## Braided fusion categories, gravitational anomalies, and the mathematical framework for topological orders in any dimensions

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Topological order describes a new kind of order in gapped quantum liquid states of matter that correspond to patterns of long-range entanglement, while gravitational anomaly describes the obstruction that a seemingly consistent low energy effective theory cannot be realized by any well defined quantum model in the same dimension. Amazingly, topological order and gravitational anomaly have a very direct relation: gravitational anomalies can be realized on the boundary of topologically ordered states in one higher dimension and are described by topological orders in one higher dimension. In this paper, we try to develop a general theory for topological order and gravitational anomaly in any dimensions. (1) We introduce the notion of BF category to describe the braiding and fusion properties of topological excitations that can be point-like, string-like, etc. A subset of BF categories - closed BF categories - classify topological orders in any dimensions, while generic BF categories classify (potentially) anomalous topological orders that can appear at a boundary of a gapped quantum liquid in one higher dimension. (2) We introduce topological path integral based on tensor network to realize those topological orders. (3) Bosonic topological orders have an important topological invariant: the vector bundles of the degenerate ground states over the moduli spaces of closed spaces with different metrics. They may fully characterize topological orders. (4) We conjecture that a topological order has a gappable boundary iff the above mentioned vector bundles are flat. (5) We find a holographic phenomenon that every topological order with a gappable boundary can be uniquely determined by the knowledge of the boundary. As a consequence, BF categories in different dimensions form a (monoid) cochain complex, that reveals the structure and relation of topological orders and gravitational anomalies in different dimensions. We also studied the simplest kind of bosonic topological orders that have no non-trivial topological excitations. We find that this kind of topological orders form a  $\mathbb{Z}$  class in 2+1D (with gapless edge), a  $\mathbb{Z}_2$  class in 4+1D (with gappable boundary), and a  $\mathbb{Z} \oplus \mathbb{Z}$  class in 6+1D (with gapless boundary).

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# I. INTRODUCTION OF TOPOLOGICAL ORDER

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In 1989, through a theoretical study of chiral spin liquid,  $^{1,2}$  we realized that there exists a new kind of order – topological order  $^{3-5}$  – beyond Landau symmetry breaking theory. Topological order cannot be characterized by the local order parameters associated with the symmetry breaking. However, topological order can be characterized/defined by (a) the topology-dependent ground state degeneracy  $^{3,4}$  and (b) the non-Abelian geometric phases of the degenerate ground states  $^{5,6}$ , where both of them are robust against any local perturbations that can break any symmetries. This is just like superfluid order is characterized/defined by zero-viscosity and quantized vorticity that are robust against any local perturbations that preserve the U(1) symmetry.

We know that, microscopically, superfluid order is originated from boson or fermion-pair condensation. Then, what is the microscopic origin of topological order? What is the microscopic origin of robustness against any local perturbations? Recently, it was found that, microscopically, topological order is related to long-range entanglement.<sup>7,8</sup> In fact, we can regard topological order as pattern of long-range entanglement in many-body ground states,<sup>9</sup> which is defined as the equivalent classes of stable gapped quantum liquid<sup>10</sup> states under local unitary transformations.<sup>11–13</sup> The notion of topological orders and many-body quantum entanglement leads to a new point of view of quantum phases and quantum phase transitions: 9 for bosonic Hamiltonian quantum systems without any symmetry, their gapped quantum liquid phases<sup>10</sup> can be divided into two classes: short-range entangled (SRE) states and long-range entangled (LRE) states.

SRE states are states that can be transformed into tensor-product states via local unitary transformations. All SRE states belong to the same phase. LRE states are states that cannot be transformed into tensor-product states via local unitary transformations. LRE states can belong to different quantum phases, which are nothing but the topologically ordered phases. Chiral spin liquids  $^{1,2}$ , integral/fractional quantum Hall states  $^{14-16}$ ,  $\mathbb{Z}_2$  spin liquids  $^{17-19}$ , non-Abelian fractional quantum Hall states  $^{20-23}$ , etc., are examples of topologically ordered phases.

Topological order and long-range entanglement are truly new phenomena. They require new mathematical language to describe them. It appears that tensor category theory  $^{9,11,24-27}$  and simple current algebra  $^{20,28-30}$  (or pattern of zeros  $^{31-39}$ ) may be part of the new mathematical language. Using tensor category theory, we have developed a systematic and quantitative theory for topological orders with gapped edge for 2+1D interacting boson and fermion systems.  $^{9,11,25,27}$  For 2+1D topo-

logical orders (with gapped or gapless edge) that have only Abelian statistics, we find that we can use integer K-matrices to classify them and use the following U(1) Chern-Simons theory to describe them<sup>40–45</sup>

$$\mathcal{L} = \frac{K_{IJ}}{4\pi} a_{I\mu} \partial_{\nu} a_{J\lambda} \epsilon^{\mu\nu\lambda}. \tag{1}$$

# II. A SUMMARY OF MAIN RESULTS/CONJECTURES

In this paper, we try to develop a general theory for topological order and gravitational anomaly for local bosonic quantum systems in any dimensions. <sup>46</sup> We would like to consider the following basic issues:

- 1. How to classify topological orders in any dimensions. (Previous works have classified topological orders in 1+1 space-time dimensions: there is no nontrivial topological order in 1+1D.<sup>12,47</sup>. The 2+1D topological orders with gappable boundary are classified by unitary fusion category.<sup>9,11,26,48,49</sup>)
- 2. How to classify anomalous topological orders that can only appear on a boundary of a gapped system, but cannot be realized by any well-defined system in the same dimension? (In this paper, we follow the tradition to use the term "topological order" to mean anomaly-free topological order.)
- 3. Given a low energy effective theory (for example given the data that describes the fusion and braiding of topological excitations), how to determine if it is anomalous or anomaly-free? Can we realize a given set of fusion and braiding properties by a well defined model in the same dimension?
- 4. Given an anomaly-free topological order, how to determine if its boundary can be gapped or not? (See Ref. 45, 50–52 for discussions about the gappable boundary of 2+1D Abelian topological orders and Ref. 48 for general 2+1D topological orders.)
- 5. Given two bosonic Hamiltonians, how to determine if their ground states have the same topological order or not? $^{5,6,53-58}$

In this paper, we try to address the above issues. Let us first summarize the main results of this paper. They include

1. We define BF category, closed BF category, and exact BF category in various dimensions based on higher category theory. We will explain why the structures of an higher category automatically encode the information of the fusion and braiding of topological excitations which can be point-like, string-like, etc (see Sections IV and XI). These definitions are based on many intuitive physical consideration, and are conjectural and incomplete. The

precise definition is not important to us at the current stage. What is important to us is the general framework we provide, and how conjectures and physically important questions can be formulated in this framework. These conjectures and questions will serve as a blueprint for future studies.

We argue that the closed BF categories (defined in higher category theory) classify anomaly-free topological orders (defined in many-body wave functions). The exact BF categories classify topological orders with gappable boundary, and the BF categories classify all potentially anomalous topological orders (that can be realized on the boundary of well defined quantum models in one higher dimension), except a small class of anomalous topological orders described by unquantized gravitational Chern-Simons terms in 4k+3 space-time dimensions. See Section VI A.

- 2. We show that the above three kinds of BF categories in different dimensions form a (commutative-monoid-valued) cochain complex (just like closed and exact differential forms form a cochain complex). As a result, all the topological orders with gappable boundary are fully characterized (in a many-to-one fashion) and classified by anomalous topological orders in one lower dimension.
- 3. The perturbative and global gravitational anomalies (except those described by unquantized gravitational Chern-Simons terms in 4k + 3 space-time dimensions) are classified by closed BF categories in one higher dimension (see Section VIB).
- 4. We develop a tensor network approach that produce a large class of exact topological orders in any dimensions. We also use tensor networks in one higher dimension to produce (a large class of) topological orders and anomalous topological orders in any dimensions (see Section XII).
- 5. As an application of the developed theory, we studied the simplest bosonic topological orders that have no non-trivial topological excitations. We find that this kind of topological orders form a  $\mathbb{Z}$  class in 2+1D, a  $\mathbb{Z}_2$  class in 4+1D,<sup>59,60</sup> and a  $\mathbb{Z} \oplus \mathbb{Z}$  class in 6+1D (see Section XV).<sup>61-63</sup> The boundary of  $\mathbb{Z}$ -class topological orders must be gapless (with perturbative gravitational anomalies), while the boundary of  $\mathbb{Z}_2$ -class topological orders can be gapped. But such a gapped boundary must be topological, which contains non-trivial topological excitations and has global gravitational anomalies.
- 6. As another application, we show that, for a 2+1D bosonic topological order, the chiral central charge c of the edge state must satisfy  $cD_g/2 \in \mathbb{Z}$  for g > 2, where  $D_g$  is the ground state degeneracy on genus g surface.

The above main results are built upon many new concepts and results. In the following, we will summarize them in detail.

#### A. Braided fusion category

In this paper, we will only consider local (shortrange interacting) bosonic quantum systems with a finite gap. To develop a theory of topological order in n+1-dimensional space-time, we assume that such a topologically ordered phase (a gapped phase) is characterized by the gravitational responses, as well as the topological properties of its topological excitations of spatial dimension p for  $0 \le p \le n-1$  (such as particle-like, string-like, and membrane-like excitations). The gravitational responses includes the thermal Hall effect in 2+1D (which is related to the chiral central charge of the edge states). The topological properties of the topological excitations include their fusion and braiding properties. The collection of all those topological properties defines a categorical notion, which generalizes the usual mathematical notion of braided tensor category and will be called a  $\mathbf{BF}_{n+1}$  category (see Section IV), where "B" stands for "braiding", "F" stands for fusion and the subscript always means the space-time dimension. In physics, the term "BF category" is synonymous to "gapped effective theory". In other word, a "gapped effective theory" is really a collection of data that describes the fusion and the braiding of topological excitations.

The main result of this paper is to develop an mathematical definition of  $BF_{n+1}$  category, which will allow us to develop a general theory for topological order and gravitational anomaly (perturbative and global) in any dimensions. We will first try to define  $BF_{n+1}$  category physically, trying to bring in relevant concepts for the definition (see Section V). Then we will define  $BF_{n+1}$  category mathematically using the n-category theory  $^{64-67}$  (see Section XI).

To have a simple understanding of the mathematical definition of  $BF_{n+1}$  category, we can start with a class of 0-categories – Hilbert spaces. A 1-category is a category enriched by 0-categories. Namely, it has a set of objects and a hom space hom(a, b) (or a space of arrows  $\{a \rightarrow b\}$ ) for each ordered pair of objects (a, b), and each hom space is a 0-category, i.e. a Hilbert space. A 1-category with only one object \* can describe a 0+1D quantum system, and the space of morphisms hom(\*,\*) is the local operator (observables) algebra of the quantum system (see Section XA). An morphism in hom(a,b) (or an arrow  $a \rightarrow b$ ) can also be viewed as a defect in the time direction, i.e. an instanton. A 2-category is a category enriched by 1-categories. More precisely, a 2-category consists of a set of objects (or 0-morphisms), a set of 1morphisms  $\{a \to b\}$  from object a to object b and a set of 2-morphisms  $\{x \Rightarrow y\}$  for 1-morphisms  $x, y : a \rightarrow b$ . The full hom space hom(a, b) between a and b, consisting all 1-morphisms from a to b and all 2-morphisms

between these 1-morphisms, form a 1-category. A 2category with one object \* and additional assumptions on unitarity describes a 1+1D gapped quantum systems. It contains the information of the fusion of point-like excitations in the 1+1D systems. The point-like excitations are described by the 1-morphisms from  $* \rightarrow *$  and the fusion of point-like excitations are described by composition of arrows (see Section XB). 2-morphisms are fusion/splitting channels of the point-like excitations and can also be viewed as instantons as they are defects in the time direction. A 3-category is a category enriched by 2categories. A 3-category with one object and additional assumptions describes a 2+1D gapped quantum system. It contains the information of the fusion and the braiding of string-like excitations (1-morphisms), point-like excitations (2-morphisms) and instantons (3-morphisms) in the 2+1D systems (see Section XC). More generally, the notion of an (n+1)-category automatically includes the fusion and braiding structures of excitations of all codimensions in an (n+1) space-time dimensional system (see Section XID). One does not need to mention fusion and braiding at all.

In addition to  $\mathrm{BF}_{n+1}$  category, we will also introduce the notion of a  $\mathrm{BF}_{n+1}^{pre}$ -category, which can describe (not in a minimal way) many interesting constructions of topological orders from concrete models, and that of a  $\mathrm{MBF}_{n+1}^{pre}$ -category, which can describe multiple phases connected by gapped domain walls, including the gapped boundary cases.

Terminology of dimensions: Both space-time dimensions and spatial dimension will be used in this work. To avoid confusion, we will always try to make it clear which one we mean. In general, by a p-dimensional topological excitation or defect, we always mean the spatial dimension; by an n-dimensional topological order, we always mean the space-time dimension. To avoid confusion, we will also use the term: an l-codimensional excitation. The subscript n in  $BF_n$ -category always means the space-time dimension. Sometimes we will use n+1 for space-time dimensions instead of n for the obvious reason.

# B. Simple, composite, and elementary topological excitations

The excitations above a topologically ordered ground state play a key role in developing a definition  $BF_n$ -category (and topological order). To use those excitations to define a  $BF_n$ -category, we introduced the notion of topological excitations which can be point-like, string-like, membrane-like, etc. We discussed the notions of simple and composite topological excitations (see Section IV) We also introduced the notion of elementary topological excitations (see Section VB7.)

We conjecture that topological orders are fully determined via the fusion and braiding properties of the elementary topological excitations alone (plus the gravi-

tational responses). (See Section XI.) This allows us to identify a special type of n-categories –  $BF_n$ -categories – that describe/define the topological orders.

## C. Stacking operation and tensor product ⊠

Let us use  $TO_n$  to denote a (anomaly-free) topologically ordered phase and  $aTO_n$  to denote a potentially anomalous topologically ordered phase, in n-dimensional space-time. Clearly, the set of (anomaly-free) topologically ordered phases  $\{TO_n\}$  is a subset of potential anomalous topologically ordered phases  $\{aTO_n\}$ . Both sets  $\{TO_n\}$  and  $\{aTO_n\}$  admit a multiplication operation: we can stack two physical systems that realize two topological orders  $(\mathcal{C}_n^1 \text{ and } \mathcal{C}_n^2)$  to obtain a double layer system that realizes another topological order. Such a stacking operation is a symmetric tensor product  $\boxtimes$ :  $\mathcal{C}_n^1 \boxtimes \mathcal{C}_n^2 = \mathcal{D}_n$ . In general, a topological order may not have an inverse. So the two sets  $\{TO_n\}$  and  $\{aTO_n\}$ , with the stacking  $\boxtimes$ , form commutative monoids. (A monoid is like a group except that some elements may not have inverse. See Section VIII A).

We like to point out that some topological orders do have an inverse under the stacking operation  $\boxtimes$ , which are called invertible.<sup>61,62</sup> The collection of all invertible topological orders form a group under the stacking  $\boxtimes$ .

An example of invertible anomalous topological orders, is described by an effective theory given by a gravitational Chern-Simons 3-form with a unquantized coefficient  $\kappa_{gCS}$ .  $\kappa_{gCS}$  generates a unquantized thermal Hall conductivity. We denote such anomalous topological orders as  $\text{aTO}_3^{\kappa_{gCS}}$  and call them gCS anomalous topological orders. Under the stacking, we have  $\text{aTO}_3^{\kappa_{gCS}} \boxtimes \text{aTO}_3^{\tilde{\kappa}_{gCS}} = \text{aTO}_3^{\kappa_{gCS} + \tilde{\kappa}_{gCS}}$ . So such invertible anomalous topological orders form an Abelian group isomorphic to the real numbers.

The gCS anomalous topological orders only appear in 4k+3 space-time dimensions. They are all described by gravitational Chern-Simons forms which exist only in 4k+3 space-time dimensions. It is not entirely clear to us how to include such gCS anomalous topological orders in our BF category approach. So in this paper, we will take a quotient. More precisely, we will use the term "anomalous topological orders" to refer to the quotient  $\{aTO_n\}/\{gCS \text{ anomalous topological orders}\}$ , which is also a monoid. The set of BF categories, defined as higher categories, form a monoid as well. We conjecture that the two monoids are isomorphic:

$$\{ \text{BF categories} \} \\ \simeq \frac{\{ \text{potentially anomalous topological orders} \}}{\{ \text{gCS anomalous topological orders} \}}. \tag{2}$$

This is a key expression of this paper. It relates a mathematical construction (BF category) to a physical phenomenon (topological order on a boundary).

Group structure can also be recovered on the certain quotient of the monoid of the BF<sub>n</sub> categories. We introduce two equivalence relations  $\sim$  and  $\stackrel{\text{w}}{\sim}$  between two BF<sub>n</sub> categories. Two BF<sub>n</sub> categories  $\mathcal{C}_n$  and  $\mathcal{D}_n$  are called quasi-equivalent if  $\mathcal{C}_n \sim \mathcal{D}_n$  and Witt equivalent if  $\mathcal{C}_n \stackrel{\text{w}}{\sim} \mathcal{D}_n$ . Witt equivalence  $\mathcal{C}_n \stackrel{\text{w}}{\sim} \mathcal{D}_n$  means that two corresponding phases  $\mathcal{C}_n$  and  $\mathcal{D}_n$  can be connected by a gapped domain wall. If the domain wall is not only gapped, but its topological excitations also all come from the dimension reduction of the topological excitations in an n-dimensional BF categories, then we say  $\mathcal{C}_n \stackrel{\text{w}}{\sim} \mathcal{D}_n$ . Under  $\boxtimes$  operation, the equivalence classes of BF<sub>n</sub> categories under  $\sim$  or  $\stackrel{\text{w}}{\sim}$  form Abelian groups (see Section VIII D).

#### D. Two versions of quantum theories

We point out that local bosonic quantum theory has two versions: Hamiltonian version and the Lagrangian version. The two versions are really different theories. The Hamiltonian version will be called local bosonic Hamiltonian quantum (lbH) theory which are described by lattice bosonic Hamiltonian with short range interactions. The Lagrangian version will be referred to as local bosonic Lagrangian quantum (lbL) theory, which are described by local bosonic path integral with short range interactions (see Appendix A). As a result, there are two version of BF categories, which will be referred to as H-type BF $_n^{\rm H}$  category (for lbH theory) and L-type BF $_n^{\rm L}$  category (for lbH theory).

In this paper, topological order and long-range entanglement belong to the Hamiltonian version of quantum theory and are described by  $\mathrm{BF}_n^{\mathrm{H}}$  category. While topological quantum field theories (TQFT) studied in high energy physics and mathematics mostly belong to the Lagrangian version of quantum theory and are associated with  $\mathrm{BF}_n^{\mathrm{L}}$  category.

## E. The boundary-bulk relation

Not all the locally consistent sets of topological properties (i.e. not all  $BF_n^H$  categories or not all gapped effective theories) can be realized by lattice qubit models (i.e. the lbH systems) in the same dimension (see Section VIB). Those  $BF_n^H$  categories, which are not realizable by lattice models in the same dimension, are called anomalous  $BF_n^H$  categories. We argue that (1) a generic  $BF_n^H$  category  $\mathcal{C}_n$  (or a potentially anomalous gapped effective theory) in n-dimensional space-time can always be realized by a boundary of a lattice qubit model in (n+1)-dimensional space-time whose bulk realizes another  $BF_{n+1}^H$  category  $\mathcal{C}_{n+1}$ . (2)  $\mathcal{C}_{n+1}$  is uniquely determined by  $\mathcal{C}_n$ . Therefore, we introduce the notion of the <u>bulk</u> of  $\mathcal{C}_n$  (see Definition 20), denoted by  $\mathcal{Z}_n(\mathcal{C}_n)$  and defined by  $\mathcal{Z}_n(\mathcal{C}_n) := \mathcal{C}_{n+1}$  (see Lemma 2). Clearly,

under such a definition, the <u>bulk</u> of a <u>bulk</u> is trivial:  $\mathcal{Z}_{n+1}(\mathcal{Z}_n(\mathcal{C}_n)) = \mathbf{1}_{n+2}$  (see Section VII C).

Physically, an topological order in n + 1-dimensional space-time is defined as an equivalent class of many-body wave functions (see Section III). If a topological order can have a gapped boundary, then it can have many different types of gapped boundary, described by different anomalous topological orders in n-dimensional spacetime. However, for a given boundary anomalous topological order, there can be only one unique bulk topological order. This has a flavour of holographic principle: the topological class of the surface part of a many-body wave functions determines the topological class of the whole bulk many-body wave functions. Thus we have a mapping  $\underline{bulk}$ : boundary topological orders  $\rightarrow$  bulk topological orders. We see that the bulk operator has a geometric meaning of describing a boundary-bulk relation of a many-body wave function.

Since topological order can be described by an algebraic structure – BF categroy. The geometrically or physically defined  $\underline{bulk}$  operator corresponds to an algebraic construction of center in category theory. In fact, we will show in Ref. 70 that the notion of the  $\underline{bulk}$  is equivalent to a mathematical and a purely algebraic notion of the center. Such a connection between a geometric notion of  $\underline{bulk}$  and an algebraic notion of center is quite amazing and deep, and was confirmed in  $2+1D.^{26,49,71}$ 

Similarly, we can also define a notion of the <u>bulk</u> for BF<sub>n</sub><sup>L</sup> categories, which also satisfy  $\mathcal{Z}_{n+1}(\mathcal{Z}_n(\mathcal{C}_n)) = \mathbf{1}_{n+2}$ .

## F. Closed and exact BF categories

If a BF $_n^H$  category  $\mathcal{C}_n$  can be realized by a lattice model in the same dimension (i.e. the <u>bulk</u>  $\mathcal{C}_{n+1}$  is trivial), such a BF $_n^H$  category is said to be <u>closed</u> (and the corresponding gapped effective theory is said to be <u>free of anomaly</u>). In other words,  $\mathcal{C}_n$  is closed iff  $\mathcal{Z}_n(\mathcal{C}_n) = \mathbf{1}_{n+1}$ . If the qubit model that realizes the closed BF $_n^H$  category  $\mathcal{C}_n$  also has a gapped boundary, which is described by a BF $_{n-1}^H$  category  $\mathcal{C}_{n-1}$  in one lower dimension, then the BF $_n^H$  category  $\mathcal{C}_n$  is said to be exact. In other words,  $\mathcal{C}_n$  is exact iff there exists a (n-1)-dimensional BF $_{n-1}^H$  category  $\mathcal{C}_{n-1}$  such that  $\mathcal{C}_n = \mathcal{Z}_{n-1}(\mathcal{C}_{n-1})$  (see Section VI A). Similarly, we can also define closed/exact BF $_n^L$  categories.

**Remark 1.** A closed/exact  $BF_n^L$  category is automatically a closed/exact  $BF_n^H$  category. More precisely, we have the monoid homomorphism

$$\{\text{closed/exact BF}_n^L \text{ Cat.}\} \rightarrow \{\text{closed/exact BF}_n^H \text{ Cat.}\}$$

A non-trivial closed BF<sup>L</sup> category might correspond to a trivial closed BF<sup>H</sup><sub>n</sub> category. Mathematically, a closed BF<sup>L</sup><sub>n</sub> category may correspond to an n-(n-1)- $\cdots$ -1-0 fully extended TQFT, while a closed BF<sup>H</sup><sub>d</sub> category may correspond to an "x-(n-1)- $\cdots$ -1-0 extended TQFT", <sup>72-75</sup> although the definitions of the those concepts are quite

different. An "x-(n-1)-···-1-0 extended TQFT" is defined as a theory where we can assign a Hilbert space (the space of degenerate ground states) to every closed orientable d-manifold, but we do not require the path integral to be well defined for every closed orientable d-dimensional space-time manifold. We only require the path integral to be well defined for every closed orientable d-dimensional mapping torus (a mapping torus is a fiber bundle over  $S^1$ , where the fiber is the space and  $S^1$  is the time). In Ref. 72–74, it was shown that any finite unitary x-(n-1)-···-1-0 extended TQFT extends to an n-(n-1)-···-1-0 fully extended TQFT. So it is also possible that a closed BF $_d^L$  category is the same as a closed BF $_d^H$  category.

Although the BF categories sounds abstract, in low dimensions, they correspond to some well known tensor categories. Let us give some examples. In 1+1D, the closed and the exact BF $_2^H$  categories are always trivial (see Section XIIB2). The generic BF $_{1+1}^H$  categories are unitary fusion 1-categories (UFC) which are always anomalous except the trivial UFC, i.e. the category  $\mathcal{H}$ ilb of finite dimensional Hilbert spaces (see Section C 3 a and Example 3).

In 2+1D, we believe that the closed BF $_3^{\rm H}$  categories are classified by the unitary modular tensor categories (UMTC), up to some  $E_8$  quantum Hall states  $\mathcal{C}_3^{E_8}$ . To more precisely, there is a many-to-one surjective map that maps the set of closed BF $_3^{\rm H}$  categories to the set of UMTCs, and the kernel of the map is the set of the  $E_8$  quantum Hall states:

$$\begin{aligned} \mathbf{1}_3 &\to \{(\mathfrak{C}_3^{E_8})^{\boxtimes n}\} &\to \{\text{closed BF}_3^{\text{H} \text{ categories}}\} \\ &\to \{\text{UMTCs}\} \to \mathbf{1}_3 \quad (3) \end{aligned}$$

where the arrows are monoid homomorphisms.

The exact  $BF_3^H$  and  $BF_3^L$  categories are the monoidal center of UFC's. The monoidal center  $\mathcal Z$  actually maps a  $BF_2^H$  category to an exact  $BF_3^H$  category. The unitary braided fusion categories (which may not be modular) are examples of generic (potentially anomalous)  $BF_3^H$  and  $BF_3^L$  categories (see Section XC).

## G. Gravitational anomaly and its classification

In this paper, as in Ref. 46, we define gravitational anomaly as the obstruction that a set of seemly consistent low energy properties cannot be realized by any well defined quantum models in the same dimension. If the low energy properties cannot be realized by any well defined bosonic Hamiltonian models, we say there is a H-type gravitational anomaly. If the low energy properties cannot be realized by any well defined local bosonic path integrals, we say there is a L-type gravitational anomaly (see Section VIB).

Because a potentially anomalous theory (gapped or gapless) can always be realized as a boundary of a gapped

state in one-higher dimension (see Corollary 4), and because the theory in one-higher dimension are described by closed BF category, we see that gravitational anomalies and closed BF categories (*i.e.* topological orders) in one-higher dimension are closely related. More precisely, anomaly-free topological orders (or closed BF categories), gravitational anomalies, and gCS anomalous topological orders form three monoids, and we have a short exact sequence of monoid homomorphisms

$$\begin{aligned} \mathbf{1} \to & \{d+1\text{D gCS anomalous topological orders}\} \\ \to & \{d+1\text{D gravitational anomalies}\} \\ \to & \{\text{closed BF}_{d+2} \text{ categories}\} \to \mathbf{1} \end{aligned}$$
 (4)

Note that d+1D gCS anomalous topological orders only appear in d+1=4k+3. In other dimensions, d+1D gravitational anomalies are fully classified by closed BF<sub>d+2</sub> categories. In 2+1D, the only gravitational anomaly that is not classified by closed BF<sub>4</sub> categories is the one described by a unquantized gravitational Chern-Simons term.

## H. A classification of topological order

Since all possible topological orders in lattice qubit models are described by the closed  $\mathrm{BF}_n^{\mathrm{H}}$  categories, the closed  $\mathrm{BF}_n^{\mathrm{H}}$  categories classify bosonic topological orders (and anomaly-free gapped effective theories). Restricting eqn. (2) to closed  $\mathrm{BF}^{\mathrm{H}}$  categories, we obtain a monoid isomorphism

$$\{ \text{Topological orders} \} \simeq \{ \text{closed BF}_n^H \text{ categories} \}. \quad (5)$$

Again, we have an expression that relates a mathematical construction (closed BF<sup>H</sup> category) to a physical phenomenon (topologically ordered phases or long-range entanglement).

Similarly, the exact  $\mathrm{BF}_n^{\mathrm{H}}$  categories classify topological orders with gappable boundary. We have a monoid isomorphism

{Topological orders with gappable boundary} 
$$\simeq \{\text{exact BF}_n^H \text{ categories}\}. \tag{6}$$

## I. Monoid-cochain complex and cochain complex

The sequence  $\cdots \stackrel{\mathcal{Z}_{n-1}}{\to} \, \mathcal{C}_n \stackrel{\mathcal{Z}_n}{\to} \, \mathcal{C}_{n+1} \stackrel{\mathcal{Z}_{n+1}}{\to} \cdots$  and the fact that  $\mathcal{Z}_n\mathcal{Z}_{n-1}=0$  imply that the BF categories (i.e. BF<sup>H</sup> categories or BF<sup>L</sup> categories) in different dimensions form a monoid-cochain complex,<sup>77</sup> where taking the <u>bulk</u> (or the center)  $\mathcal{Z}_n(\cdot)$  acts like the "differential" operator that maps a BF category to another BF category in one-higher dimension. We also show that the equivalence classes of BF categories in different dimensions (under the equivalence relation  $\sim$  mentioned

in Section II C) form a cochain complex. Such monoid-cochain complex and cochain complex reveal the structure and connection between gravitational anomalies, BF categories, and topological orders in different dimensions (see Section IX). The cohomology classes of the cochain complex,  $H^n = \ker(\mathcal{Z}_n)/\mathrm{img}(\mathcal{Z}_{n-1})$ , describe types of boundaries (including gapless ones) of the topological states.

# J. Tensor network approach to topological order in any dimensions

We also develop tensor network path integrals, hopefully to realize all generic/closed/exact  $\mathrm{BF}_n^\mathrm{L}$  categories. This in turn allows us to realize a large subset of  $\mathrm{BF}_n^\mathrm{H}$  categories. Here we collect some important conjectures:

- 1. All exact  $BF_n^L$  categories can be realized via topological path integrals on space-time complex, which can be expressed as tensor network (see Section XII A). (Topological path integrals are defined as path integrals that produce topological invariant partition functions for arbitrary closed space-time. They are the fixed points of renormalization flow for gapped quantum liquids. Topological path integrals can always be described by tensor network.)
- 2. Different exact BF $_n^{\rm L}$  categories always have different topological partition functions. But the reverse is not true. Partition functions differ by a factor  $W^{{\rm ch}(M^n)} {\rm e}^{{\rm i} \sum_{\{n_i\}} \phi_{n_1 n_2 \cdots} \int_{M^n} P_{n_1 n_2 \cdots}}$  describe the same BF $^{\rm L}$  category. Here  ${\rm ch}(M^n)$  is the Euler character and  $\int_{M^n} P_{n_1 n_2 \cdots}$  are the Pontryagin numbers of the space-time  $M^n$ . The trivial BF $_n^{\rm L}$  category is described by partition functions of form  $Z(M^n) = W^{{\rm ch}(M^n)} {\rm e}^{{\rm i} \sum_{\{n_i\}} \phi_{n_1 n_2 \cdots} \int_{M^n} P_{n_1 n_2 \cdots}}$  (see eqn. (102)).
- 3. Not all topological path integrals describe exact  $\mathrm{BF}_n^\mathrm{L}$  categories. Only stable topological path integrals describe exact  $\mathrm{BF}_n^\mathrm{L}$  categories. A topological path integral in  $(n+1)\mathrm{D}$  space-time is stable iff  $|Z(S^1\times S^n)|=1$  (see Section XII A).
- 4. n-dimensional (potentially anomalous) BF $_n^{\rm L}$  categories can be described by topological path integrals in one higher dimensions (see Section XII D).
- 5. n-dimensional closed  $BF_n^L$  categories can be described by "trivial" topological path integrals in one higher dimensions. There can be many different "trivial" topological path integrals that describe the same trivial  $BF_n^L$  category. Those different "trivial" topological path integrals describe different closed  $BF_n^L$  categories in one lower dimension (see Section XII E).  $^{76,79-81}$

We note that the above results give us a concrete, practical, and constructive definition of exact  ${\rm BF^L}$  category

via topological path integorals (or tensor network) in any dimensions. Then the generic (closed) BF<sup>L</sup> categories can be defined as the boundary of exact BF<sup>L</sup> category (trivial BF<sup>L</sup> category) in one higher dimension.

### K. Probing and measuring topological orders

We propose some ways to probe and measure topological orders (see Section XIV):

- 1. For a L-type quantum system defined by a path integral, we can compute its imaginary time partition function on closed space-time  $M^{d+1}$ . The system describes an exact  $\mathrm{BF}^{\mathrm{L}}_{d+1}$  category (i.e. the topological order has a gappable boundary) iff the corresponding volume independent part of the partition function  $Z_0(M^{d+1})$  is a topological invariant of the space-time  $M^{d+1}$ .
- 2. For a H-type quantum system described by a Hamiltonian on a closed space  $\Sigma^d$ , the degenerate ground states form a vector space V. As we change the metrics on  $\Sigma^d$ , we obtain the moduli space  $\mathcal{M}_{\Sigma^d}$  of  $\Sigma^d$ . Together with the vector space V for each point in  $\mathcal{M}_{\Sigma^d}$ , we obtain a vector bundle on the moduli space  $\mathcal{M}_{\Sigma^d}$ . For different space topologies, we will get different vector bundles. The collection of those vector bundles should fully characterize the closed  $\mathrm{BF}^H_{d+1}$  category (i.e. the topological order).
- 3. A BF $_{d+1}^{H}$  category is exact iff the above mentioned vector bundle is flat (see Conjecture 26).
- 4. A BF $_n^H$  category is closed iff every nontrivial topological excitation in it has a nontrivial mutual braiding property (or a nontrivial mutual statistics) with at least one topological excitation. This is the condition for a gapped effective theory to be free of H-type gravitational anomaly (see Conjecture 28). This principle was discussed in detail in Ref. 82.

As an application of the above conjectures, let us consider the simplest topological orders that have no nontrivial topological excitations and no degenerate ground states. We find that this kind of topological orders have two defining properties: i) their partition functions on closed space-time can always be chosen to be a pure U(1)phase; ii) they are invertible under the stacking  $\boxtimes$  operation. Using those properties, one can try to classify those invertible topological orders. <sup>61–63</sup> We find that there is no non-trivial invertible BF<sup>L</sup> categories in 3+1D, and 5+1D. The invertible BF<sup>L</sup> categories in 2+1D form an Abelian group  $\mathbb{Z}$  generated by the  $E_8$  bosonic fractional quantum Hall state.<sup>83</sup> The invertible BF<sup>L</sup> categories in 6+1D form an Abelian group  $\mathbb{Z} \times \mathbb{Z}$ . The boundary of those  $\mathbb{Z}$ -class topological orders must be gapless. The invertible BF<sup>L</sup> categories in 4+1D form an Abelian group  $\mathbb{Z}_2$ . (This result has been obtained in Ref. 60). The boundary of

the non-trivial  $\mathbb{Z}_2$ -class topological order can be gapped but must carry a anomalous topological order with non-trivial topological excitations on the boundary. The invertible BF<sup>L</sup> categories are also invertible BF<sup>H</sup> categories (see Remark 1), and we show that the non-trivial invertible BF<sup>L</sup> categories are also non-trivial when viewed as BF<sup>H</sup> categories. Thus the invertible BF<sup>H</sup> categories contains a  $\mathbb{Z}$  class in 2+1D, a  $\mathbb{Z}_2$  class in 4+1D, and a  $\mathbb{Z} \oplus \mathbb{Z}$  class in 6+1D.

## III. PHYSICAL DEFINITION OF TOPOLOGICALLY ORDERED PHASE

The topologically ordered states that we will discuss in this paper are ground states of bosonic local gapped Hamiltonians. However, not all gapped ground states are topologically ordered states. Only the special gapped ground states, called gapped quantum liquids,  $^{10}$  are topologically ordered states. For example, 2+1D FQH states and 3+1D  $\mathbb{Z}_2$  gauge theory are gapped quantum liquids (i.e. topologically ordered states), while the 3+1D gapped state formed by layers of 2+1D FQH states is not a gapped quantum liquids. The notion of gapped quantum liquids (i.e. topologically ordered states) are discussed in detail in Ref. 10. We will not repeat them here.

Now we are ready to define topological order (or topologically ordered phase). We will give two definitions:

- (1) Let us call the Hamiltonian that realizes a gapped quantum liquid an l-gapped Hamiltonian. If the ground state degeneracy of a l-gapped Hamiltonian is robust against any perturbations, then the l-gapped Hamiltonian is said to be stable. Let  $M_{slgH}$  be the space of stable l-gapped Hamiltonians. Then the elements of  $\pi_0(M_{slgH})$  define the topologically ordered phases.
- (2) We can also use local unitary transformations to define topologically ordered phases: two *stable* gapped quantum liquids belong to the same topologically ordered phase iff they are connected by a local unitary transformation.<sup>9</sup>

Local unitary (LU) transformation <sup>9,11–13</sup> is an important concept which is directly related to the definition of quantum phases .<sup>9</sup> To explain LU transformation, let us first introduce *local unitary evolution*. A LU evolution is defined as the following unitary operator that act on the degrees of freedom in a lbH system:

$$\mathcal{T}[e^{-i\int_0^1 dg\,\tilde{H}(g)}]\tag{7}$$

where  $\mathcal{T}$  is the path-ordering operator and  $\tilde{H}(g) = \sum_{i} O_{i}(g)$  is a sum of local Hermitian operators. Two gapped quantum states belong to the same phase if and only if they are related by a LU evolution. 9,84,85 Note that, in this paper, the term "a gapped quantum state" really means "the subspace of the degenerate ground states".

The LU evolutions are closely related to quantum circuits with finite depth. To define quantum circuits, let us

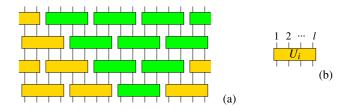


FIG. 1. (Color online) (a) A graphic representation of a quantum circuit, which is form by (b) unitary operations on blocks of finite size l. The green shading represents a causal structure.

introduce piecewise local unitary operators. A piecewise local unitary operator has a form

$$U_{pwl} = \prod_{i} U^{i}$$

where  $\{U^i\}$  is a set of unitary operators that act on non overlapping regions. The size of each region is less than a finite number l. A quantum circuit with depth M is given by the product of M piecewise local unitary operators:

$$U_{circ}^{M} = U_{pwl}^{(1)} U_{pwl}^{(2)} \cdots U_{pwl}^{(M)}$$

We will call  $U^M_{circ}$  a LU transformation. In quantum information theory, it is known that a finite time unitary evolution with a local Hamiltonian (a LU evolution defined above) can be simulated with a constant depth quantum circuit (i.e. a LU transformation) and vice-verse:

$$\mathcal{T}[e^{-i\int_0^1 dg\,\tilde{H}(g)}] = U_{circ}^M. \tag{8}$$

So two gapped quantum states belong to the same phase if and only if they are related by a LU transformation.

Using the LU transofrmations, we can define the concept of short-range and long-range entanglement.<sup>9</sup>

### Definition 1. Short-range entanglement

A state is short-range entangled (SRE) if it can be transformed into product state by a LU transformation of a fixed depth regardless how large the system is.

We can show that all short-range entangled states belong to the same phase:

**Corollary 1:** All short-range entangled states can be transformed into each other via LU transformations.

We can also show that

**Corollary 2:** For any short-range entangled state  $|\Psi\rangle$ , there exists a gapped local Hamiltonian H such that  $|\Psi\rangle$  is the only ground state of H.

## Definition 2. Long-range entanglement

A stable gapped state is long-range entangled if it is not short-range entangled.

Here "stable" means that the ground state degeneracy is robust against any small perturbations.

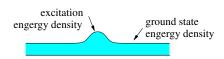


FIG. 2. The energy density distribution of a particle-like excitation.

## Definition 3. Topologically ordered states

Topologically ordered states are LRE gapped liquid states. In other words, a gapped liquid state has a non-trivial topological order iff it cannot be transformed to a product state by any LU transformations of finite depth.

Not all long-range entangled (LRE) gapped liquid states can be transformed into each other via LU transformations. Thus LRE states can belong to different phases: i.e. the LRE states that are not connected by LU transformations belong to different phases. Those different phases are nothing but the topologically ordered phases: $^{3-6}$ 

## Definition 4. Topologically ordered phases

LU transformations are equivalence relations. Topologically ordered phases are equivalence classes of topologically ordered states under the LU transformations.

In this paper, we plan to study topologically ordered phases (i.e. topological orders) in any dimensions. We would like to find a way to classify all topologically ordered phases. As we have stressed above, our study is limited to gapped quantum liquids, and does not apply to other more complicated gapped quantum states.

# IV. EXCITATIONS IN TOPOLOGICALLY ORDERED STATES

Topological orders (or patterns of long-range entanglement) can be characterized by the appearance of the "topological excitations". In this section, we will discuss/define the notion of topological excitations.

In higher dimensions, the allowed topological excitations can be quite complicated. They can be "particle-like", "string-like", "membrane-like", etc. Many concepts need to be introduced to describe and understand these excitations.

#### A. Particle-like excitations

First we define the notion of "particle-like" excitations. Consider a gapped system with translation symmetry. The ground state has a uniform energy density. If we have a state with an excitation, we can measure the energy distribution of the state over the space. If for some local area, the energy density is higher than ground state, while for the rest area the energy density is the same as ground state, one may say there is a "particle-like" excitation, or a quasiparticle, in this area (see Figure 2).

Quasiparticles defined like this can be further divided into two types. The first type can be created or annihilated by local operators, such as a spin flip. So the first type of the particle-like excitations is called local quasiparticle excitations. The second type cannot be created or annihilated by any finite number of local operators (in the infinite system size limit). In other words, the higher local energy density cannot be created or removed by any local operators in that area. The second type of the particle-like excitations is called topological quasiparticle excitations. They are characterized by the modules over the local operator algebras  $^{26,49,86}$ 

From the notions of local quasiparticles and topological quasiparticles, we can also introduce a notion of topological quasiparticle type, or simply, quasiparticle type. We say that local quasiparticles are of the trivial type, while topological quasiparticles are of nontrivial types. Also two topological quasiparticles are of the same type if and only if they differ by local quasiparticles. In other words, we can turn one topological quasiparticle into the other one of the same type by applying some local operators.

#### B. p-dimensional topological excitations

In the above, we only discussed the notion of "particle-like" topological excitations. Similarly, we can also introduce the notion of "string-like" topological excitations, or even more general p-dimensional topological excitations, where p is the spatial dimension. To define a p-dimensional topological excitations, let us first define

## Definition 5. p-dimensional excitations:

Consider a gapped lbH system defined by a local bosonic Hamiltonian  $H_0$  in n spatial dimensions. For p < n, a p-dimensional excitation is the gapped ground state of  $H_0 + \Delta H$  where  $\Delta H$  is a local hermitian operator which is non-zero only on a p-dimensional subspace  $M^p$  and is almost uniform on  $M^p$  in the large  $M^p$  limit.

**Remark 2.** (1) A p-dimensional excitation is defined only for large  $M^p$  if p > 0.

(2) If the ground state of  $H_0 + \Delta H$  has gapless modes for large  $M^p$ , then, by definition,  $\Delta H$  does not create a p-dimensional excitation.

We note that we can view a p-dimensional excitation as a gapped system with p spatial dimensions, which has a thermal dynamical limit when  $M^p$  is large. This allows us to define the equivalence relation between p-dimensional excitations:

**Definition 6.** Two p-dimensional excitations on  $M^p$  are equivalent if they can be

- (1) transformed into each other via a local unitary transformation of finite depth (see Section III)<sup>9</sup> on a neighborhood of  $M^p$  or
- (2) transformed into each other via a tensor product of an unentangled state on  $M_p$ .

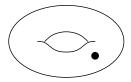


FIG. 3. The 2-dimensional excitation has a torus topology and carries a 0-dimensional (point-like) wall excitation (or a sub-defect).

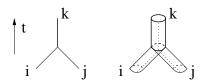


FIG. 4. The fusion of topological particles (the 0-branes) and topological loops (the 1-branes). Here *i*- and *j*-types of topological excitations are fused into a *k*-type topological excitation. The time direction points upwards.

The equivalence class is called the type of topological excitations.

When we refer to a topological excitation, we usually mean its equivalence class (or type). We have the following conjecture.

Conjecture 1: Two p-dimensional excitations localized on the two submanifolds  $M^p$  and on  $N^p$  are equivalent if we can deform them into each other smoothly without phase transition in the large  $M_p$  and  $N_p$  limit.

### C. Wall and pure excitations

We also like to point out that the p-dimensional topological excitations discussed above are not all the topological excitations that can appear in a topologically ordered state. We can have more general topological excitations, such as a p-dimensional topological excitation nested in a p'-dimensional topological excitations with p'>p (see Fig. 3). We will call the p-dimensional topological excitation as a wall excitation (or a sub-defect) of the p-dimensional topological excitation. We will call p-dimensional topological excitations that only nested in a trivial higher dimensional excitation as pure excitations. In this paper, we usually use the term "topological excitation" to refer to pure excitation.

# V. UNIVERSAL LOW ENERGY PROPERTIES AND A PHYSICAL DEFINITION OF $BF_n^H$ CATEGORY

## A. A physical definition of BF<sup>H</sup> category

The p-dimensional topological excitations can have some universal properties (or topological properties),



FIG. 5. The braiding between topological particles and topological loops in 3 dimensional space.

which, by definition, are robust against any local perturbations and can be used to physically characterize topological phases.

## Definition 7. $BF^H$ category

The collection of all topological (or universal) properties of all the topological excitations in n space dimensions, as well as the perturbative gravitational responses defines a (n+1)-dimensional braided fusion (BF<sup>H</sup> or BF<sup>H</sup><sub>n+1</sub>) category. (Here n+1 is the space-time dimension.)

So, in physics, the term "BFH category" and the term "the set of topological properties" can be used interchangeably. In physics, a set of topological properties also defines a gapped low energy effective theory, so the term "BFH category" and the term "gapped low energy effective theory" can also be used interchangeably. In this paper, we will mainly use the term "BFH category".

In Section XI, we will give a more detailed definition of BF<sup>H</sup> category, trying to describe a subset of universal properties that completely specify a BF<sup>H</sup> category, *i.e.* completely specify all other universal (or topological) properties. In the following, we will describe some of the simple universal (or topological) properties for p-dimensional topological excitations.

## B. Universal low energy properties

What is the subset of universal properties that completely specify a topological order? Here we propose that **Conjecture 2:** The fusion and the braiding properties of topological excitations, plus the universal correlations of energy-momentum tensor completely characterize the topological order (i.e. the BF<sup>H</sup> category.)

In this section, we will explain the above conjecture, in particular, what are the fusion and the braiding properties. Also, we only need to include the fusion and the braiding of a subset of topological excitations to fully define the topological order. We will discuss what this subset is.

## 1. The fusion space

The first and the most important universal property is

#### Definition 8. the generalized Fusion space:

If we put p-dimensional topological excitations (labeled by  $i, j, k, \cdots$ ) on an n-dimensional closed space  $M^n$ , a generalized fusion space is the

 $D(M^n, \mathcal{T}_{ijk...}; i, j, k, \cdots)$  dimensional (nearly) degenerate space  $\mathcal{V}^F(M^n, \mathcal{T}_{ijk...}; i, j, k, \cdots)$  of the lowest energy eigenstates.

A p-dimensional topological excitation can have a nontrivial topology and linking with other topological excitations described by  $\mathcal{T}_{ijk}$ .... Also  $i, j, k, \cdots$  label different types of topological excitations which can have different dimensions p. We also fixed the locations of the topological excitations, and assume that the topological excitations have a large size and are well separated. In this limit, the (near) degeneracy is well defined.

 $D(M^n, \mathcal{T}_{ijk\cdots}; i, j, k, \cdots)$  reduces to the ground state degeneracy  $GSD(M^n)$  on a closed topological space  $M^n$ , when there is no topological excitation:

$$GSD(M^n) = D(M^n). (9)$$

On a sphere  $S^n$ ,  $D(M^n, \mathcal{T}_{ijk...}; i, j, k, \cdots)$  reduces to

$$D_{\mathbf{1}}(\mathcal{T}_{ijk\cdots}; i, j, k, \cdots) = D(S^{n}, \mathcal{T}_{ijk\cdots}; i, j, k, \cdots)$$
 (10)

which is called the dimension of the fusion space of topological excitations  $i, j, k, \cdots$  into the trivial one **1** (see Fig. 4). The *fusion space* is the degenerate space of the lowest energy eigenstates with topological excitations  $i, j, k, \cdots$  on a sphere  $S^n$ , with the fixed positions and shapes of the topological excitations.

## 2. A "local" description of a topological excitation

The key to understand topological ordered states is to understand the fusion space  $\mathcal{V}^F(M^d, \mathcal{T}_{i_1,i_2,\cdots}; i_1, i_2,\cdots)$ . To understand the structure of the fusion space, it s very tempting to assign a vector space  $\mathcal{V}^F_{i_k}$  for each excitation  $i_k$ , and a vector space  $\mathcal{V}^F_{M^d}$  for the closed space, and view the fusion space as a tensor product of those spaces  $\mathcal{V}^F(M^d, \mathcal{T}_{i_1,i_2,\cdots}; i_1, i_2,\cdots) = \mathcal{V}^F_{M^d} \otimes_k \mathcal{V}^F_{i_k}$ . If this is true, we may regard  $\mathcal{V}^F_{i_k}$  as the space for the local degree of freedom carried by the excitation  $i_k$ . However, in general, the fusion space  $\mathcal{V}^F(M^d, \mathcal{T}_{i_1,i_2,\cdots}; i_1,i_2,\cdots)$  does not have the above tensor product structure. This makes it very difficult to understand the structure of the fusion space from the local properties of the excitation  $i_k$ .

