

Frobenius algebras and planar open string topological field theories

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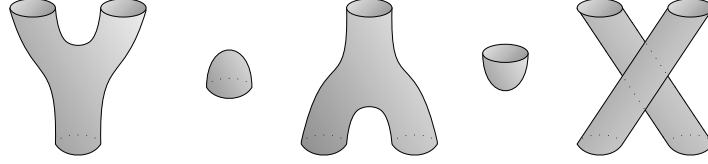
Abstract

Motivated by the Moore-Segal axioms for an open-closed topological field theory, we consider planar open string topological field theories. We rigorously define a category **2Thick** whose objects and morphisms can be thought of as open strings and diffeomorphism classes of planar open string worldsheets. Just as the category of 2-dimensional cobordisms can be described as the free symmetric monoidal category on a commutative Frobenius algebra, **2Thick** is shown to be the free monoidal category on a noncommutative Frobenius algebra, hence justifying this choice of data in the Moore-Segal axioms. Our formalism is inherently categorical allowing us to generalize this result. As a stepping stone towards topological membrane theory we define a 2-category of open strings, planar open string worldsheets, and isotopy classes of 3-dimensional membranes defined by diffeomorphisms of the open string worldsheets. This 2-category is shown to be the free (weak monoidal) 2-category on a ‘categorified Frobenius algebra’, meaning that categorified Frobenius algebras determine invariants of these 3-dimensional membranes.

1 Introduction

It is well known that 2-dimensional topological quantum field theories are equivalent to commutative Frobenius algebras [1, 16, 29]. This simple algebraic characterization arises directly from the simplicity of **2Cob**, the 2-dimensional cobordism category. In fact, **2Cob** admits a completely algebraic description as the free symmetric monoidal category on a commutative Frobenius algebra. This description is apparent when one examines the generating cobordisms in

2Cob:



These cobordisms equip the circle with a commutative algebra and coalgebra structure, and the relations on the generators of **2Cob** make the circle into a commutative Frobenius algebra.

This universal property simplifies the construction of functors from **2Cob** into other symmetric monoidal categories, such as the category of vector spaces or graded vector spaces. Simply find an example of a commutative Frobenius algebra in a symmetric monoidal category \mathcal{C} , and the universal property of **2Cob** determines a functor $\mathbf{2Cob} \rightarrow \mathcal{C}$. Since a 2-dimensional topological quantum field theory is a functor from **2Cob** into a symmetric monoidal category, it is clear that the universal property of **2Cob** greatly simplifies the study of 2-dimensional topological quantum field theories.

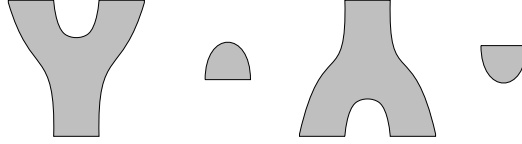
Topological string theory has recently been a topic of considerable interest to physicists, and it presents interesting new problems for mathematicians as well. The celebrated Segal axioms [46] of a conformal field theory yield 2-dimensional topological quantum field theories as the simplest examples, namely those that do not depend on the conformal structure of the string backgrounds [47]. In this context, the circle is interpreted as a ‘closed string’, and 2-dimensional cobordisms as ‘closed string worldsheets’. Topological string theory for open strings, which inevitably also contains a closed string sector, can be described as an open *and* closed topological field theory coupled to topological gravity. In this case, interactions between the open and closed strings induce a much richer 2-dimensional topology.

In an open-closed topological field theory, closed strings can evolve into open strings as demonstrated by the cobordism below:



The topological sewing conditions for these surfaces were first analyzed by Cardy and Lewellen [9, 37], and a category of such surfaces has also been constructed [3]. Our primary interest will be a subset of the conditions imposed on the open strings. In particular, the topology generated by worldsheets that can be

embedded in the plane:



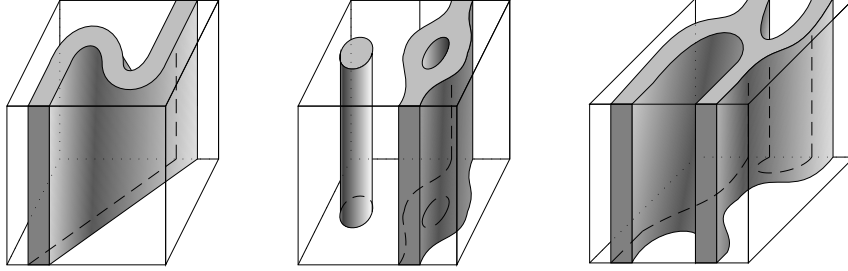
A sketch of the axioms for an open-closed field theory was given by Moore and Segal [40] and has been refined in the more recent work of Lazaroiu [35, 36]. In these axioms, the open strings are represented algebraically by a *noncommutative* Frobenius algebra. This axiom is intended to express the topological difference between the circle and the interval, or closed string and open string. One of the results of this paper is a justification of this particular axiom.

We define a topological category **2Thick** whose objects are open strings and whose morphisms are diffeomorphism classes of planar worldsheets. More precisely, the objects of this category are the natural numbers, and the morphisms are diffeomorphism classes of smooth oriented compact surfaces whose boundary is equipped with disjoint distinguished intervals. We will sometimes refer to **2Thick** as the category of ‘2-dimensional thick tangles’ since it is a ‘thickened’ version of the category of tangles embedded in the plane. We then define a planar open string topological field theory as a functor from **2Thick** into a monoidal category \mathcal{C} . We prove that **2Thick** is the monoidal category freely generated by a (noncommutative) Frobenius algebra. This implies that a planar open string topological field theory is equivalent to a (not necessarily commutative) Frobenius algebra, thereby rigorously establishing the Moore-Segal axioms for this portion of the open string data. It also implies that a Frobenius algebra in an arbitrary monoidal category determines an invariant of 2-dimensional thick tangles.

The idea that all of the known string theories might arise as the effective limit of a higher-dimensional theory known as *M*-theory [55, 57] has prompted many to consider ‘topological membrane theories’ [10, 30]. In topological membrane theory the 2-dimensional string worldsheets arise as boundaries of 3-dimensional membranes. This approach has been used to show that various types of string theories (e.g. heterotic, open, unoriented) arise from the choice of boundary conditions on a topological membrane [13, 19]. These important results suggest that one should consider a topological theory that describes open strings and their worldsheets as part of a larger 3-dimensional topology which reduces to the usual theory on the boundary. In this paper we take some first steps towards realizing such a theory.

Our description of open strings and their worldsheets has the advantage that it can easily be generalized to distinguish diffeomorphic worldsheets. Thus, rather than considering worldsheets only up to diffeomorphism, our formalism can be extended to describe open strings, *all* planar open string worldsheets, *and* the 3-dimensional membranes defined by isotopy classes of diffeomorphisms of planar open string worldsheets that preserve the initial and final states. Some

examples are given below:



Because of the three levels of structure present, this type of topology is most naturally described using 2-categories rather than categories.

Extensions of this sort have already been considered in the symmetric case of **2Cob**. Using what we will call **2Cob**₂, a 2-categorical extension of the usual two-dimensional cobordism category **2Cob**, Ulrike Tillmann has considered extended 2-dimensional topological quantum field theories consisting of 2-functors from **2Cob**₂ into a generalized version of linear categories – the 2-category of linear abelian categories [54]. By encoding 3-dimensional information into a 2-dimensional topological quantum field theory, Tillmann has established interesting connections between 2-dimensional TQFT's extended in this way and traditional 3-dimensional topological field theories.

Analogous to Tillmann's construction of extended (closed) string topological field theories, we consider extended planar open string topological field theories. We begin by rigorously defining the 2-category of open strings, planar open string worldsheets, and isotopy classes of worldsheets embedded in the cube, that we denote as **3Thick**. We will sometimes refer to this 2-category as the 2-category of '3-dimensional thick tangles' since it is a categorification of the category **2Thick**.

We will show that **3Thick** can be described by a universal property that completely characterizes these particular extended open string field theories. More precisely, we show that **3Thick** is the (semistrict monoidal) 2-category freely generated by a 'categorified Frobenius algebra'. A categorified Frobenius algebra, is just a category with the structure of a Frobenius algebra where all the usual axioms only hold up to coherent isomorphism. This description of **3Thick** implies that categorified Frobenius algebras, or pseudo Frobenius algebras as we will refer to them, determine invariants of 3-dimensional thick tangles.

The fact that our restricted choice of open string worldsheets can be described as the free monoidal category on a noncommutative Frobenius algebra is perhaps not that surprising to the expert. What is interesting is the fact that this result arises quite naturally from higher-dimensional category theory. Even more interesting is that, once seen from a categorical perspective, a universal property for the 2-category of open string membranes in the cube is obtained by a straightforward categorification of the lower dimensional result.

All of the results in this paper are obtained using the relationship between Frobenius algebras and adjunctions [33]. An adjunction is just the abstraction of the definition of adjoint functors between categories. Indeed, an adjunction in the 2-category **Cat** whose objects are categories, morphisms are functors, and whose 2-morphisms are natural transformations, produces the usual notion of adjoint functors. By considering adjunctions in arbitrary 2-categories we are able to rephrase the definition of a Frobenius algebra in the language of category theory. Using 2-categorical ‘string diagrams’, adjunctions provide a bridge that illuminates the relationship between Frobenius algebras and thick tangles.

This abstract approach has the advantage that, phrased in an intrinsically categorical way, it can easily be generalized to define pseudo Frobenius algebras and establish their relationship with 3-dimensional thick tangles. Using pseudoadjunctions, the 3-categorical analogs of adjunctions, we are able to define pseudo Frobenius algebras. Furthermore, using a version of string diagrams for 3-categories we are able to show that the 2-category **3Thick** is the 2-category freely generated by a pseudo Frobenius algebra. Thus, our results not only provide an algebraic description of **2Thick** and **3Thick**, but also provide an abstract framework with which these results naturally arise.

2 The two-dimensional case

We begin with the 2-dimensional case. In Section 2.1 we review some of the definitions of a Frobenius algebra that will appear in a categorified form later in the paper. In Section 2.2 we review the theory of 2-categorical string diagrams and we use them in Sections 2.3 and 2.4 to establish the relationship between Frobenius algebras and adjunctions. In Section 2.5 we define the monoidal category of 2-dimensional thick tangles and state one of the main theorems of the paper. This theorem follows from the categorified version that will be proven later on.

2.1 Frobenius Algebras

Frobenius algebras appear throughout many branches of mathematics. This can be attributed to the robustness of the definition and the many ways it can be formulated. Originally, the term Frobenius algebra referred to an algebra A with the property that $A \simeq A^*$ as right A -modules. Later, Nakayama provided many equivalent definitions of Frobenius algebras, including the characterization of a Frobenius algebra as a finite dimensional \mathbb{k} -algebra equipped with a linear functional $\varepsilon: A \rightarrow \mathbb{k}$ whose nullspace contains no nontrivial ideals [42, 43]. Another equivalent definition, motivated by topological considerations, defines a Frobenius algebra as an algebra A equipped with a coalgebra structure where the comultiplication is a map of A -modules. This topologically motivated definition arose in order to rigorously establish the theorem that a two-dimensional topological quantum field theory is essentially the same as a commutative Frobenius algebra. This fact was first observed by Dijkgraaf [16], but it was not rigor-

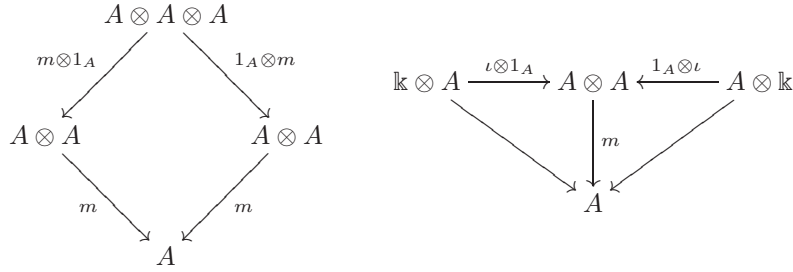
ously shown until Frobenius algebras were reformulated in terms of a coalgebra structure [1]. A modern proof of this result is given by Kock [29].

It is clear that there are many equivalent definitions of Frobenius algebras, each suited for various applications. We make no attempt to describe them all nor establish their equivalence. Rather, we will focus on a few that are relevant to our main theorem relating Frobenius algebras to thick tangles. Each of these definitions has a topological nature that we will explain using a diagrammatic shorthand notation for the relevant maps in each definition. These maps are, in a sense, the ‘topological building blocks’ that each definition provides. Loosely speaking, our theorem states that this shorthand notation is much more than a convenient device. In fact, we will show that the topological picture is exactly equivalent to the algebraic picture in the sense that any topological manipulation of the diagrams corresponds to algebraic manipulations of a Frobenius algebra compatible with its axioms.

Proposition 1 *Let A be a vector space equipped with morphisms:*

- $m: A \otimes A \rightarrow A$, and
- $\iota: \mathbb{k} \rightarrow A$,

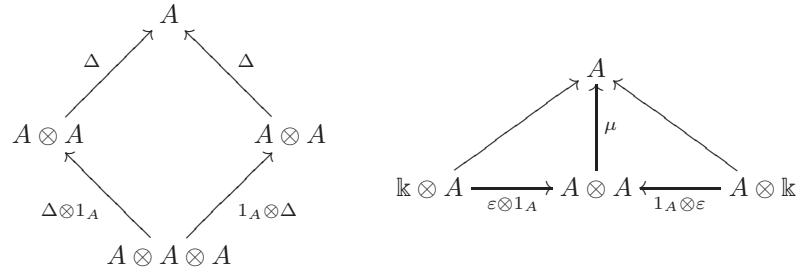
satisfying the algebra axioms:



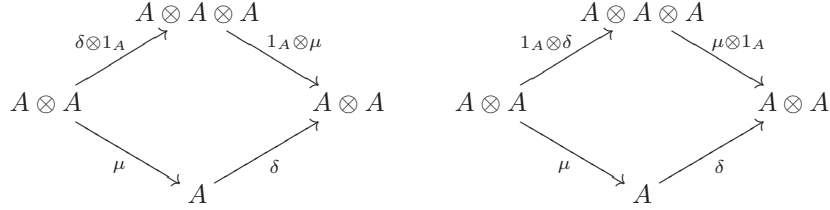
Then the following conditions on a form $\varepsilon: A \rightarrow \mathbb{k}$ are equivalent:

- i.) *There exists a map $\Delta: A \rightarrow A \otimes A$ that, together with $\varepsilon: A \rightarrow \mathbb{k}$, defines a coalgebra structure on A satisfying the Frobenius identities. Or more explicitly, the algebra A is equipped with a map $\Delta: A \rightarrow A \otimes A$ such that the following diagrams commute:*

- *the coalgebra axioms:*

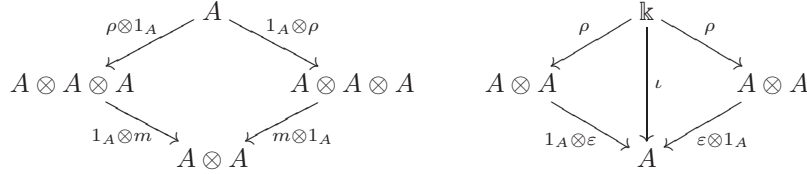


- the Frobenius identities:



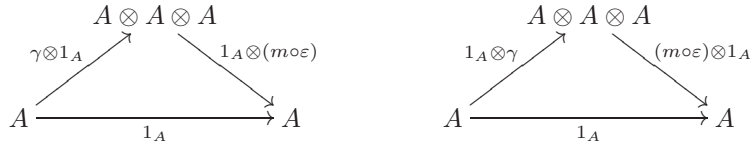
- ii.) There exists a copairing $\rho: \mathbb{k} \rightarrow A \otimes A$ that equips A with two equivalent comultiplications and counits. That is to say, the following diagrams commute:

- the equivalence of the two coalgebra structures:



- iii.) The form $\varepsilon: A \rightarrow \mathbb{k}$ is nondegenerate. Or more explicitly, the algebra A is equipped with copairing $\gamma: \mathbb{k} \rightarrow A \otimes A$ making the following diagrams commute:

- the nondegeneracy of the pairing:

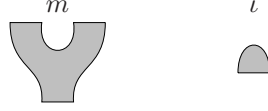


An algebra A equipped with a form $\varepsilon: A \rightarrow \mathbb{k}$ satisfying any of these equivalent conditions is called a Frobenius algebra.

To prove this proposition it is helpful to invoke a short hand notation for the morphisms and axioms in each of the above characterizations. Each formulation of Frobenius algebra has a topological interpretation where the specified maps represent the topological building blocks and the axioms are topologically motivated. We will see later on that these diagrams can be made completely rigorous using string diagrams from 2-category theory. Nevertheless, for the reader hesitant in using diagrams for mathematical proofs, we will explain how to translate this shorthand notation into the more traditional commutative diagrams.

To begin, notice that the multiplication $m: A \otimes A \rightarrow A$ for the algebra structure takes as input the tensor product of two copies of A and outputs one

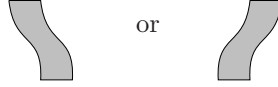
copy of A . Similarly, if we regard the ground field \mathbb{k} as the tensor product of no copies of A , then the unit for the multiplication $\iota: \mathbb{k} \rightarrow A$ takes no copies of A and produces a single copy of A . We draw these morphisms in our shorthand notation as follows:



The pictures are read from top to bottom and the horizontal lines at the top and bottom of each diagram are thought of as a copy of A . Since the ground field \mathbb{k} is thought of as no copies of A , it is represented by no horizontal line. The identity map is drawn as:

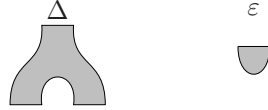


or sometimes as:

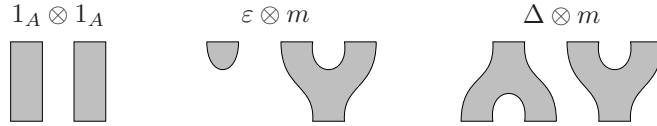


for aesthetic purposes.

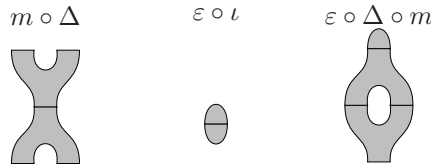
Similarly, we draw the coalgebra maps as upsidedown versions of the algebra maps:



Notice from the diagram for the multiplication and comultiplication that the tensor product $A \otimes A$ is represented by placing the diagrams side by side. In a similar manner, we draw the tensor product of morphisms by placing them side by side in the diagrams:



To compose morphisms the diagrams are stacked on top of each other. Here are a few illustrative examples:



Using these simple rules we can draw the axioms in the definition of a Frobenius algebra. The associativity axiom is depicted as:



and the unit laws as:

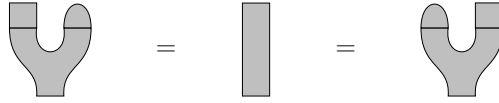
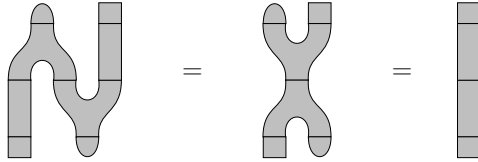


Figure 2.1 summarizes the different formulations of Frobenius algebra presented in Proposition 1. Now we are ready to prove the proposition.

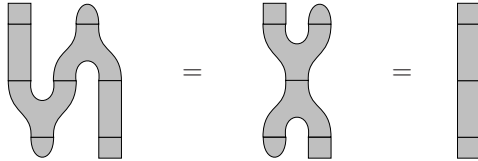
Proof of Proposition 1. (i. \Rightarrow iii.) Given an algebra and coalgebra structure on A define the map $\gamma: \mathbb{k} \rightarrow A \otimes A$ as follows:



This copairing makes the form ε nondegenerate because the equalities:



and



follow from the Frobenius identities.

