ON PICTURE (2+1)-TQFTS

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Dedicated to the memory of Xiao-Song Lin

ABSTRACT. The goal of the paper is an exposition of the simplest $(2 + 1)$-TQFTs in a sense following a pictorial approach. In the end, we fell short on details in the later sections where new results are stated and proofs are outlined. Comments are welcome and should be sent to the 4th author.

1. INTRODUCTION

Topological quantum field theories (TQFTs) emerged into physics and mathematics in the 1980s from the study of three distinct enigmas: the infrared limit of $1 + 1$ dimensional conformal field theories, the fractional quantum Hall effect (FQHE), and the relation of the Jones polynomial to $3-$manifold topology. Now 25 years on, about half the literature in $3-$dimensional topology employs some "quantum" view point, yet it is still difficult for people to learn what a TQFT is and to manipulate the simplest examples. Roughly (axioms will follow later), a $(2+1)-$dimensional TQFT is a functor which associates a vector space $V(Y)$ called "modular functor" to a closed oriented surface Y (perhaps with some extra structures); sends disjoint union to tensor product, orientation reversal to dual, and is natural with respect to transformations (diffeomorphisms up to isotopy or perhaps a central extension of these) of Y. The empty set $\emptyset$ is considered to be a manifold of each dimension: $\{0, 1, \cdots \}$. As a closed surface, the associated vector space is $\mathbb{C}$, i.e., $V(\emptyset) = \mathbb{C}$. Also if $Y = \partial X$, X an oriented $3-$manifold (also perhaps with some extra structure), then a vector $Z(X) \in V(Y)$ is determined (surfaces Y with boundary also play a role but we pass over this for now.) A closed $3-$manifold X determines a vector $Z(X) \in V(\emptyset) = \mathbb{C}$, that is a number. In the case X is the $3-$sphere with "extra structure" a link L, then Witten's "$SU(2)-$family" of TQFTs yields a Jones polynomial evaluation $Z(S^3, L) = J_L(e^{2\pi i/r})$, $r = 3, 4, 5, \ldots$, as the "closed $3-$manifold" invariants, which mathematically are the Reshetikhin-Turaev invariants based on quantum groups [Jo1][Witt][RT]. This is the best known example. Note that physicists tend to index the same family by the levels $k = r - 2$. The shift 2 is the dual Coxeter number of $SU(2)$. We will use both indices. Most of the "quantum" literature in topology focuses on such closed

3−manifold invariants but there has been a growing awareness that a deeper understanding is locked up in the representation spaces $V(Y)$ and the "higher algebras" associated to boundary (Y) (circles) and points [FQ][Fd]. Let us explain this last statement. While invariants of 3−manifolds may be fascinating in their interrelations there is something of a shortage of work for them within topology. Reidemeister was probably the last topologist to be seriously puzzled as to whether a certain pair of 3−manifolds were the same or different and, famously, solved his problem by the invention of "torsion". (In four dimensions the situation is quite the opposite, and the closed manifold information from (3+1) dimensional TQFTs would be most welcome. But in this dimension, we do not yet know interesting examples of TQFTs.) So while the subject in dimension 3 seems to be maturing away from the closed case it is running into a pedological difficulty. It is hard to develop a solid understanding of the vector spaces $V(Y)$ even for simple examples. Our goal in these notes is, in a few simple examples to provide an intuition and understanding on the same level of "admissible pictures" modulo relations, just as we understand homology as cycles modulo boundaries. This is the meaning of "picture" in the title. A picture TQFT is one where $V(Y)$ is the space of formal $\mathbb{C}$−linear combinations of "admissible" pictures drawn on Y modulo some <u>local</u> (i.e. on a disk) linear relations. We will use the terms formal links, or formal tangles, or formal pictures , etc. to mean $\mathbb{C}$-linear combinations of links, tangles, pictures, etc. Formal tangles in 3-manifolds are also commonly referred to as "skein"s. Equivalently, we can adopt a dual point of view: take the space of linear functionals on multicurves and impose linear constraints for functionals. This point of view is closer to the physical idea of "amplitude of an eigenstate": think of a functional f as a wavefunction and its value $f(\gamma)$ on a multicurve γ as as the amplitude of the eigenstate γ. Then quotient spaces of pictures become subspaces of wavefunctions.

Experts may note that central charge $c \neq 0$ is an obstruction to this "picture formulation": the mapping class group $\mathcal{M}(Y)$ acts directly on pictures and so induces an action on any $V(Y)$ defined by pictures. As c determines a central extension $\widetilde{\mathcal{M}}^c(Y)$ which acts in place of $\mathcal{M}(Y)$, the feeling that all interesting theories must have $c \neq 0$ may have discouraged a pictorial approach. However this is not true: for any $V(Y)$ its endomorphism algebra End $V \cong V^* \otimes V$ has central charge $c = 0$ $(c(V^*) = -c(V))$ and remembers the original projective representation faithfully. In fact, all our examples are either of this form or slightly more subtle quantum doubles or Drinfeld centers in which the original theory V violates some axiom (the nonsingularity of the S−matrix) but this deficiency is "cured" by doubling [K][Mu]. Although those notes focus on picture TQFTs based on variations of the Jones-Wenzl projectors, the approach can be generalized to an arbitrary spherical tensor category. The Temperley-Lieb categories are generated by a single "fundamental" representation, and all fusion spaces are of dimension 0

or 1, so pictures are just 1-manifolds. In general, 1-manifolds need to be replaced by tri-valent graphs whose edges carry labels. But $c = 0$ is not sufficient for a TQFT to have a picture description. Given any two TQFTs with opposite central charges, their product has $c = 0$, e.g. TQFTs with $\mathbb{Z}_n$ fusion rules have $c = 1$, so the product of any theory with the mirror of a different one has $c = 0$, but such a product theory does not have a picture description in our sense.

While these notes describe the mathematical side of the story, we have avoided jargon which might throw off readers from physics. When different terminologies prevail within mathematics and physics we will try to note both. Within physics, TQFTs are referred to as "anyonic systems" [Wil][DFNSS]. These are 2-dimensional quantum mechanical systems with point like excitations (variously called "quas-particle" or just "particle", anyon, or perhaps "nonabelion") which under exchange exhibit exotic statistics: a nontrival representation of the braid groups acting on a finite dimensional Hilbert space V consisting of "internal degrees of freedom". Since these "internal degrees of freedom" sound mysterious, we note that this information is accessed by fusion: fuse pairs of anyons along a well defined trajectory and observe the outcome. Anyons are a feature of the fractional quantum Hall effect; Laughlin's 1998 Nobel prize was for the prediction of an anyon carrying change $e/3$ and with braiding statistics $e^{2\pi i/3}$. In the FQHE central charge $c \neq 0$ is enforced by a symmetry breaking magnetic field B. It is argued in [Fn] that solid state realizations of doubled or "picture" TQFTs may - if found - be more stable (larger spectral gap above the degenerate ground state manifold) because no symmetry breaking is required. The important electron - electron interactions would be at a lattice spacing scale $\sim 4\mathring{A}$ rather than at a "magnetic length" typically around $150\mathring{A}$. So it is hoped that the examples which are the subject of these notes will be the low energy limits of certain microscopic solid state models. Picture TQFTs have a Hamiltonian formulation, and describe string-net condensation in physics, which serve as a classification of non-chiral topological phases of matter. An interesting mathematical application is the proof of the asymptotic faithfulness of the representations of the mapping class groups.

As mentioned above, these notes are primarily about examples either of the form $V^* \bigotimes V$ or with a related but more general doubled structure $\mathcal{D}(V)$. In choosing a path through this material there seemed a basic choice: (1) present the picture (doubled) theories in a self contained way in two dimensions with no reference to their twisted ($c \neq 0$) and less tractable parent theories V or (2) weave the stories of $\mathcal{D}(V)$ and V together from the start and exploit the action of $\mathcal{D}(V)$ on V in analyzing the structure of $\mathcal{D}(V)$. In the end, the choice was made for us: we did not succeed in finding purely combinatorial "picture-proofs" for all the necessary lemmas — the action on V is indeed very useful so we follow course (2). We do recommend to some interested brave reader that she produce her own article hewing to course (1).

In the literature [BHMV] comes closest to the goals of the notes, and [Wal2] exploits deeply the picture theories in many directions. Actually, a large part of the notes will follow from a finished [Wal2]. If one applies the set up of [BHMV] to skeins in surface cross interval, $Y \times I$, and then resolves crossings to get a formal linear combination of $1-$submanifolds of $Y = Y \times \frac{1}{2} \subset Y \times I$ one arrives at (an example of) the "pictures" we study. In this doubled context there is no need for the p_1-structure (or "two-framing") intrinsic to the other approaches. To readers familiar with [BHMV] one should think of skeins in a handle body H, $\partial H = Y$, when an undoubled theory $V(Y)$ is being discussed, and skeins in $Y \times I$ when $\mathcal{D}V(Y)$ is under consideration.

By varying pictures and relations we produce many examples, and in the Temperley-Lieb-Jones context give a complete analysis of the possible local relations. Experts have long been troubled by certain sign discrepancies between the $S-$matrix arising from representations (or loop groups or quantum groups)[MS][Witt][KM] on the one hand and from the Kauffman bracket on the other [Li][Tu][KL]. The source of the discrepancy is that the fundamental representation of $SU(2)$ is antisymmetrically self dual whereas there is no room in Kauffman's spin-network notation to record the antisymmetry. We rectify this by amplifying the pictures slightly, which yields exactly the modular functor V coming from representation theory of $SU(2)_q$.

The content of each section is as follows. In Sections 2, 3, we treat diagram TQFTs for closed manifolds. In Sections 4, 5, 7.1, we handle boundaries. In Sections 7, 9, 8, we cover the related Jones-Kauffman TQFTs, and the Witten-Reshetikhin-Turaev $SU(2)$-TQFTs which have anomaly, and non-trivial Frobenius-Schur indicators, respectively. In Section 10, we first prove the uniqueness of TQFTs based on Jones-Wenzl projectors, and then classify them according to the Kauffman variable A. A theory V or $\mathcal{D}(V)$ is unitary if the vector spaces V have natural positive definite Hermitian structures. Only unitary theories will have physical relevance so we decide for each theory if it is unitary.

2. Jones representations

2.1. **Braid statistics.** Statistics of elementary particles in 3-dimensional space is related to representations of the permutation groups S_n. Since the discovery of the fractional quantum Hall effect, the existence of anyons in 2-dimensional space becomes a real possibility. Statistics of anyons is described by unitary representations of the braid groups B_n. Therefore, it is important to understand unitary representations of the braid groups B_n. Statistics of n anyons is given by unitary representation of the n-strand braid group B_n. Since statistics of anyons of different numbers n is governed by the same local physics, unitary representations of B_n have to be compatible for different n's in order to become possible statistics of

anyons. One such condition is that all representations of B_n come from the same unitary braided tensor category.

There is an exact sequence of groups: $1 \longrightarrow PB_n \longrightarrow B_n \longrightarrow S_n \longrightarrow 1$, where PB_n is the n-strand pure braid group. It follows that every representation of the permutation group S_n gives rise to a representation of the braid group B_n. An obvious fact for such representations of the braid groups is that the images are always finite. More interesting representations of B_n are those that do not factorize through S_n, in particular those with infinite images.

To construct representations of the braid groups B_n, we recall the construction of all finitely dimensional irreducible representations (irreps) of the permutation groups S_n: the group algebra $\mathbb{C}[S_n]$, as a representation of S_n, decomposes into irreps as $\mathbb{C}[S_n] \cong \bigoplus_i \mathbb{C}^{\dim V_i} \otimes V_i$, where the sum is over all irreps V_i of S_n. This construction cannot be generalized to B_n because B_n is an infinite group for $n \geq 2$. But by passing to various different finitely dimensional quotients of $\mathbb{C}[B_n]$, we obtain many interesting representations of the braid groups. This class of representations of B_n is Schur-Weyl dual to the the class of braid group representations from the quantum group approach and has the advantage of being manifestly unitary. This approach, pioneered by V. Jones [Jo1], provides the best understood examples of unitary braid group representations besides the Burau representation, and leads to the discovery of the celebrated Jones polynomial of knots [Jo2]. The theories in this paper are related to the quantum $SU(2)_q$ theories.

2.2. **Generic Jones representation of the braid groups.** The n-strand braid group B_n has a standard presentation with generators $\{\sigma_i, i = 1, 2, \cdots, n-1\}$ and relations:

$$(2.1) \qquad \sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{if} \quad |i - j| \geq 2,$$

$$(2.2) \qquad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

If we add the relations $\sigma_i^2 = 1$ for each i, we recover the standard presentation for S_n. In the group algebra $k[B_n]$, where k is a field (in this paper k will be either $\mathbb{C}$ or some rational functional field $\mathbb{C}(A)$ or $\mathbb{C}(q)$ over variables A or q), we may deform the relations $\sigma_i^2 = 1$ to linear combinations (superpositions in physical parlance) $\sigma_i^2 = a\sigma_i + b$ for some $a, b \in k$. By rescaling the relations, it is easy to show that there is only 1-parameter family of such deformations. The first interesting quotient algebras are the Hecke algebras of type A, denoted by $H_n(q)$, with generators $1, g_1, g_2, \cdots, g_{n-1}$ over $\mathbb{Q}(q)$ and relations:

$$(2.3) \qquad g_i g_j = g_j g_i, \quad \text{if} \quad |i - j| \geq 2,$$

$$(2.4) \qquad g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}.$$

and

$$(2.5) \qquad g_i^2 = (q - 1)g_i + q.$$

FIGURE 1. Kauffman bracket

The Hecke relation 2.5 is normalized to have roots $\{-1, q\}$ when the corresponding quadratic equation is solved. The Hecke algebras $H_n(q)$ at $q = 1$ become $\mathbb{C}[S_n]$, hence they are deformations of $\mathbb{C}[S_n]$. When q is a variable, the irreps of $H_n(q)$ are in one-to-one correspondence with the irreps of $\mathbb{C}[S_n]$.

To obtain the Hecke algebras as quotients of $\mathbb{C}[B_n]$, we set $q = A^4$, and $g_i = A^3\sigma_i$, where A is a new variable, called the Kauffman variable since it is the conventional variable for the Kauffman bracket below. Note that $q = A^2$ in [KL]. The prefactor A^3 is introduced to match the Hecke relation 2.5 exactly to a relation in the Temperley-Lieb algebras using the Kauffman bracket. In terms of the new variable A, and new generators σ_i's, the Hecke relation 2.5 becomes

$$(2.6) \qquad \sigma_i^2 = (A - A^{-3})\sigma_i + A^{-2}.$$

The Kauffman bracket $<>$ is defined by the resolution of a crossing in Figure 1

As a formula, $\sigma_i = A \cdot \mathrm{id} + A^{-1}U_i$, where U_i is a new generator. The Hecke algebra $H_n(q)$ in variable A and generators $1, U_1, U_2, \cdots, U_{n-1}$ is given by relations:

$$(2.7) \qquad U_iU_j = U_jU_i, \quad \text{if} \quad |i - j| \geq 2,$$

$$(2.8) \qquad U_iU_{i+1}U_i - U_i = U_{i+1}U_iU_{i+1} - U_{i+1},$$

and

$$(2.9) \qquad U_i^2 = dU_i,$$

where $d = -A^2 - A^{-2}$.

The relation 2.9 is the same as relation 2.6, which is the Hecke relation 2.5. The relation 2.8 is the braid relation 2.4.

The Temperley-Lieb (TL) algebras, denoted as $\mathrm{TL}_n(A)$, are further quotients of the Heck algebras. In the TL algebras, we replace the relations 2.8 by

$$(2.10) \qquad U_iU_{i\pm1}U_i = U_i,$$

i.e., both sides of relation 2.8 are set to 0.

Proposition 2.11. *The Kauffman bracket $<>$: $k[B_n] \longrightarrow \mathrm{TL}_n(A)$ is a surjective algebra homomorphism, where $k = \mathbb{C}(A)$.*

The proof is a straightforward computation.

When A is generic, the TL algebras $\mathrm{TL}_n(A)$ are semi-simple, hence $\mathrm{TL}_n(A) \cong \oplus_i\mathrm{Mat}_{n_i}(\mathbb{C}(A))$, where Mat_{n_i} are $n_i \times n_i$ matrices over $\mathbb{C}(A)$ for some n_i's.

The generic Jones representation of the braid groups B_n is defined as follows:

Definition 2.12. By the decomposition $\mathrm{TL}_n(A) \cong \oplus_i \mathrm{Mat}_{n_i}(\mathbb{C}(A))$, each braid $\sigma \in B_n$ is mapped to a direct sum of matrices under the Kauffman bracket. It follows from Prop. 2.11 that the image matrix of any braid is invertible and the map is a group homomorphism when restricted to B_n.

It is an open question whether or not the generic Jones representation is faithful, i.e., are there non-trivial braids which are mapped to the identity matrix?