However, we still insist on using local properties of the excitation  $i_1$  to understand and to construct the total fusion space  $\mathcal{V}^F(M^d, \mathcal{T}_{i_1,i_2,\cdots}; i_1, i_2, \cdots)$ . To achieve this, we simply define the "local properties" of the excitation  $i_1$  as a map that defines the notion of a topological excitation as follows:

**Definition 9.** A topological excitation i is a map that maps a collection of topological excitations  $i_2, i_3, \cdots$  to a fusion space:  $i: i_2, i_3, \cdots \to \mathcal{V}^F(M^d, \mathcal{T}_{i,i_2,\cdots}; i, i_2, \cdots)$ .

Such a map i represents a "local" description of topological excitation i. In mathematical language, it is nothing but the Yoneda Lemma.

## 3. Simple type and composite type

To understand the notion of simple type and composite type, remember that a gapped lbH system is defined by a local Hamiltonian  $H_0$  in d dimensional space  $X^d$  without boundary. A collection of excitations labeled by  $i_1$ ,  $i_2$ , etc and located at  $M_1$ ,  $M_2$  etc can be produced as gapped ground states of  $H_0 + \Delta H$  where  $\Delta H$  is non-zero only near  $M_i$ 's. Here  $M_i$  is a sub manifold of  $X^d$ . By choosing different  $\Delta H$  we can create all kinds of excitations.

The gapped ground states of  $H_0 + \Delta H$  may have a degeneracy  $D(M^d, \mathcal{T}_{i_1, i_2, \dots}; i_1, i_2, \dots)$  as discussed above The degeneracy is not exact, but becomes exact in the large  $M_i$  space and large excitation separation limit.

If the Hamiltonian  $H_0 + \Delta H$  is not gapped, we will say  $D(M^d, \mathcal{T}_{i_1,i_2,\cdots}; i_1,i_2,\cdots) = 0$  (i.e.  $\mathcal{V}(M^d, \mathcal{T}_{i_1,i_2,\cdots}; i_1,i_2,\cdots)$  has zero dimension). If  $H_0 + \Delta H$  is gapped, but if  $\Delta H$  also creates excitations away from  $M_i$ 's (indicated by the bump in the energy density away from  $M_i$ 's), we will also say  $D(M^d, \mathcal{T}_{i_1,i_2,\cdots}; i_1,i_2,\cdots) = 0$ . (In this case excitations at  $M_i$ 's do not fuse to trivial excitations.) So if  $D(M^d, \mathcal{T}_{i_1,i_2,\cdots}; i_1,i_2,\cdots) > 0$ ,  $\Delta H$  only creates excitations at  $M_i$ 's.

## Definition 10. Simple and composite types:

If the degeneracy  $D(M^d, \mathcal{T}_{i_1,i_2,\cdots}; i_1, i_2, \cdots)$  cannot be lifted by any small local perturbation near  $M_1$ , then the topological type  $i_1$  at  $M_1$  is said to be simple. Otherwise, the topological type  $i_1$  at  $M_1$  is said to be composite.

The degeneracy  $D(M^d, \mathcal{T}_{i_1, i_2, \dots}; i_1, i_2, \dots)$  for simple topological types  $i_i$  is a universal property (i.e. a topological invariant) of the topologically ordered state.

When  $i_1$  is composite, then the space of the degenerate ground states  $\mathcal{V}^F(M^d; i_1, i_2, i_3, \cdots)$  will have a direct sum decomposition:

$$\mathcal{V}^{F}(M^{d}; \mathcal{T}_{i_{1}, i_{2}, \dots}; i_{1}, i_{2}, i_{3}, \dots) 
= \mathcal{V}^{F}(M^{d}; \mathcal{T}_{j_{1}, i_{2}, \dots}; j_{1}, i_{2}, i_{3}, \dots) 
\oplus \mathcal{V}^{F}(M^{d}; \mathcal{T}_{k_{1}, i_{2}, \dots}; k_{1}, i_{2}, i_{3}, \dots) 
\oplus \mathcal{V}^{F}(M^{d}; \mathcal{T}_{l_{1}, i_{2}, \dots}; l_{1}, i_{2}, i_{3}, \dots) \oplus \dots$$
(11)

where  $j_1, k_1, l_1$ , etc. are simple types. To see the above result, we note that when  $i_1$  is composite the ground state degeneracy can be split by adding some small perturbations near  $M_1$ . After splitting, the original degenerate ground states become groups of degenerate states, each group of degenerate states span the space  $\mathcal{V}(M^d; \mathcal{T}_{j_1,i_2,\cdots}; j_1,i_2,i_3,\cdots)$  or  $\mathcal{V}(M^d; \mathcal{T}_{k_1,i_2,\cdots}; k_1,i_2,i_3,\cdots)$  etc., where  $j_1$  and  $k_1$  correspond to simple quasiparticle types at  $M_1$ . The decomposition eqn. (11) is valid for any choices of  $i_2,i_3,\cdots$ . From the discussion in Section VB2, we see that a composite excitation  $i_1$  can be written as  $i_1 = j_1 \oplus k_1 \oplus \cdots$ , where  $j_1 \oplus k_1 \oplus \cdots$  is a map that maps a collection of topological excitations  $i_2,i_3,\cdots$  to a fusion space  $\mathcal{V}^F(M^d,\mathcal{T}_{j_1,i_2,\cdots};j_1,i_2,\cdots) \oplus \mathcal{V}^F(M^d,\mathcal{T}_{k_1,i_2,\cdots};k_1,i_2,\cdots) \oplus \cdots$ .

## 4. Quasiparticle fusion algebra

When we fuse two topological excitations, i and j, of simple types together, it may become a topological excitation of a composite type:

$$i \otimes j = q = k_1 \oplus k_2 \oplus \cdots, \tag{12}$$

where  $i, j, k_i$  are simple types and q is a composite type. Here the fusion is denoted as  $i \otimes j$  which represents a map that maps a collection of topological excitations  $i_2, i_3, \cdots$  to a fusion space:  $\mathcal{V}^F(M^d, \mathcal{T}_{i,j,i_2,\cdots}; i, j, i_2, i_3, \cdots)$ . We can also use an integer tenser  $N_{ij}^k$  to describe

We can also use an integer tenser  $N_{ij}^k$  to describe the quasiparticle fusion, where i,j,k label simple types. When  $N_{ij}^k = 0$ , the fusion of i and j does not contain k. When  $N_{ij}^k = 1$ , the fusion of i and j contains one k:  $i \otimes j = k \oplus k_1 \oplus k_2 \cdots$ . When  $N_{ij}^k = 2$ , the fusion of i and j contains two k's:  $i \otimes j = k \oplus k \oplus k_1 \oplus k_2 \cdots$ . This way, we can denote that fusion of simple types as

$$i \otimes j = \bigoplus_k N_{ij}^k k. \tag{13}$$

In physics, the quasiparticle types always refer to simple types. The fusion tensor  $N_{ij}^k$  is another universal property of the topologically ordered state. The degeneracy  $D(S^d; i_1, i_2, \cdots)$  is determined completely by the fusion tensor  $N_{ij}^k$  if we only have quasiparticles.

## 5. Braiding properties

If the spatial dimension is higher than 1, we can braid the topological excitations of codimension 2 or higher (see Fig. 5), which will induce an non-Abelian geometric phase described by  $N_1^{i,j,k,\cdots}$ -dimensional unitary matrix. Since the overall phase of the unitary matrix is path dependent (which may depend on the size of the excitations i,j,k, etc.,), the unitary matrices from different braidings form a projective unitary representation of the "braid group" of the p-dimensional excitations. Such a projective unitary representation is also an universal property which is independent of (homologous) braiding paths and local perturbations to the Hamiltonian.

For particle-like topological excitations, even the overall phase of the unitary matrix is well defined and path independent. The projective unitary representation of the braid group becomes a unitary representation, which describes the statistics of the topological quasiparticles.

## 6. Universal perturbative gravitational responses

The above fusion and braiding properties of topological excitations are not enough to characterize topological orders. The 2+1D  $E_8$  bosonic quantum Hall state (see Example 4), containing no non-trivial 0-dimensional and 1-dimensional topological excitations, is a counter example. However, if we put the  $E_8$  bosonic quantum Hall state

on a curved space-time and integrate out the bosons, we will obtain an effective theory that contains gravitational Chern-Simons term, whose coefficient is proportional to the chiral central charge  $c_R-c_L$  of the edge state of the  $E_8$  bosonic quantum Hall state. The gravitational Chern-Simons term is an example of the universal perturbative gravitational responses. We also need such gravitational responses to characterize an topological order.

This consideration motivates us to introduce

# Definition 11. universal perturbative gravitational responses:

Putting a system on curved space-time and integrating out all matter fields will produce an effective Lagrangian that depends on the vielbein 1-form and the Lorentz connection 1-form.<sup>87</sup> The universal perturbative gravitational responses correspond to terms that only depend on the Lorentz connection 1-form and independent of the vielbein 1-form.

The universal perturbative gravitational responses are given by the Chern-Simons forms of the gravity. They correspond to the volume independent but shape dependent partition function discussed in Section XIV. They describe the perturbative gravitational anomalies of the corresponding boundary theory.

It is known that gravitational Chern-Simons terms exist only in 4k+3 space-time dimensions. In 2+1D, there is only one kind of gravitational Chern-Simons term, which correspond to the thermal Hall effect. In 6+1D, there are two kinds of gravitational Chern-Simons terms.

## 7. Elementary and finite topological excitations

We like to point out that, according to the Definition 6 for type of topological excitations, even trivial topological states (i.e. the product states) can have (infinitely many) non-trivial types of p-dimensional topological excitations for p>1, since non-trivial topologically ordered states can exist for spatial dimension p>1.9 Adding (stacking) a nontrivial topologically ordered state (such as a FQH state) defined on the subspace  $M^p$  to the d dimensional ground state will create a nontrivial type of p-dimensional topological excitation. This makes the description and definition of BF category very difficult.

To fix this problem, we note that most of the topological excitations are descendant. They come from other lower dimensional topological excitations. We can exclude them without hurting our ability to characterize the topologically ordered state. So in the following, we will describe ways to exclude those "descendant" topological excitations.

There are three way to create "descendant" topological excitations:

(A) Adding a p-dimensional topological state of a qubit system to a p-dimensional subspace  $M^p$  creates a "descendant" p-dimensional excitation.

- (B) Proliferating pure topological excitations with dimensions less than p on subspace  $M^p$  creates a "descendant" p-dimensional excitation.
- (C) Assume we have p-dimensional topological excitation on  $M^p$  which may carry wall excitations with dimensions less than p. Proliferating those wall excitations on  $M^p$  creates a "descendant" p-dimensional excitation.

So the goal is to exclude the above three types of "descendant" topological excitations. In fact, if we allow to proliferate trivial wall excitations, the case C include the case A and case B. So in the following, we will only discuss the case C.

Let consider a p-dimensional topological excitation labeled by i on a p-dimensional subspace  $M^p$ . We can create a "descendant" p-dimensional excitation i' on  $M^p$  by proliferating the wall excitations (or sub-defects) of i on  $M^p$ . Since we can always choose to proliferate the wall excitations (or sub-defects) of i in a subregion of  $M^p$ , the excitations i and i' must have a property that they can be connected by a (p-1)-dimensional domain wall. Such excitations i and i' are also called "Witt equivalent". It indeed defines an equivalence relation and the associated equivalence class will be called the Witt class. So we can exclude the "descendant" excitations by not allowing the excitations that can be connected by a domain wall. This will solve our problem.

However, since the domain wall between i and i' can be gapless, it is hard to describe/define condition within tensor category theory, which only deal with gapped states. So we have to use a weaker condition:

(1) we do not allow excitations that can be joined alone a *gapped* domain wall.

But the condition (1) is too weak to exclude all "descendant" excitations. So we will add another condition: (2) we only allow finite excitations.

Here the notions of *finite* is defined below:

**Definition 12.** A topological excitation x is finite if the set  $\{x^{\otimes n}\}_{n=1}^{\infty}$  contains only finite number of simple excitations as direct summands.

The finiteness is a powerful condition, as implied by the following corollary:

Corollary 3: Stacking a non-invertible topologically ordered state repeatedly always generate infinitely many different non-trivial topological orders.

So most topological excitations generated by proliferating local excitations (i.e. trivial excitations) are not finite, and can be excluded by the finiteness condition.

More generally, we like to conjecture that

Conjecture 3: If topological excitations i and i' described above are both finite, then the domain wall between i and i' must be gappable.

To understand the above conjecture, let us assume that the wall excitations that create i' form a "short-range entangled"  $^{9,10}$  state on  $M_p$ . In this case, the domain wall between i and i' can be gapped. If the domain wall must be gapless, then the wall excitations on  $M^p$  must form a

"long-range entangled"  $^{9,10}$  state, and such a state must be invertible in order for i' to be finite. The invertible topological orders that belong to  $Z_2$ -class are discussed in Section XV, which first appear in 4+1D and always have a gappable boundary. So the finiteness of i' implies that the domain wall can be gapped.

The above discussion allow us to introduce

## Definition 13. Elementary topological excitation:

The set of all topological excitations is closed under the fusion operations. Let us consider the maximal subset of topological excitations that is closed under the fusion operations, and also satisfies the following conditions:

- (1) Any two different simple topological excitations with the same dimension in the subset cannot be joined by a gapped domain wall.
- (2) All the topological excitations in the subset are finite. The simple topological excitations in the subset are called elementary topological excitations.

We believe that all the topological excitations can be obtained by fusing the elementary topological excitations with the "descendant" topological excitations. This motivates the following conjecture

Conjecture 4: The fusion and the braiding properties of the elementary topological excitations (plus the universal perturbative gravitational responses) fully characterize the topological order (or the corresponding BF category).

Therefore, we can limit ourselves to consider only elementary topological excitations, and use their fusion and braiding properties to define BF category. We believe that all the topological orders contain only a finite number of elementary topological excitations. This makes the task of defining the BF category a finite problem. In other words, we can use a finite amount of data to define a BF category.

### 8. Examples

We see that topological excitations in high dimensions can be very complicated. In this section, let us give some simple examples of topological excitations.

**Example 1.** Consider a  $\mathbb{Z}_2$  topologically ordered state<sup>17–19,88</sup> in 2+1 dimensions whose effective theory is a  $\mathbb{Z}_2$  gauge theory. The  $\mathbb{Z}_2$  charge, denoted as e, is a particle-like topological excitation. The  $\mathbb{Z}_2$  vortex, denoted as v, is another particle-like topological excitation. The bound state of e and v, denoted as  $\epsilon$ , is the third particle-like topological excitation. Since the number of e-excitations (the  $\mathbb{Z}_2$  gauge charge) is conserved mod 2, which leads to an effective  $\mathbb{Z}_2$  symmetry, so if the e excitations form a 1D gas, such a 1D system may spontaneously break the  $\mathbb{Z}_2$  effective symmetry. In this case, the 1D gas of e becomes an string-like topological excitation of a nontrivial type. Similarly, the 1D gas of v and

1D gas of  $\epsilon$  can also form two other non-trivial string-like excitations. The above three nontrivial string-like excitations are all Witt equivalent to the trivial string-like excitation. Thus, the 2+1D  $\mathbb{Z}_2$  topological order has only three nontrivial elementary topological excitations, e, v, and  $\epsilon$ , which are all "particle-like". It does not have any nontrivial elementary string-like topological excitations. But it has (at least) three nontrivial non-elementary string-like topological excitations.

**Example 2.** Consider a  $\mathbb{Z}_2$  topologically ordered state in 3+1 dimensions. The  $\mathbb{Z}_2$  charge, denoted as e, is a particle-like topological excitation. The  $\mathbb{Z}_2$  vortex-line, denoted as s, is a string-like topological excitation. There is also a trivial string-like excitation, denoted as  $\mathbf{1}_1$ . We note that the non-trivial string s and the trivial string  $\mathbf{1}_1$  cannot be connected by a gapped domain wall. In other words, the non-trivial string s cannot have an end. Thus s is an elementary excitation. In fact, e and s are the only two pure elementary topological excitations which are non-trivial. On the other hand, the s-1D  $\mathbb{Z}_2$  topologically ordered state can have many nontrivial non-elementary string-like and membrane-like topological excitations, generated by the condensation of s or s on the string or membrane.

In the rest of this paper (except Section XI), when we consider topological excitations, we will only consider elementary topological excitations. When we say there are N topological excitations in a topologically ordered state, we mean there are N elementary topological excitations.

## C. A complete characterization of topological order

In the above, we discussed several universal properties of pure p-dimensional topological excitations:

- (1) the number of elementary topological types for each p,
- (2) the fusion spaces,
- (3) the projective unitary representations of the "braid group" acting on the fusion spaces, as well as
- (4) the gravitational Chern-Simons terms.

Those topological data are needed to define a BF<sup>H</sup> category (or a gapped low energy effective theory). We hope that the above definition is complete. Certainly, our description is not rigorous. It just illustrates the physical ideas to develop a rigorous definition. In Section XI, we will give a more rigorous definition of BF<sup>H</sup> category. In the Section V D, we will discuss some simple examples.

## D. Examples of BF<sup>H</sup> categories

Now let us list some examples of BF<sup>H</sup> categories, to gain a more intuitive understanding of BF<sup>H</sup> category (or set of topological properties). In those examples (and in the rest of this paper), we will only consider pure elementary topological excitations. So when we say "topolog-

ical excitations", we mean "pure elementary topological excitations". Those examples are both  $\rm BF^H$  and  $\rm BF^L$  categories.

## 1. Examples of $BF_2^H$ categories in 1+1D

First let us consider 1+1D gapped systems. In this case, we can only have particle-like topological excitations.

**Example 3.** A 1+1D system with only one type of particle-like topological excitation labeled by e (not including the trivial type). The fusion of two e's gives rise to a trivial excitations  $e \otimes e = 1$ . In 1-dimensional space, particle-like excitations cannot braid and there is no braiding property. We will denote such a BF $_{1+1}^{H}$  category as  $\mathcal{C}_{2}^{F\mathbb{Z}_{2}}$ .

## 2. Examples of $BF_3^H$ categories in 2+1D

In 2+1D, we can have both particle-like and string-like topological excitations.

**Example 4.** A 2+1D system that contains no non-trivial particle-like or string-like topological excitations. Such a system correspond to a few copies of  $E_8$  bosonic quantum Hall states. The  $E_8$  bosonic quantum Hall state is described by the following wave function with 8 kinds of bosons:

$$\prod_{I;i < j} (z_i^I - z_j^I)^{K_{II}} \prod_{I < J;i,j} (z_i^I - z_j^J)^{K_{IJ}} e^{-\frac{1}{4} \sum_{i,I} |z_i^I|^2}, (14)$$

whose low energy effective theory is given by eqn. (1) with

$$K = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$
 (15)

Despite there is no non-trivial topological excitations (since  $\det(K) = 1$ ), the system has a non-trivial thermal Hall effect<sup>68</sup> and chiral edge states<sup>89,90</sup> with chiral central charge  $c_R - c_L = 8 \times$  integer. In other word, the system has a non-trivial perturbative gravitational response (i.e. the non-trivial thermal Hall effect). (In some papers, <sup>83</sup> the  $E_8$  bosonic quantum Hall state is called short-range entangled state. However, according to our definition based on the local unitary or local invertible transformations, <sup>9,10</sup> the  $E_8$  bosonic quantum Hall state is long-range entangled.)

**Example 5.** A 2+1D system whose only topological excitation is particle-like which is labeled by e. The fusion of two e's gives rise to a trivial excitation  $e \otimes e = 1$ . The braiding property is given by the Bose statistics of e. There are no other string-like topological excitations, except the one formed by 1D gas of e's. In fact, the  $\mathbb{Z}_2$  conserved boson e may form a  $\mathbb{Z}_2$  symmetry breaking state. We will say such a string is formed by the condensation of e and denoted by e. It corresponds to the trivial Witt class of string-like topological excitation. The fusion of e is given by e0 so e1. The only nontrivial type of topological excitation is e1. We will denote such a BFe1 category as e1 so e2.

**Example 6.** A 2+1D system whose only particle-like topological excitation is labeled by e. The fusion of two e's gives rise to a trivial excitation  $e \otimes e = \mathbf{1}$ . The braiding property is given by the Fermi statistics of e. There are no other string-like topological excitations, except the one formed by 1D gas of e's. In fact, the Majorana fermions e may form a 1D BCS p-wave condensed state, with Majorana zero-modes at the ends of the 1D system. Such a string can be viewed as formed by the condensation of  $e^{92,93}$  and denoted by s. It is Witt equivalent to the trivial string-like topological excitation. The fusion of s is given by  $s \otimes s = \mathbf{1}$ . Again, the only nontrivial type of topological excitation is e. We will denote such a  $\mathrm{BF}^{\mathrm{H}}_{2+1}$  category as  $\mathrm{C}^{F\mathbb{Z}_2f}_3$ .

**Example 7.** A 2+1D system whose only topological excitation is the particle-like topological excitation labeled by e. There are no other string-like topological excitations. The fusion of two e's gives rise to a trivial excitation  $e \otimes e = 1$ . The braiding property is given by the semion statistics of e. We will denote such a BF $_{2+1}^{H}$  category as  $\mathcal{C}_{3}^{F\mathbb{Z}_{2}s}$ .

In the above three examples, the only particle-like topological excitation e is assigned a Bose, a Fermi, or a semion statistics. Such three choices are consistent with the fusion rule  $e \otimes e = 1$ , since the bond state of two bosons, two fermions, or two semions is a boson. We also discussed the string-like topological excitations, which are all Witt equivalent to the trivial excitation. In the following, we will only discuss excitations in nontrivial Witt classes, which are sufficient to characterize BF<sup>H</sup><sub>3</sub> categories (up to  $E_8$  bosonic quantum Hall states in Example 4).

**Example 8.** A 2+1D system whose only topological excitations are three types of particle-like topological excitations labeled by e, v, and  $\epsilon$ . The fusion rules of the particle-like excitations are given by  $e \otimes e = v \otimes v = \epsilon \otimes \epsilon = 1$ ,  $e \otimes v = \epsilon$ ,  $e \otimes \epsilon = v$ , and  $v \otimes \epsilon = e$ . The braiding properties are described by (1) e and v are bosons and  $\epsilon$  is a fermion; (2) moving  $\epsilon$  around e or v will induce a phase factor -1. We will denote such a BF $_{2+1}^{H}$  category as  $\mathcal{C}_{3}^{\mathbb{Z}_{2}}$ .

**Example 9.** A 2+1D system whose only topological excitations are three types of particle-like topological excitations labeled by e, v, and  $\epsilon$ . The fusion of those excitations is given by  $e \otimes e = v \otimes v = \epsilon \otimes \epsilon = 1$ ,  $e \otimes v = \epsilon$ ,  $e \otimes \epsilon = v$ , and  $v \otimes \epsilon = e$ . The braiding properties are described by (1) e and v are semions with statistics  $\pm \pi/2$  respectively and  $\epsilon$  is a boson; (2) moving  $\epsilon$  around e or v will induce a phase factor -1. We will denote such a BF $_{2+1}^H$  category as  $C_3^{\mathbb{Z}_2 ds}$ .

**Example 10.** A 2+1D system whose only topological excitations are three types of particle-like topological excitations labeled by e, v, and  $\epsilon$ . The fusion of those excitations is given by  $e \otimes e = v \otimes v = \epsilon \otimes \epsilon = 1$ ,  $e \otimes v = \epsilon$ ,  $e \otimes \epsilon = v$ , and  $v \otimes \epsilon = e$ . The braiding properties are described by (1) e, v, and  $\epsilon$  are all fermions; (2) they all have a mutual  $\pi$  statistics. We will denote such a BF $_{2+1}^{\mathrm{H}}$  category as  $\mathbb{C}_{3}^{\mathbb{Z}_{2}^{f}}$ .

**Example 11.** A 2+1D system whose only topological excitations are one type of string-like topological excitations denoted as s, and there is no particle-like topological excitations. The fusion of the two string excitations give rise to a trivial string  $s \otimes s = 1$ . There is no nontrivial braiding property between the strings. We will denote such a BF<sup>H</sup><sub>2+1</sub> category as  $\mathcal{C}_{s}^{sF\mathbb{Z}_{2}}$ .

## 3. Examples of $BF_4^H$ categories in 3+1D

In 3+1D, we can have particle-like, string-like and membrane-like (2-brane-like) topological excitations.

**Example 12.** A 3+1D system whose only topological excitations are one type of particle-like topological excitations denoted as e and one type of string-like topological excitations denoted as s. The fusion of those excitations is given by  $e \otimes e = s \otimes s = 1$ . The only nontrivial braiding property is the phase factor -1 as we move the particle e around the string s. We will denote such a  $\mathrm{BF}^{H}_{3+1}$  category as  $\mathbb{C}^{\mathbb{Z}_2}_4$ .

**Example 13.** A 3+1D system whose only topological excitations are one type of string-like topological excitations denoted as s. The fusion of the two string excitations give rise to a trivial string  $s \otimes s = 1$ . There is no nontrivial braiding property between the strings. We will denote such a BF $_{3+1}^{H}$  category as  $\mathcal{C}_{4}^{sF\mathbb{Z}_{2}}$ .

**Example 14.** A 3+1D system whose only topological excitations are one type of membrane-like topological excitations denoted as m. The fusion of the two membrane-like excitations give rise to a trivial membrane  $m \otimes m = 1$ . There is no nontrivial braiding property between the membranes. We will denote such a BF $_{3+1}^{H}$  category as  $\mathcal{C}_{4}^{mF\mathbb{Z}_{2}}$ .

### 4. Summary

In the above simple examples in various dimensions, we have described some topological properties of particle-like, string-like, and membrane-like excitations. Those properties are needed to define a BF<sup>H</sup> category. We may also need some additional topological properties, such as the non-Abelian geometric phases of the degenerate ground states  $^{5,6,44,45}$  and the linear relation between the fusion spaces (see Section X), to completely define a BF<sup>H</sup> category.

A natural question is "can those topological properties be realized by well defined lbH model in the same space-time dimensions?". In fact, some topological properties can be realized in the same dimension and the corresponding BF<sup>H</sup> category (or topological phase) is anomaly-free, while some other topological properties cannot be realized in the same dimension and the correspond BF<sup>H</sup> category (or topological phase) is anomalous. This is a issue of gravitational anomaly, which will be discussed in the next section.

## VI. A GENERAL DISCUSSION OF GRAVITATIONAL ANOMALY

## A. Closed and exact BF<sup>H</sup> categories

We are now ready to have a general discussion of gravitational anomaly. Usually, gravitational anomaly is defined through the variance of the path integral under the homeomorphic transformations of the space-time. Here, we will introduce a more general definition as in Ref. 46, which are closely related to anomaly inflow (the first examples were discovered in Ref. 95 and 96).

Let us consider an (n+1)-dimensional topologically ordered state x and a spatial p-dimensional (p < n) defect y in x (e.g. a gapped boundary of x). Let us assume that the excitations on the defect are also gapped. Such a p-dimensional defect y can have lower dimensional wall excitations (or sub-defects). These sub-defects in y have the universal properties, which again are described by a gapped low energy effective theory, i.e a  $\mathrm{BF}_{p+1}^{\mathrm{H}}$  category. We now ask, can we realize such a (p+1)-dimensional effective theory on the defect y by a well-defined local p-dimensional lattice model without a higher dimensional bulk? The answer can be yes or no. This leads to two kinds of  $\mathrm{BF}_{p+1}^{\mathrm{H}}$  categories (or two kinds of gapped low energy effective theories). This line of thinking leads to the notion of a closed  $\mathrm{BF}_{n+1}^{\mathrm{H}}$  category.

## Definition 14. Closed $BF_{n+1}^H$ category

If the topological properties of a gapped state in n spacetime dimensions, described by a  $BF_{n+1}^H$  category  $C_{n+1}$ , can be realized by a well-defined lbH system in the same dimension, then  $C_{n+1}$  is said to be closed. **Conjecture 5:** The closed  $BF_{n+1}^H$  categories classify the topological orders (i.e. the patterns of long-range entanglement). In other words, the topological excitations and gravitational responses in two gapped states are described by the same closed  $BF_{n+1}^H$  category iff the two gapped states are in the same phase.

Since a closed  $BF_{n+1}^H$  category  $\mathcal{C}_{n+1}$  can be realized by a lbH system in the same dimension, this allows us to consider the boundary of the lbH system. If the boundary of such a system can be gapped, the topological properties of the boundary will define a  $BF_n^H$  category  $\mathcal{C}_n$  in one-lower dimension. This leads to the concept of

## Definition 15. Exact $BF_{n+1}^H$ category

If a (n+1) space-time dimensional gapped lbH system, which realizes a closed  $BF_{n+1}^H$  category  $C_{n+1}$ , can have a gapped boundary, then the  $BF_{n+1}^H$  category  $C_{n+1}$  is said to be exact.

Similarly, we can also use lbL system to define the notions of a  $BF^L$  category and a closed/exact  $BF^L$  category.

## Definition 16. BF<sup>L</sup> category

The collection of all topological (or universal) properties of the instantons, the world line of particle-like topological excitations, the world sheet of string-like topological excitations, etc., in an (n+1)-dimensional space-time defines a (n+1)-dimensional BF<sup>L</sup> or BF<sup>L</sup><sub>n+1</sub> category.

## Definition 17. Closed/exact $BF_{n+1}^{L}$ category

If the topological properties in n+1 space-time dimensions, described by a  $\mathrm{BF}^L_{n+1}$  category  $\mathfrak{C}^L_{n+1}$ , can be realized by a well-defined bL system in the same dimension, then  $\mathfrak{C}^L_{n+1}$  is said to be closed. If the bL system also has a short-range correlated boundary, then  $\mathfrak{C}^L_{n+1}$  is said to be exact.

## B. A definition of gravitational anomaly

In the above, we have discussed whether a gapped low energy effective theory (i.e. the fusion and braiding properties of gapped topological excitations) can be realized by a lbH system in the same dimension or has to appear as a boundary theory of a gapped lbH system in one-higher dimension. More generally, a "low energy effective theory" is a collection of all the low energy properties, which may or may not be gapped. We want to consider when a low energy effective theory can be realized by a local Hamiltonian system in the same dimension or has to appear as a boundary theory of a gapped lbH system in one-higher dimension. This leads to the following concept:

## Definition 18. H-type gravitational anomaly

If we can realize a low energy effective theory (gapped or gapless) by a lbH system in the same space-time dimension, we say the low energy effective theory is free of H-type gravitational anomaly.

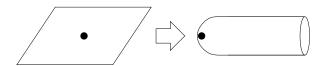


FIG. 6. Dimensional reduction: A point defect in 2D looks like a boundary of an effective 1D system, if we wrap the 2D space into a cylinder.

Let us assume that

**Conjecture 6:** A potentially anomalous n-dimensional low energy effective theory (gapped or gapless) can always be realized on an n-dimensional defect in a  $\tilde{n}$ -dimensional lbH system with an energy gap, where  $\tilde{n}$  is finite and  $\tilde{n} > n$ .

Note that given an n-dimensional defect  $M^n$  in a higher  $\tilde{n}$ -dimensional space  $\tilde{M}^{\tilde{n}}$ , we can always deform the higher dimensional space  $\tilde{M}^{\tilde{n}}$  so that the defect looks like a boundary when viewed from far away (see Fig. 6).<sup>46</sup> This process will be called *dimensional reduction*. We have the following result of dimensional reduction.

**Corollary 4:** A potentially anomalous n-dimensional low energy effective theory (gapped or gapless) can always be realized by a boundary of a (n+1)-dimensional local Hamiltonian system with an energy gap.

**Remark 3.** We see that a p-dimensional excitation can be viewed as an anomalous  $\mathrm{BF}^{\mathrm{H}}_{p+1}$  category in (p+1) dimensional space-time. A simple p-dimensional excitation may, however, correspond to an composite  $\mathrm{BF}^{\mathrm{H}}_{p+1}$  category.

It is clear that the low energy effective theory on the boundary of short-range entangled state can be realized as a pure boundary theory without the bulk. So the low energy effective theory on the boundary of short-range entangled state is always free of gravitational anomaly, while the low energy effective theory on the boundary of long-range entangled state always have gravitational anomaly. This line of thinking allows us to show that

Corollary 5: (1) The H-type gravitational anomalies in n space-time dimensions are classified by topological orders<sup>3,5</sup> (i.e. patterns of long-range entanglement<sup>9</sup>) in one-higher dimension. In other words, the H-type gravitational anomalies in n space-time dimensions are classified by closed  $BF_{n+1}^H$  categories  $C_{n+1}^{closed}$  in one-higher dimension

- (2) The gapped H-type gravitational anomalies in n space-time dimensions are classified by exact  $BF_{n+1}^H$  categories  $\mathcal{C}_{n+1}^{\text{exact}}$  in one-higher dimension.
- (3) A gapped system described by a  $BF_n^H$  category  $\mathfrak{C}_n$  has a H-type gravitational anomaly if  $\mathfrak{C}_n$  is not closed. So a non-closed  $BF_n^H$  category  $\mathfrak{C}_n$  describes a gravitationally anomalous theory of H-type. We also call a non-closed  $BF_n^H$  category as an anomalous  $BF_n^H$  category.

Similarly, we can also define

## Definition 19. L-type gravitational anomaly

If we can realize a low energy effective theory (gapped or gapless) by a lbL system in the same space-time dimension, we say the low energy effective theory is free of L-type gravitational anomaly.

We also have

**Conjecture 7:** a potentially anomalous n-dimensional low energy effective theory (gapped or gapless) can always be realized by a boundary of a (n+1)-dimensional lbL system with an energy gap.

Thus

Corollary 6: (1) The L-type gravitational anomalies in n space-time dimensions are classified by closed  $BF_{n+1}^L$  categories  $\mathfrak{C}_{n+1}^{L,\operatorname{closed}}$  in one-higher dimension.

(2) The short-range correlated L-type gravitational anomalies in n space-time dimensions are classified by exact  $BF_{n+1}^L$  categories  $\mathcal{C}_{n+1}^{L,\mathrm{exact}}$  in one-higher dimension. (3) A system described by a  $BF_n^L$  category  $\mathcal{C}_n^L$  has a gravitational anomaly if  $\mathcal{C}_n^L$  is not closed. So a non-closed  $BF_n^L$  category  $\mathcal{C}_n$  describes a gravitationally anomalous theory of L-type.

We have listed many examples of BF categories in Section VD. In Appendix C, we will discuss those simple examples further to illustrate the notions of exact, closed, and anomalous BF categories, and to see how those simple examples fit into the above three classes of BF categories.

## VII. BOUNDARY-BULK RELATION FOR BF CATEGORIES IN DIFFERENT DIMENSIONS

The results in this section apply to both BF<sup>H</sup> and BF<sup>L</sup> categories. We will refer them as BF categories.

### A. The boundary of a given bulk

We have introduced  $\mathrm{BF}_n$  category to describe a set of topological excitations in n-dimensional space-time. Those topological excitations have a property that they are closed under the local fusion and braiding operations (see Fig. 4 and Fig. 5). Their fusion rules braiding properties are consistent among themselves. We also introduced the notions of closed/exact  $\mathrm{BF}_n$  category.

The (generic) BF categories in n space-time dimension are closely related to the exact BF categories in n+1 space-time dimension. In this section, we will explore this relation in details.

Consider a well-defined gapped state in n+1 spacetime dimension, whose topological excitations are described by an exact BF category  $\mathcal{C}_{n+1}$ . Since  $\mathcal{C}_{n+1}$  is exact, the gapped state in n+1 space-time dimension has a gapped boundary of n space-time dimension. Some of the topological excitations on the gapped boundary come from the topological excitations in the bulk, while others are confined on the boundary and only appear on the boundary. Since the boundary topological excitations can still fuse and braid within the n-dimensional boundary, they are described by a BF category  $\mathcal{C}_n$ . In general, such  $\mathcal{C}_n$  is not unique. Moreover, even if  $\mathcal{C}_n$  is fixed, we still can not fix the boundary type. For example, in toric code model, there are two types of gapped boundaries: a rough boundary and a smooth boundary (see Fig. 13). The boundary excitations in both cases are given by the unitary fusion category  $\Re \mathcal{C}_{\mathbb{Z}_2}$ , which is the category of representations of  $\mathbb{Z}_2$  group.

## B. The bulk-to-boundary map

The additional data that is needed to determine the boundary is the so-called the bulk-to-boundary map, which is a functor  $f: \mathcal{C}_{n+1} \to \mathcal{C}_n$  that maps the bulk topological excitations into a subset of boundary topological excitations. Those bulk topological excitations, that are mapped into the trivial excitations on the boundary. are said to be condensed on the boundary. For example, in the toric code model, the smooth boundary in Fig. 13 corresponds to the condensation of e-particles; the rough boundary in Fig. 13 corresponds to the condensation of m-particles. In general, there might be different sets of boundary excitations  $\mathcal{C}_n$  for a given  $\mathcal{C}_{n+1}$  as in Levin-Wen types of lattice models $^{26}$ . So an n-dimensional boundary of a given (n+1)-dimensional  $\mathcal{C}_{n+1}$  is determined by a pair  $(\mathcal{C}_n, \mathcal{C}_{n+1} \xrightarrow{f} \mathcal{C}_n)$ . The functor f can not be arbitrary. It must satisfy some consistency conditions. We will return to this point later.

Remark 4. As we will argue later that  $\mathcal{C}_n$  determines the <u>bulk</u>  $\mathcal{C}_{n+1}$  uniquely up to isomorphisms. If  $\mathcal{C}_{n+1}$  has non-trivial automorphisms, then a bulk-to-boundary map can be twisted by these automorphisms to give different bulk-to-boundary maps. The non-trivial automorphism is physically detectable. So we will treat  $\mathcal{C}_n$  associated to a different bulk-to-boundary maps as different boundary types. There is no contradiction between different boundary types associated to the same  $\mathcal{C}_n$  and the uniqueness of the <u>bulk</u> up to isomorphisms in Lemma 2.

Another way to characterize a boundary is to specify the condensation (of the bulk excitations) that can create a trivial condensed phase  $\mathbf{1}_{n+1}$  and a gapped boundary. For example, in the Levin-Wen models, the boundary types can be classified by Lagrangian algebras in the tensor category of bulk excitations<sup>48</sup>. Two types of boundary in toric model corresponds to two different condensations<sup>48</sup>. What is really important to us is that if an n+1 dimensional BF category  $\mathcal{D}_{n+1}$  is exact, it is reasonable that one can always create the trivial phase and a gapped boundary via a condensation of the bulk excitations in  $\mathcal{D}_{n+1}$ . We assume this for the rest of this section. We will use it, in particular, in the proof of Theorem 4.

## C. The bulk of a given boundary

Now we consider the bulk-boundary relation in the reversed order. Given an n space-time dimensional boundary BF category  $\mathcal{C}_n$ , it turns out that it determines uniquely an (n+1) space-time dimensional <u>bulk</u> BF category  $\mathcal{C}_{n+1}$ . Here, we must make it very clear what we mean by "a bulk". A given n space-time dimensional BF category can always be realized as an n-dimensional defect in a higher dimensional (possibly trivial) topological order. But such realization is almost never unique. However, by the dimensional reduction given in Fig. 6 and Corollary 4, we can always reduce such a realization down to an exact (n+1) space-time dimensional BF category  $\mathcal{C}_{n+1}$  with a gapped boundary given by  $\mathcal{C}_n$ . Such BF category  $\mathcal{C}_{n+1}$  is unique. This will be our first important result (Lemma 2), which leads to many interesting consequences.

Before we state Lemma 2, let us first state a generalization of the results in Ref. 9 to the anomalous topological phases.

**Lemma 1:** If there are two n+1-dimensional lbH systems H and H' realizing the same n-dimensional topological phase as their boundaries, then there is a neighborhood U of the boundary such that the restriction of H in U, denoted by  $H|_U$ , can be deformed smoothly to  $H'|_U$  without closing the gap. In other words, there is a smooth family  $H_t$  for  $t \in [0,1]$  without closing the gap such that  $H_0 = H$ , and  $H_r$  and  $H_s$  differ only in U for  $s, t \in [0,1]$  and  $H_1|_U = H'|_U$ .

**Lemma 2:** Two exact BF categories  $\mathcal{C}_{n+1}$  and  $\mathcal{C}'_{n+1}$  must be equivalent  $\mathcal{C}_{n+1} = \mathcal{C}'_{n+1}$  if they can have gapped boundaries described by the same BF category  $\mathcal{C}_n$ .

Proof. We need use Lemma 1. Let H be a local Hamiltonian qubit system that realizes the topological bulk phase  $\mathcal{C}_{n+1}$ , and H' be the one that realizes the topological bulk phase  $\mathcal{C}'_{n+1}$ . By Lemma 1, we are able to deform  $H_0 = H$  smoothly only in a neighborhood U of the boundary such that  $H_t$  does not close the gap for all  $t \in [0,1]$  and  $H_1|_U = H'|_U$ . Therefore, we can connect two local Hamiltonian qubit systems H and H' by adding a region which contains only a neighborhood V of the boundary depicted in Fig. 7 as the dotted box. In the region V, the lbH system is smoothly deformed from  $H|_U$  to  $H'|_U$ . The remaining bulk are glued by brutal force, which creates a domain wall labeled by X between two bulk phases as shown in Fig. 7.

If the domain wall is trivial, then we are done. Assume that the domain X is non-trivial. Then the domain wall must end near the boundary but outside the region V and create a non-trivial defect junction (a defect of codimension 2). Since, in the dotted neighborhood of the boundary (see Fig. 7),  $H_t$  does not close the gap, all observables  $\langle O \rangle (t)$ , including the topological excitations, can cross from the left side of V to the right side of V smoothly (without crossing any singularities). Consequentially, there is no macroscopic detectable defects

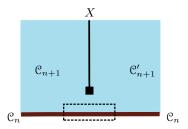


FIG. 7. The bulk phase of a given boundary is unique. The dotted box V on the boundary represents the region where the Hamiltonian is smoothly deformed from  $H_0$  to  $H_1$ .

between the boundary and the defect junction.

Since the bulk phase and the wall phase are topological, we can move the X-wall up and create a larger neighborhood and continue this move until the domain wall X is completely removed. As a consequence, two exact BF categories  $\mathcal{C}_{n+1}$  and  $\mathcal{C}'_{n+1}$  must be equivalent.

In other words, two gapped topological states belong to the same phase if they can have gapped boundaries which are of the same type (i.e. described by the same BF category). As a consequence, we introduce the following definition.

**Definition 20.** If an n-dimensional BF category  $\mathcal{C}_n$  describes the topological excitations on the boundary of a gapped lbH system in (n+1)-dimensional space-time, then we call the unique BF $_{n+1}$  category determined by this (n+1)-dimensional space-time gapped lbQH system as the <u>bulk</u> of  $\mathcal{C}_n$ , denoted by  $\mathcal{Z}_n(\mathcal{C}_n)$ .

Remark 5. We choose the letter " $\mathcal{Z}$ " because it is also used in algebra for the notion of center. In particular,  $\mathcal{Z}_n$ is somewhat similar to the so-called  $E_n$ -center<sup>97</sup>. We will show in the next paper 70 that the bulk is exactly equivalent to the mathematical notion of center<sup>97</sup>. In this work, we don't need this result in such generality. We only need it in a few lower dimensional cases. In the case of 3 space-time dimensional topological phases with a gapped boundary, this result has already been rigorously proved. Indeed, consider a Levin-Wen type of lattice model with a bulk lattice constructed from a unitary fusion category C and a boundary lattice from a C-module M. It was proved rigorously in Ref. 26 that the excitations on the boundary constructed on an  $_{\mathbb{C}}\mathcal{M}$ -lattice, are given by the unitary fusion category  $\mathcal{C}_{\mathfrak{M}}^{\vee}:=\mathfrak{F}\mathrm{un}_{\mathfrak{C}}(\mathfrak{M},\mathfrak{M})$  of  $\mathfrak{C}$ module functors. The bulk excitations is given by  $\mathcal{Z}(\mathcal{C})$ which is the monoidal center of C. 11,26. And we have  $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{C}_{\mathcal{M}}^{\vee})$  as unitary modular tensor categories<sup>98</sup>. A model independent proof of this boundary-bulk relation for 3 space-time dimensional theories was also given in Ref. 71. For this reason, it is harmless for readers to take the bulk Z simply as a synonym of the center.

It is clear that  $\mathcal{Z}_n(\mathbf{1}_n) \simeq \mathbf{1}_{n+1}$ , where  $\mathbf{1}_{n/n+1}$  is the n/n+1-dimensional trivial phase. Using this notation, a closed BF<sub>n</sub> category  $\mathfrak{C}$  means  $\mathcal{Z}_n(\mathfrak{C}_n) \simeq \mathbf{1}_{n+1}$ , and an

exact  $BF_{n+1}$  category is equivalent to  $\mathcal{Z}_n(\mathcal{C}_n)$  for some  $BF_n$  category  $\mathcal{C}_n$ .

The following result follows from Lemma 2 immediately.

Corollary 7: The <u>bulk</u> of the <u>bulk</u> of a BF<sub>n</sub> category  $\mathcal{C}_n$  is trivial, i.e.  $\mathcal{Z}_{n+1}(\mathcal{Z}_n(\mathcal{C}_n)) = \mathbf{1}_{n+2}$ .

Remark 6. As we remarked in Remark 5 that the <u>bulk</u> is equivalent to the center, Corollary 7 also means that the center of a center is trivial. This is a very interesting and non-trivial result, which is the dual of the statement that the boundary of a boundary is empty. As the physical or geometric intuition is so obvious, its mathematical meaning is very non-trivial <sup>70</sup>. We believe that the triviality of the center of a center is also a robust phenomena which can be proved in many different contexts using different notions of center in mathematics. For example, it seems also plausible to establish this result in the framework of factorization algebras <sup>97</sup>.

# VIII. MONOIDAL AND GROUP STRUCTURE OF $BF_n$ CATEGORIES IN THE SAME DIMENSION

## A. A tensor product of $BF_n$ categories

Let a closed BF category  $\mathcal{C}_n^1$  be realized by a gapped lbH system  $\Lambda_1$ , and a closed BF category  $\mathcal{C}_n^2$  by another gapped lbH system  $\Lambda_2$ . If we stack the two lbH systems together to form an lbH system  $\Lambda_{12}$ , then the gapped model  $\Lambda_{12}$  will give rise to a new closed BF category, denoted by  $\mathcal{C}_n^1 \boxtimes \mathcal{C}_n^2$ .

This defines a tensor product among all closed BF categories. It turns out that such tensor product can be generalized to generic BF categories. Let an n-dimensional BF category  $\mathcal{C}_n^1$  be realized by the boundary of a gapped lbH system  $\Lambda_1$  in (n+1)-dimensional space-time. The topological excitations in the model  $\Lambda_1$  is described by an exact (n+1)-dimensional BF category  $\mathcal{C}_{n+1}^1$ . Similarly, let another n-dimensional BF category  $\mathcal{C}_n^2$  be realized by the boundary of a gapped lbH system  $\Lambda_2$ . The topological excitations in  $\Lambda_2$  is described by an exact BF category  $\mathcal{C}_{n+1}^2$ . If we stack the two local Hamiltonian systems together to form the third lbH system  $\Lambda_{12}$ , then the gapped boundary of  $\Lambda_{12}$  will give rise to a BF category denoted by  $\mathcal{C}_n^1 \boxtimes \mathcal{C}_n^2$ .

Let  $\mathcal{M}^n$  be the set of BF<sub>n</sub> categories. The tensor product  $\boxtimes$  defines a multiplication on the set  $\mathcal{M}^n$ . It is clear that  $\mathbf{1}_n \boxtimes \mathcal{C}_n \simeq \mathcal{C}_n \simeq \mathcal{C}_n \boxtimes \mathbf{1}_n$ , and the multiplication is associative, i.e.  $(\mathcal{C}_n \boxtimes \mathcal{D}_n) \boxtimes \mathcal{E}_n \simeq \mathcal{C}_n \boxtimes (\mathcal{D}_n \boxtimes \mathcal{E}_n)$ , and commutative, i.e.  $\mathcal{C}_n \boxtimes \mathcal{D}_n \simeq \mathcal{D}_n \boxtimes \mathcal{C}_n$ .

**Lemma 3:** The multiplication  $\boxtimes$  and the unit  $\mathbf{1}_n$  provide the set  $\mathcal{M}^n$  a structure of commutative monoid.

Notice that the tensor product commutes with  $\mathcal{Z}$ . More precisely, we have

$$\mathcal{Z}_n(\mathcal{C}_n \boxtimes \mathcal{D}_n) \simeq \mathcal{Z}_n(\mathcal{C}_n) \boxtimes \mathcal{Z}(\mathcal{D}_n) \tag{16}$$

as BF<sub>n+1</sub> categories. In other words,  $\mathcal{Z}_n : \mathcal{M}^n \to \mathcal{M}^{n+1}$  is a homomorphism between monoids.

Remark 7. We also like to make a remark on the action of the center functor on composite BF categories  $\mathcal{C}_n \oplus \mathcal{D}_n$ . In general,  $\mathcal{Z}_n(\mathcal{C}_n \oplus \mathcal{D}_n) \neq \mathcal{Z}_n(\mathcal{C}_n) \oplus \mathcal{Z}_n(\mathcal{D}_n)$ . When  $\mathcal{C}_n = \mathcal{D}_n$ , we have  $\mathcal{Z}_n(\mathcal{C}_n \oplus \mathcal{C}_n) = \mathcal{Z}_n(\mathcal{C}_n) \times M_{2\times 2}$ , where  $M_{2\times 2}$  is the  $2\times 2$  matrix algebra. The phase  $\mathcal{Z}_n(\mathcal{C}_n) \times M_{2\times 2}$  is unstable and can flow to the stable one  $\mathcal{Z}_n(\mathcal{C}_n)$ . The composition  $\oplus$  and the tensor product  $\boxtimes$ , together with the tensor unit  $\mathbf{1}_n$  and zero category  $\mathbf{0}_n$ , give a commutative ring structure to all BF categories. We will not discuss it further in this work. More details will be given Ref. 70.