(iii. \Rightarrow ii.) We only need to check that the axioms of ii. are satisfied since ii. and iii. have the same morphisms. The first axiom is proved as follows:

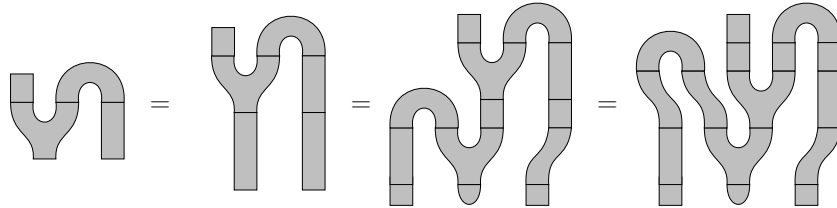



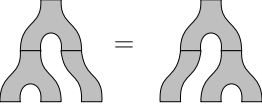

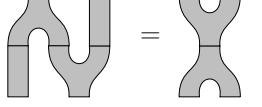
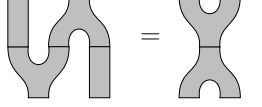
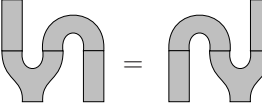
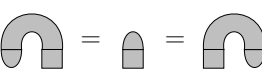


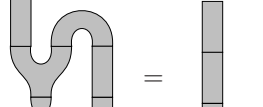
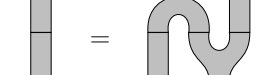


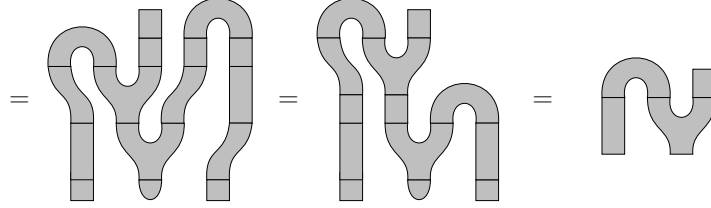


Figure 1: Definitions of Frobenius Algebra

| | | | |
|--------------------|---|---|---|
| Description | <i>An algebra equipped with a coalgebra structure satisfying the Frobenius identities</i> | <i>An algebra equipped with a copairing inducing two equivalent coalgebra structures</i> | <i>An algebra with a non-degenerate form</i> |
| Morphisms |  |  |  |
| Axioms | <p><i>algebra axioms and</i></p>     | <p><i>algebra axioms and</i></p>     | <p><i>algebra axioms and</i></p>     |



For the reader who finds these topological manipulations a bit too cavalier, we will translate this proof into a traditional commutative diagram. Although, the equivalence between the algebraic and topological pictures will be established later on, an explicit example will make this relationship more apparent.

We want to show that the map

$$A \xrightarrow{1_A \otimes \rho} A \otimes A \otimes A \xrightarrow{m \otimes 1_A} A \otimes A$$

is equal to the map

$$A \xrightarrow{\rho \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes m} A \otimes A.$$

In the diagrammatic proof, the first equality, where the bottom of the diagram is stretched, corresponds to composing with the identity $1_A \otimes 1_A$. The second equality is an application of the nondegeneracy axiom:

$$\begin{array}{ccccccc}
 A & \xrightarrow{1_A \otimes \rho} & A^{\otimes 3} & \xrightarrow{m \otimes 1_A} & A^{\otimes 2} & \xrightarrow{1_A \otimes 1_A} & A^{\otimes 2} \\
 & & & & \downarrow \rho \otimes 1_A \otimes 1_A & & \nearrow 1_A \otimes \varepsilon \otimes 1_A \\
 & & & & A^{\otimes 4} & \xrightarrow{1_A \otimes m \otimes 1_A} & A^{\otimes 3}
 \end{array}$$

where $A^{\otimes n}$ is short hand for the n -fold tensor product of A .

The next equality holds in any monoidal category as a consequence of the tensor product being functor. What it amounts to is the fact that in a monoidal category it does not matter in which order we apply maps between the tensor product of objects. In particular, whenever we have morphisms $f: A \rightarrow A'$ and $g: B \rightarrow B'$ in a monoidal category we get a commuting square:

$$\begin{array}{ccc}
 A \otimes B & \xrightarrow{A \otimes g} & A \otimes B' \\
 \downarrow f \otimes B & & \downarrow f \otimes B' \\
 A' \otimes B & \xrightarrow{A' \otimes g} & A' \otimes B'
 \end{array}$$

In our case we are interested in the special case when $f = \rho$ and $g = m \otimes 1_A$. Hence our commutative diagram becomes:

$$\begin{array}{ccccccc}
 A & \xrightarrow{1_A \otimes \rho} & A^{\otimes 3} & \xrightarrow{m \otimes 1_A} & A^{\otimes 2} & \xrightarrow{1_A^{\otimes 2}} & A^{\otimes 2} \\
 & & \downarrow \rho \otimes 1_A^{\otimes 3} & & \downarrow \rho \otimes 1_A^{\otimes 2} & & \nearrow 1_A \otimes \varepsilon \otimes 1_A \\
 & & A^{\otimes 5} & \xrightarrow{1_A^{\otimes 2} \otimes m \otimes 1_A} & A^{\otimes 4} & \xrightarrow{1_A \otimes m \otimes 1_A} & A^{\otimes 3}
 \end{array}$$

Next we use the associativity of the multiplication:

$$\begin{array}{ccccccc}
 A & \xrightarrow{1_A \otimes \rho} & A^{\otimes 3} & \xrightarrow{m \otimes 1_A} & A^{\otimes 2} & \xrightarrow{1_A^{\otimes 2}} & A^{\otimes 2} \\
 & & \downarrow \rho \otimes 1_A^{\otimes 3} & & \downarrow \rho \otimes 1_A^{\otimes 2} & & \nearrow 1_A \otimes \varepsilon \otimes 1_A \\
 & & A^{\otimes 5} & \xrightarrow{1_A^{\otimes 2} \otimes m \otimes 1_A} & A^{\otimes 4} & \xrightarrow{1_A \otimes m \otimes 1_A} & A^{\otimes 3} \\
 & & & \searrow 1_A \otimes m \otimes 1_A^{\otimes 2} & \nearrow 1_A \otimes m \otimes 1_A & & \\
 & & & & A^{\otimes 4} & &
 \end{array}$$

Now we use two applications of the property of monoidal categories mentioned above:

$$\begin{array}{ccccccc}
 A & \xrightarrow{1_A \otimes \rho} & A^{\otimes 3} & \xrightarrow{m \otimes 1_A} & A^{\otimes 2} & \xrightarrow{1_A^{\otimes 2}} & A^{\otimes 2} \\
 \downarrow \rho \otimes 1_A & & \downarrow \rho \otimes 1_A^{\otimes 3} & & \downarrow \rho \otimes 1_A^{\otimes 2} & & \nearrow 1_A \otimes \varepsilon \otimes 1_A \\
 A^{\otimes 3} & \xrightarrow{1_A^{\otimes 3} \otimes \rho} & A^{\otimes 5} & \xrightarrow{1_A^{\otimes 2} \otimes m \otimes 1_A} & A^{\otimes 4} & \xrightarrow{1_A \otimes m \otimes 1_A} & A^{\otimes 3} \\
 & \searrow 1_A \otimes m & \searrow 1_A \otimes m \otimes 1_A^{\otimes 2} & \nearrow 1_A \otimes m \otimes 1_A & & & \\
 & & A^{\otimes 2} & \xrightarrow{1_A^{\otimes 2} \otimes \rho} & A^{\otimes 4} & &
 \end{array}$$

Finally, we apply the other nondegeneracy axiom.

$$\begin{array}{ccccccc}
 A & \xrightarrow{1_A \otimes \rho} & A^{\otimes 3} & \xrightarrow{m \otimes 1_A} & A^{\otimes 2} & \xrightarrow{1_A^{\otimes 2}} & A^{\otimes 2} \\
 \downarrow \rho \otimes 1_A & & \downarrow \rho \otimes 1_A^{\otimes 3} & & \downarrow \rho \otimes 1_A^{\otimes 2} & & \nearrow 1_A \otimes \varepsilon \otimes 1_A \\
 A^{\otimes 3} & \xrightarrow{1_A^{\otimes 3} \otimes \rho} & A^{\otimes 5} & \xrightarrow{1_A^{\otimes 2} \otimes m \otimes 1_A} & A^{\otimes 4} & \xrightarrow{1_A \otimes m \otimes 1_A} & A^{\otimes 3} \\
 & \searrow 1_A \otimes m & \searrow 1_A \otimes m \otimes 1_A^{\otimes 2} & \nearrow 1_A \otimes m \otimes 1_A & & & \\
 & & A^{\otimes 2} & \xrightarrow{1_A^{\otimes 2} \otimes \rho} & A^{\otimes 4} & & \\
 & & & \nearrow 1_A^{\otimes 2} & & \nearrow 1_A^{\otimes 2} &
 \end{array}$$

For the second axiom of ii. we use the unit laws for the algebra together with the nondegeneracy axioms.

(ii. \Rightarrow i.) To show that ii. implies i. we must define the comultiplication $\Delta: A \rightarrow A \otimes A$ map. We do this as follows:

We leave it as an exercise to the reader to verify that this defines a coalgebra structure on A . We will verify the Frobenius identities. The first Frobenius identity is proved by the string of equalities:

which follows from the associativity of the multiplication and the axioms of a monoidal category. With just a little more work, the second Frobenius identity is proved similarly:

□

The observant reader will have noticed that in proving Proposition 1 we never used the fact that A was a vector space. In fact, if we translate all of the pictures above into commutative diagrams it is clear that the proof of Proposition 1 relied only on the abstract properties of the maps in each characterization. This means that if we were to take all of the diagrams in the discussion above and place them in some other category \mathcal{C} ‘sufficiently like **Vect**’ then the proof would still be valid, and we can define Frobenius algebras in the category \mathcal{C} using any of the above descriptions. This process of writing mathematical objects using only commutative diagrams and placing them in other categories where they make sense is what category theorists call *internalization*. An internalized Frobenius algebra is sometimes referred to as a *Frobenius object*.

Let’s consider what kind of additional structure the category \mathcal{C} must have in order to be ‘sufficiently like **Vect**’, that is, in order to define a Frobenius object in \mathcal{C} . Since all of our diagrams required the tensor product $A \otimes A$ our category \mathcal{C} should have a multiplication. Furthermore, since we used the ‘unit vector space’, or the ground field \mathbb{k} , our category \mathcal{C} should have a unit for the above multiplication. But all this just amounts to the definition of a monoidal category. So the notion of a Frobenius algebra makes sense in any monoidal category \mathcal{C} , and our diagrammatic proof of Proposition 1 shows that all three characterizations are equivalent in \mathcal{C} . Eventually we will provide yet another equivalent definition of Frobenius algebra in terms of adjunctions, but first we will need to develop some categorical language.

2.2 String diagrams for 2-categories

Now we begin the process of rephrasing the definition of a Frobenius algebra in the language of higher-dimensional category theory. Our goal is to define Frobenius algebras using the notion of an adjunction in a 2-category. In order to understand the relationship between adjunctions and Frobenius algebras we will use string diagrams from 2-category theory. Once we have defined Frobenius algebras in terms of adjunctions we can then study the corresponding string diagrams and the topology of 2-dimensional thick tangles will begin to emerge.

Before we get too far ahead of ourselves we recall the definition of a 2-category. Speaking colloquially the idea is that, just as categories have objects, morphisms between objects, and various axioms regarding composites and identities, 2-categories have objects, morphisms between objects, and *2-morphisms* between morphisms together with axioms for the composites and identities of both morphisms and 2-morphisms. What makes 2-categories so interesting is that the axioms for the 1-morphisms can now hold only up to coherent isomorphism. Thus, rather than having composition be associative on the nose, we can instead require that composition be associative up to isomorphism satisfying laws of its own. Similarly with the identity constraints. When all the axioms of a 2-category hold up to isomorphism it is called a weak 2-category or bicategory [8].

Below we discuss string diagrams for *strict* 2-categories. That is, 2-categories where composition is strictly associative and the identity constraints hold as

equations. We justify this choice with two reasons. First, there is a coherence theorem for bicategories which states that every bicategory is biequivalent to a strict 2-category [51]. Biequivalence is a notion of equivalence between bicategories in which the usual axioms of an equivalence only hold up to coherent isomorphism. This notion of equivalence is often the most natural one to use between bicategories. The second reason for considering strict 2-categories rather than bicategories is because this notion provides the correct framework to understand the topology of 2-dimensional thick tangles. However, in Section 4 we will see that the topology of 3-dimensional thick tangles does require the more general notion of a bicategory.

Typically, the objects of a category are represented geometrically as little bullets, and the morphisms of a category as arrows between the bullets:

$$\bullet \xrightarrow{F} \bullet$$

The composite of two morphisms is usually drawn as:

$$\bullet \xrightarrow{F} \bullet \xrightarrow{G} \bullet$$

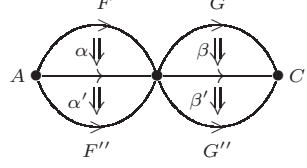
In a 2-category we have objects and morphisms as before, but now there are 2-morphisms going between morphisms:

There are two types of composites coming from the two ways we can glue these pictures together. We have a *vertical composite*:

and a *horizontal composite*:

The axioms of a 2-category require that there exist an identity 2-morphism for vertical and horizontal composition, but perhaps the most interesting axiom of

a 2-category is the *interchange law*:



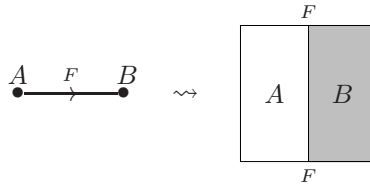
The interchange law asserts that the above diagram is unambiguously defined. That is, the result of vertically composing and then horizontally composing is the same as first horizontally composing and then vertically composing.

This sort of notation, often referred to as globular notation, is common among 2-category theorists. But for our purposes, we will be interested in a different sort of diagram related to 2-categories — string diagrams. String diagrams are just the Poincaré duals of the usual globular diagrams. We construct string diagrams from globular diagrams by inverting the dimensions of the picture. The objects, typically represented in globular notation as 0-dimensional points (or bullets), become 2-dimensional surfaces in the new notation. The morphism represented by 1-dimensional edges remain 1-dimensional, and the 2-dimensional globes representing 2-morphisms become 0-dimensional in the string diagram.

To see how this works, let \mathcal{D} be a 2-category. We depict objects A and B of \mathcal{D} as surfaces:

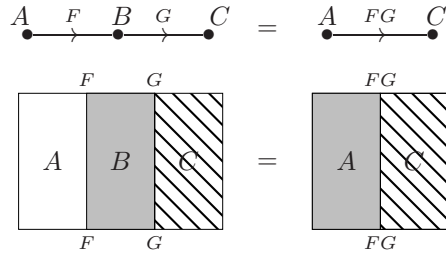


where we have shaded the surface corresponding to B in order to easily distinguish it from A . Below we show the process of Poincaré dualizing a morphism $F: A \rightarrow B$ in \mathcal{D} from the globular notation into the string notation:



On the left is a morphism drawn in the usual globular notation, and on the right the same morphism drawn in string notation.

The composite of morphisms $F: A \rightarrow B$ and $G: B \rightarrow C$ in \mathcal{D} is drawn as:



As a convenient convention, the identity morphism of objects in \mathcal{D} are not drawn. This convention allows the identification:

$$\begin{array}{|c|} \hline A \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1_A \\ \hline A & A \\ \hline 1_A \end{array}$$

of string diagrams.

If $F, F': A \rightarrow B$ are morphism of \mathcal{D} and $\alpha: F \Rightarrow F'$ is a 2-morphism, then we depict this as:

$$\begin{array}{c} \begin{array}{ccc} & F & \\ & \curvearrowright & \\ A & \alpha & B \\ & \curvearrowleft & \\ & F' & \end{array} & \rightsquigarrow & \begin{array}{|c|c|} \hline F \\ \hline A & \alpha & B \\ \hline F' \end{array} \end{array}$$

where the circle surrounding α is thought of as being 0-dimensional. We include it only as a means to label the 2-morphism. Following the convention that the identity morphisms are not drawn in string diagrams we omit identity 2-morphisms as well. This allows the identification:

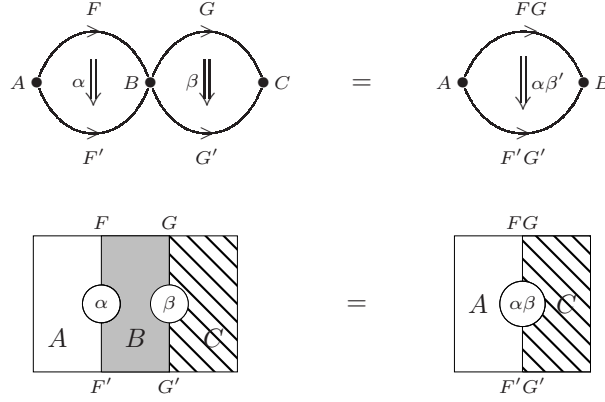
$$\begin{array}{|c|c|} \hline F \\ \hline A & 1_F & B \\ \hline F \end{array} = \begin{array}{|c|c|} \hline F \\ \hline A & B \\ \hline F \end{array}$$

of string diagrams.

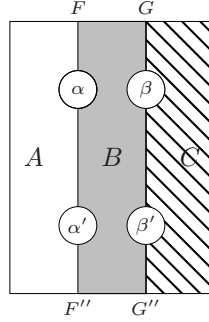
Horizontal and vertical composition is achieved in the obvious way. If $F, F', F'': A \rightarrow B$ are morphisms and $\alpha: F \Rightarrow F'$, $\alpha': F' \Rightarrow F''$ are 2-morphisms, then the vertical composite of α and α' is:

$$\begin{array}{c} \begin{array}{ccc} & F & \\ & \curvearrowright & \\ A & \alpha & B \\ & \curvearrowleft & \\ & F' & \\ & \curvearrowright & \\ & \alpha' & \\ & \curvearrowleft & \\ & F'' & \end{array} & = & \begin{array}{ccc} & F & \\ & \curvearrowright & \\ A & \alpha\alpha' & B \\ & \curvearrowleft & \\ & F'' & \end{array} \\ \\ \begin{array}{|c|c|} \hline F \\ \hline \alpha \\ \hline A & B \\ \hline \alpha' \\ \hline F'' \end{array} & = & \begin{array}{|c|c|} \hline F \\ \hline A & \alpha\alpha' & B \\ \hline F'' \end{array} \end{array}$$

If $F, F': A \rightarrow B$ and $G, G': B \rightarrow C$ with $\alpha: F \Rightarrow F'$ and $\beta: G \Rightarrow G'$, then the horizontal composite is depicted as follows:



The interchange law in the 2-category \mathcal{D} tells us that each string diagram such as:



can be uniquely interpreted as a diagram in \mathcal{D} . Later we will see that, together with the identity morphisms, the interchange law justifies vertical and horizontal topological deformations of these string diagrams. Of course, theorems proving that any topological deformation of a string diagram produces an equivalent diagram in \mathcal{D} have been proved by Joyal and Street [22, 23], but we prefer to demonstrate the needed deformations directly from the axioms.

Before moving on, as an application of this string diagram technology, we would like to translate the definition of an adjunction into string diagrams. An adjunction is a concept that makes sense in any 2-category, although most of us learn about adjunctions in the 2-category **Cat** consisting of categories, functors, and natural transformations. Adjunctions in this 2-category are just adjoint functors which are prevalent throughout mathematics. We will be interested in adjunctions because they provide a categorical framework for understanding Frobenius algebras. Once we set up this framework we will be able to categorify it and arrive at a definition of a pseudo Frobenius algebra.

