation with an axis has a unique center, and the only invariant lines under the collineation are the axis and the lines through the center. (This statement is the dual of the assertion that every non-identity collineation with a center has a unique axis, and the only invariant points are the center and the points of the axis — see the remarks following Theorem 23.) The extended collineation induced by M has axis ℓ and therefore has a center. Because it fixes the four real lines through r, the center must in fact be r. These observations are summarized in the following theorem (see [17], pp. 133-134, and in particular Theorem 5.1.5). It will play a critical role in determining the collineations and dilatations of affine planes derived from Ψ.

THEOREM 45. *The transformations of Ψ induced by real, non-singular matrices are collineations mapping real points to real points and real lines to real lines. An elation or homology of Δ_3 extends to an elation or homology of Ψ with the same axis and center.*

COROLLARY 46. *There are 5,616 collineations of Ψ induced by real, non-singular matrices.*

PROOF. The matrix-induced collineations of Ψ correspond exactly to the matrix-induced collineations of Δ_3. It is well known that every collineation of the projective plane of order 3 is matrix induced, and vice versa (by the Fundamental Theorem of Projective Geometry; compare also [17], p. 43, Theorems 2.4.4 and 2.4.5). There are exactly 5,616 collineations of the projective plane over the three element field, by Theorem 21, so there must be the same number of matrix-induced collineations of Ψ. ⊣

Besides matrix-induced collineations, the plane Ψ also has collineations that are induced by automorphisms of the near-field Q. An automorphism of Q must leave 0 and 1, and -1 fixed. Therefore, it is uniquely determined by its action on the element i. The image of i must be complex (since real elements are mapped to themselves), so there are six possibilities for an image. Every one of them determines an automorphism. For instance, an automorphism δ of Q is determined by specifying that $\delta(i) = -1 - i$. Indeed,

$$\delta(a + b \cdot i) = \delta(a) + \delta(b) \cdot \delta(i) = a + b \cdot (-1 - i).$$

In particular,

$$\delta(1 + i) = -i, \qquad \delta(1 - i) = -1 + i,$$

and δ is its own inverse. It is straightforward to check that each of the six automorphisms leaves no complex point fixed.

Every automorphism γ of Q induces a bijection of vectors: the image of a vector x under γ is defined to be the vector whose coordinates are the images of the coordinates of x under γ. Since γ is an automorphism of the near-field, the induced bijection preserves vector addition and scalar multiplication. The mapping γ also induces a transformation of the plane Ψ: the image of a point p under γ is defined to be the image of the vectors in p under γ, and the image of a line ℓ under γ is defined to be the image of the points on ℓ under γ. The points and lines of Ψ are defined in terms of the vector operations, so the transformation induced by γ on Ψ must be a collineation. (See [17], p. 135, Theorem 5.1.7.)

THEOREM 47. *The six automorphisms of Q induce collineations of Ψ that fix all the real points and real lines. The collineations induced by non-identity automorphisms of Q leave no complex points fixed.*

PROOF. We have already seen that the transformation induced by an automorphism of Q is a collineation. Since automorphisms of Q fix all real numbers, they must fix real vectors, and hence also real points. Real lines contain four real points, so such lines must also be left fixed.

Suppose that γ is an automorphism of Q different from the identity. Let q be a complex point. The vectors in q are non-zero — say, their first coordinates are non-zero — and not real. Multiply any such vector by a suitable scalar to obtain a vector of the form $x = (1, x_1, x_2)$ in q. This is the only vector in q with first coordinate 1, since every other vector in q is a non-zero scalar multiple of x. The image of q under γ is determined by the vector

$$\gamma(x) = (\gamma(1), \gamma(x_1), \gamma(x_2)) = (1, \gamma(x_1), \gamma(x_2)).$$

This image vector is certainly different from x, since at least one of the two coordinates x_1 and x_2 must be complex, and γ leaves no complex number fixed. But x is the only vector in q with first coordinate 1. Thus, the image $\gamma(x)$ must determine a point different from q. ⊣

COROLLARY 48. *The only central collineation of Ψ that is induced by an automorphism of Q is the identity.*

PROOF. Suppose γ is a central collineation of Ψ induced by an automorphism of Q. Then γ fixes all real points and all real lines. A central collineation is completely determined by what it does to one point different from its center and not on its axis. No matter what the center and axis of γ are, there is certainly a real point that is different from the center and not on the axis. Of course, γ maps any such real point to itself, so γ must be the identity collineation. ⊣

The next theorem is the analogue for Ψ of the Fundamental Theorem of Projective Geometry. It is stated and proved in [17], Chapter 5 (see, in particular, Theorem 5.1.8), and the proof is rather involved.