2.3. **Unitary Jones representations.** The TL algebras $\mathrm{TL}_n(A)$ have a beautiful picture description by L. Kauffman, inspired by R. Penrose's spin-networks, as follows: fix a rectangle $\mathcal{R}$ in the complex plane with n points at both the top and the bottom of $\mathcal{R}$ (see Fig.2), $\mathrm{TL}_n(A)$ is spanned formally as a vector space over $\mathbb{C}(A)$ by embedded curves in the interior of $\mathcal{R}$ consisting of n disjoint arcs connecting the $2n$ boundary points of $\mathcal{R}$ and any number of simple closed loops. Such an embedding will be called a diagram or a multi-curve in physical language, and a linear combination of diagrams will be called a formal diagram. Two diagrams that are isotopic relative to boundary points represent the same vector in $\mathrm{TL}_n(A)$. To define the algebra structure, we introduce a multiplication: vertical stacking from bottom to top of diagrams and extending bilinearly to formal diagrams; furthermore, deleting a closed loop must be compensated for by multiplication by $d = -A^2 - A^{-2}$. Isotopy and the deletion rule of a closed trivial loop together will be called "d-isotopy".

FIGURE 2. Generators of TL

For our application, the variable A will be evaluated at a non-zero complex number. We will see later that when $d = -A^2 - A^{-2}$ is not a root of a Chebyshev polynomial Δ_i, $\mathrm{TL}_n(A)$ is semi-simple over $\mathbb{C}$, therefore, isomorphic to a matrix algebra. But when d is a root of some Chebyshev polynomial, $\mathrm{TL}_n(A)$ is in general not semi-simple. Jones discovered a semi-simple quotient by introducing local relations, called the Jones-Wenzl projectors [Jo4][We][KL]. Jones-Wenzl projectors have certain rigidity. Represented by formal diagrams in TL algebras, Jones-Wenzl projectors make it possible to describe two families of TQFTs labelled by integers. Conventionally the integer is either $r \geq 3$ or $k = r - 2 \geq 1$. The integer r is related to the order of A, and k is the level related to the $SU(2)$-Witten-Chern-Simons theory. One family is related to the $SU(2)_k$-Witten-Reshetikhin-Turaev (WRT) TQFTs, and will be called the Jones-Kauffman TQFTs. Although Jones-Kauffman TQFTs are commonly stated as the same as WRT TQFTs, they are really not. The other family is related to the quantum double of Jones-Kauffman TQFTs, which are of the Turaev-Viro type. Those doubled TQFTs, labelled by a
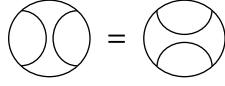
FIGURE 3. Z_2 homology

level $k \geq 1$, are among the easiest in a sense, and will be called diagram TQFTs. The level $k = 1$ diagram TQFT for closed surfaces is the group algebras of $\mathbb{Z}_2$-homology of surfaces. Therefore, higher level diagram TQFTs can be thought as quantum generalizations of the $\mathbb{Z}_2$-homology, and the Jones-Wenzl projectors as the generalizations of the homologous relation of curves in Figure 3.

The loop values $d = -A^2 - A^{-2}$ play fundamental roles in the study of Temperley-Lieb-Jones theories, in particular the picture version of $\mathrm{TL}_n(A)$ can be defined over $\mathbb{C}(d)$, so we will also use the notation $\mathrm{TL}_n(d)$. In the following, we focus the discussion on d, though for full TQFTs or the discussion of braids in $\mathrm{TL}_n(A)$, we need A's. Essential to the proof and to the understanding of the exceptional values of d is the trace tr: $\mathrm{TL}_n(d) \longrightarrow \mathbb{C}$ defined by Fig.4. This Markov trace is defined on diagrams by (and then extended linearly) connecting the endpoints at the top to the endpoints at the bottom of the rectangle by n non crossing arcs in the complement of the rectangle, counting the number $\#$ of closed loops (deleting the rectangle), and then forming $d^{\#}$.

FIGURE 4. Markov Trace

The Markov trace $(x, y) \rightarrowtail \mathrm{tr}(\overline{x}y)$ extends to a sesquilinear pairing on $\mathrm{TL}_n(d)$, where bar (diagram) is reflection in a horizontal middle-line and bar(coefficient) is complex conjugation.

Define the nth Chebyshev polynomial $\triangle_n(x)$ inductively by $\triangle_0 = 1, \triangle_1 = x$, and $\triangle_{n+1}(x) = x\triangle_n(x) - \triangle_{n-1}(x)$. Let c_n be the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$. There are c_n different diagrams $\{D_i\}$ consisting of n disjoint arcs up to isotopy in the rectangle $\mathcal{R}$ to connect the $2n$ boundary points of $\mathcal{R}$. These c_n diagrams generate $\mathrm{TL}_n(d)$ as a vector space. Let $M_{c_n \times c_n} = (m_{ij})$ be the matrix of the Markov trace Hermitian pairing in a certain order of $\{D_i\}$, i.e. $m_{ij} = \mathrm{tr}(\overline{D_i}D_j)$, then we have:

$$(2.13) \qquad \mathrm{Det}(M_{c_n \times c_n}) = \prod_{i=1}^{n} \Delta_i(d)^{a_{n,i}},$$

where $a_{n,i} = \binom{2n}{n-i-2} + \binom{2n}{n-i} - 2\binom{2n}{n-i-1}$.

FIGURE 5. Jones Wenzl projectors

This is derived in [DGG].

As a quick consequence of this formula, we have:

Lemma 2.14. *The dimension of* $TL_n(d)$ *as a vector space over* $\mathbb{C}(d)$ *is* c_n *if* d *is not a root of the Chebyshev polynomials* $\Delta_i, 1 \leq i \leq n$, *where* $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Proof. By the formula 2.13, if d is not a root of $\Delta_i, 1 \leq i \leq n$, then $\{D_i\}$ are linearly independent. As a remark, since each D_i is a monomial of U_i's, it follows that $\{U_i\}$ generate $TL_n(d)$ as an algebra.

$\square$

Next we show the existence and uniqueness of the Jones-Wenzl projectors.

Theorem 2.15. *For* $d \in \mathbb{C}$ *that is not a root of* $\triangle_k$ *for all* $k < n$, *then* $TL_n(d)$ *contains a unique element* p_n *characterized by:* $p_n^2 = p_n \neq 0$ *and* $U_i p_n = p_n U_i = 0$ *for all* $1 \leq i \leq n-1$. *Furthermore* p_n *can be written as* $p_n = 1 + U$ *where* $U = \sum c_j h_j$, h_j *a product of* $U_i's$, $1 \leq i \leq n-1$ *and* $c_j \in \mathbb{C}$.

Proof. Suppose p_n exists and can be expanded as $p_n = a1 + U$, then $p_n^2 = p_n(a1 + U) = p_n(a1) = ap_n = a^2 1 + aU$, so $a = 1$. Now check uniqueness by supposing $p_n = 1 + U$ and $p_n' = 1 + V$ both have the properties above and expand $p_n p_n'$ from both sides:

$$p_n' = 1 \cdot p_n' = (1 + U)p_n' = p_n p_n' = p_n(1 + V) = p_n \cdot 1 = p_n.$$

The proof is completed by H. Wenzl's [We] inductive construction of p_{n+1} from p_n which also reveals the exact nature of the "generic" restriction on d. The induction is given in Figure 5, where $\mu_n = \frac{\Delta_{n-1}(d)}{\Delta_n(d)}$.

Tracing the inductive definition of p_{n+1} yields $\text{tr}(p_{n+1}) = d\,\text{tr}(p_n) - \frac{\triangle_{n-1}}{\triangle_n}\text{tr}(p_n)$ showing $\text{tr}(p_n)$ satisfies the Chebyshev recursion (and the initial data). Thus $\text{tr}(p_n) = \triangle_n$.

It is not difficult to check that $U_i p_n = p_n U_i = 0$, $i < n$. (The most interesting case is U_{n-1}.) Consult [KL] or [Tu] for details. $\square$

The idempotent p_n is called the *Jones-Wenzl idempotent*, or the Jones-Wenzl projector, and plays an indispensable role in the pictorial approach to TQFTs.

Theorem 2.16. *(1): If $d \in \mathbb{C}$ is not a root of Chebyshev polynomials $\Delta_i, 1 \le i \le n$, then the TL algebra $\mathrm{TL}_n(d)$ is semisimple.*

(2): Fixing an integer $r \ge 3$, a non-zero number d is a root of $\triangle_i$, $i < r$ if and only if $d = -A^2 - A^{-2}$ for some A such that $A^{4l} = 1, l \le r$. If $d = -A^2 - A^{-2}$ for a primitive $4r$-th root of unity A for some $r \ge 3$ or a primitive $2r$th or rth for r odd, then the TL algebras $\{\mathrm{TL}_n(d)\}$ modulo the Jones-Wenzl idempotent p_{r-1} are semi-simple.

Proof. (1): $\mathrm{TL}_n(d)$ is a $*$-algebra. By formula 2.13, the determinant of the Markov trace pairing is $\prod_{i=1}^n \Delta_i(d)^{a_{n,i}}$, hence the $*$-structure is non-degenerate. By Lemma B.5, $\mathrm{TL}_n(d)$ is semi-simple.

(2): The first part follows from $\Delta_n(d) = (-1)^n \frac{A^{2n+2} - A^{-2n-2}}{A^2 - A^{-2}}$. In Section 5, we will show that the kernel of the Markov trace Hermitian pairing is generated by p_{r-1}, and the second part follows.

$\square$

The semi-simple quotients of $\mathrm{TL}_n(d)$ in the above theorem will be called the Temperley-Lieb-Jones (TLJ) algebras or just Jones algebras, denoted by $\mathrm{TLJ}_n(d)$. The TLJ algebras are semi-simple algebras over $\mathbb{C}$, therefore it is isomorphic to a direct sum of matrix algebras, i.e.,

$$(2.17) \qquad\qquad \mathrm{TLJ}_n(d) \cong \oplus_i Mat_{n_i}(\mathbb{C}).$$

As in the generic Jones representation case, the Kauffman bracket followed by the decomposition yields a representation of the braid groups.

Proposition 2.18. *(1): When the Markov trace Hermitian paring is $\pm$-definite, then Jones representations are unitary, but reducible. When $A = \pm ie^{\pm\frac{2\pi i}{4r}}$, the Markov trace Hermitian pairing is $+$-definite for all n's.*

(2): Given a braid $\sigma \in B_n$, the Markov trace is a weighted trace on the matrix decomposition 2.17, and when multiplied by $(-A)^{-3\sigma}$ results in the Jones polynomial of the braid closure of σ evaluated at $q = A^4$.

Unitary will be established in Section 10, and reducibility follows from the decomposition 2.17. That the Markov trace, normalized by the framing-dependence factor, is the Jones polynomial follows from direct verification of invariance under Reidermeister moves or Markov's theorem (see e.g. [KL]).

2.4. **Uniqueness of Jones-Wenzl projectors.** Fix an $r \ge 3$ and a primitive $4r$th root of unity or a primitive $2r$th or rth root of unity for r odd, and $d = -A^2 - A^{-2}$. In this section, we prove that TL_d has a unique ideal generated by p_{r-1}. When A is a primitive $4r$th root of unity, this is proved in the Appendix of [Fn] by F. Goodman and H. Wenzl. Our elementary argument works for all A as above.

Notice that TL_d admits the structure of a (strict) monoidal category, with the tensor product given by horizontal "stacking", e.g., juxtaposition of diagrams.

This tensor product (denoted $\otimes$) is clearly associative, and 1_0, the identity on 0 vertices or the empty object, serves as a unit. The tensor product and the original algebra product on TL_d satisfy the interchange law, $(f \otimes g) \cdot (f' \otimes g') = (f \cdot f') \otimes (g \cdot g')$, whenever the required vertical composites are defined.

We may use this notation to recursively define the projectors p_k: $p_{k+1} = p_k \otimes 1_1 - \mu_k(p_k \otimes 1_1)U_k^{k+1}(p_k \otimes 1_1)$. We define $p_0 = 1_0$, $p_1 = 1_1$ and $\mu_k = \frac{\Delta_{k-1}}{\Delta_k}$. Using this we can prove a sort of "decomposition theorem" for projectors:

Proposition 2.19. $p_k = \left(\otimes_{i=1}^{\lfloor \frac{k}{r} \rfloor} p_r \right) \otimes p_{(k \bmod r)}.$

Proof. We proceed by induction, using the recursive definition of the Jones-Wenzl projectors. For p_1 the statement is trivial. Assuming the assertion holds for p_k, we then have (let $m = k \bmod r$):

$$p_{k+1} = p_k \otimes 1_1 - \mu_k(p_k \otimes 1_1)U_k^{k+1}(p_k \otimes 1_1)$$

$$= (\bigotimes_{i=1}^{\lfloor \frac{k}{r} \rfloor} p_r) \otimes p_m \otimes 1_1 - \mu_k((\bigotimes_{i=1}^{\lfloor \frac{k}{r} \rfloor} p_r) \otimes p_m \otimes 1_1)U_k^{k+1}((\bigotimes_{i=1}^{\lfloor \frac{k}{r} \rfloor} p_r) \otimes p_m \otimes 1_1)$$

Then, if $m \neq 0$,

$$= (\bigotimes_{i=1}^{\lfloor \frac{k}{r} \rfloor} p_r) \otimes p_m \otimes 1_1 -$$

$$\mu_k((\bigotimes_{i=1}^{\lfloor \frac{k}{r} \rfloor} p_r) \otimes p_m \otimes 1_1)(1_{k-m} \otimes U_m^{m+1})((\bigotimes_{i=1}^{\lfloor \frac{k}{r} \rfloor} p_r) \otimes p_m \otimes 1_1)$$

$$= (\bigotimes_{i=1}^{\lfloor \frac{k}{r} \rfloor} p_r) \otimes \left(p_m \otimes 1_1 - \mu_m(p_m \otimes 1_1)U_m^{m+1}(p_m \otimes 1_1) \right)$$

(The $\lfloor \frac{k}{r} \rfloor$ copies of p_r can be factored out of the second term by $p_r p_r = p_r$.)

$$= (\bigotimes_{i=1}^{\lfloor \frac{k}{r} \rfloor} p_r) \otimes p_{m+1}$$

If $m < r - 1$, then $m + 1 = (k+1) \bmod r$; if $m = r - 1$, then we get one more copy of p_r, as needed. So it remains to consider the case above where $k \bmod r = 0$. But then $\mu_k = \mu_{k \bmod r} = \mu_0 = 0$, so that

$$p_{k+1} = (\bigotimes_{i=1}^{\frac{k}{r}} p_r) \otimes 1_1 = (\bigotimes_{i=1}^{\lfloor \frac{k+1}{r} \rfloor} p_r) \otimes p_1$$

as desired.   □

In analogy with the standard notion from ring theory, an ideal in TL is defined to be a class of morphisms which is internally closed under addition, and externally

closed under both the vertical product (composition) $\cdot$ and the horizontal product $\otimes$. Given such an ideal I, we may form the quotient category TL/I, which has the same objects as TL, and hom-sets formed by taking the usual quotient of $\text{Hom}(\mathbf{m}, \mathbf{n})$ by those morphisms in $I \cap \text{Hom}(\mathbf{m}, \mathbf{n})$.

We can prove that $<p_{r-1}>$ is an ideal.

Lemma 2.20. *The ideal $\mathcal{R}_d = <p_{r-1}>$ is a proper ideal.*

Proof. It suffices to show that the $\otimes$-identity 1_0 is not in the ideal. In order for 1_0 to be in the ideal, it would have to be obtained from some closed network (e.g., element of $\text{Hom}(\mathbf{0}, \mathbf{0})$) which contains at least one copy of p_{r-1}. Fixing such a projector, we expand all other terms in the network (this includes getting rid of closed loops), so that we are left with a linear combination of closed networks, each having exactly one $r-1$ strand projector. Now, considering each term seperately, if there are any strands that leave and re-enter the projector on the same side, then the network is null (since $p_{r-1}U_i^{r-1} = 0$). So the only remaining terms will be strand closures of p_{r-1}; but by the above, these are null as well, so that every term in the expansion vanishes.

Since every closed network with p_{r-1} is null, it follows that $1_0 \notin \mathcal{R}_d$, and therefore $\mathcal{R}_d$ is a proper ideal of TL. $\square$

In fact, this same ideal is generated by any p_k for $k \geq r-1$; this is established via a sequence of lemmas.