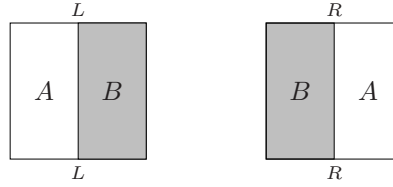
Let \mathcal{D} be a 2-category. An *adjunction* (A, B, L, R, i, e) in \mathcal{D} consists of objects A and B , morphism $L: A \rightarrow B$ and $R: B \rightarrow A$, and 2-morphisms $i: 1_A \Rightarrow LR$

and $e: RL \Rightarrow 1_B$ called the *unit* and *counit* of the adjunction, such that the following diagrams:

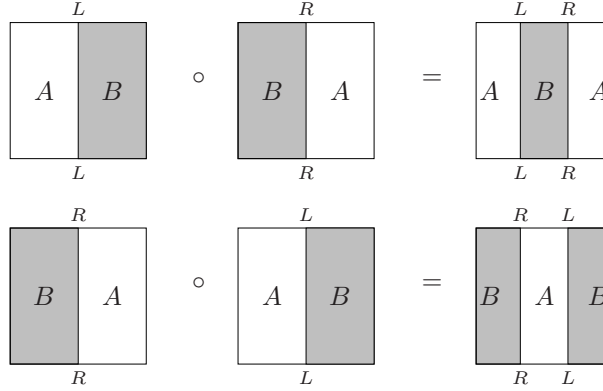
$$\begin{array}{ccc}
 & LRL & \\
 iL \nearrow & & \searrow Le \\
 L & \xrightarrow{1_L} & L
 \end{array}
 \qquad
 \begin{array}{ccc}
 & RLR & \\
 Ri \nearrow & & \searrow eR \\
 R & \xrightarrow{1_R} & R
 \end{array}$$

commute. We sometimes refer to these identities as the *zig-zag* identities for reasons that will soon become apparent.

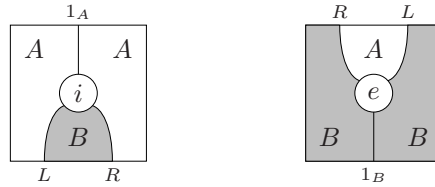
In string notation the morphisms L and R are depicted as:



and their composites as:

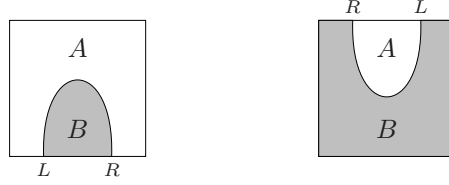


With the above diagrams in mind it is easy to see that the unit and counit of the adjunction can be depicted as:

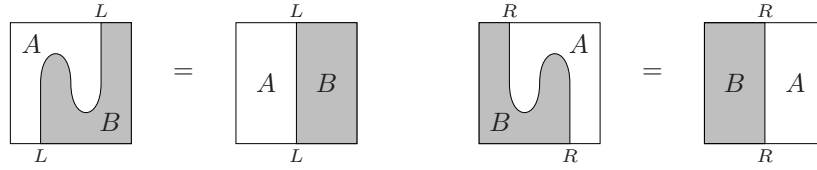


However, applying the convention that identity morphisms are not depicted in the string diagrams, these pictures can be simplified. Further, we can omit the labels of i and e when no confusion is likely to arise, in which case the unit and

counit become:



and in this notation the zig-zag identities become:



which explains their name. These identities say that a ‘zig-zag’ can be ‘straightened out’.

2.3 The walking adjunction

In the previous section we defined the notion of an adjunction in a 2-category. In this section we will study a very special adjunction — the ‘walking adjunction’. Before defining the walking adjunction we feel obliged to motivate this seemingly strange terminology. Imagine you are sitting in a small table in the back of a crowded pub enjoying a beer with a close friend, when in walks a fellow with enormous bushy eyebrows. His eyebrows are in fact so large that it seems his entire body serves no other purpose than to provide a frame for these enormous eyebrows to perch on. In that case, you might be tempted to comment to your friend: “Look, there goes the walking pair of eyebrows”. In the same way, the walking adjunction is the minimal amount of structure needed in order to have an adjunction; it is the 2-category freely generated by an adjunction. The 2-category is merely the frame upon which the adjunction ‘perches’. This ‘walking’ terminology was coined by James Dolan and our explanation of it is adapted from the expository writings of John Baez [4]. In general, we will refer to the free X on generating data Y as the *walking* Y .

We begin by explaining what it means for some structure to ‘generate a 2-category’.

Definition 2 *Let Y be a structure that can be defined in an arbitrary 2-category. If Y consists of objects Y_1, Y_2, \dots, Y_n , morphisms $F_1, F_2, \dots, F_{n'}$, and 2-morphisms $\alpha_1, \alpha_2, \dots, \alpha_{n''}$ for $n, n', n'' \in \mathbb{Z}^+$, then the 2-category X is generated by Y if:*

- (i.) *Every object of X is some Y_i .*
- (ii.) *Every 1-morphism of X can be obtained by compositions from the F_i ’s and 1_{Y_i} ’s.*

(iii.) Every 2-morphism of X is obtained by horizontal and vertical composition from the 2-morphisms α_i , and identity 2-morphisms 1_F for arbitrary 1-morphisms F .

We say that X is freely generated¹ by Y if the set of Y objects in \mathcal{C} are in bijection with 2-functors from X into \mathcal{C} , for every 2-category \mathcal{C} .

Definition 3. The walking adjunction Adj is the 2-category freely generated by:

- objects A and B ,
- morphisms $L: A \rightarrow B$ and $R: B \rightarrow A$, and
- 2-morphisms $i: 1_A \rightarrow L \circ R$ and $e: R \circ L \rightarrow 1_B$

such that the following diagrams:

$$\begin{array}{ccc}
 L & \xrightarrow{iL} & LRL \\
 & \searrow 1_L & \downarrow Le \\
 & & L
 \end{array}
 \qquad
 \begin{array}{ccc}
 R & \xrightarrow{Ri} & RLR \\
 & \searrow 1_R & \downarrow eR \\
 & & R
 \end{array}$$

commute.

Thus, the walking adjunction Adj , or the 2-category freely generated by an adjunction, has the property that every adjunction in a 2-category \mathcal{C} corresponds to a 2-functor $\text{Adj} \rightarrow \mathcal{C}$. The walking adjunction was first studied under the name of the free adjunction [45]. Another description of the walking adjunction can be obtained from its categorification explicitly constructed by Lack [31].

The walking adjunction turns out to be intimately related to the walking monoid. The notion of a monoid makes sense in any monoidal category, so the walking monoid is just the monoidal category freely generated by a monoid. Recall that a monoidal category is a category equipped with a multiplication functor and a unit object. However, it is sometimes useful to think of monoidal category as a special kind of 2-category. More precisely, a monoidal category is just a one object 2-category:

| \mathcal{M} – monoidal category | \mathcal{C} – 2-category |
|-----------------------------------|---------------------------------------|
| objects | morphisms |
| tensor product of objects | composition of morphisms |
| morphisms | 2-morphisms |
| composition | vertical composition of 2-morphisms |
| tensor product of morphisms | horizontal composition of 2-morphisms |

¹This can be stated more elegantly using globular sets [7], but for our purposes this definition suffices.

The objects of the monoidal category are just the morphisms of the 2-category \mathcal{C} . The tensor product of objects comes from the composition of morphisms in \mathcal{C} . Note that every morphism of \mathcal{C} is composable since it only has one object. The morphisms of the monoidal category \mathcal{M} are the 2-morphisms of the 2-category \mathcal{C} . The composition of morphisms in \mathcal{M} comes from the vertical composition of morphisms in \mathcal{C} , and the tensor product of morphisms in \mathcal{M} comes from the horizontal composition of morphisms in \mathcal{C} .

We can also apply this same trick in reverse. Given a monoidal category \mathcal{M} , we can regard \mathcal{M} as a 2-category $\Sigma(\mathcal{M})$ with one object by applying the above procedure in reverse. We sometimes refer to the 2-category $\Sigma(\mathcal{M})$ as the *suspension* of the monoidal category \mathcal{M} . Since a monoidal category is just a one object 2-category, we can use Definition 2 to define the monoidal category freely generated by a monoid and dually the monoidal category freely generated on a comonoid.

Below we define the walking monoid **Mon** and the walking comonoid **Comon**, not to be confused with **Mon** the category whose objects are monoids, and **Comon** the category whose objects are comonoids.

Definition 4 *The walking monoid **Mon** is the monoidal category freely generated by:*

- objects A and I , and
- morphisms $m: A \otimes A \rightarrow A$ and $\iota: I \rightarrow A$

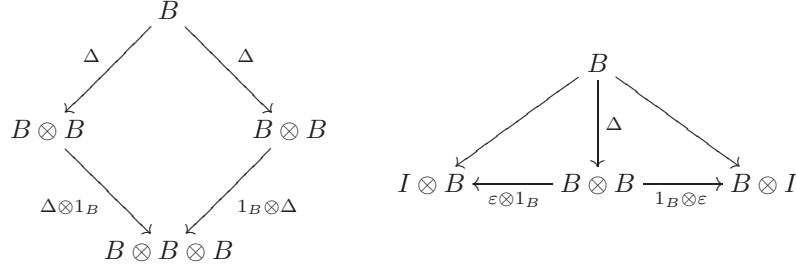
such that

$$\begin{array}{ccc}
 & A \otimes A \otimes A & \\
 m \otimes 1_A \swarrow & & \searrow 1_A \otimes m \\
 A \otimes A & & A \otimes A \\
 m \searrow & & \swarrow m \\
 & A &
 \end{array}
 \qquad
 \begin{array}{ccccc}
 I \otimes A & \xrightarrow{\iota \otimes 1_A} & A \otimes A & \xleftarrow{1_A \otimes \iota} & A \otimes I \\
 & \searrow & \downarrow m & \swarrow & \\
 & & A & &
 \end{array}$$

commute. Dually, the walking comonoid **Comon** is the monoidal category freely generated by:

- objects B and I , and
- morphisms $\Delta: B \rightarrow B \otimes B$ and $\varepsilon: B \rightarrow I$

such that



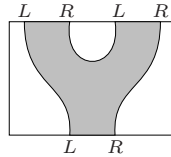
commute.

The relationship between the walking adjunction and the walking monoid and comonoid is summed up by:

Theorem 5 *The monoidal category $\text{Hom}(A, A)$ in the walking adjunction is the walking monoid and the monoidal category $\text{Hom}(B, B)$ in the walking adjunction is the walking comonoid.*

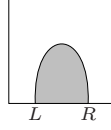
Before proving this theorem we pause briefly to explain why this theorem is morally true from a topological perspective. Using string diagrams it will be clear that the walking monoid is contained in $\text{Hom}(A, A)$ in the walking adjunction. The argument we use was originally developed by Müger [41] and was later elaborated on by Baez [5]. Unfortunately, the main difficulty of this proof is showing the opposite inclusion: that the walking monoid arises as $\text{Hom}(A, A)$ in the walking adjunction. Or to put it another way, the difficulty lies in showing that $\text{Hom}(A, A)$ has *only* the relations of the walking monoid and no additional relations. We will see that both inclusions can be proved using a bit of abstract category theory; but the topological arguments provide the right intuition for understanding the more abstract result.

In the walking adjunction, all of the objects in the monoidal category $\text{Hom}(A, A)$ are generated by the morphism LR . We will show that LR is equipped with the structure of a monoid. The multiplication on the object LR is defined, using the counit of the adjunction, as $m := LeR: LRLR \rightarrow LR$. We depict this in string notation as follows:



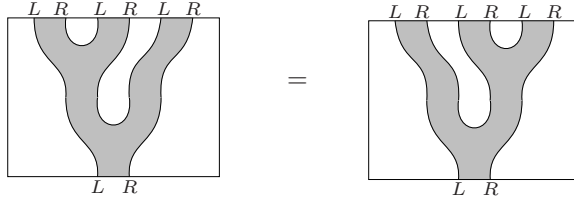
where we have been slightly artistic with the strings representing the identity morphisms on L and R . This diagram is meant to be reminiscent of the short

hand notation used in Section 2.1. The unit ι for the monoid defined to be:

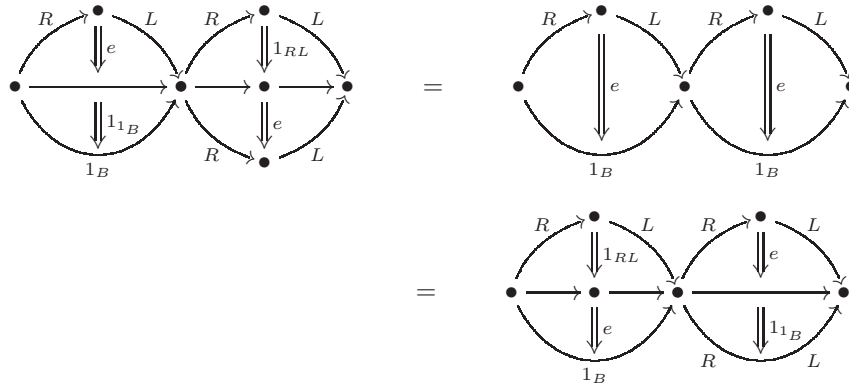


the unit for the adjunction $\iota := i: 1_A \rightarrow LR$.

Now we can use the axioms of a 2-category, together with the axioms for an adjunction, to show that this multiplication is associative and that the unit satisfies the unit axioms. For the multiplication to be associative we must have an equality of string diagrams:

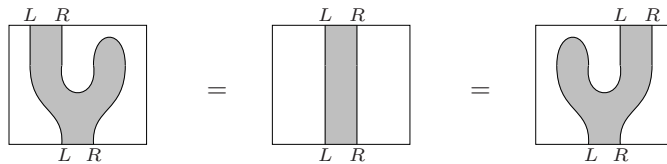


To prove that these two string diagrams are equal we can convert them back into the more traditional globular notation. Notice that nothing interesting occurs with the identity morphisms, 1_L on the far left, and 1_R on the far right so the interesting part is what is happening in the middle. Translating this into globular notation the proof is as follows:



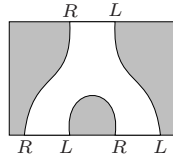
which amounts to nothing more than the interchange law and the axiom for vertical composition of identities in a 2-category. In this case, $1_{RL} \cdot e = e = e \cdot 1_{1_B}$.

The unit axioms for the monoid require the following equations of string diagrams:

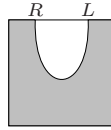


But these axioms follow directly from the zig-zag axioms in the definition of an adjunction. Thus, it is clear that the walking monoid is contained in $\text{Hom}(A, A)$ within the walking adjunction.

Similarly, we will show that the morphism RL is a comonoid in the monoidal category $\text{Hom}(B, B)$. We define a comultiplication for RL to be the morphism $\Delta := RiL: RL \rightarrow RLRL$, drawn diagrammatically as:



The counit for the comultiplication is:



the counit for the adjunction $\varepsilon := e: 1_B \rightarrow RL$. By similar arguments as those above, it follows that the walking comonoid is contained in $\text{Hom}(B, B)$ in the walking adjunction.

As we mentioned above, the main difficulty in proving Theorem 5 is not in showing that the walking monoid is contained in $\text{Hom}(A, A)$ within the walking adjunction, but rather, in showing that the walking monoid actually arises as $\text{Hom}(A, A)$ in the walking adjunction. Doing so requires one to prove that the monoid generating the walking monoid is LR , where (A, B, L, R, i, e) is the adjunction generating the walking adjunction. As a consequence, this means that for every monoid T in a monoidal category \mathcal{D} there exists an adjunction (A, B, L, R, i, e) in some 2-category \mathcal{C} , where \mathcal{D} is a subcategory of $\text{Hom}(A, A)$ and $T = LR$. Loosely speaking, we must show that every monoid arises from an adjunction.

The problem of showing that every monoid arises from an adjunction turns out to be very related to the problem of showing that every *monad* arises from an adjunction. Recall that a monad on an object A in a 2-category \mathcal{C} is just a monoid in the monoidal category $\text{Hom}(A, A)$. Given a monoid in a monoidal category \mathcal{D} , we can regard \mathcal{D} as a 2-category $\Sigma(\mathcal{D})$ with only one object, say \bullet . Then a monoid T in \mathcal{D} becomes a monad \mathbb{T} on the object \bullet in the 2-category $\Sigma(\mathcal{D})$. Hence, showing that every monoid arises from an adjunction can be deduced from showing that every monad arises from an adjunction.

When monads were first discovered in the 1950's this question as to whether every monad comes from an adjunction was raised by Hilton and others [38]. At this time, people were mostly interested in monads that arose from adjoint functors, or adjunctions in the 2-category **Cat**. If \mathcal{A} is a category and $T: A \rightarrow A$

is a functor defining a monad on \mathcal{A} , then in this context two well known solutions appeared, the Kleisli construction $\mathcal{A} \xrightleftharpoons[\perp]{} \mathcal{A}_{\mathbb{T}}$ [28], and the Eilenberg-Moore construction $\mathcal{A} \xrightleftharpoons[\perp]{} \mathcal{A}^{\mathbb{T}}$ [18]. These two solutions are, in a certain sense, the initial and terminal solution to the problem of constructing such an adjunction. Similarly, a comonoid in $\text{Hom}(\mathcal{A}, \mathcal{A})$ is known as a comonad, and these constructions work equally well to create a pair of adjoint functors where the functor $\mathcal{A}^{\mathbb{T}} \rightarrow \mathcal{A}$ is now the left adjoint.

For our purposes we will need to consider monads in 2-categories other than **Cat**. In particular, we would like to consider the 2-category $\Sigma(\text{Mon})$, the suspension of the walking monoid. Unfortunately, the Eilenberg-Moore and the Kleisli construction do not work in the completely general context of an arbitrary 2-category. While it is true that every adjunction in a 2-category \mathcal{C} produces a monad, it is not always true that one can find an adjunction in \mathcal{C} generating a given monad. The failure of this construction can be attributed to the lack of an object in \mathcal{C} to play the role of the Eilenberg-Moore category of algebras (or the lack of a Kleisli object, but we will focus on Eilenberg-Moore objects in this paper). When such an object does exist we call it an Eilenberg-Moore object for the monad \mathbb{T} . The existence of Eilenberg-Moore objects in a 2-category \mathcal{C} is a completeness property of the 2-category in question. In particular, \mathcal{C} has Eilenberg-Moore objects if it is finitely complete as a 2-category [49, 50].

Since the 2-category $\Sigma(\text{Mon})$ has only one object it is obvious that there will not be an object in $\Sigma(\text{Mon})$ to play the role of an Eilenberg-Moore object for the monad \mathbb{T} . Fortunately, there is a categorical construction known as the free completion under Eilenberg-Moore objects that takes a 2-category \mathcal{C} and enlarges it into a 2-category $\mathbf{EM}(\mathcal{C})$ that has Eilenberg-Moore objects for every monad \mathbb{T} in \mathcal{C} . This means that every monad \mathbb{T} in \mathcal{C} arises from an adjunction in $\mathbf{EM}(\mathcal{C})$. Furthermore, there is a fully faithful embedding $Z: \mathcal{C} \rightarrow \mathbf{EM}(\mathcal{C})$ with the property that for any other 2-category \mathcal{C}' with Eilenberg-Moore objects, composition with Z induces an equivalence of categories between the functor category $[\mathcal{C}, \mathcal{C}']$ and the full subcategory of the functor category $[\mathbf{EM}(\mathcal{C}), \mathcal{C}']$ consisting of those 2-functors that preserve Eilenberg-Moore objects [32].

This completion is possible because Eilenberg-Moore objects can be described as a weighted limit [50] whose weight is finite in the sense of [24]. This also means that if we are only interested in an Eilenberg-Moore object for a single monad \mathbb{T} in \mathcal{C} , then we do not have to complete \mathcal{C} under Eilenberg-Moore objects for *every* monad \mathbb{T} in \mathcal{C} . We can instead define the 2-category $\mathbf{EM}_{\mathbb{T}}(\mathcal{C})$, the free completion of \mathcal{C} under an Eilenberg-Moore object for the monad \mathbb{T} . If \mathbb{T} is a monad on the object A of \mathcal{C} , this 2-category will contain A (identified with its image under the embedding), and an Eilenberg-Moore object $A^{\mathbb{T}}$ for the monad \mathbb{T} . Hence, in $\mathbf{EM}_{\mathbb{T}}(\mathcal{C})$ the monad \mathbb{T} is generated by an adjunction $A \xrightleftharpoons[\perp]{} A^{\mathbb{T}}$.