Remark 8. A general tensor product between two n-dimensional BF categories can be defined. We can stack one topological order  $\mathcal{C}_n$  on the top of the other  $\mathcal{D}_n$  and glue them by inserting between them a (n+1)-dimensional "glue", the topological type of which is given by a (n+1)-dimensional BF category  $\mathcal{E}_{n+1}$ . Such a physical gluing process creates a new (possibly anomalous) n-dimensional topological phase, denoted by  $\mathcal{C}_n \boxtimes_{\mathcal{E}_{n+1}} \mathcal{D}_n$ , where  $\boxtimes_{\mathcal{E}_{n+1}}$  defines a new type of tensor product. It actually contains the old tensor product  $\boxtimes$  as a special case, i.e.  $\boxtimes = \boxtimes_{\mathbf{1}_{n+1}}$ .

Let  $\mathcal{M}^n_{\mathrm{closed}}$  and  $\mathcal{M}^n_{\mathrm{exact}}$  be the subsets of  $\mathcal{M}^n$  consisting of the equivalence classes of closed and exact  $\mathrm{BF}_n$  categories, respectively. Clearly, the multiplication  $\boxtimes$  is closed on the subsets  $\mathcal{M}^n_{\mathrm{closed}}$  and  $\mathcal{M}^n_{\mathrm{exact}}$ . Therefore,  $\mathcal{M}^n_{\mathrm{closed}}$  and  $\mathcal{M}^n_{\mathrm{exact}}$  are two sub-monoids of  $\mathcal{M}^n$ .

A monoid is not a group since the inverse may not exist. In our case, there is no group structure on  $\mathcal{M}^n$ . Because the long range entanglement on different layer can not cancel each other, a double layer system  $\mathcal{C}_n \boxtimes \mathcal{D}_n$  has no long range entanglement if and only if each factor has no long range entanglement. In other words (see also Conjecture 3),

$$\mathfrak{C}_n \boxtimes \mathfrak{D}_n \simeq \mathbf{1}_n$$
 iff  $\mathfrak{C}_n$  and  $\mathfrak{D}_n$  have no non-trivial elementary topological excitations. (17)

Equivalently, all elements in  $\mathcal{M}^n$ , that have non-trivial elementary topological excitations, are not invertible. On the other hand, the elements in  $\mathcal{M}^n$ , that have no non-trivial elementary topological excitations, are invertible.

In order to obtain a group structure, we have to consider certain quotient sets of  $\mathcal{M}^n$ , and to obtain the quotient sets, we need to introduce a few new concepts.

## B. Dimension reduction of a BF category

A *n*-dimensional BF category (*i.e.* a BF<sub>n</sub><sup>H</sup> or BF<sub>n</sub><sup>L</sup> category) can be viewed as a BF<sub>p</sub> category for p < n:

## Definition 21. The project functor $P_D$

Consider an *n*-dimensional BF category  $\mathcal{C}_n$  on  $X^n$  which

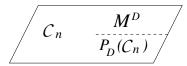


FIG. 8. Consider a BF category  $\mathcal{C}_n$  realized by a boundary  $M^n$  of a system in (n+1)-dimensional space-time. Now consider a D-dimensional sub space-time  $M^D$  on the n-dimensional boundary. The topological excitations in the D-dimensional sub space-time  $M^D$  all come from the BF category  $\mathcal{C}_n$ . All topological excitations and their topological properties define a D-dimensional category  $P_D(\mathcal{C}_n)$ .



FIG. 9. (a) A local Hamiltonian qubit system  $\Lambda$  defined by a path integral in n space-time dimensions. Its topological excitations are described by a n-dimensional BF category  $\mathbb{C}_n$ . (b) If we fold the time direction, we obtain a local Hamiltonian qubit system  $\overline{\Lambda}$  which is the time-reversal transformation of the local Hamiltonian qubit system  $\Lambda$ . Its topological excitations are described by a n-dimensional BF category  $\overline{C}_n$ . The gapped boundary of  $\mathbb{C}_n \boxtimes \overline{\mathbb{C}}_n$  is described by a (n-1)-dimensional BF category  $P_{n-1}(\mathbb{C}_n)$ :  $\mathcal{Z}_n[P_{n-1}(\mathbb{C}_n)] = \mathbb{C}_n \boxtimes \overline{\mathbb{C}}_n$ .

is a boundary of (n+1) dimensional space-time. Let  $M^D$  be a D-dimensional sub space-time  $M^D \subset M^n$  (see Fig. 8). If we view  $M^D$  as a subsystem whose topological excitations all come from the n-dimensional  $\mathcal{C}_n$ , then topological excitations on  $M^D$  define a D-dimensional BF category which is denoted as  $P_D(\mathcal{C}_n)$ .  $P_D$  is a functor that maps  $\mathcal{C}_n$  to  $\mathcal{C}_D$ .

We call  $P_D(\mathcal{C}_n)$  a projection of  $\mathcal{C}_n$  from n-dimensions to D-dimensions. We know that if  $M^D$  contains no topological excitations, then the BF category on  $M^D$  is trivial. The BF category  $P_D(\mathcal{C}_n)$  on  $M^D$  does contain topological excitations and thus nontrivial. But all the topological excitations come trivially from its higher dimensional parent. So, in some sense,  $P_D(\mathcal{C}_n)$  is "trivial". In the following, we are going to introduce an equivalence relation  $\sim$  between BF categories that makes  $P_D(\mathcal{C}_n)$  equivalent to trivial BF category when  $\mathcal{C}_n$  is closed.

## C. Dual BF category

In order to obtain a group structure, we have to consider certain quotient sets of  $\mathcal{M}^n$ . Before we do that, we need first introduce the dual of a BF category.

## Definition 22. Dual BF category

(1) Let  $\Lambda$  be a lbH system in an (n+1)-dimensional space-time. The lbH system can always be described by a path integral. Then the dual lbH system  $\overline{\Lambda}$  is the lbH

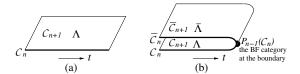


FIG. 10. (a) A local Hamiltonian qubit system  $\Lambda$  defined by a path integral in n+1 space-time dimensions. Its topological excitations are described by an exact n + 1-dimensional BF category  $\mathcal{C}_{n+1}$ . The topological excitations on its gapped boundary (represented by the thick line) are described by an n-dimensional BF category  $\mathcal{C}_n$ . (b) If we fold the time direction, we obtain a local Hamiltonian qubit system  $\overline{\Lambda}$  which is the time-reversal transformation of the local Hamiltonian qubit system  $\Lambda$ . Its topological excitations are described by an exact n+1-dimensional BF category  $\overline{\mathbb{C}}_{n+1}$ . The topological excitations on its gapped boundary (represented by the thick line) are described by a *n*-dimensional BF category  $\overline{\mathbb{C}}_n$ . If we stack the two boundaries  $\mathcal{C}_n$  and  $\overline{\mathcal{C}}_n$  together, we will obtain a new boundary  $\mathcal{C}_n \boxtimes \overline{\mathcal{C}}_n$ . We see that the boundary between  $\mathcal{C}_n \boxtimes \overline{\mathcal{C}}_n$  and  $\mathbf{1}_n$  is described by  $P_{n-1}(\mathcal{C}_n)$ .

system described by the time-reversal transformation of the path integral that defines the first lbH system  $\Lambda$  (see Fig. 9). For more details, see Appendix A.

(2) Let  $\mathcal{C}_n$  be the BF category realized by a boundary of a lbH system  $\Lambda$ . Then the dual  $\overline{\mathbb{C}}_n$  is the BF category realized by the boundary of the dual lbH system  $\overline{\Lambda}$  (see Fig. 10).

We collect a few basic properties of the dual BF categories below.

**Lemma 4:** Let  $C_n$  and  $D_n$  be two  $BF_n$  categories. We have:

- 1.  $\overline{1}_n = 1_n$ ,
- 2.  $\mathcal{Z}_n(\overline{\mathcal{C}}_n) = \overline{\mathcal{Z}_n(\mathcal{C}_n)}$ .
- 3. If  $\mathcal{D}_n$  is closed, then  $\underline{\mathcal{D}}_n \boxtimes \overline{\mathcal{D}}_n$  is exact and  $\mathcal{Z}_{n-1}[P_{n-1}(\mathcal{D}_n)] = \mathcal{D}_n \boxtimes \overline{\mathcal{D}}_n$

The last result can be proved by folding an ndimensional topological phase defined by  $\mathfrak{D}_n$  along a codimension 1 hyperplane (see Fig. 9).

From Lemma 4, we can easily see that if  $\mathcal{C}_n$  is a closed BF category realized by a lbH system  $\Lambda$ , then the dual  $\mathcal{C}_n$ is also closed which can be realized by the time-reversal transformed lbH system  $\Lambda$ . By folding a  $\mathcal{C}_{n+1}$ -phase and by (17), we obtain an interesting corollary of Lemma 4.

Corollary 8: For a BF<sub>n+1</sub> category  $\mathcal{C}_{n+1}$ ,  $\mathcal{Z}(\mathcal{C}_{n+1})$  is invertible if  $\mathcal{Z}_n[P_n(\mathfrak{C}_{n+1})] \simeq \mathfrak{C}_{n+1} \boxtimes \overline{\mathfrak{C}}_{n+1}$ .

**Remark 9.** If the only invertible  $BF_{n+2}$  category is the trivial one for certain n (possibly for n < 6 and  $n \neq 1,3$  see Section XV),  $\mathcal{C}_{n+1}$  is closed if and only if  $\mathcal{Z}_n[P_n(\mathfrak{C}_{n+1})] \simeq \mathfrak{C}_{n+1} \boxtimes \overline{\mathfrak{C}}_{n+1}$ . When n=2, this result reproduces a well-known mathematical result, which says that a premodular category C is modular if and only if

 $Z(\mathcal{C}) \simeq \mathcal{C} \boxtimes \overline{\mathcal{C}}$ , where  $Z(\mathcal{C})$  is the monoidal center of  $\mathcal{C}$ , and  $\overline{\mathbb{C}}$  is the same category as  $\mathbb{C}$  but with the braiding given by the anti-braiding in C. This also justifies our notation  $\mathcal{Z}(\cdot)$  in this case.

## Quasi-equivalence relation and a group structure

In an attempt to obtain a group structure, we introduce an equivalence relation  $\sim$  between BF categories, hoping that the equivalence classes of such equivalence relation form a group.

**Definition 23.** Two BF<sub>n</sub> categories,  $\mathcal{C}_n$  and  $\mathcal{C}'_n$ , are called quasi-equivalent, denoted by  $\mathcal{C}_n \sim \mathcal{C}'_n$ , if there exist n-dimensional closed BF categories  $\mathcal{D}_n$  and  $\mathcal{D}'_n$  such that  $\mathcal{C}_n \boxtimes \mathcal{D}_n \boxtimes \overline{\mathcal{D}}_n = \mathcal{C}'_n \boxtimes \mathcal{D}'_n \boxtimes \overline{\mathcal{D}}'_n$ 

 $\sim$  is indeed an equivalence relation since it satisfies

## Lemma 5:

- (1)  $\mathcal{C}_n \sim \mathcal{C}_n$ .
- (2)  $\mathcal{C}_n \sim \mathcal{C}'_n$  implies that  $\mathcal{C}'_n \sim \mathcal{C}_n$ . (3)  $\mathcal{C}_n \sim \mathcal{C}'_n$  and  $\mathcal{C}'_n \sim \mathcal{C}''_n$  implies that  $\mathcal{C}_n \sim \mathcal{C}''_n$ .

*Proof.* We give the proof of the third condition.  $\mathcal{C}_n \sim \mathcal{C}'_n$ and  $\mathcal{C}'_n \sim \mathcal{C}''_n$  implies that  $\mathcal{C}_n \boxtimes \mathcal{D}_n \boxtimes \overline{\mathcal{D}}_n = \mathcal{C}'_n \boxtimes \mathcal{D}'_n \boxtimes \overline{\mathcal{D}}_n$  $\overline{\mathcal{D}}'_n$  and  $\mathcal{C}'_n \boxtimes \mathcal{E}_n \boxtimes \overline{\mathcal{E}}_n = \mathcal{C}''_n \boxtimes \mathcal{E}'_n \boxtimes \overline{\mathcal{E}}'_n$ . So, we have  $\mathfrak{C}_n \boxtimes \mathfrak{D}_n \boxtimes \overline{\mathfrak{D}}_n \boxtimes \mathfrak{E}_n \boxtimes \overline{\mathfrak{E}}_n = \mathfrak{C}'_n \boxtimes \mathfrak{D}'_n \boxtimes \overline{\mathfrak{D}}'_n \boxtimes \mathfrak{E}_n \boxtimes \overline{\mathfrak{E}}_n =$  $\mathbb{C}''_n \boxtimes \mathbb{E}'_n \boxtimes \overline{\mathbb{E}}'_n \boxtimes \mathcal{D}'_n \boxtimes \overline{\mathbb{D}}'_n$ . Thus  $\mathbb{C}_n \sim \mathbb{C}''_n$ .

We denote the set of equivalence classes of ndimensional BF categories under the quasi-equivalence relation  $\sim$  by  $\mathcal{A}^n$ , i.e.  $\mathcal{A}^n := \mathcal{M}^n / \sim$ . We denote the equivalence class of  $\mathcal{C}_n$  in  $\mathcal{A}^n$  by  $[\mathcal{C}_n]$ . We hope that the set  $\mathcal{A}^n$  can form a group under the tensor product  $\boxtimes$ .

From the definition of  $\sim$  we can easily see that

- (1) If  $\mathfrak{C}_n \sim \mathfrak{C}'_n$  and  $\mathfrak{D}_n \sim \mathfrak{D}'_n$ , then  $\mathfrak{C}_n \boxtimes \mathfrak{D}_n \sim \mathfrak{C}'_n \boxtimes \mathfrak{D}'_n$ . (2) For a closed BF category  $\mathfrak{C}_n$ ,  $\mathfrak{C}_n \boxtimes \overline{\mathfrak{C}}_n \sim \mathbf{1}_n$ .

Thus, the tensor product  $\boxtimes$  is compatible with the equivalence relation. In other words,  $[\mathcal{C}_n] \boxtimes [\mathcal{D}_n] := [\mathcal{C}_n \boxtimes \mathcal{D}_n]$  is a well-defined binary multiplication  $\boxtimes : \mathcal{A}^n \times \mathcal{A}^n \to \mathcal{A}^n$ , and the quotient map  $\mathcal{M}^n \to \mathcal{A}^n$  is a homomorphism between two commutative monoids. Moreover, for a closed BF category  $\mathcal{C}_n$ , the inverse of  $[\mathcal{C}_n]$  is given by  $[\overline{\mathcal{C}}_n]$ .

Also, we note that taking the bulk  $\mathcal{Z}_n(\cdot)$  is compatible with the equivalence relation that defines  $\mathcal{A}^n$ :

**Lemma 7:** If  $\mathfrak{C}_n \sim \mathfrak{C}'_n$ , then  $\mathcal{Z}_n(\mathfrak{C}_n) = \mathcal{Z}_n(\mathfrak{C}'_n)$ .

*Proof.*  $C_n \sim C'_n$  implies that  $C_n \boxtimes D_n \boxtimes \overline{D}_n = C'_n \boxtimes D'_n \boxtimes \overline{D}_n$  $\overline{\mathbb{D}}'_n$ . Thus  $\mathcal{Z}_n(\mathfrak{C}_n \boxtimes \mathfrak{D}_n \boxtimes \overline{\mathfrak{D}}_n) = \mathcal{Z}_n(\mathfrak{C}'_n \boxtimes \mathfrak{D}'_n \boxtimes \overline{\mathfrak{D}}'_n)$ , which implies that  $\mathcal{Z}_n(\mathcal{C}_n) \boxtimes \mathcal{Z}_n(\mathcal{D}_n \boxtimes \overline{\mathcal{D}}_n) = \mathcal{Z}_n(\mathcal{C}'_n) \boxtimes$  $\mathcal{Z}_n(\mathcal{D}'_n \boxtimes \overline{\mathcal{D}}'_n)$ . Since  $\mathcal{D}_n$  and  $\mathcal{D}'_n$  are closed, we have  $\mathcal{Z}_n(\mathfrak{D}_n \boxtimes \overline{\mathfrak{D}}_n) = \mathcal{Z}_n(\mathfrak{D}'_n \boxtimes \overline{\mathfrak{D}}'_n) = \mathbf{1}_{n+1}$ . Thus  $\mathcal{Z}_n(\mathfrak{C}_n) =$ 

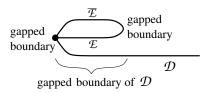


FIG. 11.  $\mathcal{D}_n \boxtimes \mathcal{E}_n \boxtimes \overline{\mathcal{E}}_n$  has a gapped boundary implies that  $\mathcal{D}_n$  has a gapped boundary if  $\mathcal{E}_n$  is closed.

As a result,  $\mathcal{Z}_n$  canonically induces a monoid homomorphism, also denoted by  $\mathcal{Z}_n$ , from  $\mathcal{A}^n$  to  $\mathcal{A}^{n+1}$ , such that the following diagram:

$$\mathcal{M}^{n} \xrightarrow{\mathcal{Z}_{n}} \mathcal{M}^{n+1}$$

$$\sim \bigvee_{\mathcal{A}^{n}} \xrightarrow{\mathcal{Z}_{n}} \mathcal{A}^{n}$$

is commutative.

From the definition of  $\sim$ , we can easily see that if  $\mathcal{C}_n \sim \mathcal{D}_n$  and  $\mathcal{C}_n$  is closed, then  $\mathcal{D}_n$  is also closed. This allows us to show that

**Lemma 8:** Let  $\mathcal{A}_{closed}^n$  be a subset consisting of the quasi-equivalence classes of all closed BF<sub>n</sub> categories. Then,  $\mathcal{A}_{closed}^n$  is an Abelian group under the stacking  $\boxtimes$  operation.

We can also show that

**Lemma 9:** If  $C_n \sim D_n$  and  $C_n$  is exact, then  $D_n$  is exact.

*Proof.* We note that  $\mathcal{C}_n \sim \mathcal{D}_n$  implies that there exist closed  $\mathcal{E}_n$  and  $\mathcal{E}'_n$  such that that  $\mathcal{C}_n \boxtimes \mathcal{E}'_n \boxtimes \overline{\mathcal{E}}'_n = \mathcal{D}_n \boxtimes \mathcal{E}_n \boxtimes \overline{\mathcal{E}}_n$ . Since  $\mathcal{C}_n$  is closed, so is  $\mathcal{D}_n$ . Since  $\mathcal{C}_n \boxtimes \mathcal{E}'_n \boxtimes \overline{\mathcal{E}}'_n$  is exact, so is  $\mathcal{D}_n \boxtimes \mathcal{E}_n \boxtimes \overline{\mathcal{E}}_n$  (*i.e.* has a gapped boundary). Then from Fig. 11, we can show that  $\mathcal{D}_n$  is exact.

Since the center of the dual is the dual of the center:  $\mathcal{Z}(\overline{\mathbb{C}}_n) = \overline{\mathcal{Z}(\mathbb{C}_n)}$ , therefore, for exact BF categories, we also have:

**Lemma 10:** Let  $\mathcal{A}_{\text{exact}}^n$  be the quasi-equivalence classes of all exact BF categories. Then,  $\mathcal{A}_{\text{exact}}^n$  is an Abelian group under the stacking  $\boxtimes$  operation.

Recall an elementary result in mathematics.

**Lemma 11:** Let  $f: X \to Y$  be a surjective homomorphism between two commutative monoids X and Y. If both ker f and Y are Abelian groups, so is X.

*Proof.* It is enough to show that any element x in X has a right inverse x', i.e. xx' = 1 (by the commutativity, x'x = xx' = 1). Notice that if such x' exists, it must be unique. Otherwise, let x' and x'' be so that xx' = 1 = xx''. Then we have x' = x'xx'' = x'' (using the associativity and commutativity). For any x in X, let y be the inverse of f(x) and let z be an element in X such that f(z) = y. Then we have f(xz) = f(x)f(z) = x'

f(x)y = 1. Therefore,  $xz \in \ker f$ . Since  $\ker f$  is an Abelian group, there is an element d such that xzd = 1. Hence zd is the right inverse of x.

Therefore, we obtain an important result.

**Proposition 1:** The set  $\mathcal{A}^n$  of quasi-equivalence classes of BF<sub>n</sub> categories form an Abelian group under the stacking  $\boxtimes$  operation. Moreover,  $\mathcal{Z}_n : \mathcal{A}^n \to \mathcal{A}^{n+1}$  is a group homomorphism.

## E. Witt equivalence relation

The quasi-equivalence relation is not the only equivalence relation available. In this subsection, we will discuss more equivalence relations, among which Witt equivalence relation is the most important one.

**Definition 24.** Two *n*-dimensional BF categories  $C_n$  and  $C'_n$  are called *k*-equivalent for  $k \geq n-1$  and denoted by  $C_n \stackrel{k}{\sim} C'_n$  if there exist *k*-dimensional BF categories  $D_k$  and  $D'_k$  such that

$$\mathfrak{C}_n \boxtimes \mathcal{Z}_{n-1}[P_{n-1}(\mathfrak{D}_k)] \simeq \mathfrak{C}'_n \boxtimes \mathcal{Z}_{n-1}[P_{n-1}(\mathfrak{D}'_k)].$$

An immediate consequence of above definition is  $\mathcal{Z}_n(\mathcal{C}_n) \simeq \mathcal{Z}_n(\mathcal{D}_n)$  if  $\mathcal{C}_n \stackrel{k}{\sim} \mathcal{D}_n$ . Namely, two k-equivalent BF categories  $\mathcal{C}_n$  and  $\mathcal{D}_n$  must share the same <u>bulk</u>.

**Remark 10.** When n=3, the 2-equivalence  $\stackrel{2}{\sim}$  for closed BF<sub>3</sub>-categories is the usual Witt equivalence relation<sup>71,99</sup>.

The k-equivalence  $\stackrel{k}{\sim}$  is indeed a well-defined equivalence relation follows from the following Lemma.

**Lemma 12:** For  $k \ge n$ , we have

- (1)  $\mathcal{C}_n \stackrel{k}{\sim} \mathcal{C}_n$ .
- (2)  $\mathcal{C}_n \stackrel{k}{\sim} \mathcal{C}'_n$  implies that  $\mathcal{C}'_n \stackrel{k}{\sim} \mathcal{C}_n$ .
- (3)  $\mathcal{C}_n \stackrel{k}{\sim} \mathcal{C}'_n$  and  $\mathcal{C}'_n \stackrel{k}{\sim} \mathcal{C}''_n$  imply that  $\mathcal{C}_n \stackrel{k}{\sim} \mathcal{C}''_n$ .

The k-equivalence relation is different for different k. A k-equivalence class is larger than a k+1-equivalence class. Namely, two (k+1)-equivalent BF $_n$ -categories is automatically k-equivalent. Namely,

$$\cdots \overset{k+1}{\sim} \Rightarrow \overset{k}{\sim} \Rightarrow \cdots \Rightarrow \overset{n}{\sim} \Rightarrow \overset{n-1}{\sim}.$$

Note that the *n*-equivalence relation is not the quasi-equivalence relation  $\sim$ . Instead, we have  $\sim \Rightarrow \stackrel{n}{\sim}$ . The  $\stackrel{n-1}{\sim}$  is also called the Witt equivalence, also denoted by  $\stackrel{\text{w}}{\sim}$ 

Using equation (16) and Lemma 4, we obtain immediately a few results from Definition 24:

**Lemma 13:** For  $k \ge n - 1$ ,

- (1) If  $\mathcal{C}_n \stackrel{k}{\sim} \mathcal{C}'_n$  and  $\mathcal{D}_n \stackrel{k}{\sim} \mathcal{D}'_n$ , then  $\mathcal{C}_n \boxtimes \mathcal{D}_n \stackrel{k}{\sim} \mathcal{C}'_n \boxtimes \mathcal{D}'_n$ .
- $(2) \, \mathcal{Z}_{n-1}(P_{n-1}(\mathfrak{C}_k)) \stackrel{k}{\sim} \mathbf{1}_n.$
- (3) If  $C_n$  is closed and  $C_n \stackrel{k}{\sim} D_n$ , then  $D_n$  is closed.

(4) If  $C_n$  is closed, then  $C_n \boxtimes \overline{C}_n \stackrel{k}{\sim} \mathbf{1}_n$  for k = n, n - 1. (5) If  $C_n$  is exact, i.e.  $C_n \simeq \mathcal{Z}_{n-1}(\mathcal{D}_{n-1})$  for some  $\mathcal{D}_{n-1}$ , then  $C_n \boxtimes \mathcal{Z}_{n-1}(\overline{\mathcal{D}_{n-1}}) \stackrel{k}{\sim} \mathbf{1}_n$  for k = n, n - 1.

We give a sufficient condition for the equivalence relation  $\stackrel{n}{\sim}$  among closed BF categories below.

**Proposition 2:** Let  $\mathcal{C}_n$  and  $\mathcal{D}_n$  be n-dimensional BF categories. If  $\mathcal{C}_n$  (or  $\mathcal{D}_n$ ) is closed and there exists an n-dimensional BF category  $\mathcal{E}_n$  such that  $\mathcal{C}_n \boxtimes \overline{\mathcal{D}}_n \simeq \mathcal{Z}_{n-1}(P_{n-1}(\mathcal{E}_n))$ , then  $\mathcal{D}_n$  (or  $\mathcal{C}_n$ ) is closed and  $\mathcal{C}_n \stackrel{n}{\sim} \mathcal{D}_n$ .

*Proof.* If  $C_n$  is closed, then we have

$$\mathcal{Z}_n(\mathfrak{D}_n) \simeq \mathcal{Z}_n(\mathfrak{C}_n \boxtimes \overline{\mathfrak{D}}_n) \simeq \mathcal{Z}_n(\mathcal{Z}_{n-1}(P_{n-1}(\mathfrak{E}_n))) \simeq \mathbf{1}_{n+1}.$$

Namely,  $\mathcal{D}_n$  is closed. Multiplying  $\mathcal{D}_n$  on the both side of the condition  $\mathcal{C}_n \boxtimes \overline{\mathcal{D}}_n \simeq \mathcal{Z}_{n-1}(\mathcal{E}_n)$  and applying the 3rd result in Lemma 4, we obtain

$$\mathfrak{C}_n \boxtimes \mathcal{Z}_{n-1}(P_{n-1}(\mathfrak{D}_n)) \simeq \mathfrak{D}_n \boxtimes \mathcal{Z}_{n-1}(P_{n-1}(\mathfrak{E}_n)),$$

i.e. 
$$\mathcal{C}_n \stackrel{n}{\sim} \mathcal{D}_n$$
.

For Witt equivalence relation, we denote the set of equivalence classes of n-dimensional BF categories under the equivalence relation  $\overset{\text{w}}{\sim}$  as  $\mathcal{A}^n_{\text{w}}$ , i.e.  $\mathcal{A}^n_{\text{w}} := \mathcal{M}^n / \overset{\text{w}}{\sim}$ . The set  $\mathcal{A}^n_{\text{w}}$  is a quotient of  $B^n$ . We denote the equivalence class of  $\mathcal{C}_n$  in  $\mathcal{A}^n_{\text{w}}$  by  $[\mathcal{C}_n]_{\text{w}}$ .

**Proposition 3:** The set  $\mathcal{A}_w^n$ , together with the multiplication given by the stacking  $\boxtimes$  operation and the unit element  $[\mathbf{1}_n]_w$ , is an Abelian group. The inverse is given by the dual of a BF category. Moreover,  $\mathcal{Z}: \mathcal{A}_w^n \to \mathcal{A}_w^{n+1}$  is a group homomorphism.

The Witt equivalence between two closed  $BF_n$  categories can also be understood in a different way. We introduce another equivalence relation.

**Definition 25.** Two BF<sub>n</sub> categories  $\mathcal{C}_n$  and  $\mathcal{D}_n$  are called gw-equivalent, denoted by  $\mathcal{C}_n \stackrel{\text{gw}}{\sim} \mathcal{D}_n$ , if there exists an (n-1)-dimensional BF category  $\mathcal{E}_{n-1}$  such that

$$\mathfrak{C}_n \boxtimes \overline{\mathfrak{D}}_n \simeq \mathcal{Z}_{n-1}(\mathcal{E}_{n-1}). \tag{18}$$

**Remark 11.** When both  $\mathcal{C}_n$  and  $\mathcal{D}_n$  are closed, the physical meaning of gw-equivalence is clear. It means that  $\mathcal{C}_n$  and  $\mathcal{D}_n$  are gw-equivalent if and only if they can be connected by a gapped domain wall.

**Lemma 14:** Three defining properties of an equivalence relation (recall Lemma 12) hold. Namely,  $\stackrel{gw}{\sim}$  is a well-defined equivalence relation.

Proposition 2 implies the following result.

**Lemma 15:** If  $C_n$  (or  $D_n$ ) is closed, then  $C_n \stackrel{\text{gw}}{\sim} D_n$  implies that  $D_n$  (or  $C_n$ ) is closed and  $C_n \stackrel{\text{w}}{\sim} D_n$ .

Conversely, we will show that  $\stackrel{\text{w}}{\sim}$  also implies  $\stackrel{\text{gw}}{\sim}$ . We will start first prove an important lemma.

**Lemma 16:**  $\mathcal{C}_{n+1} \stackrel{\text{w}}{\sim} \mathbf{1}_{n+1}$  if and only if  $\mathcal{C}_{n+1}$  is exact.

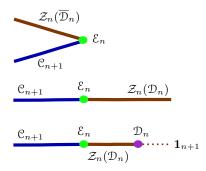


FIG. 12. The three steps in the proof of Lemma 16, i.e.  $C_n \stackrel{\text{W}}{\sim} 1_n$  iff  $C_n$  is exact.

*Proof.* If  $C_{n+1}$  is exact, it is obvious that  $C_{n+1} \stackrel{\text{w}}{\sim} \mathbf{1}_{n+1}$ . Conversely, by definition,  $C_{n+1} \stackrel{\text{w}}{\sim} \mathbf{1}_{n+1}$  means that there are  $D_n$  and  $\mathcal{E}_n$  such that

$$\mathcal{C}_{n+1} \boxtimes \mathcal{Z}_n(\overline{\mathcal{D}}_n) \simeq \mathcal{Z}_n(\mathcal{E}_n).$$

It implies  $\mathcal{C}_{n+1}$  is closed. Moreover, its physical meaning is that the (n+1)-dimensional topological order  $\mathcal{Z}_n(\mathcal{E}_n)$ , as the <u>bulk</u> of an *n*-dimensional boundary  $\mathcal{E}_n$ , can be factorized as a double-layered system  $\mathcal{C}_{n+1} \boxtimes \mathcal{Z}_n(\mathcal{D}_n)$  (see Fig. 12). By unfolding this double-layered system along its  $\mathcal{E}_m$ -boundary, we obtain two (n+1)-dimensional topological orders  $\mathcal{C}_{n+1}$  and  $\mathcal{Z}_n(\mathcal{D}_n)$ , which are connected by a gapped n-dimensional domain wall  $\mathcal{E}_n$ . Since the topological order  $\mathcal{Z}_n(\mathfrak{D}_n)$  itself allows a gapped boundary, we are able to condense  $\mathcal{Z}_n(\mathcal{D}_n)$  to the trivial phase  $\mathbf{1}_{n+1}$ . This condensation creates a gapped boundary given by  $\mathfrak{D}_n$  and a narrow band bounded by  $\mathcal{E}_n$  and  $\mathfrak{D}_n$ . This narrow band connects the topological order  $\mathcal{C}_n$  to the trivial order  $\mathbf{1}_{n+1}$  (see Fig. 12). Therefore, this narrow band should be viewed as an n-dimensional gapped boundary, which is of type  $\mathcal{E}_n \boxtimes_{\mathcal{Z}_n(\mathcal{D}_n)} \mathcal{D}_n$ , of an (n+1)dimensional bulk phase  $\mathcal{C}_{n+1}$ . Therefore, we must have  $\mathcal{C}_{n+1} \simeq \mathcal{Z}_n(\mathcal{E}_n \boxtimes_{\mathcal{Z}_n(\mathcal{D}_n)} \mathcal{D}_n).$ 

**Proposition 4:** For two closed BF categories  $\mathcal{C}_n$  and  $\mathcal{D}_n$ ,  $\mathcal{C}_n \stackrel{\text{w}}{\sim} \mathcal{D}_n$  if and only if  $\mathcal{C}_n \stackrel{\text{gw}}{\sim} \mathcal{D}_n$ , or equivalently, if and only if they are connected by an (n-1)-dimensional gapped domain wall.

*Proof.* By Lemma 15, it is enough to show that  $\overset{\text{w}}{\sim}$  implies  $\overset{\text{gw}}{\sim}$ . By Proposition 3,  $\mathcal{C}_n \overset{\text{w}}{\sim} \mathcal{D}_n$  implies that  $[\mathcal{C}_n]_{\text{w}} \boxtimes [\overline{\mathcal{D}}_n]_{\text{w}} = [\mathbf{1}_n]_{\text{w}}$ . By Lemma 16, there exists an (n-1)-dimensional BF category  $\mathcal{E}_{n-1}$  such that  $\mathcal{C}_n \boxtimes \overline{\mathcal{D}}_n \simeq \mathcal{Z}_{n-1}(\mathcal{E}_{n-1})$ , which means  $\mathcal{C}_n \overset{\text{gw}}{\sim} \mathcal{D}_n$ .

# IX. THE COCHAIN COMPLEX OF THE BF CATEGORIES

The BF categories in the same dimension form a commutative monoid, which is denoted by  $\mathcal{M}^n$ . The bulk of a boundary defines a homomorphism  $\mathcal{Z}_n: \mathcal{M}^n \to \mathcal{M}^{n+1}$  between commutative monoids. These homomorphisms

 $\mathcal{Z}_n$  for all non-negative integers n satisfy the property that  $\mathcal{Z}_{n+1}(\mathcal{Z}_n(\mathcal{M}^n)) \simeq \mathbf{1}_{n+2} \in \mathcal{M}^{n+2}$ . In other words,  $\mathcal{Z}_n$  is a differential operator in a cochain complex. Therefore, we obtain a commutative-monoid-valued cochain complex

$$\cdots \xrightarrow{\mathcal{Z}_{n-1}} \mathcal{M}^n \xrightarrow{\mathcal{Z}_n} \mathcal{M}^{n+1} \xrightarrow{\mathcal{Z}_{n+1}} \mathcal{M}^{n+2} \xrightarrow{\mathcal{Z}_{n+2}} \cdots . \tag{19}$$

Because a non-trivial BF category does not have an inverse in  $\mathcal{M}^n$ ,  $\mathcal{M}^n$  is not an Abelian group in general. Therefore, (19) is not a usual cochain complex (which is valued in Abelian groups).

However, the sets  $\mathcal{A}^n$  of the equivalence classes of BF<sub>n</sub> categories in different dimensions do form cochain complex because  $\mathcal{A}^n$  are Abelian groups:

**Theorem 1:** The sets  $\mathcal{A}^n$  of the equivalence classes of BF categories for all n, together with the group homomorphisms  $\mathcal{Z}_n$ , form a cochain complex:

$$\cdots \xrightarrow{\mathcal{Z}_{n-1}} \mathcal{A}^n \xrightarrow{\mathcal{Z}_n} \mathcal{A}^{n+1} \xrightarrow{\mathcal{Z}_{n+1}} \mathcal{A}^{n+2} \xrightarrow{\mathcal{Z}_{n+2}} \cdots$$
 (20)

Remark 12. Since we don't yet have a precise mathematical definition of a BF category, above theorem should be understood as a physical theorem of the equivalence classes of topological orders.

We can define the n-th cohomology group as usual.

**Definition 26.**  $H^n := \ker(\mathcal{Z}_n)/\operatorname{img}(\mathcal{Z}_{n-1}).$ 

Similarly, we have the following result for  $\mathcal{A}_{w}^{n}$ :

**Theorem 2:** The sets  $\mathcal{A}_{w}^{n}$  of the equivalence classes of BF categories for all n, together with the group homomorphisms  $\mathcal{Z}_{n}$ , form a cochain complex:

$$\cdots \xrightarrow{\mathcal{Z}_{n-1}} \mathcal{A}_{\mathbf{w}}^{n} \xrightarrow{\mathcal{Z}_{n}} \mathcal{A}_{\mathbf{w}}^{n+1} \xrightarrow{\mathcal{Z}_{n+1}} \mathcal{A}_{\mathbf{w}}^{n+2} \xrightarrow{\mathcal{Z}_{n+2}} \cdots . \tag{21}$$

We can define the n-th cohomology group as usual.

**Definition 27.**  $H_w^n := \ker(\mathcal{Z}_n)/\operatorname{img}(\mathcal{Z}_{n-1}).$ 

In fact  $\mathcal{Z}_n(\mathcal{A}_{\mathrm{w}}^n) = \mathbf{1}_{n+1}$  and  $\mathcal{H}_{\mathrm{w}}^n = \mathcal{A}_{\mathrm{w}}^n$ . By Theorem 4, two closed BF<sub>n</sub> categories  $\mathcal{C}_n$  and  $\mathcal{D}_n$  belong to the same class in  $\mathcal{H}_{\mathrm{w}}^n$  iff the boundary between  $\mathcal{C}_n$  and  $\mathcal{D}_n$  can be gapped. Then it is clear that the *n*-th cohomology group  $\mathcal{H}_{\mathrm{w}}^n$  classify the types of (n-1) space-time dimensional gapless boundaries.

When n = 3, closed BF<sub>3</sub> categories are unitary modular tensor categories (UMTC), and exact BF<sub>3</sub> categories are those monoidal centers of unitary fusion categories. So the cohomology group  $H_w^3$  is nothing but the Witt group<sup>99</sup> for UMTCs. It classifies the types of 2 spacetime dimensional gapless boundaries. It is not surprising that the Witt group was originally introduced to classify 2-dimensional rational conformal field theories<sup>99</sup>.

# X. GENERAL EXAMPLES OF LOW DIMENSIONAL BF<sup>H</sup> CATEGORIES WITH ONLY PARTICLE-LIKE EXCITATIONS

In this section, we will discuss some general examples of BF<sup>H</sup> categories in low dimensions. Since only defects of codimension 2 are detectable by braiding with other excitations, in the cases that spatial dimension is not more than 2, we can only detect particle-like excitations via braiding. So we will restrict ourselves to only those particle-like excitations in a BF<sup>H</sup> category and ignore higher dimensional defects. For simplicity, we will also ignore all those 0-dimensional defect nested on non-trivial higher dimension defects, such as the one depicted as the blue point in Fig.13. Higher dimensional excitations or defects will be studied in Section XI.

## A. 0+1D topological phases

A 0+1D topological phase is just a quantum mechanics system. It is given by a finite dimensional Hilbert space V equipped with the local operator algebra  $A = \operatorname{End}(V)$ . The data V is redundant and can be recovered from A. Therefore, a 0+1D topological phase can be described by a category with a single object \* and  $\operatorname{hom}(*,*) = A$ . Equipping A with the operator normal, we can turn it into a  $C^*$ -algebra.

## B. Particles in 1+1D topological phases

In a 1+1D phase, there is no braiding between topological particles. So this phase is characterized by the following data: (for a detailed discussion see Ref. 11, 100, and 101):

- (1) An integer N that describes the number of nontrivial types of particle-like topological excitations.
- (2) An one-to-one map  $i \to i^*$ ,  $i, i^* = 0, 1, \dots, N$  that satisfy  $0 = 0^*$  and  $(i^*)^* = i$ .
- (3) A rank-3 tensor  $N_k^{ij}$  that describes dimension of the fusion spaces of the topological excitations. Moreover,  $N_{0i}^j = \delta_{ij} = N_{i0}^j$  and  $N_k^{ij}$  satisfies an associativity property.
- (4) A rank-10 tensor  $F_{l,n\gamma\lambda}^{ijk,m\alpha\beta}$  satisfies the pentagon identities. It describes the linear relations between the fusion spaces of the topological excitations.

Using categorical language, above data amounts to a unitary fusion category  $\mathbb C$  of topological excitations (simple types or or composite types), with only finite many simple types  $i,j,k\in I$  where |I|=N+1. The hom space  $\hom_{\mathbb C}(X,Y)$  is a finite dimensional Hilbert space for any two excitations X and Y. Moreover, fusion of two simple excitations i and j give the tensor product  $i\otimes j$ . This fusion product  $i\otimes j$  is completely determined by

$$N_{ij}^k = \dim \hom_{\mathcal{C}}(i \otimes j, k). \tag{22}$$

In particular,  $N_{0i}^j = \delta_{ij} = N_{i0}^j$  implies that  $0 \otimes i = i = i \otimes 0$ . 0 is the tensor unit **1** of C. In particular,  $N_{00}^0 = 1$  means that the vacuum degeneracy of the vacuum is trivial. In more categorical language, we have

$$\dim \hom_{\mathcal{C}}(\mathbf{1}, \mathbf{1}) = 1. \tag{23}$$

These structures  $(\mathcal{C}, \otimes, 0)$ , together with the rank-10 tensor  $F_{l,n\gamma\lambda}^{ijk,m\alpha\beta}$  satisfying the pentagon identities, equip the unitary category  $\mathcal{C}$  with a structure of a monoidal or tensor category. The existence of anti-particle  $i^*$  for all i further implies that  $\mathcal{C}$  is also rigid. Combining all of these results, we have shown that  $\mathcal{C}$  has a structure of a unitary fusion category (UFC). Notice that the hom space should be viewed as instantons in time direction.

Among all UFC's, the most trivial one is the category  $\mathcal{H}$ ilb of finite dimensional Hilbert spaces. By Ref. 26, any UFC  $\mathcal{C}$  can be realized as the boundary excitations of a Levin-Wen type of lattice model with bulk excitations given by the category  $\mathcal{Z}(\mathcal{C})$ , which is the monoidal center of  $\mathcal{C}$ . Mathematically, it is well-known that  $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{H}$ ilb if and only if  $\mathcal{C} \simeq \mathcal{H}$ ilb. According to our general theory of bulk-boundary relation in Section VII, the only anomalous free (or closed) BF<sub>2</sub> category is  $\mathcal{H}$ ilb. It is also clear that the trivial 1+1-dimensional phase is also the bulk of a trivial 0+1-dimensional phase. Therefore,  $\mathcal{H}$ ilb is the only closed and exact BF<sub>2</sub> category (which can be composite BF<sub>2</sub> categories. See Appendix B).

### C. Particles in 2+1D topological phases

In a 2+1D topological phase, particles can fuse with each other and also braid with each other. We will now list the ingredients in these fusion and braiding structures (for a more physical description of some of the following properties, see Section  $\overline{VB}$ ):

- 1. There is a finite set I of anyons. An anyon  $i \in I$  corresponds to a simple object in a category  $\mathbb{C}$ . A generic object is a direct sum of simple objects, e.g.  $i \oplus j \oplus k$  for  $i, j, k \in I$ , which corresponds to superposition of anyons.
- 2. Between two generic objects X and Y, there are fusion-splitting channels which forms a vector space over  $\mathbb{C}$ :  $\hom(X,Y)$ . In particular, the vector space  $\hom(i,X)$  tells us how many ways the simple anyon i can fuse into a generic object X and the vector space  $\hom(X,i)$  tells us how many splitting channels from X to i.
- 3. The unitarity of the anyon system is a physical requirement. It immediately implies that the category  $\mathcal{C}$  has to be semisimple.
- 4. The fusion of two objects X and Y gives arise to a tensor product  $X \otimes Y$  which must be associative and unital. The tensor unit **1** is given by the vacuum.
- 5. The mutual statistics among anyons is given by the braiding  $c_{X,Y}: X \otimes Y \to Y \otimes X$  for all  $X,Y \in \mathcal{C}$ . This information of braiding is encoded in the physically measurable linear map:  $c_{X,Y}: \hom(i,X \otimes Y) \to \hom(i,Y \otimes X)$ .

- 6. Anyons can be created or annihilated from the vacuum in pairs. In particular, we need a dual object  $X^{\vee}$  for each anyon  $X \in \mathcal{C}$ , together with morphisms  $\operatorname{ev}_X : X^{\vee} \otimes X \to \mathbf{1}$ ,  $\operatorname{coev}_X : \mathbf{1} \to X \otimes X^{\vee}$  and their adjoints  $\operatorname{ev}_X^*$  and  $\operatorname{coev}_X^*$ , satisfying the some natural condition. This says that  $\mathcal{C}$  must be a rigid tensor category.
- 7. Each anyon has spins. It amounts to an automormphism  $\theta_X: X \to X$  satisfying some properties. This requires  $\mathcal{C}$  to be a ribbon category.

In summary, above braiding-fusion structures of a system of anyons amount to a unitary premodular category. The unitary premodular categories are BF<sub>3</sub> categories. However, unitary premodular categories only represent a subset of 3-dimensional BF<sub>3</sub> category. The anomalous BF<sub>3</sub> category  $\mathcal{C}_3^{sF\mathbb{Z}_2}$  with only string-like topological excitations (see Example 11 and Section C 3 c) is not a unitary premodular category. We will discuss those string-like excitations in Section XI.

In general, a unitary premodular category is anomalous (*i.e.* not realizable by 2+1D qubit models). We have the following result characterizing the anomalous free unitary premodular categories.

**Theorem 3:** If a unitary premodular category  $\mathbb{C}$  is not anomalous, *i.e.* if it can be realized by a 2 + 1D local Hamiltonian qubit system, then  $\mathbb{C}$  must be modular.

*Proof.* If a 2+1 anyon system can be defined in 2+1-dimension, all its particles should be detectable by the braiding among themselves. As a consequence, if an anyon X is such that its mutual braidings with all other anyons are trivial, i.e.  $c_{X,Y} \circ c_{Y,X} = \mathrm{id}_{X \otimes Y}$  for all  $Y \in \mathcal{C}$ , then X must be uniquely fixed by this property. On the other hand, we know that the mutual braidings between the vacuum  $\mathbf{1}$  and any other object Y is trivial. Therefore, we must have  $X = \mathbf{1}$ . In other words,  $\mathcal{C}$  is modular.

**Remark 13.** We note that the UMTC's are closed  $BF_3^H$  categories (the path integral can be defined on mapping tori – fiber bundles over  $S^1$ ). They are also closed  $BF_3^L$  categories (the path integral can be defined for any oriented space-time topologies). This fact is very subtle and will be discussed in Sections XIII A 3 and XIII C.

## XI. A MATHEMATICAL DEFINITION OF A $BF_{n+1}$ -CATEGORY

We have described the examples and the general structures of a BF category without giving it a precise mathematical definition. In this section, we will try to outline a mathematical definition of a BF category. Since our understanding of topological orders in high dimensions at the current stage is very limited, many assumptions and conjectures are imposed in order to proceed. The mathematical definition we obtained is conjectural and

far from being complete. In particular, coherence morphisms in higher categories and their properties are ignored. As we will show, many physically interesting and important questions, which can be studied or formulated in this framework, are quite irrelevant to the structures we have ignored. Indeed, by focusing on the fusion and braiding structures alone, we are leading to a very rich theory. The precise definition of a BF category is not important to us at this stage. But many important questions and conjectures formulated in our framework will serve as a blueprint for future studies.

#### A. Defects of all dimensions and higher categories

Topological excitations can also be viewed as defects in a topological phase. In this section, we will use the term "excitation" and "defect" interchangeably. We will use "a domain wall" to refer to a 1-codimensional defect, or more generally, a defect of 1-lower dimension (1-higher codimension), and domain walls between domain walls for defects of 2-lower dimension.

The main difficulty in describing an (n+1)-dimensional BF-category precisely is the existence of topological excitations in different dimensions. Excitations in different dimensions carry different level of richness of structures. For example, a p-dimensional excitation can have particle-like excitations nested in it. Moreover, they can be fused and braided within the p-dimensional excitation. An higher dimensional excitation has much richer structures than a particle-like excitation. So it is clear that defects of different dimensions should belong to different layers in a multi-layered structure. It suggests us to arrange topological excitations according to their codimensions: at the 0-th level, there is a unique n-spatial dimensional bulk phase; the first level, there are domain walls or defects of codimension 1; at the second-level, there are walls between walls (or defects of codimension 2); at the n-th level, there are particle-like excitations (or n-codimensional defects); at the (n + 1)-th level, there are instantons (or (n + 1)-codimensional defects). This multi-layered structure coincides exactly with that of an (n+1)-category with one object. More precisely, the unique object (or 0-morphism) corresponds to the bulkphase; 1-morphisms correspond to the domain walls; 2morphisms between a pair of 1-morphisms correspond to walls between walls; ..., (n+1)-morphisms correspond to instantons.

**Remark 14.** The only reason that the notion of a phase (or order) of matter was invented is because there are phase transitions. A topological order x should be uniquely determined by its relation or "phase transition" to all topological orders, including x itself. This relation between two (n+1)-dimensional phases x and y can be characterized by all possible n-dimensional domain walls. In this work, we only consider gapped domain walls because gapless phases are much richer than the finite category theory. In category theory, the relation between

two objects is encoded by morphisms between them, and an object x in a category  $\mathcal{C}$  can be determined uniquely (up to isomorphisms) by a family of sets of morphisms  $\{\hom_{\mathcal{C}}(y,x)\}_{y\in\mathcal{C}}$  and maps between them. This is called Yoneda Lemma in category theory. Therefore, we should consider the category of (n+1)-dimensional topological orders with morphisms given by domain walls. According to the philosophy of Yoneda lemma, a topological order should be characterized completely by the domain walls between itself and all topological orders. Moreover, notice that a domain wall is itself a topological phase. There are domain wall between domain walls and domain walls between domain walls between domain walls. Each time the dimension of the domain walls is reduced by one until we reach the instantons which is a localized defect in the time direction. As a consequence, the category of all (n+1)-dimensional topological orders, denoted by  $\mathfrak{BF}_{n+1}$ , must be an (n+1)-category with 1-morphisms given by domain walls, 2-morphisms given by domain walls between domain walls, ..., (n+1)-morphisms given by instantons. To determine a given topological order x, the information of all domain walls, although sufficient, are too large to work with. A small part of it is given by all the domain walls between a phase x and itself, or the full subcategory of  $\mathfrak{BF}_{n+1}$  supported on x, denoted by  $\hat{x}$ . This small part is nothing but a phase x with gapped defects of all dimensions discussed in the previous paragraph. We believe that it is rich enough to characterize the topological phase x uniquely.