Lemma 2.21. $<p_r>=<p_{r-1}>$

Proof. It is clearly sufficient to show $p_{r-1} \in <p_r>$. Set $x = p_r \otimes 1_1$, and expand p_r in terms of p_{r-1} according to the recursive definition. Then connect the rightmost two strands in a loop: e.g., by pre- and post-multiplying by the appropriate elements of $\text{Hom}(\mathbf{r-1}, \mathbf{r+1})$ and $\text{Hom}(\mathbf{r+1}, \mathbf{r-1})$, respectively. Using the fact that $p_{r-1}p_{r-1} = p_{r-1}$, the resulting diagram simplifies to $(d - \mu_{r-1})p_{r-1}$; and since $\mu_{r-1} \neq d$, the coefficient is invertible, so that $p_{r-1} \in <p_r>$. $\square$

Lemma 2.22. *For any integer $k \geq 1$, $<p_{kr}>=<p_{r-1}>$.*

Proof. By induction; the base case is established in the previous Lemma. For $k \geq 2$, we can write $p_{kr} = p_{(k-2)r} \otimes p_r \otimes p_r$, and then consider the tangle $p_{kr} \otimes 1_r$. By again pre- and post-multiplying by appropriate tangles, and using $p_r p_r = p_r$, we see that $p_{(k-1)r} \in <p_{kr}>$. $\square$

Lemma 2.23. *For any $k \geq r-1$, $<p_k>=<p_{r-1}>$.*

Proof. This basically uses the same technique as the previous lemma, combined with the fact that $p_r(p_l \otimes 1_{r-l}) = p_r$ (see [KL]).

Let $m = k \bmod r$; if $m = 0$, then this falls under the case of the previous lemma, so $0 < m < r$. Now consider $x = p_k \otimes 1_{2r-m}$; we can use the technique of the previous lemma to merge the last three groups of r strands into one, so that the

resulting element $x' = p_{\lfloor \frac{k}{r} \rfloor r} \otimes (p_r(p_l \otimes 1_{r-l})1_r)$. But $p_r(p_l \otimes 1_{r-l}) = p_r$, so that $x' = p_{(\lfloor \frac{k}{r} \rfloor + 1)r}$, whence, by the previous lemma, $<p_{r-1}>=<p_k>$. $\qquad\square$

Thus, in the quotient category $\mathrm{TL}/\mathcal{R}_d$, all k-projectors, for $k \geq r-1$, are null.

We have shown that $\mathcal{R}_d$ is an ideal; our strategy in showing that $\mathcal{R}_d$ is unique will be to show that it has no proper ideals, and that the quotient $\mathrm{TL}/\mathcal{R}_d$ has no nontrivial ideals. To show the latter fact, we will show that the ideal (in the quotient) generated by any element is in fact all of $\mathrm{TL}/\mathcal{R}_d$.

We note also that $\mathrm{TL}/\mathcal{R}_d$ may be described succinctly as the subcategory of TL whose tangles have less than $r-1$ "through-passing" strands. This subcategory does not close under $\otimes$ as described above, but can be shown to be well-defined under the reduction $1_{r-1} \rightsquigarrow (1_{r-1} - p_{r-1})$. This view is not necessary in what follows, so we do not pursue it further; but it may be useful in thinking about the quotient category.

A preliminary observation is that $\mathrm{TL}/\mathcal{R}_d$ has no zero divisors:

Lemma 2.24. *Let $x, y \in TL/\mathcal{R}_d$. If $x \otimes y = 0$, then $x = 0$ or $y = 0$.*

Proof. The statement clearly holds in TL; so the only way it could fail in the quotient is if p_{r-1} had a tensor decomposition.

So, suppose, $x \otimes y = p_{r-1}$, where x is a tangle on $k > 0$ strands and y is a tangle on $l > 0$ strands, both nontrivial (that $\mathrm{dom}(x) = \mathrm{cod}(x)$ and $\mathrm{dom}(y) = \mathrm{cod}(y)$ follows from the fact that $p_{r-1}p_{r-1} = p_{r-1}$). Then the properties of projectors and the interchange law give:

$$x \otimes y = (x \otimes y)(x \otimes y) = xx \otimes yy \Longrightarrow xx = x, yy = y$$

Further $(x \otimes y)U_i^{k+l} = 0$ for all i, so that $xU_i^k \otimes y = 0 \Longrightarrow xU_i^k = 0$, and likewise $yU_i^l = 0$. Thus both x and y are projectors. But the strand closure of $p_k \otimes p_l$ is $\Delta_k \Delta_l$, which are both nonzero, and the strand closure of p_{r-1} is zero, so we have reached a contradiction. $\qquad\square$

The next lemma introduces an algorithm that is the key to the rest of the proof:

Lemma 2.25. *Any nonzero ideal $I \subset TL/\mathcal{R}_d$ contains at least one element of* $\mathrm{Hom}(\mathbf{r-3}, \mathbf{r-3})$.

Proof. Let $x \neq 0 \in I$, say $x \in \mathrm{Hom}(\mathbf{m}, \mathbf{n})$. First, if $m \neq n$, then we can tensor with the unique basis element in either $\mathrm{Hom}(\mathbf{0}, \mathbf{2})$ or $\mathrm{Hom}(\mathbf{2}, \mathbf{0})$ the appropriate number of times so that we get an $x' \in \mathrm{Hom}(\mathbf{k}, \mathbf{k}) \in \mathbf{I}$, where $k = \max\{m, n\}$. (By the previous lemma, $x' \neq 0$.) If $k \leq r-3$, then $x' \otimes 1_{r-3-k} \in \mathrm{Hom}(\mathbf{r-3}, \mathbf{r-3})$ is an element of the ideal; so it remains to show the case where $k > r-3$.

First, assume k and $r-3$ have the same parity; if not, use $x' \otimes 1_1$ instead of x'. Then let $k_0 = k, x'_0 = x'$, and use the following algorithm (starting with $i = 0$):

  (1) If $k_i = r-3$, then stop: $x'_i \in \mathrm{Hom}(\mathbf{r-3}, \mathbf{r-3})$ is in the ideal.

(2) Since $k_i \geq r - 1$, and $x_i' \neq 0$, it follows that $x_i' \neq \alpha p_{k_i}$, since all $r - 1$ and above projectors are null in $\mathrm{TL}/\mathcal{R}_d$. Recall that p_{k_i} is the unique element in $\mathrm{Hom}(\mathbf{k_i}, \mathbf{k_i})$ such that (i) $U_j^{k_i} p_{k_i} = p_{k_i} U_j^{k_i} = 0$ for $1 \leq j < k_i$; and (ii) $p_{k_i} p_{k_i} = p_{k_i}$. From this it follows that the only elements which satisfy (i) are αp_{k_i}, for some $\alpha \in \mathbb{C}$. Therefore, since $x_i' \neq \alpha p_{k_i}$, there exists some $U_i = U_{j_i}^{k_i}$ such that $U_i x' \neq 0$.

(3) Using an argument similar to the above, there exists some $U_i' = U_{j_i'}^{k_i}$ such that $(U_i x') U_i' \neq 0$.

(4) Set V_i to be the unique basis element in $\mathrm{Hom}(\mathbf{k_i} - \mathbf{2}, \mathbf{k_i})$ which connects the j_i and $j_i + 1$ vertices on the top (codomain) objects, and connects the remaining $k - 2$ vertices on top and bottom to each other. Then $V_i U_i x' U_i'$ can be described as being exactly like $U_i x' U_i'$, except that the top half-loop of the U_i has been factored out as d, thus reducing the domain object by two vertices. It is thus clear that $V_i U_i x' U_i' \neq 0$.

(5) Similarly, choose V_i' to be the unique element in $\mathrm{Hom}(\mathbf{k_i}, \mathbf{k_{i-2}})$ connecting the j_i' and $j_i' + 1$ vertices of the domain object, thus closing the half loop of U_i'. Then $V_i U_i x' U_i' V_i' \neq 0$.

(6) Set $x_{i+1}' = V_i U_i x' U_i' V_i'$, $k_{i+1} = k_i - 2$, and return to step (1).

After $j = \frac{1}{2}(k_0 - (r - 3))$ passes through the algorithm, the desired element $x_j' \in I$ is produced. □

The proof of the previous Lemma is useful in establishing that $\mathcal{R}_d$ has no proper sub-ideals.

Lemma 2.26. *For any $x \in \mathcal{R}_d, x \neq 0$, then $<x> = \mathcal{R}_d$.*

Proof. Use the techniques previous Lemma to get an element $x' \in <x>$ such that $x' \in \mathrm{Hom}(\mathbf{k}, \mathbf{k})$, and $k \equiv r - 1 \bmod 2$. Then follow the algorithm, except for on steps (2) and (3): for, since $x_i' \in \mathcal{R}_d$, it is possible that $x_i' = \alpha p_{k_i}$. If this is not the case, proceed with the algorithm as it is stated. However, if $x_i' = \alpha p_{k_i}$, then it follows that $<x> = \mathcal{R}_d$, by Lemma 2.23. So it only remains to show that this does happen at some point before the algorithm terminates: e.g., that for some i, $x_i' = \alpha p_{k_i}$.

But, suppose this didn't happen; then, the algorithm goes through to completion, yielding an element $y \in <x>$ such that $y \in \mathrm{Hom}(\mathbf{r} - \mathbf{3}, \mathbf{r} - \mathbf{3})$, $y \neq 0$. But then $y \notin \mathcal{R}_d$, since every nonzero element of $\mathcal{R}_d$ must have at least $r - 1$ strands. This contradicts the fact that $y \in <x> \subset \mathcal{R}_d$; therefore, there must be some i such that $x_i' = \alpha p_{k_i}$, and so the lemma follows. □

Now we can put all of this together to obtain our desired result:

Theorem 2.27. *TL_d has a unique proper nonzero ideal when A is as in Lemma 3.4.*

Proof. By Lemma 2.20, $\mathcal{R}_d = <p_{r-1}>$ is a proper ideal, which, by Lemma 2.26, has no proper sub-ideals. To prove the theorem, therefore, it suffices to show that the quotient category $\text{TL}/\mathcal{R}_d$ has no proper nonzero ideals.

Consider $<x>$, for any $x \in \text{TL}/\mathcal{R}_d$. By Lemma 2.25, there exists some $y \in <x>$ such that $x \in \text{Hom}(\mathbf{r-3}, \mathbf{r-3})$. But now, instead of stopping at this point in the algorithm, we continue the loop, with the possibility that x_i' might actually be a projector. So we again modify steps (2) and (3), as below:

- (1') If $k_i = 0$, stop; set $x' = x_i'$.
- (2') If $x_i' = \alpha p_{k_i}$ for some constant α, then stop, with $x' = x_i'$. Otherwise, proceed with step (2) of the original algorithm.
- (3') If $U_i x_i' = \alpha p_{k_i}$ for some constant α, then stop, with $x' = x_i'$. Otherwise, proceed with step (3) of the original algorithm.

So now, when the algorithm terminates, we are left with some $x' \in <x>$, with either: (a) $x' = \alpha 1_0$, $\alpha \neq 0$; or (b) $x' = \alpha p_k$ for some $1 \leq k < r-1$, $\alpha \neq 0$. In case (a), we have that $1_0 \in <x>$, so that $<x> = \text{TL}/\mathcal{R}_d$. In case (b), consider the element $y = \frac{1}{\alpha} x' \otimes 1_k = p_k \otimes 1_k \in <x>$. We can then pre- and post-multiply by the elements of $\text{Hom}(\mathbf{0}, \mathbf{2k})$ and $\text{Hom}(\mathbf{2k}, \mathbf{0})$, respectively, which join the left group of k strands to the right group of k strands. In other words, the resulting element is simply Δ_k, the strand closure of p_k, times 1_0. Since $\Delta_k \neq 0$ for $1 \leq k \leq r-2$, it follows that $1_0 \in <x>$, so that we still have $<x> = \text{TL}/\mathcal{R}_d$.

So we have shown that $\text{TL}/\mathcal{R}_d$ has no proper nonzero ideals, and therefore, that TL has the unique ideal $\mathcal{R}_d$. $\qquad\qquad\square$

As a corollary, we have the following:

Theorem 2.28. *1):If d is not a root of any Chebyshev polynomial $\Delta_k, k \geq 1$, then the Temperley-Lieb category TL_d is semisimple.*

2): Fixing an integer $r \geq 3$, a non-zero number d is a root of $\triangle_k$, $k < r$ if and only if $d = -A^2 - A^{-2}$ for some A such that $A^{4l} = 1, l \leq r$. If $d = -A^2 - A^{-2}$ for a primitive $4r$-th root of unity A or $2r$-th r odd or r-th r odd for some $r \geq 3$, then the tensor category TLJ_d has a unique nontrivial ideal generated by the Jones-Wenzl idempotent p_{r-1}. The quotient categories TLJ_d are semi-simple.

3. Diagram TQFTs for closed manifolds

3.1. **"d-isotopy", local relation, and skein relation.** Let Y be an oriented compact surface, and $\gamma \subset Y$ be an imbedded unoriented 1-dimensional submanifold. If $\partial Y \neq \phi$ then fix a finite set F of points on ∂Y and require $\partial \gamma = F$ transversely. That is, γ a disjoint union of non-crossing loops and arcs, a "multicurve". Let $\mathcal{S}$ be the set of such γ's. To "linearize" we consider the complex span $\mathbb{C}[\mathcal{S}]$ of $\mathcal{S}$, and then impose linear relations. We <u>always</u> impose the "isotopy" constraint $\gamma' = \gamma$, if γ' is isotopic to γ. We also always impose a constraint of the form $\gamma \cup \mathbf{O} = d \cdot \gamma$ for some $d \in \mathbb{C} \backslash \{0\}$, independent of γ (see an example below
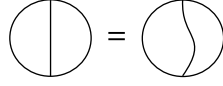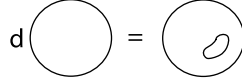
FIGURE 6. Isotopy

FIGURE 7. d constraint

that we do not impose this relation). The notation $\gamma \cup \mathbf{O}$ means a multi curve made from γ by adding a disjoint loop $\mathbf{O}$ to γ where $\mathbf{O}$ is "trivial" in the sense that it is the boundary of a disk B^2 in the interior of Y. Taken together these two constraints are "$d-$isotopy" relation: $\gamma' - \frac{1}{d} \cdot (\gamma \cup \mathbf{O}) = 0$ if γ' is isotopic to γ.

A diagram *local relation* or just a local relation is a linear relation on multicurves $\gamma_1, \ldots, \gamma_m$ which are identical outside some disk B^2 in the interior of Y, and intersect ∂B^2 transversely. By a disk here, we mean a topological disk, i.e., any diffeomorphic image of the standard 2-disk in the plane. Local relations are usually drawn by illustrating how the γ_i differ on B^2. So the "isotopy" constraint has the form:

and the "$d-$constraint" has the form:

Local relations have been explored to a great generality in [Wal2] and encode information of topologically invariant partition functions of a ball. We may filter a local relation according to the number of points of $\gamma_i \cap \partial B^2$ which may be $0, 2, 4, 6, \ldots$ since we assume γ transverse to ∂B^2. "Isotopy" has degree $= 2$ and "d-constraint" degree $= 0$.

Formally, we define a local relation and a skein relation as follows:

Definition 3.1. (1) Let $\{D_i\}$ be all the diagrams on a disk B^2 up to diffeo-morphisms of the disk and without any loops. The diagrams $\{D_i\}$ are filtered into degrees $= 2n$ according to how many points of $D_i \cap \partial B^2$, and there are Catalan number c_n many diagrams of degree $2n$ ($c_0 = 1$ which is the empty diagram). A degree $= 2n$ diagram local relation is a formal linear equation of diagrams $\sum_i c_i D_i = 0$, where $c_i \in \mathbb{C}$, and $c_i = 0$ if D_i is not of degree $= 2n$.

(2) A skein relation is a resolution of over-/under-crossings into formal pictures on B^2. If the resolutions of crossings for a skein relation are all formal diagrams, then the skein relation induces a *set* map from $\mathbb{C}[B_n]$ to $TL_n(d)$.

The most interesting diagram local relations are the Jones-Wenzl projectors (the rectangle $\mathcal{R}$ is identified with a disk B^2 in an arbitrary way). When we impose a local relation on $\mathbb{C}[\mathcal{S}]$, we get a quotient vector space of $\mathbb{C}[\mathcal{S}]$ as follows: for any multi-curve γ and a disk B^2 in the interior of Σ, if γ intersects B^2 transversely
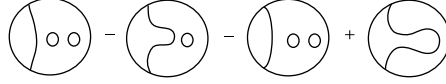
FIGURE 8. Singular arc
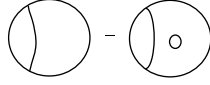

FIGURE 9. Resolution relation
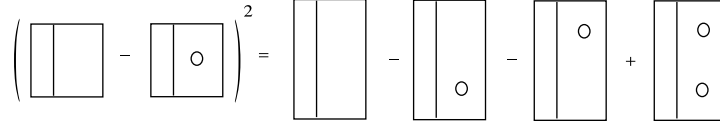

FIGURE 10. Two term relation

FIGURE 11. Two term squared

and the part $\gamma \cap B^2$ of γ in B^2 matches one of the diagram D_j topologically in the local relation $\sum_i c_i D_i = 0$, and $c_j \neq 0$, then we set $\gamma = -\sum_{i \neq j} \frac{c_i}{c_j} \gamma_i'$ in $\mathbb{C}[\mathcal{S}]$ where γ_i' is obtained from γ by replacing the part $\gamma \cap B^2$ of γ in B^2 by the diagram D_i.

Kauffman bracket is the most interesting skein relation in this paper. More general skein relations can be obtained from minimal polynomials of R-matrices from a quantum group. Kauffman bracket is an unoriented version of the $SU(2)_q$ case.