THEOREM 49. *Every collineation of Ψ can be written in a unique way as a composition of a collineation induced by an automorphism of Q with a collineation induced by a real, non-singular matrix (and, conversely, every such composition is a collineation of Ψ).*

An immediate consequence of this theorem, together with Corollary 46 and Theorem 47, is a computation of the exact number of collineations of Ψ (see [17], p. 136, Exercise 5.1.3).

COROLLARY 50. *The plane Ψ has exactly*

$$6 \cdot 5{,}616 = 33{,}696$$

collineations.

Another consequence of Theorem 49 is that every collineation of Ψ maps the subplane Δ_3 onto itself (see [17], p. 136, Lemma 2).

COROLLARY 51. *Every collineation of Ψ maps real points to real points, and real lines to real lines.*

PROOF. Every collineation is the composition of a collineation induced by an automorphism of Q with a collineation induced by a real, non-singular matrix. By Theorems 45 and 47, both types of collineations map real points to real points, and real lines to real lines. The composition of two such collineations obviously has the same property. ⊣

The goal of the next two theorems is to show that central collineations of Ψ are just extensions of the central collineations of Δ_3. This leads directly to a determination of the dilatations of the two derived affine planes. The first theorem deals with the case when the collineation has a real center.

THEOREM 52. *The collineations of Ψ with a real center are just the extensions of the central collineations of Δ_3. In particular, they all have a real axis.*

PROOF. The direction of the theorem from right to left is contained in Theorem 45. For the reverse direction, suppose σ is a collineation of Ψ. Use Theorem 49 to write σ as a composition of collineations

$$\sigma = \gamma \circ \tau,$$

where γ is induced by an automorphism of Q, and τ by a real, non-singular matrix — that is, τ is the extension of a collineation of Δ_3. The collineation γ fixes all real points and lines, by Theorem 47, so the collineation

$$\tau = \gamma^{-1} \circ \sigma$$

must agree with σ on all real points and lines.

Assume σ has a real center p, that is, σ fixes all lines passing through p. Then it must fix the four real lines passing through p. It follows that τ fixes the four real lines through p. Thus, the restriction of τ to Δ_3 is a central collineation with a real center. Consequently, this restriction has a real axis. The extension τ must have the same real center and real axis, by Theorem 45. Because σ and τ both fix all lines through p, so does the composition

$$\gamma = \sigma \circ \tau^{-1}.$$

But the only automorphism-induced collineation with a center is the identity, by Corollary 48. Thus, γ is the identity collineation, so σ coincides with τ. Since τ has a real axis, so does σ. In other words, σ is the extension of a (matrix-induced) collineation of Δ_3 with a real center and a real axis. ⊣

The next theorem deals with the case of a complex center.

THEOREM 53. *No non-identity collineation of Ψ has a complex center.*

PROOF. Let σ be a collineation of Ψ with a complex center q. The goal is to prove that σ must be the identity. Use Theorem 49 to write σ as a composition of collineations

$$\sigma = \gamma \circ \tau, \tag{1}$$

where γ is induced by an automorphism of Q, and τ by a real, non-singular matrix. Through every complex point passes exactly one real line, so there are nine complex lines and one real line passing through q. On each of the nine complex lines there is exactly one real point, and on the real line there are four real points. The ten lines are concurrent in a complex point, so the 13 real points on all of the lines are distinct from one another.

By assumption, the collineation σ maps each line through q to itself. Since every collineation of Ψ maps real points to real points, σ must map the unique real point on each complex line through q to itself. Thus, σ fixes at least nine real points. It also maps the real line through q to itself, and therefore permutes the four real points on it. In other words, the restriction of σ to Δ_3 fixes one line ℓ of Δ_3 and leaves invariant all of the points of Δ_3 not on this line.

A central collineation is completely determined by its action on one point not on the axis and different from the center. There is certainly a real point that is different from the center, and not on the axis, of σ, and that is also not on ℓ. (The line ℓ has four real points, and the axis of σ has at most four real points. Thus, there are at least five other real points to choose from.) Any such point is left fixed by σ, so σ must be the identity. ⊣

The two theorems just proved combine to give the desired analysis of the central collineations of the plane Ψ.

COROLLARY 54. *The central collineations of the plane Ψ are precisely the extensions of the central collineations of the subplane Δ_3. In particular, they all have real centers and real axes.*

PROOF. The center of a non-identity central collineation of Ψ cannot be complex, by the previous theorem, so it must be real. Therefore, Theorem 52 applies to yield the desired conclusion. ⊣

There are collineations of Ψ that map any given real line to any other. (Take the extension to Ψ of any collineation of Δ_3 that maps the restriction of the first real line to the restriction of the second. See [17], p. 134–135.) It is also true that there are collineations of Ψ mapping any given complex line to any other. (See [17], pp. 134–135, Theorem 5.1.6, for a proof.) A collineation never maps a real line to a complex line, or vice versa, by Corollary 51.