At the current stage, an n-category is nothing but a name for a multi-layered structure. This mathematical notion contains many more structures. But whether these extra structures are relevant to topological order is not entirely clear. We will explain what additional structures are needed for a topological order by first looking at a simple example: the toric model<sup>88</sup> (a  $\mathbb{Z}_2$  spin liquid<sup>17–19</sup>).

### B. The toric code model enriched by defects

In this section, we will illustrate the additional structures that are needed, in particular, the fusion and braiding structures, in the toric code model. We will also explain its relation to other topological phases in  $\mathcal{BF}_{2+1}$  such as the trivial phase. For convenience, we will refer to a subcategory of  $\mathcal{BF}_{n+1}$  consists of a finite many objects as a multi-BF $_{n+1}$ -category or an MBF $_{n+1}$ -category.

The toric code model is a 2-dimensional lattice model depicted in Fig. 13. In this subsection, we will review the results in Ref. 26 in terms of a 3-category. We will also use the language used in Ref. 26 freely. Let  $\Re p_{\mathbb{Z}_2}$  be the category of representations of the  $\mathbb{Z}_2$  group. It is a unitary fusion category. In the language of Levin-Wen model, the bulk lattice is, by construction, determined by the unitary fusion category  $\Re p_{\mathbb{Z}_2}$ , thus will be referred to as an  $\Re p_{\mathbb{Z}_2}$ -bulk. If there is a domain wall, it was

shown in Ref. 26 that the lattice model near the domain wall can be constructed from an indecomposable  $\Re \exp_{\mathbb{Z}_2}$ -bimodule category X. All (bi)modules over a unitary fusion category are assumed to be semisimple. We will refer to such lattice near the domain wall as an X-wall. Similarly, if there is a gapped boundary, or equivalently, a gapped domain wall between the toric code model and the trivial phase, the lattice model near the boundary can be constructed from a  $\Re \exp_{\mathbb{Z}_2}$ -module Y and will be referred to as a Y-boundary.

The trivial phase can be viewed as a Levin-Wen model based on the unitary fusion category  $\mathcal{H}$ ilb of finite dimensional Hilbert spaces.

The toric code model gives a 3-category, denoted by  $TC_3$ , in its full complexity. Actually, it is quite convenient and perhaps more illustrative to describe a slightly larger 3-category,  $TC_3^b$ , which contains two objects: the toric code model and the trivial phase.  $TC_3$  can be obtained as sub-category of  $TC_3^b$ . There are four layers of structures in  $TC_3^b$ .

- Objects or 0-morphisms: the trivial phase, denoted by 1 and toric code denoted by tc.
- 1-morphism: There are 4 types of 1-morphisms given by various types of defect lines or domain walls:
  - 1. 1-morphisms  $tc \to tc$  are given by domain walls between two  $\Re \operatorname{ep}_{\mathbb{Z}_2}$ -bulks. They are classified by  $\Re \operatorname{ep}_{\mathbb{Z}_2}$ -bimodules. The trivial wall is the  $\Re \operatorname{ep}_{\mathbb{Z}_2}$ -wall, where  $\Re \operatorname{ep}_{\mathbb{Z}_2}$  is viewed as an  $\Re \operatorname{ep}_{\mathbb{Z}_2}$ -bimodule. An example of non-trivial domain wall is given by the  $\Re \operatorname{ep}_{\mathbb{Z}_2}$ -bimodule:

$$_{\mathcal{R}ep_{\mathbb{Z}_2}}\mathcal{H}ilb_{\mathcal{R}ep_{\mathbb{Z}_2}},$$

which is the category of finite dimensional Hilbert spaces and is depicted as the dotted line in Fig. 13. The trivial  $\Re p_{\mathbb{Z}_2}$ -wall can be any line other than the dotted line in the lattice, in particular it can be the vertical line connecting to the dotted line via the blue point in Fig. 13.

There are more simple  $\Re \operatorname{\mathsf{Rep}}_{\mathbb{Z}_2}$ -bimodules. For example, there is another  $\Re \operatorname{\mathsf{Rep}}_{\mathbb{Z}_2}$ -bimodule structure on  $\Re \operatorname{\mathsf{Hilb}}$ .

- 2. There are two simple 1-morphisms tc  $\rightarrow$  1. These two 1-morphisms correspond to two types of boundaries: the  $_{\mathcal{R}ep_{\mathbb{Z}_2}}\mathcal{H}ilb_{\mathcal{H}ilb}$ -boundary, which is called "rough boundary" in Fig. 13, and the  $_{\mathcal{R}ep_{\mathbb{Z}_2}}(\mathcal{R}ep_{\mathbb{Z}_2})_{\mathcal{H}ilb}$ -boundary, which is called "smooth boundary" in Fig. 13.
- 3. There are two simple 1-morphisms from  $\mathbf{1} \to \mathrm{tc}$  given by the bimodule  $\mathrm{Hilb}\mathrm{Hilb}_{\mathrm{Rep}_{\mathbb{Z}_2}}^{\mathrm{op}}$  and  $\mathrm{Hilb}(\mathrm{Rep}_{\mathbb{Z}_2}^{\mathrm{op}})_{\mathrm{Rep}_{\mathbb{Z}_2}}$ .
- 4. 1-morphisms  $\mathbf{1} \to \mathbf{1}$  are domain walls in the trivial phase. The only simple one is the trivial wall given by the  $\mathcal{H}$ ilb-bimodule  $\mathcal{H}$ ilb $\mathcal{H}$ ilb $\mathcal{H}$ ilb $\mathcal{H}$ ilb.

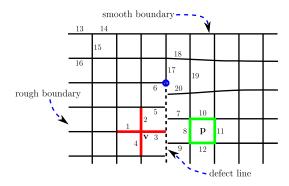


FIG. 13. Toric code with boundaries of two types, a transparent domain wall (the dotted line) and a 0-d defect depicted by the blue point at the corner of edge labeled by 6 and 17.

- 2-morphisms: 2-morphisms are defects of codimension 2. These point-like defects (for example the bulk point in Fig. 13) are completely classified in Ref. 26 by bimodule functors. For example,
  - 2-morphism 1-morphism 1. a from the  $_{\mathbb{R}\mathrm{ep}_{\mathbb{Z}_2}}(\mathbb{R}\mathrm{ep}_{\mathbb{Z}_2})_{\mathbb{R}\mathrm{ep}_{\mathbb{Z}_2}}$  to itself is given by a bimodule functor from  $\mathbb{R}\mathrm{ep}_{\mathbb{Z}_2}$  to  $\mathbb{R}\mathrm{ep}_{\mathbb{Z}_2}$ . Since  $_{\mathbb{R}ep_{\mathbb{Z}_2}}(\mathbb{R}ep_{\mathbb{Z}_2})_{\mathbb{R}ep_{\mathbb{Z}_2}}$  is a trivial defect line, a pointlike defect on such defect line should be nothing but a bulk excitation. Therefore, the bulk excitations, or defects of codimension 2 on the trivial domain wall  ${\mathcal R}\mathrm{ep}_{{\mathbb Z}_2},$  are classified by the objects in the category  $\widetilde{\mathcal{F}}\mathrm{un}_{\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}|\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}}(\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2},\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2})$  of bimodule functors from  $\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}$  to  $\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}$ . There are 4 such bimodule functors that are simple. They are denoted by  $1, e, m, \epsilon$ . The category  $\begin{array}{lll} \operatorname{\mathcal{F}un}_{\operatorname{\mathcal{R}ep}_{\mathbb{Z}_2}\mid\operatorname{\mathcal{R}ep}_{\mathbb{Z}_2}}(\operatorname{\mathcal{R}ep}_{\mathbb{Z}_2},\operatorname{\mathcal{R}ep}_{\mathbb{Z}_2}) & \text{is also called} \\ \operatorname{monoidal\ center\ of\ the\ monoidal\ category\ } \operatorname{\mathcal{R}ep}_{\mathbb{Z}_2}, \end{array}$ often denoted by  $Z(\operatorname{Rep}_{\mathbb{Z}_2})$ .
  - 2. Similarly, 2-morphisms from 1-morphism  $\Re_{\operatorname{Rep}_{\mathbb{Z}_2}} \operatorname{Hilb}_{\operatorname{Rep}_{\mathbb{Z}_2}}$  to itself is given by objects in  $\operatorname{Fun}_{\operatorname{Rep}_{\mathbb{Z}_2}|\operatorname{Rep}_{\mathbb{Z}_2}}(\operatorname{Hilb},\operatorname{Hilb})$ , which is actually equivalent to  $Z(\operatorname{Rep}_{\mathbb{Z}_2})$  as monoidal categories. Namely, it also contains four simple objects, corresponding to four simple wall excitations.
  - 3. 2-morphisms from 1-morphism  $\Re \operatorname{ep}_{\mathbb{Z}_2}$  to  $\mathcal{H}$ ilb are given by objects in  $\operatorname{\mathcal{F}un}_{\operatorname{\mathfrak{R}ep}_{\mathbb{Z}_2}|\operatorname{\mathcal{R}ep}_{\mathbb{Z}_2}}(\operatorname{\mathcal{R}ep}_{\mathbb{Z}_2}, \operatorname{\mathcal{H}ilb})$ . An example of such 2-morphism is depicted as the lattice configuration around the blue point in Fig. 13. There is a stabilizer operator

$$Q = \sigma_6^x \sigma_{17}^y \sigma_{18}^z \sigma_{19}^z \sigma_{20}^z, \tag{24}$$

which commutes with other stabilizers (see eq.(8) in Ref. 26). Two eigenvalues of Q correspond to two distinct simple 2-morphisms, or two simple objects  $F_{\pm}$  in  $\operatorname{\mathcal{F}un}_{\operatorname{\mathfrak{Rep}}_{\mathbb{Z}_2}|\operatorname{\mathfrak{Rep}}_{\mathbb{Z}_2}}(\operatorname{\mathfrak{Rep}}_{\mathbb{Z}_2}, \operatorname{\mathcal{H}ilb})$ .

4. 2-morphisms from 1-morphism  $\mathcal{H}$ ilb to  $\mathbb{R}ep_{\mathbb{Z}_2}$  are given by objects in  $\mathbb{F}un_{\mathbb{R}ep_{\mathbb{Z}_2}|\mathbb{R}ep_{\mathbb{Z}_2}}(\mathcal{H}ilb, \mathbb{R}ep_{\mathbb{Z}_2})$ .

There are again two simple 2-morphisms given by  $\overline{F}_{\pm}$  which is the two-sides adjoint of  $F_{\pm}$ .

- 5. 2-morphisms from 1-morphisms  $_{\mathcal{R}ep_{\mathbb{Z}_2}}(\mathcal{R})_{\mathcal{H}ilb}$  to itself, if  $\mathcal{R}=\mathcal{R}ep_{\mathbb{Z}_2}$  or  $\mathcal{H}ilb$ , are given by objects in  $\mathcal{F}un_{\mathcal{R}ep_{\mathbb{Z}_2}}(\mathcal{R},\mathcal{R})\simeq\mathcal{R}ep_{\mathbb{Z}_2}$  for both cases.
- 3-morphisms: 3-morphisms are given by instantons which can be viewed as defects in the time direction. More precisely, in this case, they are natural transformation between bimodule functors. We recall a typical example of an instanton given in Ref. 26. Imagine a dotted vertical interval Hilb-wall in Fig. 13, with a upper end  $F_{+}$  and a lower end  $\overline{F}_{+}$ , shrinking in the time direction and finally disappeared. This is given by an instanton. More precisely, two defect junction  $F_{+}$  and  $F_{+}$ , when viewed from far away, fuse into a single defect junction on the trivial defect, i.e. a bulk excitation which was shown to be  $1 \oplus \epsilon$ . Then the instanton describe above is the morphism  $1 \oplus \epsilon \rightarrow 1$  in  $Z(\operatorname{Rep}_{\mathbb{Z}_2})$ . Moreover, it is clear that all morphisms in the categories:  $Z(\text{Rep}_{\mathbb{Z}_2})$ ,  $\text{Fun}_{\text{Rep}_{\mathbb{Z}_2}|\text{Rep}_{\mathbb{Z}_2}}(\text{Hilb}, \text{Hilb})$  and  $\operatorname{Fun}_{\operatorname{Rep}_{\mathbb{Z}_2}|\operatorname{Rep}_{\mathbb{Z}_2}}(\operatorname{Rep}_{\mathbb{Z}_2},\operatorname{Hilb})$ , etc. are instantons or 3morphisms in  $TC_3^b$ .

In addition to above 4 layers of structures: 0-,1-,2-,3-morphisms, there are much more structures naturally required by physics. We will illustrate them one by one.

- Composition of morphisms
  - 1. Physically, when two instantons move closer to each other in the time direction, they can be viewed as a single instanton. This says that 3-morphisms can be composed. Since they are given by morphisms in an ordinary 1-category, they can be composed just as usual.
  - 2. 2-morphisms, or the defect junctions, can be fused as particles. Indeed, mathematically, these particle-like excitations are given by module functors. So the fusion among these particles is exactly given by the composition of module functors. For example, the four bulk excitations,  $1, e, m, \epsilon$  fuse exactly as the composition of functors in  $Z(\Re \exp_{\mathbb{Z}_2}) = \Im \Re \Re e_{\mathbb{Z}_2}|\Re e_{\mathbb{Z}_2}(\Re e_{\mathbb{Z}_2}, \Re e_{\mathbb{Z}_2})$ . This gives arise to a monoidal structure on  $Z(\Re e_{\mathbb{Z}_2})$ . Notice that when two particles fuse, the two instantons living on the time line (about the same time), which ends at these two particles, also move close to each other. This process gives arise to a potentially new composition of 3-morphisms.

$$(e \xrightarrow{f} e, m \xrightarrow{g} m) \mapsto (e \otimes m \xrightarrow{f \otimes g} e \otimes m).$$

Mathematically, it is achieved by the fact that  $\otimes$  is a functor which automatically fuse the instantons.

Moreover, a defect junction from  $\operatorname{Rep}_{\mathbb{Z}_2}$  to  $\operatorname{Hilb}$  can fuse with a defect junction from  $\operatorname{Hilb}$  to  $\operatorname{Rep}_{\mathbb{Z}_2}$  to give a defect junction from  $\operatorname{Rep}_{\mathbb{Z}_2}$  to  $\operatorname{Rep}_{\mathbb{Z}_2}$  (a bulk excitation), or from  $\operatorname{Hilb}$  to  $\operatorname{Hilb}$ . More precisely, by Eq. (35) in Ref. 26, we have

$$F_{+} \circ \overline{F}_{+} = F_{-} \circ \overline{F}_{-} \simeq 1 \oplus \epsilon,$$
  
$$F_{+} \circ \overline{F}_{-} = F_{-} \circ \overline{F}_{+} \simeq e \oplus m.$$

3. 1-morphisms can be composed. For example, consider two domain walls  $_{\mathcal{R}ep_{\mathbb{Z}_2}}\mathcal{M}_{\mathcal{R}ep_{\mathbb{Z}_2}}$  and  $_{\mathcal{R}ep_{\mathbb{Z}_2}}\mathcal{N}_{\mathcal{R}ep_{\mathbb{Z}_2}}$  sitting parallel and next to each other. Then, when viewed from far away, they simply fuse into a single domain wall, which is given by

$$\mathcal{M} \boxtimes_{\mathcal{R}ep_{\mathbb{Z}_2}} \mathcal{N}$$

where the tensor product  $\boxtimes_{\Re \operatorname{ep}_{\mathbb{Z}_2}}$  is well-defined mathematically  $^{102}$  and the resulting category is again an  $\Re \operatorname{ep}_{\mathbb{Z}_2}$ -bimodule. Notice that  $\Re \operatorname{ep}_{\mathbb{Z}_2}$  is the trivial domain wall. It is trivial in the sense that if we replace  $\mathbb{M}$  by  $\Re \operatorname{ep}_{\mathbb{Z}_2}$ , then viewed from far away the fused wall  $\Re \operatorname{ep}_{\mathbb{Z}_2} \boxtimes_{\Re \operatorname{ep}_{\mathbb{Z}_2}} \mathbb{N}$  must be the same as a single  $\mathbb{N}$ -wall. Yes, indeed,  $\Re \operatorname{ep}_{\mathbb{Z}_2} \boxtimes_{\Re \operatorname{ep}_{\mathbb{Z}_2}} \mathbb{N} \simeq \mathbb{N}$  is guaranteed mathematically by the defining properties of the tensor product  $\boxtimes_{\Re \operatorname{ep}_{\mathbb{Z}_2}}$ . In other words, under the tensor product the trivial  $\Re \operatorname{ep}_{\mathbb{Z}_2}$ -wall acts like an identity 1-morphism. It is also clear that the composition of 1-morphisms are associative, i.e.

$$\begin{split} & (\mathcal{L} \boxtimes_{\Re \operatorname{ep}_{\mathbb{Z}_2}} \mathfrak{M}) \boxtimes_{\Re \operatorname{ep}_{\mathbb{Z}_2}} \mathfrak{N} \\ & \simeq \mathcal{L} \boxtimes_{\Re \operatorname{ep}_{\mathbb{Z}_2}} (\mathfrak{M} \boxtimes_{\Re \operatorname{ep}_{\mathbb{Z}_2}} \mathfrak{N}). \end{split}$$
 (25)

Also notice that fusion of domain wall also fuse excitations on different wall horizontally. This process provides a (potentially) new composition of 2-morphisms, and at the same time, it provides a (potentially) new composition of 3-morphisms.

• Bulk-to-wall maps: This structure can be viewed as a substructure of the composition of morphisms. But due to its importance in our study later, it is beneficial to discuss them in detail now.

Let  $\mathcal{M}$  be a  $\Re \mathsf{Pp}_{\mathbb{Z}_2}$ -bimodule. The  $1, e, m.\epsilon$ -particles can fuse into the  $\mathcal{M}$ -wall and becoming wall excitations. This process gives arise to two maps, called left/right bulk-to-wall maps, which are given by two monoidal functors L and R:

$$Z(\operatorname{Rep}_{\mathbb{Z}_2}) \xrightarrow{L} Z(\mathfrak{M}) \xleftarrow{R} Z(\operatorname{Rep}_{\mathbb{Z}_2}),$$
 (26)

where  $Z(\mathcal{M}) := \mathfrak{Fun}_{\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}|\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}}(\mathcal{M},\mathcal{M})$  is a unitary fusion category of  $\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}$ -bimodule functors from  $\mathcal{M}$  to  $\mathcal{M}$  and describes the excitations on the  $\mathcal{M}$ -wall. More precisely, the functor L and R are defined as follows:

$$L: \quad 1/e/m/\epsilon \mapsto 1/e/m/\epsilon \otimes_{\Re \operatorname{ep}_{\mathbb{Z}_2}} \operatorname{id}_{\mathbb{M}}.$$

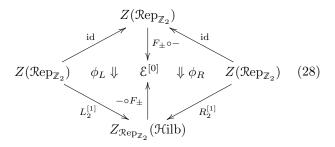
$$R: \quad 1/e/m/\epsilon \mapsto \operatorname{id}_{\mathbb{M}} \otimes_{\Re \operatorname{ep}_{\mathbb{Z}_2}} 1/e/m/\epsilon$$

where we have used the fact that the anyons  $1, e, m, \epsilon$  can be viewed as bimodule endo-functors on  $\operatorname{Rep}_{\mathbb{Z}_2}$  and that their images can be viewed as an endo-functors on  $\operatorname{Rep}_{\mathbb{Z}_2} \otimes_{\operatorname{Rep}_{\mathbb{Z}_2}} \mathfrak{M} \simeq \mathfrak{M}$  for L and  $\mathfrak{M} \otimes_{\operatorname{Rep}_{\mathbb{Z}_2}} \operatorname{Rep}_{\mathbb{Z}_2} \simeq \mathfrak{M}$  for R. The functors L and R can be combined into a two-side bulk-to-wall map:

$$Z(\operatorname{Rep}_{\mathbb{Z}_2}) \boxtimes Z(\operatorname{Rep}_{\mathbb{Z}_2}) \xrightarrow{L \boxtimes R} Z(\mathfrak{M}).$$
 (27)

The bulk-to-wall map (27) is a dominant functor, which means that any object in  $Z(\mathcal{M})$  appear as a subobject of an object in the image of  $L \boxtimes R$ . Moreover, the functors L and R are also central<sup>48,71,99</sup>.

- Braidings: The bulk excitations can be braided. It is true for all defect junctions living in a trivial defect line. We will show later that the braiding structure is automatic for endo 2-morphisms of the identity 1-morphism in an n-category for  $n \geq 2$ .
- Half Braidings: A bulk excitation can be half-braided with a wall-excitation. The general braiding between bulk excitations  $e/m/\epsilon$  and  $F_{\pm}$  (or  $\overline{F}_{\pm}$ ) is encoded in the following commutative diagrams up to isomorphisms  $\phi_L$  and  $\phi_R$ :



where  $\mathcal{E}^{[0]} = \mathcal{F}un_{\mathcal{R}ep_{\mathbb{Z}_2}|\mathcal{R}ep_{\mathbb{Z}_2}}(\mathcal{R}ep_{\mathbb{Z}_2},\mathcal{H}ilb)$  and both functors  $L_2^{[1]}$  and  $R_2^{[1]}$  are invertible. These isomorphisms  $\phi_L$  and  $\phi_R$  give the so-called half-braiding between the bulk excitation and defects  $F_{\pm}$  in  $\mathcal{E}^{[0]}$ .

**Remark 15.** In above case, since both functors  $L_2^{[1]}$  and  $R_2^{[1]}$  are invertible, the isomorphisms  $\phi_L$  and  $\phi_R$  actually gives the following full braiding:

$$F_+ \otimes e \mapsto m \otimes F_+, \quad F_+ \otimes m \mapsto e \otimes F_+,$$
 (29)

which has an interesting  $\mathbb{Z}_2$ -crossed braiding structure. The toric code enriched by the transparent domain wall as shown in Figure 13 actually gives an example of symmetry enriched topological order<sup>9</sup>. The  $\mathbb{Z}_2$ -crossed braiding in (29) is a part of the structure of a  $\mathbb{Z}_2$ -crossed

braided fusion category, which is obtained by a  $\mathbb{Z}_2$ -extension of  $Z(\operatorname{Rep}_{\mathbb{Z}_2})$  and describes an symmetry enriched topological order. Both the modular tensor category  $Z(\operatorname{Rep}_{\mathbb{Z}_2})$  and its  $\mathbb{Z}_2$ -extension can be viewed as two different minimal descriptions of the toric code model from two different points of view.

In summary, the defects in the toric code model form a 4-layered structure: only one object or 0-morphism, which can be labeled by  $\mathbb{R}ep_{\mathbb{Z}_2}$ , 1-morphisms given by  $\mathbb{R}_{\mathbb{Z}_2}$ -bimodules, 2-morphisms given by bimodule functors and 3-morphisms given by natural transformations between bimodule functors. This 4-layered structure needs to be enriched in order to describe a physical topological order. In particular, the composition of imorphisms should be introduced for i > 0, and all compositions are associative and unital. Certain braiding structures, including the half-braiding, are needed to describe a physical topological order. We will show in later sections that the notion of n-category automatically encode these structures, thus can be used as a proper mathematical language to model the topological properties of excitations in a topological order.

Remark 16. For general Levin-Wen models, one simply replace the only 0-morphism  $\Re \mathrm{ep}_{\mathbb{Z}_2}$  by a unitary fusion category  $\mathbb{C}$ , everything else remains the same. We obtain a new 3-category with one object  $\mathbb{C}$ . More generally, we have a 3-category  $\mathcal{F}$ us with objects given by unitary fusion categories, 1-morphisms by bimodules, 2-morphisms by bimodule functors and 3-morphisms by natural transformations between bimodule functors.

## C. Defects in an (n+1)-dimensional topological order

In this subsection, we will generalize topological properties of defects in toric code model to an arbitrary (n+1)-dimensional topological order. These properties summarized in this subsection should serve as a guide to formulate a mathematical definition of a BF<sub>n+1</sub>-category.

1. Defects of all codimensions: In each codimension l $(1 \le l \le n+1)$ , there are only finite types of simple topological excitations (or defects), labeled by  $i^{[l]}, j^{[l]}, k^{[l]}$ , etc. Notice that the superscript of  $i^{[l]}$  represent the codimension of the defect and will be omitted if it is clear from the context. The trivial pure l-codimensional defect is denoted by  $1^{[l]}$ . We assume that  $1^{[l]}$  is simple. A general defect can be composite (not simple). These topological excitations are instantons for l = n + 1; particle-like excitations for l = n; string-like excitations for l = n - 1; surface-like excitations for l = n - 2, etc. This is a very rough way of labeling these excitations. In general, a (l+1)-codimensional defect can be a domain wall between two (not necessarily different) l-codimensional defects, each of which again can be a domain wall between two (l-1)-codimensional defects, so on and so forth.

All the gapped domain walls between two l-codimensional defect  $x^{[l]}$  and  $y^{[l]}$  and domain wall between domain walls, etc., form an interesting mutlilayered structure, which will be denoted by  $\hom(x,y)$ . It will become clear later that  $\hom(x^{[l]},y^{[l]})$  is an (n-l)-category.

2. Ground-state degeneracy: We fixed the locations of all the topological defects, and assume that the topological excitations are well separated. In this limit, the topological degeneracy is robust against any local perturbations of the Hamiltonian.

The ground state is obtained by decorating the space  $\Sigma$  with trivial defects  $1^{[l]}$  for all  $0 \leq l \leq n$ . Then we obtain the space of ground states, denoted by  $\hom_{\Sigma}(1^{[n]}, 1^{[n]})$ , which is also called *ground-state degeneracy*. It depends on the topology of  $\Sigma$  in general. When  $\Sigma = S^n$ , if the ground-state degeneracy is trivial, i.e.

$$hom_{S^n}(1^{[n]}, 1^{[n]}) \simeq \mathbb{C}, \tag{30}$$

then the theory is called *stable*. By the stable condition (30), this degeneracy is independent of how many trivial defect  $1^{[n]}$  we introduce. Otherwise, it is clearly not true. This tells us why the stable condition (30) is natural. We will return to this point later.

We can also define a general space of lowest energy states with the appearance of nontrivial higher dimensional defects  $i_1, \dots, i_k$ . The ground-state degeneracy in this case is given by the Hilbert space:  $\bigotimes_{j=1}^k \hom_{\Sigma}(1_{1\cdots i_j}^{[n]}\dots,1_{1\cdots i_j}^{[n]}\dots)$ , where  $1_{1\cdots i_j}^{[n]}\dots$  represents the trivial sub-defect on  $i_j$ .

3. Fusion between defects of the same codimension: Defects of the same codimension can be fused (see Fig. 4). For example, any two adjacent domain walls, when viewed from far away, simply fuse to a single domain wall. This gives arise to a fusion product  $i^{[l]} \otimes j^{[l]}$ . The trivial domain walls act like the unit under the fusion product. More precisely, if f is domain wall between two defects  $x^{[l]}$  and  $y^{[l]}$ , and the trivial domain walls  $1_{x^{[l]}}, 1_{y^{[l]}}$  inside the defect  $x^{[l]}$  and  $y^{[l]}$ , respectively, act like the units for the fusion product, i.e.  $1_{x^{[l]}} \otimes f = f$  and  $f \otimes 1_{y^{[l]}} = f$ . These structures match exactly with the composition of higher morphisms in a higher category. The trivial domain wall behave like the identity higher morphism in a higher category.

The information of fusion product is encoded in the so-called "hom space" or "fusion rules", denoted by  $\operatorname{hom}(i^{[l]} \otimes j^{[l]}, k^{[l]})$ , which contains the information of all gapped domain walls between  $i \otimes j$  and k and domain walls between domain walls, etc. This generalized fusion rule  $\operatorname{hom}(i^{[l]} \otimes j^{[l]}, k^{[l]})$  is actually an (n-l)-category. For example, for l=n+1 (instantons), the "fusion rules"  $\operatorname{hom}(i^{[n+1]} \otimes j^{[n+1]}, k^{[n+1]})$  is empty; for l=n (particle-like defects), the fusion rules  $\operatorname{hom}(i^{[n]} \otimes j^{[n]}, k^{[n]})$  is a finite dimensional Hilbert space; for l=n-1 (defect lines), the fusion rule  $\operatorname{hom}(i^{[n-1]} \otimes j^{[n-1]}, k^{[n-1]})$  is a unitary 1-category.

For example, in Levin-Wen type of lattice models constructed in Ref. 26, if three domain walls, associated to three bimodule categories  $e_1(\mathcal{M}_{13})e_3$ ,  $e_1(\mathcal{M}_{12})e_2$  and  $e_2(\mathcal{M}_{23})e_3$ , respectively, are connected by a defect junction as follows:

$$\begin{array}{ccc}
\mathcal{M}_{13} \\
\mathcal{C}_1 & \mathcal{C}_3 \\
\mathcal{M}_{12} & \mathcal{C}_2 & \mathcal{M}_{23}
\end{array} \tag{31}$$

then the defect junctions are classified by the unitary 1-category  $\operatorname{\mathcal{F}un}_{\mathcal{C}_1|\mathcal{C}_3}(\mathcal{M}_{12}\otimes_{\mathcal{C}_2}\mathcal{M}_{23},\mathcal{M}_{13})$  of  $\mathcal{C}_1$ - $\mathcal{C}_3$ -bimodule functors from  $\mathcal{M}_{12}\otimes_{\mathcal{C}_2}\mathcal{M}_{23}$  to  $\mathcal{M}_{13}$ . <sup>26</sup>

In general, the hom space  $\hom(i^{[l]} \otimes j^{[l]}, k^{[l]})$  is an (n-l)-category. But once we select the particle-like excitations a,b,c on defects  $i^{[l]},j^{[l]}$  and  $k^{[l]}$ , we should always obtain a finite dimensional Hilbert space

$$\mathrm{hom}(a^{[n]}_{\cdots i^{[l]} \cdots} \otimes b^{[n]}_{\cdots j^{[l]} \cdots}, c^{[n]}_{\cdots k^{[l]} \cdots}) \cong \mathbb{C}^N$$

for some finite  $N \in \mathbb{Z}_{\geq 0}$ . These "(n-l)-categorical fusion rules" define a fusion (or tensor) product  $\otimes$  among l-codimensional defects. By natural physical requirements, these fusion products must be associative, i.e. existing an associator:  $\alpha_{i,j,k}^{[l]}: i^{[l]} \otimes (j^{[l]} \otimes k^{[l]}) \xrightarrow{\simeq} (i^{[l]} \otimes j^{[l]}) \otimes k^{[l]}$ . Moreover, the trivial type  $1^{[l]}$  fuses as a tensor unit, i.e.  $1^{[l]} \otimes i^{[l]} \simeq i^{[l]} \otimes 1^{[l]} \otimes 1^{[l]}$ . We require that these associators and unit isomorphisms are unitary. By that we mean, for arbitrary decoration of particle-like excitations on l-dimensional excitations  $i^{[l]}, j^{[l]}$  and  $k^{[l]}$ , the data of the associators and unit isomorphisms boil down to finite number of linear maps between finite dimensional Hilbert spaces. We require these linear maps to be unitary. In categorical language, these fusion rules provide a monoidal structure among all l-codimensional defects.

- 4. General fusions: Topological defects of different dimensions can also fuse. For example, for  $l \geq l'$ , a pure l-codimensional defect  $x^{[l]}$  in the bulk can fuse into an l'-codimensional defect  $y^{[l']}$  and becomes an l-codimensional defect nested in y. The information of this kind of fusion is automatically included in the fusion between  $1_{1\cdots 1}^{[l']}$  and  $y^{[l']}$  in the same dimension.
- 5. 1-dimensional bulk-to-wall maps: A special case of the general fusion will be important to us later. It is called bulk-to-wall maps. Consider two gapped domain wall  $f^{[l+1]}, g^{[l+1]} \in \text{hom}(x^{[l]}, y^{[l]})$  between  $x^{[l]}$  and  $y^{[l]}$ . An (l+2)- (or higher) codimensional defect nested in  $x^{[l]}$  or  $y^{[l]}$  can fuse into the wall f and become a defect nested in f. Notice that non-trivial domain walls in  $x^{[l]}$  or  $y^{[l]}$  can not be included because they might change the type of the domain wall f. This process give left/right 1-dimensional bulk-to-wall maps:

$$hom^{>1}(x,x) \xrightarrow{L_f} hom(f,f) \xleftarrow{R_f} hom^{>1}(y,y),$$

or equivalently, the two-side bulk-to-wall map:

$$L_f \boxtimes R_f : \text{hom}^{>1}(x, x) \boxtimes \text{hom}^{>1}(y, y) \to \text{hom}(f, f).$$
 (32)

Note that the notation hom<sup>>1</sup> $(\cdot, \cdot)$  simply means that all non-trivial (l+1)-codimensional defects are excluded.

The set of (l+2)- (and higher) codimensional defects nested in  $x^{[l]}$  or  $y^{[l]}$  acts on the set of defects between  $f^{[l+1]}$  and  $g^{[l+1]}$  (including domain walls and wall between walls). More explicitly, for a sub-defect  $a^{[k]}$  in x,  $b^{[k]}$  in y and a sub-defect  $m^{[k]}$  in hom(f,g) and  $k \geq l+2$ , the action

$$\hom^{>1}(x,x)\boxtimes \hom(f,g)\boxtimes \hom^{>1}(y,y)\to \hom(f,g)$$
$$(a,m,b)\mapsto a\otimes m\otimes b,$$

The action is clearly associative and unital. Therefore,  $\hom(f,g)$  as a topological phase is a  $\hom^{>1}(x,x)$ - $\hom^{>1}(y,y)$ -bimodule in some sense. The two-side bulk-to-wall map in (32) can be recovered from this action by taking m to be the trivial sub-defect  $1_{1\cdots f}^{[k]}$  in f, i.e. g=f and

$$L \boxtimes R : (a,b) \mapsto a \otimes 1^{[k]}_{1\cdots_f} \otimes b.$$
 (33)

In general, for fixed m, this action defines the left/right 1-dimensional bulk-to-wall maps:

$$\hom^{>1}(x,x) \xrightarrow{L_m} \hom^{>1}(f,g) \xleftarrow{R_m} \hom^{>1}(y,y), \quad (34)$$

and a two-side 1-dimensional bulk-to-wall map

$$\hom^{>1}(x,x) \boxtimes \hom^{>1}(y,y) \xrightarrow{L_m \boxtimes R_m} \hom^{>1}(f,g). (35)$$

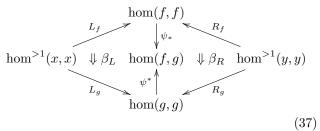
6. k-dimensional bulk-to-wall maps: In general, all (l+k+1)- and higher codimensional defects nested in a defect  $x^{[l]}$  can be fused into a defect  $z^{[l+k]}$  directly as long as z is sitting adjacent to x in the sense that either z is nested in x or z lies in a boundary of x connected to another defect  $y^{[l]}$ . We assume the later situation as it includes the former one as a special case. For simplicity, we will refer to the fusion map

$$hom^{>k}(x^{[l]}, x^{[l]}) \boxtimes hom^{>k}(y^{[l]}, y^{[l]}) 
\to hom(z^{[l+k]}, z^{[l+k]})$$
(36)

as the two-side k-dimensional bulk-to-wall map from  $x^{[l]} \otimes y^{[l]}$  to  $z^{[l+k]}$ . The two-side 1-dimensional bulk-to-wall map in (35) can also be generalized to higher dimensions by replacing the second  $z^{[l+k]}$  in (36) by another defect  $u^{[l+k]}$ .

7. Anti-excitations: For each l-codimensional defect x, there is an l-dimensional anti-excitation  $\bar{x}$  such that a pair of such excitations can be created from (or annihilated to) the vacuum  $1^{[l]}$ . In categorical language, this amounts to a rigidity or duality structure on the fusion among all l-codimensional defects. For example, in the case of a 2+1-topological phase, an anyon has a dual given by the anti-anyon. The category of anyons is, in particular, a rigid tensor category or tensor category with duals.

- 6. Braiding between defects: Any defects of codimension 2 or higher nested in the trivial lower codimensional defects can be braided. Moreover, any (l+2)- (or higher) codimensional defects nested in an l-codimensional defect x can be braided within x.
- 7. Half braidings: If  $f^{[l+1]}$  is a gapped domain wall between  $x^{[l]}$  and  $y^{[l]}$ . Then an (l+2)- (or higher) codimensional defect nested in  $x^{[l]}$  or  $y^{[l]}$  can be half-braided with defects in f. More explicitly, we consider two gapped domain wall  $f^{[l+1]}, g^{[l+1]} \in \text{hom}(x^{[l]}, y^{[l]})$  between x and y. For a gapped domain wall  $\psi^{[l+2]}$  in hom(f,g), defects in hom(f,f) and hom(g,g) can fuse onto  $\psi$ . This gives arise to two maps denoted by  $\psi_*$  and  $\psi^*$ . Thus we obtain the following diagram (recall the diagram (28)):



where  $\beta_L$  represents the physical process of deforming the fusion path from  $\psi_* \circ L_f$  to  $\psi^* \circ L_g$  and  $\beta_R$  is similar. These processes are nothing but the (left/right) halfbraidings.

Notice that each domain wall  $\psi$  between f and g defines a 1-dimensional bulk-to-wall map

$$\hom^{>1}(x,x) \xrightarrow{L_{\psi}} \hom(f,g) \xleftarrow{R_{\psi}} \hom^{>1}(y,y), \quad (38)$$

where  $L_{\psi} := \psi_* \circ L_f \simeq \psi^* \circ L_g$  and  $R_{\psi} := \psi_* \circ R_g \simeq \psi^* \circ R_f$ . The defects nested in  $x^{[l]}$  or  $y^{[l]}$  are not allowed to cross the domain wall, unless the wall f is invertible (or transparent), e.g. x = y and f is the trivial domain wall  $1_x^{[l+1]}$ .

- 8. Half braiding between wall excitations: The most general half-braiding occurs between two wall excitations. It is illustrated in Figure. 14 (see Ref. 103 for a discussion of this braiding in 1+1D conformal field theory). The half braiding discussed before is a special case of this general half braiding. It is actually a special case of equ. (39). Half braidings generate all full braidings.
- 9. Compatibility of fusions in different directions: Actually, the half braiding depicted in Figure 14 is a consequence of the compatibility of between the horizontal and vertical fusions, which is illustrated in Figure 15. If we denote the vertical fusion by  $\bullet$  and the horizontal fusion by  $\circ$ , we must have

$$(\phi' \circ \psi') \bullet (\phi \circ \psi) \simeq (\phi' \bullet \phi) \circ (\psi' \bullet \psi). \tag{39}$$

In summary, an (n+1)-dimensional topological order with defects of all codimensions has an (n+2)-layered structure similar to that of an (n+1)-category, together

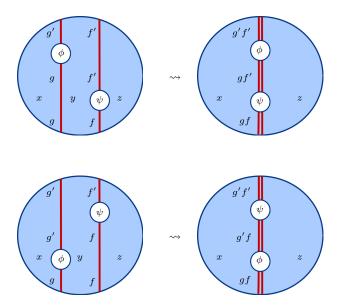


FIG. 14. Above four picture illustrate the half braiding between two walls between walls. More precisely, let  $x^{[l]}, y^{[l]}, z^{[l]}$  be three l-codimensional defects, g, g' domain walls between x and y, f, f' are walls between y and z, and  $\phi^{[l+2]}$  is a wall between g and g' and  $\psi^{[l+2]}$  a wall between f and f'. The two squigarrows  $\leadsto$  represents fusion of (l+1)-codimensional walls. The compatibility between the first column implies the compatibility of the second column, which is a half braiding.

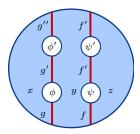


FIG. 15. Let  $x^{[l]}$ ,  $y^{[l]}$ ,  $z^{[l]}$  be three l-codimensional defects, g, g', g'' domain walls between x and y, f, f', f'' are walls between y and z, and  $\phi, \phi'$  and  $\psi, \psi'$  are (l+2)-codimensional defects. First fusing horizontally then vertically must be compatible with first fusing vertically then horizontally. This compatibility leads to the equation (39).

with additional fusion structures, which also correspond to the composition of i-morphisms in an higher category, and certain (half-)braiding structures. We will show in the next subsection that these structures are automatically encoded in an (n+1)-category.

## D. Fusion and braiding structures in n-categories

In this subsection, we will show how the notion of n-categories automatically encodes all the fusion and braiding structures. This result is well-known in mathematics. We will follows Baez-Dolan in Ref. 64 and Baez's note Ref. 65. A physics oriented reader should keep in mind the following dictionary: an (n+1)-category with only one object roughly corresponds to an (n+1)-dimensional topological order; 1-morphisms correspond to defects of codimension 1; and 2-morphisms correspond to defects of codimension 2, so on and so forth, (n+1)-morphisms are defects of codimension (n+1) lying in the time direction and are also called instantons.

For convenience, we introduce some notations. Let  $x^{[l]}$  (or x for simplicity) be an l-morphism in an n-category (n > l). We denote the full sub-(n - l)-category consists of one object x and hom(x, x) as  $\hat{x}$ . The notation  $\mathrm{id}_x^k$  means  $\mathrm{id}_x^1 := \mathrm{id}_x$ ,  $\mathrm{id}_x^2 := \mathrm{id}_{\mathrm{id}_x}$ , so on and so forth. For l > 0, an l-morphism  $x^{[l]} : \mathrm{id}_x^{l-1} \to \mathrm{id}_x^{l-1}$  is called a pure l-morphism. A 1-morphism is automatically pure.

A 0-category is just a set. We define a  $\mathbb{C}$ -linear 0-category to be a set with a  $\mathbb{C}$ -linear structure, i.e. a vector space over  $\mathbb{C}$ .

In a 1-category  $\mathcal{C}$  (see also Section  $\mathbb{D}$ ), there are the set of objects  $Ob(\mathcal{C})$  and the set of morphisms  $hom_{\mathcal{C}}(x,x)$ for  $x \in Ob(\mathcal{C})$ . The composition of 1-morphisms are required to satisfy the associativity and unit properties, i.e.  $(h \circ g) \circ f = h \circ (g \circ f), f \circ id_x = f, id_y \circ f = f$ for  $f: x \to y$ ,  $g: y \to z$  and  $h: z \to u$ . Therefore, the set  $hom_{\mathcal{C}}(x,x)$  is a monoid with the unit given by the identity 1-morphism  $id_x$  and the multiplication given by the composition of 1-morphisms. 1-categories that are relevant to physics are often C-linear Abelian 1categories, which requires, in particular, each hom space  $hom_{\mathcal{C}}(x,y)$  to be a finite dimensional vector space over  $\mathbb{C}$ , and the direct sum (or the coproduct)  $x \oplus y$  is welldefined. The direct sum is characterized by the fact that  $hom_{\mathcal{C}}(z, x \oplus y) \simeq hom_{\mathcal{C}}(z, x) \oplus hom_{\mathcal{C}}(z, y)$  as vector spaces. In a C-linear 1-category with one object \*, hom(\*,\*) is a 0-category with a composition, i.e. an algebra over  $\mathbb{C}$ .

Examples of  $\mathbb{C}$ -linear Abelian 1-categories are abundant. We list a few that are familiar to physicists.

- 1. The category of vector spaces over  $\mathbb{C}$ , denoted by  $\operatorname{Vect}$  with 1-morphisms given by linear map, i.e.  $\operatorname{hom}_{\operatorname{Vect}}(x,x) = \operatorname{End}_{\mathbb{C}}(x)$ .
- 2. The category of representations of a group G, denoted by  $\Re \operatorname{ep}_G$ , with 1-morphisms given by linear maps that intertwine the G-action.

In a 2-category  $\mathcal{C}$  (see also Section D), there are the set of object  $\mathrm{Ob}(\mathcal{C})$  and a set of 1-morphisms  $\{f: x \to y\}$  for  $x, y \in \mathrm{Ob}(\mathcal{C})$  and a set of 2-morphisms  $\{\phi: f \Rightarrow g\}$  for

two 1-morphisms  $f, g: x \to y$  (see the diagram below).



The 2-morphisms can be composed. Namely, for  $\phi: f \Rightarrow g$  and  $\psi: g \Rightarrow h$ , their composition is also called *vertical composition* and will be denoted by  $\psi \bullet \phi$ , i.e.

$$x \xrightarrow{\psi \phi} y \quad \stackrel{\bullet}{\to} \quad x \xrightarrow{\psi \bullet \phi} y$$

The associativity and unit properties hold on the nose. Namely, we have

$$\psi \bullet (\phi \bullet \chi) = (\psi \bullet \phi) \bullet \chi, \quad \psi \bullet \mathrm{id}_f = \psi, \quad \mathrm{id}_g \bullet \phi = \phi.$$

Therefore, these 1-morphisms and 2-morphisms form a 1-category  $\hom_{\mathbb{C}}(x,y)$  with objects being 1-morphisms in  $\mathbb{C}$  and 1-morphisms being 2-morphisms in  $\mathbb{C}$ .

The 1-morphisms in C can also be composed. This composition is, by definition, a functor

$$\operatorname{hom}_{\mathfrak{C}}(x,y) \times \operatorname{hom}_{\mathfrak{C}}(y,z) \xrightarrow{\circ} \operatorname{hom}_{\mathfrak{C}}(x,z)$$

which is illustrated in the following diagrams:

$$\underbrace{x \underbrace{\psi \phi}_{f'} y \underbrace{\psi \psi}_{g'} z}_{g'} \quad \mapsto \quad x \underbrace{\psi \psi \circ \phi}_{g' \circ f'} z \qquad (40)$$

which induces a new composition, called *horizontal composition*, among 2-morphisms, and is denoted by  $\psi \circ \phi$ .

The composition  $\circ$  of 1-morphisms also satisfies similar associativity and unit properties. The difference is that they do not hold on the nose, but hold up to invertible 2-morphisms. For example, for  $x,y,z\in \mathrm{Ob}(\mathfrak{C})$  and  $x\xrightarrow{f}y\xrightarrow{g}z\xrightarrow{h}u$ , there are an associator 2-isomorphism

$$\alpha_{h,q,f}: h \circ (g \circ f) \Rightarrow (h \circ g) \circ f$$

and unit 2-isomorphisms

$$l_f: \mathrm{id}_u \circ f \Rightarrow f, \quad r_f: f \circ \mathrm{id}_x \Rightarrow f.$$

The associator and unit 2-isomorphisms satisfy the usual pentagon relations and triangle relations. Then it is easy to see that the 1-category  $\hom_{\mathbb{C}}(x,x)$  is a monoidal category with the tensor unit given by  $\mathrm{id}_x$  and the tensor product  $\otimes$  given by the composition of 1-morphisms. The unit properties of horizontal composition  $\circ$  does not play an explicit role in physics. For convenience, we assume that the unit properties of horizontal composition hold on the nose, i.e.

$$id_u \circ f = f = f \circ id_x \tag{41}$$

We give a few examples of 2-categories below:

- 1. Any 1-category can always be lifted to a 2-category by adding identity 2-morphisms.
- 2. Any monoidal 1-category can be viewed as a 2-category with a single object.
- 3. The category of 1-categories is a 2-category with 1-morphisms given by functors and 2-morphisms by natural transformations.
- 4. The category of algebras in Vect with 1-morphisms given by bimodules and 2-morphisms by bimodule maps. More explicitly, for algebras a, b, c in Vect, the bimodule  ${}_am_b, {}_bn_c$  are two 1-morphisms  $a \to b$  and  $b \to c$ , respectively. The composition of m and n is defined by the tensor product  $m \otimes_b n$ .

In a 2-category, the set of 2-morphisms satisfying an interesting commutativity property, which follows from an argument of Eckmann-Hilton for the commutativity of the higher homotopy groups and generalizes to higher categories<sup>64</sup>. Consider four 2-morphisms in the following diagram:

where the arrow  $\leadsto$  represents two vertical composition followed by a horizontal composition. By the unit property, we can switch the order of the 2-morphisms in each row in above diagram, we obtain

where the arrow  $\leadsto$  represents again two vertical composition followed by a horizontal composition. The axiom of 2-category requires certain compatibility between the vertical and horizontal composition. For  $x \stackrel{\langle g,g',g'' \rangle}{\leftarrow} y \stackrel{\langle f,f',f'' \rangle}{\leftarrow} z$  and  $q \stackrel{\phi}{\rightarrow} q' \stackrel{\phi'}{\rightarrow} q''$ , it says

$$(\psi \circ \phi) \bullet (\psi' \circ \phi') = (\psi \bullet \psi') \circ (\phi \bullet \phi'), \tag{44}$$

which is nothing but the equation (39) and further leads to the following identities:

$$\psi \circ \phi = (\psi \bullet 1) \circ (1 \bullet \phi) = (\psi \circ 1) \bullet (1 \circ \phi)$$

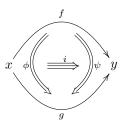
$$= \psi \bullet \phi$$

$$= (1 \circ \psi) \bullet (\phi \circ 1) = (1 \bullet \phi) \circ (\psi \bullet 1)$$

$$= \phi \circ \psi.$$
(45)

As a consequence, the set  $hom_{\mathcal{C}}(id_x, id_x)$  must be a commutative monoid. We will refer to this commutativity as 2-dimensional commutativity.

In a 3-category C, there are 0, 1, 2, 3-morphisms (see the diagram below),



where 0-morphisms are objects. Note that, for  $x, y \in \text{Ob}(\mathcal{C})$ ,  $\text{hom}_{\mathcal{C}}(x,y)$  is a 2-category; and, for  $f,g:x\to y$ ,  $\text{hom}_{\mathcal{C}}(f,g)$  is a 1-category; and, for  $\phi,\psi:f\Rightarrow g$ ,  $\text{hom}_{\mathcal{C}}(\phi,\psi)$  is a 0-category.