As a digression we describe an unusual example where we impose "isotopy" but <u>not</u> the "d-constraint". It is motivated by the theory of finite type invariants. A singular crossing (outside $\mathcal{S}$) suggests the "type 1" relation in Figure 3.

This relation is closely related to $\mathbb{Z}_2$−homology and is compatible with the choice $d = 1$. We will revisit it again under the name $\mathbb{Z}_2$−gauge theory.

Now consider the "type 2" relation Figure 9 which comes by resolving the arc in Figure 8 using either arrow along the arc. (Reversing the arrow leaves the relation Figure 9 on unoriented diagrams unchanged.)

Formally we may write the resolution relation Figure 9 as the square of the 2 term relation drawn in Figure 10.

Interpreting "times" as "vertical stacking" makes the claim immediate as shown in Figure 11.

Since the two term relation Figure 10 does not appear to be a consequence of the resolution relation Figure 9, dividing by the resolution relation induces nilpotence in the algebra (of degree $= 2$ diagrams under vertical stacking). By imposing the "d" relation we find that only semi-simple algebras are encountered. This is closer to the physics (the simple pieces are symmetries of a fixed particle type or "super-selection sector") and easier mathematically so henceforth we <u>always</u> assume a "$d-$constraint" for some $d \in \mathbb{C}\backslash\{0\}$.

3.2. **Picture classes.** Fix a local relation $R = 0$. Given an oriented closed surface Y. The vector space $\mathbb{C}[\mathcal{S}]$ is infinitely dimensional. We define a finitely dimensional quotient of $\mathbb{C}[\mathcal{S}]$ by imposing the local relation R as in last section: $\mathbb{C}[\mathcal{S}]$ modulo the local relation. The resulting quotient vector space will be called the picture space, denoted as $\mathrm{Pic}^R(Y)$. Elements of $\mathrm{Pic}^R(Y)$ will be called picture classes. We will denote $\mathrm{Pic}^R(Y)$ as $\mathrm{Pic}(Y)$ when R is clear or irrelevant for the discussion.

Proposition 3.2.   (1) $Pic(Y)$ is independent of the orientation of Y.
   (2) $Pic(S^2) = \mathbb{C}\emptyset$, so it is either 0 or $\mathbb{C}$.
   (3) $Pic(Y_1 \amalg Y_2) \cong Pic(Y_1) \otimes Pic(Y_2)$.
   (4) $Pic(Y)$ is a representation of the mapping class group $\mathcal{M}(Y)$. Furthermore, the action of $\mathcal{M}(Y)$ is compatible with property (3).

Proof. Properties (1) (3) and (4) are obvious from the definition. For (2), since every simple closed curve on S^2 bounds a disk, a multicurve with m loops is $d^m\emptyset$ by "d-isotopy". Therefore, if $\emptyset$ is not 0, it can be chosen as the canonical basis. $\square$

For any choice of $A \neq 0$, we may impose the Jones-Wenzl projector as a local relation. The resulting finitely dimensional vector spaces $\mathrm{Pic}(Y)$ might be trivial. For example, if we choose a $d \neq \pm 1$ and impose the Jones-Wenzl projector $p_2 = 0$ as the local relation. To see that the resulted picture spaces $= 0$, we reconnect two adjacent loops in a disk into one using $p_2 = 0$; this gives the identity $(d^2 - 1)\emptyset = 0$. If $d \neq \pm 1$, then $\emptyset = 0$, hence $\mathrm{Pic}(S^2) = 0$. Even if $\mathrm{Pic}(Y)$'s are not 0, they do not necessarily form a TQFT in general. We do not know any examples. If exist, such non-trivial vector spaces might have interesting applications because they are representations of the mapping class groups. In the cases of Jones-Wenzl projectors, only certain special choices of A's lead to TQFTs.

3.3. **Skein classes.** Fix a $d \in \mathbb{C}\backslash\{0\}$, a skein relation $K = 0$ and a local relation $R = 0$. Given an oriented 3-manifold X (possibly with boundaries). Let $\mathcal{F}$ be all the non-crossing loops in X, i.e., all links l's in the interior of X, and $\mathbb{C}[\mathcal{F}]$ be their linear span. We impose the "d-isotopy" relation on $\mathbb{C}[\mathcal{F}]$, where a knot is trivial if it bounds a disk in X. For any 3-ball B^3 inside X and a link l, the part $l \cap B^3$ of l can be projected onto a proper rectangle $\mathcal{R}$ of B^3 using the orientation of X (isotopy l if necessary). Resolving all crossings with the given skein relation

$K = 0$, we obtain a formal diagram in $\mathcal{R}$, where the local relation $R = 0$ can be applied. Such operations introduce linear relations onto $\mathbb{C}[\mathcal{F}]$. The resulting quotient vector space will be called the skein space, denoted by $S_{d,K,R}(X)$ or just $S(X)$, and elements of $S(X)$ will be called skein classes.

As mentioned in the introduction, the empty set $\emptyset$ has been regarded as a manifold of each dimension. It is also regarded as a multicurve in any manifold Y or a link in any X, and many other things. In the case of skein spaces, the empty multicurve represents an element of the skein space $S(X)$. For a closed manifold X, this would be the canonical basis if the skein space $S(X) \cong \mathbb{C}$. But the empty skein is the 0 vector for some closed 3-manifolds. In these cases, we do not have a canonical basis for the skein space $S(X)$ even if $S(X) \cong \mathbb{C}$.

Skein spaces behave naturally with respect to disjoint union, inclusion of spaces, orientation reversal, and self-diffeomorphisms: the skein space of a disjoint union is isomorphic to the tensor product; an orientation preserving embedding from $X_1 \to X_2$ induces a linear map from $S(X_1)$ to $S(X_2)$, orientation reversal induces a conjugate-linear map on $S(X)$, and diffeomorphisms of X act on $S(X)$ by moving pictures around, therefore $S(X)$ is a representation of the orientation preserving diffeomorphisms of X up to isotopy.

Proposition 3.3.   (1) *If Y is oriented, then $Pic(Y)$ is an algebra.*
  (2) *If $\partial X = Y$, then $Pic(Y)$ acts on $S(X)$. If Y is oriented, then $S(X)$ is a representation of $Pic(Y)$.*

Proof. (1): Given two multicurves x, y in Y, and consider $Y \times [-1, 1]$, draw x in $Y \times 1$ and y in $Y \times -1$. Push x into the interior of $Y \times [0, 1]$ and y into $Y \times [-1, 0]$. Isotope x, y so that their projections onto $Y \times 0$ are in general position. Resolutions of the crossings using the given skein relation result in a formal multicurve in Y, which is denoted by xy. We define $[x][y] = [xy]$, where $[\cdot]$ denotes the picture class. Suppose the local relation is $R = 0$, and let $\hat{R}$ be a multicurve obtained from the closure of R arbitrarily outside a rectangle $\mathcal{R}$ where the local relation resides. To show that this multiplication is well-defined, it suffices to show that $\hat{R}y = 0$. By general position, we may assume that y miss the rectangle $\mathcal{R}$. Then by definition, $\hat{R}y = 0$ no matter how we resolve the crossings away from the local relation R. It is easy to check that this multiplication yields an algebra structure on $Pic(Y)$.

(2): The action is defined by gluing a collar of the boundary and then re-parameterizing the manifold to absorb the collar. Let Y_ϵ be $Y \times [0, \epsilon]$, which can be identified with a small collar neighborhood of Y in X. Given a multicurve x in X and γ in Y, draw γ on $Y \times 0$ and push it into Y_ϵ. Then the union $\gamma \cup x$ is a multicurve in $X_+ = Y_\epsilon \cup_Y X$. Absorbing Y_ϵ of X_+ into X yields a multicurve $\gamma \cup x$ in X, which is defined to be $\gamma.x$.

$\square$

3.4. **Recoupling theory.** In this section, we recall some results of the recoupling theory in [KL], and deduce some needed results for later sections.

Fix a $A \in \mathbb{C}\backslash\{0\}$, two families of numbers are important for us: the Chebyshev polynomials $\Delta_n(d)$ and the quantum integers $[n]_A = \frac{A^{2n}-A^{-2n}}{A^2-A^{-2}}$. When A is clear from the context, we will drop the A from $[n]_A$. The Chebyshev polynomials and quantum integers are related by the formula $\Delta_n(d) = (-1)^n[n+1]_A$.

Note that $[-n]_A = -[n]_A, [n]_{-A} = [n]_{\bar{A}} = [n]_A, [n]_{iA} = (-1)^{n+1}[n]_A$. Some other relations of quantum integers depend on the order of A.

Lemma 3.4. *Fix $r \geq 3$.*

(1) *If A is a primitive $4r$th root of unity, then $[n+r] = -[n]$ and $[r-n] = [n]$. The Jones-Wenzl projectors $\{p_i\}$ exist for $0 \leq i \leq r - 1$, and $\mathrm{Tr}(p_{r-1}) = \Delta_r = 0$.*

(2) *If r odd and A is a primitive $2r$th root of unity, then $[n+r] = [n]$ and $[r-n] = -[n]$. The Jones-Wenzl projectors $\{p_i\}$ exist for $0 \leq i \leq r - 1$, and $\mathrm{Tr}(p_{r-1}) = \Delta_r = 0$.*

(3) *If r odd and A is a primitive rth root of unity, then $[n+r] = [n]$ and $[r-n] = -[n]$. The Jones-Wenzl projectors $\{p_i\}$ exist for $0 \leq i \leq r - 1$, and $\mathrm{Tr}(p_{r-1}) = \Delta_r = 0$.*

The proof is obvious using the induction formula for p_n in Lemma 2.15, and $[n] \neq 0$ for $0 \leq n \leq r - 1$ for such A's.

Fix an r and A as in Lemma 3.4, and let I be the range that p_i exists and $\mathrm{Tr}(p_i) \neq 0$. Let $L_A = \{p_i\}_{i\in I}$, then $I = \{0, 1, \cdots, r - 2\}$. Both L_A and I will be called the label set. Note that if A is a primitive $2r$th root of unity and r is even, then $\{p_i\}$ exist for $0 \leq i \leq \frac{r-2}{2}$, and $\mathrm{Tr}(p_{\frac{r-2}{2}}) = 0$.

Given a ribbon link l in S^3, i.e. each component is a thin annulus, also called a framed link, then the Kauffman bracket of l, i.e. the Kauffman bracket and "d-isotopy" skein class of l, is a framed version of the Jones polynomial of l, denoted by $< l >_A$. The Kauffman bracket can be generalized to colored ribbon links: ribbon links that each component carries a label from L_A; the Kauffman bracket of a colored ribbon link l is the Kauffman bracket of the formal ribbon link obtained by replacing each component a of l with its label p_i inside the ribbon a and thickening each component of p_i inside a into small ribbons. Since S^3 is simply-connected, the Kauffman bracket of any colored ribbon link is a Laurent polynomial in A, hence a complex number.

Let H_{ij} be the colored ribbon Hopf link in the plane labelled by Jones-Wenzl projectors p_i and p_j, then the Kauffman bracket of H_{ij} is

(3.5) $$\tilde{s}_{ij} = (-1)^{i+j}[(i+1)(j+1)]_A.$$

The matrix $\tilde{s} = (\tilde{s}_{ij})_{i,j\in I}$ is called the modular $\tilde{s}$-matrix. Let $\tilde{s}_{\mathrm{even}}$ be the restriction of $\tilde{s}$ to even labels. Define $\bar{i} = k - i = r - 2 - i$.

Lemma 3.6. (1) *If A is a primitive $4r$th root of unity, then the modular $\tilde{s}$ matrix is non-singular.*

(2) *If r is odd and A is a primitive $2r$th or rth root of unity, then $\tilde{s}_{\bar{i}j} = \tilde{s}_{ij}$.*

(3) *If r odd, and A is a primitive $2r$th root of unity or rth root of unity, then the modular $\tilde{s}$ has rank $= \frac{r-1}{2}$. Moreover, $\tilde{s} = \tilde{s}_{\mathrm{even}} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.*

Proof. Since $\tilde{s}$ is a symmetric real matrix, so the rank of $\tilde{s}$ is the same as $\tilde{s}^2$.

By the formula 3.6, we have $(\tilde{s}^2)_{ij}$

$$= \frac{(-1)^{i+j}}{(A^2 - A^{-2})^2} \sum_{l=0}^{r-2} [A^{2(i+1)(l+1)} - A^{-2(i+1)(l+1)}][A^{2(l+1)(j+1)} - A^{-2(l+1)(j+1)}]$$

$$= \frac{(-1)^{i+j}}{(A^2 - A^{-2})^2} \sum_{l=0}^{r-2} [A^{2(i+1)(l+1)+2(l+1)(j+1)} + A^{-2(i+1)(l+1)-2(l+1)(j+1)}$$

$$- A^{2(i+1)(l+1)-2(l+1)(j+1)} - A^{-2(i+1)(l+1)+2(l+1)(j+1)}].$$

The first sum $\sum_{l=0}^{r-2} A^{2(i+1)(l+1)+2(l+1)(j+1)}$ is a geometric series $= \frac{A^{2(i+j+2)}-(A^{2(i+j+2)})^r}{1-A^{2(i+j+2)}}$ if $A^{2(i+j+2)} \neq 1$. The second sum $\sum_{l=0}^{r-2} A^{-2(i+1)(l+1)-2(l+1)(j+1)}$ is the complex conjugate of the first sum.

The third sum $-\sum_{l=0}^{r-2} A^{2(i+1)(l+1)-2(l+1)(j+1)}$ is also a geometric series $= -\frac{A^{2(i-j)}-(A^{2(i-j)})^r}{1-A^{2(i-j)}}$ if $A^{2(i-j)} \neq 1$. The 4th sum $-\sum_{l=0}^{r-2} A^{-2(i+1)(l+1)+2(l+1)(j+1)}$ is the complex conjugate of the third sum.

If A is a $4r$th primitive, since $0 \leq i,j \leq r-2$, we have $4 \leq 2(i+j+2) \leq 4r-4$ and $-(r-2) \leq i-j \leq r-2$. Hence, $A^{2(i+j+2)} \neq 1$ and $A^{2(i-j)} \neq 1$ unless $i = j$. The first sum and the second sum add to $\frac{A^{2(i+j+2)}-(-1)^{i+j}-1+(-1)^{i+j}A^{2(i+j+2)}}{1-A^{2(i+j+2)}}$. So if $i+j$ is even, then $= -2$; if $i+j$ is odd, then $= 0$. If $A^{2(i-j)} \neq 1$, then the third and 4th add to $-\frac{A^{2(i-j)}-(-1)^{i-j}-1+(-1)^{i-j}A^{2(i-j)}}{1-A^{2(i-j)}}$. It is $= 2$ if $i - j$ is even and $= 0$ if $i - j$ odd. Therefore, if $i \neq j$, the four sums add to 0 and if $i = j$, then they add to $-2 - 2(r-1) = -2r$. It follows that $\tilde{s}^2$ is a diagonal matrix with diagonal entries $= \frac{-2r}{(A^2-A^{-2})^2}$.

If A is a $2r$th or rth primitive, if $A^{2(i+j+2)} \neq 1$, then the first sum -1. The second sum is also -1 since it is the complex conjugate. If $A^{2(i-j)} \neq 1$, then the third sum is $= 1$ and so is the 4th sum. It follows that if neither $A^{2(i+j+2)} = 1$ nor $A^{2(i-j)} = 1$, then the (i,j)th entry of $\tilde{s}^2$ is 0.

If $A^{2(i+j+2)} = 1$, then $2(i+j+2) = r, 2r, 3r$ as $0 \leq i,j \leq r-2$ and $4 \leq 2(i+j+2) \leq 4r-4$. When r is odd, $2(i+j+2) = 2r$, so $i = \bar{j}$. Therefore, if $i+j+2 \neq r$, the first and second sum is -1. If $i+j+2 = r$, then the first and second sum both are $r-1$. If $A^{2(i-j)} = 1$, then $2(i-j) = -r, 0, r$ as $-2(r-2) \leq 2(i-j) \leq 2(r-2)$. It follows that $i = j$ as r is odd. If $i = j$, the third and 4th sum both are $= -(r-1)$. Put everything together, we have if $i \neq j$

or $i + j \neq r - 2$, then $(\tilde{s}^2)_{ij} = 0$. If $i = j$, then $i + j \neq r - 2$ because $r - 2$ is odd, and $(\tilde{s}^2)_{ij} = \frac{-2r}{(A^2 - A^{-2})^2}$. If $i + j = r - 2$, then $i \neq j$, and $(\tilde{s}^2)_{ij} = \frac{-2r}{(A^2 - A^{-2})^2}$. Hence $\tilde{s}^2 = \frac{-2r}{(A^2 - A^{-2})^2}(m_{ij})$, where $m_{ij} = 0$ unless $i = j$ or $i + j = k = r - 2$.