This construction is particularly well suited for the problem at hand. The Eilenberg-Moore completion works just as well, when the 2-category \mathcal{C} has only one object: that is, when $\mathcal{C} = \Sigma(\mathcal{D})$ is the suspension of a monoidal category \mathcal{D} . As we mentioned above, a monad in the 2-category $\Sigma(\mathcal{D})$ is just a monoid

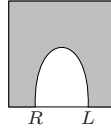
in the monoidal category \mathcal{D} . Hence, we have shown that every monoid T in a monoidal category \mathcal{D} arises from an adjunction in the 2-category $\mathbf{EM}_T(\Sigma(\mathcal{D}))$. For more on the Eilenberg-Moore completion see [32], or [33] for an explicit description of $\mathbf{EM}(\mathbf{Vect})$.

We are now ready to prove Theorem 5. Note that this theorem follows as a decategorification of a theorem due to Lack [31]. We present the proof for completeness.

Proof of Theorem 5. Let T be the monoid generating the walking monoid. We will show that the 2-category $\mathbf{EM}_T(\Sigma(\mathbf{Mon}))$ is isomorphic to the walking adjunction \mathbf{Adj} . By the universal property of the walking adjunction, adjunctions in a 2-category \mathcal{C} correspond bijectively to 2-functors $\mathbf{Adj} \rightarrow \mathcal{C}$. Since the 2-category $\mathbf{EM}_T(\Sigma(\mathbf{Mon}))$ contains an adjunction $A \xrightleftharpoons[\perp]{} A^\mathbb{T}$ we get a 2-functor $\Lambda: \mathbf{Adj} \rightarrow \mathbf{EM}_T(\Sigma(\mathbf{Mon}))$. If \mathbf{Adj} is generated by the adjunction (A, B, L, R, i, e) , then Λ maps this generating adjunction to the adjunction $A \xrightleftharpoons[\perp]{} A^\mathbb{T}$ in $\mathbf{EM}_T(\Sigma(\mathbf{Mon}))$.

We now construct the inverse of the 2-functor Λ . We have already shown that given an adjunction (A, B, L, R, i, e) , then the map LR is a monad on A (equivalently LR is a monoid in $\mathbf{Hom}(A, A)$). Hence, by the universal property of the walking monoid we get a 2-functor $\Sigma(\mathbf{Mon}) \rightarrow \mathbf{Adj}$. In the walking adjunction this monad has an Eilenberg-Moore object, namely B . Thus, the universal property of the Eilenberg-Moore completion determines a 2-functor $\bar{\Lambda}: \mathbf{EM}_T(\Sigma(\mathbf{Mon})) \rightarrow \mathbf{Adj}$ that preserves Eilenberg-Moore objects. By their construction, it is clear that the composites of Λ and $\bar{\Lambda}$ are equal to the identity. Hence, $\mathbf{EM}_T(\Sigma(\mathbf{Mon})) \cong \mathbf{Adj}$. \square

From our topological perspective it is clear that we could not define a comultiplication in $\mathbf{Hom}(A, A)$ since this would require a map:



which does not exist in the walking adjunction. In the next section we will consider the categorical framework where this map is given, the walking ambidextrous adjunction.

2.4 The walking ambidextrous adjunction

In this section we further examine adjunctions by looking at adjunctions that are 2-sided. This means that in addition to the 2-morphisms $i: 1_A \Rightarrow LR$ and $e: RL \Rightarrow 1_B$, there are also 2-morphisms $j: 1_B \Rightarrow RL$ and $k: LR \Rightarrow 1_A$ satisfying the triangle identities. Sometimes category theorists will specify which of the two possible adjunctions they mean by referring to one as a left adjunction and the other as a right adjunction. Since we will consider adjunctions that are

both left and right adjunctions, we will call them *ambidextrous adjunctions* to indicate their ‘two-handedness’. Sometimes we call an ambidextrous adjunction an *ambijunction* for short.

Our primary interest is the walking ambijunction. This has also been referred to as the free biadjunction. The authors is hesitant to use this terminology because of possible confusion that may arise when considering morphisms of bicategories.

Definition 6. *The walking ambidextrous adjunction Ambi is the 2-category freely generated by:*

- objects A and B ,
- morphisms $L: A \rightarrow B$ and $R: B \rightarrow A$, and
- 2-morphisms $i: 1_A \rightarrow L \circ R$, $e: R \circ L \rightarrow 1_B$, $j: 1_B \rightarrow R \circ L$, and $k: L \circ R \rightarrow 1_A$

such that

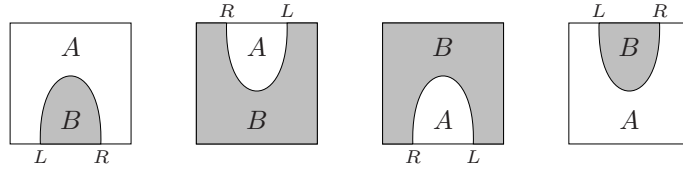
$$\begin{array}{ccc} & LRL & \\ iL \nearrow & & \searrow Le \\ L & \xrightarrow{1_L} & L \end{array} \qquad \begin{array}{ccc} & RLR & \\ Ri \nearrow & & \searrow eR \\ R & \xrightarrow{1_R} & R \end{array}$$

and

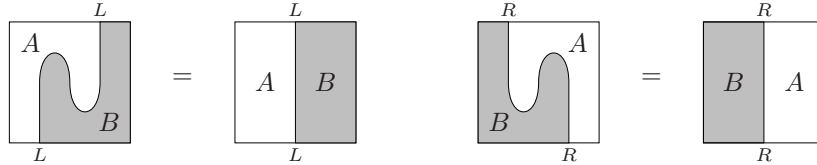
$$\begin{array}{ccc} & RLR & \\ jR \nearrow & & \searrow Rk \\ R & \xrightarrow{1_R} & R \end{array} \qquad \begin{array}{ccc} & LRL & \\ Lj \nearrow & & \searrow kL \\ L & \xrightarrow{1_L} & L \end{array}$$

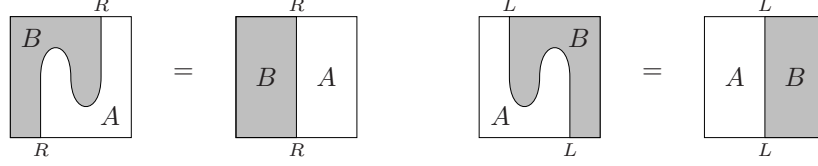
commute.

For later convenience we depict the 2-morphisms in the walking ambidextrous adjunction in string notation:



The zig-zag laws for the four maps above are depicted in string notation as:





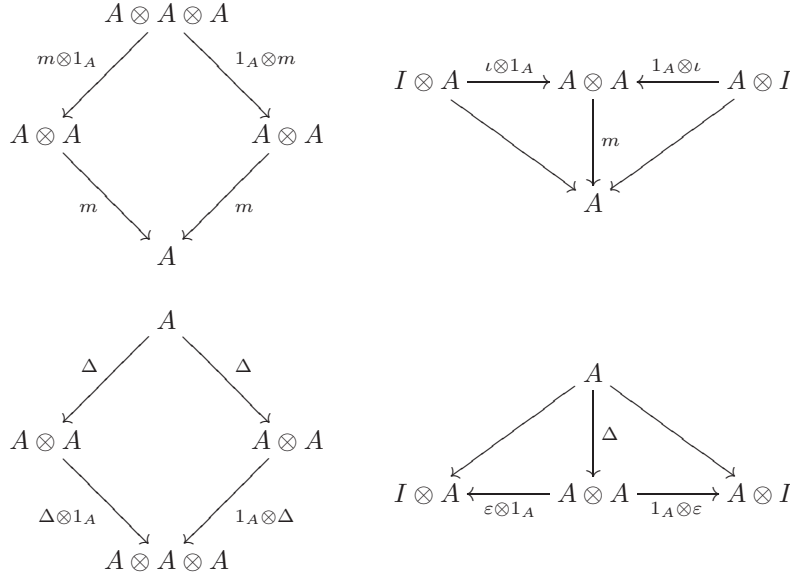
Notice that the walking ambidextrous adjunction has the same maps and axioms as the walking adjunction but with the color inverted versions as well.

As eluded to in the previous section, the importance of the walking ambidextrous adjunction is its relationship to the walking Frobenius algebra. Understanding the relationship between the two provides a characterization of Frobenius algebras that, phrased in an intrinsically categorical way, easily admits categorification to provide a definition of a pseudo Frobenius algebra. Below we define the walking Frobenius algebra based on Proposition 1 (i.), although any of the three definitions would produce equivalent monoidal categories.

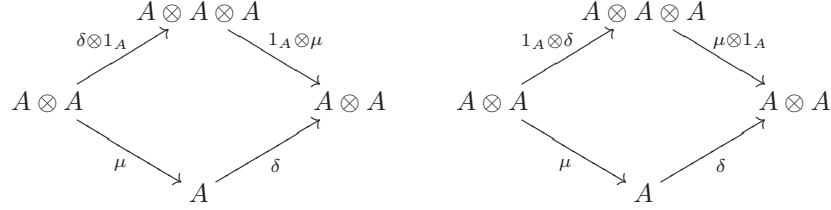
Definition 7 *The walking Frobenius algebra Frob is the monoidal category freely generated by:*

- *objects A and I , and*
- *morphisms $m: A \otimes A \rightarrow A$, $\iota: I \rightarrow A$, $\Delta: A \rightarrow A \otimes A$ and $\varepsilon: A \rightarrow I$*

such that



and

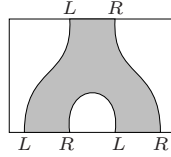


commute.

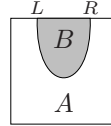
For more on the walking Frobenius algebra see [5, 29].

Theorem 8. *The monoidal category $\text{Hom}(A, A)$ in the walking ambidextrous adjunction is the walking Frobenius algebra (equivalently $\text{Hom}(B, B)$).*

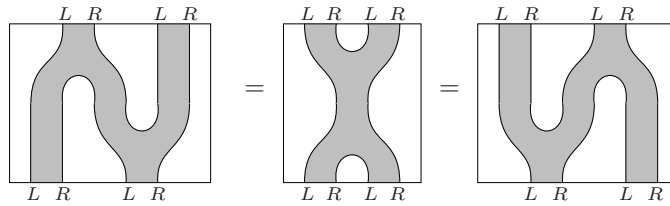
Proof. In Theorem 5 we saw that the object LR in $\text{Hom}(A, A)$ had a monoidal structure given by $LeR: LRLR \rightarrow LR$ with unit $i: 1_A \rightarrow LR$. Define a comultiplication on LR by the morphism $LjR: LR \rightarrow LRLR$.



The counit for this comultiplication is $k: LR \rightarrow 1_A$.



To show that the comultiplication is coassociative take the proof that $\text{Hom}(B, B)$ is a comonoid object and invert the colors of the shaded regions. All that remains to be shown is that the monoid and comonoid structures are compatible, that is, they must satisfy the Frobenius identities:



This follows directly from the identity axioms and the interchange law relating vertical and horizontal composition in a 2-category. The first equality is proved in globular notation below:

$$\begin{array}{c}
\text{Diagram 1} \\
= \\
\text{Diagram 2} \\
= \\
\text{Diagram 3}
\end{array}$$

and the other Frobenius identity follows similarly. Hence, $\text{Hom}(A, A)$ in the walking ambidextrous adjunction contains the walking Frobenius algebra.

To prove the converse we will again borrow some results from monad theory. Extending the work of Street [53] and Eilenberg and Moore [18], the author has shown that every Frobenius algebra F in an arbitrary monoidal category \mathcal{D} is generated by an ambidextrous adjunction in the 2-category $\mathbf{EM}(\Sigma(\mathcal{D}))$. This construction uses the fact that a Frobenius algebra F in the monoidal category \mathcal{D} defines a Frobenius monad \mathbb{F} [34, 53], or a monad and a comonad on the one object of the 2-category $\Sigma(\mathcal{D})$. This monad and comonad arising from a Frobenius object has a special property that makes the Eilenberg-Moore object for the monad isomorphic to the Eilenberg-Moore object for the comonad. Thus, freely completing \mathcal{D} under Eilenberg-Moore objects for the monad suffices to produce an ambidextrous adjunction in $\mathbf{EM}(\Sigma(\mathcal{D}))$ that generates F .

We do not have to complete the monoidal category \mathcal{D} under Eilenberg-Moore objects for every Frobenius algebra in \mathcal{D} . Indeed, if F is a Frobenius algebra in \mathcal{D} then we can define the free completion of $\Sigma(\mathcal{D})$ under an Eilenberg-Moore object for the single Frobenius monad \mathbb{F} , denoted $\mathbf{EM}_{\mathbb{F}}(\Sigma(\mathcal{D}))$. In $\mathbf{EM}_{\mathbb{F}}(\Sigma(\mathcal{D}))$ the monad \mathbb{F} is generated by an ambidextrous adjunction $A \xrightleftharpoons[\tau]{\eta} A^{\mathbb{F}}$. Hence, by similar arguments as the proof of Theorem 5, we have that the 2-category \mathbf{Ambi} is isomorphic to the 2-category $\mathbf{EM}_{\mathbb{F}}(\Sigma(\mathbf{Frob}))$, so that \mathbf{Frob} really is $\text{Hom}(A, A)$ in the walking ambidextrous adjunction. \square

We then have the following corollary:

Corollary 9 *Every 2D topological quantum field theory, in the sense of Atiyah [2], arises from an ambijunction in the 2-category $\mathbf{EM}(\Sigma(\mathbf{Vect}))$.*

Proof. A 2D topological quantum field theory is a monoidal functor from $\mathbf{2Cob}$ into \mathbf{Vect} . It is well known that such a functor amounts to a commutative Frobenius algebra [29]. Hence, the result follows. \square

This result implies that all of the known 2-dimensional topological quantum field theories that have been constructed in the axiomatic sense can be understood as arising from an ambijunction in some 2-category.

2.5 Two-dimensional thick tangles

Category theory has been used as a language to describe relationships in topology, especially in algebraic topology; but perhaps the most exciting interplay between category theory and topology comes from understanding various types of topology as categories with extra structure. By seeing a category as a structure rather than a means to describe structure, progress has been made in fields that had at first seemed quite mysterious. Aside from the description of **2Cob** as the free symmetric monoidal category on a commutative Frobenius algebra, the most notable instance is the category of tangles in 3-dimensional space. This category has a completely algebraic description as the free braided monoidal category with duals on one object [56, 20, 23, 48]. Using this universal property, it is easy to construct functors from the category of tangles into other braided monoidal categories with duals, such as the category of representations of a quantum group. Furthermore, any such functor determines an invariant of tangles, and in particular, a knot invariant. This categorical description plays a vital role in understanding the Jones polynomial and other ‘quantum invariants’ of knots [44].

In this section we provide yet another example of this phenomenon. First we define the topological category **2Thick** of two-dimensional thick tangles. Analogous to the description of **2Cob** as the free symmetric monoidal category on a commutative Frobenius algebra, we will prove that **2Thick** is the free monoidal category on a noncommutative Frobenius algebra.

Definition 10 *The monoidal category of two-dimensional thick tangles denoted **2Thick** has nonnegative integers as objects. The 1-morphisms from k to l are boundary preserving diffeomorphism classes of smooth oriented compact surfaces X with boundary ∂X equipped with disjoint distinguished intervals $i_j^s: I \hookrightarrow \partial X$, $1 \leq j \leq k$, $i_m^t: \hookrightarrow \partial X$, $1 \leq m \leq l$, equipped with a smooth embedding $d: X \hookrightarrow \mathbb{R} \times [0, 1]$ such that*

$$d^{-1}(\mathbb{R} \times 0) = I_1^s \sqcup I_2^s \sqcup \cdots \sqcup I_k^s, \quad I_j^s = i_j^s(I), \quad d(I_j^s) = [j - \frac{1}{3}, j + \frac{1}{3}] \times 0,$$

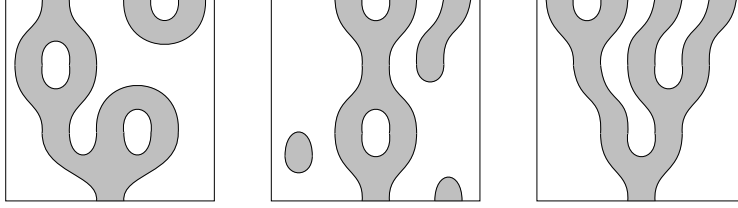
$$d^{-1}(\mathbb{R} \times 1) = I_1^t \sqcup I_2^t \sqcup \cdots \sqcup I_l^t, \quad I_j^t = i_j^t(I), \quad d(I_j^t) = [j - \frac{1}{3}, j + \frac{1}{3}] \times 1.$$

The image $d(X)$ is called a diagram of two-dimensional thick tangles.

Composition $Y \circ X$ of 1-morphisms $k \xrightarrow{X} l \xrightarrow{Y} m$ is defined by sewing of surfaces at boundary intervals $I_j^t(X)$ and $I_j^s(Y)$. The identity 1-morphism $1_k: k \rightarrow k$ is the union $\coprod_{j=1}^k [j - \frac{1}{3}, j + \frac{1}{3}] \times [0, 1]$. The identity axioms follow from the isomorphisms $1_l \circ X \xrightarrow{\sim} X$ obtained by taking a neighborhood $(U, I_j^s) \simeq ([0, 1] \times [0, 1], [0, 1], [0, 1] \times 0)$ of the distinguished interval $I_j^s \subset X$ and by taking any isomorphism $[0, 1] \times [0, 1] \bigcup_{[0, 1] \times 1 \sim I_j^s} U \simeq U$. The tensor product is the disjoint union. The unit object is 0.

This monoidal category is actually a decategorified version of the category of ‘planar thick tangles’ defined by Kerler and Lyubashenko [26]. Some examples

of two-dimensional thick tangles are shown below²:



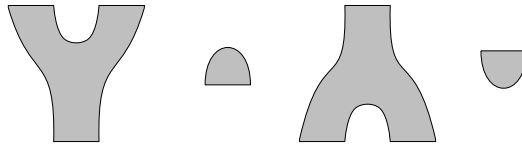
We mentioned in the introduction that **2Thick** has the alternative description as the category whose objects are open strings and whose morphisms are diffeomorphism classes of planar open string worldsheets.

We now state one of the main theorems of this paper. The proof follows as a corollary of a result proven in Section 4.2. We will however sketch a proof for the eager reader.

Theorem 11 *The category of two-dimensional thick tangles is equivalent to the free monoidal category on a Frobenius object.*

Sketch of Proof. The height function $\text{pr}_2 \circ d: \partial X \rightarrow [0, 1]$ defines a Morse function on the surface X . This allows a decomposition of X into the elementary building blocks defining the data of a Frobenius algebra. The equality of two surfaces related by the sliding of handles implies the associativity of the multiplication, the coassociativity of the comultiplication, and the Frobenius identities. The equality of those surfaces obtained by the cancellation of local maxima with a local minima imply the unit and counit axioms. One can check that these are the only allowed Morse moves for these surfaces so that **2Thick** is freely generated by a Frobenius algebra. \square

This result therefore justifies the assignation of a noncommutative Frobenius algebra to the image of the generators



in the Moore-Segal axioms of an open-closed topological field theory. The idea that the free monoidal category on a noncommutative Frobenius algebra might be related to a topological category of this sort has been suggested by Baez [5] and Kock [29].

Corollary 12 *A Frobenius algebra in a monoidal category determines an invariant of 2-dimensional thick tangles.*

²Note that here we are depicting the surfaces in a slightly artistic fashion.