The 3-morphisms can be composed with the associativity and unit properties that hold on the nose. The 2-morphisms can be composed so that the associativity and unit properties hold up to 3-isomorphisms that satisfy the pentagon and triangle relations. The 1-morphisms can be composed and satisfy a new kind of associativity and unit properties such that the associator and unit isomorphisms satisfying the pentagon relations and the triangle relations only up to 3-isomorphisms which satisfy further coherence properties<sup>26,49,86</sup>.

The composition of 1-morphisms (or 2-morphisms) defines a tensor product of any pair of 1-morphisms (or 2-morphisms). Therefore, in a 3-category  $\mathcal{C}$ , the 2-category hom $_{\mathcal{C}}(x,x)$  is a monoidal 2-category for  $x\in \mathrm{Ob}(\mathcal{C})$ , and the 1-category hom $_{\mathcal{C}}(f,f)$ , for  $f:x\to y$ , is a monoidal 1-category. We denote  $\psi\circ\phi$ , for  $\phi,\psi:f\Rightarrow f$ , by  $\phi\otimes\psi$ . Similar to 2-category case, 2-morphisms in 3-category also satisfy some commutativity when  $f=\mathrm{id}_x$ . The difference is that, due to the existence of 3-morphisms, the 2-dimensional commutativity does not hold on the nose but only up to a 3-isomorphism. In other words, there is an 3-isomorphism

$$c_{\phi,\psi}:\phi\otimes\psi\xrightarrow{\simeq}\psi\otimes\phi$$

for each  $\phi, \psi$ :  $\mathrm{id}_x \Rightarrow \mathrm{id}_x$ . These 3-isomorphisms are required to satisfy certain coherence properties, which includes the usual hexagon relations for a braiding tensor category. As a consequence, the 1-category  $\mathrm{hom}_{\mathcal{C}}(\mathrm{id}_x,\mathrm{id}_x)$  is a braided tensor category.

We give two examples of 3-category below:

- the category of 2-categories is a 3-category with objects given by 2-categories, 1-morphisms given by 2-functors, 2-morphisms given by 2-natural transformations and 3-morphisms given by modifications.
- 2. the category Fus of fusion categories is a 3-category with objects given by fusion categories, 1-morphisms by bimodule categories, 2-morphisms by bimodule functors and 3-morphisms by natural transformations between bimodule functors.

In the 3-category  $\mathcal{F}$ us, let  $\mathcal{C}$  be an object, or equivalently, a fusion category. Then the identity 1-morphism id $_{\mathcal{C}}$  is simply the trivial  $\mathcal{C}$ - $\mathcal{C}$ -bimodule

 $_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$ . Then the 1-category  $\mathrm{hom}_{\mathfrak{Fus}}(\mathrm{id}_{\mathcal{C}},\mathrm{id}_{\mathcal{C}})$  is nothing but the category  $\mathfrak{Fun}_{\mathcal{C}|\mathcal{C}}(\mathcal{C},\mathcal{C})$  of  $\mathcal{C}$ -C-bimodule functors from  $_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$  to  $_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$ . This category, denoted by  $Z(\mathcal{C}) := \mathcal{Fun}_{\mathcal{C}|\mathcal{C}}(\mathcal{C},\mathcal{C})$ , is a braided monoidal category, which is also called the monoidal center of  $\mathcal{C}$ .

For an object x in a 3-category  $\mathcal{C}$ , the 3-morphisms in  $\mathrm{hom}_{\mathcal{C}}(\mathrm{id}_{\mathrm{id}_x},\mathrm{id}_{\mathrm{id}_x})$  satisfy a new kind of commutativity. More precisely, two 3-morphisms can be composed in 3 different ways, in which  $\circ_3$  arise from the usual composition of 3-morphisms,  $\circ_2$  from the composition of 2-morphisms and  $\circ_1$  from the composition of 1-morphisms. If we illustrate these three compositions by a diagram similar to the first diagram in (42), it will give a 3-dimensional diagram in which  $\circ_1$  is horizontal (x-direction),  $\circ_2$  is vertical (y-direction) and  $\circ_3$  is in the third direction (z-direction). This leads to a new kind of commutativity which contains three independent 2-dimensional commutativity's in xy-plane, xz-plane and yz-plane. We will refer to this kind of commutativity as 3-dimensional commutativity.

An n-category  $\mathcal E$  can be viewed as a category enriched by (n-1)-categories. Namely, the hom spaces of an n-category are (n-1)-categories. All i-morphisms for i>0 can be composed. Therefore,  $\operatorname{hom}_{\mathcal E}(x,x)$  is a monoidal (n-1)-category for  $x\in\operatorname{Ob}(\mathcal E)$ ; and the (n-2)-category  $\operatorname{hom}_{\mathcal E}(\operatorname{id}_x,\operatorname{id}_x)$  is monoidal and equipped with a 2-dimensional braiding structure which arises from the compositions of 1,2-morphisms; and the (n-3)-category  $\operatorname{hom}_{\mathcal E}(\operatorname{id}_{\operatorname{id}_x},\operatorname{id}_{\operatorname{id}_x})$  is monoidal and equipped with a 3-dimensional braiding structure which arises from the compositions 1, 2, 3-morphisms. For example,

- 1. when n = 4,  $hom_{\mathcal{E}}(id_x, id_x)$  is a braided monoidal 2-category<sup>104,105</sup>,
- 2. when n = 4, the 3-morphisms in  $hom_{\mathcal{E}}(id_{id_x}, id_{id_x})$  form a 1-category equipped with a 3-dimensional braiding structure. This 1-category is nothing but symmetric monoidal 1-category.
- 3. when n=5,  $\hom_{\mathcal{E}}(\mathrm{id}_{\mathrm{id}_x},\mathrm{id}_{\mathrm{id}_x})$  is a 2-category equipped with a 3-dimensional braiding structure. This 2-category is called a weakly involuntary monoidal 2-category<sup>64</sup>.

More examples of this type was shown in Ref. 64 and was called k-tuply monoidal n-categories.

For us, we will use (n+1)-category to describe a (n+1)-dimensional topological order. The 1-morphisms correspond to defects of codimension 1; and the 2-morphisms correspond to defects of codimension 2, so on and so forth. Notice that only defects of codimension 2 or higher can be braided. This coincides with the commutativity of 2- or higher morphisms in  $\hom_{\mathcal{E}}(\mathrm{id}_x,\mathrm{id}_x)$  in an n-category  $\mathcal{E}$ .

Note that braiding between defects of different dimensions are automatically encoded in the braiding between

defects of the same dimension. For example, one can braid a particle-like excitation with a string-like excitation. In order to see the braiding, the particle should be viewed as a 0-dimensional defect nested in a trivial string; and the string-like excitation should be viewed as a string equipped with the trivial particle. Then this braiding can be encoded by the braiding between the string and the trivial string.

We will illustrate this structure in the case when the spatial dimension n=3. In this case, string-like excitations, or 2-codimensional defects nested in the trivial 1-codimensional defect, form a braided monoidal 2-category  $\mathcal{D}$ . The braiding in  $\mathcal{D}$  is a pseudo-natural isomorphism  $R: \otimes \to \otimes^{\mathrm{op}}$  between two tensor product functors, where  $\otimes^{\mathrm{op}}$  is defined by  $a \otimes^{\mathrm{op}} b := b \otimes a$  for  $a, b \in \mathrm{Ob}(\mathcal{D})$ . The defining data of such a pseudo-natural isomorphism includes  $a \otimes a \otimes a \otimes a$ 

- 1. equivalences  $R_{a,b}: a \otimes b \xrightarrow{\cong} b \otimes a$  for  $a, b \in \mathrm{Ob}(\mathfrak{D})$
- 2. invertible 2-morphisms  $R_{f,q}$ :

for each pair of 1-morphisms  $(a \xrightarrow{f} a', b \xrightarrow{g} b')$ .

 $R_{f,g}$  is the braiding between the particle-like excitation f, which connects the string a and a', and the particle-like excitation g, which connects the string b and b'. The braiding discussed in the previous paragraph is a special case of the pair  $(R_{a,b}, R_{f,g})$  when  $a = 1_{\mathcal{D}} = a'$ , b = b' and  $g = \mathrm{id}_b$ .

Similar to  $R_{f,g}$ , in an (n+1)-topological order, the braiding of an i-codimensional defect x and a j-dimensional defect y with arbitrary decoration of higher codimensional defects for  $i \geq j$  is encoded in the braiding between the trivial j-codimensional defect and y. If no decoration of higher codimensional defect is specified, it automatically means that the default decorations by the trivial higher dimensional defects up to codimension n (excluding the instantons) are chosen. Therefore, all the information of braiding is encoded in the braiding among defects of the same codimension.

**Remark 17.** Notice that the data in the braiding  $R_{x,y}$ :  $x \otimes y \to y \otimes x$  boils down to a family of k-isomorphisms for  $j < k \le n+1$ . Since, for j < k < i, the k-morphisms are trivial due to our convention (41), the braiding data further boils down to a family of k-isomorphisms for  $i \le k \le n+1$ . If we assume that Yoneda lemma holds for any properly defined m-categories for all  $m \in \mathbb{Z}_+$ , then we can further reduce the data of a k-isomorphisms to a set of data of (n+1)-isomorphisms, which is an uncontrollable large set in general. But if we assume certain finiteness to our (n+1)-category as we will do in the

next subsection, the braiding data can be reduced to a finite number of (n+1)-isomorphisms, which are directly measurable in physics. More precisely, these finite number of (n+1)-isomorphisms are invertible linear maps between two spaces of lowest energy states associated to two configurations of excitations including at least the i-codimensional defect x and y-codimensional defect y, and these two configurations can be deformed from one to the other by moving x around y without crossing other defects.

Recall the half-braiding discussed in Section XI C. This structure is also automatically encoded in the structure of higher category. To see this, consider the following diagrams. The composition is obtained by first composed horizontally then vertically,

$$x \xrightarrow{\text{id}_{x}} x \xrightarrow{\text{f}} y \sim x \xrightarrow{\text{id}_{x}} x \qquad (47)$$

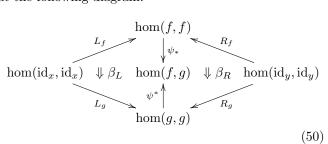
where  $\phi$  in the second diagram is actually a abbreviation for  $\mathrm{id}_f \circ \phi$ . Due to the unit property of the identity (l+2)-morphisms composed vertically, we can exchange the (l+2)-morphisms in the first row in the first diagram in (47) with the second row. We have he following diagrams and composition.

where  $\phi$  in the second diagram is an abbreviation for  $\mathrm{id}_g \circ \phi$ . The axioms of the (n+1)-category requires that there is a half-braiding

$$c_{\phi,\psi}: \phi \bullet \psi \xrightarrow{\simeq} \psi \bullet \phi.$$
 (49)

More precisely,  $c_{\phi,\psi}$  is a natural transformation between the two functors  $\hom(\mathrm{id}_x,\mathrm{id}_x)\boxtimes \hom(f,g)\to \hom(f,g)$  described by (42) and (43).

Another way to visualize the half braiding is to look at the following diagram:



where  $L_{f/g}$  and  $R_{f/g}$  are functors defined by the composition (also called left/right bulk-to-wall maps) and

 $\psi_* = \psi \circ -$  and  $\psi^* = -\circ \psi$ . Above diagram is commutative up to two isomorphisms

$$\beta_L: \psi_* \circ L_f \xrightarrow{\simeq} \psi^* \circ L_q \tag{51}$$

$$\beta_R: \psi_* \circ R_q \xrightarrow{\simeq} \psi^* \circ R_f \tag{52}$$

which are nothing but the half-braidings.

Notice that  $c_{\psi,\phi}$  can not be defined in general. That is why it is called the half-braiding. Another important difference with a full braiding, comparing it to (42) and (43), is that only the vertical composition  $\bullet$  (not the horizontal one!) is related to the half-braiding. This fact corresponds exactly to the half-braiding between defects nested in an l'-dimensional defect and those in a (l-1)-dimensional wall which is only well-defined the l'-th direction that is normal to the wall.

When f and g are invertible (corresponding to transparent domain walls), the bulk-to-wall maps  $L_f, L_g, R_f, R_g$  are all invertible, then we can fully braid morphisms in  $\hom(\mathrm{id}_{x/y}, \mathrm{id}_{x/y})$  with those in  $\hom(f, g)$ . More explicitly, we obtain the following two double braidings:

$$\psi_* \circ L_f \circ L_g^{-1} \circ R_g \xrightarrow{\beta_L^{-1} \beta_R} \psi_* \circ R_f,$$
 (53)

$$\psi^* \circ L_g \circ L_f^{-1} \circ R_f \xrightarrow{\beta_L \beta_R^{-1}} \psi^* \circ R_g. \tag{54}$$

An example of such double braiding and transparent domain wall is given in (29) in the toric code model.

More general half braidings can be obtained by replacing the "=" in equation (44) by an higher isomorphism  $\simeq$ . This is nothing but the categorical description of the compatibility of two different orders of fusions in different direction (recall equation (39) and Figure 15).

In the remaining part of this subsection, we will discuss k-dimensional bulk-to-wall maps in an (n+1)-category. We assume that certain additive structure to an n-category and a symmetric tensor product  $\boxtimes$  and the direct sum  $\oplus$  for n-category is well-defined. When n=1, the symmetric tensor product  $\boxtimes$  is the Deligne tensor product.

Recall the 1-dimensional bulk-to-wall map (32)(33) introduced in Section XI C. Since the defect  $x^{[l]}$  and  $y^{[l]}$  correspond to two l-morphisms and  $f^{[l+1]}$  is an (l+1)-morphism  $f: x \to y$ . The two-side 1-dimensional bulk-to-wall map in an (n+1)-category is the following (n-l-1)-functor (i.e. a functor between two (n-l-1)-categories):

$$\hom(\mathrm{id}_x,\mathrm{id}_x)\boxtimes \hom(\mathrm{id}_y,\mathrm{id}_y)\xrightarrow{L_f\boxtimes R_f} \hom(f,f) \quad (55)$$

defined by the horizontal composition induced from the composition of (l+1)-morphisms as shown in the following diagrams:

$$x \underbrace{\downarrow \downarrow \phi}_{id_{x}} x \boxtimes y \underbrace{\downarrow \downarrow \psi}_{id_{y}} y \mapsto x \underbrace{\downarrow \psi \circ id_{f} \circ \phi}_{f} y. (56)$$

And the functor  $L_f/R_f$  is called left/right 1-dimensional bulk-to-wall map. More generally, we will also refer to the following map

$$\operatorname{hom}(\operatorname{id}_x,\operatorname{id}_x)\boxtimes\operatorname{hom}(\operatorname{id}_y,\operatorname{id}_y)\xrightarrow{L_\psi\boxtimes R_\psi}\operatorname{hom}(f,g),$$

where  $L_{\psi} = \psi_* \circ L_f$   $(R_{\psi} = \psi_* \circ R_f)$  is the left (right) 1-dimensional bulk-to-wall map.

The k-dimensional bulk-to-wall map introduced in (36) is also automatically encoded in the horizontal composition induced from the composition of (l+1)-morphisms in an (n+1)-category. For example, let  $f,g:x\to y$  be two (l+1)-morphisms and  $z:f\Rightarrow g$  an (l+2)-morphism, a two-side 2-dimensional bulk-to-wall map from two  $x\boxtimes y$  to z is an (n-l-2)-functor

$$\operatorname{hom}(\operatorname{id}_{\operatorname{id}_x},\operatorname{id}_{\operatorname{id}_x}) \boxtimes \operatorname{hom}(\operatorname{id}_{\operatorname{id}_y},\operatorname{id}_{\operatorname{id}_y}) \to \operatorname{hom}(z,z), (57)$$

the definition of which is naturally included in the definition of composition of (l+1)-morphisms. More generally, if z is a (l+k)-morphism, the left/right k-dimensional bulk-to-wall maps (57) are two (n-l-k)-functors:

$$\hom(\mathrm{id}_x^k,\mathrm{id}_x^k) \xrightarrow{L} \hom(z,z) \xleftarrow{R} \hom(\mathrm{id}_y^k,\mathrm{id}_y^k) \qquad (58)$$

or equivalently, a two-side bulk-to-wall map:

$$\hom(\mathrm{id}_x^k, \mathrm{id}_x^k) \boxtimes \hom(\mathrm{id}_y^k, \mathrm{id}_y^k) \xrightarrow{L \boxtimes R} \hom(z, z), \quad (59)$$

where the notation  $\mathrm{id}_x^k$  means  $\mathrm{id}_x^1 := \mathrm{id}_x$ ,  $\mathrm{id}_x^2 := \mathrm{id}_{\mathrm{id}_x}$ , so on and so forth. More generally, if z' is another (l+k)-morphism, for a fixed morphism in  $\mathrm{hom}(z,z')$ , there is a two-side k-dimensional bulk-to-wall map from the right hand side of (59) to  $\mathrm{hom}(z,z')$ .

In summary, the rich inner structures of higher category exactly catch the information of the fusion and braiding of defects in a topological order.

### E. Unitary (n+1)-categories and $BF_{n+1}^{pre}$ -categories

In this section, we introduce the notion of unitary (n+1)-category and that of a BF $_{n+1}^{pre}$ -category.

A unitary 0-category is a finite dimensional Hilbert space.

Now we introduce the definition of a unitary 1-category. We recommend Müger's review Ref. 106.

**Definition 28.** A \*-1-category  $\mathcal{C}$  is a  $\mathbb{C}$ -linear category equipped with a functor  $*:\mathcal{C}\to\mathcal{C}^{\mathrm{op}}$  which acts trivially on the objects and is antilinear and involutive on morphisms, i.e.  $*: \hom_{\mathbb{C}}(x,y) \to \hom_{\mathbb{C}}(y,x)$  is defined so that

$$(g \circ f)^* = f^* \circ g^*, \qquad (\lambda f)^* = \bar{\lambda} f^*, \qquad f^{**} = f. \quad (60)$$
  
for  $f: u \to v, \ g: v \to w, \ h: x \to y, \ \lambda \in \mathbb{C}^\times$ .

**Definition 29.** A 1-category  $\mathbb{C}$  is called *finite* if  $\mathbb{C}$  is Abelian  $\mathbb{C}$ -linear and semisimple so that there are only finite number of simple objects and hom spaces are all finite dimensional vector spaces over  $\mathbb{C}$ .

**Definition 30.** A \*-1-category is called *unitary* if it is finite and \* satisfies the positivity condition:  $f \circ f^* = 0$  implies f = 0.

**Definition 31.** A finite 1-category is a  $\mathbb{C}$ -linear Abelian semisimple category with only finitely many inequivalent simple objects and finite dimensional hom spaces. A fusion category is a finite monoidal category such that  $hom(1,1) \simeq \mathbb{C}$ , where 1 is the tensor unit.

A unitary fusion 1-category has a lot of nice properties. In particular, it is automatically spherical, and automatically a  $C^*$ -category  $^{107}$ , and the left dual of an object automatically coincides with the right dual.

In a 2-category  $\mathfrak C$ , the direct sum  $x\oplus y$  (or the coproduct) of two objects x and y is characterized by the property that  $\hom(z,x\oplus y)\simeq \hom(z,x)\oplus \hom(z,y)$  as 1-categories for all z.

**Definition 32.** A 2-category  $\mathcal{C}$  is called *finite* if there are only finite number of simple objects; every object is a direct sum of simple objects; and all hom spaces are finite 1-categories.

Remark 18. Although we need the additive structure in a finite 2-category, we don't require it to be Abelian. In particular, we expect to have non-trivial morphisms between two simple objects.

**Definition 33.** A 2-category  $\mathcal C$  said to have adjoints for 1-morphisms if for every 1-morphism  $f:X\to Y$ , there exists another 1-morphism  $g:Y\to X$ , the unit 2-morphisms  $\eta:\operatorname{id}_X\Rightarrow g\circ f$  and  $\tilde\eta:\operatorname{id}_Y\Rightarrow f\circ g$  and the co-unit 2-morphisms  $\epsilon:f\circ g\Rightarrow\operatorname{id}_Y$  and  $\tilde\epsilon:g\circ f\Rightarrow\operatorname{id}_X$  such that they form an ambidextrous adjunction, i.e. left and right adjoints exist and coincide.

**Remark 19.** The above notion is stronger than the usual notion of a 2-category  $\mathcal{C}$  having adjoints for 1-morphisms, which only requires the existence of the right adjoint and the left adjoint<sup>108</sup>. The 2-morphisms  $\eta, \tilde{\eta}, \epsilon, \tilde{\epsilon}$  are called duality maps.

**Definition 34.** A unitary 2-category is a finite 2-category having adjoints for 1-morphisms such that all hom spaces are unitary 1-categories and all coherence isomorphisms are unitary, i.e.

$$\alpha_{f,g,h}^* = \alpha_{f,g,h}^{-1}, \quad l_f^* = l_f^{-1}, \quad r_f^* = r_f^{-1}$$
 
$$\tilde{\epsilon} = \eta^*, \quad \tilde{\eta} = \epsilon^*,$$

for 1-morphisms f, g, h, where  $\alpha$ , l, r and  $\eta, \tilde{\eta}, \epsilon, \tilde{\epsilon}$  are the associator, the left unit isomorphism, the right unit isomorphism and duality maps, respectively.

It is well-known that a 2-category with a single object is a monoidal category (see Section XID). It is very easy to obtain the following result.

**Lemma 17:** For a unitary 2-category with a single object, the 1-category of 1-morphisms is a unitary fusion 1-category, in which the tensor product is given by the composition of 1-morphisms and the tensor unit is the identity 1-morphism.

We would like to define a unitary n-category recursively. We assume that a good definition of n-category is chosen with certain additive structure such that the direct sums (or coproducts) are well-defined. In an n-category, the direct sum  $x \oplus y$  (or the coproduct) of two objects x and y (if defined) is characterized by the property that  $hom(z, x \oplus y) \simeq hom(z, x) \oplus hom(z, y)$  as (n-1)-categories for all z. An i-morphism that is not a direct sum of two non-zero i-morphisms is called simple. In the n-categories that we are interested in, finite coproducts (or direct sum) of i-morphisms always exist.

### **Definition 35.** An n-category $\mathcal{C}$ is called *finite* if

- 1. the homotopy 1-category h $\mathcal{C}$  obtained from  $\mathcal{C}$  by taking the same set of objects and defining 1-morphisms as the equivalence classes of 1-morphisms in  $\mathcal{C}$  is (not Abelian in general) semisimple with only finite many simple objects, and the identity morphism  $\mathrm{id}_f$  is simple for all simple morphisms f;
- 2. for any pair of object  $x, y \in \mathcal{C}$ , the (n-1)-category hom $_{\mathcal{C}}(x, y)$  is finite.

**Remark 20.** In a finite (n+1)-category, the fusion and braiding data can all be encoded efficiently by considering only the simple l-morphisms for  $1 \le l \le n+1$ , which can be further reduced to a finite set of (n+1)-isomorphisms (recall Remark 17), which should be directly measurable in a physical realization of this category.

We will follow the footsteps of Lurie in Ref. 108. Let  $\mathcal C$  be an n-category for  $n \geq 2$ . Its homotopy 2-category  $h_2\mathcal C$  is a 2-category with the same objects and 1-morphisms as those in  $\mathcal C$  and 2-morphisms being the isomorphism classes of those in  $\mathcal C$ .  $\mathcal C$  has adjoints for 1-morphisms if  $h_2\mathcal C$  has adjoints for 1-morphisms. For 1 < l < n,  $\mathcal C$  has adjoints for k-morphisms if, for any pair of objects  $x,y\in \mathcal C$ , the (n-1)-category  $\hom_{\mathcal C}(x,y)$  has adjoints for all (l-1)-morphisms.  $\mathcal C$  is said to have adjoints if  $\mathcal C$  has adjoints for l-morphisms for all 0 < l < n.

Remark 21. If the n-category  $\mathcal C$  has a monoidal structure,  $\mathcal C$  has duals for the objects if the homotopy 1-category  $h\mathcal C$  is a rigid monoidal category such that two side duals coincide. Then  $\mathcal C$  is said to have duals if  $\mathcal C$  has duals for objects and adjoints for l-morphisms for all 0 < l < n.

We propose an incomplete definition of a unitary n-category recursively as follows.

**Definition 36.** A unitary n-category  $\mathcal{C}$  is a finite n-category such that it has adjoints and, for any pair  $x, y \in \mathcal{C}$ ,  $\hom_{\mathcal{C}}(x, y)$  is a unitary (n-1)-category and all coherence isomorphisms are unitary. If a unitary n-category  $\mathcal{C}$  has a monoidal structure with simple tensor unit and duals, it is called a unitary fusion n-category.

Remark 22. It is unclear to us how to generalize the definition the unitarity for all the coherence isomorphisms in Definition 34 to higher categories. This drawback does not play any role in our later discussion.

Remark 23. We have assumed that the tensor unit is simple in the unitary fusion n-category. If this condition is not satisfied, we will call such n-category as unitary multi-fusion n-category. This notion is also very useful. Multi-fusion n-categories can be naturally produced in the process of dimensional reduction or by the general tensor product  $\boxtimes_{\mathcal{E}_{n+1}}$  discussed in Remark 8. For example, consider a toric code model defined on a strip of lattice such that boundaries on both left and right side of the strip are smooth (see Fig. 13). When viewed from far away, this strip narrows down to a line, a closed 1 + 1Dtopological order. Excitations on this line are given by the multi-fusion 1-category  $\operatorname{Fun}_{\operatorname{Hilb}}(\operatorname{Rep}_{\mathbb{Z}_2}, \operatorname{Rep}_{\mathbb{Z}_2})$ , the center of which is again trivial<sup>102</sup>. More generally, for any l-morphism x in a unitary n-category, the category  $hom(x \oplus x, x \oplus x)$  is a multi-fusion (n-l-1)-category. The topological phases associated to multi-fusion n-categories are not stable. For example, the stable condition (30) is violated for such multi-fusion n-categories. For this reason, we choose not to discuss them further in this work. We will do that in Ref. 70. Also notice that fusion ncategories are closed under the stacking operation:  $\boxtimes$ .

**Definition 37.** A unitary n-functor f between two unitary n-categories is an n-functor preserving the adjoints and the additive structures (e.g. direct sums).

Similar to Lemma 17, we have the following result.

**Lemma 18:** The fully subcategory of a unitary (n+1)-category consisting of a single object is automatically a unitary fusion n-category. These two notions are equivalent.

A unitary (n+1)-category with one object can describe a (n+1)-dimensional topological order with a given (not necessarily complete) set of defects. The 1-morphisms are 1-codimensional defects, 2-morphisms are 2-codimensional defects that are connecting two (not necessarily distinct) 1-codimensional defects, so on and so forth. (n+1)-isomorphisms are instantons. For this reason, we will introduce a new terminology.

**Definition 38.** A pre-BF<sub>n+1</sub>-category or a BF<sup>pre</sup><sub>n+1</sub>-category is an (n + 1)-category with a single object \* such that hom(\*, \*) is a unitary fusion n-category.

**Remark 24.** A BF $_{n+1}^{pre}$ -category is not a unitary (n+1)-category because we do not assume an additive structure on objects. An alternative definition is to define a

 $\mathrm{BF}_{n+1}^{pre}$ -category as a unitary (n+1)-category with one simple object. So all finite coproducts (direct sums) are included. There is no essential difference between these two approaches. But our approach makes some later discussion easier.

**Example 15.** For each n > 0, there is a trivial and the smallest  $\mathrm{BF}_{n+1}^{pre}$ -category  $\mathbf{1}_{n+1}$ , which is defined as the smallest  $\mathrm{BF}_{n+1}^{pre}$ -category that contains  $\{*,\mathrm{id}_*,\mathrm{id}_{\mathrm{id}_*},\cdots,\mathrm{id}_*^n\}$  and  $\mathrm{hom}(\mathrm{id}_*^n,\mathrm{id}_*^n) = \mathbb{C}$ . More explicitly, we have

- 1. for n = 0,  $\mathbf{1}_{0+1}$  is the 1-category with a single object \* and  $hom(*,*) = \mathbb{C}$ ;
- 2. for n = 1,  $\mathbf{1}_{1+1}$  is the 2-category with a single object and hom(\*,\*)  $\simeq \mathcal{H}$ ilb;
- 3. for n=2,  $\mathbf{1}_{1+1}$  is a 3-category with a single object \* and a unique simple 1-morphism id $_*$  and hom(id $_*$ , id $_*$ )  $\simeq \mathcal{H}$ ilb. An non-simple 1-morphism is a finite direct sum of id $_*$ .

**Example 16.** Some non-trivial examples of  $BF_{n+1}^{pre}$ -categories in low dimensions:

- 1. A BF $_{0+1}^{pre}$ -category is just a simple (because we require id\* to be "simple")  $C^*$ -algebra over  $\mathbb{C}$ , i.e. a matrix algebra over  $\mathbb{C}$ .
- 2. A  $\mathrm{BF}_{1+1}^{pre}$ -category is a 2-category with one object \* such that  $\mathrm{hom}(*,*)$  is a unitary fusion 1-category.
- 3. A unitary braided fusion 1-category  $\mathcal{D}$  determines a BF $_{2+1}^{pre}$ -category, which consists of one object \* and only one simple 1-morphism  $\mathrm{id}_*$  such that  $\mathrm{hom}(\mathrm{id}_*,\mathrm{id}_*)=\mathcal{D}.$
- 4. A unitary fusion category  $\mathcal{C}$  determines a  $\mathrm{BF}_{2+1}^{pre}$  category, which consists of one object  $\mathcal{C}$ , 1-morphisms given by  $\mathcal{C}$ -bimodules, 2-morphisms given by  $\mathcal{C}$ - $\mathcal{C}$ -bimodule functors and 3-morphisms by natural transformations between bimodule functors, i.e. the full subcategory of  $\mathcal{F}$ us consisting of a single object  $\mathcal{C}$  (recall Remark 16).

Due to the connection to topological order, an l-morphism f in a  $\mathrm{BF}_{n+1}^{pre}$ -category will also be called an l-codimensional defect. Higher morphisms in  $\mathrm{hom}(f,f)$  are higher codimensional defects nested on f. The only object will be denoted by \* unless we specify it otherwise.

In order to study situations that involve multi topological phases, such as a topological phase with a gapped boundary, we would like to introduce the notion of an  $\mathrm{MBF}_{n+1}^{pre}$ -category, where the letter "M" stands for "multi topological phases".

**Definition 39.** An MBF $_{n+1}^{pre}$ -category is an (n+1)-category with only finitely many objects such that the hom space hom(x,y) for a pair of objects (x,y) is a unitary n-category and a unitary fusion n-category if x=y.

**Remark 25.** Notice that in an MBF $_{n+1}^{pre}$ -category, a 1-morphism in hom(x,y) might not have an adjoint if  $x \neq y$  (not required by definition). But all other 1-morphisms and higher morphisms have adjoints. Also note that there is no additive structure on objects. It is a good thing. It allows us to describe real physical situations, in which there are finite many topological phases connected by gapped domain walls. But the additive structure of gapped domain walls (as objects in hom(x,y)) is unavoidable because it can be generated automatically from the additive structure of hom(x,x) by the composition of morphisms.

**Example 17.** Examples of  $MBF_{2+1}^{pre}$ -category are full subcategories of the 3-category  $\mathcal{F}us$  (see Remark 16) consisting of only finite number of objects.

Remark 26. We have ignored the coherence properties. Subtleness may arise in the study of the universal perturbative gravitational responses in one dimension higher (recall Section V B 6). For example, it is possible that there is no non-trivial topological excitation in an n space-time dimensional topological order (e.g. the  $E_8$ quantum Hall system), corresponding to a trivial  $BF_n^{pre}$ category, but it has non-trivial gravitational responses, which can be captured by (n+1)-dimensional mathematical structures (see discussion in Section VB6 and XV). In other words, an n-category is not adequate to describe an n-dimensional topological order with non-trivial gravitational responses. We believe that the universal perturbative gravitational responses can have non-trivial effects on the coherence property. Namely, the coherence properties of n-morphisms might not hold on the nose due to the gravitational anomalies. Then we have to introduce higher invertible morphisms. For this reason, perhaps, a more complete framework to describe topological order is to use  $(\infty, n)$ -category<sup>108</sup> instead of n-category. In an  $(\infty, n)$ -category, l-morphisms exist for all  $l \in \mathbb{Z}_+$ , and all *l*-morphisms are invertible if l > n. This is beyond the scope of this paper. We leave it to the future.

# F. Conceptual gaps between a $\mathrm{BF}^{pre}$ -category and a topological order

The reason that a pre-BF-category is not yet a BF-category is because a BF $_{n+1}^{pre}$ -category describes an (n+1)-dimensional topological order equipped with a chosen (not necessarily complete) set of defects. We can choose the set to contain only the trivial defects or choose one containing the large amount of defects, which are obtained from condensation and closed by fusion. Then it creates an ambiguity because different choices of the set of defects will give different BF $_{n+1}^{pre}$ -categories, which are all associated to the same topological order. We give an explicit example below.

**Example 18.** We can associate the following two different  $BF_{2+1}^{pre}$ -categories to the same toric code model:

- 1. the modular tensor category  $Z(\Re \operatorname{ep}_{\mathbb{Z}_2})$ , which is viewed as a unitary 3-category with only one object  $\Re \operatorname{ep}_{\mathbb{Z}_2}$ , only one 1-morphism  $\Re \operatorname{ep}_{\mathbb{Z}_2}$  (viewed as the trivial  $\Re \operatorname{ep}_{\mathbb{Z}_2}$ -bimodule), 2-morphisms given by  $\Re \operatorname{ep}_{\mathbb{Z}_2}$ -bimodule functors and 3-morphisms given by natural transformations between bimodule functors. Notice that the 1-category of 2-morphisms is nothing but the modular tensor category  $Z(\Re \operatorname{ep}_{\mathbb{Z}_2}) = \operatorname{Fun}_{\Re \operatorname{ep}_{\mathbb{Z}_2}|\Re \operatorname{ep}_{\mathbb{Z}_2}}(\Re \operatorname{ep}_{\mathbb{Z}_2}, \Re \operatorname{ep}_{\mathbb{Z}_2})$ .
- 2. the full subcategory of  $\mathcal{F}$ us with the only object  $\mathcal{R}ep_{\mathbb{Z}_2}$ , denoted by  $\mathcal{F}us|_{\mathcal{R}ep_{\mathbb{Z}_2}}$ . Its 1-morphisms includes not only the trivial  $\mathcal{R}ep_{\mathbb{Z}_2}$ -bimodule  $\mathcal{R}ep_{\mathbb{Z}_2}$  but also all other bimodules such as  $_{\mathcal{R}ep_{\mathbb{Z}_2}}\mathcal{H}ilb_{\mathcal{R}ep_{\mathbb{Z}_2}}$ . This is the toric code model enriched by gapped domain walls and defects of higher codimensions<sup>26</sup>.

Notice that  $Z(\Re \operatorname{ep}_{\mathbb{Z}_2})$  viewed as a 3-category is a proper subcategory of the second 3-category  $\operatorname{Fus}|_{\operatorname{\Re ep}_{\mathbb{Z}_2}}$ . Moreover, when  $Z(\operatorname{\Re ep}_{\mathbb{Z}_2})$  is viewed as a 2-category with one object, it is the looping of the second 3-category  $\operatorname{Fus}|_{\operatorname{\Re ep}_{\mathbb{Z}_2}}$ , often denoted by  $\Omega(\operatorname{Fus}|_{\operatorname{\Re ep}_{\mathbb{Z}_2}})$ .

This is a general phenomenon in Levin-Wen type of lattice models. More precisely, we can replace  $\Re \exp_{\mathbb{Z}_2}$  by any unitary fusion 1-category  $\mathbb{C}$ . The category  $\mathbb{C}$  determines a lattice model and all its extensions by defects. One can associate to this Levin-Wen model either by its bulk excitations which are given by the unitary modular tensor category  $Z(\mathbb{C})$ , or by the full-subcategory of  $\mathcal{F}$ us consisting of only one object  $\mathbb{C}$ . These two associated categories are different as  $\mathrm{BF}_3^{pre}$ -categories.

The point of above examples is that one can associate different  $\mathrm{BF}^{pre}_{n+1}$ -categories to the same topological order. More generally, one can also paste a finite many quantum Hall systems to a (2+1)-dimensional defect in a higher dimensional topological phase without changing the phase. This process creates further ambiguities. We want to find a way to characterize topological orders in a unique way that is also minimal (or efficient) and complete.

Ideally, we would like to find a minimal generating set of all excitations. In order to do that, we would like find precise mathematical descriptions of those *condensed* excitations that can be obtained from lower dimensional excitations by condensations, and those called *elementary excitations* and the mixture of these two types called *quasi-elementary excitations*. Our hope is to use elementary excitations only to characterize the topological phase.

# G. Condensed/elementary excitations and the definition of a BF-category

In this subsection, we will discuss how to characterize condensed/elementary topological excitations in physical language. We will also propose a definition of a  $BF_{n+1}$ -category. But we can not say that we have achieved our

goal. We will discuss a few problems naturally associated to this definition.

In general, a simple l-codimensional excitation  $x^{[l]}$  in a topological phase can be very complicated. For example, it can contain a quantum Hall system for l=2, or a closed (l+1)-dimensional topological order in general. More precisely, x can be factorized as follows:

$$x^{[l]} = x_0^{[l]} \boxtimes x_1^{[l]} \boxtimes \cdots \boxtimes x_k^{[l]},$$

where  $x_1, \dots, x_k$  are all simple closed topological orders and  $\boxtimes$  is the stacking operation. We will refer to  $x_1, \dots, x_k$  as closed factors of x. The fusion product between two excitations both with non-trivial closed factors is shown in the following equation:

$$(x_0^{[l]} \boxtimes x_1^{[l]}) \otimes (y_0^{[l]} \boxtimes y_1^{[l]}) = (x_0^{[l]} \otimes y_0^{[l]}) \boxtimes x_1^{[l]} \boxtimes y_1^{[l]}.$$

If  $x_0^{[l]}, x_1^{[l]}$  and  $x_2^{[l]}$  are simple, so is  $x_0^{[l]} \boxtimes x_1^{[l]} \boxtimes x_2^{[l]}$ . Since the closed factors  $x_1^{[l]}$  and  $y_1^{[l]}$  are long-range entangled, they can not cancel each other via the stacking operation. For this reason, this closed factor are of infinite type. Namely, by applying fusion product repeatedly, we obtain infinite number of such factors. If we assume that there are only finitely many simple l-codimensional defects, then an l-codimensional defect can not contain any closed factor. We will assume this finiteness from now on so that we can ignore the closed factors completely for  $\mathrm{BF}_{n+1}$ -category.

If an excitation x can be obtain from a condensation of other excitations, then we would like to say that xis condensed. But there are a lot of ambiguities in this terminology. For example, it is possible that a set  $S_1$  of excitations can be obtained from another set  $S_2$  of excitations via condensations. At the same time,  $S_2$  can be obtained from  $S_1$  via condensations. Let's consider a more explicit situation. If a defect x can be obtained from a condensation involving an p-dimensional defect y, it seems reasonable that x is at least (p+1)-dimensional. Our intuition is that this condensation involves a large number of y, one can fine tune the system before the condensation so that all of these y-defects are arranged sitting next to (but separated from) each other. They fill a subspace of dimension at least (p+1). When you turn on the local interaction (or condensation), this subspace simply turns into a new excitation of dimension at least (p+1). But what makes the situation much more complicated is that this new excitation can be a trivial one. This means that a pure excitation might be obtained from a condensation of defects that are domain walls between higher dimensional defects. On the other hand, at least some of the later defects can also be generated from pure excitations via condensation and fusion. So it does not make any sense to say that an excitation is condensed in general. We must specify where it is condensed from. In other words, the generating sets for all topological excitations are not unique in general. We have to make some

choices. A natural choice is to include pure excitations in the generating set and hope that they can generate other defects.

**Definition 40.** An excitation (or defect) x is called *condensed* if x can be obtained from pure excitations of lower dimensions via nontrivial condensations. An excitation that is not condensed is called *quasi-elementary*.

Remark 27. In above definition, a condensed excitation can be a pure excitation or a domain wall between two other defects, which can be domain walls between higher dimensional defects.

A pure particle-like excitation is automatically quasielementary. The trivial defect  $1^{[l]}$  is quasi-elementary by definition because it can be reproduced only by the trivial condensation. In the 3+1D  $\mathbb{Z}_2$  gauge theory (recall Example 12), a particle and a vortex line are both quasielementary.

The direct sum of two condensed/quasi-elementary excitations is still condensed/quasi-elementary. The fusion product of two condensed excitations is also condensed. In general, the fusion product of a condensed excitation and a non-trivial quasi-elementary excitation is quasi-elementary. The fusion product of two quasi-elementary excitations can be either condensed or quasi-elementary.

Remark 28. A more complicated examples of quasielementary excitations can be obtained by considering the defect junctions. More precisely, in (31), if one of the three defect lines is condensed and one is quasielementary, then the defect junction is quasi-elementary. These generic defects can be obtained from the set of all elementary excitations via condensations followed by fusions, thus can be ignored in a minimal description of a topological order.

Our main goal is to find a characterization of a subset of quasi-elementary excitations called elementary excitations. In general, a simple l-codimensional quasi-elementary excitation  $x^{[l]}$  can be factorized as follows:

$$x^{[l]} = x_0^{[l]} \otimes c^{[l]}, \tag{61}$$

where  $\otimes$  is the fusion product, c is a simple condensed excitation, which will be called a *condensed factor* of x. It is tempting to say that  $x_0$  in (61) does not contain any condensed factor and in some sense "elementary". Unfortunately, such factorization is not unique in general. It is possible that  $e_0 = e_0' \otimes e_0'$  and  $e_0' \otimes e_0' \otimes e_0 = 1$ , where  $1^{[l]}$  is the trivial l-codimensional excitation. Therefore, it makes no sense to say that an excitation is free of condensed factors. We must find a better way to characterize an elementary excitation.

According to the discussion in Section VB7, a condensed and finite l-codimensional pure excitation should have gapped domain walls with the trivial l-codimensional excitation. If the condensation produces a gapless domain wall that can not be gapped, similar

to quantum Hall systems, we believe that the condensed excitation, which can be viewed as an anomalous topological order, is long range entangled and can not be canceled by fusion products. Therefore, it is reasonable to believe that such a condensed excitation can not be finite and should be ignored according to the finiteness of a  $\mathrm{BF}^{pre}$ -category. We will also refer to such condensation as of infinite type.

A condensation is of finite type if the domain wall between the condensed and uncondensed phases can be gapped. The connection by gapped domain wall defines an equivalence relation, called Witt equivalence, on the set of simple l-codimensional excitations. For an l-codimensional defect x, we denote its equivalence class by [x], which is called  $Witt\ class\ of\ x$ . Different equivalence classes are disconnected. It is clear that different elementary excitations must be disconnected.

This picture of Witt class is very important. Although it does not tell us how to select the elementary excitation from each Witt class. It immediately implies that in a real  $\mathrm{BF}_{n+1}$  category, containing only elementary excitations, simple objects are all disconnected. Therefore, it suggests us to give a definition of BF-category simply as follows:

**Definition 41.** A BF<sub>n+1</sub>-category is a BF<sup>pre</sup><sub>n+1</sub>-category such that hom(x, y) = 0 for any pair of non-equivalent  $(x \ncong y)$  simple l-morphisms  $x^{[l]}$  and  $y^{[l]}$ .

**Remark 29.** A BF<sub>n+1</sub>-category is still not Abelian because hom(x, x) is nontrivial for a simple l-morphism x in general. Each simple l-morphism in a BF<sub>n+1</sub>-category corresponds to an elementary excitation.

# H. A $\mathrm{BF}_{n+1}$ -category as the core of a $\mathrm{BF}_{n+1}^{pre}$ -category

There are a lot of natural questions associated to Definition 41. For example, given a  $BF_{n+1}^{pre}$ -category, can we determine which  $BF_{n+1}$ -category is associated to it? Is it unique? It is not hard to see that the uniqueness will be a serious problem. For example, the dotted domain wall in Figure 13 can have gapped domain wall with the trivial domain wall as shown as the blue dot in Figure 13. But we can also chose not to include the blue dot in our lattice model and create a  $BF_3^{pre}$ -category, in which the morphism associated to the dotted wall in Figure 13 is not connected (by gapped walls) to that associated to the trivial domain wall. This choice is very arbitrary. So it seems that it is impossible to associated a BF-category uniquely to a BF<sup>pre</sup>-category unless the BF<sup>pre</sup>-category satisfies certain maximal condition, i.e. all possible gapped domain walls and walls between walls are included in the  $BF^{pre}$ -category. Even if the  $BF^{pre}$ category we start with satisfies the maximal condition, there is still a problem of selecting which one in a Witt class to be the elementary excitation.

Given an l-codimensional excitation y, condensations of higher codimensional subdefects in y will produce many different *l*-codimensional excitations, which can be further condensed. The condensation provides the excitations in a Witt class of y a partial order  $\leq$ . As shown in Section 6.2 in Ref. 48, in 2+1D, two anyon systems, which are given by two modular tensor categories and connected by a gapped domain wall, can be obtained from another anyon system via two condensations. We believe that this phenomenon generalizes to higher dimensions. More precisely, we believe that if two l-codimensional excitations can be connected by gapped domain wall, these two excitations together with the gapped domain wall can be obtained from condensations of the higher codimensional sub-defects in a possibly third excitations. This picture suggests the following conjecture.

Conjecture 8: The excitations in a Witt class with the particle order  $\leq$  form a lattice, which is a mathematical notion and means a partially ordered set such that every two elements have a least upper bound and a greatest lower bound. Moreover, in each equivalence class, there is a unique (up to isomorphisms) minimum of the whole class, called the root of the class. In other words, all other defects in the class and the associated gapped domain walls can be obtained from the root via condensations. The trivial l-codimensional defect  $1^{[l]}$  is the root of the class  $[1^{[l]}]$ .

The existence of least upper bound is a little hard to see. Its evidence comes from the theory of anyon condensation in 2+1D, in which such lattice structures have already appeared in the Witt classes of non-degenerate braided fusion categories (or modular tensor categories)<sup>48,99</sup>. We will call such lattice a rooted lattice. Notice that a rooted lattice is not a rooted tree because the former contains loops in general.

**Definition 42.** A simple l-codimensional excitation x called *elementary* if x is the root of its class [x].

**Remark 30.** Using above picture, condensed l-codimensional excitations are those excitations in the Witt class  $[1^{[l]}]$  except the root  $1^{[l]}$ . All the rest are quasi-elementary.

A pure particle-like excitation is automatically elementary. The trivial defect  $1^{[l]}$  is elementary by definition. In the 3+1D  $\mathbb{Z}_2$  gauge theory (recall Example 12), a particle and a vortex line are both elementary. If x is (quasi-)elementary, then its anti-particle  $\bar{x}$  is also (quasi-)elementary. The direct sum of elementary excitations are also called elementary.

**Conjecture 9:** The fusion product of two elementary excitations is also elementary.

As a consequence, the set of all elementary excitations is closed under fusion product.

**Remark 31.** By our assumption of the finiteness, the condensation can not go forever. This gives another interesting set of excitations in a given Witt class. They

are sitting at the other ends of the rooted lattice, thus will be called *leaves* of the class. They can be obtained from the root excitation via a maximal condensation such that no further condensation is possible (see Section 6.1 in Ref. 48 for closed 2+1D cases).

The next important question is how to determine the root of each Witt class. We believe that if the complete set of observable data is given for a topological order, in principle, one should be able to determine the root. In order to achieve it, we need a yet-to-be-developed theory of condensation for higher dimensional topological order.

Conjecture 10: A full-fledged condensation theory allow us to identify the elementary excitation as the root of each lattice associated to each Witt class.

A  $BF^{pre}$ -category might not be maximal, but it already contains all the elementary defects. We propose the following conjecture.

Conjecture 11: For a given  $BF^{pre}$ -category  $\mathbb{C}$ , a full-fledged condensation theory allows us to find all new defects obtained from condensations (of finite type, i.e. with gapped domain walls) of excitations in  $\mathbb{C}$ . Then we can extend  $\mathbb{C}$  by adding all these new defects so that we obtain a maximal  $BF^{pre}$ -category  $\mathbb{C}^{max}$  which contains  $\mathbb{C}$  as a sub- $BF^{pre}$ -category.

Remark 32. For practical purpose, it is not necessary to work out the maximal  $BF^{pre}$ -category before we identify the root of a Witt class. But it is important that the entire Witt class can be recovered in principle.

Assuming this, it provides us a new way to define a  $\mathrm{BF}_{n+1}$ -category from a  $\mathrm{BF}_{n+1}^{pre}$ -category.

**Definition 43.** For a given  $\mathrm{BF}_{n+1}^{pre}$  category  $\mathcal{C}$ , the core of a  $\mathcal{C}$  is the smallest sub- $\mathrm{BF}_{n+1}^{pre}$ -category containing all elementary morphisms, each of which is defined by the root of a Witt class of morphisms in  $\mathcal{C}^{max}$ . We denote the core of  $\mathcal{C}$  by  $\mathrm{core}(\mathcal{C})$ .

**Definition 44.** A BF<sub>n+1</sub>-category is a BF<sup>pre</sup><sub>n+1</sub>-category that is equivalent to the core of another BF<sup>pre</sup><sub>n+1</sub>-category.

Remark 33. Let S be a set of morphisms in an (n+1)-category  $\mathcal{D}$ . We will call the smallest sub-(n+1)-category that contains the set S as the closure of S. In this language,  $\operatorname{core}(\mathcal{C})$  is the closure of the set of elementary morphisms in  $\mathcal{C}$ . The notion of closure only make sense in a given  $\mathcal{C}$ . But when the set S itself is already a special kind (n+1)-category, it is sometimes clear what it means by a smallest unitary  $\operatorname{sub-}(n+1)$ -category containing it without referring to a large category in which it lives. For example, in the third example in Example 19, the extension only involves adding the additive structure to S. From now on, we will refer to such extension as a  $\operatorname{BF}^{pre}$ -closure of S.