THEOREM 55. *Any real line of Ψ can be mapped to any other real line — but not to a complex line — by a collineation. Similarly, any complex line of Ψ can be mapped to any other complex line — but not to a real line — by a collineation.*

The theorem can be used to compute the orbits of lines of Ψ, and consequently the number of collineations of the affine restrictions. It also shows that, up to isomorphism, there are only two affine restrictions of Ψ.

COROLLARY 56. *All affine restrictions of Ψ obtained by deleting real lines are isomorphic, and all affine restrictions obtained by deleting complex lines are isomorphic.*

Let Ψ_0 be the affine restriction of Ψ obtained by deleting some fixed real line k, and Ψ_1 the affine restriction obtained by deleting some fixed complex line m.

COROLLARY 57. *The affine plane Ψ_0 has* 2,592 *collineations and* 18 *dilatations.*

PROOF. The plane Ψ has 13 real lines, and k can be mapped to any one of them (but not to a complex line) by a collineation, by Theorem 55. Therefore, the orbit of k has size 13. There are 33,696 collineations of Ψ, by Corollary 50. Consequently, the affine restriction Ψ_0 has

$$33{,}696\,/\,13 = 2{,}592$$

collineations, by Theorem 22.

There are $3 \cdot 3 \cdot 2 = 18$ collineations of Δ_3 whose axis is (the restriction of) k, by Theorem 24. The extensions of these collineations to Ψ continue to have axis k, by Theorem 45, and they are the only collineations of Ψ with axis k, by Corollary 54 and Theorem 45. ⊣

COROLLARY 58. *The affine plane* Ψ_1 *has* 432 *collineations and* 1 *dilatation.*

PROOF. The plane Ψ has 78 complex lines, and m can be mapped to any one of them (but not to a real line) by a collineation, by Theorem 55. Therefore, the orbit of m has size 78. There are 33,696 collineations of Ψ, by Corollary 50. Consequently, the affine restriction Ψ_1 has

$$33{,}696 \,/\, 78 = 432$$

collineations, by Theorem 22.

No non-identity collineation of Ψ has a complex axis, by Corollary 54. Therefore, the plane Ψ_1 has only one dilatation: the identity. ⊣

COROLLARY 59. Ψ *has two affine restrictions, up to isomorphism*: Ψ_0 *and* Ψ_1.

PROOF. Every affine restriction of the plane Ψ is isomorphic to either Ψ_0 or Ψ_1, by Corollary 56. These two latter affine planes cannot be isomorphic to each other: their collineation groups have different sizes, by Corollary 57 and Corollary 58. ⊣

4.5. Affine planes of order nine. We are now in a position to enumerate the affine planes of order nine. This enumeration is certainly known; see, for instance, [2], p. 117.

THEOREM 60. *The affine planes of order nine are, up to isomorphism,*

$$\Delta_0, \qquad \Omega_0, \qquad \Omega_1, \qquad \Omega_0^d, \qquad \Omega_1^d, \qquad \Psi_0, \qquad \Psi_1.$$

PROOF. The projective planes

$$\Delta, \qquad \Omega, \qquad \Omega^d, \qquad \Psi \tag{1}$$

are mutually non-isomorphic. For instance, the sizes of the collineation groups of Δ and Ψ are different from each other and from the size of the collineation groups of Ω and Ω^d; see Theorems 21 and 27, Corollary 50, and the remarks preceding Corollary 41. The collineation groups of Ω and Ω^d have the same size. However, Ω has a line that is fixed by every collineation, whereas Ω^d does not; see Theorem 27 and Corollary 28. It was shown by Lam, Kolesova, and Theil, using a computer, that these are the only projective planes of order nine (see [12]). Therefore, every affine plane of order nine must be an affine restriction of one of these projective planes.

An affine restriction of one of the projective planes in (1) cannot be isomorphic to an affine restriction of any of the others. For an isomorphism between two such affine restrictions would extend to an isomorphism between the projective planes, by Theorem 1. On the other hand, the non-isomorphic affine restrictions of the projective planes in (1) have been completely determined (in Corollaries 26, 40, 44, and 59): they are precisely the planes formulated in the statement of the theorem. ⊣

4.6. Extensions of isomorphisms. There is one more known geometric result, of a different nature, that is needed in the present paper. It was shown in Theorem 1 that every collineation of an affine geometry has a (unique) extension to a collineation of the geometry at infinity. The converse is, in general, not true: there may be isomorphisms between the geometries at infinity that do not extend to isomorphisms between the affine geometries. However, in dimensions greater than two, this cannot happen.