Proof. This follows from the universal property of the monoidal category of thick tangles. \square

We can define a planar open string topological field theory as a monoidal functor from **2Thick** into a monoidal category \mathcal{C} . We then have the following as a simple restatement of Theorem 11:

Theorem 13 *A planar open string topological field theory is equivalent to a (not necessarily commutative) Frobenius algebra in the monoidal category \mathcal{C} .*

3 Categorification

We may now reap the rewards of our characterization of Frobenius algebras in terms of the walking adjunction. We will see that, by assuming a similar relationship holds in higher dimensions, we can use the existing definition of a categorified adjunction to define a categorified Frobenius algebra. We will then show that this definition of categorified Frobenius algebra is equivalent to what one might expect by replacing the equations with coherent isomorphisms in Proposition 1. The advantage of this approach is that the usual hassle of figuring out coherence conditions for these isomorphisms can be avoided by utilizing what is known about categorified adjunctions.

We start off in Section 3.1 by defining some categorical notions that we will need later on. We then describe the generalization of string diagrams to the context of 3-categories in Section 3.2. Then we define the walking pseudo monoid and the walking pseudo Frobenius algebra in terms of pseudoadjunctions in Sections 3.3 and 3.4. In Section 3.5 we provide some examples of pseudo Frobenius algebras before moving on to prove the main theorem that the walking pseudo Frobenius algebra is triequivalent to the monoidal bicategory of 3-dimensional thick tangles.

3.1 Preliminaries for higher categories

Adjunctions make sense in a 2-category and even in the more general context of bicategories or weak 2-categories. To categorify the notion of an adjunction we will need to climb one categorical dimension to the level of 3-categories. Intuitively, the idea of a 3-category is not hard to grasp. Categories have objects, morphisms and some axioms; 2-categories have objects, morphisms, 2-morphisms and some axioms; and 3-categories have objects, morphisms, 2-morphisms, *and* 3-morphisms together with some axioms.

Since 3-categories have an extra level of structure, it is possible for the composition and unit axioms for both 1-morphisms and 2-morphisms to hold only up to coherent isomorphism. Similarly, we can require that the interchange law for 2-morphisms is only satisfied up to isomorphism as well. When all levels of composites and identities up the level of 3-morphisms satisfy the usual axioms up to isomorphism, we call this type of 3-category a weak 3-category or tricategory. If the axioms hold as equations at each level rather than up to

isomorphism, then we call this type of 3-category a strict 3-category. The most exciting thing about 3-categories is that this is the first level in which ‘weak’ structures are no longer equivalent in some sense to the strict version. Thus, every tricategory *is not* triequivalent to a strict 3-category. Triequivalence is the weakest notion of equivalence that can be defined for 3-categories. This notion is the one most naturally suited for relating tricategories. The standard reference for tricategories and triequivalences is Gordon, Power and Street [21].

Although every tricategory is not triequivalent to a strict 3-category, there is a notion of 3-category which every tricategory is triequivalent to — a semistrict 3-category. Semistrict 3-categories are a hybrid notion between strict 3-categories and tricategories. They represent the strictest known class of 3-categories that remain triequivalent to tricategories. Hence, it is this notion, that of a semistrict 3-category, that will provide a sufficiently general context to consider categorified adjunctions or pseudoadjunctions.

A semistrict 3-category is defined using enriched category theory [25] as a category enriched in **Gray** [21]. For this reason, semistrict 3-categories are sometimes referred to as **Gray**-categories. We will use both terms interchangeably. **Gray** is the symmetric monoidal closed category whose underlying category is **2-Cat**; the category whose objects are 2-categories, and whose morphisms are 2-functors. **2-Cat** has a natural monoidal structure given by the cartesian product. However, enriching over $(\mathbf{2-Cat}, \times)$ only produces *strict* 3-categories [17].

Enriching in **Gray** produces a more interesting notion of 3-category because **Gray** has a more interesting monoidal structure, namely the ‘Gray’-tensor product. This has the effect of equipping a **Gray**-category \mathcal{K} with a cubical functor $M: \mathcal{K}(A, B) \times \mathcal{K}(B, C) \rightarrow \mathcal{K}(A, C)$ for all objects A, B, C in \mathcal{K} . This means that if $f: F \Rightarrow F'$ in $\mathcal{K}(A, B)$, and $g: G \Rightarrow G'$ in $\mathcal{K}(B, C)$, then, rather than commuting on the nose, we have an invertible 3-cell in the following square:

$$\begin{array}{ccc} FG & \xRightarrow{Fg} & FG' \\ fG \Downarrow & \Downarrow f_g & \Downarrow fG' \\ F'G & \xRightarrow{F'g} & F'G' \end{array}$$

where we write the 3-morphism $M_{f,g}$ as f_g following Marmolejo [39]. A bit more concretely, in a **Gray**-category composites are still strictly associative, and identities behave as identities on the nose, but the interchange law now holds only up to coherent isomorphism.

Definition 14 *A semistrict 3-category is a category enriched in **Gray**.*

Using enriched category theory it is also possible to define the morphisms between two semistrict 3-categories. A semistrict 3-functor between two semistrict 3-categories is just a **Gray**-enriched functor, or a **Gray**-functor as it is often called. We will also use these two terminologies interchangeably.

Definition 15 A pseudoadjunction $i, e, I, E: L \dashv_p R: B \rightarrow A$ in a **Gray**-category \mathcal{K} consists of:

- morphisms $R: A \rightarrow B$ and $L: B \rightarrow A$,
- 2-morphisms $i: 1 \Rightarrow LR$ and $e: RL \Rightarrow 1$, and
- coherence 3-isomorphisms

$$\begin{array}{ccc}
 & RLR & \\
 Ri \nearrow & \Downarrow I & \searrow eR \\
 R & \xrightarrow{1} & R
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 & LRL & \\
 iL \nearrow & \Downarrow E & \searrow Le \\
 L & \xrightarrow{1} & L
 \end{array}$$

such that the following two diagrams are both identities:

$$\begin{array}{ccccc}
 & & RL & & \\
 & \nearrow 1 & & \searrow e & \\
 RL & \xrightarrow{RiL} & RLRL & \xrightarrow{e^{-1}} & 1 \\
 & \searrow RE & & \nearrow RLe & \\
 & & RL & &
 \end{array}
 \quad
 \begin{array}{ccccc}
 & LR & & 1 & \\
 & \nearrow i & & \searrow 1 & \\
 1 & \xrightarrow{i^{-1}} & LRLR & \xrightarrow{LeR} & LR \\
 & \searrow i & & \nearrow iLR & \\
 & & LR & &
 \end{array}$$

Having the aim of defining a pseudo Frobenius algebra to be $LR \in \text{Hom}(A, A)$ in an ambidextrous pseudoadjunction, it is important to understand the structure of $\text{Hom}(A, A)$. Just as fixing one object in 2-category produces a monoidal category, fixing one object in a semistrict 3-category produces a semistrict monoidal 2-category or a **Gray**-monoid as it is sometimes referred.

Definition 16 A semistrict monoidal 2-category is a one object **Gray**-category.

For later convenience we write this definition in more elementary terms below. We employ the convention that the 2-morphism $M_{f,g}$ will be written as f_g in the context of semistrict 3-categories, and will be written more indicatively as $\otimes_{f,g}$, or simply as \otimes , in the context of semistrict monoidal 2-categories.

Proposition 17. A semistrict monoidal 2-category consists of a 2-category \mathcal{C} together with:

- 1) An object $I \in \mathcal{C}$.
- 2) For any two objects $A, B \in \mathcal{C}$ an object $A \otimes B$ in \mathcal{C} .
- 3) For any 1-morphism $f: A \rightarrow A'$ and any object $B \in \mathcal{C}$ a 1-morphism $f \otimes B: A \otimes B \rightarrow A' \otimes B$.
- 4) For any 1-morphism $g: B \rightarrow B'$ and any object $A \in \mathcal{C}$ a 1-morphism $A \otimes g: A \otimes B \rightarrow A \otimes B'$.

- 5) For any object $B \in \mathcal{C}$ and any 2-morphism $\alpha: f \rightarrow f'$ a 2-morphism $\alpha \otimes B: f \otimes B \Rightarrow f' \otimes B$.
- 6) For any object $A \in \mathcal{C}$ and any 2-morphism $\beta: g \Rightarrow g'$ a 2-morphism $A \otimes \beta: A \otimes g \Rightarrow A \otimes g'$.
- 7) For any two 1-morphisms $f: A \rightarrow A'$ and $g: B \rightarrow B'$ a 2-isomorphism

$$\begin{array}{ccc}
 A \otimes B & \xrightarrow{A \otimes g} & A \otimes B' \\
 f \otimes B \downarrow & \Downarrow \otimes_{f,g} & \downarrow f \otimes B' \\
 A' \otimes B & \xrightarrow{A' \otimes g} & A' \otimes B'
 \end{array}$$

This data is subject to the following conditions.

- i) For any object $A \in \mathcal{C}$ we have $A \otimes -: \mathcal{C} \rightarrow \mathcal{C}$ and $- \otimes A: \mathcal{C} \rightarrow \mathcal{C}$ are 2-functors.
- ii) For x any object, morphism or 2-morphisms of \mathcal{C} we have $x \otimes I = I \otimes x = x$.
- iii) For x any object, morphism or 2-morphism of \mathcal{C} , and for all objects $A, B \in \mathcal{C}$ we have $A \otimes (B \otimes x) = (A \otimes B) \otimes x$, $A \otimes (x \otimes B) = (A \otimes x) \otimes B$ and $x \otimes (A \otimes B) = (x \otimes A) \otimes B$.
- iv) For any 1-morphism $f: A \rightarrow B$, $g: B \rightarrow B'$ and $h: C \rightarrow C'$ in \mathcal{C} we have $\otimes_{A \otimes g, h} = A \otimes \otimes_{g, h}$, $\otimes_{f \otimes B, h} = \otimes_{f, B \otimes h}$ and $\otimes_{f, g \otimes C} = \otimes_{f, g} \otimes C$.
- v) For any objects $A, B \in \mathcal{C}$ we have $1_A \otimes B = A \otimes 1_B = 1_{A \otimes B}$, and for any 1-morphism $f: A \rightarrow B$, $g: B \rightarrow B'$ in \mathcal{C} we have $\otimes_{1_A, g} = 1_{A \otimes g}$ and $\otimes_{f, 1_{B'}} = 1_{f \otimes B'}$.
- vi) For any 1-morphisms $f, h: A \rightarrow A'$, $g, k: B \rightarrow B'$, and any 2-morphisms $\alpha: f \Rightarrow h$, and $\beta: g \Rightarrow k$,

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A \otimes B & \xrightarrow{f \otimes B} & A' \otimes B \\
 \left(\begin{array}{c} \swarrow A \otimes k \\ \searrow A \otimes g \end{array} \right) & \Downarrow \otimes_{f,g}^{-1} & \downarrow A' \otimes g \\
 A \otimes B' & \xrightarrow{f \otimes B'} & A' \otimes B' \\
 \left(\begin{array}{c} \swarrow A \otimes \beta \\ \searrow h \otimes B' \end{array} \right) & \Downarrow \alpha \otimes B' & \\
 \end{array} & = & \begin{array}{ccc}
 A \otimes B & \xrightarrow{f \otimes B} & A' \otimes B \\
 \left(\begin{array}{c} \swarrow A \otimes k \\ \searrow h \otimes B \end{array} \right) & \Downarrow \alpha \otimes B & \downarrow A' \otimes g \\
 A \otimes B' & \xrightarrow{h \otimes B'} & A' \otimes B' \\
 \left(\begin{array}{c} \swarrow A' \otimes k \\ \searrow A' \otimes \beta \end{array} \right) & \Downarrow \otimes_{h,k}^{-1} & \\
 \end{array}
 \end{array}$$

vii) For any 1-morphisms $f: A \rightarrow A'$, $g: B \rightarrow B'$, $f': A' \rightarrow A''$, $g': B' \rightarrow B''$,

$$\begin{array}{ccc}
 A \otimes B & \xrightarrow{f \otimes B} & A' \otimes B \xrightarrow{f' \otimes B} A'' \otimes B \\
 \downarrow A \otimes g & \Downarrow \otimes f, g & \downarrow A' \otimes g \Downarrow \otimes f', g \downarrow A'' \otimes g \\
 A \otimes B' & \xrightarrow{f \otimes B'} & A' \otimes B' \xrightarrow{f' \otimes B'} A' \otimes B'' \\
 \downarrow A \otimes g' & \Downarrow \otimes f, g' & \downarrow A' \otimes g' \Downarrow \otimes f', g' \downarrow A'' \otimes g' \\
 A \otimes B'' & \xrightarrow{f \otimes B''} & A' \otimes B'' \xrightarrow{f' \otimes B''} A'' \otimes B''
 \end{array}
 =
 \begin{array}{ccc}
 A \otimes B & \xrightarrow{f \circ f' \otimes B} & A'' \otimes B \\
 \downarrow A \otimes g \circ g' & \Downarrow \otimes f \circ f', g \circ g' & \downarrow A'' \otimes g \circ g' \\
 A \otimes B'' & \xrightarrow{f \circ f' \otimes B''} & A'' \otimes B''
 \end{array}$$

Proof. This is a straightforward verification. \square

Just as monoids can be defined in any monoidal category, pseudomonoids can be defined in any semistrict monoidal 2-category. A pseudomonoid is just a monoid where the equations describing the associativity and unit constraints are replaced by coherent isomorphism [14].

Definition 18 A pseudomonoid M in the semistrict monoidal 2-category \mathcal{C} consists of:

- An object M of \mathcal{C} .
- A multiplication morphism: $m: M \otimes M \rightarrow M$.
- A unit for multiplication: $\iota: 1 \rightarrow M$, and
- coherence 2-isomorphisms

$$\begin{array}{ccc}
 M & \xrightarrow{M \otimes \iota} & M \otimes M \xleftarrow{\iota \otimes M} M \\
 & \searrow r & \downarrow m \swarrow \ell \\
 & & M
 \end{array}$$

and

$$\begin{array}{ccc}
 M \otimes M \otimes M & & M \otimes M \otimes M \\
 \downarrow M \otimes m & & \downarrow m \otimes M \\
 M \otimes M & \xrightarrow{a} & M \otimes M \\
 \downarrow m & & \downarrow m \\
 M & & M
 \end{array}$$

such that the following two equations are satisfied:

$$\begin{array}{ccc}
 M^4 & \xrightarrow{m \otimes M^2} & M^3 \\
 \downarrow M^2 \otimes m & \searrow M \otimes m \otimes M & \downarrow m \otimes M \\
 M^3 & \xrightarrow{Ma} & M^2 \\
 \downarrow M \otimes m & \searrow M \otimes m & \downarrow m \\
 M^2 & \xrightarrow{m} & M
 \end{array}
 =
 \begin{array}{ccc}
 M^4 & \xrightarrow{m \otimes M^2} & M^3 \\
 \downarrow M^2 \otimes m & \searrow \otimes_{m, m} & \downarrow m \otimes M \\
 M^3 & \xrightarrow{m \otimes M} & M^2 \\
 \downarrow M \otimes m & \searrow a & \downarrow m \\
 M^2 & \xrightarrow{m} & M
 \end{array}$$

$$\begin{array}{c}
\begin{array}{ccccc}
& & M^2 & & \\
& m \otimes M & \nearrow & m & \\
M^2 & \xrightarrow{M \otimes \iota \otimes M} & M^3 & \xrightarrow{a^{-1}} & M \\
& \searrow M \otimes \ell^{-1} & \downarrow & \nearrow m & \\
& & M^2 & & \\
& \swarrow 1 & \searrow M \otimes m & &
\end{array}
& = &
\begin{array}{ccccc}
& & M^3 & & \\
& M \otimes \iota \otimes M & \nearrow & m \otimes M & \\
M^2 & \xrightarrow{r \otimes M} & M^2 & \xrightarrow{m} & M \\
& \searrow & \downarrow & &
\end{array}
\end{array}$$

As an example, note that a weak monoidal category is a pseudomonoid in **Cat**. We now move on to describe how some of these definitions can be understood diagrammatically.

3.2 String diagrams for higher categories

In this section we will sketch a version of string diagrams for 3-categories. Unfortunately we will only scratch the surface of this beautiful subject. We will focus on those aspects needed to see the relationship between pseudo Frobenius algebras and 3-dimensional thick tangles. The idea of string diagrams for 3-categories is given in [52]. Carter and Saito also use 3-categorical string diagrams implicitly in their work and explain its relationship to singularity theory [12].

In the notation that we will use, objects, morphisms and 2-morphisms will be depicted in exactly the same way as before. What's new is the 3-morphisms which we will simply draw as arrows *going between string diagrams*. For example, the morphism $I: 1_R \Rightarrow R \cdot e R$ in the definition of a pseudoadjunction is depicted as:

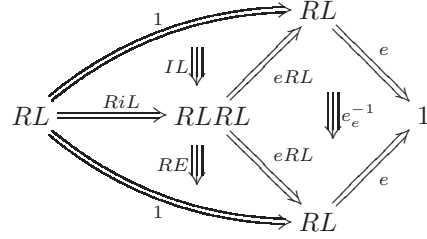
$$\begin{array}{ccc}
\begin{array}{|c|c|} \hline R \\ \hline B & A \\ \hline R \end{array} & \xRightarrow{I} & \begin{array}{|c|} \hline R \\ \hline \begin{array}{c} A \\ \text{hook} \\ B \end{array} \\ \hline R \end{array}
\end{array}$$

Notice that the source and target of 3-morphisms in the diagram are the 2-morphism represented by the string diagrams. The 3-morphism $E: i L \cdot L e \Rightarrow 1_L$ is depicted similarly as:

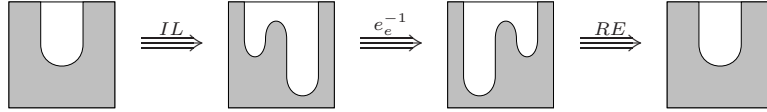
$$\begin{array}{ccc}
\begin{array}{|c|} \hline L \\ \hline \begin{array}{c} A \\ \text{hook} \\ B \end{array} \\ \hline L \end{array} & \xRightarrow{E} & \begin{array}{|c|c|} \hline L \\ \hline A & B \\ \hline L \end{array}
\end{array}$$

From now on we will omit the labels on the string diagrams representing the 2-morphisms. As usual, the gray colored region is meant to represent the object B and the white area the object A . All the other labels can be deduce from the diagrams. Using this notation we can even depict the pseudoadjunction axioms

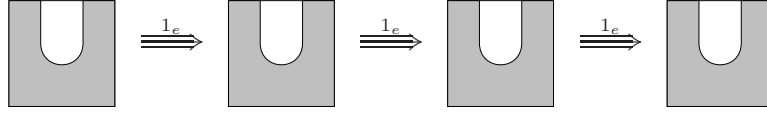
as commutative diagrams of string diagrams. For example, the assertion that



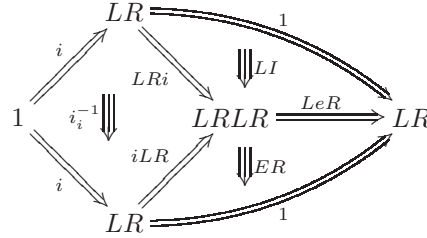
is equal to the identity can be translated into diagrams. This axiom says that the composite:



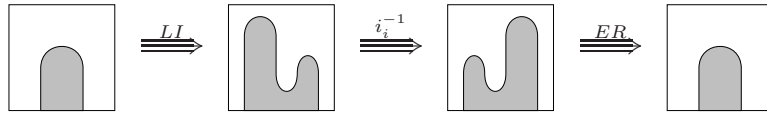
is equal to the composite:



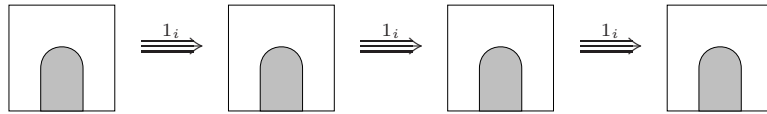
Similarly, the other pseudoadjunction axiom:



just says that the composite:



is equal to the composite:



We sometimes refer to the coherence laws for a pseudoadjunction as the *triangler identities*.