### I. Closed/anomalous $BF_{n+1}$ -categories

In this subsection, we want to give a mathematical definition of a closed/anomalous  $BF_{n+1}$ -category.

We have shown in Section XID that if we view a  $\mathrm{BF}_{n+1}^{pre}$ -category as a topological phase with defects of all codimensions, i-morphisms corresponding to i-codimensional defect, the information of the braiding between any two defects  $R_{x,y}: x \otimes y \to y \otimes x$  is automatically included in the defining structure of an (n+1)-category. Without lose of generality, we can assume x and y are of the same dimension. What motivates our definition of a closed  $\mathrm{BF}_{n+1}^{pre}$ -category is the conjecture that all excitations in a closed topological order can be detected by braiding with other defects. Therefore, if an excitation double braids with any other excitations does not give any physically detectable difference, this excitation must be the trivial excitation.

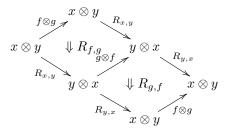
**Definition 45.** In a BF $_{n+1}^{pre}$ -category with the object \*, two pure morphisms  $x^{[l]}$  and  $y^{[l]}$  are said to be *mutually symmetric* if one of the following conditions is satisfied:

- 1. at least one of the braidings  $x \otimes y \xrightarrow{\cong} y \otimes x$  and  $y \otimes x \xrightarrow{\cong} x \otimes y$ , or simply the double braiding, is not defined as a sub-structure of a BF<sup>pre</sup><sub>n+1</sub>-category,
- 2. both of the braidings  $R_{x,y}: x \otimes y \to y \otimes x$  and  $R_{y,x}: y \otimes x \to x \otimes y$  are well-defined and double braiding  $R_{y,x} \circ R_{x,y}$  is trivial in all higher homotopies. By "trivial in all higher homotopies", we mean that  $R_{y,x} \circ R_{x,y} = \mathrm{id}_{x \otimes y}$  if i = n, or for 1 < i < n, there are  $r_{i+2}^{[i+2]}$  such that

$$R_{y,x} \circ R_{x,y} \stackrel{r_{i+2}^{[i+2]}}{\simeq} \mathrm{id}_{x \otimes y},$$
 (62)

and  $r'_{i+2}$  such that  $r_{i+2} \circ r'_{i+2} \stackrel{r_{i+3}}{\simeq} \mathrm{id}_{\mathrm{id}_{x \otimes y}}$ , so on and so forth, until  $r_{n+1} \circ r'_{n+1} = \mathrm{id}_{x \otimes y}^{n-i+1}$ .

Moreover, for all  $x \xrightarrow{f} x$  and  $y \xrightarrow{g} y$ , we require that  $R_{g,f} \circ R_{f,g}$ , as shown in the following diagram,



is trivial in all higher homotopies. And the same is true for all higher morphisms in hom(x, x) and hom(y, y).

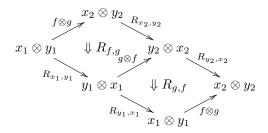
The two (l+1)-morphisms f and g above are said to be mutually symmetric if both  $R_{y,x} \circ R_{x,y}$  and  $R_{g,f} \circ R_{f,g}$  are trivial to higher homotopies and the same is true for all higher morphisms in hom(f,f) and hom(g,g).

**Remark 34.** The intuition for being trivial in all higher homotopies is that there should not be any detectable difference between  $R_{y,x} \circ R_{x,y}$  and  $\mathrm{id}_{x \otimes y}$  even with decoration by higher codimensional defects.

More generally, in a  $\mathrm{BF}_{n+1}^{pre}$ -category with the object \*, the braiding between two (not necessarily pure) morphisms  $f^{[l+1]}: x_1^{[l]} \to x_2^{[l]}$  and  $g^{[l+1]}: y_1^{[l]} \to y_2^{[l]}$  can also be automatically defined by the axioms of an (n+1)-category. We want to define when they can be said to be mutually symmetric. For simplicity, we assume that  $x_1, x_2$  and  $y_1, y_2$  are pure. More general cases are similar. We omit the details.

**Definition 46.** f and g are said to be mutually symmetric if one of the following conditions is satisfied:

- 1. the double braiding is not defined.
- 2. All three double braidings:  $R_{y_1,x_1} \circ R_{x_1,y_1}$ ,  $R_{y_2,x_2} \circ R_{x_2,y_2}$  and  $R_{g,f} \circ R_{f,g}$  in the following diagram:



are trivial in all higher homotopies. The same is true for all higher morphisms in hom(f, f) and hom(g, g).

In a  $\mathrm{BF}_{n+1}^{pre}$ -category or a  $\mathrm{BF}_{n+1}$ -category, the pure identity n-morphism  $\mathrm{id}_*^n$  is automatically mutually symmetric to all other morphisms. The set of all excitations that are mutually symmetric to all excitations including themselves forms a sub- $\mathrm{BF}^{pre}$ -category which is called the braiding-center of  $\mathcal{C}$ , denoted by  $Z(\mathcal{C})$ . Notice that for any  $\mathrm{BF}^{pre}$ -category, its braiding-center must contain the trivial  $\mathrm{BF}^{pre}$ -category  $\mathbf{1}_{n+1}$ , which is the smallest  $\mathrm{BF}_{n+1}$ -category that contains  $\{*,\mathrm{id}_*,\mathrm{id}_{\mathrm{id}_*},\cdots,\mathrm{id}_n^*\}$ .

**Remark 35.** The notation for the braiding-center  $Z(\mathcal{C})$  is slightly different from the bulk (or center)  $\mathcal{Z}(\mathcal{C})$  of  $\mathcal{C}$  because the bulk  $\mathcal{Z}(\mathcal{C})$  is a  $\mathrm{BF}_{n+2}^{pre}$ -category. But this two notions are related. We will study them in Ref. 70.

**Definition 47.** A closed BF<sub>n+1</sub>-category  $C_{n+1}$  is a BF<sub>n+1</sub>-category such that  $Z(C_{n+1}) \simeq \mathbf{1}_{n+1}$ . A BF<sub>n+1</sub>-category that is not closed is called *anomalous*.

The simplest closed  $BF_{n+1}$ -category is just the trivial one  $\mathbf{1}_{n+1}$ . The following result follows immediately from the definition.

**Lemma 19:** In a closed BF<sub>n+1</sub>-category, the only simple 1-morphism is id<sub>\*</sub>.

**Example 19.** We give a few families of examples of closed  $BF_{n+1}$ -categories.

- 1. All  $BF_{0+1}$ -categories are closed.
- 2. We know that any unitary 2-category with one object is equivalent to a unitary fusion 1-category. Since simple 1-morphisms in a closed  $BF_2$  mutually symmetric to all 1-morphisms because there is no braiding, the only closed  $BF_2$ -category must be the trivial one  $\mathbf{1}_2$ , or equivalently, the unitary 1-category  $\mathfrak{H}$ ilb. Any unitary  $BF_2$ -category with at least one non-trivial simple 1-morphisms is anomalous.
- 3. An non-degenerate braided fusion category (including all modular tensor categories) C can be viewed as 3-category with a single object \* and a single 1-morphism id\* and with hom(id\*, id\*) = C. Its BF\*\*pre\*-closure determines a closed BF\*\*3\* category by adding additive structures to the 1-morphisms and associated 2,3-morphisms. This family of examples includes all quantum Hall systems. A braided fusion 1-category that is not non-degenerate gives an anomalous BF\*\*\*pre\*-category.
- 4. The 3 + 1-dimensional  $\mathbb{Z}_2$  gauge theory gives a closed BF<sub>4</sub><sup>pre</sup>-category. In this case, all the particle-like excitations are mutually symmetric to each other, but they can be distinguished by braiding them with the vortex line.

**Lemma 20:** If  $\mathbb C$  and  $\mathbb D$  are closed  $\mathrm{BF}_{n+1}^{pre}$ -categories, then  $\mathbb C\boxtimes \mathbb D$  is also closed. If one of them is anomalous, then  $\mathbb C\boxtimes \mathbb D$  is anomalous.

The following conjecture is one of the main goal of this work. We believe that it is true up to additional anomalies such as the spins and universal perturbative gravitational responses.

Conjecture 12: There is a one-to-one correspondence between the equivalence classes of closed/anomalous  $BF_{n+1}$ -categories and the set of closed/anomalous topological orders.

### J. Closed $BF_{n+1}^{pre}$ -categories

Sometimes, it is also convenient to have a notion of a closed  $\mathrm{BF}_{n+1}^{pre}$ -category because they can describe many  $\mathrm{BF}_{n+1}^{pre}$ -categories constructed from lattice models.

Since a  $\mathrm{BF}_{n+1}^{pre}$ -category is not a minimal way of describing topological order, we expect to have non-trivial domain walls and walls between walls in a closed  $\mathrm{BF}_{n+1}^{pre}$ -category. But these walls and walls between walls must be condensed. By Conjecture 8, all simple 1-morphisms in a closed  $\mathrm{BF}_{n+1}^{pre}$ -category should be connected to the trivial domain wall id\* by gapped domain walls. But in a generic  $\mathrm{BF}_{n+1}^{pre}$ -category, gapped walls between walls are randomly included. In other words, a condensed domain wall might be superficially disconnected to id\*. Therefore, we would like to find an alternative description of a condensed domain wall.

A necessary condition for a domain wall to be condensed x is that the two-side 1-dimensional bulk-to-wall maps

$$\operatorname{hom}(\operatorname{id}_*, \operatorname{id}_*) \boxtimes \operatorname{hom}(\operatorname{id}_*, \operatorname{id}_*) \xrightarrow{L \boxtimes R} \operatorname{hom}(x, x)$$

are dominant. For a generic l-codimensional defect x, the dominance of the two-side l-dimensional bulk-to-wall map to x is not a sufficient condition for x to be condensed. For example, the vortex line in 2+1D  $\mathbb{Z}_2$  gauge theory satisfies this dominance condition, but it is an elementary excitation and is detectable by double braiding with particles.

Situation is quite different for a sub-defect in a 1-codimensional domain wall. It can only be half-braided (but not double-braid) with pure excitations in the bulk. This half-braiding is not enough to detect such a sub-defect by pure excitations. However, motivated by Levin's work Ref. 82, we propose the following conjecture.

Conjecture 13: In a domain wall, if an sub-excitation lies in the image of the two-side bulk-to-wall map, it is detectable by bulk pure excitations, otherwise, it is not detectable by bulk pure excitations.

As a consequence, the dominance of bulk-to-wall maps to a domain wall x is sufficient for x to be a condensed domain wall. Otherwise, the topological order can not be closed. Similar results should also hold for walls between walls. Therefore, we propose the following definition of a closed  $\mathrm{BF}_{n+1}^{pre}$ -category.

**Definition 48.** A BF $_{n+1}^{pre}$ -category  $\mathcal{C}$  is called closed if the only simple morphism in  $i\hat{d}_* \cap Z(\mathcal{C})$  is  $id_*^n$ , and all domain walls and walls between walls satisfy the dominance condition.

**Example 20.** Any BF $_3^{pre}$ -category arises from Levin-Wen type of lattice models enriched by defects of all codimensions $^{26}$  is closed. More precisely, such a BF $_3^{pre}$ -category is the full subcategory of  $\mathcal{F}$ us consisting of a single unitary fusion category  $\mathcal{C}$ . The 1-morphisms are  $\mathcal{C}$ -bimodules. They are condensed from bulk excitations $^{26,48}$ . This follows from the mathematical fact that the monoidal functor  $L \boxtimes R$ :  $Z(\mathcal{C})^{\boxtimes 2} \to \mathcal{F}$ un $_{\mathcal{C}|\mathcal{C}}(\mathcal{M},\mathcal{M})$  is dominant for any indecomposable semisimple  $\mathcal{C}$ -bimodule  $\mathcal{M}$ .  $^{99}$  The gapped domain walls (2-morphisms) between the 1-morphisms  $\mathcal{M}$  and  $\mathcal{C}$  is given by the category  $\mathcal{F}$ un $_{\mathcal{C}|\mathcal{C}}(\mathcal{C},\mathcal{M})$  (see the blue dot in Fig. 13 in the toric code model for an example of wall of this kind).

According to Lemma 19, the only simple 1-morphism in a core of a closed  $BF_{n+1}^{pre}$ -category is id<sub>\*</sub>. For n=2, the core of a closed  $BF_3^{pre}$ -category is the  $BF^{pre}$ -closure of \*,id<sub>\*</sub> and hom(id<sub>\*</sub>,id<sub>\*</sub>), which is a non-degenerate unitary braided fusion 1-category. In other words, the core of an closed  $BF_3^{pre}$ -category is equivalent to an non-degenerate unitary braided fusion 1-category.

**Definition 49.** Two closed  $\mathrm{BF}_{n+1}^{pre}$ -categories  $\mathfrak C$  and  $\mathfrak D$  are called *core equivalent* if  $\mathrm{core}(\mathfrak C) \simeq \mathrm{core}(\mathfrak D)$  as  $\mathrm{BF}_{n+1}^{pre}$ -categories.

**Conjecture 14:** The set of core equivalence classes of closed  $BF_{n+1}^{pre}$ -categories are one-to-one corresponding to the set of closed  $BF_{n+1}$ -categories.

### K. Closed $MBF_{n+1}^{pre}$ -category

Using the same idea in the last subsection, we can define a closed  $MBF_{n+1}^{pre}$  as follows:

**Definition 50.** An MBF $_{n+1}^{pre}$ -category  $\mathcal{C}$  is closed if

- 1.  $\hat{x}$  is a closed  $\mathrm{BF}_{n+1}^{pre}$ -category for all  $x \in \mathrm{Ob}(\mathcal{C})$ ,
- 2. all two-side bulk-to-wall maps to walls and walls between walls lying outside of  $\hat{x}$  for all  $x \in \text{Ob}(\mathcal{C})$  are dominant.

Remark 36. The intuition is that a closed MBF $_{n+1}^{pre}$ -category should contain only condensed domain walls. According to Conjecture 13, we believe that the dominance of the two-side bulk-to-wall maps guarantees that all domain walls and walls between walls in a closed MBF $_{n+1}^{pre}$ -category are condensed. If the left/right bulk-to-wall maps to a domain wall  $f^{[1]} \in \text{hom}(x,y)$  are dominant and also fully faithful, then two maps are equivalences, and f is a transparent domain wall between x and y. All of these results are not necessarily true if the MBF $^{pre}$ -category  $\mathcal C$  is not closed.

**Definition 51.** Two closed MBF $_{n+1}^{pre}$ -categories  $\mathcal{C}$  and  $\mathcal{D}$  are equivalent if there is an (n+1)-functor which gives bijection on objects and unitary n-equivalences on  $hom(x,y) \to hom(F(x),F(y))$  for all  $x,y \in Ob(\mathcal{C})$ .

Note that a closed  $\mathrm{MBF}_{n+1}^{pre}$ -category is not necessarily connected. For example, it is known that two quantum Hall systems can not be connected by gapped domain walls if these two systems belong to different Witt classes  $^{48,71,99}$ . More generally, it is not possible to connect a closed (n+1)-dimensional topological phase to an anomalous (n+1)-dimensional phase by a gapped domain wall because they must share the same (n+2)-dimensional bulk.

**Example 21.** We give a few families of closed  $MBF_{n+1}^{pre}$ -categories below.

1. Any MBF<sub>3</sub><sup>pre</sup>-category arising from Levin-Wen models enriched by domain walls and defects are closed. More precisely, any full subcategories of the 3-category Fus (see Remark 16) consisting of only finite number of objects are closed MBF<sub>3</sub><sup>pre</sup>-categories. In particular, the 3-category TC<sub>3</sub><sup>b</sup> from the toric code model with gapped boundaries is an example of closed MBF<sub>3</sub><sup>pre</sup>-category.

2. A few quantum Hall systems with a gapped domain walls form a closed  $\mathrm{MBF}_{n+1}^{pre}$ -category. If all quantum Hall systems selected are Witt equivalent  $^{48,71,99}$ , then such a closed  $\mathrm{MBF}_{n+1}^{pre}$ -category is also connected.

**Definition 52.** For a closed BF $_{n+1}^{pre}$ -category  $\mathcal{C}$ , a gapped boundary of  $\mathcal{C}$  is a closed MBF $_{n+1}^{pre}$ -category  $\mathcal{C}_f^b$  consisting of only two objects x and e, only one 1-morphism  $f: x \to e$  and no 1-morphism from e to x except the zero morphism, such that

- 1.  $\hat{x} = \mathcal{C}$  and  $\hat{e} = \mathbf{1}_{n+1}$  as  $\mathrm{BF}_{n+1}^{pre}$ -categories,
- 2.  $\hat{f}$  is a BF $_n^{pre}$ -category and  $\hat{f} \simeq \text{Im}(L) \boxtimes \mathcal{D}$ , where  $\mathcal{D}$  is a closed BF $_n^{pre}$ -category and L is the bulk-to-wall map L: hom(id\*, id\*)  $\to$  hom(f, f), which is a dominant central monoidal (n-1)-functor. "Central" means that there are half braidings (recall (49))

$$c_{L(a),b}: L(a) \bullet b \xrightarrow{\simeq} b \bullet L(a)$$

for  $a \in \text{hom}(id_*, id_*)$  and  $b \in \text{hom}(f, f)$ .

**Remark 37.** Above definition is motivated from the anyon condensation theory in  $2+1D^{48}$ . The dominance condition in Definition 52 is automatic by the definition of an MBF $_{n+1}^{pre}$ -category. It gives a good control of higher morphisms lying in the domain wall f. Note that the category hom(id\*, id\*) is a braided monoidal (n-1)-category. That L is a central monoidal (n-1)-functor implies that the functor L factors through the monoidal center of the monoidal (n-1)-category hom(f, f).

$$\begin{array}{ccc} \hom(\mathrm{id}_*,\mathrm{id}_*) & \xrightarrow{\exists ! \ \tilde{L}} & \mathcal{Z}(\hom(f,f)) \\ & & & \downarrow^{forget} \\ & & & & \hom(f,f) \, . \end{array}$$

By the dominance and the centralness, we conjecture that the monoidal (n-1)-category hom(f,f) can be recovered from hom $(\mathrm{id}_*,\mathrm{id}_*)$  as the category of modules over the commutative algebra  $L^\vee(\mathrm{id}_f)$  in hom $(\mathrm{id}_*,\mathrm{id}_*)$ , where  $L^\vee$  is the two-side adjoint functor of L. One can see that this is nothing but a higher categorical analog of the condensation theory of 2+1D developed in Ref. 48. We will give more details in the future.

**Definition 53.** If a closed  $BF_{n+1}$ -category  $\mathcal{C}$  has a gapped boundary, it is called *exact*.

# XII. TENSOR NETWORK (TN) APPROACH TO TOPOLOGICAL PHASES IN ANY DIMENSIONS

We have discussed BF<sup>H</sup> category and topological order in any dimensions via their universal properties (*i.e.* their topological invariants). In this section, we are going to discuss how to realize those BF<sup>H</sup> categories in physical systems defined by path integrals.

However, path integrals described by tensor network naturally describe a lbL system, and  $\mathrm{BF}^{\mathrm{L}}$  categories. This is because the path integrals constructed using tensors can be defined on space-time of any topologies. The path integrals that describe  $\mathrm{BF}^{\mathrm{H}}$  categories are only required to be well defined on space-time which is a mapping torus. So it is more natural to realize  $\mathrm{BF}^{\mathrm{L}}$  categories through tensor network and path integrals.

Therefore, we will first concentrate on the tensor network and path integral construction of  $\mathrm{BF}^{\mathrm{L}}$  categories. We will try to write down the most general form of path integrals using tensors, hoping our construction can produce all the  $\mathrm{BF}^{\mathrm{L}}$  categories. This way, we can obtain an alternative definition and a classification of  $\mathrm{BF}^{\mathrm{L}}$  categories in any dimensions. We have seen that is quite difficult to define  $\mathrm{BF}^{\mathrm{L}}$  category via their topological invariants (such as the fusion and the braiding properties). The tensor network and path-integral way to define and to classify  $\mathrm{BF}^{\mathrm{L}}$  categories may be more practical. Since  $\mathrm{BF}^{\mathrm{L}}$  categories form a subset of  $\mathrm{BF}^{\mathrm{H}}$  category, we can at least understand a subset of  $\mathrm{BF}^{\mathrm{H}}$  categories this way.

### A. TN realization of exact BF<sup>L</sup> category

A BF<sup>L</sup> category is a description of the topological properties of p-dimensional topological excitations, such as their fusion and braiding properties. It is quite difficult to formulate such a theory. On the other hand, we may use TN and path integrals  $^{109,110}$  to realize exact BF<sup>L</sup> categories in a concrete physical way. Thus it is possible to study the exact BF<sup>L</sup> category via its TN realization or its path integral realization, without directly involving the fusion and braiding of p-dimensional topological excitations. In this section, we will describe such an approach.

All lbL systems, include L-type topologically ordered states, are described by path-integrals. A path-integral can be described by a TN with finite dimensional tensors defined on a space-time lattice (or a space-time complex) (see Appendix A). Even though L-type topologically ordered states are all gapped, only some of them can be described by fixed-point path-integrals which are topological path integrals:

#### Definition 54. Topological path integral

- (1) A topological path integral has an action amplitude that can be described by a TN with *finite dimensional* tensors.
- (2) It is a sum of the action amplitudes for all the paths. (The summation corresponds to the tensor contraction.)
- (3) Such a sum (called the partition function Z(M)) on a closed space-time M only depend on the topology of the space-time. The partition function is invariant under the local deformations and reconnections of the TN.
- (4) To describe a lbL system, we require the partition function to be well defined on space-time with any topology.

(5) To describe a local Hamiltonian qubit system, we only require the partition function to be well defined on space-time which is a mapping torus.

In the next a few sections, we will give concrete examples of the topological path integrals. The topological path integrals are closely related to exact BF<sup>L</sup> categories. <sup>11,111–113</sup> We like to conjecture that

Conjecture 15: All exact  $BF^L$  categories (i.e. topological states with gapped boundary) are described by topological path integrals.

We make such a conjecture because we believe that the tensor network representation that we are going to discuss is the most general one. It can capture all possible fixed-point tensors under renormalization flow, and those fixed-point tensors give rise to topological path integrals. Note that the fixed point tensors with a finite dimensions always produce entanglement spectrums<sup>114</sup> that have a finite gap. As a result, the topological path integrals (described by tensors of finite dimensions) can only produce the exact BF<sup>L</sup> categories. This is because exact BF<sup>L</sup> categories have gapped boundary and gapped entanglement spectrum, while closed BF<sup>L</sup> categories that are not exact have gapless boundary and gapless entanglement spectrum.

We also like to remark that we cannot say that all topological path integrals describe exact  $\mathrm{BF^L}$  categories, since some topological path integrals are stable while others are unstable (see Definition 56). We will see that only the stable topological path integrals describe exact  $\mathrm{BF^L}$  categories, and unstable topological path integrals do not describe exact  $\mathrm{BF^L}$  categories. The definition of stable topological path integrals is given in Definition 56. Here we like to point out that

Conjecture 16: A topological path integral in (n+1)-dimensional space-time constructed with finite dimensional tensors is stable iff  $|Z(S^1 \times S^n)| = 1$ .

Note that  $Z(S^1 \times S^n)$  is the ground state degeneracy on n-dimensional space  $S^n$ . If a system has a gap and the ground degeneracy is 1, a small perturbation cannot do much to destabilize the state. So  $Z(S^1 \times S^n) = 1$  is the sufficient condition for a stable topological path integral. This argument implies that if the ground degeneracy is 1 on  $S^n$ , then the system has no locally distinguishable ground state, and the ground state degeneracy on space with other topologies are all robust against any small perturbations. In Ref. 78, it is shown that if the entanglement entropy of the region M has the following area-law structure  $S_M = cA + \gamma A^0 + o(1/A)$ , then the ground state degeneracy is indeed 1 on  $S^n$ . Here A is the "area" of the boundary of the region M.

Since the topological path integrals are independent of re-triangulation of the space-time, the partition function on a closed space-time only depends on the topology of the space-time.

Conjecture 17: Two exact BF<sup>L</sup> categories are the same iff their topological path integrals produce the topology-

dependent partition functions that belong to the same connected component in the space of topological partition functions  $V^L_{Z(M)}$ .

Here  $V_{Z(M)}^L$  is defined as

**Definition 55.** A stable topological path integral produces a topology-dependent partition function Z(M) for closed space-time M with any orientable topologies.  $V_{Z(M)}^{L}$  is the space of all such topological partition functions.

In fact, we know that each connected component in  $V_{Z(M)}^L$  contains many topology-dependent partition functions (see Section XIII A 1). Two topological path integrals, Z(M) and Z'(M), can belong to the same connected component in the space of topological partition functions if the two topological path integrals differ by

$$Z'(M)/Z(M) = W^{\chi(M)} e^{i \sum_{\{n_i\}} \phi_{n_1 n_2 \dots} \int_M P_{n_1 n_2 \dots}},$$
 (63)

where  $\chi(M)$  is the Euler number of M and  $P_{n_1n_2...}$  are combinations of Pontryagin classes:  $P_{n_1n_2...} = p_{n_1} \wedge p_{n_2} \wedge \cdots$  on M. Z(M) and Z'(M) are connected since complex numbers W and  $\phi_{n_1n_2...}$  are not quantized.

Since the path integrals are local, thus the ratio Z'(M)/Z(M) is also local. This means that if we triangulate the space-time manifold M into a complex C, then Z'(M)/Z(M) can be expressed as a product of the amplitudes from each simplex in C. Eqn. (63) may be the only topological invariant local path integral that is not quantized (i.e. W and  $\phi_{n_1n_2...}$  can be any complex numbers). Thus

Conjecture 18: It is possible that Z(M) and Z'(M) are connected iff they are related by eqn. (63).

In other words, if two topological path integrals produce two topology-dependent partition functions that differ by a factor  $W^{\chi(M)} e^{i\sum_{\{n_i\}}\phi_{n_1n_2}...\int_M P_{n_1n_2}...}$ , then the two topological path integrals describe the same exact BF<sup>L</sup> category.

Summarizing the above discussions:

- (1) All exact BF<sup>L</sup> categories (*i.e.* all L-type topological states with gapped boundary) are described by *stable* topological path integral constructed with finite dimensional tensors.
- (2) All stable topological path integrals describe exact  $\rm BF^L$  categories.

So, we may view the stable topological path integrals as a concrete definition of the exact  $BF^L$  categories. The stable topological path integrals also classify the exact  $BF^L$  categories. Since exact  $BF^L$  categories are also exact  $BF^H$  categories, the above topological path integrals and TN also describe a subset of exact  $BF^H$  categories.

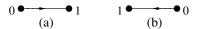


FIG. 16. The tensor  $A_{\pm v_0 v_1}$  is associated with a segment, with a branching structure. The branching structure gives the vertices a local order: the  $i^{th}$  vertex has i incoming edges. The segment in (a) has an orientation  $s_{01} = +$  and the segment in (b) has an orientation  $s_{01} = -$ .

$$0 - 2 = 0 - 2$$

FIG. 17. A triangulation of a 1D complex.

#### B. Examples of TN realization of $BF_n^L$ category

### 1. TN realization of 0+1D exact $BF_1^L$ category

The topological path integral that describes a 0+1D topologically ordered state can be constructed from two complex tensors  $A_{\pm v_0v_1}$ . The tensor  $A_{\pm v_0v_1}$  can be associated with a segment, which has a branching structure. (For details about the branching structure, see Appendix A 1.) A branching structure is a choice of orientation of each edge (see Fig. 16). Here the  $v_0$  index is associated with the vertex-0 (See Fig. 16). It represents the degrees of freedom on the vertices.

Using the tensors, we can define the topological path integral on any 1-complex that has no boundary:

$$Z = \sum_{v_0, \dots \text{ edge}} A_{s_{01} v_0 v_1} \tag{64}$$

where  $\sum_{v_0,...}$  sums over all the vertex indices,  $s_{01} = +$  or - depending on the orientation of edge(see Fig. 16), and

$$A_{+v_0v_1} = A_{v_1}^{v_0}, A_{-v_0v_1} = A_{v_0}^{v_1}.$$
(65)

We want to choose the tensors  $A_{\pm_{v_0v_1}}$  such that the path integral is re-triangulation invariant. Such a topological path integral describes a topologically ordered state in 1-space-time dimension and also define an exact BF $_1^{\rm H}$  category in 1 dimension.

The invariance of Z under the re-triangulation in Fig. 17 requires that

$$A^{v_0}_{v_2} = \sum_{v_1} A^{v_0}_{v_1} A^{v_1}_{v_2}. \tag{66}$$

We would like to mention that there are other similar conditions for different choices of the branching structures.

We obtain a total of three conditions

$$A_{v_{2}}^{v_{0}} = \sum_{v_{1}} A_{v_{1}}^{v_{0}} A_{v_{2}}^{v_{1}},$$

$$A_{v_{1}}^{v_{0}} = \sum_{v_{2}} A_{v_{2}}^{v_{0}} A_{v_{1}}^{v_{2}},$$

$$A_{v_{2}}^{v_{1}} = \sum_{v_{0}} A_{v_{0}}^{v_{1}} A_{v_{2}}^{v_{0}},$$
(67)

If we view  $A^{v_0}_{v_1}$  as a matrix A, and  $A^{v_1}_{v_0}$  as a matrix  $\tilde{A}$ , the above can be rewritten as

$$A = AA = A\tilde{A} = \tilde{A}A. \tag{68}$$

We see that the fixed point A is a matrix with a form

$$A = \tilde{A} = U^{-1} \begin{pmatrix} I_{n \times n} & 0_{n \times m} \\ 0_{m \times n} & 0_{m \times m} \end{pmatrix} U$$
 (69)

Here, we would like to introduce a notion of stable fixed point. If A is not a fixed point, when we combine two segments into one segment, A transforms as

$$\sum_{v_1} A^{v_0}_{v_1} A^{v_1}_{v_2} \to \Gamma A^{v_0}_{v_2}, 
\sum_{v_2} A^{v_0}_{v_2} A^{v_2}_{v_1} \to \Gamma A^{v_0}_{v_1}, 
\sum_{v_0} A^{v_1}_{v_0} A^{v_0}_{v_2} \to \Gamma A^{v_1}_{v_2},$$
(70)

where  $\Gamma$  is a rescaling factor. Such a transformation defines a renormalization flow. Then, A is a stable fixed point if  $A+\delta A$  will alway flow back to fixed-point tensor A' that have the same topology-dependent partition function for any  $\delta A$  and a proper rescaling factor  $\Gamma$ . This leads to the following definition of stable topological path integral:

**Definition 56.** A topological path integral described by a finite dimensional tensor T produces a partition function Z(M) that only depend on the topology of the closed space-time. If we perturb the tensor  $T \to T + \delta T$ , the perturbed tensor may flow to another fixed-point tensor T' under a renormalization flow which give rise to another topological path integral. The new topological path integral produce a topological partition function Z'(M). If Z'(M) and Z(M) belong to the same connected component of  $V_{Z(M)}^H$  for any perturbation  $\delta T$ , then the topological path integral described by T is a stable topological path integral, and T is a stable fixed-point tensor.

The above definition describes the physical meaning of stable topological path integral. However, it is very hard to use such a definition to determine if a topological path integral is stable or not. However, if the Conjecture 16 is true, it will allow us to determine which topological path integrals are stable and which are not.

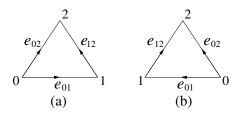


FIG. 18. The tensor  $B_{\pm}{}^{e_{01}e_{12}e_{02}}_{v_0v_1v_2}$  is associated with a triangle, with a branching structure. The branching structure gives the vertices a local order: the  $i^{th}$  vertex has i incoming edges. The triangle in (a) has an orientation  $s_{012} = +$  and the triangle in (b) has an orientation  $s_{012} = -$ .

Most of the fixed points in (69) are not stable. The only stable fixed point A has a form

$$A = \tilde{A} = U^{-1} \begin{pmatrix} 1 & 0_{1 \times m} \\ 0_{m \times 1} & 0_{m \times m} \end{pmatrix} U \tag{71}$$

A for different U give rise to the same topological partition function, and describe the same exact  $\mathrm{BF}^{\mathrm{L}}_{1}$  category. Thus there is only a trivial topological order in  $0+1\mathrm{D}$ .

**Corollary 9:** All 1-dimensional exact  $BF_1^L$  categories and exact  $BF_1^H$  categories are trivial.

### 2. TN realization of 1+1D exact $BF_2^L$ category

The topological path integral that describes a 1+1D topologically ordered state can be constructed from a tensor set  $T_2$  of two complex tensors  $T_2 = (w_{v_0}, B_{\pm v_0 v_1 v_2}^{e_{01}e_{12}e_{02}})$ . The tensor  $B_{v_0 v_1 v_2; e_{02}}^{e_{01}e_{12}}$  is complex and can be associated with a triangle, which has a branching structure (see Fig. 18).  $w_{v_0}$  is real and can be associated with a vertex. A branching structure is a choice of orientation of each edge in the complex so that there is no oriented loop on any triangle (see Fig. 18 and Fig. 21). Here the  $v_0$  index is associated with the vertex-0, the  $e_{01}$  index is associated with the edge-01 (See Fig. 18). They represent the degrees of freedom on the vertices and the edges.

Using the tensors, we can define the topological path integral on any 3-complex that has no boundary:

$$Z = \sum_{v_0, \dots; e_{01}, \dots \text{ vertex}} W_{v_0} \prod_{\text{face}} B_{s_{012}} \frac{e_{01}e_{12}e_{02}}{v_0v_1v_2}$$
(72)

where  $\sum_{v_0,\dots;e_{01},\dots}$  sums over all the vertex indices and the edge indices,  $s_{012} = +$  or - depending on the orientation of triangle (see Fig. 18), and

$$B_{+v_0v_1v_2}^{e_{01}e_{12}e_{02}} = B_{v_0v_1v_2;e_{02}}^{e_{01}e_{12}},$$

$$B_{-v_0v_1v_2}^{e_{01}e_{12}e_{02}} = B_{e_{01}e_{12}}^{v_0v_1v_2;e_{02}}.$$
(73)

We want to choose the tensors  $(w_{v_0}, B_{\pm v_0 v_1 v_2}^{e_{01}e_{12}e_{02}})$  such that the path integral is re-triangulation invariant. Such

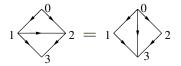


FIG. 19. A triangulation of a 2D complex.

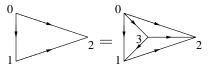


FIG. 20. A triangulation of another 2D complex.

a topological path integral, if stable, describes a topologically ordered state in 2-space-time dimensions and also define an exact  $\mathrm{BF}_2^\mathrm{L}$  category.

The invariance of Z under the re-triangulation in Fig. 19 requires that

$$\sum_{e_{12}} B^{e_{01}e_{12}}_{v_0v_1v_2;e_{02}} B^{v_1v_2v_3;e_{13}}_{e_{12}e_{23}} = \sum_{e_{03}} B^{e_{01}e_{13}}_{v_0v_1v_3;e_{03}} B^{v_0v_2v_3;e_{03}}_{e_{02}e_{23}}.$$
(74)

We would like to mention that there are other similar conditions for different choices of the branching structures. The branching structure of a tetrahedron affects the labeling of the vertices.

The invariance of Z under the triangulation in Fig. 20 requires that

$$B_{v_0v_1v_2;e_{02}}^{e_{01}e_{12}} = \sum_{e_{03}e_{13}e_{23},v_3} w_{v_3} B_{e_{03}e_{31}}^{v_0v_3v_1;e_{01}} B_{v_3v_1v_2;e_{32}}^{e_{31}e_{12}} B_{v_0v_3v_2;e_{02}}^{e_{03}e_{32}}.$$
(75)

Again there are other similar conditions for different choices of the branching structures.

The above two types of the conditions are sufficient for producing a topologically invariant partition function Z.

Here we would like to point out that two different solutions are regarded as equivalent (*i.e.* describe the same exact  $BF^L$  category) if

- (1) they can be connected by a one parameter family of the solitons continuously.
- (2) they can be mapped into each other by the relabeling of the indices  $i \to \tilde{i} = f(i)$ .

There may be additional equivalence relations. In general, two fixed-point tensor sets  $T_2$  and  $T_2'$  are regarded as equivalent if their corresponding topological partition functions for any closed orientable space-time mapping tori are the same: Z(M) = Z'(M).

It turns out that in 1+1D, all the stable solutions have a trivial topology-dependent partition function Z(M)=1, since there is no nontrivial topological order in 1+1D. <sup>12,47</sup> Thus

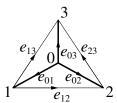


FIG. 21. The tensor  $C_{\pm v_0 v_1 v_2 v_3; \phi_{123} \phi_{013} \phi_{023} \phi_{012}}$  is associated with a tetrahedron, which has a branching structure. If the vertex-0 is above the triangle-123, then the tetrahedron will have an orientation  $s_{0123} = -$ . If the vertex-0 is below the triangle-123, the tetrahedron will have an orientation  $s_{0123} = +$ . The branching structure gives the vertices a local order: the  $i^{th}$  vertex has i incoming edges.

Corollary 10: all 2-dimensional exact  $BF_2^L$  categories and exact  $BF_2^H$  categories are trivial.

Since the boundary of 1+1D gapped state can always be gapped,

Corollary 11: all 2-dimensional closed  $BF_2^L$  categories and closed  $BF_2^H$  categories are exact.

As a result,

**Corollary 12:** all 2-dimensional closed  $BF_2^L$  categories and closed  $BF_2^H$  categories are trivial.

### 3. TN realization of 2+1D exact $BF_3^H$ category

The topological path integral that describes a 2+1D topologically ordered state with a gapped boundary can be constructed from a tensor set  $T_3$  of two real and one complex tensors  $T_3$  =

 $(w_{v_0},A_{v_0v_1}^{e_{01}},C_{\pm v_0v_1v_2v_3;\phi_{123}\phi_{013}\phi_{023}\phi_{012}})$ . The complex tensor  $C_{\pm v_0v_1v_2v_3;\phi_{123}\phi_{013}\phi_{023}\phi_{012}}^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23}}$  can be associated with a tetrahedron, which has a branching structure (see Fig. 21). A branching structure is a choice of orientation of each edge in the complex so that there is no oriented loop on any triangle (see Fig. 21). Here the  $v_0$  index is associated with the vertex-0, the  $e_{01}$  index is associated with the edge-01, and the  $\phi_{012}$  index is associated with the triangle-012. They represents the degrees of freedom on the vertices, edges, and the triangles.

Using the tensors, we can define the topological path integral on any 3-complex that has no boundary:

$$Z = \sum_{v_0, \dots; e_{01}, \dots; \phi_{012}, \dots \text{ vertex}} W_{v_0} \prod_{\text{edge}} A_{v_0 v_1}^{e_{01}} \times$$

$$\prod_{\text{tetrs}} C_{s_{0123} v_0 v_1 v_2 v_3; \phi_{123} \phi_{013} \phi_{023} \phi_{012}}$$
(76)

where  $\sum_{v_0,\dots;e_{01},\dots;\phi_{012},\dots}$  sums over all the vertex indices, the edge indices, and face indices,  $s_{0123} = +$  or - depending on the orientation of tetrahedron (see Fig. 21), and

$$C_{+v_0v_1v_2v_3;\phi_{123}\phi_{013}\phi_{023}\phi_{012}}^{+e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}} = C_{v_0v_1v_2v_3;\phi_{013}\phi_{123}}^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}$$

$$C_{-v_0v_1v_2v_3;\phi_{123}\phi_{013}\phi_{023}\phi_{012}}^{-e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}} = C_{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}^{v_0v_1v_2v_3;\phi_{123}\phi_{013}\phi_{023}\phi_{012}} = C_{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}^{-e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}$$

$$(77)$$

We want to choose the tensors  $(w_{v_0}, A_{v_0v_1}^{e_{01}}, C_{\pm v_0v_1v_2v_3;\phi_{123}\phi_{013}\phi_{023}\phi_{012}})$  such that the path integral is re-triangulation invariant. Such a topological path integral describes a L-type topologically ordered state in 3-space-time dimensions and also define an exact BF $_3^{\rm L}$  category.

The invariance of Z under the re-triangulation in Fig. 22 requires that

$$\sum_{\phi_{123}} C_{v_0 v_1 v_2 v_3; \phi_{013} \phi_{123}}^{e_{01} e_{02} e_{03} e_{12} e_{13} e_{23}; \phi_{012} \phi_{023}} C_{v_1 v_2 v_3 v_4; \phi_{124} \phi_{234}}^{e_{12} e_{13} e_{14} e_{23} e_{24} e_{34}; \phi_{123} \phi_{134}} \\
= \sum_{e_{04}} A_{v_0 v_4}^{e_{04}} \sum_{\phi_{014} \phi_{024} \phi_{034}} C_{v_0 v_1 v_2 v_4; \phi_{014} \phi_{124}}^{e_{01} e_{02} e_{04} e_{12} e_{14} e_{24}; \phi_{012} \phi_{024}} C_{e_{01} e_{03} e_{04} e_{13} e_{14} e_{34}; \phi_{013} \phi_{034}}^{e_{014} \phi_{124} e_{23} e_{24} e_{34}; \phi_{023} \phi_{034}}. \tag{78}$$

We would like to mention that there are other similar conditions for different choices of the branching structures. The branching structure of a tetrahedron affects the labeling of the vertices.

The invariance of Z under the re-triangulation in Fig. 23 requires that

$$C_{v_{0}v_{2}v_{3}v_{4};\phi_{024}\phi_{234}}^{e_{02}e_{02}e_{02}e_{024}e_{234};\phi_{023}\phi_{034}} = \sum_{e_{01}e_{12}e_{13}e_{14},v_{1}} w_{v_{1}}A_{v_{0}v_{1}}^{e_{01}}A_{v_{1}v_{2}}^{e_{12}}A_{v_{1}v_{3}}^{e_{13}}A_{v_{1}v_{4}}^{e_{14}} \sum_{\phi_{012}\phi_{013}\phi_{014}\phi_{123}\phi_{124}\phi_{134}} C_{v_{0}v_{1}v_{2}v_{3};\phi_{013}\phi_{012}}^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}} C_{v_{0}v_{1}v_{2}v_{4};\phi_{014}\phi_{124}}^{v_{01}v_{1}\phi_{12}}C_{v_{0}v_{1}v_{2}v_{3};\phi_{013}\phi_{133}}^{e_{014}\phi_{123}\phi_{124}\phi_{134}} C_{v_{1}v_{2}v_{3}v_{4};\phi_{014}\phi_{134}}^{e_{014}\phi_{123}\phi_{134}}C_{v_{1}v_{2}v_{3}v_{4};\phi_{134}\phi_{134}}^{e_{12}e_{14}e_{23}e_{24}e_{34};\phi_{123}\phi_{134}}$$

$$(79)$$

Again there are other similar conditions for different choices of the branching structures.

The above two types of the conditions are sufficient for producing a topologically invariant partition function Z, which is nothing but the topological invariant for three

manifolds introduced by Turaev and Viro. <sup>111</sup> Again, two different solutions are regarded as equivalent if they produces the same topology-dependent partition function for

any closed space-time.

#### C. The unitarity condition

Let we choose the space-time to have a form  $M^d \times I$ , where  $M^d$  is the space and the 1D segment I is the time. The path integral on  $M^d \times I$  gives rise to a transfer matrix  $U = \mathrm{e}^{-\tau H}$ . The indices on one boundary of  $M^d \times I$  correspond of the row index of U and the indices on the other boundary of  $M^d \times I$  correspond of the column index of U. Since H is hermitian, U must has non-negative eigenvalues. This is the unitarity condition for the path integral. One can show that if the two real and one complex tensors  $(w_{v_0}, A^{e_{01}}_{v_0v_1}, C_{\pm}^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23}}_{v_0v_1v_2v_3;\phi_{123}\phi_{013}\phi_{023}\phi_{012}})$  satisfy (see Appendix A 3)

$$w_{v_0} > 0, \quad A_{v_0 v_1}^{e_{01}} > 0, \tag{80}$$

$$C_{+v_0 v_1 v_2 v_3; \phi_{123} \phi_{013} \phi_{023} \phi_{012}} = (C_{-v_0 v_1 v_2 v_3; \phi_{123} \phi_{013} \phi_{023} \phi_{012}}^{-e_{01} e_{02} e_{03} e_{12} e_{13} e_{23}})^*,$$

then the path integral described by  $(w_{v_0}, A^{e_{01}}_{v_0v_1}, C_{\pm v_0v_1v_2v_3;\phi_{123}\phi_{013}\phi_{023}\phi_{012}})$  is unitary. The above is for 2+1D path integral. We have similar unitarity condition for TN path integral in other dimensions.

Now it is clear that the above construction of topological path integrals can be easily generalized to any dimensions. Such a construction can be viewed as concrete definition of exact BF<sup>L</sup> categories (or L-type topological orders with gappable boundaries).

### D. TN realization of generic BF<sup>L</sup> category

After using TNs and topological path integral to define/construct exact BF<sup>L</sup> categories, in this section, we would like to use TNs and topological path integrals to construct generic BF<sup>L</sup> categories. The idea is every simple, for a TN defined by a stable topological path integral, its boundary can give us an explicit construction of a generic BF<sup>L</sup> category.

Let us try to construct a path integral that gives rise to a generic topological theory described by a generic BF<sup>L</sup> category. Note that such a topological theory may be anomalous. So the topological theory may not be described by a path integral in the same dimension. The trick is try to describe the topological theory using a path integral in higher dimension.

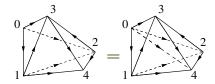


FIG. 22. A triangulation of a 3D complex.

To define a generic and potentially anomalous topological theory (i.e. a generic  $\mathrm{BF}_n^\mathrm{L}$  category  $\mathcal{C}_n$ ) in n-dimensional space-time, let us assume that space-time cell-complex  $M^n$  has a topology of  $S^1 \times S^{n-1}$ . We then view  $S^{n-1}$  as a boundary of n-dimensional solid ball  $D^n$  and extend  $M^n$  to  $M^{n+1} = S^1 \times D^n$ . (For a more general discussion, see Appendix E.) Now we can give  $M^n$  a triangulation, and then extend that triangulation to  $M^{n+1}$ .

Next, we put a topological path integral described by a tensor set  $T_{n+1}$  of finite-dimensional tensors on cell-complex  $M^{n+1}$ . Such a topological path integral defines a (n+1)-dimensional exact  $\mathrm{BF}^{\mathrm{L}}_{n+1}$  category  $\mathfrak{C}^{\mathrm{exact}}_{n+1}$ . More generally, we can also modify the tensor set on the boundary from  $T_{n+1}$  to  $T^b_{n+1}$ , such that the path integral on  $M^{n+1}$  is still a topological path integral (*i.e.* retriangularization independent).

We note that the topological path integral on  $M^{n+1}$ , defined by the pair of tensor sets  $(T_{n+1}, T_{n+1}^b)$ , only depends on the fields (the indices of the tensors) on the boundary  $M^n$ , since the topological path integral does not change under the re-triangularization on  $M^{n+1}$ , as long as we fix the triangularization and the fields on the boundary  $M^n$ . So the topological path integral on  $M^{n+1}$  also defines a path integral (a quantum theory) on  $M^n$  (an n-dimensional space-time). Since the topological path integral is invariant under the retrianglations both in the bulk and on the boundary, we like to make the following conjecture:

Conjecture 19: Assume that the tensor set  $T_{n+1}$  describes a stable topological path integral in (n+1)-dimensional space-time, and the pair  $(T_{n+1}, T_{n+1}^b)$  describes a topological path integral in (n+1)-dimensional space-time  $M^{n+1}$  with a boundary  $M^n$ . Such a theory is gapped in the bulk and on the boundary.

Since the boundary is gapped. The topological path integral on  $M^{n+1}$  also defines the gapped topological excitations on the boundary  $M^n$  which is described by an n-dimensional  $\mathrm{BF}_n^{\mathrm{L}}$  category  $\mathfrak{C}_n$ . We have  $\mathfrak{C}_{n+1}^{\mathrm{exact}} = \mathcal{Z}_n(\mathfrak{C}_n)$ . Now, we can see that

Conjecture 20: The theory on  $M^n$ , defined by a pair of stable tensor sets  $(T_{n+1}, T_{n+1}^b)$  as outlined above, describes a generic  $BF_n^L$  category  $\mathfrak{C}_n$  in n dimension. The center of  $\mathfrak{C}_n$  is given by  $\mathfrak{C}_{n+1}^{\text{exact}} = \mathcal{Z}_n(\mathfrak{C}_n)$ , where  $\mathfrak{C}_{n+1}^{\text{exact}}$  is the  $BF_{n+1}^H$  category realized by the (n+1)-dimensional topological path integral defined by the tensor set  $T_{n+1}$ . Note that the n-dimensional theory is only defined on

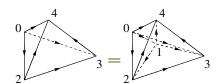


FIG. 23. A triangulation of another 3D complex.

 $M^n = S^1 \times S^{n-1}$  with a particular extension  $M^{n+1} = S^1 \times D^n$ . Since  $\mathfrak{C}_{n+1}^{\mathrm{exact}}$  is nontrivial, the different extensions of  $M^n$  to different (n+1)-dimensional manifolds may leads to different values of the path integral (see discussions in the next section). This is a sign of gravitational anomaly.

We also note that the path integral defined above on  $M^n$  is enough to give rise to p-dimensional topological excitations on the space  $S^{n-1}$  and determine their fusion and braiding properties. This is why we believe the Conjecture 20. We like to further conjecture that

Conjecture 21: All generic  $BF_n^L$  categories  $\mathcal{C}_n$  in n dimension can be realized this way by pairs of tensor sets  $(T_{n+1}, T_{n+1}^b)$  of finite dimensional tensors.

### E. TN realization of closed BF<sup>L</sup> category

With the tensor formulation of generic BF<sup>L</sup> categories discussed in last section, it is quite natural to reduce it into a tensor formulation of closed BF<sup>L</sup> category. We just need to figure out which (generic) BF<sup>L</sup> categories constructed above are closed BF<sup>L</sup> categories.