$\square$

We define a colored tangle category Δ_A based on a label set L_A. Consider $\mathbb{C} \times I$, the product of the plane $\mathbb{C}$ with an interval I, the objects of Δ_A are finitely many labelled points on the real axis of $\mathbb{C}$ identified with $\mathbb{C} \times \{0\}$ or $\mathbb{C} \times \{1\}$. A morphism between two objects are formal tangles in $\mathbb{C} \times I$ whose arc components connect the objects in $\mathbb{C} \times \{0\}$ and $\mathbb{C} \times \{1\}$ transversely with same labels, modulo Kauffman bracket and Jones-Wenzl projector p_{r-1}. Horizontal juxtaposition as a tensor product makes Δ_A into a strict monoidal category.

The quantum dimension d_i of a label i is defined to the Kauffman bracket of the 0-framed unknot colored by the label i. So $d_i = \Delta_i(d)$. The total quantum order of Δ_A is $D = \sqrt{\sum_i d_i^2}$, so $D = \sqrt{\frac{-2r}{(A^2 - A^{-2})^2}}$. The Kauffman bracket of the 1-framed unknot is of the form $\theta_i d_i$, where $\theta_i = A^{-i(i+2)}$ is called the twist of the label i. Define $p_\pm = \sum_{i \in I} \theta_i^{\pm 1} d_i^2$, then $D^2 = p_+ p_-$.

A triple (i, j, k) of labels is admissible if $\mathrm{Hom}(p_i \otimes p_j, p_k)$ is not 0. The theta symbol $\theta(i, j, k)$ is the Kauffman bracket of the theta network, see [KL].

Lemma 3.7.   (1) $\mathrm{Hom}(p_i \otimes p_j, p_k)$ *is not 0 if and only if the theta symbol* $\theta(i, j, k)$ *is non-zero, then* $\mathrm{Hom}(p_i \otimes p_j, p_k) \cong \mathbb{C}$.
   (2) $\theta(i, j, k) \neq 0$ *if and only if* $i + j + k \leq 2(r - 2)$, $i + j + k$ *is even and* $i + j \geq k, j + k \geq i, k + i \geq j$.

3.5. **Handles and S-matrix.** There are various ways to present an n-manifold X: triangulation, surgery, handle decomposition, etc. The convenient ways for us are the surgery description and handle decompositions.

Handle decomposition of a manifold X comes from from a Morse function of X. Fix a dimension$= n$, a k-handle is a product structure $I^k \times I^{n-k}$ on the n-ball B^n, where the part of boundary $\partial I^k \times I^{n-k} \cong S^{k-1} \times I^{n-k}$ is specified as the attaching region. The basic operations in handlebody theory are handle attachment, handle slide, stabilization, and surgery. They correspond to how Morse functions pass through singularities in the space of smooth functions on X. Let us discuss handle attachment and surgery here. Given an n-manifold X with a sub-manifold $S^{k-1} \times I^{n-k}$ specified in its boundary, and an attach map $\phi : \partial I^k \times I^{n-k} \to S^{k-1} \times I^{n-k}$, we can attach a k-handle to X via ϕ to form a new manifold $X' = X \cup_\phi I^k \times I^{n-k}$. The new manifold X' depends on ϕ, but only on its isotopy class. It follows from Morse theory or triangulation that every smooth manifold X can be obtained from 0-handles by attaching handles successively, i.e., has a handle decomposition. Moreover, the handles can be arranged to be attached in the order of their indices, i.e., from 0-handles, first all 1-handles attached, then all 2-handles, etc.

Given an n-manifold X, a sub-manifold $S^k \times I^{n-k}$ and a map $\phi : \partial I^{k+1} \times S^{n-k-1} \to S^k \times S^{n-k-1}$, we can change X to a new manifold X' by doing index k surgery on $S^k \times I^{n-k}$ as follows: delete the interior of $S^k \times I^{n-k}$, and glue in $I^{k+1} \times S^{n-k-1}$ via ϕ along the common boundary $S^k \times S^{n-k-1}$. Of course the resulting manifold X' depends on the map ϕ, but only on its isotopy class. Handle decompositions of $n + 1$-manifolds are related to surgery of n-manifolds as their boundaries.

It is fundamental theorem that every orientable closed 3-manifold can be obtained from surgery on a framed link in S^3; moreover, if two framed links give rise to the same 3-manifold, they are related by Kirby moves, which consist of stabilization and handle slides. This is extremely convenient for constructing 3-manifold invariants from link invariants: it suffices to write down a magic linear combination of invariants of the surgery link so that the combination is invariant under Kirby moves. The Reshetikhin-Turaev invariants were discovered in this way.

The magic combination is provided by the projector ω_0 from the first row of the S matrix: given a surgery link L of a 3-manifold, if every component of L is colored by ω_0, then the resulting link invariant is invariant under handle slides. Moreover, a certain normalization using the signature of the surgery link produces a 3-manifold invariant as in Theorem 3.9 below.

The projector ω_0 is a ribbon tensor category analogue of the regular representation of a finite group, and is related to surgery as below. In general, all projectors ω_i are related to surgery in a sense, which is responsible for the gluing formula for the partition function Z of a TQFT.

Lemma 3.8.   (1) *Given a 3 manifold X with a knot K inside, if K is colored by ω_0, then the invariant of the pair (X, K) is the same as the invariant of X', which is obtained from X by 0-surgery on K.*
   (2) *Let $S^2 \subset X$ be an embeded 2-sphere, then any labeled multicurve γ interests S^2 transversely must carry the trivial label. In other words, non-trivial particle type cannot cross an embeded S^2.*

The colored tangle category Δ_A has natural braidings, and duality, hence is a ribbon tensor category. An object a is simple if $\mathrm{Hom}(a, a) = \mathbb{C}$. A point marked by a Jones-Wenzl projector p_i is a simple object of Δ_A. Therefore, the label set L_A can be identified with a complete set of simple object representatives of Δ_A. A ribbon category is premodular if the number of simple object classes is finite, and is called modular if furthermore, the modular S-matrix $S = \frac{1}{D}\tilde{s}$ is non-singular. A non-singular S-matrix $S = (s_{ij})$ can be used to define projectors $\omega_i = \frac{1}{D}\sum_{j \in I} s_{ij}p_j$, which projects out the ith label.

Given a ribbon link l, $< \omega_0 * l >$ denotes the Kauffman bracket of the colored ribbon link l that each component is colored by ω_0.

Theorem 3.9.   (1) *The tangle category Δ_A is a premodular category, and is modular if and only if A is a primitive $4r$th root of unity.*
   (2) *Given a premodular category Λ, and X an oriented closed 3-manifold with an m-component surgery link l, then $Z_{JK}(X) = \frac{1}{D^{m+1}}(\frac{p_-}{D})^{\sigma(l)} < \omega_0 * l >$ is a 3-manifold invariant, where $\sigma(l)$ is the signature of the framing matrix of l.*

3.6. **Diagram TQFTs for closed manifolds.** In this section, fix an integer $r \geq 3$, A as in Lemma 3.4. For these special values, the picture spaces $\text{Pic}^A(Y)$ form a modular functor which is part of a TQFT. These TQFTs will be called diagram TQFTs. In the following, we verify all the applicable axioms for diagram TQFTs for closed manifolds after defining the partition function Z.

   The full axioms of TQFTs are given in Section 6.3. The applicable axioms for closed manifolds are:

   (1) Empty surface axiom: $V(\emptyset) = \mathbb{C}$
   (2) Sphere axiom: $V(S^2) = \mathbb{C}$. This is a consequence of the disk axiom and gluing formula.
   (3) Disjoint union axiom for both V and Z:
   (4) Duality axiom for V:
   (5) Composition axiom for Z: This is a consequence of the gluing axiom.

   These axioms together form exactly a tensor functor as follows: the category $\mathcal{X}^{2,cld}$ of oriented closed surfaces Y as objects and oriented bordisms up to diffeomorphisms between surfaces as morphisms is a strict rigid tensor category if we define disjoint union as the tensor product; $-Y$ as the dual object of Y; for birth/death, given an oriented closed surface Y, let $Y \times S^1_- : \emptyset \longrightarrow -Y \coprod Y$ be the birth operator, and $Y \times S^1_+ : -Y \coprod Y \longrightarrow \emptyset$ the death operator, here $S^1_\mp$ are the lower/up semi-circles.

Definition 3.10. A (2+1)-anomaly free TQFT for closed manifolds is a *nontrivial* tensor functor $V : \mathcal{X}^{2,cld} \longrightarrow \mathcal{V}$, where $\mathcal{V}$ is the tensor category of finite dimensional vector spaces.

   Non-triviality implies $V(\emptyset) = \mathbb{C}$ by the disjoint union axiom. Since $\emptyset \coprod \emptyset = \emptyset$, $V(\emptyset) = V(\emptyset) \otimes V(\emptyset)$. Hence $V(\emptyset) \cong \mathbb{C}$ because otherwise $V(\emptyset) = 0$ the theory is trivial. The empty set picture $\emptyset$ is the canonical basis, therefore, $V(\emptyset) = \mathbb{C}$.

   The disjoint union axiom and the trace formula for Z in Prop. 6.2 fixes the normalization of 3-manifold invariants. Given an invariant of closed 3-manifolds, then multiplication of all invariants by scalars leads to another invariant. Hence Z on closed 3-manifolds can be changed by multiplying any scalar k. But this freedom is eliminated from TQFTs by the disjoint union axiom which implies $k = k^2$, hence $k = 1$ since otherwise the theory is trivial. The trace formula implies $Z(S^2 \times S^1) = 1$. We set $Z(S^3) = \frac{1}{D}$, and D is the total quantum order of the theory.

Recall that the picture space $\text{Pic}^A(Y)$ is defined even for unorientable surfaces. When Y is oriented, $\text{Pic}(Y)$ is isomorphic to $K_A(Y \times I)$. Given a bordism X from Y_1 to Y_2, we need to define $Z_D(X) \in \text{Pic}^A(-Y_1 \coprod Y_2)$. It follows from the disjoint union axiom and the duality axiom, $Z_D(X)$ can be regarded as a linear map $\text{Pic}^A(Y_1) \longrightarrow \text{Pic}^A(Y_2)$.

Given a closed surface Y, let $V_D(Y) = \text{Pic}^A(Y)$. For a diffeomorphism $f : Y \to Y$, the action of f on pictures is given by moving them in Y. This action on pictures descends to an action of mapping classes on $V(Y)$. To define $Z_D(X)$ for a bordism X from Y_1 to Y_2, fix a relative handle decomposition of X from Y_1 to Y_2.

Suppose that Y_2 is obtained from Y_1 by attaching a single handle of indices $= 0, 1, 2, 3$. For indices $= 0, 3$, the linear map is just a multiplication by $\frac{1}{D_{JK}}$, where $D_{JK} = \sqrt{\frac{-2r}{(A^2 - A^{-2})^2}}$. Since S^3 is a 0-handled followed by a 3-handle, $Z_D(S^3) = \frac{1}{D_{JK}^2}$. This is not a coincidence, but a special case of a theorem of K. Walker and V. Turaev that $Z_D(X) = |Z_{JK}(X)|^2$ for any oriented closed 3-manifold.

Given a multicurve γ in Y_1,

1): If a 1-handle $I \times B^2$ is attached to Y_1, isotopy γ so that it is disjoint from the attaching regions $\partial I \times B^2$ of the 1-handle. Label the co-core circle $\frac{1}{2} \times \partial B^2$ of the 1-handle by ω_0 to get a formal multicurve in Y_2. This defines a map from $\text{Pic}^A(Y_1)$ to $\text{Pic}^A(Y_1)$ by linearly extending to pictures classes.

2): If a 2-handle $B^2 \times I$ is attached to Y_1, isotopy γ so that it intersects the attaching circle $\partial B^2 \times \frac{1}{2}$ of the 2-handle transversely. Expand this attaching circle slightly to become a circle s just outside the 2 handle and parallel to the attaching circle $\partial B^2 \times \frac{1}{2}$. Label s by ω_0. Fuse all strands of γ so that a single labeled curve intersects the attaching circle $\partial B^2 \times \frac{1}{2}$; only the 0-labeled curves survive the projector ω_0 on s. By drawing all remaining curves on the Y_1 outside the attaching region plus the two disks $B^2 \times \{0\}$ and $B^2 \times \{1\}$, we get a formal diagram in Y_2.

We need to prove that this definition is independent of handle-slides and cancellation pairs, which is left to the interested readers.

Now we are ready to verify all the axioms one by one:

The empty surface axiom: this is true as we have a non-trivial theory.

The sphere axiom: by the "d-isotopy" constraint, every multicurve with m loops $= d^m \emptyset$. If $\emptyset$ picture on S^2 is $= 0$ in $\text{Pic}^A(S^2)$, then $Z_D(B^3) = 0$ which leads to $Z_D(S^3) = 0$. But $Z_D(S^3) \neq 0$, it follows that $\text{Pic}^A(S^2) = \mathbb{C}$.

The disjoint union axioms for both V and Z are obvious since both are defined by pictures in each connected component.

$\text{Pic}(-Y) = \text{Pic}(Y)$ is the identification for the duality axiom. To define a functorial identification of $\text{Pic}^A(-Y)$ with $\text{Pic}^A(Y)^*$, we define a Hermitian paring: $\text{Pic}^A(Y) \times \text{Pic}^A(Y) \longrightarrow \mathbb{C}$. Since $\text{Pic}^A(Y)$ is an algebra, and semi-simple, it is a matrix algebra. For any $x, y \in \text{Pic}^A(Y)$, we identify them as matrices, and define

$< x, y >= \text{Tr}(x^\dagger y)$. This is a non-degenerate inner product. The conjugate linear map $x \to< x, \cdot >$ is the identification of $\text{Pic}^A(-Y)$ with $\text{Pic}^A(Y)^*$.

Summarizing, we have;

Theorem 3.11. *The pair (V_D, Z_D) is a $(2+1)$-anomaly free TQFT for closed manifolds.*

3.7. **Boundary conditions for picture TQFTs.** In Section 3.1, we consider $\mathbb{C}[\mathbb{S}]$ for surfaces Y even with boundaries. Given a surface Y with m boundary circles with n_i fixed points on the ith boundary circle, by imposing Jones-Wenzl projector p_{r-1} away from the boundaries, we obtain some pictures spaces, denoted as $\text{Pic}^A(Y; n_1, \cdots, n_m)$. To understand the deeper properties of the picture space $\text{Pic}^A(Y)$, we need to consider the splitting and gluing of surfaces along circles. Given a simple closed curve (scc) s in the interior of Y, and a multicurve γ in Y, isotope s and γ to general position. If Y is cut along s, the resulted surface Y_{cut} has two more boundary circles with n points on each new boundary circle, where n is the number of intersection points of $s \cap \gamma$, and $n \in \{0, 1, 2, \cdots, \}$. In the gluing formula, we like to have an identification of $\oplus \text{Pic}^A(Y_{\text{cut}})$ with all possible boundary conditions with $\text{Pic}^A(Y)$, but this sum consists of infinitely many non-trivial vector space,which contradicts that $\text{Pic}^A(Y)$ is finitely dimensional. Therefore, we need more refined boundary conditions. One problem about the crude boundary conditions of finitely many points is due to bigons resulted from the "d-isotopy" freedom: we may introduce a trivial scc intersecting s at many points, or isotope γ to have more intersection points with s. The most satisfactory solution is to define a picture category, then the picture spaces become modules over these categories. Picture category serves as crude boundary conditions. To refine the crude boundary conditions, we consider the representation category of the picture category as new boundary conditions. The representation category of a picture category is naturally Morita equivalent to the original picture category. The gluing formula can be then formulated as the Morita reduction of picture modules over the representation category of the picture category. The labels for the gluing formula are given by the irreps of the picture categories. This approach will be treated in the next two sections. In this section, we content ourselves with the description of the labels for the diagram TQFTs, and define the diagram modular functor for all surfaces. In Section 6.3, we will give the definition of a TQFT, and later verify all axioms for diagram TQFTs.

The irreps of the non-semi-simple TL annular categories at roots of unity were contained in [GL], but we need the irreps of the semi-simple quotients of TL annular categories, i.e., the TLJ annular categories.

The irreps of the TLJ annular will be analyzed in Sections 5.5 5.6. In the following, we just state the result. By Theorem B.8 in Appendix B, each irrep can be represented by an idempotent in a morphism space of some object. Fix h $(0 \le h \le k)$ many points on S^1, and let $\omega_{i,j;h}$ be the following diagram in the

annulus $\mathcal{A}$: the two circles in the annulus are labeled by ω_i, ω_j, and $h = 3$ in the diagram.
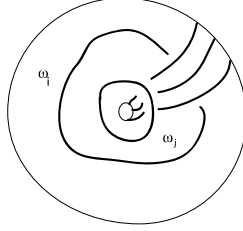

FIGURE 12. Annular projector

The labels for diagram TQFTs are the idempotents $\omega_{i,j;h}$ above. Given a surface Y with boundary circles $\gamma_i, i = 1, .., m$. In the annular neighborhood $\mathcal{A}_i$ of γ_i, fix an idempotent $\omega_{i,j;h}$ inside $\mathcal{A}_i$. Let $\mathrm{Pic}_D^A(Y; \omega_{i,j;h})$ be the span of all multicurves that within $\mathcal{A}_i$ agree with $\omega_{i,j;h}$ modulo p_{r-1} .