Notice that in the 2-categorical context we were able to distort the diagrams for a given 2-morphisms and get a diagram *equal* to one that we started with, whereas in this context the semistrictness makes these types of manipulations appear as isomorphisms in our diagrams. For instance the arrows corresponding to e_e^{-1} and i_i^{-1} above.

It will be helpful to imagine the 3-morphisms as tracing out a surface starting from the source and extending to the target. In this way the source and target are thought of as representing different time slices of a Morse function on these surfaces. There is much more to say about 3-categorical string diagrams but we will stop here.

3.3 The walking pseudoadjunction

In this section we study a very special pseudoadjunction, the walking pseudoadjunction. The existence of the walking pseudoadjunction was first proven by Lack [31]. This means that any pseudoadjunction in a **Gray**-category \mathcal{K} corresponds to a **Gray**-functor from the walking pseudoadjunction into \mathcal{K} . Alternatively, the walking pseudoadjunction can be thought of as the semistrict 3-category freely generated by the data of a pseudoadjunction. Analogous to Section 2.3 we will show that the semistrict monoidal category $\text{Hom}(A, A)$ is the walking pseudomonoid.

We begin by defining what it means to freely generate a semistrict 3-category.

Definition 19 *Let Y be a structure that can be defined in an arbitrary semistrict 3-category. If Y consists of objects Y_1, Y_2, \dots, Y_n , morphisms $F_1, F_2, \dots, F_{n'}$, 2-morphisms $\alpha_1, \alpha_2, \dots, \alpha_{n''}$, and 3-morphisms $\phi_1, \phi_2, \dots, \phi_{n'''}^{\prime\prime}$ for $n, n', n'', n''' \in \mathbb{Z}^+$, then the semistrict 3-category X is generated by Y if:*

- (i.) *Every object of X is some Y_i .*
- (ii.) *Every 1-morphism of X can be obtained by compositions from the F_i 's and 1_{Y_i} 's.*
- (iii.) *Every 2-morphism of X is obtained by taking Gray-tensor products and by vertical composition from:*
 - *the 2-morphisms α_i , and*
 - *the identity 2-morphisms 1_F for arbitrary 1-morphisms F .*
- (iv.) *Every 3-morphism is obtained from the Gray-tensor product of, and vertical and horizontal compositions from:*
 - *the 3-morphisms ϕ_i ,*
 - *the coherence morphisms f_g for arbitrary 2-morphisms f and g , and*

- the identity 3-morphisms 1_F for arbitrary 2-morphisms F .

We say that X is freely generated by Y if the set of Y objects in \mathcal{K} are in bijection with **Gray**-functors from X into \mathcal{K} , for every semistrict 3-category \mathcal{K} .

Definition 20 The walking pseudoadjunction \mathbf{pAdj} is the semistrict 3-category freely generated by:

- morphisms $R: A \rightarrow B$ and $L: B \rightarrow A$,
- 2-morphisms $i: 1 \Rightarrow LR$ and $e: RL \Rightarrow 1$, and
- coherence 3-isomorphisms

$$\begin{array}{ccc}
 & RLR & \\
 Ri \nearrow & & \searrow eR \\
 R & \xrightarrow{1} & R \\
 & \Downarrow I &
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 & LRL & \\
 iL \nearrow & & \searrow Le \\
 L & \xrightarrow{1} & L \\
 & \Downarrow E &
 \end{array}$$

such that the following two diagrams are both identities:

$$\begin{array}{ccc}
 & RL & \\
 \curvearrowright 1 \nearrow & & \searrow e \\
 RL & \xrightarrow{RiL} & RLRL \\
 \searrow RE \curvearrowright 1 & & \nearrow eRL \\
 & RL & \\
 & \Downarrow e_e^{-1} &
 \end{array}
 \quad
 \begin{array}{ccc}
 LR & \xrightarrow{1} & LR \\
 i \nearrow & & \searrow Li \\
 1 & \xrightarrow{i_i^{-1}} & LRLR \\
 \searrow i & & \nearrow iLR \\
 LR & \xrightarrow{1} & LR \\
 & \Downarrow E_R &
 \end{array}$$

We mentioned in Section 3.1 that fixing an object of a semistrict 3-category produces a semistrict monoidal 2-category. Thus, using Definition 19 it makes sense to talk about the semistrict monoidal 2-category freely generated by a pseudomonoid.

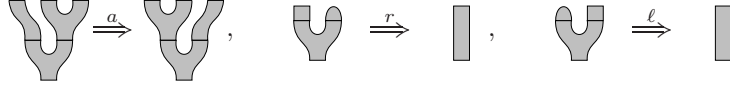
Definition 21 The walking pseudomonoid is the semistrict monoidal 2-category freely generated by the data defining a pseudomonoid.

Lack has also explicitly construct the walking pseudomonoid in the context of pseudomonads [31].

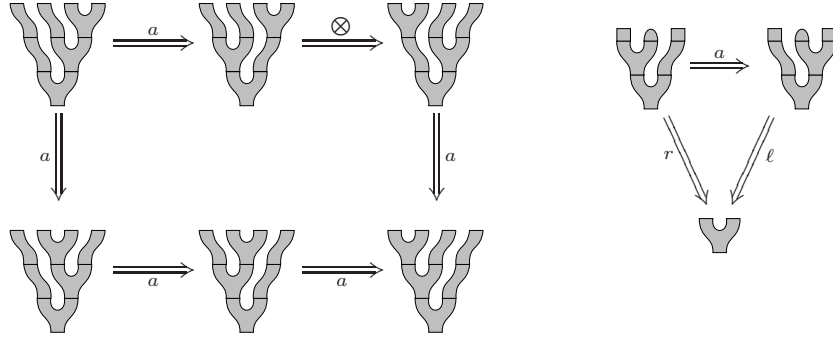
To better understand the walking pseudomonoid and its relationship to pseudoadjunctions it will be helpful to describe pseudomonoids using an extension of our shorthand notation from Section 2.1. In this notation, a pseudomonoid is an object of a semistrict monoidal 2-category represented by the interval, equipped with morphisms:



and coherence 2-isomorphisms:



such that



commute. With this topological description of pseudomonoids it will be much easier to prove:

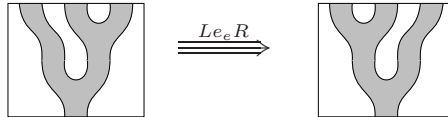
Theorem 22 *The walking pseudomonoid is $\text{Hom}(A, A)$ in the walking pseudoadjunction.*

Proof. Since all of the morphisms in $\text{Hom}(A, A)$ are generated by LR , we will show that LR is a pseudomonoid. We will again appeal to string diagrams for the proof. We define the multiplication and unit map as in Section 2.3.



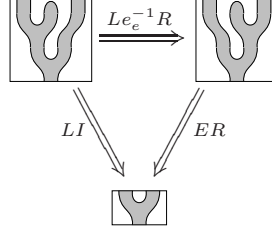
All that remains is to define the coherence 2-isomorphisms and show that they satisfy the appropriate coherence axioms.

The map $a: (mLR).m \Rightarrow (LRm).m$ is given by:



In this shorthand notation the first axiom for a pseudomonoid requires that

grams:



commutes. This axiom follows from the triangulator identities in the definition of a pseudoadjunction. Thus, $\text{Hom}(A, A)$ in the walking pseudoadjunction contains the walking pseudomonoid.

To prove the converse we again appeal to monad theory. In this case, we are interested in pseudomonads. A pseudomonad on an object A of a semistrict 3-category is just a pseudomonoid in the semistrict monoidal 2-category $\text{Hom}(A, A)$ [39]. Since every semistrict monoidal 2-category \mathcal{C} can be regarded as a one object semistrict 3-category $\Sigma(\mathcal{C})$, a pseudomonoid in \mathcal{C} amounts to a pseudomonad on the one object of $\Sigma(\mathcal{C})$. Thus, the problem of showing that a semistrict monoidal 2-category generated by a pseudomonoid extends to a semistrict 3-category generated by a pseudoadjunction amounts to showing that every pseudomonad arises from a pseudoadjunction.

Again, the Eilenberg-Moore construction saves the day. Since Eilenberg-Moore objects can be defined as a weighted limit, this notion can be extended using enriched category theory to define Eilenberg-Moore objects in a **Gray**-category. While it is not true that an arbitrary **Gray**-category always possesses an Eilenberg-Moore object for every pseudomonad, using enriched category theory we can define the free Eilenberg-Moore completion $\mathbf{EM}(\mathcal{K})$ of a **Gray**-category \mathcal{K} [32, 33]. By the theory of such completions we obtain a **Gray**-functor $Z: \mathcal{K} \rightarrow \mathbf{EM}(\mathcal{K})$ with the property that for any **Gray**-category \mathcal{L} with Eilenberg-Moore objects, composition with Z induces an equivalence of categories between the **Gray**-functor category $[\mathcal{K}, \mathcal{L}]$ and the full subcategory of the **Gray**-functor category $[\mathbf{EM}(\mathcal{K}), \mathcal{L}]$ consisting of those **Gray**-functors which preserve Eilenberg-Moore objects [32]. Furthermore, Z will be fully faithful.

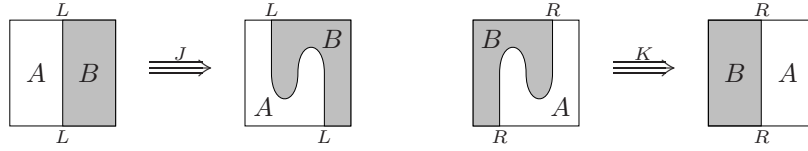
In the Eilenberg-Moore completion $\mathbf{EM}(\mathcal{K})$ every pseudomonad in the **Gray**-category \mathcal{K} arises from a pseudoadjunction. If \mathbb{T} is the pseudomonad in $\Sigma(\mathbf{pMon})$ generating the suspension of the walking pseudomonoid, then we can freely complete $\Sigma(\mathbf{pMon})$ under an Eilenberg-Moore object for just the pseudomonad \mathbb{T} . Again, we denote the completion under an Eilenberg-Moore object for the pseudomonad \mathbb{T} as $\mathbf{EM}_{\mathbb{T}}(\mathcal{K})$, and in this **Gray**-category the pseudomonad \mathbb{T} is generated by a pseudoadjunction $A \xrightleftharpoons[\perp_p]{} A^{\mathbb{T}}$. By the universal property of the walking pseudomonoid, this determines a **Gray**-functor $\Lambda: \mathbf{pAdj} \rightarrow \mathbf{EM}_{\mathbb{T}}(\mathcal{K})$.

To construct an inverse to the **Gray**-functor Λ , note that we have shown that the morphism LR in $\text{Hom}(A, A)$ of \mathbf{pAdj} is a pseudomonad. The universal property of the walking pseudomonoid then determines a **Gray**-functor $\mathbf{pMon} \rightarrow \mathbf{pAdj}$. Further, one can check that the pseudomonad LR has an Eilenberg-Moore

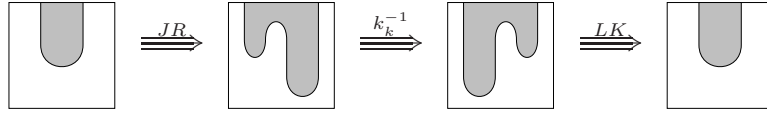
object in \mathbf{pAdj} , namely B . Hence, by the universal property of the Eilenberg-Moore completion, we get a **Gray**-functor $\bar{\Lambda}: \mathbf{EM}_{\mathbb{T}}(\mathcal{K}) \rightarrow \mathbf{pAdj}$ that preserves Eilenberg-Moore objects. It is easy to see that Λ and $\bar{\Lambda}$ define an isomorphism of **Gray**-categories. \square

3.4 The walking pseudo ambijunction

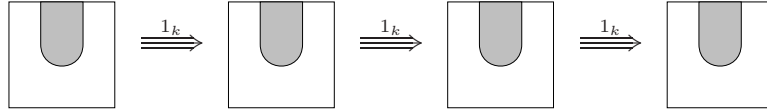
An ambidextrous pseudoadjunction, or pseudo ambijunction for short, is a 2-sided pseudoadjunction. This means that we have the additional 2-morphisms $j: 1_B \Rightarrow RL$ and $k: LR \rightarrow 1_A$ and the additional 3-morphisms



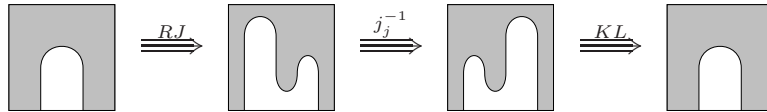
These 3-morphisms must satisfy the coherence conditions that the composite:



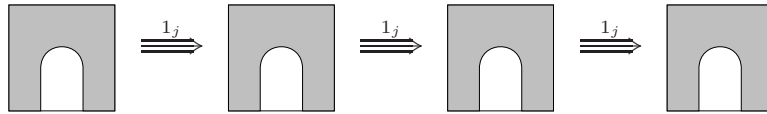
is equal to the composite:



and that the composite:



is equal to the composite:



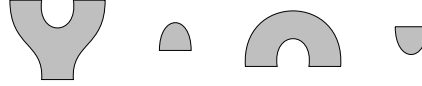
Definition 23 *The walking ambidextrous pseudoadjunction \mathbf{pAmbi} is semistrict 3-category freely generated by a pseudo ambijunction.*

We now define a pseudo Frobenius algebra by categorifying the relationship between Frobenius algebras and adjunctions.

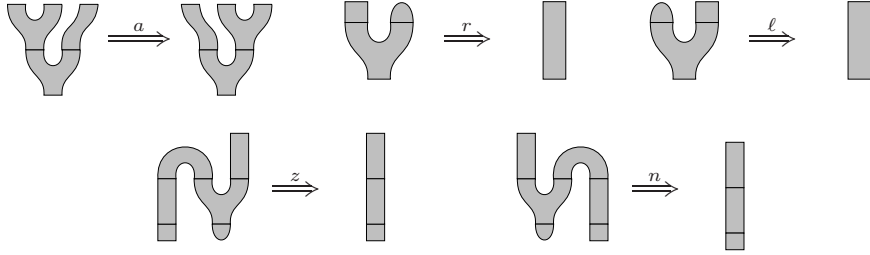
Definition 24 *The walking pseudo Frobenius algebra is the semistrict monoidal 2-category $\text{Hom}(A, A)$ in the walking pseudo ambijunction. Hence, a pseudo Frobenius algebra is $LR \in \text{Hom}(A, A)$ for some pseudo ambijunction $L \dashv_p R: B \rightarrow A$.*

Although we obtained this definition by categorifying the relationship between Frobenius algebras and adjunctions, it is equivalent to a definition that can be obtained by replacing equations with isomorphisms and determining the correct coherence conditions. Below we provide an equivalent definition that can be viewed as a categorification of Proposition 1 (iii.). In the next section we will see categorifications of the descriptions given by Proposition 1 (i.) and (ii.) and later we will show that these are also equivalent to the description given above. The definition we present below is the simplest and most easily related to pseudo ambijunctions. It might be described as a pseudomonoid equipped with a form defining a ‘weakly nondegenerate’ pairing. It is also very related to the notion of a Frobenius pseudomonoid defined by Street [53].

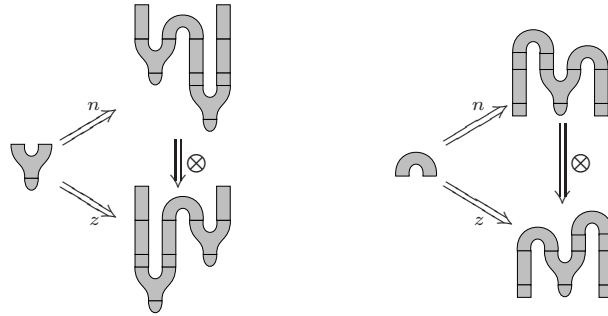
Proposition 25 *A pseudo Frobenius algebra can be equivalently defined as follows: A pseudo Frobenius algebra is an object F of a semistrict monoidal 2-category \mathcal{C} equipped with morphisms:*



and 2-isomorphisms:



satisfying the pseudomonoid axioms and making the following diagrams commute:

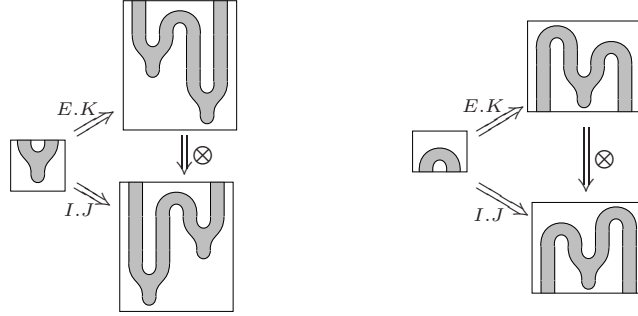


Note that in the first diagram above the 2-isomorphisms \otimes is actually two applications of \otimes .

Proof. In the previous section we saw that a pseudomonoid can always be defined as $LR \in \text{Hom}(A, A)$ for some pseudoadjunction $A \xrightleftharpoons[\underset{R}{\perp p}]{\overset{L}{\perp p}} B$. This follows from the fact that the walking pseudomonoid is $\text{Hom}(A, A)$ in the walking pseudoadjunction. In order to prove the proposition, we need to show that in an ambidextrous pseudoadjunction the morphisms LR has the additional structure described by the diagrams and axioms above. The 2-isomorphisms z and n in $\text{Hom}(A, A)$ are defined by the pasting composites depicted below:



The precise definition of the pasting composites can be deduced from the diagrams. These 2-isomorphisms satisfy the specified axioms since:



commute from the fact that each set (I, E) and (J, K) satisfy the triangulator identities.

The converse is proven, again, using monad theory. Given a pseudo Frobenius object as defined by the diagrams and axioms in the proposition, it is clear that the object F of the semistrict monoidal 2-category \mathcal{C} is a pseudomonoid in \mathcal{C} . Hence, by definition \mathbb{F} is a pseudomonad on the one object of the semistrict 3-category $\Sigma(\mathcal{C})$. The maps z and n provide this pseudomonad with some additional structure. In particular, these maps make \mathbb{F} into a *Frobenius pseudomonad* [33, 53].

Using an enriched version of the Eilenberg-Moore completion, the author has shown that every Frobenius pseudomonad in a semistrict 3-category \mathcal{K} arises from an ambidextrous pseudoadjunction in $\mathbf{EM}(\mathcal{K})$, where $\mathbf{EM}(\mathcal{K})$ is the free completion of \mathcal{K} under Eilenberg-Moore objects for every pseudomonad in \mathcal{K} [33]. Since a pseudo Frobenius algebra F in \mathcal{C} is the same thing as a Frobenius pseudomonad \mathbb{F} on the one object of $\Sigma(\mathcal{C})$, it follows that F arises from some pseudo ambijunction in $\mathbf{EM}(\Sigma(\mathcal{C}))$. \square

We have actually done more than provided an equivalent characterization of pseudo Frobenius algebras. It is a simple extension to show:

Corollary 26 *The walking pseudo Frobenius algebra is the semistrict monoidal 2-category freely generated by a single object, and the morphisms and 2-isomorphisms of Proposition 25 subject to the axioms given.*

Proof. One can check that if F is the pseudo Frobenius algebra generating the walking pseudo Frobenius algebra, then the semistrict 3-category $\mathbf{EM}_{\mathbb{F}}(\Sigma(\text{pFrob}))$ is isomorphic to the walking ambidextrous pseudoadjunction. Hence, pFrob is $\text{Hom}(A, A)$ in pAdj . \square

3.5 Examples

We now provide a brief survey of the literature where higher-dimensional analogs of Frobenius algebras have appeared.