THEOREM 61. *For affine geometries of dimension at least* 3, *every isomorphism between their geometries at infinity can be extended to an isomorphism between the projective extensions, and hence between the affine geometries themselves.*

PROOF. We sketch the main ideas of the proof. The various facts used about affine and projective geometries may be found in Chapters 6–9 of [2]. Let D be an affine geometry of dimension $n \geq 3$. Then D and its projective extension D^* are Desarguesian (every affine and projective geometry of dimension at least three is Desarguesian), and hence so is the geometry at infinity. A skew field F can be constructed on a line — any line — using two points on the line as parameters, and using an auxiliary point not on the line, to define the operations of addition and multiplication and to verify the skew field properties. The skew fields constructed on different lines are all isomorphic, independently of the points chosen as parameters and as auxiliary points.

The geometry D^* can be coordinatized using a vector space V of dimension $n+1$ over the skew field F. Its points are the 1-dimensional subspaces of V,

$$[v] = \{v \cdot a : a \in F\}$$

where v is a non-zero vector from V (with coordinates in F), and $v \cdot a$ is the scalar product of v with a. The affine points — the points of D — have the form

$$[(v_0, v_1, v_2, \dots)],$$

where $v_0 \neq 0$, and the points at infinity have the form

$$[(0, v_1, v_2, v_3, \dots)],$$

where not all v_i are zero. The lines of D^* are the 2-dimensional subspaces of V:

$$[u, v] = \{u \cdot a + v \cdot b : a, b \in F\},$$

where u and v are non-zero, linearly independent vectors in V. In the case of lines at infinity, both the vectors u and v are points at infinity.

By the Fundamental Theorem of Projective Geometry applied to the geometry at infinity, every collineation σ of the geometry at infinity can be written as the composition of a collineation induced by an automorphism α of the skew field, and a collineation induced by a linear transformation τ of the subspace of V that determines the points at infinity, the subspace of vectors of the form

$$v = (0, v_1, v_2, v_3, \dots).$$

The collineation induced by α — say, γ_α — is defined on points $[v]$ of the geometry at infinity by the formula

$$\gamma_\alpha([v]) = \gamma_\alpha([(0, v_1, v_2, \dots)]) = [(0, \alpha(v_1), \alpha(v_2), \alpha(v_3), \dots)].$$

The automorphism induced by τ — say, δ_τ — is defined by the rule

$$\delta_\tau([v]) = [\tau(v)]$$

for all such points.

Let τ^* be the mapping on V defined by

$$\tau^*(v_0, v_1, v_2, \dots) = (v_0, u_1, u_2, \dots),$$

where

$$\tau(0, v_1, v_2, \dots) = (0, u_1, u_2, \dots).$$

It is easy to check that τ^* is a linear transformation of V extending τ. The collineations γ_α and δ_τ can be extended to collineations on all of D^* by requiring

$$\gamma_\alpha([(v_0, v_1, v_2 \dots)]) = [(\alpha(v_0), \alpha(v_1), \alpha(v_2), \dots)]$$

and

$$\delta_\tau([(v_0, v_1, v_2, \dots)]) = [\tau^*(v_0, v_1, v_2, \dots)]$$

for all non-zero vectors $(v_0, v_1, v_2, \dots)$ in V. The composition of these two extensions is clearly an extension of σ. This proves that *every collineation of the geometry at infinity can be extended to a collineation of all of* D^*. Of course, this extension maps the affine space D to itself.

Now suppose that two affine geometries of dimension at least 3 have isomorphic geometries at infinity, say σ is such an isomorphism. Then the skew fields coordinatizing the two affine geometries are isomorphic, and the two geometries at infinity have the same dimension. It follows that the affine geometries also have the same dimension: it is one more than the dimension of the geometries at infinity. Affine geometries of the same dimension and coordinatized by the same skew field are isomorphic, so there is an isomorphism λ between the two affine geometries. By Theorem 1, λ extends to an isomorphism between the projective extensions of the two affine geometries. The composition

$$\mu = \lambda^{-1} \circ \sigma$$

is a collineation of the geometry at infinity of D. By the observations of the previous paragraph, μ extends to a collineation of D^*. (We use the same symbol to denote this extension). Then the composition $\lambda \circ \mu$ is an isomorphism from the projective extension of D to that of D' which agrees with σ on the points at infinity. ⊣

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