Theorem 3.12. *If A is as in Lemma 3.4, then the pair $(Pic^A(Y), Z_D)$ is an anomaly-free TQFT.*

3.8. **Jones-Kauffman skein spaces.** In this section, fix an integer $r \geq 3$, A as in Lemma 3.4, and $d = -A^2 - A^{-2}$.

Definition 3.13. Given any closed surface Y, let $\mathrm{Pic}^A(Y)$ be the picture space of pictures modulo p_{r-1}. Given an oriented 3-manifold X, the skein space of p_{r-1} and the Kauffman bracket is called the Jones-Kauffman skein space, denoted by $K_A(X)$.

The following theorem collects the most important properties of the Jones-Kauffman skein spaces. The proof of the theorem relies heavily on handlebody theory of manifolds.

Theorem 3.14. *a): Let A be a primitive 4rth root of unity. Then*
  (1) $K_A(S^3) = \mathbb{C}$.
  (2) *There is a canonical isomorphism of $K_A(X_1 \amalg X_2) \cong K_A(X_1) \# K(X_2)$.*
  (3) *If $\partial X_1 = \partial X_2$, then $K_A(X_1) \cong K_A(X_2)$, but not canonically.*
  (4) *If the $\emptyset$ link is not 0 in $K_A(X)$ for a closed manifold X, then $K_A(X) \cong \mathbb{C}$ canonically. The $\emptyset$ link in $K_A(\#_{r=1}^m S^1 \times S^2)$ and $K_A(Y \times S^1)$ is not 0 for oriented closed surface Y.*
  (5) $K_A(-X) \times K_A(X) \longrightarrow K_A(DX)$ *is non-degenerate. Therefore, $K_A(\overline{X})$ is isomorphic to $K_A(X)^*$, but not canonically.*
  (6) $K_A(Y \times I) \longrightarrow End(K_A(X))$ *is an isomorphism if $\partial X = Y$.*
  (7) $Pic^A(Y)$ *is canonically isomorphic to $K_A(Y \times I)$ if Y is orientable, hence also isomorphic to $End(K_A(X))$.*

b): If A is a primitive $2r$th root of unity or rth root of unity, then (2) does not hold, and it follows that the rest fail for disconnected manifolds.

Proof. (1) Obvious.

(2) The idea here in physical terms is that non-trivial particles cannot cross an S^2.

The skein space $K_A(X_1 \amalg X_2)$ is a subspace of $K_A(X_1) \# K(X_2)$ by inclusion. So it suffices to show this is onto. Given any skein class x in $K_A(X_1) \# K(X_2)$, by isotopy we may assume x intersects the connecting S^2 transversely. Put the projector ω_0 on S^2 disjoint from x, then ω_0 encircle x from outside. Apply ω_0 to x to project out the 0-label, we split x into two skein classes in $K_A(X_1 \amalg X_2)$, therefore the inclusion is onto.

(3) This is an important fact. For example, combining with (1), we see that the Jones-Kauffman skein space of any oriented 3-manifold is $\cong \mathbb{C}$.

We will show below that any bordism W^4 from X_1 to X_2 induces an isomorphism. Moreover, the isomorphism depends only on the signature of the 4-manifold W^4.

Pick a 4-manifold W such that $\partial W = -X_1 \cup_Y (Y \times I) \cup_Y X_2$ (W exists since every orientable 3 manifold bounds a 4 manifold), and fix a handle-decomposition of W. 0-handles, and dually 4-handles, induce a scalar multiplication. 1-handles, or dually 3-handles, also induce a scalar multiplication by (2). By (2), we may assume that $X_i, i = 1, 2$ are connected. Therefore, we will fix a relative handle decompositions of W with only 2-handles, and let $L_{X_i}, i = 1, 2$ be the attaching links for the 2-handles in X_i, respectively. Then $X_1 \setminus L_{X_1} \cong X_2 \setminus L_{X_2}$. The links $L_{X_i}, i = 1, 2$ are dual to each other in a sense: let $L_{X_i}^{\text{dual}}$ be the link consists of cocores of the 2-handles on X_i, then surgery on L_{X_1} in X_1 results X_2, while surgery on $L_{X_1}^{\text{dual}}$ in X_1 results X_1, and vice versa.

Define a map $h_1 : K_A(X_1) \to K_A(X_2)$ as follows: for any skein class representative γ_1, isotope γ_1 so that it misses L_{X_1}. Note that in the skein spaces, labeling a component L_1 of a link L by ω_0, denoted as $\omega_0 * L_1$, is equivalent to surgering the component; then $h(\gamma_1) = \gamma_1 \amalg \omega_0 * L_{X_2}$, where γ_1 is now considered as a link in X_2. Formally, we write this map as:

$$(X_1; \gamma_1) \to (X_1; L_{X_1} \coprod L_{X_1}^{\text{dual}} \coprod \gamma_1) \to (X_2; L_{X_2} \coprod \gamma_2),$$

where γ_2 is γ_1 regarded as a skein class in X_2. In this map, L_{X_1} is mapped to the empty skein as it has been surged out, while $L_{X_1}^{\text{dual}}$ is mapped to L_{X_2}. Then define h_2 similarly. The composition of h_1 and h_2 is the link invariant of the colored link L_{X_i} union a small linking circle for each component plus a parallel copy of $L_{X_i}^{\text{dual}}$ union its small linking circles as in the Fig. 3.8, which is clearly a scalar, hence an isomorphism.

Now we see that a pair, (W, a handle decomposition), induces an isomorphism. Using Cerf theory, we can show that the isomorphism is first independent of the
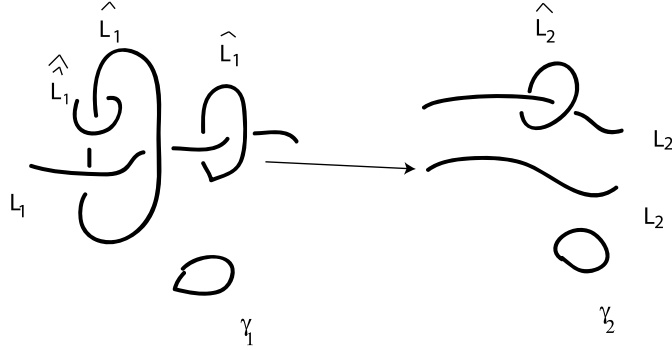
FIGURE 13. Skein space maps

handle decomposition; secondly it is a bordism invariant: if there is a 5-manifold N which is a relative bordism from W to W', then W and W' induces the same map. Hence the isomorphism depends only on the signature of the 4-manifold W. The detail is a highly non-trivial exercise in Cerf theory.

(4) follows from (1)–(3) easily.

(5) The inner product is given by doubling. By (3) $K_A(X)$ is isomorphic to $K_A(H)$, where H is a handlebody with the same boundary. By (4), the same inner product is non-singular for $K_A(H)$. Chasing through the isomorphism in (3) shows that the inner product on $K_A(X)$ is also non-singular. Since $K_A(DX) \cong \mathbb{C}$, hence $K_A(\overline{X})$ is isomorphic to $K_A(X)^*$.

(6) $K_A(Y \times I)$ is isomorphic to $K_A(-X \coprod X)$ by (3). It follows that the action of $K_A(Y \times I)$ on $K_A(X)$: $K_A(X) \otimes K_A(Y \times I) \to K_A(X \cup_Y (Y \times I)$ becomes an action of $K_A(-X \coprod X)$ on $K_A(X)$: $K_A(X) \otimes K_A(-X \coprod X) \to K_A(DX \coprod X)$. By the paring in (5), we identify the action as the action of $\mathrm{End}(X) = K_A(-X) \otimes K_A(X)$ on $K_A(X)$.

(7) follows from (6) easily.

$\square$

The pairing $K_A(-X) \times K_A(X) \longrightarrow K_A(DX)$ allows us to define a Hermitian product on $K_A(X)$ as follows:

Definition 3.15. Given an oriented closed 3-manifold X, and choose a basis e of $K_A(DX)$. Then $K_A(DX) = \mathbb{C}e$. For any multicurves x, y in X, consider x as a multicurve in $-X$, denoted as $\bar{x}$. Then define $\bar{x} \cup y = <x, y> e$, i.e. the ratio of the skien $\bar{x} \cup y$ with e. If $\emptyset$ is not 0 in $K_A(DX)$, then Hermitian pairing is canonical by choosing $e = \emptyset$.

Almost all notations are set up to define the Jones-Kauffman TQFTs. We see in Theorem 3.14 that if two 3-manifolds $X_i, i = 1, 2$ have the same boundary, then $K_A(X_1)$ and $K_A(X_2)$ are isomorphic, but not canonically. We like to define the modular functor space $V(Y)$ to be a Jones-Kauffman skein space. The dependence

on X_i is due to a framing-anomaly, which also appears in Witten-Restikhin-Turaev $SU(2)$ TQFTs. To resolve this anomaly, we introduce an extension of surfaces. Recall by Poincare duality, the kernel λ_X of $H_1(\partial X; \mathbb{R}) \to H_1(X; \mathbb{R})$ is a Lagrangian subspace $\lambda \subset H_1(Y; \mathbb{R})$. This Lagrangian subspace contains sufficient information to resolve the framing dependence. Therefore, we define an extended surface as a pair $(Y; \lambda)$, where λ is a Lagrangian subspace of $H_1(Y; \mathbb{R})$. The orientation, homology and many other topological property of an extended surface $(Y; \lambda)$ mean that of the underlying surface Y.

The labels for the Jones-Kauffman TQFTs are the Jones-Wenzl projectors $\{p_i\}$. Given an extended surface $(Y; \lambda)$ with boundary circles $\gamma_i, i = 1, .., m$. Glue m disks B^2 to the boundaries to get a closed surface $\widehat{Y}$ and choose a handlebody H such that $\partial H = \widehat{Y}$, and the kernel λ_H of $H_1(Y; \mathbb{R}) \to H_1(H; \mathbb{R})$ is λ. In a small solid cylinder neighborhood $B_i^2 \times [0, \epsilon]$ of each boundary circle γ_i, fix a Jones-Wenzl projector p_{i_j} inside some $\text{arc} \times [0, \epsilon]$, where the arc is any fixed diagonal of B_i^2. Let $V_{JK}^A(Y; \lambda, \{p_{i_j}\})$ be the Jones-Kauffman skein space of H of all pictures within the solid cylinders $B_i^2 \times [0, \epsilon]$ agree with $\{p_{i_j}\}$.

Lemma 3.16. *Let B^2 be a 2-disk, $\mathcal{A}$ an annulus and P a pair of pants, and (Y, λ) an extended surface with m punctures labelled by $p_{i_j}, j = 1, 2, \cdots, m$, then*

   (1) *$V_{JK}^A(B^2; p_i) = 0$ unless $i = 0$, and $V_{JK}^A(B^2; p_0) = \mathbb{C}$.*
   (2) *$V_{JK}^A(\mathcal{A}; p_i, p_j) = 0$ unless $i = j$, and $V_{JK}^A(\mathcal{A}; p_i, p_i) = \mathbb{C}$*
   (3) *$V_{JK}^A(P; p_i, p_j, p_k) = 0$ unless i, j, k is admissible, and $V_{JK}^A(P; p_i, p_j, p_k) \cong \mathbb{C}$ if i, j, k is admissible.*
   (4) *Given an extended surface $(Y; \lambda)$, and let H be a genus=g handlebody such that $\partial H = (\widehat{Y}; \lambda)$ as extended manifolds. Then admissible labelings of any framed trivalent spine dual to a pants decomposition of Y with all external edges labelled by the corresponding boundary label p_{i_j} is a basis of $V_{JK}^A(Y; \lambda, \{p_{i_j})\}$.*
   (5) *$V_{JK}^A(Y)$ is generated by bordisms $\{X | \partial X = Y\}$ if Y is closed and oriented.*

Given an extended surface $(Y; \lambda)$, to define the partition function $Z_{JK}(X)$ for any X such that $\partial X = Y$, let us first assume that $\partial X = (Y; \lambda)$ as an extended surface. Find a handlebody H such that $\lambda_H = \lambda$, and a link L in H such that surgery on L yields X. Then we define Z_X as the skein in $V_D(H)$ given by the L labeled by ω_0 on each component of L. If λ_X is not λ, then choose a 4-manifold W such that $\partial W = -X \coprod X$ and the Lagrange space λ and λ_X extended through W. W defines an isomorphism between $K_A(X)$ and itself. The image of the empty skein in $K_A(X)$ is $Z(X)$. Given $f : (Y_1; \lambda_1) \to (Y_2; \lambda_2)$, the mapping cylinder I_f defines an element in $V(Y_1 \coprod Y_2) \cong V(Y_1) \otimes V(Y_2)$ by the disjoint union axiom. This defines a representation of the mapping class group $\mathcal{M}(Y)$, which might be a projective representation.

Theorem 3.17. *If A is a primitive $4r$th root of unity for $r \geq 3$, then the pair (V_{JK}, Z_{JK}) is a TQFT.*

This theorem will be proved in Section 7.

There is a second way to define the projective representation of $\mathcal{M}(Y)$. Given an oriented surface Y, the mapping class group $\mathcal{M}(Y)$ acts on $\mathrm{Pic}(Y)$ by moving pictures in Y. This action preserves the algebra structure of $\mathrm{Pic}(Y)$ in Prop. 3.3. The algebra $\mathrm{Pic}(Y) \cong \mathrm{End}(K_A(Y))$ is a simple matrix algebra, therefore any automorphism ρ is given by a conjugation of an invertible matrix M_ρ, where M_ρ is only defined up to a non-zero scalar. It follows that for each $f \in \mathcal{M}(Y)$, we have an invertible matrix $V_f = M_f$, which forms a projective representation of the mapping class group $\mathcal{M}(Y)$.

4. Morita equivalence and cut-paste topology

Temperley-Lieb-Jones algebras can be generalized naturally to categories by allowing different numbers of boundary points at the top and bottom of the rectangle $\mathcal{R}$. Another interesting generalization is to replace the rectangle by an annulus $\mathcal{A}$. Those categories provide crude boundary conditions for $V(Y)$ when Y has boundary, and serve as "scalars" for a "higher" tensor product structure which provides the formal framework to discuss relations among $V(Y)$'s under cut-paste of surfaces. The vector spaces $V(Y)$ of a modular functor V can be formulated as bimodules over those picture categories. An important axiom of a modular functor is the gluing formula which encodes locality of a TQFT, and describes how a modular functor $V(Y)$ behaves under splitting and gluing of surfaces along boundaries. The gluing formula is best understood as a Morita reduction of the crude picture categories to their representation categories, which provides refined boundary conditions for surfaces with boundaries. Therefore, the Morita reduction of a picture category amounts to the computation of all its irreps. The use of bimodules and their tensor products over linear categories to realize gluing formulas appeared in [BHMV]. In this section, we will set up the formalism. The irreducible representations of our examples will be computed in the next section.

We work with the complex numbers $\mathbb{C}$ as the ground ring. Let Λ denote a linear category over $\mathbb{C}$ meaning that the morphisms set of Λ are vector spaces over $\mathbb{C}$ and composition is bilinear. We consider two kinds of examples: "rectangular" and "annular" Λ's. (The adjectives refer to methods for building examples rather than additional axioms.) We think of rectangles($\mathcal{R}$) as oriented vertically with a "top" and "bottom" and annuli($\mathcal{A}$) has an "inside" and an "outside". Sometimes, we draw an annulus as a rectangle, and interpret the rectangles as having their left and right sides glued. The objects in our examples are finite collections of points, or perhaps points marked by signs, arrows, colors, etc., on "top" or "bottom" in the rectangular case, and on "inside" or "outside" in the annular case. The morphisms are formal linear combinations of "pictures" in $\mathcal{R}$ or $\mathcal{A}$ satisfying some linear

relations. The most important examples are the Jones-Wenzl projectors. Pictures will variously be unoriented submanifolds (i.e. multicurves), 1-submanifolds with various decorations such as orienting arrows, reversal points, transverse flags, etc., and trivalent graphs. Even though the pictures are drawn in two dimensions they may in some theories be allowed to indicate over-crossings in a formal way. A morphism is sometimes called an "element" as if Λ had a single object and were an algebra.

Our $\mathcal{R}$'s and $\mathcal{A}$'s are parameterized, i.e. not treated merely up to diffeomorphism. One crucial part of the parameterization is that a base point arc $* \times I \subset S^1 \times I = \mathcal{A}$ be marked. The $*$ marks the base point on S^1 and brings us to a technical point. Are the objects of Λ the continuously many collections of finitely many points in I (or S^1) or are they to be simply one representative example for each non-negative integer m. The second approach makes the category feel bit more like an algebra (which has only one object) and the linear representations have a simpler object grading. One problem with this approach is that if an annulus $\mathcal{A}$ factored as a composition of two by drawing a degree=1 scc $\gamma \subset \mathcal{A}$ (and parameterizing both halves), even if γ is transverse to an element x in $\mathcal{A}$ $\gamma \cap x$ may not be the representative set of its cardinality. This problem can be overcome by picking a base point preserving re-parameterization of γ. This amounts to "skeletonizing" the larger category and replacing some "strict" associations by "weak" ones. Apparently a theorem of S. MacLane guarantees that no harm follows, so either viewpoint can be adopted [Ma]. We will work with the continuously many objects version.