First, let us start with the tensor formulation of a generic n-dimensional BF<sup>L</sup><sub>n</sub> category  $\mathcal{C}_n$  defined by a pair of tensor sets  $(T_{n+1}, T_{n+1}^b)$ . We conjecture that

Conjecture 22:  $(T_{n+1}, T_{n+1}^b)$  describes a closed  $BF_n^L$  category if  $T_{n+1}$  gives rise to a topological partition function  $Z(M^{n+1})$  that describes a trivial topological phase. Every closed  $BF_n^L$  category can be obtained this way.

To understand the conjecture, we note that the topological partition function for a trivial phase has a form

$$Z(M^{n+1}) = W^{\chi(M^{n+1})} e^{i \sum_{\{n_i\}} \phi_{n_1 n_2 \cdots} \int_{M^{n+1}} P_{n_1 n_2 \cdots}}$$
(81)

We can introduce a trivial topological state described by the following partition function

$$Z^{\text{tri}}(M^{n+1}) = W^{-\chi(M^{n+1})} e^{-i\sum_{\{n_i\}} \phi_{n_1 n_2 \dots} \int_{M^{n+1}} P_{n_1 n_2 \dots}}$$

(The boundary of such a trivial topological state is the gCS anomalous topological state discussed in Section II C.) We then stack the two systems. The partition function for the combined system satisfies

$$Z^{\text{combined}}(M^{n+1}) = 1. (82)$$

We see that for the combined system, the (n + 1)-dimensional path integral on any two (n+1)-dimensional cell complex,  $N^{n+1}$  and  $N'^{n+1}$ , will be the same, as long as

- (1) the boundaries of  $N^{n+1}$  and  $N'^{n+1}$  are the same:  $\partial N^{n+1} = \partial N'^{n+1} = M^n$ ,
- (2) the triangulations on  $N^{n+1}$  and  $N'^{n+1}$  reduce to the same triangulations on  $M^n$ ,

Thus, the combined (n+1)-dimensional path integral on  $N^{n+1}$  only depend on the fields on the n-dimensional

boundary  $M^n$ . The path integral does not depend on how we extend  $M^n$  into  $N^{n+1}$ . Therefore the (n+1)-dimensional path integral on  $N^{n+1}$  define an n-dimensional path integral on  $M^n$ , which in turn describes a well defined topological theory in n-dimensional spacetime, whose excitations are described by a closed  $\mathrm{BF}_n^\mathrm{L}$  category in n dimensions.

Remark 38. We note that although the (n+1)-dimensional path integral on  $N^{n+1}$  described by the tenser set  $(T_{n+1}, T_{n+1}^b)$  is topological (i.e. independent of the retriangulizations both in the bulk and on the boundary), Only the path integral of the combined system leads to a well defined n-dimensional path integral on  $M^n = \partial N^{n+1}$  which describes a gapped topological state, a closed  $BF_n^L$  category. Such an n-dimensional path integral on  $M^n$  for the combined system may not be retriangularization invariant. Thus the n-dimensional path integral on  $M^n$  may not describe an exact  $BF_n^L$  category.

#### XIII. EXAMPLES OF TN REALIZATION OF TOPOLOGICALLY ORDERED STATES

## A. TN realization of exact BF<sup>L</sup> categories

1. TN realization of the trivial  $BF^L$  category in any dimensions

One way to obtain topological path integral in any dimensions is to assume that all tensors in the tensor set  $T_n$  are 1-dimensional (i.e. all the indices in the tensor have a range 1). Such kind of tensor set describes a trivial topological state. Here we like to use such a trivial example to perform some nontrivial check for some of our conjectures.

When all the tensor are 1-dimensional, we assign a weight  $W_k$  to each k-cell in the n-dimensional space-time complex  $M^n$ , where  $W_k$  is real for  $k = 0, \dots, n-1$  and is complex for k = n. The partition function has a form

$$Z(M^n) = W_n^{N_n^+} (W_n^*)^{N_n^-} \prod_{k=0}^{n-1} W_k^{N_k}.$$
 (83)

where  $N_k$  is the number of k-cells and  $N_n^{\pm}$  is the number of n-cells with the  $\pm$  orientation.

The re-triangulation invariance requires that

$$W_k = W^{(-)^k}, \quad W \in \mathbb{R}. \tag{84}$$

In this case

$$Z(M^n) = W^{\chi(M^n)} \tag{85}$$

where  $\chi(M)$  is the Euler characteristics of cell complex M. We see that a trivial topological theory can give rise to a nontrivial partition function with a nontrivial dependence on the topology of the space-time. Such a

seemingly non-trivial "topological" partition function actually describes a trivial topological order since W is not quantized.  $W \neq 1$  and W = 1 correspond to the same phase.

## 2. TN realization of a 1+1D unstable topological path integral

One way to construct 1+1D topological path integral is to use the elements of a finite group G to label the edge degrees of freedom and assume there is no degrees of freedom on the vertices (i.e. the range of the vertex index is 1:  $v_i = 1$ ). We choose  $B_{v_0v_1v_2;e_{02}}^{e_{01}e_{12}}$ ,  $B_{e_{01}e_{12}}^{v_0v_1v_2;e_{02}}$ , and  $w_{v_0}$  in (72) as

$$B_{111;e_{02}}^{e_{01}e_{12}} = B_{e_{01}e_{12}}^{111;e_{02}} = 1 \text{ if } e_{01}e_{12} = e_{02},$$

$$B_{111;e_{02}}^{e_{01}e_{12}} = B_{e_{01}e_{12}}^{111;e_{02}} = 0 \text{ otherwise, } w_1 = |G|^{-1}, \quad (86)$$

where  $e_{ij} \in G$  and |G| is the number of the elements in G. The resulting path integral is a topological path integral (*i.e.* re-triangulation invariant).

We find that the partition function on  $S^1 \times S^1$  is

$$Z(S^1 \times S^1) = |G|. \tag{87}$$

According to the Conjecture 16, the topological path integral is unstable. This is a correct result. The topological path integral actually describes a 1+1D gauge theory in zero coupling limit, where G is the gauge group. In 1+1D, a gauge theory alway confine for any finite coupling even for discrete gauge group. Thus 1+1D gauge theory is unstable and does not describe a topological phase.

#### 3. TN realization of the 3+1D trivial $BF_4$ category

In Ref. 72, 76, and 115, a 3+1D topological path integral is constructed using the data of a UMTC. The partition function of the 3+1D topological theory on a closed 3+1D space-time is given by

$$Z(M^4) = e^{\frac{2\pi i}{8}(c_R - c_L)\sigma(M^4)} W^{\chi(M^4)}, \tag{88}$$

where  $c_R - c_L$  is the chiral central charge of the UMTC,  $\sigma(M^4)$  the signature of  $M^4$ , and W an arbitrary real number that can be continuously deformed to 1.

Now we would like to show that the construction using UMTC data give rise to a trivial  $BF_4$  category (i.e. a trivial 3+1D topological order). First, we can have 3+1D bosonic lattice model that breaks the time-reversal symmetry and produces an effective action

$$S = i\kappa_{gCS} \int_{M^4} p_1, \tag{89}$$

where  $p_1$  is the first Pontryagin class. The lattice model has no topological excitations and is a trivial topologically ordered state, since it is continuously connected to the product state as  $\kappa_{gCS} \to 0$ .

If we stack the lattice model with the 3+1D topological path integral constructed from UMTC, we can make the combined theory to have a trivial partition function  $Z(M^4) = 1$ , if we choose  $\kappa_{gCS} = -\frac{2\pi i}{24}(c_R - c_L)$ , since the signature  $\sigma(M^4)$  can be expressed as

$$\sigma(M^4) = \int_{M^4} p_1/3. \tag{90}$$

According to Conjecture 17, the combined theory realize a trivial topological order (*i.e.* a trivial BF category). The original theory must be trivial since its stacking with a trivial phase is trivial.

By choosing different UMTC's, we can construct many different topological path integrals that describe the same trivial category. Later in Section XIII C, we will take advantage of this many-to-one representation of trivial BF category and use them to construct closed BF categories in one lower dimension.

### 4. TN realization of a 2+1D exact $BF_3^L$ category

One way to construct a 2+1D topological path integral is to use the elements of a finite group G to label the edge degrees of freedom and assume there is no degrees of freedom on the vertices and the faces (i.e. the range of the index is 1:  $v_i = 1$  and  $\phi_i = 1$ ). We choose  $C_{v_0v_1v_2v_3;\phi_{012}\phi_{123}}^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}, C_{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}^{v_{00}v_1v_2v_3;\phi_{013}\phi_{123}}, C_{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}^{e_{00}v_0v_1}, w_{v_0},$  and  $A_{v_0v_1}^{e_{01}}$  in (76) as

$$C_{v_0v_1v_2v_3;\phi_{013}\phi_{123}}^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}} = C_{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}^{v_0v_1v_2v_3;\phi_{013}\phi_{123}}$$

$$= 1, \text{ if every } e_{ij}e_{jk} = e_{ik},$$

$$C_{v_0v_1v_2v_3;\phi_{013}\phi_{123}}^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}} = C_{v_0v_1v_2v_3;\phi_{013}\phi_{123}}^{v_0v_1v_2v_3;\phi_{013}\phi_{123}}$$

$$= 0, \text{ otherwise,}$$

$$w_{v_0} = |G|^{-1}, \quad A_{v_0v_1}^{e_{01}} = 1. \tag{91}$$

where  $e_{ij} \in G$ . The resulting path integral is a topological path integral (*i.e.* re-triangulation invariant).

We find that the partition function on  $S^1 \times S^2$  is

$$Z(S^1 \times S^2) = 1. \tag{92}$$

According to the Conjecture 16, the topological path integral is stable. This is a correct result. The topological path integral actually describes a 2+1D gauge theory in zero coupling limit, where G is the gauge group. In 2+1D, a discrete gauge theory is alway in the deconfined phase for small enough coupling. Thus 2+1D gauge theory is stable and describe a topological phase.

We can construct a more general 2+1D exact BF<sub>3</sub><sup>L</sup> category by twisting the above topological path integral by the cocycle  $\omega(g_0, g_1, g_2)$  in the group cohomology class

$$H^3(G, \mathbb{R}/\mathbb{Z}): ^{116-118}$$

$$C_{v_0v_1v_2v_3;\phi_{013}\phi_{123}}^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}} = \omega(e_{01}, e_{12}, e_{23}),$$
if every  $e_{ij}e_{jk} = e_{ik},$ 

$$C_{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}^{v_0v_1v_2v_3;\phi_{013}\phi_{123}} = \omega^*(e_{01}, e_{12}, e_{23}),$$
if every  $e_{ij}e_{jk} = e_{ik},$ 

$$C_{v_0v_1v_2v_3;\phi_{013}\phi_{123}}^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}} = C_{v_0v_1v_2v_3;\phi_{013}\phi_{123}}^{v_0v_1v_2v_3;\phi_{013}\phi_{123}} = C_{v_0v_1v_2v_3;\phi_{013}\phi_{123}e_{23};\phi_{012}\phi_{023}}^{v_0v_1v_2v_3;\phi_{013}\phi_{123}} = 0, \text{ otherwise,}$$

$$w_{v_0} = |G|^{-1}, \quad A_{v_0v_1}^{e_{01}} = 1. \tag{93}$$

Certainly, such a construction can also be easily generalized to any higher dimensions.

### B. TN realization of generic BF<sup>L</sup> categories

Eq. 93 gives rise to an exact  $BF^L$  category in 2+1D. Its boundary will give rise an generic  $BF^L$  category in 1+1D. So using the TN realization of exact  $BF^L$  categories, we can obtain the TN realization of generic  $BF^L$  categories in one lower dimensions.

### C. TN realization of closed BF<sup>L</sup> categories

In Ref. 72, 76, and 115, a 3+1D topological path integral is constructed using the data of a UMTC, which describe a trivial topologically ordered state in 3+1D as discussed before (see Section XIII A 3). However, the natural boundary of the 3+1D topological path integral is very interesting. It is a 2+1D topologically ordered state whose particle-like topological excitations are described by the same UMTC that was used to construct the 3+1D topological path integral. <sup>76,79–81</sup> So all the closed 2+1D BF<sub>3</sub><sup>L</sup> categories that correspond to the UMTC can be described by TN and its topological path integral in one higher dimensions. Since the 3+1D topological path integral describes a trivial topologically ordered state in 3+1D, we believe that we can also use a path integral in 2+1D to describe the closed 2+1D BF<sub>3</sub><sup>L</sup> categories that correspond to UMTC's.

Suppose we have a 2+1D path integral that describe a UMTC, what is the nature of its fixed-point partition function (i.e. the volume-independent partition function  $Z_0(M^3)$ ). Since the UMTC describes a gapped topological state, should we expect the fixed-point partition function  $Z_0(M^3)$  to be topological, i.e. independent of the deformation of the shape of space-time  $M^3$ . The answer is no if the chiral central charge  $c_R - c_L$  of the UMTC is not multiple of 8. The volume-independent partition function  $Z_0(M^3)$  must contain a gravitational Chern-Simons term

$$Z_0(M^3) == e^{i\frac{2\pi(c_R - c_L)}{24} \int_{M^{d+1}} \omega_3}$$
 (94)

which make  $Z_0(M^3)$  not topological.

On the other hand, if we introduce a framing to  $M^3$  and allow the fixed-point partition function to depend on the framing, or we extend  $M^3$  to a  $M^4$  with  $\partial M^4 = M^3$ , then we can obtain a fixed-point partition function that is topological (see eqn. (88)). But such a topological partition function for the 2+1D UMTC is "anomalous" since it either depends on the framing, or depends the signature of its 4-dimensional extension. However, such an anomaly can be canceled by stacking with an invertible gCS anomalous topological order described in Section II C. The price we pay is that the combined anomaly free partition function (see eqn. (94)) cannot be topological unless  $c_R - c_L = 0 \mod 8$ .

#### XIV. PROBING AND MEASURING BF CATEGORIES (I.E. TOPOLOGICAL ORDERS AND GRAVITATIONAL ANOMALIES)

In this paper, we pointed out a direct connection between gravitational anomalies and topological orders in one higher dimension. Using such a connection, we have developed a systematic theory of topological order and gravitational anomaly in any dimensions. In this section, we will discuss another important issue: How to probe and measure different topological orders and gravitational anomalies. Or in other words, how to probe and measure different BF categories. Here "probe and measure" means the methods in experiments and/or numerical calculations that allow us to distinguish different topological orders and gravitational anomalies.

# A. How to probe and measure the closed BF<sup>L</sup> categories described by non-fixed-point path integrals

If the path integral described by a TN has no long range correlations, it will describe a closed BF<sup>L</sup> category. But how to determine which closed BF<sup>L</sup> category the path integral can produce? How to determine if two path integrals give rise to the same closed BF<sup>L</sup> category or not?

Let us consider, for simplicity, a 2+1D path integral defined by a TN on a 2+1D space-time complex M. We assume that the path integral has no long range correlations. We consider the limit where the space-time complex is formed by many 3-cells (the thermal dynamical limit). In this limit, the partition function have a form

$$Z_{\text{path}}(M^3) = e^{c_0 N_0 + c_1 N_1 + c_2 N_2 + c_3 N_3} Z_0(M)$$

$$e^{O(1/N_0) + O(1/N_1) + O(1/N_2) + O(1/N_3)}.$$
(95)

where  $N_d$  is the number of the d-cells in the space-time complex. Note that that term  $c_iN_i$  is proportional to the volume of space-time. So  $c_i$ 's correspond to the density of ground state energy and are not universal. On the other hand,  $Z_0(M^3)$  is the volume-independent part

of the partition function which contains universal structures.

To understand the universal structures in  $Z_0(M^{d+1})$ , let us use  $\mathcal{M}_{M^{d+1}}$  to denote the moduli space the closed space-time  $M^{d+1}$  with different metrics but the same topology. Then the volume-independent part of the partition function  $Z_0(\cdot)$  can be viewed as a map from  $\mathcal{M}_{M^{d+1}}$  to  $\mathbb{C}$ .

Conjecture 23: If  $Z_0(M_0^{d+1}) \neq 0$  for a point  $M_0^{d+1}$  in  $\mathcal{M}_{M^{d+1}}$ , then  $Z_0(M^{d+1}) \neq 0$  for every point  $M^{d+1}$  in  $\mathcal{M}_{M^{d+1}}$ .

We note that  $\mathcal{M}_{M^{d+1}}$  is connected. So  $Z_0(M_0^{d+1})$  and  $Z_0(M_1^{d+1})$  for two points  $M_0^{d+1}, M_1^{d+1} \in \mathcal{M}_{M^{d+1}}$  are partition functions of topological states that belong to the same gapped phase. As a result  $Z_0(M_0^{d+1})/Z_0(M_1^{d+1}) = W^{\chi(M)} e^{i \sum_{\{n_i\}} \phi_{n_1 n_2 \cdots} \int_M P_{n_1 n_2 \cdots}}$  (see Conjecture 18). So the partition function  $Z_0(\cdot)$  is actually a map  $Z_0: \mathcal{M}_{M^{d+1}} \to \mathbb{C} - \{0\} \sim U(1)$ . If  $\pi_1(\mathcal{M}_{M^{d+1}}) \neq 0$ , such map may have a non-trivial winding number.

To understand the winding number, let us use  $G_{\text{homeo}}(M^{d+1})$  to denote the homeomorphism group of the space-time  $M^{d+1}$ . Note that  $G_{\text{homeo}}(M^{d+1})$  only depends on the topology of  $M^{d+1}$  and is the same for every point  $M^{d+1} \in \mathcal{M}_{M^{d+1}}$ . Let us use  $G^0_{\text{homeo}}(M^{d+1})$  to denote the subgroup of  $G_{\text{homeo}}(M^{d+1})$  which is the connected component of  $G_{\text{homeo}}(M^{d+1})$  that contain identity. The mapping class group is formed by the discrete components of the homeomorphism group:

Definition 57. mapping class group  $\begin{array}{ll} \operatorname{MCG}(M^{d+1}) &\equiv & G_{\operatorname{homeo}}(M^{d+1})/G^0_{\operatorname{homeo}}(M^{d+1}) &= \\ \pi_0[G_{\operatorname{homeo}}(M^{d+1})]. \end{array}$ 

We note that every homeomorphism  $f: M^{d+1} \to M^{d+1}$  in  $\mathrm{MCG}(M^{d+1})$  defines a mapping torus  $M^{d+1} \rtimes_f S^1$  that describes how  $M^{d+1}$  deform around a loop  $S^1$ , and correspond to an element in  $\pi_1(\mathcal{M}_{M^{d+1}})$ .

Since  $\pi_1(\mathcal{M}_{M^{d+1}}) = \text{MCG}(M^{d+1})$ , the winding number is a group homomorphism  $\text{MCG}(M^{d+1}) \to \mathbb{Z}$ . So the winding numbers (i.e. the group homomorphisms) always form integer classes  $\mathbb{Z}$ . This leads us to believe that the winding numbers (or the group homomorphism  $\text{MCG}(M^{d+1}) \to \mathbb{Z}$ ) are always realized by the partition function  $Z_0(M^{d+1})$  that contains the gravitational Chern-Simons term  $\omega_{d+1}$ 

$$Z_0(M^{d+1}) \sim e^{i \kappa_{gCS} \int_{M^{d+1}} \omega_{d+1}}$$
 (96)

where  $\mathrm{d}\omega_{d+1}=P_{n_1n_2\cdots}$  is a combination of Pontryagin classes which are the only integer characteristic classes of oriented manifolds. If the values of  $P_{n_1n_2\cdots}$  on mapping tori are not always non-zero, then  $\kappa_{gCS}$  is quantized since we require

$$e^{i \kappa_{gCS} \int_{M^{d+1} \rtimes_f S^1} P_{n_1 n_2 \cdots}} = 1$$
 (97)

for any mapping torus  $M^{d+1}\rtimes_f S^1$ . In this case,  $\frac{\kappa_{gCS}}{2\pi}\int_{M^{d+1}\rtimes_f S^1}P_{n_1n_2...}$  is the winding number for the

loop in  $\mathcal{M}_{M^{d+1}}$  described by the mapping torus  $M^{d+1} \rtimes_f S^1$ 

Such type of winding numbers and the partition function exist only when d+1=4k+3. We also note that there is always one and only one combination of Pontryagin classes for each d+1=4k+3 whose value on mapping torus is always zero. (They correspond to the signature  $\sigma$  of the manifold.) For such Pontryagin class, the corresponding gravitational Chern-Simons term  $\omega_{d+1}^{\sigma}$  can have a unquantized coefficient.

Clearly, two bosonic systems that give rise to partition functions with different winding numbers must belong to two different phases. So the winding numbers of partition functions are a type of topological invariants that can be used to probe and measure the closed  $\mathrm{BF}^\mathrm{L}_{d+1}$  categories.

To gain a better understanding of what part of the  $\mathrm{BF}_{d+1}^{\mathrm{L}}$  categories that the winding numbers characterize, we note that invertible topological order are described by partition functions that are pure U(1) phase. In particular the  $\mathbb{Z}$ -class of invertible topological order (see Section XV), such as the  $E_8$  quantum Hall state in d+1=3, are described by

$$Z_0(M^{d+1}) = e^{i \kappa_{gCS} \int_{M^{d+1}} \omega_{d+1}}$$
 (98)

and  $\kappa_{qCS}$  is quantized since we require

$$e^{i\kappa_{gCS}\int_{M^{d+2}}P_{n_1n_2\cdots}} = 1 \tag{99}$$

for any closed  $M^{d+2}$ . We note that even  $\omega_{d+1}^{\sigma}$  is required to have a quantize coefficient in order to be diffeomorphic invariant. For example, in 2+1D

$$Z_0(M^3) = e^{i \kappa_{gCS} \int_{M^{d+1}} \omega_3} = e^{i \frac{2\pi c}{24} \int_{M^{d+1}} \omega_3}$$
 (100)

where  $c \equiv 12\kappa_{gCS}/\pi$  must be quantized as 0 mod 8. In fact c is the chiral central charge of the edge states and the above partition function describes the stacking of c/8  $E_8$  quantum Hall states.

We note that the group homomorphism  $\mathrm{MCG}(M^{d+1}) \to \mathbb{Z}$  is additive under the stacking  $\boxtimes$  operation. By comparing eqn. (96) and eqn. (100), we find that

**Theorem 4:** For any  $BF_{d+1}^L$  category  $\mathfrak{C}_{d+1}$ , there always exists an invertible  $BF_{d+1}^L$  category  $\mathfrak{C}_{d+1}^{invertible}$ , such that the partition function  $Z_0(\cdot)$  for the combined  $BF_{d+1}^L$  category  $\mathfrak{C}_{d+1} \boxtimes \mathfrak{C}_{d+1}^{invertible}$  has vanishing winding numbers.

We like to stress that having vanishing winding numbers does not imply the partition function must be constant locally. In fact, the  $E_8$  quantum Hall state is an example that the partition function has zero winding numbers (since the Pontryagin number for  $p_1$  is always zero for mapping torus), but the partition function is not a constant due to the non-zero thermal Hall effect.

# B. How to probe and measure the exact BF<sup>L</sup> categories described by non-fixed-point path integrals

We know that an exact BF<sup>L</sup> can be described by a topological path integral that is independent of retriangulation of space-time and independent of local change of space-time metrics (i.e.  $Z_0(M^{d+1})$ ) is constant on  $\mathcal{M}_{M^{d+1}}$  locally). Such a topological path integral is a fixed-point of the renormalization group transformation.

For a non-fixed-point path integral that describes an exact BF<sup>L</sup> category, it will flow to a fixed-point path integral under renormalization group transformation. <sup>109,110</sup> Since the renormalization group transformation change the volume of the space-time, the fixed-point path integral has no volume dependent part and correspond to the volume-independent partition function  $Z_0(M^{d+1})$ . The fixed-point path integral should be closely related to the topological path integral:

Conjecture 24: The topological path integral that describes an exact  $BF^L$  category coincide with the volume-independent part  $Z_0(M^{d+1})$  of the partition function that realizes the  $BF^L$  category. Thus we can use the volume-independent part of the partition functions,  $Z_0(M^{d+1})$ , to probe the topological orders described by exact  $BF^L$  categories.

This conjecture has lead to some related researches and is confirmed for simple exact BF<sup>L</sup> categories. <sup>58,119,120</sup> Since the topological path integral is re-triangulation invariant, we see that  $Z_0(M^{d+1})$  is not only independent of volume, it is also independent of shape. It only depends on the topology of  $M^{d+1}$ . Therefore, the topological partition function  $Z_0(M^{d+1})$  is a topological invariant for d+1-manifold  $M^{d+1}$ , and different BF<sup>L</sup><sub>d+1</sub> categories give different topological invariants for  $M^{d+1}$ . In 2+1D, the topological invariants from exact BF<sup>L</sup><sub>a</sub> categories are the Turaev-Viro invariants for 3-manifolds. <sup>111</sup>

We have introduced several related concepts, non-fixed point partition functions  $Z(M^{d+1})$  of local bosonic path integral, volume-independent partition functions  $Z_{0}(M^{d+1})$ , topological partition functions  $Z_{\text{top}}(M^{d+1})$  (assumed to be stable here), and closed/exact  $\mathrm{BF}^{L}_{d+1}$  categories. We will summarize their relations here. We first note that all of them are monoids under the stacking operation  $\boxtimes$ . Thus we can describe their relations using surjective monoid homomorphisms

non-fixed point partition functions 
$$Z(M^{d+1}) o$$
volume-independent partition functions  $Z_0(M^{d+1}) o$ closed BF $_{d+1}^{L}$  categories (101)

The reduction from non-fixed point partition functions  $Z(M^{d+1})$  to volume-independent partition functions  $Z_0(M^{d+1})$  is the renormalization group flow. Volume-independent partition functions  $Z_0(M^{d+1})$  may have non-zero winding numbers that force them to have a non-trivial dependence on the metrics of space-time (via

the gravitational Chern-Simons terms). The relation between volume-independent partition functions  $Z_0(M^{d+1})$  and the closed  $\mathrm{BF}^{\mathrm{L}}_{d+1}$  categories is many-to-one.

We also have a short exact sequence

$$1 \to \{ W^{\chi(M)} e^{i \sum_{\{n_i\}} \phi_{n_1 n_2 \cdots} \int_M P_{n_1 n_2 \cdots}} \} \to$$
topological partition functions  $Z_{\text{top}}(M^{d+1}) \to$ exact BF<sup>L</sup><sub>d+1</sub> categories  $\to 1$  (102)

The relation between volume-independent partition functions  $Z_0(M^{d+1})$  and exact  $\mathrm{BF}^{\mathrm{L}}_{d+1}$  categories is one-to-one only if we mod out the factor like  $W^{\chi(M)} \operatorname{e}^{\operatorname{i} \sum_{\{n_i\}} \phi_{n_1 n_2 \cdots} \int_M P_{n_1 n_2 \cdots}}$ . Last, we have

 $1 \to \text{topological partition functions } Z_{\text{top}}(M^{d+1}) \to \text{volume-independent partition functions } Z_0(M^{d+1}).$ 

# C. How to probe and measure the closed $BF^H$ categories

A closed  $\mathrm{BF}^{\mathrm{L}}_{d+1}$  category is described by a local bosonic path integral that is required to be well defined for arbitrary space-time  $M^{d+1}$ , while a closed  $\mathrm{BF}^{\mathrm{H}}_{d+1}$  category is described by a local bosonic Hamiltonian that is required to be well defined for arbitrary space  $\Sigma^d$ . Since closed  $\mathrm{BF}^{\mathrm{H}}_{d+1}$  categories are gapped, we require the Hamiltonian on a closed space  $\Sigma^d$  to be gapped, whose degenerate ground states form a finite dimensional vector space V which is a subspace of the total Hilbert space  $H_{\Sigma^d}$  of the boson system. Let  $\mathcal{M}_{\Sigma^d}$  be the moduli space for closed space  $\Sigma^d$  with different metrics and  $\mathcal{M}$  the disjoint union of these moduli spaces. We see that we have a ground-state vector space V for every point in  $\mathcal{M}_{\Sigma^d}$ . Therefore, for each  $\Sigma^d$ , a closed  $\mathrm{BF}^{\mathrm{L}}_{d+1}$  category gives us a complex vector bundle on  $\mathcal{M}_{\Sigma^d}$ , which is a sub-bundle of the trivial bundle  $\mathcal{M}_{\Sigma^d} \times H_{\Sigma^d}$ .

Conjecture 25: The complex vector bundle of degenerate ground states on  $\mathcal{M}$  may fully characterize the closed  $BF_{d+1}^H$  category.

We note that,  $\pi_1(\mathcal{M}_{\Sigma^d}) = \text{MCG}(\Sigma^d)$ . Along a loop g in  $\pi_1(\mathcal{M}_{\Sigma^d})$ , the fiber bundle gives us a monodromy U(g) which is a unitary matrix acting on the ground state vector space V. We may view g as an element in the group  $\text{MCG}(\Sigma^d)$ . So U(g) gives an projective representation of  $\text{MCG}(\Sigma^d)$ .

To understand why we only get a projective representation, we note that the topological robustness of the ground state degeneracy implies that the unitary matrix  $U_0$  for contractible loop must be a pure over-all phase (which can be path dependent), so that  $U_0$  cannot distinguish (or split) the degenerate ground states. Similarly, U(g) may also have a path-dependent overall phase, which leads to the projective representation of  $MCG(\Sigma^d)$ . We also like to mention that the trace of U(g) is the partition function on the corresponding mapping

torus:

$$\operatorname{Tr} U(g) = Z_0(\Sigma^d \rtimes_g S^1).$$

As a result, we obtain

$$|Z_0(\Sigma^d \times S^1)| = \text{ground state degeneracy on } \Sigma^d.$$

For space with different topologies, we will get different projective representations. Those finite dimensional projective representations are the non-Abelian geometric phases of the degenerate ground states introduced in Ref. 5 and 6. Certainly, the non-Abelian geometric phases contain more information than the projective representations. They contain all the information about the vector bundle  $\mathcal{E}_{\Sigma^d}$  on  $\mathcal{M}_{\Sigma^d}$ , and thus fully characterize the closed BF<sup>H</sup><sub>d+1</sub> category.

If the vector bundle  $\mathcal{E}_{\Sigma^d}$  is not flat, the partition function on mapping torus  $Z_0(\Sigma^d \rtimes_g S^1)$  cannot be topological. It will depend on the metrics of the space-time  $\Sigma^d \rtimes_g S^1$ . It is very strange since the bosonic system has short range correlation and a finite energy gap. In the thermal dynamical limit, the space-time becomes flat, and bosonic system should not be able to sense the geometry of the space-time. The fact that the partition function does depend on the metrics of the space-time means that the entanglement in the ground state can still sense the geometry of the space in the flat limit. We like to link such a geometry sensitivity to the gapless nature of boundary excitations and entanglement spectrum:

Conjecture 26: The boundary of a closed  $BF_{d+1}^H$  category is gappable iff the ground state vector bundle  $\mathcal{E}_{\Sigma^d}$  over  $\mathcal{M}_{\Sigma^d}$  is flat.

What is the obstruction that prevent the vector bundle to be flat? First, for a contractible loop g, U(g) is a pure U(1) phase. So the non-flat part is only contained in the U(1) phase of the complex vector bundle. We can examine it by considering the determinant bundle  $\mathcal{E}_{\Sigma^d}^{\text{det}}$  of the vector bundle  $\mathcal{E}_{\Sigma^d}$ , which is a complex line bundle over  $\mathcal{M}_{\Sigma^d}$ . Let us consider closed submanifold  $B \subset \mathcal{M}_{\Sigma^d}$ . Then, Chern number of the line bundle  $\mathcal{E}_{\Sigma^d}^{\text{det}}$  on B should be given by a certain Pontryagin number on  $\Sigma^d \rtimes B$ :

$$\int_{B} C = \int_{\Sigma^{d} \times B} P_{n_1 n_2 \dots} \tag{103}$$

due to some localness consideration. Here  $\Sigma^d \rtimes B$  is a fiber bundle with the space  $\Sigma^d$  as the fiber and B as the base manifold. We see that the Pontryagin classes in all dimensions could be the obstructions to have a flat vector bundle  $\mathcal{E}_{\Sigma^d}$ .

Let us consider an example of 2+1D theory whose gravitational response contain the gravitational Chern-Simons term:

$$Z_0(\Sigma^2 \rtimes S^1) = e^{i\frac{2\pi c}{24} \int_{\Sigma^2 \rtimes S^1} \omega_3}$$
 (104)

where c is the chiral central charge of the edge states. For such a theory, the Chern number in eqn. (103) is given

by

$$\int_{B^2} C = \frac{c}{24} D_g \int_{\Sigma^2 \rtimes B^2} p_1 = \text{integer}, \tag{105}$$

for any surface bundle  $\Sigma^2 \rtimes B^2$ , where  $D_g$  is the ground state degeneracy on  $\Sigma^2$ , and g is the genus of  $\Sigma^2$ .

Since  $\int_{\Sigma^2 \times B^2} p_1 \neq 0$  for some surface bundle,  $\int_{B^2} C \neq 0$  for some B and the vector bundle  $\mathcal{E}_{\Sigma^d}$  is not flat if  $c \neq 0$ . So the appearance of the gravitational Chern-Simons term implies the gapless edge excitations.

It was shown that  $\int_{\Sigma^2 \rtimes B^2} p_1 = 0 \mod 12$  for any orientable surface bundles. Let  $^{121,122}$  If the genus of the fiber  $\Sigma^2$  is less than 2, then  $\int_{\Sigma^2 \rtimes B^2} p_1 = 0$ . Let  $^{121,123}$  If the genus of the fiber  $\Sigma^2$  is greater than 2, then we can always find a base manifold  $B^2$  with a genus equal or less than 111, such that there is a surface bundle  $\Sigma^2 \rtimes B^2$  with  $\int_{\Sigma^2 \rtimes B^2} p_1 = \pm 12$ . Thus

**Theorem 5:** For a 2+1D gapped quantum liquid (i.e. a closed  $BF_3^H$  category), the chiral central charge of the edge state is quantized as  $cD_g/2 =$  an integer, for each g > 2.

**Application 1.** For a bosonic quantum Hall state with one branch of edge mode (*i.e.* c = 1), the ground state degeneracy  $D_q$  must be even for q > 2.

**Application 2.** For closed BF<sub>3</sub><sup>H</sup> categories with fusion rule  $i \otimes j = \bigoplus_k N_{ij}^k k$ , the ground state degeneracy  $D_g$  is given by<sup>33</sup>

$$D_g = \sum_{i} (N_i N_{\bar{i}})^{g-1} \tag{106}$$

where  $\bar{i}$  is the antiparticle of i and the matrix  $N_i$  is given by  $(N_i)_j^k = N_{ij}^k$ . For  $\nu = 1$  bosonic Pfaffian quantum Hall state, we have

$$N_{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_{\psi} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_{\sigma} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

$$(107)$$

We find that  $D_1=3$ ,  $D_2=10$ ,  $D_3=36$ ,  $D_4=136$ ,  $D_5=528$ , etc. Therefore the chiral central charge must be quantized as  $c=0 \mod 1/2$ , which agrees with c=3/2. We also see that  $cD_g/2=$  integer is not valid for g=2. This allows us to prove that

Corollary 13: for a 4-dimensional orientable surface bundle E with fiber of genus 2,  $\int_E p_1 = 0 \mod 24$  (or the signature is  $0 \mod 8$ ).

**Application 3.** The chiral central charge of invertible  $BF_3^H$  category is quantized as  $c=0 \mod 2$ , since  $D_g=1$ . However, at the moment, we do not know if the minimal chiral central charge c=2 can be realized by an invertible  $BF_3^H$  category. In contrast, the chiral central charge of invertible  $BF_3^L$  category is quantized as  $c=0 \mod 8$ , where the minimal chiral central charge c=8 is realized by the  $E_8$  quantum Hall state.

If we have a fermionic system in 2+1D, both  $\Sigma^d$  and  $\Sigma^2 \rtimes B^2$  should be chosen to be spin manifolds. In this case  $\int_{\Sigma^2 \rtimes B^2} p_1 = 0 \mod 48$  for any spin surface bundles. <sup>121,123</sup> We find that

**Theorem 6:** For fermionic invertible topological orders, the chiral central charge is quantized as  $c = 0 \mod 1/2$ .

The minimal chiral central charge c=1/2 can be realized by  $p+\mathrm{i}\,p$  superconductor, which contain no non-trivial topological excitations.

Next let us consider bosonic 1+1D topological orders (i.e. closed BF<sub>2</sub><sup>H</sup> categories). Since MCG( $S^1$ ) is trivial,  $\mathcal{M}_{\Sigma^1}$  is simply connected. Since the Pontryagin classes for circle bundle  $S^1 \rtimes B$  all vanishes, the determinant bundle of the vector bundle  $\mathcal{E}_{\Sigma^1}^{\text{det}}$  over  $\mathcal{M}_{\Sigma^1}$  is flat. Thus the vector bundle  $\mathcal{E}_{\Sigma^1}$  is flat, and the vector bundle is trivial since  $\mathcal{M}_{\Sigma^1}$  is simply connected. Therefore, all bosonic closed BF<sub>2</sub><sup>H</sup> categories are trivial (if we assume that all non-trivial closed BF<sub>2</sub><sup>H</sup> categories have non-trivial vector bundle  $\mathcal{E}_{\Sigma^d}$ ).

It appears that the vector bundle  $\mathcal{E}_{\Sigma^d}$  on  $\mathcal{M}_{\Sigma^d}$  is a high resolution characterization of the closed  $\mathrm{BF}^{\mathrm{H}}_{d+1}$  category. The non-trivial closed  $\mathrm{BF}^{\mathrm{H}}_{d+1}$  category should lead to a non-trivial vector bundle  $\mathcal{E}_{\Sigma^d}$ . On the other hand, since the structure of the vector bundle can be so rich, it is very likely that not every allowed vector bundle  $\mathcal{E}_{\Sigma^d}$  on  $\mathcal{M}_{\Sigma^d}$  can be realized by closed  $\mathrm{BF}^{\mathrm{H}}_{d+1}$  categories.

# D. How to probe and measure the exact $BF^H$ categories

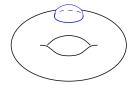
For an exact BF<sup>H</sup> category, the ground state vector bundle is always flat and the partition function on mapping torus always topological. The Conjecture 26 implies the reverse: a flat vector bundle always correspond to an exact BF<sup>H</sup> category. Also, for a flat vector bundle, the unitary matrices U(g) form a representation of the mapping class group  $MCG(\Sigma^d)$  which fully characterize the flat bundle. Thus

Conjecture 27: An exact  $BF_{d+1}^H$  category is fully characterized by a collection of representations of the mapping class groups  $MCG(\Sigma^d)$  for various spatial topologies. In particular, the representations of  $MCG(\Sigma^d)$  can be computed via the universal wave function overlap<sup>58,119,120</sup> or tensor network calculations. <sup>54–57</sup>

# E. How to probe the gravitational anomaly through quasiparticle statistics

We also have the following two useful conjectures. The first one is

Conjecture 28:  $C_n$  is a closed  $BF_n^H$  category iff (1) any nontrivial pure p-dimensional topological excitations in  $C_n$  can be detected by their nontrivial mutual



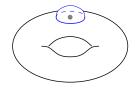


FIG. 24. The thick blue line is a string-like topological excitation in d-dimensional space created at the boundary of a membrane operator. The 1-dimensional topological excitations condense on a membrane-like topological excitation which form a torus. A particle-like topological excitation on the membrane-like topological excitation can be probed by calculating the average of the membrane operator with its boundary on the torus which may or may not enclose the particle-like topological excitation.

braiding properties with some other topological excitations.

(2) any nontrivial pure p'-dimensional topological excitations on a p-dimensional topological excitation  $M^p$  can be detected by their nontrivial mutual braiding properties with some other topological excitations on  $M^D$  or by their different "mutual half-braiding" properties with some other topological excitations in  $\mathcal{C}_n$  which condense on  $M^p$ .

We like to point out that the above conjecture is not fully formulated. We state it here just to illustrate an idea.

Here the mutual braiding mean that we fix one topological excitation and move other topological excitations around the first one. A nontrivial mutual braiding property means that the mutual braiding generate a nontrivial (non-Abelian) geometric phase. See Section XII for a mathematical description.

Let us explain what is the "mutual half-braiding" property (see Fig. 24). 82 We know that a d-dimensional excitation in the bulk  $\mathcal{C}_n$  can be created at the boundary of a (d+1)-brane operator  $\hat{O}_{d+1}$ . If a d-dimensional excitation condense on the subspace  $M^p$ , then we have  $\langle \Psi_{M^p,0}|\hat{O}_{d+1}|\Psi_{M^p,0}\rangle \neq 0$  if the boundary of the (d+1)-brane operator  $\hat{O}_{d+1}$  lie within  $M^p$ . Here  $|\Psi_{M^p,0}\rangle$  is the wave function of the system where the describe a pure p-dimensional topological excitation on  $M^p$ . Let  $|\Psi_{M^p,i}\rangle$  be the wave function of the system where  $M^p$  contains some other topological excitations. Then a topological excitation on  $M^D$  can be distinguished by their different "mutual half-braiding" properties with some other topological excitations in  $\mathcal{C}_n$  if

$$\frac{\langle \Psi_{M^p,i} | \hat{O}_{D+1} | \Psi_{M^p,i} \rangle}{\langle \Psi_{M^p,0} | \hat{O}_{D+1} | \Psi_{M^p,0} \rangle} \neq 1 \tag{108}$$

when the topological excitation i on  $M^p$  is enclosed by the boundary of  $\hat{O}_{p+1}$ .

The second one is a generalization of a result by Levin $^{82}$ :

Conjecture 29: An  $C_n$  is the bulk (or center) of a  $BF_n^H$  category  $C_{n-1}$  iff they satisfy the following condition: all the topological excitations in  $C_{n-1}$  can be distinguished

by their different mutual braiding properties with some other topological excitations in  $\mathcal{C}_{n-1}$ , or by their different "mutual half-braiding" properties with some other topological excitations in  $\mathcal{C}_n$  which condense on the boundary.

# XV. TOPOLOGICAL ORDERS THAT HAVE NO NON-TRIVIAL TOPOLOGICAL EXCITATIONS

As an application of the above conjectures, in this section, we are going to try to classify a very simple class of topological orders that has no non-trivial topological excitations in the bulk. One may wonder, without any non-trivial topological excitations, such class of topological orders may only contain the trivial one. In fact, even without any non-trivial topological excitations in the bulk, the topological order can still be non-trivial since the boundary may be non-trivial. The  $E_8$  bosonic quantum Hall state in  $2+1\mathrm{D}$  is an example of such kind of topological order, whose boundary must be gapless.

We know that in 2+1D, the number of point-like topological excitations is equal to the ground state degeneracy on  $T^2$ . In higher dimensions, the ground state degeneracy on  $S^1 \times S^n$  and on other spatial topologies are directly related to the number of point-like and other topological excitations. Thus we have the following result.

**Theorem 7:** A H-type topological order (a closed  $BF_n^H$  category) has no non-trivial elementary topological excitations iff it has no ground state degeneracy on any closed spaces.

For such a topological order  $\mathcal{C}_n$  (a closed BF<sub>n</sub>-category), due to the absence of ground state degeneracy, topological partition function  $Z(M^n)$  on the space-time  $M^n$  (with/without boundaries) is an non-zero  $\mathbb{C}$ -number. Due to unitarity, it must be a pure U(1) phase on any closed space-time  $M^n$  which is a mapping torus for H-type theory. We have a parallel result for L-type.

**Theorem 8:** A L-type topological order (a closed  $BF_n^L$  category) has no non-trivial elementary topological excitations iff its topological partition function  $Z(M^n)$  is alway a pure U(1) phase on any closed orientable spacetime, up to a factor  $W^{\chi(M^n)} e^{i\sum_{\{n_i\}}\phi_{n_1n_2}...\int_{M^n}P_{n_1n_2}...}$ .

Given such a topological phase  $\mathcal{C}_n$  without any ground state degeneracy, if we stack the time-reversed system  $\overline{\mathcal{C}}_n$  on the top of  $\mathcal{C}$ , all the phases are canceled, and we must obtain the trivial topological order, in which all topological partition functions are  $1 \in \mathbb{C}$ . Namely, we must have  $\mathcal{C}_n \boxtimes \overline{\mathcal{C}}_n = \mathbf{1}_n$ . So such kind of topological orders are invertible. For a generic closed  $\mathrm{BF}_n^L$  category  $\mathcal{C}_n$ , the inverse of its partition function,  $1/Z_{\mathcal{C}_n}(M^n)$ , may not be the partition function of any topological order. But when  $Z_{\mathcal{C}_n}(M^n)$  is a pure U(1) phase,  $1/Z_{\mathcal{C}_n}(M^n)$  will be a partition function of a topological order. In fact  $1/Z_{\mathcal{C}_n}(M^n) = Z_{\overline{\mathcal{C}}_n}(M^n)$ . So when the partition function  $Z_{\mathcal{C}_n}(M^n)$  is a pure U(1) phase, the corresponding topological order  $\mathcal{C}_n$  is invertible. An non-zero quantum field theory (L-type theory) with 1-dimensional state spaces

is also called *invertible* by Freed and Teleman<sup>61-63</sup>.

Let us use  $C_n$  to denote the n-complex obtained by triangulating the space-time. Due to locality, we require that, at least for some simple space-time topologies, the U(1) phase  $Z(M^n)$  comes from the product of local U(1) phases for each n-simplex:<sup>46</sup>

$$Z(M^n) = \langle C_n, \omega_n \rangle = \prod_{i \in C_n} \langle S_n^{(i)}, \omega_n \rangle, \quad \langle S_n^{(i)}, \omega_n \rangle \in U(1),$$

where  $S_n^{(i)}$  is the  $i^{\text{th}}$  n-simplex in the complex  $C_n$ , and  $\omega_n$  is a U(1)-valued n-cochain. Such a partition function will be called local. In general  $\omega_n$  may depend on some local geometric structures (such as connections and vielbein  $^{87}$  on the n-complex) that can still affect the gapped ground state.

To find the *n*-cochains  $\omega_n$  that can describe invertible closed L-type BF<sub>n</sub><sup>L</sup> categories, let us consider a partition function constructed via the Pontryagin classes  $P_{n_1 n_2 \dots} = p_{n_1} \wedge p_{n_2} \wedge \dots$ :

$$Z(M^n) = \langle C_n, \omega_n \rangle = e^{i \sum_{n_1 n_2 \dots \phi_{n_1 n_2} \dots \int_{M^n} P_{n_1 n_2} \dots}}$$

$$(109)$$

Such a partition function is local and is a pure U(1) phase that does not depend on the volume of the space-time. So it describes an invertible  $\mathrm{BF}_n^\mathrm{L}$  category. But such an invertible  $\mathrm{BF}_n^\mathrm{L}$  category is trivial since the partition function can be continuously deformed to 1. We see that, although Pontryagin classes can give rise to local topological partition functions, since the coefficients  $\phi_{n_1 n_2 \dots}$  of the Pontryagin classes are not quantized, they do not give rise to non-trivial invertible  $\mathrm{BF}_n^\mathrm{L}$  categories. So a key to obtain non-trivial invertible  $\mathrm{BF}_n^\mathrm{L}$  categories is to find topological terms with quantized coefficients.

We note that the cobordism group of 5-dimensional closed oriented manifolds is  $\Omega_5^{SO} = \mathbb{Z}_2$  (see Appendix F). It was proposed recently in Ref. 60, that there is a corresponding quantized topological term given by a Stiefel-Whitney class  $w_2 \wedge w_3$ :

$$Z(M^5) = \langle C_5, \omega_5 \rangle = e^{\pi i \int_{M^5} w_2 \wedge w_3}.$$
 (110)

Let us assume that there exists a 4+1D gapped local bosonic theory, integrating out the matter field will produce the above partition function. Such a model realizes a non-trivial exact BF<sub>5</sub><sup>L</sup> category  $\mathcal{C}_5^{L,w_2w_3}$  since the value of the partition function is a non-trivial -1 on  $M^5 = SU(3)/SO(3)$  (see Appendix F) and the partition function is a topological invariant. Such a model also realizes a non-trivial exact BF<sub>5</sub><sup>H</sup> category  $\mathcal{C}_5^{H,w_2w_3}$  since the value of the partition function is non-trivial on a 5-dimensional mapping torus  $\mathbb{C}P^2 \rtimes_* S^1$  generated by the complex conjugation  $*: \mathbb{C}P^2 \to \mathbb{C}P^2$  (see Appendix F). The above local topological partition function, being a pure U(1) phase, describes an invertible BF<sub>5</sub><sup>L</sup> category  $\mathcal{C}_5^{L,w_2w_3}$  (also an invertible BF<sub>5</sub><sup>H</sup> category  $\mathcal{C}_5^{H,w_2w_3}$ ), which is its own inverse, i.e.  $\mathcal{C}_5^{L,w_2w_3} \boxtimes \mathcal{C}_5^{L,w_2w_3} = \mathbf{1}_5$ . We believe that  $e^{\pi \, i \, \int_M w_2 \wedge w_3}$  is the only quantized topological

term in 5-dimensional space-time. Thus in 4+1D, the invertible  $BF_5^L$  categories form a group  $\mathbb{Z}_2$ .

The boundary of the exact BF<sub>5</sub><sup>L</sup> category  $\mathcal{C}_5^{L,w_2w_3}$  gives rise to an anomalous BF<sub>4</sub><sup>L</sup> category  $\mathcal{C}_4^{L,w_2w_3}$ . The partition function for  $\mathcal{C}_4^{L,w_2w_3}$  is not gauge invariant on  $\mathbb{C}P^2$ . Under the complex conjugation  $*:\mathbb{C}P^2\to\mathbb{C}P^2$ , it changes sign  $Z_0(\mathbb{C}P^2)\to -Z_0(\mathbb{C}P^2)$ , since the phase change of the partition function is given by  $\mathrm{e}^{\mathrm{i}\pi\int_{\mathbb{C}P^2\rtimes_*S^1}w_2w_3}=-1$ . This represents a new type of global gravitational anomaly in a 3+1D bosonic theory.