Trivial examples. Since every monoidal category can be regarded as a semistrict monoidal 2-category with only identity 2-morphisms, every Frobenius algebra is a pseudo Frobenius algebra with only identity 2-morphisms and trivial coherences.

***-autonomous categories.** Note that a pseudo Frobenius algebra in the 2-category \mathbf{Cat} is a weak monoidal category with some extra structure. Street has shown that the condition that a monoidal category be Frobenius is equivalent to the condition that the monoidal category be *-autonomous [53]. These *-autonomous monoidal categories are known to have an interesting relationship with quantum groups and quantum groupoids [15]. Combined with our result relating Frobenius pseudomonoids to pseudo ambijunctions, the relationship with *-autonomous categories may have implications to quantum groups, as well as the field of linear logic where *-autonomous categories are used extensively.

Khovanov's Frobenius functors. In *A functor-valued invariant of tangles*, Khovanov sketches a definition of a topological quantum field theory with corners and suggests that the useful examples of these structures arise from functors with a 2-sided adjoint, or what he calls ‘Frobenius functors’ [27]. In our language we would say that these TQFT’s with corners arise from ambijunctions in \mathbf{Cat} . He then lists and describes in detail some categories with many Frobenius functors, that is, he lists many examples of ambijunctions in various 2-subcategories of the 2-category \mathbf{Cat} . We repeat his list below although we will not describe in detail the ambijunctions. For details the reader is referred to Khovanov’s paper [27].

Categories with many Frobenius functors:

- Categories of modules over symmetric and Frobenius algebras and their derived categories.
- Categories of highest weight modules over simple Lie algebras and their derived categories.
- Derived categories of coherent sheaves on Calabi-Yau manifolds.
- Fuyaya-Floer categories of lagrangians in a symplectic manifold.

Tillmann’s Frobenius categories [54] Tillmann suggests that in order to encode 3-dimensional information into a 2-dimensional topological quantum field theory one must consider a more interesting version of the 2-dimensional cobordism category, namely $\mathbf{2Cob}_2$. The objects of $\mathbf{2Cob}_2$ are closed, oriented, compact 1-manifolds, and the morphisms are oriented, compact 2-manifolds. The 2-morphisms of this 2-category are the connected components of orientation preserving diffeomorphisms of the 2-manifolds. This cobordism 2-category was first studied by Carmody [11].

By extending the category $\mathbf{2Cob}$ to the 2-category $\mathbf{2Cob}_2$, Tillmann defines a modular functor as a monoidal 2-functor from $\mathbf{2Cob}_2 \rightarrow \mathbb{k}\text{-}\mathbf{Cat}$, where $\mathbb{k}\text{-}\mathbf{Cat}$ is the 2-category of linear categories, linear functors and linear natural transformations. Tillmann calls the image of such a 2-functor a ‘Frobenius category’. She goes on to show that these Frobenius categories are related to 3-dimensional topological quantum field theory. In our terminology, these ‘Frobenius categories’ are a symmetric version of pseudo Frobenius algebras in the 2-category of \mathbb{k} -linear categories.

This example is particularly related to the results of this paper. In the next section we will discuss the nonsymmetric version of the cobordism 2-category described above. The effect of removing the symmetry requirement amounts to ‘smashing the cobordisms flat’ into what we call $\mathbf{3Thick}$, the 2-category of 3-dimensional thick tangles. This 2-category turns out to be an extension of the category $\mathbf{2Thick}$ defined in Section 2.5. Three-dimensional thick tangles are the most important example of pseudo Frobenius algebras since we will also show that the monoidal bicategory of 3-dimensional thick tangles is triequivalent to the walking pseudo Frobenius algebra. This means that all of the examples above are the image of a **Gray**-functor from the monoidal bicategory of 3-dimensional thick tangles into the relevant semistrict monoidal categories.

4 Thick tangles

We now define an extension of the monoidal category of 2-dimensional thick tangles. It is perhaps not surprising that this extension will involve an extra level of categorical structure. In fact there are numerous examples of higher categories providing an algebraic description generalizing various kinds of algebraically

defined topological categories. For instance, just as the category of tangles in 3-dimensions is the free braided monoidal category with duals on one object, Baez and Langford have shown that the 2-category of 2-tangles in 4-dimensions is the free semistrict braided monoidal 2-category with duals on one object [6]. We will give yet another example of this phenomenon by showing that the monoidal bicategory of 3-dimensional thick tangles defined below is the walking pseudo Frobenius algebra. Note that a monoidal bicategory is just a one object tricategory. Hence, by the coherence theorem of Gordon-Power-Street, every monoidal bicategory is triequivalent to a semistrict monoidal 2-category.

Definition 27. (Kerler and Lyubashenko [26]) *The monoidal bicategory of 3-dimensional thick tangles denoted $\mathbf{3Thick}$ has nonnegative integers as objects. The 1-morphisms from k to l are smooth oriented compact surfaces X with boundary ∂X equipped with disjoint distinguished intervals $i_j^s: I \hookrightarrow \partial X$, $1 \leq j \leq k$, $i_m^t: \hookrightarrow \partial X$, $1 \leq m \leq l$, equipped with a smooth embedding $d: X \hookrightarrow \mathbb{R} \times [0, 1]$ such that*

$$d^{-1}(\mathbb{R} \times 0) = I_1^s \sqcup I_2^s \sqcup \cdots \sqcup I_k^s, \quad I_j^s = i_j^s(I), \quad d(I_j^s) = [j - \frac{1}{3}, j + \frac{1}{3}] \times 0,$$

$$d^{-1}(\mathbb{R} \times 1) = I_1^t \sqcup I_2^t \sqcup \cdots \sqcup I_l^t, \quad I_j^t = i_j^t(I), \quad d(I_j^t) = [j - \frac{1}{3}, j + \frac{1}{3}] \times 1.$$

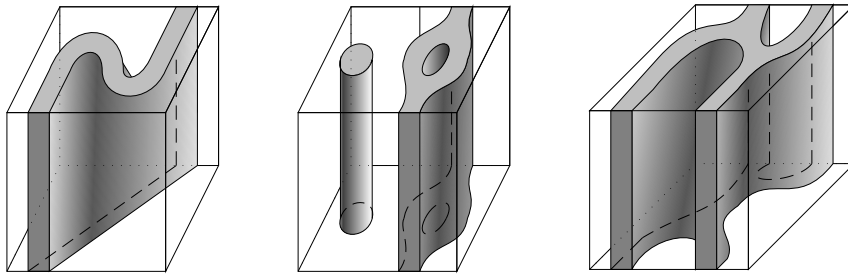
The image $d(X)$ is called a diagram of thick tangles.

The 2-morphisms $\phi: X \Rightarrow Y: k \rightarrow l$ of $\mathbf{3Thick}$ are isotopy classes of oriented homeomorphisms $f: X \rightarrow Y$, which preserve the distinguished intervals,

i.e., $i_{jY}^{s/t} = (I \xrightarrow{i_{jY}^{s/t}} X \xrightarrow{f} Y)$. (Each homeomorphism $f_t: X \rightarrow Y$, $t \in [0, 1]$ in the isotopy family also preserves the distinguished intervals.) Composition

$Y \circ X$ of 1-morphisms $k \xrightarrow{X} l \xrightarrow{Y} m$ is defined by sewing of surfaces at boundary intervals $I_j^t(X)$ and $I_j^s(Y)$. The unit 1-morphism $1_k: k \rightarrow k$ is the union $\coprod_{j=1}^k [j - \frac{1}{3}, j + \frac{1}{3}] \times [0, 1]$. The isomorphism $1_l \circ X \xrightarrow{\sim} X$ are obtained by taking a neighborhood $(U, I_j^s) \simeq ([0, 1] \times [0, 1], [0, 1], [0, 1] \times 0)$ of the distinguished interval $I_j^s \subset X$ and by taking any isomorphism $[0, 1] \times [0, 1] \bigcup_{[0, 1] \times 1 \sim I_j^s} U \simeq U$. The tensor product is the disjoint union. The unit object is 0. The associativity constraints are obvious.

Here are some examples:



These surfaces can be interpreted as those topological membranes that arise from diffeomorphisms of planar open string worldsheets.

We are now ready to state the main theorem, but first, note that since a monoidal bicategory is just a one object tricategory, and a semistrict monoidal 2-category can also be regarded as a one object tricategory, it makes sense to talk about a triequivalence between them.

Theorem 28 *The monoidal bicategory of 3-dimensional thick tangles is triequivalent to the walking pseudo Frobenius algebra.*

We will prove this theorem in Section 4.2, but first we give a few corollaries.

Corollary 29 *A pseudo Frobenius object in a semistrict monoidal 2-category determines an invariant of 3-dimensional thick tangles.*

Proof. This follows from the universal property of the monoidal bicategory of thick tangles. \square

Corollary 30 *The monoidal category of two-dimensional thick tangles is equivalent to the walking Frobenius algebra.*

Proof. This result follows as a decategorification of Theorem 28. \square

If we define an extended planar open string topological field theory as a monoidal 2-functor from **3Thick** into a monoidal 2-category \mathcal{C} , then Theorem 28 can be rephrased as follows:

Theorem 31 *An extended planar open string topological field theory is equivalent to a pseudo Frobenius algebra in the monoidal 2-category \mathcal{C} .*

4.1 Some lemmas and related definitions

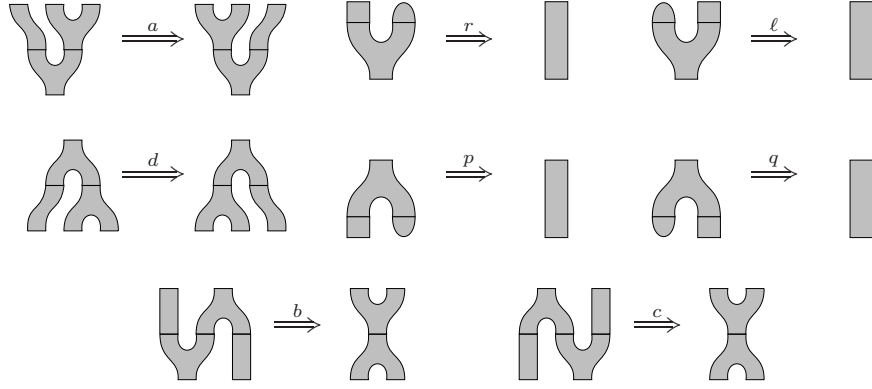
In this section we state a few lemmas due to Kerler and Lyubashenko that provide a generators and relations description of the monoidal bicategory of 3-dimensional thick tangles. Although some of these definitions are quite long, we will see in the next section that the semistrict monoidal 2-category with these generators and relations is monoidally 2-equivalent to the semistrict monoidal 2-category generated by a pseudo Frobenius algebra.

In a sense, all of the definitions below can be viewed as equivalent descriptions of the free semistrict monoidal 2-category generated by a pseudo Frobenius algebra.

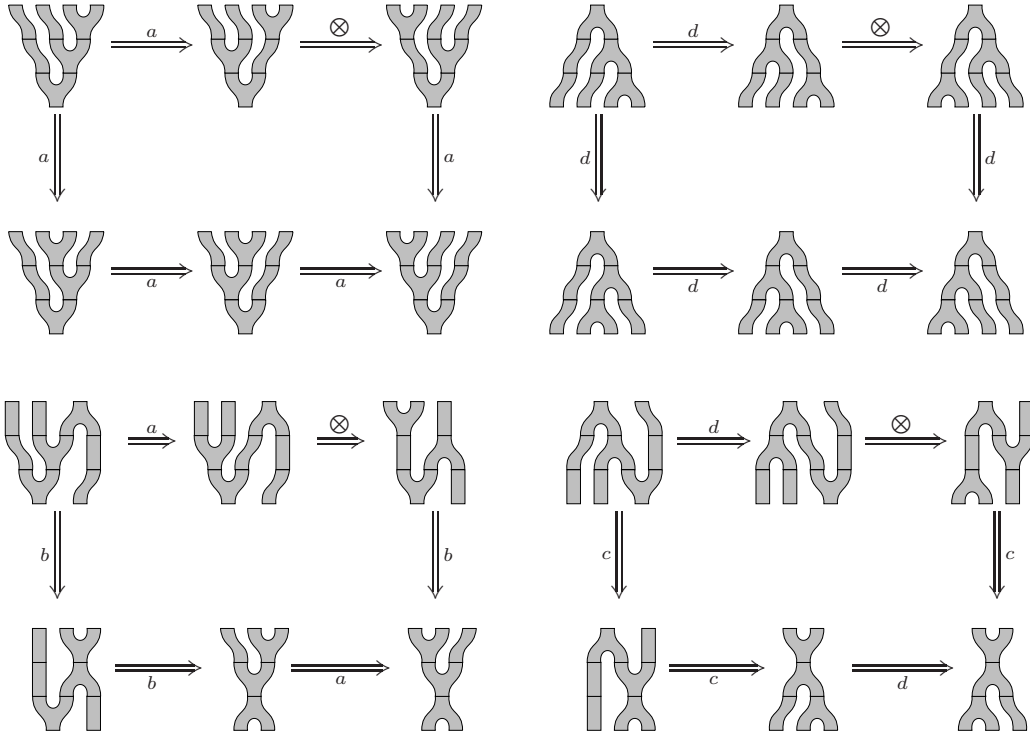
Lemma 32 (Kerler and Lyubashenko [26]) *The monoidal bicategory of 3-dimensional thick tangles is triequivalent to the semistrict monoidal 2-category \mathcal{F}_1 generated by one object, morphisms:*

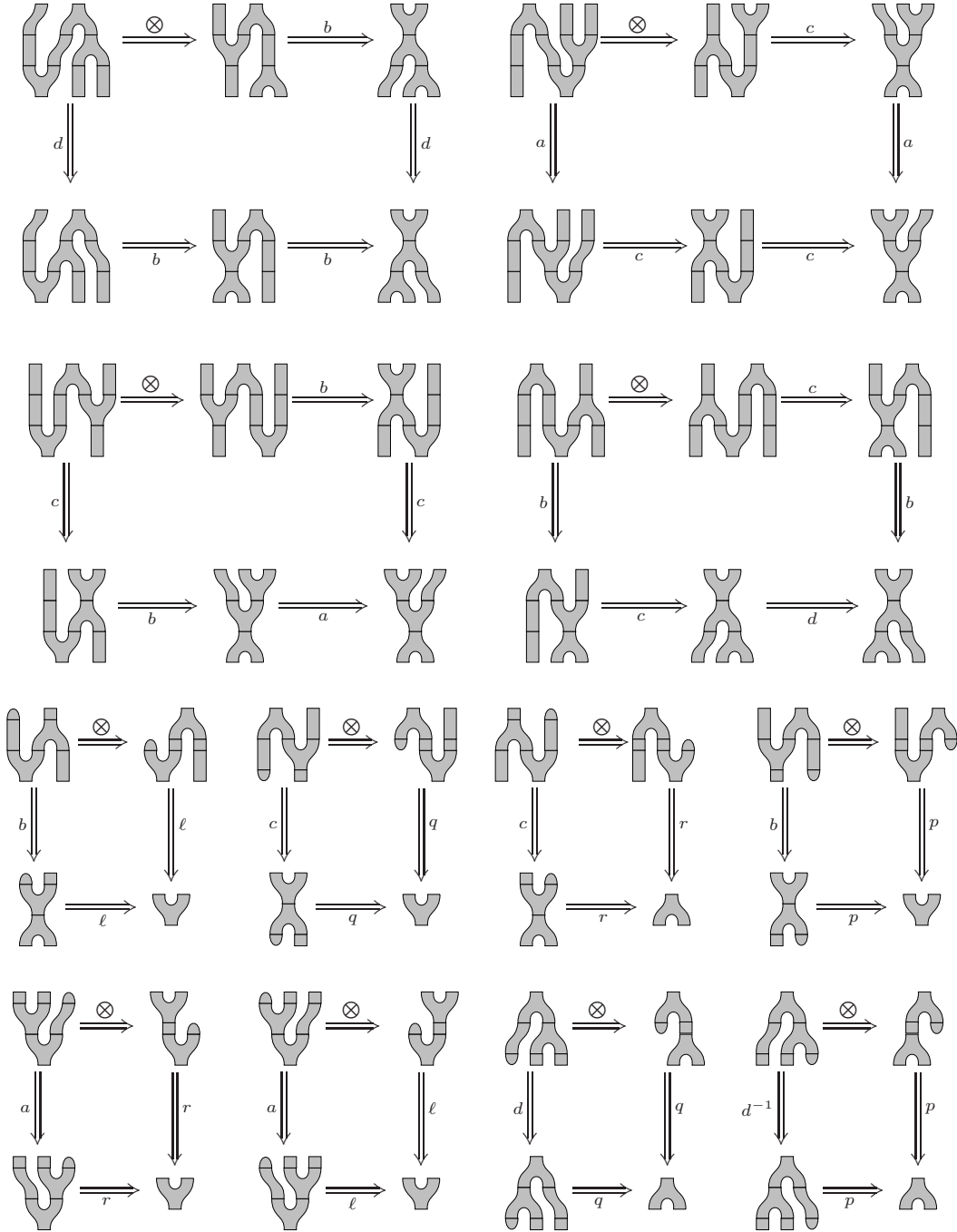


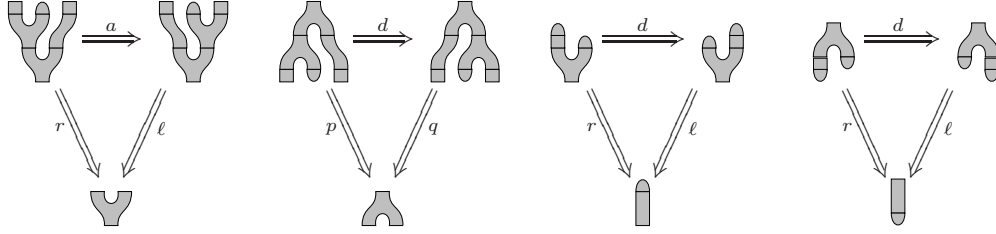
and 2-isomorphisms:



such that the following twenty diagrams:







commute.

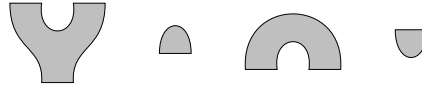
Proof. See Kerler and Lyubashenko [26]. \square

Note that this triequivalence is not a triequivalence in its weakest form. First of all, **3Thick** and \mathcal{F}_1 have only one object viewed as tricategories. Hence, the trifunctors defining this triequivalence are the identity on objects. Furthermore, since both 2-categories are freely generated by one object it is clear that these trifunctors must also be the identity on 1-morphisms. Thus, we are not using the term ‘trieuivalence’ in its weakest form.

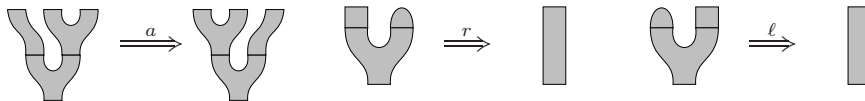
This definition should remind the reader of the definition of Frobenius algebra given in Proposition 1 (i.). Notice that all of the axioms that held as equalities are now replaced by coherent isomorphisms. It might be described as a pseudomonoid and pseudocomonoid satisfying the Frobenius identities up to coherent isomorphism. Notice also the tremendous number of coherence laws for this definition. Since the coherence laws describe how each of generating 2-morphisms behave with respect to each other, the large number of generating 2-morphisms means a large number of coherence conditions.

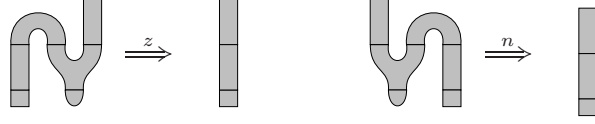
In the next lemma we show that the definition given above is monoidally 2-equivalent to a definition with a few less coherence conditions. To clarify what is meant by monoidal 2-equivalence, let \mathcal{D} and \mathcal{D}' be monoidally 2-equivalent categories. We can regard \mathcal{D} and \mathcal{D}' as one object **Gray**-categories. In this case, the monoidal 2-equivalence translates to a 3-equivalence of **Gray**-categories. Hence, we will sometimes say a 3-equivalence of semistrict monoidal 2-categories by which we simply mean a 3-equivalence of their respective suspensions.