Recall the following definition from Appendix B:

Definition 4.1. A representation of a linear category Λ is a functor $\rho : \Lambda \to \mathcal{V}$, where $\mathcal{V}$ is the category of finite dimensional vector spaces. The action is written on the right: $\rho(a) = V_a$ and given $m \in \mathrm{Mor}(a, b), \rho(m) : V_a \to V_b$. We write on the left to denote a representation of Λ^{op}.

Let us track the definitions with the simplest pair of examples, temporarily denote $\Lambda^{\mathcal{R}}$ and $\Lambda^{\mathcal{A}}$ the $\mathcal{R}$ and $\mathcal{A}$-categories with objects finite collections of points and morphisms transversely embedded un-oriented 1-submanifolds with the marked points as boundary data. Let us say that 1) we may vary multicurves by "d-isotopy" for $d = 1$, and 2) to place ourselves in the simplest case let us enforce the skein relation: $p_2 = 0$ for $d = 1$. This means that we allow arbitrary recoupling of curves. This is the Kauffman bracket relation associated to $A = e^{\frac{2\pi i}{6}}, d = -A^2 - A^{-2} = 1$. The admissible pictures may be extended to over-crosssings by the local Kauffman bracket rule in Figure 1 in Section 2.2.

In these theories, which we call the rectangular and annular Temperley-Lieb-Jones categories TLJ at level $= 1, d = 1$, over-crossings are quite trivial, but at higher roots of unity they are more interesting. In schematic Figures 14, 15 give examples of $\Lambda^{\mathcal{R}}$ and $\Lambda^{\mathcal{A}}$ representations.
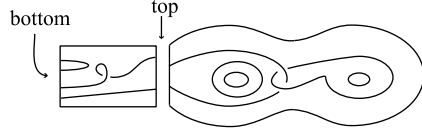
FIGURE 14. $\Lambda^{\mathcal{R}}$ acts (represents) on vector space of pictures in twice punctured disk (or genus=2 handlebody)
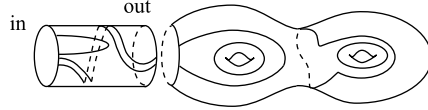

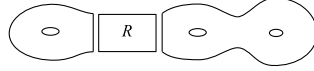FIGURE 15. $\Lambda^{\mathcal{A}}$ acts on vector space of pictures in punctured genus=2 surface

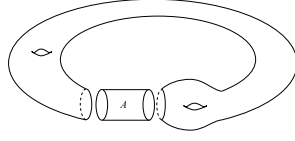
FIGURE 16. $\Lambda^{\mathcal{R}}$ acts on both sides

It is actually the morphism between objects index by 4 to 2 points, resp. $_4\Lambda_2^R$, which is acting in Figure 14, Figure 15.

The actions above are by regluing and then re-parameterizing to absorb the collar. For each object a in $\text{TLJ}_{d=1}$, the functor assigns the vector space of pictures lying in a given fixed space with boundary data equal the object a. Given a fixed picture in $\mathcal{R}$ and $\mathcal{A}$, i.e. an element e of Λ, gluing and absorbing the collar defines a restriction map: $f(e) : V_4 \to V_2$ (in the case illustrated) between the vector spaces with the bottom (in) and the top (out) boundary conditions. To summarize the annular categories act on vector spaces which are pictures on a surface by gluing on an annulus. The rectangular categories, in practice, act on handlebodies or other 3-manifolds with boundary by gluing on a solid cylinder; Figure 14 is intentionally ambiguous and may be seen as a diagram or 3−manifolds. Because we can use framing and overcrossing notations in the rectangle we are free to think of $\mathcal{R}$ either as 2-dimensional, $I \times I$, or 3−dimensional $I \times B^2$.

4.1. **Bimodules over picture category.** Because a rectangle or annulus can be glued along two sides, we need to consider $\Lambda^{\text{op}} \times \Lambda$ actions: $\Lambda^{\text{op}} \times \Lambda \xrightarrow{\rho} \mathcal{V}$. The composition $\Lambda \underset{m \mapsto (m^{\text{op}}, m)}{\xrightarrow{\triangle}} \Lambda^{\text{op}} \times \Lambda \xrightarrow{\rho} \mathcal{V}$ describes the action of gluing an $\mathcal{R}$ or $\mathcal{A}$ on two sides (Figure 16, Figure 17).

We refer to the action of Λ^{op} as "left" and the action of Λ as "right".

Definition 4.2. 1) Let M be a right Λ-representation (or "module") and N a left Λ-module. Denote by $M \bigotimes_\Lambda N$ the $\mathbb{C}$-module quotient of $\bigoplus_a M_a \bigotimes_\mathbb{C} {}_a N$ by all

FIGURE 17. $\Lambda^{\mathcal{A}}$ acts on both sides

relations of the form $u\alpha \otimes v = u \otimes \alpha v$, where "a" and "b" are general objects of Λ, $u \in M_a$, $v \in {}_bN$ and $\alpha \in {}_a\Lambda_b = \mathrm{Mor}(a, b)$.

2) A $\mathcal{C} \times \mathcal{D}$ bimodule is a functor $\rho : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathcal{V}$. Note that $\mathcal{C}$ is naturally a $\mathcal{C} \times \mathcal{C}$ bimodule, which will be called the regular representation of $\mathcal{C}$.

Suppose now that Λ is semi-simple. This means that there is a set I of isomorphism classes of (finite dimensional) irreducible representations $\rho_i, i \in I$ of Λ and every (finite dimensional) representation of ρ may be decomposed $\rho \cong \bigoplus V_i \otimes \rho_i$, where V_i is a $\mathbb{C}$-vector space with no Λ action; $\dim(V_i)$ is the multiplicity of ρ_i. (If $\rho_1(a) = M_a^1$ and $\rho_2(a) = M_a^2$, then $\rho_1 \oplus \rho_2 = M_a^1 \oplus M_a^2$, and similarly for morphisms.)

The following example is contained in the section in the general discussion, but it is instructive to see how things work in $\mathrm{TLJ}_{d=1}^{\mathcal{R}}$ and $\mathrm{TLJ}_{d=1}^{\mathcal{A}}$, the TLJ rectangular and picture categories for $d = 1$. These simple examples include the celebrated toric codes TQFT in [Ki1] or $\mathbb{Z}_2$ gauge theory, and illustrate the general techniques. Since it is almost no extra work, we will include the corresponding calculation for $\mathrm{TLJ}_{d=-1}^{\mathcal{R}}$ and $\mathrm{TLJ}_{d=-1}^{\mathcal{A}}$ where $A = e^{\frac{2\pi i}{12}}, d = -1$ and $p_2 = 0$ for $d = -1$.

A general element $x \in {}_a\Lambda_{\pm 1 b}^{\mathcal{R}}$ is determined by its coefficients of "squeezed" diagrams where only 0 and 1 arcs cross the midlevel of the rectangle such diagrams look like:
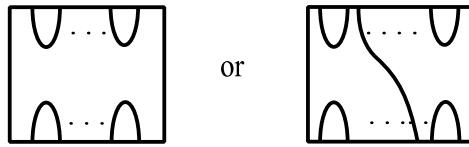

FIGURE 18. Squeezed morphisms

Similarly $x \in {}_a\Lambda_{\pm 1,b}^{\mathcal{A}}$ are determined by the coefficients of the diagrams in an annulus made by gluing the left and right sides of FIGS??. To each $a \in \Lambda^0, \Lambda = \Lambda_{\pm 1}^{\mathcal{R} \text{ or } \mathcal{A}}$, let V_a be the vector space spanned by diagrams, with a end points on the top (outside) and zero (a even) or one (a odd) end point on the bottom (inside), thus $V_a = {}_{0 \text{ or } 1}\Lambda_a$. The gluing map ${}_{0 \text{ or } 1}\Lambda_a^{\mathcal{R}} \otimes {}_a\Lambda_b^{\mathcal{R}} \xrightarrow{\rho^{\mathcal{R}}} {}_{0 \text{ or } 1}\Lambda_b^{\mathcal{R}}$ provides the two representations $\rho_0^{\mathcal{R}}$ and $\rho_1^{\mathcal{R}}$ of $\Lambda_{\pm 1}^{\mathcal{R}}$ ($\rho_{0 \text{ (or 1)}}$ sends a odd (or even) to the 0-dimensional vector space.)

Lemma 4.3. *The representation $\rho_0^{\mathcal{R}}$ and $\rho_1^{\mathcal{R}}$ are irreducible.*

Proof. Proof: Consider $\rho_0^{\mathcal{R}}$, the morphism vector space $_{2k}\Lambda_{2k}$ has dimension=1 (spanned by the empty diagram in a rectangle) so that in "grade", $2k$, $\rho_0^{\mathcal{R}}$ is automatically irreducible. There is a morphism $m \in \ _{2k}\Lambda_{2n}$, $m^\dagger \in \ _{2n}\Lambda_{2k}$:

$$m = d^{-n/2} \ \boxed{\overset{\smile\cdots\smile}{\underset{\curvearrowright\cdots\curvearrowleft}{}}}_{k}^{n} \qquad\qquad m^+ = d^{-n/2} \ \boxed{\overset{\smile\cdots\smile}{\underset{\curvearrowright\cdots\curvearrowleft}{}}}_{n}^{k}$$

FIGURE 19. Factored morphisms

and $m^\dagger m = \mathrm{id} \in \ _{2k}\Lambda_{2k}$. Thus any representation $\{V\}$ on the even grades of the categories must have equal dimension in all (even) grades since $\rho_0^{\mathcal{R}}(m)$ and $\rho_0^{\mathcal{R}}(m^\dagger)$ are inverse to each other. It follows that any proper subrepresentation of $\rho_0^{\mathcal{R}}$ must be zero dimensional in all grades. Thus $\rho_0^{\mathcal{R}}$ is irreducible.

The argument for $\rho_1^{\mathcal{R}}$ is similar, simply add a vertical line near the right margin of the rectangles in Fig. 19 to obtain the corresponding $m, m^\dagger$ in the odd grades. $\square$

Lemma 4.4. *Any irreducible representation of $\Lambda_{\pm 1}^{\mathcal{R}}$ is isomorphic to $\rho_0^{\mathcal{R}}$ or $\rho_1^{\mathcal{R}}$.*

Proof. The proof is based on "resolutions of the identity". In this case that means:

FIGURE 20. Resolution of identity

Acting by ρ on $\{V\}$ may be factored schematically as shown in Figure 21.

FIGURE 21. Picture action

By Theorem B.8, for every $a \in \Lambda^0$, V_a is a subspace of $_a\Lambda_b$ for sone $b \in \Lambda^0$. In formulas, let $l \in \ _{2n}\Lambda_{2k}$ (for the even case) $\rho(l) = \rho\left(l \cdot _{2k}m_0^\dagger \cdot _0 m_{2k}\right)$, so the action factors through $_0\Lambda_{2k}$. On the even (odd) grades the action is isomorphic to $\rho_0^{\mathcal{R}} \ \left(\rho_1^{\mathcal{R}}\right)$ tensor the subspace of $_{2n}\Lambda_0$ generated by elements of the form $l \cdot _{2k}m_0^\dagger$ with trivial action. So the general representation is isomorphic to a direct sum of

irreducibles. In this simple case it was not necessary (as it will be in other cases) to construct the Hermitian structure on Λ to derive semi-simplicity.

$\square$

Now consider representations of $\Lambda_{\pm 1}^{\mathcal{A}}$. Again $x \in {}_a\Lambda_{\pm 1,b}^{\mathcal{R}}$ is determined by diagrams with a "weight" of 0 or 1.

In the special ("principle graph") cases: ${}_0\Lambda_0$ and ${}_1\Lambda_1$ there are four diagrams (Figure 22) up to isotopy in the presence of relations $p_2 = 0$ for $d = \pm 1$.
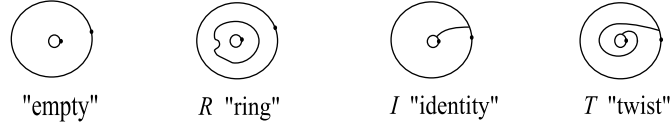

"empty" R "ring" I "identity" T "twist"

FIGURE 22. Idempotents for annular $d = \pm 1$

The reader should observe that if pictures are glued to be outside of $\emptyset$, ring R, straight arc I, or twist T they may be transformed to another picture:

$$\emptyset \bigotimes R = R, \; R \bigotimes R = \pm\emptyset, \; I \bigotimes T = T, \text{ and } T \bigotimes T = \pm T.$$

(The signs are for $d = \pm 1$). Let us call the object (i.e. number of end points) a "crude label". We have two crude labels "0" and "1" in this example. For each crude label the symmetric $\left(\frac{\emptyset + R}{2}, \text{ and } \frac{I+T}{2}\right)$ and anti-symmetric $\left(\frac{\emptyset - R}{2}, \text{ and } \frac{I-T}{2}\right)$ averages are in fact $(+1, -1)$ eigenvectors under gluing on a ring R in ${}_0\Lambda_{1,0}$ and gluing on T in ${}_1\Lambda_{1,1}$. The combinations $(\emptyset - iR)$ and $(\emptyset + iR)$ are ± 1-eigenvectors for the action of R in ${}_0\Lambda_{-1,0}$ and $(T - iT)$ and $(T + iT)$ are ± 1-eigenvectors for the action of T in ${}_0\Lambda_{-1,0}$. In all cases these vectors span a 1-dimensional representation of four algebras ${}_0\Lambda_{1,0}, {}_1\Lambda_{-1,0}, {}_1\Lambda_{1,1}, {}_1\Lambda_{-1,1}$ in which they lie. That is, ${}_0\Lambda_0$ and ${}_1\Lambda_1$ have the structure of commutative rings under gluing $(\cdot)$ and formal sum $(+)$. They satisfy ${}_0\Lambda_{\pm 1,0} \cong \mathbb{C}[R]/(R^2 = \pm\emptyset)$ and ${}_1\Lambda_{\pm 1,1} = \mathbb{C}(T)/(T^2 = \pm I)$ with $\emptyset$ and I serving as respective identities.

What is more important than the representations of these algebras is the representations of the entire category ${}_a\Lambda_{\pm 1,b}$ in which they lie. Similar to the rectangular case, these four representatives together form the "principle graph" from which the rest of the "Bratteli diagram" for full category representations follows formally.

Lemma 4.5. *These 4 representations of $\Lambda_{\pm 1}^{\mathcal{A}}$ are a complete set of irreducibles.*

The Bratteli diagram in Figure 23 explains how to extend the algebra representations to the linear category ("algebroid") in the rectangle $\mathcal{R}$ case.

All vector spaces above are 1-dimensional and spanned by the indicated picture in $\mathcal{R}$ and the $\nearrow$ is "add line on right", the $\searrow$ "bend right". The annular case is similar and is shown in Figures 24, 25.

Note that we interpret the rectangles as having their left and right sides glued.
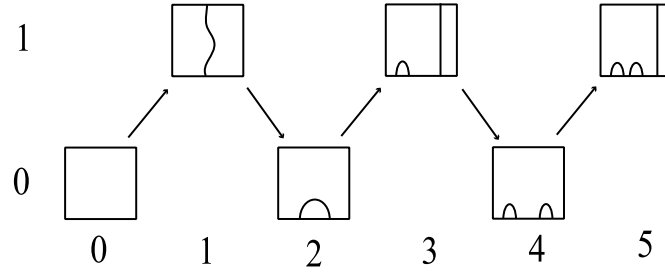
FIGURE 23. Bratteli diagram
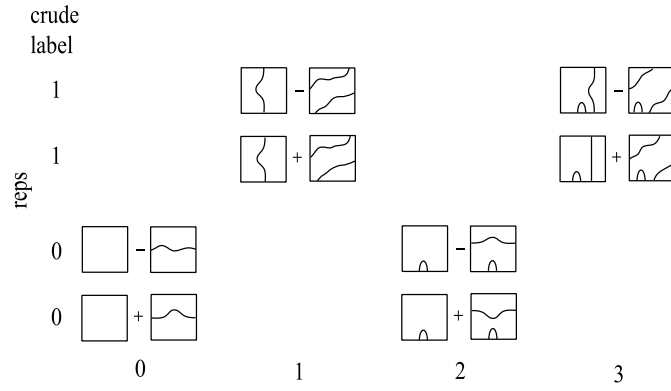
FIGURE 24. Λ^A_{+1}
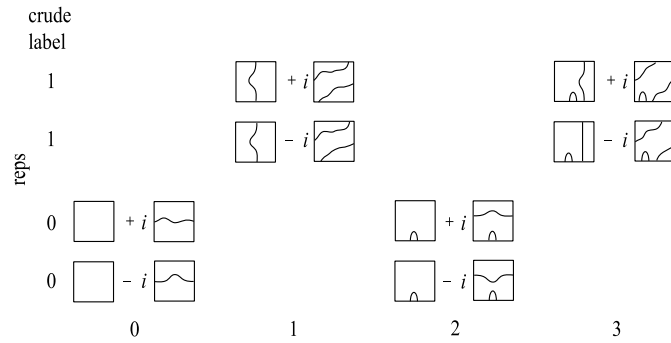
FIGURE 25. Λ^A_{-1}

In the case of annular categories there is no tensor structure (horizontal stacking) so in general the arrows present in the $\mathcal{R}$-case seems more difficult to define in generality, but should be clear in these examples. In the annular diagrams above all vector spaces of morphisms $_a\Lambda_b$ have dimension=2 if $a = b \bmod 2$ and zero otherwise. As in the rectangular case, resolutions of the identity morphisms of $\Lambda^A_{\pm 1}$ into morphism which factor through 0 or 1-strand show that all representations are

sums of the four. One dimensionality and the existence of invertible morphisms
between grades (exactly those shown in Fig. 19, but now with the convention that
the vertical sides of rectangles are glued to form an annulus) again show that the
four are irreducible.