The above example describes one class of quantized topological terms, which leads to one class of invertible  $\mathrm{BF}_n^\mathrm{L}$  categories. The partition functions for this class of topological orders is a topological invariant. Therefore, this class of topological orders is exact and has gapped boundaries which contain non-trivial topological excitations.

There is another class of quantized topological terms. Let us consider a 2+1D example. Let  $\omega_3^{p_1}$  be the three form whose derivative is the first Pontryagin class:  $\mathrm{d}\omega_3^{p_1} = p_1$ .  $\omega_3^{p_1}$  is a gravitational Chern-Simons term. We can use  $\omega_3^{p_1}$  to construct a local topological partition function integral in 2+1D:

$$Z(M^3) = \langle C_3, \omega_3 \rangle = e^{i \phi \int_{M^3} \omega_3^{p_1}}.$$
 (111)

However, for some 3-manifold  $M^3$ , gravitational Chern-Simons term  $\omega_3^{p_1}$  is only well defined on patches of  $M^3$ , with discontinuity between the patches. In this case  $\int_{M^3} \omega_3^{p_1}$  is not well defined. Since the cobordism group of 3-dimensional closed oriented manifolds is  $\Omega_3^{SO}=0$  (see Appendix F), we can view  $M^3$  as a boundary of  $M^4$ :  $\partial M^4=M^3$ , and rewrite the 2+1D topological partition function as

$$Z(M^3) = \langle C_3, \omega_3 \rangle = e^{i \phi \int_{M^4} p_1}.$$
 (112)

The above is well defined only if it does not depend on how we extend  $M^3$  to  $M^4$ . This requires  $\phi$  to be quantized as  $\phi = 0 \mod 2\pi/3$  (see Appendix F). So gravitational Chern-Simons term gives rise to a quantized topological term:

$$Z(M^3) = \langle C_3, \omega_3 \rangle = e^{2\pi i k \int_{M^3} \omega_3^{p_1}/3}, \quad k \in \mathbb{Z}.$$
 (113)

We see that, in 2+1D, the invertible BF $_3^{\rm L}$  categories form a group  $\mathbb Z$ . Such invertible BF $_3^{\rm L}$  categories are generated by the  $E_8$  quantum Hall state<sup>83</sup> (see Example 4).

Similarly, using the properties of Pontryagin classes  $p_1$  and  $p_2$  in 8-dimensions (see Appendix F):

$$\int_{M^8} \frac{p_1^2 - 2p_2}{5} \in \mathbb{Z}, \quad \int_{M^8} \frac{-2p_1^2 + 5p_2}{9} \in \mathbb{Z}.$$
 (114)

we can construct the following topological partition func-

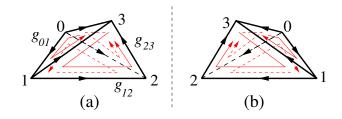


FIG. 25. (Color online) Two branched simplices with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.

tion in 6+1D:

$$Z(M^{7}) = \exp\left(2\pi i k_{1} \int_{M^{7}} \frac{\omega_{7}^{p_{1}^{2}} - 2\omega_{7}^{p_{2}}}{5}\right) \times \exp\left(2\pi i k_{2} \int_{M^{7}} \frac{-2\omega_{7}^{p_{1}^{2}} + 5\omega_{7}^{p_{2}}}{9}\right),$$

$$d\omega_{7}^{p_{1}^{2}} = p_{1}^{2}, \quad d\omega_{7}^{p_{2}} = p_{2}, \quad k_{1}, k_{2} \in \mathbb{Z}.$$
(115)

We see that the invertible BF<sub>7</sub><sup>L</sup> categories are classified by two integers  $(k_1, k_2)$  and form a group  $\mathbb{Z} \oplus \mathbb{Z}$ .

The partition functions for this second class of topological orders is a topological invariant up to U(1) phases. Thus the second class of topological orders is closed and not exact. Their boundary must be gapless.

#### XVI. ACKNOWLEDGEMENT

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# Appendix A: Lattice model defined by a path integral

#### 1. Space-time complex

To define a lattice model through a space-time path integral, we first triangulate of the n-dimensional space-time to obtain a space-time complex  $M_{\rm tri}$ . We will call a cell in the space-time complex as a simplex. In order to define a generic lattice theory on the space-time complex

 $M_{\rm tri}$ , it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure. <sup>125–127</sup> A branching structure is a choice of orientation of each edge in the n-dimensional complex so that there is no oriented loop on any triangle (see Fig. 25).

The branching structure induces a *local order* of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming edges, and the second vertex is the vertex with only one incoming edge, etc. So the simplex in Fig. 25a has the following vertex ordering: 0,1,2,3.

The branching structure also gives the simplex (and its sub simplexes) an orientation denoted by  $s_{ij\cdots k}=\pm$ . Fig. 25 illustrates two 3-simplices with opposite orientations  $s_{0123}=+$  and  $s_{0123}=-$ . The red arrows indicate the orientations of the 2-simplices which are the subsimplices of the 3-simplices. The black arrows on the edges indicate the orientations of the 1-simplices.

#### 2. Path integral on a space-time complex

The degrees of freedom of our lattice model live on the vertices (denoted by  $g_i$  where i labels the vertices), on the edges (denoted by  $h_{ij}$  where ij labels the edges), and on other high dimensional cells of the space-time complex. The action amplitude  $e^{-S_{cell}}$  for an n-cell  $(ij \cdots k)$  is complex function of  $g_i, h_{ij}, \cdots : V_{ij\cdots k}(\{g_i\}, \{h_{ij}\}, \cdots)$ . The total action amplitude  $e^{-S}$  for a configuration  $\{g_i\}, \{h_{ij}\}, \cdots$  (or a path) is given by

$$e^{-S} = \prod_{(ij\cdots k)} [V_{ij\cdots k}(\{g_i\}, \{h_{ij}\}, \cdots)]^{s_{ij\cdots k}}$$
 (A1)

where  $\prod_{(ij\cdots k)}$  is the product over all the *n*-cells  $(ij\cdots k)$ . Note that the contribution from an *n*-cell  $(ij\cdots k)$  is  $V_{ij\cdots k}(\{g_i\},\{h_{ij}\},\cdots)$  or  $V_{ij\cdots k}^*(\{g_i\},\{h_{ij}\},\cdots)$  depending on the orientation  $s_{ij\cdots k}$  of the cell. Our lattice theory is defined by the following imaginary-time path integral (or partition function)

$$Z = \sum_{\{g_i\},\{h_{ij}\},\dots (ij\cdots k)} [V_{ij\cdots k}(\{g_i\},\{h_{ij}\},\dots)]^{s_{ij\cdots k}}$$
 (A2)

We would like to point out that, in general, the path integral may also depend on some additional weighting factors  $w_{g_i}$ ,  $A_{g_i,g_j}^{h_{ij}}$ , etc (see (72) and (76)). In this section, for simplicity, we will assume those weighting factors are all equal to 1.

Here, we like to introduce an important concept:

### Definition 58. Uniform path integral

In the above path integral (A2), we have assigned the same action amplitude  $V_{ij\cdots k}(\{g_i\},\{h_{ij}\},\cdots)$  to each simplex  $(ij\cdots k)$ . Such a path integral is called a uniform path integral.

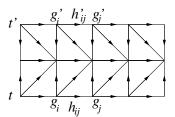


FIG. 26. Each time-step of evolution is given by the path integral on a particular form of branched graph. Here is an example in 1+1D.

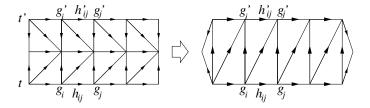


FIG. 27. The reduction of double-layer time-step to single-layer time-step on space with boundary for an 1+1D topological path integral.

In this paper, we only study systems described by uniform path integral. In physics, they correspond to "locally" translation invariants system in space and time directions, where the breaking of the exact translation symmetries only come from the global topology of spacetime. (We also like to point out that the doubling of unit cell does not break the "local" translation invariance, since after doubling the unit cell, there still some "local" translation invariance left.)

#### 3. Path integral and Hamiltonian

Consider a space-time complex of topology  $M_{\rm space} \times I$  where I = [t,t'] represents the time dimension and  $M_{\rm space}$  is a closed space complex (see Fig. 26). The space-time complex  $M_{\rm space} \times I$  has two boundaries: one at time t and another at time t'. A path integral on the space-time complex  $M_{\rm space} \times I$  give us an amplitude  $Z[\{g_i',h_{ij}',\cdots\},\{g_i,h_{ij},\cdots\}]$  from a configuration  $\{g_i,h_{ij},\cdots\}$  at t to another configuration  $\{g_i',h_{ij}',\cdots\}$  at t'. Here,  $\{g_i,h_{ij},\cdots\}$  and  $\{g_i',h_{ij}',\cdots\}$  are the degrees of freedom on the boundaries (see Fig. 26). We like to interpret  $Z[\{g_i',h_{ij}',\cdots\},\{g_i,h_{ij},\cdots\}]$  as the amplitude of an evolution in imaginary time by a Hamiltonian:

$$Z[\{g'_{i}, h'_{ij}, \cdots\}, \{g_{i}, h_{ij}, \cdots\}]$$

$$= \langle g'_{i}, h'_{ij}, \cdots | e^{-(t'-t)H} | g_{i}, h_{ij}, \cdots \rangle.$$
(A3)

However, such an interpretation may not be valid since  $Z[\{g'_i, h'_{ij}, \dots\}, \{g_i, h_{ij}, \dots\}]$  may not give raise to a Hermitian matrix. It is a worrisome realization that path

integral and Hamiltonian evolution may not be directly related.

Here we would like to use the fact that the path integral that we are considering are defined on the branched graphs with a "reflection" property (see (A1)). We like to show that such path integral are better related Hamiltonian evolution. The key is to require that each time-step of evolution is given by branched graphs of the form in Fig. 26. One can show that  $Z[\{g'_i, h'_{ij}, \dots\}, \{g_i, h_{ij}, \dots\}]$  obtained by summing over all in the internal indices in the branched graphs Fig. 26 has a form

$$Z[\{g'_{i}, h'_{ij}, \cdots\}, \{g_{i}, h_{ij}, \cdots\}]$$

$$= \sum_{\{g''_{i}, h''_{ij}, \cdots\}} U^{*}[\{g''_{i}, h''_{ij}, \cdots\}, \{g'_{i}, h'_{ij}, \cdots\}]$$

$$U[\{g''_{i}, h''_{ij}, \cdots\}, \{g_{i}, h_{ij}, \cdots\}]$$
(A4)

and represents a positive-definite Hermitian matrix. Thus the path integral of the form (A1) always correspond to a Hamiltonian evolution in imaginary time. In fact, the above  $Z[\{g_i',h_{ij}',\cdots\},\{g_i,h_{ij},\cdots\}]$  can be viewed as an imaginary-time evolution  $T=\mathrm{e}^{-\Delta\tau H}$  for a single time step.

#### 4. Time-reversal transformation

Consider a lattice model  $\Lambda$  described by a spacetime path integral defined by the action amplitude  $V_{ij\cdots k}^{\Lambda}(\{g_i\},\{h_{ij}\},\cdots)$ . If we fold the time direction as Fig. 9, we will get a time-reversal transformed lattice model  $\bar{\Lambda}$ . The lattice model  $\bar{\Lambda}$  is described by a different space-time path integral defined by the action amplitude  $V_{ij\cdots k}^{\bar{\Lambda}}(\{g_i\},\{h_{ij}\},\cdots)$ , which is given by

$$V_{ij\cdots k}^{\bar{\Lambda}}(\{g_i\},\{h_{ij}\},\cdots) = [V_{ij\cdots k}^{\Lambda}(\{g_i\},\{h_{ij}\},\cdots)]^*.$$

The above defines the time-reversal transformation.

### Appendix B: Simple and composite BFH categories

Consider a p-dimensional topological excitation on a p-dimensional subspace  $M^p$  of the space  $M^d$ . We note that the p-dimensional space  $M^p$  can support excitations whose dimensions are less than p. So we can view the p-dimensional topological excitation as a  $\mathrm{BF}^{\mathrm{H}}_{p+1}$  category (note that p is the space dimension and p+1 is the spacetime dimension). This suggests that a p-dimensional topological excitation corresponds to a  $\mathrm{BF}^{\mathrm{H}}_{p+1}$  category.

Since a p-dimensional topological excitations can be simple or composite, the  $BF_{p+1}^H$  categories can also be simple or composite. Following the definition of the simple and composite topological excitations in Section II B, we can have the following definition of simple and composite  $BF_n^H$  categories:

**Definition 59. Simple/composite BF** $_n^H$  category: A BF $_n^H$  category is simple if its ground state degeneracy on any closed space is robust against any small perturbations. Otherwise, the BF $_n^H$  category is composite.

The fractional quantum Hall states and the 2+1D  $Z_2$  spin liquid are example of simple BF<sub>3</sub><sup>H</sup> categories. To give an example of composite BF<sub>3</sub><sup>H</sup> category, let us consider a family of Hamiltonian H(g) parametrized by g. The ground state of H(0) is a product state with trivial topological order and the ground state of H(1) is the 2+1D  $Z_2$  spin liquid. At  $g=g_c$  there is a first order phase transition between the product state and the  $Z_2$  spin liquid state. Then the gapped ground state of  $H(g_c)$  (at the transition point) is an example of composite BF<sub>3</sub><sup>H</sup> category, which can be expressed as a sum  $(\oplus)$  of a trivial BF<sub>3</sub><sup>H</sup> category and a 2+1D  $Z_2$  topological order. We see that the composite BF<sup>H</sup> categories are unsta-

We see that the composite BF<sup>H</sup> categories are unstable. For simplicity, in this paper, we will use "BF category" and "topological order" to only refer simple BF category. We will use "potentially composite BF category" to refer the generic BF category that can be simple or composite.

#### Appendix C: Examples of BF categories

# 1. Examples of exact BF categories (i.e. gapped phases of qubit models with gapped boundaries)

a.  $2+1D \mathbb{Z}_2$  topological order

The 3-dimensional BF $_3^{\rm L}$  category  $\mathcal{C}_3^{\mathbb{Z}_2}$  in Example 8 is an exact BF $_3^{\rm L}$  category. (Note that an exact BF $_3^{\rm L}$  category is also an exact BF $_3^{\rm H}$  category.) It has three and only three particle-like topological excitations labeled by  $e,\ v,\$ and  $\epsilon.$  Those topological excitations are their own anti-particles (i.e. satisfy a  $\mathbb{Z}_2$  fusion rule). e and v are bosons, while  $\epsilon$  is a fermion. Such a 3-dimensional BF $_3^{\rm L}$  category  $\mathcal{C}_3^{\mathbb{Z}_2}$  can be realized by a toric code model<sup>88</sup> in 2+1 dimensions. As a topological phase, it coincides with the  $\mathbb{Z}_2$ -spin-liquid<sup>17–19</sup>. Since BF $_3^{\rm L}$  category corresponds to effective theory in physics, we write a BF $_3^{\rm L}$  category as a gapped effective theory. In fact we have

$$\mathcal{C}_3^{\mathbb{Z}_2} = 2 + 1D \, \mathbb{Z}_2$$
 gauge theory, (C1)

where e is the  $\mathbb{Z}_2$  charge, v the  $\mathbb{Z}_2$  vortex, and  $\epsilon$  the bond state of e and v. Note that  $\mathcal{C}_3^{\mathbb{Z}_2}$  also correspond to a  $U(1) \times U(1)$  Chern-Simons theory in eqn. (1) (see Ref. 40–45)

$$\mathfrak{C}_3^{\mathbb{Z}_2}=U(1)\times U(1)$$
 Chern-Simons theory with K-matrix  $K=\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$  (C2)

where e is the unit-charge of the first U(1) and v the unit-charge of the second U(1).

As an exact BF<sub>3</sub><sup>L</sup> category,  $\mathcal{C}_3^{\mathbb{Z}_2}$  must a center of some BF<sub>2</sub><sup>L</sup> category. In fact  $\mathcal{C}_3^{\mathbb{Z}_2}$  can be a center of the  $\mathcal{C}_2^{F\mathbb{Z}_2}$  category discussed in Example 3.

#### b. Double semion model

The 3-dimensional BF $_3^{\rm L}$  category  $\mathcal{C}_3^{\mathbb{Z}_2 ds}$  in Example 9 is another exact BF $_3^{\rm L}$  category. It has three and only three particle-like topological excitations labeled by  $e, \, v,$  and  $\epsilon$ . Those topological excitations are their own antiparticles. e and v are independent semions with statistics  $\pm \pi/2$ , while  $\epsilon$  is the bound state of e and v and is a boson. Such a 3-dimensional BF $_3^{\rm L}$  category  $\mathcal{C}_3^{\mathbb{Z}_2 ds}$  can be realized by the so called double-semion string-net model 11 or a double-layer (2,-2,0) fractional quantum Hall state 128 in 2+1 dimensions. In fact  $\mathcal{C}_3^{\mathbb{Z}_2 ds}$  is a  $U(1) \times U(1)$  Chern-Simons theory described in eqn. (1) (see Ref. 40–45)

$$\begin{array}{l} \mathbb{C}_3^{\mathbb{Z}_2ds}=U(1)\times U(1) \text{ Chern-Simons theory}\\ \text{with } K\text{-matrix } K=\begin{pmatrix} 2 & 0\\ 0 & -2 \end{pmatrix}, \end{array} \tag{C3} \label{eq:C3}$$

where e is the unit-charge of the first U(1) and v the unit-charge of the second U(1).

#### c. $3+1D \mathbb{Z}_2$ topological order

The 4-dimensional BF<sub>4</sub><sup>L</sup> category  $\mathcal{C}_4^{\mathbb{Z}_2}$  in Example 12 is also an exact BF<sub>4</sub><sup>L</sup> category. It has one particle-like topological excitation denoted by e and one string-like topological excitation denoted by s, and no other topological excitations. Such a 4-dimensional BF<sub>4</sub><sup>L</sup> category  $\mathcal{C}_4^{\mathbb{Z}_2}$  can be realized by a  $\mathbb{Z}_2$ -spin-liquid 129 in 3+1 dimensions. In fact

$$\mathcal{C}_4^{\mathbb{Z}_2} = 3+1D \ \mathbb{Z}_2$$
 gauge theory, (C4)

where e is the  $\mathbb{Z}_2$  charge and s the  $\mathbb{Z}_2$  vortex-line.

All the above topological states can have a gapped boundary. Thus they are exact  $\mathrm{BF}_n^\mathrm{L}$  categories. They are also exact  $\mathrm{BF}_n^\mathrm{H}$  categories, since every exact  $\mathrm{BF}_n^\mathrm{L}$  category is an exact  $\mathrm{BF}_n^\mathrm{H}$  category.

# 2. Examples of closed BF categories (*i.e.* gapped phases of qubit models)

a.  $\nu = 1/2$  bosonic Laughlin state

The 3-dimensional BF $_3^{\rm H}$  category  $\mathcal{C}_3^{F\mathbb{Z}_2 s}$  in Example 7 is a closed BF $_3^{\rm H}$  category. It has only one particle-like topological excitation labeled by e, which is its own anti-particles and has a semion statistics. Such a 3-dimensional BF $_3^{\rm H}$  category  $\mathcal{C}_3^{F\mathbb{Z}_2 s}$  can be realized by a

filling-fraction  $\nu=1/2$  fractional quantum Hall state, the Laughlin state, in 2+1 dimensions. In fact

$$C_3^{F\mathbb{Z}_2 s} = U(1)$$
 Chern-Simons theory  
with K-matrix  $K = (2)$ , (C5)

where e is the unit-charge of the U(1).

The closed BF<sub>3</sub><sup>H</sup> category  $\mathcal{C}_3^{F\mathbb{Z}_2s}$  illustrates the Conjecture 28. Every topological excitation in  $\mathcal{C}_3^{F\mathbb{Z}_2s}$  (which is the semion e) has a nontrivial mutual statistics with at least one other topological excitation (which is also e). According to the Conjecture 28, the BF<sub>3</sub><sup>H</sup> category  $\mathcal{C}_3^{F\mathbb{Z}_2s}$  should be closed

We like to point out that 2+1D topological theory with the semion as the only type of topological excitation is a closed  $\mathrm{BF}^{\mathrm{H}}_3$  category, but it is not closed  $\mathrm{BF}^{\mathrm{L}}_3$  category. It is an anomalous  $\mathrm{BF}^{\mathrm{L}}_3$  category. This is because the theory has a L-type gravitational anomaly. It cannot be defined as a lbL system in 2+1D, because the lbL system is required to be well defined on space-time with any topology that is orientable. The theory can only be defined as a boundary of a lbL system in 3+1D. So the theory corresponds to a non-closed (*i.e.* anomalous)  $\mathrm{BF}^{\mathrm{L}}_3$  category.

In contrast, the theory has no H-type gravitational anomaly. It can be realized by a qubit model on a 2D lattice. Hence, the corresponding  $\mathrm{BF}_3^{\mathrm{H}}$  category  $\mathrm{C}_3^{F\mathbb{Z}_2s}$  is a closed  $\mathrm{BF}_3^{\mathrm{H}}$  category. Note that to be a closed  $\mathrm{BF}_3^{\mathrm{H}}$  category, we only require the path integral representation of the theory to be well defined on space-time which is a mapping torus. Thus the same theory can be free of H-type gravitational anomaly but not free of L-type gravitational anomaly. For more details, see Sections XIII A 3 and XIII C.

#### b. A three-fermion $\mathbb{Z}_2$ topological state

The 3-dimensional BF $_3^{\rm H}$  category  $\mathcal{C}_3^{\mathbb{Z}_2f^3}$  in Example 10 is the second closed BF $_3^{\rm H}$  category. It has three and only three particle-like topological excitations labeled by  $e,\,v$  and  $\epsilon$ , which are their own anti-particles. All the three topological excitations are fermions with mutual  $\pi$  statistics. Such a 3-dimensional BF $_3^{\rm H}$  category  $\mathcal{C}_3^{\mathbb{Z}_2f^3}$  can be realized by a four-layer fractional quantum Hall state in 2+1 dimensions. In fact (see Ref. 94)

$$C_3^{\mathbb{Z}_2 f^3} = U^4(1)$$
 Chern-Simons theory  
with K-matrix  $K = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$ . (C6)

Again, the above topological theory has a L-type gravitational anomaly, although it has no H-type gravitational anomaly. Thus it is a closed  $BF_3^H$  category but not a closed  $BF_3^L$  category.

#### c. Gapless edge state and chiral central charge

The above two examples are not exact BF $_3^{\rm H}$  categories, since they have gapless edge excitations that are robust against any local interactions on the edge. The BF $_3^{\rm H}$  category  $\mathcal{C}_3^{F\mathbb{Z}_2 s}$  has an edge with a chiral central charge  $c_R-c_L=1$ , and  $\mathcal{C}_3^{\mathbb{Z}_2 f^3}$  has an edge with a chiral central charge implies gapless edge states. In fact, there is a quite direct relation between the chiral central charge and the statistics of the topological excitations:  $^{101,130}$ 

$$\frac{1}{\sqrt{\sum_{\alpha} d_{\alpha}^2}} \sum_{\alpha} d_{\alpha}^2 e^{i\theta_{\alpha}} = e^{i2\pi(c_R - c_L)/8}$$
 (C7)

where  $\alpha$  labels all the particle-like topological excitations (including the trivial one). Here  $\theta_{\alpha}$  is the statistical angle and  $d_{\alpha}$  the quantum dimension of the topological excitations. Such a relation can help us to determine which BF $_3^{\rm H}$  category cannot be exact. Also, when  $c_R-c_L\neq 0$  mod 8, the topological theory will have a L-type gravitational anomaly. For more details, see Sections XIII A 3 and XIII C.

# 3. Examples of anomalous BF categories (i.e. gapped anomalous theories)

a. An anomalous  $BF_2^L$  category as an edge of 2+1D  $\mathbb{Z}_2$  topological state

The 2-dimensional BF $_2^{\rm L}$  category  $\mathcal{C}_2^{F\mathbb{Z}_2}$  in Example 3 is an anomalous BF $_2^{\rm L}$  category. It has only one particle-like topological excitation labeled by e, which is its own antiparticles and has a Bose statistics. Such a 2-dimensional BF $_2^{\rm L}$  category  $\mathcal{C}_2^{F\mathbb{Z}_2}$  can be realized by a boundary of a  $\mathbb{Z}_2$ -spin-liquid $^{17-19,88}$  (described by  $\mathcal{C}_3^{\mathbb{Z}_2}$ ) in 2+1 dimensions. In other words.

$$\mathcal{Z}_2(\mathcal{C}_2^{F\mathbb{Z}_2}) = 2+1D \mathbb{Z}_2$$
 gauge theory  $=\mathcal{C}_3^{\mathbb{Z}_2}$ ,

where e is the  $\mathbb{Z}_2$  charge. The 2+1D  $\mathbb{Z}_2$ -spin-liquid has two particle-like topological excitations: the  $\mathbb{Z}_2$  charge e and the  $\mathbb{Z}_2$  vortex v. The 2+1D  $\mathbb{Z}_2$ -spin-liquid can have many different kinds of boundaries. The boundary created by the condensation of the  $\mathbb{Z}_2$  vortices v realizes the BF<sup>L</sup><sub>2</sub> category  $\mathfrak{C}_2^{F\mathbb{Z}_2}$ , which is the simplest example of UFC.

# b. An anomalous $BF_3^L$ category as a boundary of 3+1D $\mathbb{Z}_2$ topological state

Similarly, the 3-dimensional BF<sub>3</sub><sup>L</sup> category  $\mathcal{C}_3^{F\mathbb{Z}_2b}$  in Example 5 is another anomalous BF<sub>3</sub><sup>L</sup> category. It has only one particle-like topological excitation labeled by e, which is its own anti-particles and has a Bose statistics.

Such a 3-dimensional BF<sub>3</sub><sup>L</sup> category  $\mathcal{C}_3^{F\mathbb{Z}_2b}$  can be realized by the *boundary* of a  $\mathbb{Z}_2$ -spin-liquid (described by  $\mathcal{C}_4^{\mathbb{Z}_2}$ ) in 3+1 dimensions. In other words,

$$\mathcal{Z}_3(\mathcal{C}_3^{F\mathbb{Z}_2 b}) = 3+1D \mathbb{Z}_2$$
 gauge theory  $= \mathcal{C}_4^{\mathbb{Z}_2}$ ,

where e is the  $\mathbb{Z}_2$  charge. The 3+1D  $\mathbb{Z}_2$ -spin-liquid has a particle-like and a string-like topological excitations e and s. The boundary created by the condensation of the string-like topological excitations s realizes the BF $_3^{\mathrm{L}}$  category  $\mathcal{C}_3^{F\mathbb{Z}_2b}$ .

# c. Another anomalous $BF_3^L$ category as another boundary of 3+1D $\mathbb{Z}_2$ topological state

The 3-dimensional BF $_3^{\rm L}$  category  $\mathcal{C}_3^{sF\mathbb{Z}_2}$  in Example 11 is also an anomalous BF $_3^{\rm L}$  category. It has only one string-like topological excitation labeled by s, which satisfies a  $\mathbb{Z}_2$  fusion rule. Such a 3-dimensional BF $_3^{\rm L}$  category  $\mathcal{C}_3^{sF\mathbb{Z}_2}$  can be realized by the boundary of a  $\mathbb{Z}_2$ -spinliquid 129 (described by  $\mathcal{C}_4^{\mathbb{Z}_2}$ ) in 3+1 dimensions. In other words,

$$\mathcal{Z}_3(\mathcal{C}_3^{sF\mathbb{Z}_2}) = 3+1D \mathbb{Z}_2$$
 gauge theory  $= \mathcal{C}_4^{\mathbb{Z}_2}$ ,

where the above string-like topological excitation s correspond to the  $\mathbb{Z}_2$  vortex line in the  $\mathbb{Z}_2$ -spin-liquid. The boundary created by the condensation of the particle-like topological excitations e in the  $\mathbb{Z}_2$ -spin-liquid realizes the BF<sup>L</sup><sub>3</sub> category  $\mathfrak{C}_3^{sF\mathbb{Z}_2}$ .

# d. An anomalous $BF_3^L$ category as a boundary of 3+1D twisted $\mathbb{Z}_2$ topological state with emergent fermions

The 3-dimensional BF $_4^{\rm L}$  category  $\mathcal{C}_3^{F\mathbb{Z}_2f}$  in Example 6 is yet another anomalous BF $_4^{\rm L}$  category. It has only one particle-like topological excitation labeled by e, which is its own anti-particles and has a Fermi statistics. Such a 3-dimensional BF $_4^{\rm L}$  category  $\mathcal{C}_3^{F\mathbb{Z}_2b}$  can be realized by the boundary of a twisted  $\mathbb{Z}_2$  string-net state  $^{11,131}$  (described by the BF $_4^{\rm L}$  category  $\mathcal{C}_4^{t\mathbb{Z}_2}$ ) in 3+1 dimensions. In other words,

$$\mathcal{Z}_3(\mathcal{C}_3^{F\mathbb{Z}_2f})=3+1\mathrm{D}$$
 twisted  $\mathbb{Z}_2$  gauge theory  $=\mathcal{C}_4^{t\mathbb{Z}_2},$ 

where e is the  $\mathbb{Z}_2$  charge. The 3+1D twisted  $\mathbb{Z}_2$  gauge theory is a  $\mathbb{Z}_2$  gauge theory where the  $\mathbb{Z}_2$  charge carries a Fermi statistics. Such a twisted  $\mathbb{Z}_2$  gauge theory can emerge from a lattice qubit model. The 3+1D twisted  $\mathbb{Z}_2$ -spin-liquid also has a particle-like and a string-like topological excitations e and s. The boundary created by the condensation of the string-like topological excitations s realizes the BF $_3$  category  $\mathfrak{C}_3^{\mathbb{F}\mathbb{Z}_2f}$ .

It is very strange to see that a simple theory with one fermion is anomalous. We like to point out that it is easy to realize a gapped effective theory with a fermion excitation using a fermionic local Hamiltonian system in the same dimension. However, we cannot realize a gapped state, whose only type of topological excitations is fermionic, using a bosonic local Hamiltonian system (i.e. a lattice qubit model) in the same dimension. This implies that, by definition, a gapped theory with only one type of fermion excitation is anomalous. Such a theory has to be a boundary of a gapped qubit state in one-higher dimension.

The anomalous  $\mathrm{BF}_3^\mathrm{L}$  category  $\mathrm{C}_3^{F\mathbb{Z}_2f}$  also illustrate the Conjecture 28. The fermion e in  $\mathrm{C}_3^{F\mathbb{Z}_2f}$  has a trivial mutual statistics with all other topological excitations (which is also e). According to the Conjecture 28, the  $\mathrm{BF}_3^\mathrm{L}$  category  $\mathrm{C}_3^{F\mathbb{Z}_2f}$  should be anomalous.

# e. An anomalous $BF_4^L$ category as a boundary of 4+1D $\mathbb{Z}_2$ topological state

The 4-dimensional BF<sub>4</sub><sup>L</sup> category  $\mathcal{C}_4^{mF\mathbb{Z}_2}$  in Example 14 is an anomalous BF<sub>4</sub><sup>L</sup> category. It has only one membrane-like topological excitation labeled by m, which satisfies a  $\mathbb{Z}_2$  fusion rule. Such a 4-dimensional BF<sub>4</sub><sup>L</sup> category  $\mathcal{C}_4^{mF\mathbb{Z}_2}$  can be realized by the *boundary* of a  $\mathbb{Z}_2$ -spin-liquid (described by  $\mathcal{C}_5^{\mathbb{Z}_2}$ ) in 4+1 dimensions. In other words,

$$\mathcal{Z}_4(\mathcal{C}_4^{mF\mathbb{Z}_2}) = 4+1D \mathbb{Z}_2$$
 gauge theory  $=\mathcal{C}_5^{\mathbb{Z}_2}$ .

The 4+1D  $\mathbb{Z}_2$ -spin-liquid  $\mathcal{C}_5^{\mathbb{Z}_2}$  has a particle-like and a membrane-like topological excitations,  $\mathbb{Z}_2$  charge particle and  $\mathbb{Z}_2$  vortex membrane. The above membrane-like topological excitation m corresponds to the  $\mathbb{Z}_2$  vortex membrane in  $\mathcal{C}_5^{\mathbb{Z}_2}$ . The 4+1D  $\mathbb{Z}_2$ -spin-liquid can have many different kinds of boundaries. The boundary created by the condensation of the  $\mathbb{Z}_2$  charge particles realizes the  $\mathrm{BF}_4^{\mathbb{Z}}$  category  $\mathcal{C}_4^{mF\mathbb{Z}_2}$ .

# f. An anomalous $BF_4^L$ category as a boundary of a 4+1D membrane condensed state

The 4-dimensional BF $_4^{\rm L}$  category  $\mathcal{C}_4^{sF\mathbb{Z}_2}$  in Example 13 is our last example of anomalous BF $_4^{\rm L}$  category. It has only one string-like topological excitation labeled by s, which satisfies a  $\mathbb{Z}_2$  fusion rule. Such a 4-dimensional BF $_4^{\rm L}$  category  $\mathcal{C}_4^{sF\mathbb{Z}_2}$  can be realized by the boundary of a non-oriented membrane condensed state (described by  $\mathcal{C}_5^{\mathbb{Z}_2 m}$ ) in 4+1 dimensions. In other words,

$$\mathcal{Z}_4(\mathbb{C}_4^{mF\mathbb{Z}_2}) = \text{ 4+1D membrane condensed state} = \mathbb{C}_5^{\mathbb{Z}_2 m}.$$

Such a 4+1D  $BF_{4+1}^L$  category is described by the following effective Lagrangian

$$\mathcal{L} = \frac{K_{IJ}}{4\pi} b_{I\mu\nu} \partial_{\lambda} b_{J\rho\sigma} \epsilon^{\mu\nu\lambda\rho\sigma}, \tag{C8}$$

with

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{C9}$$

The 4+1D membrane condensed state  $C_5^{\mathbb{Z}_2 m}$  has two kinds of string-like topological excitations labeled by  $s_1$  and  $s_2$ . The above string-like topological excitations  $s_1$  corresponds  $s_1$  in  $C_5^{\mathbb{Z}_2 m}$ . The 4+1D membrane condensed state can have many different kinds of boundaries. The boundary created by the condensation of  $s_2$  realizes the BF<sub>4</sub> category  $C_4^{mF\mathbb{Z}_2}$ .

The gapped effective theories discussed in this section all have L-type (as well as the H-type) gravitational anomalies. They represent global gravitational anomalies

### Appendix D: A brief introduction to category theory

In this section, we will give a brief introduction to category theory, which is basically an abstract theory about relations (maps) and the composition of the relations (maps). The language of the category theory is used in the main text to define  $\mathrm{BF}_n^{\mathrm{H}}$  category as an unitary n-category.

**Definition 60.** A 1-category C consists of the following data:

- 1. A set of objects  $Ob(\mathcal{C})$ .
- 2. For each pair  $x, y \in \text{Ob}(\mathcal{C})$  a set of morphisms  $\text{hom}_{\mathcal{C}}(x, y)$ .
- 3. For each triple  $x, y, z \in \text{Ob}(\mathcal{C})$  a composition map  $\text{hom}_{\mathcal{C}}(y, z) \times \text{hom}_{\mathcal{C}}(x, y) \to \text{hom}_{\mathcal{C}}(x, z)$ , denoted as  $(f, g) \mapsto f \circ g$  for  $f \in \text{hom}_{\mathcal{C}}(x, y)$ ,  $g \in \text{hom}_{\mathcal{C}}(y, z)$ .

These data are to satisfy the following rules:

- 1. For every element  $x \in \mathrm{Ob}(\mathcal{C})$  there exists a morphism  $\mathrm{id}_x \in \mathrm{hom}_{\mathcal{C}}(x,x)$  such that  $\mathrm{id}_x \circ \phi = \phi$  and  $\psi \circ \mathrm{id}_x = \psi$  whenever these compositions make sense.
- 2. Composition is associative, i.e.,  $(f \circ g) \circ h = f \circ (g \circ h)$  whenever these compositions make sense.

We list a few examples of 1-category below:

- 1. The category Set of sets consists of sets as objects and maps as 1-morphisms. The composition of 1-morphisms is just the usual composition of maps. The identity morphism is just the identity map.
- 2. The category  $\operatorname{Vect}$  of vector spaces over  $\mathbb C$  consists of vector spaces as objects and linear maps as morphisms, i.e.  $\hom_{\operatorname{Vect}}(x,x)=\operatorname{End}_{\mathbb C}(x)$ .
- 3. The category  $\Re \operatorname{ep}_G$  of representations of a group G consists of representations of the group G as objects and linear maps that intertwine the G-action as 1-morphisms.

**Remark 39.** A morphism  $f \in \text{hom}_{\mathbb{C}}(x,y)$  is also referred as an arrow from x to y, denoted by  $x \xrightarrow{f} y$ . A morphism  $f: x \to y$  is called an *isomorphism* of the category  $\mathbb{C}$  if there exists a morphism  $g: y \to x$  such that  $f \circ g = \text{id}_y$  and  $g \circ f = \text{id}_x$ .

**Definition 61.** A subcategory of a category  $\mathcal{B}$  is a category  $\mathcal{A}$  whose objects and arrows form subsets of the objects and arrows of  $\mathcal{A}$  and such that source, target and composition in  $\mathcal{A}$  agree with those of  $\mathcal{B}$ . We say  $\mathcal{A}$  is a full subcategory of  $\mathcal{B}$  if  $\text{hom}_{\mathcal{A}}(x,y) = \text{hom}_{\mathcal{B}}(x,y)$  for all  $x,y \in \text{Ob}(\mathcal{A})$ . We say  $\mathcal{A}$  is a strictly full subcategory of  $\mathcal{B}$  if it is a full subcategory and given  $x \in \text{Ob}(\mathcal{A})$  any object of  $\mathcal{B}$  which is isomorphic to x is also in  $\mathcal{A}$ .

**Definition 62.** For any category  $\mathcal{C}$ , the *opposite category*  $\mathcal{C}^{op}$  is defined so that  $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$  and  $hom_{\mathcal{C}^{op}}(x,y) = hom_{\mathcal{C}}(y,x)$  for all  $x,y \in Ob(\mathcal{C})$ .

**Remark 40.** It follows directly from the definition that any two identity morphisms of an object x of  $\mathcal{A}$  are the same. Thus we may and will speak of *the* identity morphism  $\mathrm{id}_x$  of x.

**Definition 63.** A functor  $F : \mathcal{A} \to \mathcal{B}$  between two categories  $\mathcal{A}, \mathcal{B}$  is given by the following data:

- 1. A map  $F : Ob(\mathcal{A}) \to Ob(\mathcal{B})$ .
- 2. For every  $x, y \in \text{Ob}(\mathcal{A})$  a map  $F : \text{hom}_{\mathcal{A}}(x, y) \to \text{hom}_{\mathcal{B}}(F(x), F(y))$  such that  $f \mapsto F(f)$ .

These data should be compatible with composition and identity morphisms in the following manner:  $F(f \circ g) = F(f) \circ F(g)$  for a composable pair (f,g) of morphisms of  $\mathcal{A}$  and  $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ .

Note that there is an *identity* functor  $\mathrm{id}_{\mathcal{A}}$  for every category  $\mathcal{A}$ . In addition, given a functor  $G:\mathcal{B}\to\mathcal{C}$  and a functor  $F:\mathcal{A}\to\mathcal{B}$  there is a *composition* functor  $G\circ F:\mathcal{A}\to\mathcal{C}$  defined in an obvious manner.

**Definition 64.** Let  $F: \mathcal{A} \to \mathcal{B}$  be a functor.

1. We say F is faithful if for any objects x, y of Ob(A) the map

$$F: \hom_{\mathcal{A}}(x, y) \to \hom_{\mathcal{B}}(F(x), F(y))$$

is injective.

- 2. If these maps are all bijective then F is called fully faithful.
- 3. The functor F is called essentially surjective if for any object  $y \in \text{Ob}(\mathcal{B})$  there exists an object  $x \in \text{Ob}(\mathcal{A})$  such that F(x) is isomorphic to y in  $\mathcal{B}$ .

**Definition 65.** A natural transformation  $\beta: F \to G$  between two functors  $F, G: \mathcal{A} \to \mathcal{B}$  is a set of maps  $\{\beta_x: F(x) \to G(x)\}_{x \in \mathrm{Ob}(\mathcal{A})}$  such that the following diagram:

$$F(x) \xrightarrow{\beta_x} G(x)$$

$$F(f) \downarrow \qquad \qquad \downarrow G(f)$$

$$F(y) \xrightarrow{\beta_y} G(y)$$

is commutative.

Let  ${\bf 1}$  be the 1-category with a single object  ${ullet}$  and a single morphism id.

**Definition 66.** A 2-category  $\mathcal{E}$  consists of a set of objects and a 1-category of morphisms hom(A, B) for each pair of objects a and b together with

- 1. identity morphism: there is a functor  $\mathbf{1}_a: \mathbf{1} \to \text{hom}(a,a)$  for all  $a \in \text{Ob}(\mathcal{E})$ .
- 2. composition functor:

$$\circ_{a,b,c} : \hom(a,c) \times \hom(a,b) \to \hom(a,c)$$
  
 $(f,g) \mapsto f \circ g$ 

3. associativity isomorphisms: for  $a, b, c, d \in Ob(\mathcal{E})$ , there is a natural isomorphism:

$$\alpha: \circ_{a,c,d} \circ (\circ_{a,b,c} \times \mathrm{id}) \to \circ_{a,b,d} \circ (\mathrm{id} \times \circ_{b,c,d}).$$

4. left and right unit isomorphisms:  $l : \circ_{a,a,b} \circ (\mathbf{1}_a \times \mathrm{id}) \to \mathrm{id}$  and  $r : \circ_{a,b,b} \circ (\mathrm{id} \times \mathbf{1}_b) \to \mathrm{id}$ .

satisfying the following coherence conditions:

1. associativity coherence:

$$((e \circ f) \circ g) \circ h \xrightarrow{\alpha 1} (e \circ (f \circ g)) \circ h \qquad (D1)$$

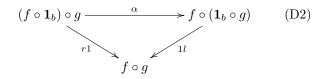
$$\alpha \downarrow \qquad \qquad \downarrow \alpha$$

$$(e \circ f) \circ (g \circ h) \qquad e \circ ((f \circ g) \circ h)$$

$$\alpha \downarrow \qquad \qquad \downarrow \alpha$$

$$e \circ (f \circ (g \circ h))$$

2. identity coherence:



Remark 41. It immediately follows from the axioms of the 2-category that the 1-category hom(x,x) is a monoidal 1-category with the tensor product given by the composition and the tensor unit by the identity 1-morphism  $\mathrm{id}_x := \mathbf{1}_x(\bullet)$ .

#### Appendix E: An extension problem

Let  $M^n$  be a fiber bundle with  $S^1$  as the base and (n-1)-dimensional manifold  $F^{n-1}$  as the fiber (i.e.  $M^n$  is a mapping torus). Assume  $M^n$  is oriented. Is  $M^n$  always a boundary of a manifold  $M^{n+1}$ , where  $M^{n+1}$  is a fiber

bundle with  $S^1$  as the base and n-dimensional manifold  $F^n$  as the fiber. The answer is no. This is because  $F^n$ needs to be the boundary of  $F^{n+1}$  and some manifolds cannot be realized as boundaries of other manifolds (such as 4-manifolds with non-zero signature).

Now let us consider a slightly different extension problem. Let  $M^n$  be a fiber bundle with  $S^1$  as the base and (n-1)-dimensional cell complex  $F^{n-1}$  as the fiber. Assume  $M^n$  is oriented. Is  $M^n$  always a boundary of a cell-complex  $M^{n+1}$ , where  $M^{n+1}$  is a fiber bundle with  $S^1$  as the base and n-dimensional cell-complex  $F^n$  as the fiber. The answer is yes, since we can take  $F^{n+1}$  to be the cone on  $F^n$ . 132

We also has the following result: the signature is multiplicative in oriented fiber bundles of odd fiber dimension. 133 Combine the above two results, we conclude that the path integral of Chern-Simons theory can always be defined in 2+1 dimension where the space-time is a mapping torus.

#### Appendix F: The oriented cobordism groups

Two oriented smooth n-dimensional manifolds M and N are said to be equivalent if  $M \cup (-N)$  is a boundary of another manifold, where -N is the N manifold with a reversed orientation. With the multiplication given by the disjoint union, the corresponding equivalence classes has a structure of an Abelian group  $\Omega_n^{SO}$ , which is called the cobordism group of closed oriented smooth manifolds. For low dimensions, we have <sup>134</sup>

 $\Omega_0^{SO}=\mathbb{Z},$  generated by a point.  $\Omega_1^{SO}=0,$  since circles bound disks.

 $\Omega_2^{\dot{S}O} = 0$ , since all oriented surfaces bound handlebodies.

 $\Omega_2^{SO} = 0$ .

 $\Omega_4^{SO} = \mathbb{Z}$ , generated by  $\mathbb{C}P^2$ .

 $\Omega_5^{SO} = \mathbb{Z}_2$ , generated by the Wu manifold SU(3)/SO(3), detected by the deRham invariant or Stiefel-Whitney number  $\int_M w_2 \wedge w_3$ .

 $\begin{array}{l} \Omega_6^{SO} = 0. \\ \Omega_7^{SO} = 0. \end{array}$ 

 $\Omega_8^{ISO} = \mathbb{Z} \oplus \mathbb{Z}$  generated by  $\mathbb{C}P^4$  and  $\mathbb{C}P^2 \times \mathbb{C}P^2$ .

The free part of the cobordism groups  $\Omega_n^{SO}$  can be fully detected by the Pontryagin numbers  $P_{n_1n_2}...(M)$ . In particular, we have  $^{135}$ 

$$P_{1}(\mathbb{C}P^{2}) = \int_{\mathbb{C}P^{2}} p_{1} = 3;$$

$$P_{1,1}(\mathbb{C}P^{2} \times \mathbb{C}P^{2}) = \int_{\mathbb{C}P^{2} \times \mathbb{C}P^{2}} p_{1}^{2} = 18,$$

$$P_{2}(\mathbb{C}P^{2} \times \mathbb{C}P^{2}) = \int_{\mathbb{C}P^{2} \times \mathbb{C}P^{2}} p_{2} = 9,$$

$$P_{1,1}(\mathbb{C}P^{4}) = \int_{\mathbb{C}P^{4}} p_{1}^{2} = 25,$$

$$P_{2}(\mathbb{C}P^{4}) = \int_{\mathbb{C}P^{4}} p_{2} = 10.$$
(F1)

We also have the following fundamental theorem:

**Theorem 9:** Two closed oriented n-manifolds  $M_0$  and  $M_1$  are cobordism equivalent iff they have the same Stiefel-Whitney and Pontryagin numbers.

In Ref. 136–138, some cobordism groups  $\Omega_n^{MT}$  of ndimensional mapping tori are obtained

$$\begin{split} \Omega_{2k}^{MT} &= \Omega_{2k-1}^{SO} \oplus \hat{\Omega}_{2k}^{SO}, \quad \text{for } k>2 \\ \Omega_{4}^{MT} &= 0, \end{split} \tag{F2}$$

where  $\hat{\Omega}_{2k}^{SO}$  is the subgroup of  $\Omega_{2k}^{SO}$  with vanishing signature. The structure of  $\Omega_{2k}^{MT}$  suggests that we can use the following two types of Stiefel-Whitney and Pontryagin numbers to detect/distinguish the elements of  $\Omega_{2k}^{MT}$ :

$$\int_{M} P_{n_{1}n_{2}\dots}, \qquad \int_{M} d\theta \wedge P_{n_{1}n_{2}\dots}, 
\int_{M} W_{n_{1}n_{2}\dots}, \qquad \int_{M} d\theta \wedge W_{n_{1}n_{2}\dots},$$
(F3)

where  $d\theta$  is the one form on the base manifold  $S^1$  which is parametrized by  $\theta \in [0, 2\pi), P_{n_1 n_2 \dots}$  are combinations of Pontryagin classes:  $P_{n_1 n_2 \dots} = p_{n_1} \wedge p_{n_2} \wedge \dots$ , and  $W_{n_1 n_2 \dots}$ are combinations of Stiefel-Whitney classes:  $W_{n_1 n_2 \dots} =$  $w_{n_1} \wedge w_{n_2} \wedge \cdots$  on M.

For mapping tori with odd dimensions, the result is more complicated: for k > 1, the homomorphism

$$\Omega_{2k+1}^{MT} \to \Omega_{2k+1}^{SO} \oplus \hat{\Omega}_{2k}^{SO} \oplus W_{(-)^k}(\mathbb{Z}, \mathbb{Z})$$
 (F4)

is an isomorphism (for k even) or is injective with cokernel  $Z_2$  (for k odd). Here  $W_{\pm}(\mathbb{Z},\mathbb{Z}) \simeq \mathbb{Z}^{\infty} \oplus \mathbb{Z}_2^{\infty} \oplus \mathbb{Z}_4^{\infty}$  is the Witt group of isometries of free finite-dimensional Zmodules with a symmetric (antisymmetric) unimodular bilinear form. We also have 138

$$\Omega_3^{MT} = \mathbb{Z}^{\infty} \oplus \mathbb{Z}_2^{\infty}, \tag{F5}$$

We note that  $\Omega_5^{MT}$  contains  $\Omega_5^{SO}$ . In fact  $\Omega_5^{SO}$  is also generated by the mapping torus of the complex-conjugation-map  $\mathbb{C}P^2 \to \mathbb{C}P^2$ . This implies that the invertible  $\mathrm{BF}_5^{\mathrm{H}}$  categories contain a  $\mathbb{Z}_2$  class, which is also the  $\mathbb{Z}_2$ class for invertible BF<sup>L</sup><sub>5</sub> categories.

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