Lemma 33 (Kerler and Lyubashenko [26]) *The semistrict monoidal 2-category \mathcal{F}_1 from Lemma 32 is 3-equivalent to the semistrict monoidal 2-category \mathcal{F}_2 generated by one object, morphisms:*

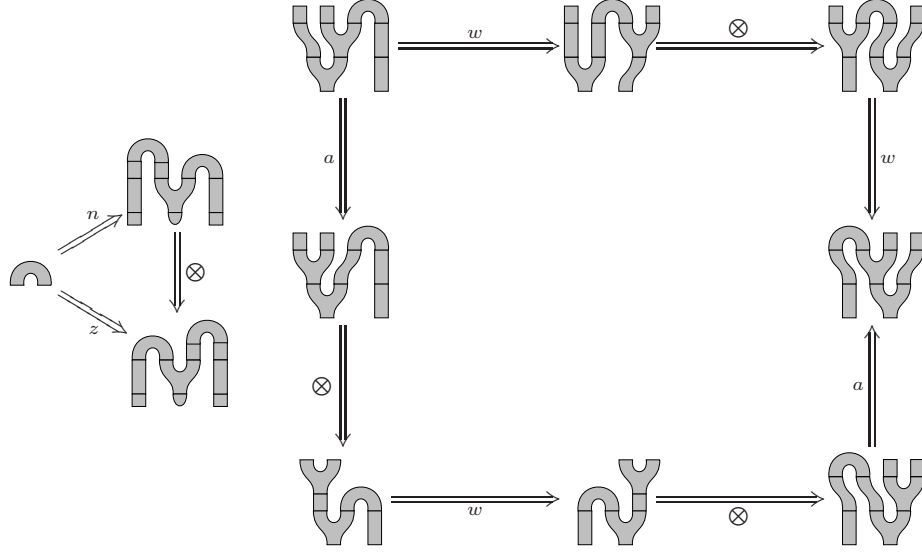


and 2-isomorphisms:

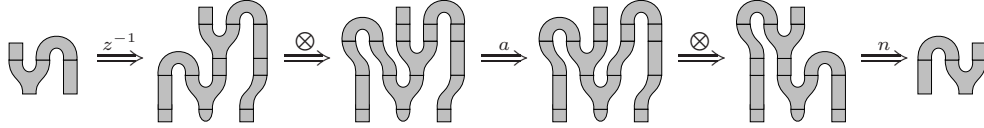




satisfying the pseudomonoid axioms and making the following diagrams commute:



where w is the following composite:



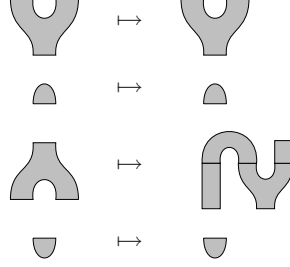
Notice that this definition is reminiscent of the definition of Frobenius algebra given in Proposition 1 (ii.). Notice in particular relation of the isomorphism w to the axioms in Proposition 1 (ii.). This definition might be described as a pseudomonoid equipped with a form and a copairing defining two coherently isomorphic comultiplications and counits. In this case, we have been careful to use a minimum number of generating 2-morphisms in order to minimize the number of coherence axioms.

To simplify diagrams, we will from now on omit inverses on the labels for the 2-morphisms. The correct label should be apparent from the diagram.

Proof. We define a strict 2-functor $\Theta: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ as follows:

- On objects Θ is the identity map.

- On the generating morphisms Θ is given as follows:



- On the coherence 2-isomorphisms in \mathcal{F}_1 given by the commutative diagram:

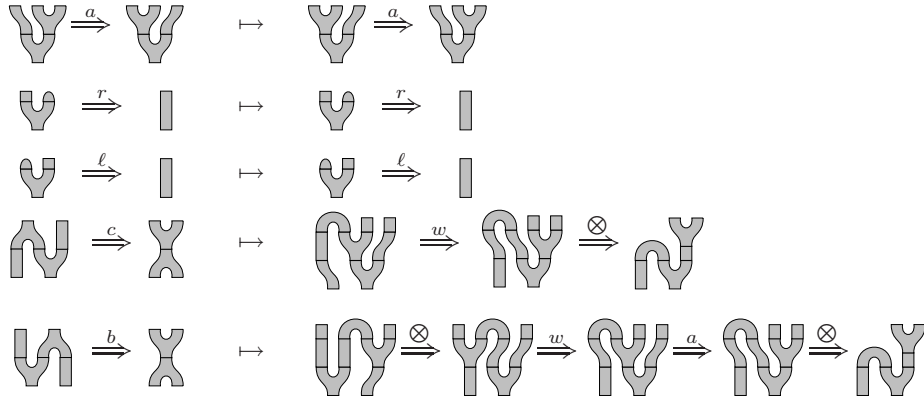
$$\begin{array}{ccc}
 A \otimes B & \xrightarrow{A \otimes g} & A \otimes B' \\
 f \otimes B \downarrow & \Downarrow \otimes_{f,g} & \downarrow f \otimes B' \\
 A' \otimes B & \xrightarrow{A' \otimes g} & A' \otimes B'
 \end{array}$$

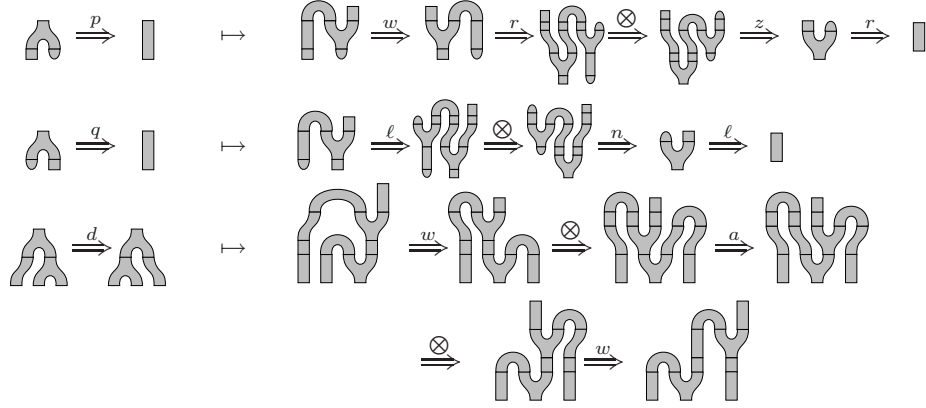
Θ maps $\otimes_{f,g}$ to the 2-cell given by the diagram:

$$\begin{array}{ccc}
 \Theta(A) \otimes \Theta(B) & \xrightarrow{\Theta(A) \otimes \Theta(g)} & \Theta(A) \otimes \Theta(B') \\
 \Theta(f) \otimes \Theta(B) \downarrow & \Downarrow \otimes_{\Theta(f), \Theta(g)} & \downarrow \Theta(f) \otimes \Theta(B') \\
 \Theta(A') \otimes \Theta(B) & \xrightarrow{\Theta(A') \otimes \Theta(g)} & \Theta(A') \otimes \Theta(B')
 \end{array}$$

This choice can consistently be made by the coherence axioms of a **Gray**-category.

- On the generating 2-isomorphisms Θ is defined as follows:

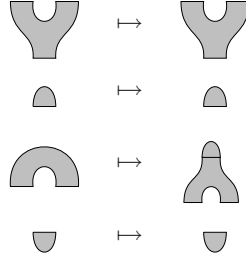




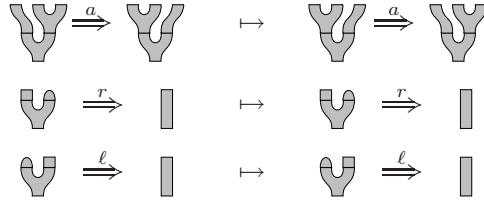
One can check that these maps satisfy the required relations making Θ into a strict 2-functor. This amounts to checking that all 20 of the axioms are satisfied by the maps in the image of Θ . Furthermore, it clear from the definition that Θ preserves the monoidal structure on the nose. That is, Θ is a strictly monoidal strict 2-functor.

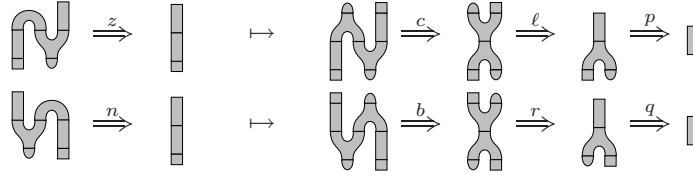
We define the other strict 2-functor $\bar{\Theta}: \mathcal{F}_2 \rightarrow \mathcal{F}_1$ defining a 2-equivalence of 2-categories as follows:

- On objects $\bar{\Theta}$ is the identity map.
- On the generating morphisms $\bar{\Theta}$ is defined as follows:



- On the coherence 2-isomorphisms \otimes , $\bar{\Theta}$ is defined analogously as Θ . Again, this assignation is well-defined by the coherence axioms for a **Gray**-category.
- On the generating 2-isomorphisms $\bar{\Theta}$ is defined as follows:





Again, it is a routine and laborious calculation to check that these maps satisfy the required relations making $\bar{\Theta}$ into a strict 2-functor. By construction, it is clear that $\bar{\Theta}$ strictly preserves the monoidal structure.

To see that Θ and $\bar{\Theta}$ define a 2-equivalence of 2-categories we must check that their composites are 2-naturally isomorphic to the identity. Since both 2-functors are the identity on objects it is clear that on objects the composites are actually equal to the identity map. One can check that on any generating morphisms, say X , the image of X under the composites of the above 2-functors is naturally isomorphic to X . Hence, the strict 2-categories \mathcal{F}_1 and \mathcal{F}_2 are 2-equivalent. Furthermore, since Θ and $\bar{\Theta}$ are strict monoidal functors it is clear that \mathcal{F}_1 and \mathcal{F}_2 are monoidally 2-equivalent. Regarding \mathcal{F}_1 and \mathcal{F}_2 as one object **Gray**-categories, we see then that Θ and $\bar{\Theta}$ define a 3-equivalence of **Gray**-categories. \square

The observant reader will have noticed that the isomorphisms defining the image of the generators in the above maps are precisely the proofs of the equivalent definitions of a Frobenius algebra with the equalities replaced by coherent isomorphisms. In the next section we will show that these two 2-categories are actually monoidally 2-equivalent to the walking pseudo Frobenius algebra. This means that an object of a semistrict monoidal 2-category equipped with the morphisms and 2-morphisms satisfying the axioms of either of the above lemmas serves as an equivalent definition of pseudo Frobenius algebra. Hence, categorifying any of the equivalent characterizations from Proposition 1 produces monoidally 2-equivalent 2-categories.

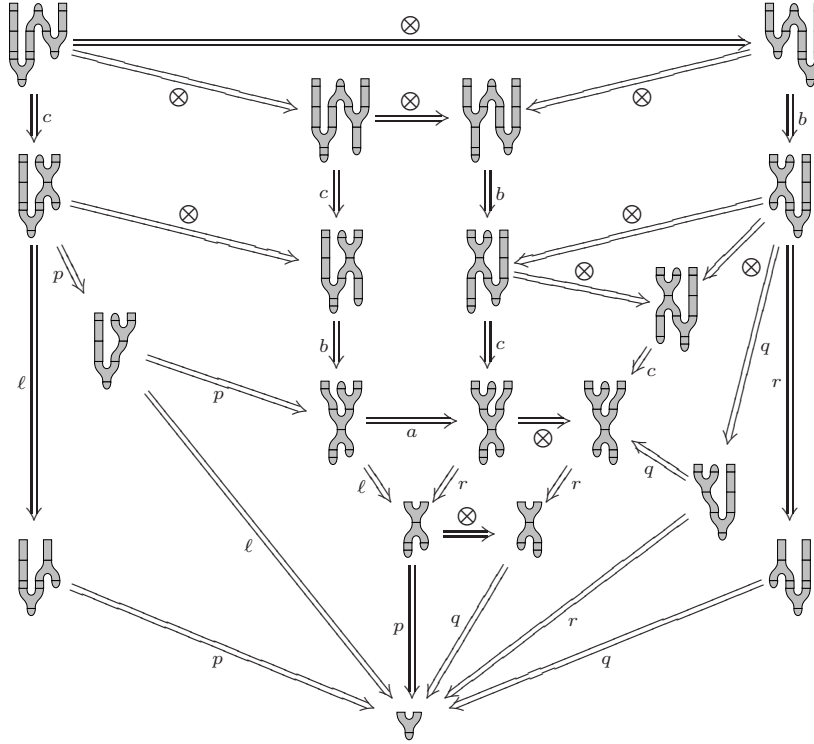
4.2 Proof of main theorem

In this section we prove the main result of this paper. We have shown in the previous section that the monoidal bicategory of 3-dimensional thick tangles is triequivalent to the semistrict monoidal category \mathcal{F}_2 defined in Lemma 33. We now show that the walking pseudo Frobenius algebra is 3-equivalent to \mathcal{F}_2 and the main result will follow.

Proof of Theorem 28. To prove that the walking pseudo Frobenius algebra is 3-equivalent to the semistrict monoidal category \mathcal{F}_2 we will use the description of the walking pseudo Frobenius algebra given in Proposition 25. Notice that the walking pseudo Frobenius algebra and the semistrict monoidal 2-category \mathcal{F}_2 are both generated by the same objects, morphisms, and 2-morphisms. Thus, it suffices to prove that the axioms are satisfied in both definitions.

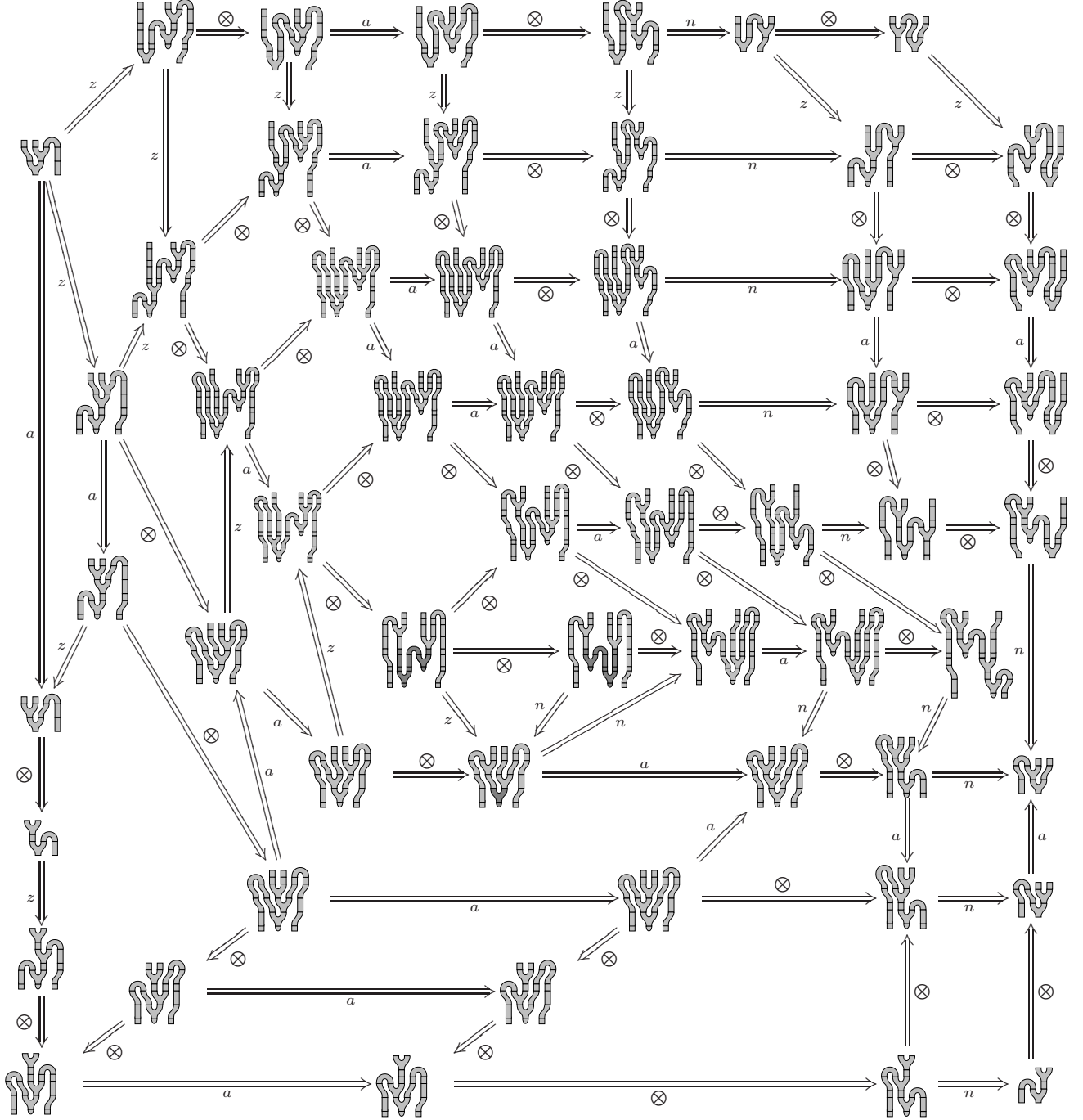
To prove that the walking pseudo Frobenius algebra satisfies the axioms for the generators of the semistrict monoidal 2-category \mathcal{F}_2 , we only need to check the axiom for the map w since the other coherence axiom is included in the description of pFrob given in Proposition 25. The proof is given in Figure 4.2. The outer rectangle of this diagram is the axiom for the isomorphism w in the definition of \mathcal{F}_2 with w expanded out. All of the outer squares commute by the properties of a semistrict monoidal 2-category. The innermost triangle with some of 1-morphisms drawn slightly darker is the first coherence condition from Proposition 25. Just below this triangle is a rather distorted rectangle that is the coherence for associativity in a pseudomonoid. Note that some of the arrows labelled by \otimes may mean multiple applications of this 2-morphism.

To show that the semistrict monoidal 2-category \mathcal{F}_2 satisfies the axioms of the walking pseudo Frobenius algebra we must show that the first coherence axiom in Proposition 25 is satisfied. To simplify the proof, we use the 3-equivalence of monoidal strict 2-categories \mathcal{F}_1 and \mathcal{F}_2 from Lemma 33. With the twenty axioms from the generators of \mathcal{F}_1 at our disposal, the proof becomes much simpler. The proof is shown below:



Since the walking pseudo Frobenius algebra has the same monoidal structure as \mathcal{F}_2 it is clear that the suspension of the walking pseudo Frobenius algebra and the suspension of \mathcal{F}_2 are 3-equivalent **Gray**-categories.

□

Figure 2: Proof that pFrob satisfies the relations of the \mathcal{F}_2 .

5 Conclusion

We used the description of the walking adjunction to understand the walking Frobenius algebra and its relation to 2-dimensional thick tangles. **2Thick** while not optimal from the perspective of studying all open string worldsheets has the advantage that it arises quite naturally from higher-dimensional category theory. Our success in algebraically characterizing **2Thick** suggests that the full category of open strings and their worldsheets might be algebraically characterized by studying more interesting instances of categorical adjunctions.

An adjunction is an intrinsically categorical concept and since this notion has already been generalized to the context of **Gray**-categories we were able to use the relationship between adjunctions and Frobenius algebras to categorify the notion of a Frobenius algebra. The description of the walking pseudo Frobenius algebra in terms of the walking pseudo ambidextrous adjunction also allowed us to see the relationship of pseudo Frobenius algebras to 3-dimensional thick tangles using string diagrams. These results suggest that the general machinery of adjunctions in n -categories may prove useful in studying string membranes and other interesting cobordism categories. In a future work we will explain how the topology of arbitrary string membranes (rather than those defined by diffeomorphisms) can be algebraically described using a generalization of what we have described in this paper.

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