By the corollary B.7 to Schur's lemma, the above decompositions into irre-
ducibles are all unique. There are direct generalizations of the categories so far
considered to Temperley-Lieb-Jones categories in the next section.

4.2. **Cutting and paste as Morita equivalence.** Crude labels for picture cat-
egories are given as finitely many points of the boundary. In the gluing formulas
for TQFTs, labels are irreps of the picture categories. The passage from the crude
labels of points to the refined labels of irreps is Morita equivalence.

Definition 4.6. Two linear categories $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent if there are
$\mathcal{C} \times \mathcal{D}$ bimodule M and $\mathcal{D} \times \mathcal{C}$ bimodule N such that $M \otimes N \cong \mathcal{C}$ and $N \otimes M \cong \mathcal{D}$
as bimodules.

Let Λ be a linear category, $\{a_i\}_{i \in I}$ a family of objects of Λ. For each $i \in I$, let e_i
be an idempotent in the algebra $_{a_i}\Lambda_{a_i}$. Define a new linear category Δ as follows:
the objects of Δ is the index set I, and the morphism set $_i\Delta_j = e_i {}_{a_i}\Lambda_{a_j}$.

Given an object a in Λ, define the $\Delta \times \Lambda$ bimodule M as $_iM_a = e_i {}_{a_i}\Lambda_a$, and the
$\Lambda \times \Delta$ bimodule N as $_aN_i = {}_a\Lambda_{a_i} e_i$.

A key lemma is the following theorem in Appendix A of [BHMV]:

Theorem 4.7. *Suppose the idempotents e_i generate Λ as a two-sided ideal. Then
the bimodule $M \otimes_\Lambda N \cong \Delta$ and $N \otimes_\Delta M \cong \Lambda$, i.e., Λ and Δ are Morita equivalence.*

*Consequently, tensoring (on the left or right), by the modules N and M, gives
rise to the Morita equivalence of Λ and Δ. Moreover, these equivalences preserves
tensor product of bimodules.*

Given two surfaces Y_1, Y_2 such that $\partial Y_1 = \bar\gamma_1 \prod \gamma, \partial Y_2 = \bar\gamma \prod \gamma_2$, and the picture
spaces $\mathrm{Pic}(Y_1), \mathrm{Pic}(Y_2)$ are bimodules over the picture category Λ, then the picture
space $\mathrm{Pic}(Y_1 \cup_\gamma Y_2)$ is the tensor product of $\mathrm{Pic}(Y_1)$ and $\mathrm{Pic}(Y_2)$ over Λ. Morita
equivalence, applied to the picture category Λ, sends the bimodule $_-\mathrm{Pic}(Y_1 \cup_\gamma Y_2)_-$
to the bimodule $_-\mathrm{Pic}(Y_1) \otimes \mathrm{Pic}(Y_2)_-$ (over the representation category Δ of the
picture category Λ) because tensor products are preserved under Morita equiva-
lence. Now the general gluing formula can be stated as a consequence of Morita
equivalence:

Theorem 4.8. *Let Y_1, Y_2 are two oriented surfaces such that $\partial Y_1 = \bar\gamma_1 \prod \gamma$ and
$\partial Y_2 = \bar\gamma \prod \gamma_2$. Then the picture bimodule $_-V(Y_1 \cup_\gamma Y_2)_-$ is isomorphic to $_-V(Y_1) \otimes_\Delta V(Y_2)_-$
as bimodules.*

As explained in Appendix B, the idempotents e_i label a complete set of irreps
of the linear category Λ. Therefore, gluing formulas for picture TQFTs need the

representation categories of the picture categories. In the axioms of TQFTs the label set was a mysterious feature, now we will see its origins in picture TQFTs.

Now let us write the Morita equivalence more explicitly. Let Λ be some $\Lambda^{\mathcal{R}}$ (or $\Lambda^{\mathcal{A}}$) and suppose Λ is semi-simple with index set I. The pictures in a fixed 3-manifold (surface) with a "left" and "right" gluing region provide a bimodule $_aB_b$ on for Λ. If the gluing region is not connected within the 3-manifold (surface) then $B \cong B^{\text{left}} \bigotimes_\Lambda B^{\text{right}}$. We treat this case first.

Lemma 4.9. $B^l \bigotimes_\Lambda B^r \cong_{i \in I} (V_i \bigotimes W_i)$, where $_aB^{\text{right}} \cong \bigoplus_i V_i \otimes \rho_i$ and $_aB^{\text{left}} \cong \bigoplus W_j \otimes \rho_j^{\text{op}}$, $\left(\rho_j^{\text{op}}(m) = (\rho_j(m))^\dagger \right)$.

Proof. Note that $\rho_i^{\text{op}} \bigotimes \rho_j \cong \text{Hom}_\Lambda(\rho_i, \rho_j) \cong \left\{ \begin{smallmatrix} \mathbb{C} \text{ if } i=j \\ 0 \text{ if } i \neq j \end{smallmatrix} \right.$. As usual $\bigoplus_\Lambda$ distributes over $\bigotimes_\Lambda$, the unusual feature is that the coefficients are vector spaces V_i and W_j, not complex numbers. They are "multiplied" by (ordinary) tensor product $\otimes$. $\square$

The manipulations above are standard in the context of 2-vector spaces [Fd][Wal2], and in fact a representation is a 2-vector in the 2-vector space of all formal representations.

Now suppose the regions to be glued to the opposite ends of $\mathcal{R}$ ($\mathcal{A}$) are part of a connected component of a 3-manifold (surface), then write the bi-module $_aB_b \cong \bigoplus_{(i,j) \in I^{\text{op}} \times I} W_{ij} \otimes (\rho_i^{\text{op}} \bigotimes_\Lambda \rho_j)$ as a bimodule. Define a 2-trace,

$$\text{tr} B = \bigoplus_{a \in \text{obj}(\Lambda)} {_aB_a}/u\alpha \cong \alpha^{\text{op}} u,$$

where $\alpha \in {_a\Lambda_b}, u \in {_bB_a}$ are arbitrary. Again, "linear algebra" yields:

Lemma 4.10. $tr(B) \cong \bigoplus_{i \in I} W_{ii}$.

Proof. Schur's lemma implies $\rho_i^{\text{opp}} \bigotimes_\mathbb{C} \rho_j \cong \mathbb{C}$ iff $i = j$. $\square$

Note that disjoint union of the spaces carries over to tensor product, $\bigotimes_\mathbb{C}$, of the modules of pictures on the space. This makes lemma 4.9 a special case of lemma 4.10. And further observe that both lemmas match the form of the "gluing formula" as expected, with $I = \mathcal{L}$, the label set, and adjoint ($\dagger$) is the involution $\hat{} : \mathcal{L} \to \mathcal{L}$.

4.3. **Annualization and quantum double.** Annular categories are closely related to the corresponding rectangle categories. In particular, there is an interesting general principle:

Conjecture: *If $\Lambda^{\mathcal{R}}$ and $\Lambda^{\mathcal{A}}$ are rectangular and annular versions of locally defined picture/relation categories, then $(\mathcal{D}(Rep(\Lambda^{\mathcal{R}})) \cong Rep(\Lambda^{\mathcal{A}})$, the Drinfeld center or quantum double of the representation category of the rectangular picture category is isomorphic to the representation category of the corresponding annular category.*

The conjecture and its higher category generalizations are proved in [Wal2].

5. Temperley-Lieb-Jones categories

To obtain the full strucure of the picture TQFTs, we need to consider surfaces with boundaries, and boundary conditions for the corresponding vector spaces $V(Y)$. The crude boundary conditions using objects in TLJ categories are not suitable for the gluing formulas. As shown in Section 3.7, Section 4, we need to find the irrpes of the TLJ categories. Two important properties of boundary condition categories needed for TQFTs are semi-simplicity and the finiteness of irreps. For TLJ categories, both properties follow from a resolution of the identity in the Jones-Wenzl projectors.

Let X be a compact *parameterized* n-manifold. The interesting cases in this paper are the unit interval $I = [0, 1]$ or the unit circle S^1. Define a category $\mathcal{C}(X)$ as follows: an object a of $\mathcal{C}(X)$ consists of finitely many points in the interior of X, and given two objects a, b, a morphism in $\mathrm{Mor}(a, b)$ is an $(n+1)$-manifold, not necessarily connected, in the interior of $X \times [0, 1]$ whose boundaries are $a \times 0, b \times 1$, and intersects the boundary of $X \times [0, 1]$ transversely. Given two morphisms $f \in {}_a\mathcal{C}_b, g \in {}_b\mathcal{C}_c$, the composition of f, g will be just the vertical stacking from bottom to top followed by the rescaling of the height to unit length 1. When X is a circle, we will also draw the vertical stacking of two cylinders as the gluing of two annuli in the plane from inside to outside. More often, we will draw the stacking of cylinders as vertical stacking of rectangles one on top of the other with periodic boundary conditions horizontally. Note the two boundary circles of a cylinder are parameterized, so they have base points and are oriented. The gluing respects both the base-point and orientation.
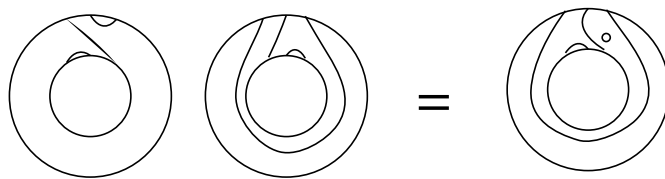
FIGURE 26. Composition of annular morphisms

Given a non-zero number $d \in \mathbb{C}$, the *Temperley-Lieb category* TL_d is the linear category obtained from $\mathcal{C}([0, 1])$ by first imposing d-isotopy in each morphism set, and then taking formal finite sums of morphisms as follows: the objects of TL_d are the same as that of $\mathcal{C}([0, 1])$, and for any two objects a, b, the vector space $\mathrm{Mor}_{\mathrm{TL}}(a, b)$ is spanned by the set $\mathrm{Mor}(a, b)$ modulo d-isotopy.

The structure of the Temperley-Lieb categories TL_d depends strongly on the values of d as we have seen in the *Temperley-Lieb algebras* $\mathrm{TL}_n(d) = \mathrm{Mor}(a, a)$ for any object $a \in \mathrm{TL}_d^0$ consisting of n points. When A is as in Lemma 3.4,

the semi-simple quotient of the Temperley-Lieb category TL_d by the Jones-Wenzl idempotent p_{r-1} is a semi-simple category. The associated semi-simple algebras $\text{TL}_n(d)$ were first discovered by Jones in [Jo4]. Therefore the semi-simple quotient categories of TL_d for a particular d will be called the *rectangular Temperley-Lieb-Jones category* $\text{TLJ}_d^{\mathcal{R}}$, where $d = -A^2 - A^{-2}$. Note that there will be several different A's which result in the same TLJ category as the coefficients of the Jones-Wenzl idempotents are rational functions of d. If we replace the interval $[0,1]$ in the definition of the Temperley-Lieb categories by the unit circle S^1, we get the *annular Temperley-Lieb categories* $\text{TL}_d^{\mathcal{A}}$, and their semi-simple quotients *the annular Temperley-Lieb-Jones categories* $\text{TLJ}_d^{\mathcal{A}}$.

5.1. **Annular Markov trace.**
In the analysis of the structure of the TL algebras, the Markov trace defined by Figure 4 in Section 2 plays an important rule. In order to analyze the annular TLJ categories, we introduce an annular version of the Markov trace and 2-category generalizations.

Recall that $\Delta_n(x)$ is the Chebyshev polynomial. Let $C_n(x)$ be the algebra $\mathbb{C}[x]/(\Delta_n(x))$. Inductively, we can check that the constant term of Δ_n is not 0 if n is odd, and is 0 if n is even. For n even, the coefficient of x is $(-1)^{\frac{n}{2}}n$. Let $n = 2m$ and $q_{2m}(x)$ be the element of $C_n(x)$ represented by $\frac{\Delta_{2m}(x)}{x}$.

Define the annular Markov trace $Tr^{\mathcal{A}}$ as follows: $Tr^{\mathcal{A}} : \text{TL}_{n,d} \to C_n(x)$ is defined exactly the same as in Figure 4 in Section 2 except instead of counting the number of simple loops in the plane, the image becoming elements in the annular algebra, where x is represented by the center circle(=called a ring sometimes).

Proposition 5.1. $Tr^{\mathcal{A}}(p_n) = \Delta_n(x)$.

It follows that the algebra $C_n(x)$ can be identified as the annular algebra when d is a simple root of $\Delta_n(x)$.

If the inside and outside of the annulus $\mathcal{A}$ are identified, we have a torus T^2. The annular Markov trace followed by this identification leads to a 2-trace from $\text{TL}_{n,d}$ to the vector space of pictures in T^2.

5.2. **Representation of Temperley-Lieb-Jones categories.**
Our goal is to find the representations of a TLJ category $\text{TLJ}_d^{\mathcal{R}}$ or $\text{TLJ}_d^{\mathcal{A}}$. The objects consisting of the same number of points in such categories are isomorphic, therefore the set of natural numbers $\{0, 1, 2, \cdots\}$ can be identified with a skeleton of the category (a complete set of representatives of the isomorphism classes of objects). Each morphism set $\text{Mor}(i, j)$ is spanned by pictures in a rectangle or an annulus.

To find all the irreps of a TLJ category, we use Theorem B.8 in Appendix B to introduce a table notation as follows: we list a skeleton $\{0, 1, \cdots, \}$ in the bottom row. Each *isomorphism* class ρ_j of irreps of the category is represented by a row of vector spaces $\{V_{j,i}\} = \{\rho_j(i)\}$. Each column of vector spaces $\{V_{i,j}\}$ determines an isomorphism class of objects of the category. The graded morphism linear

maps of any two columns will be $_i\mathrm{TLJ}_j$, in particular the graded linear maps of any column to itself give rise to the decomposition of $_i\mathrm{TLJ}_i$ into matrix algebras, i.e., $_i\mathrm{TLJ}_i = \oplus_j \mathrm{Hom}(V_{j,i}, V_{j,i})$. To find all irreps of TLJ, we look for minimal idempotents of $_i\mathrm{TLJ}_i$ starting from $i = 0$. Suppose there exists an m_0 such that the irreps $\{e_{j_i}\}$ of $_j\mathrm{TLJ}_j, j \leq m_0$ are sufficient to decompose every $_m\mathrm{TLJ}_m$ as $\oplus_{j_i} \mathrm{Hom}(V_{j_i,m}, V_{j_i,m})$ for all $m \geq m_0$, then it follows that all irreps of TLJ are found; otherwise, a non-zero new representation space $V_{k,a}$ from some new irrep ρ_k and $a \in \mathrm{TLJ}^0$ implying $\mathrm{Hom}(V_{k,a}, V_{k,a}) \subset {_a\mathrm{TLJ}_a}$ will contradict the fact that $_a\mathrm{TLJ}_a \cong \oplus_{j \neq k} \mathrm{Hom}(V_{j,a}, V_{j,a})$.

Remark: For the annulus categories, we can identify one irrep as the trivial label using the disk axiom of a TQFT. Given a particular formal picture x in an annulus, we define the disk consequences of x as all the formal pictures obtained by gluing x to a collar of the disk: given a picture y on the disk, composition x and y is a new picture in the disk. By convention, pictures with mismatched boundary conditions are 0. Then the trivial label is the one whose disk consequences form the vector space $\mathbb{C}$, while all others would result in 0.

For an object $m \in \mathrm{TLJ}^0$ if $\mathrm{id}_m = \oplus_{j<m}(\oplus_l f^l_{m,j} \cdot g^l_{j,m})$ for $f^l_{m,j} \in {_m\mathrm{TLJ}_j}, g^l_{j,m} \in {_j\mathrm{TLJ}_m}$, where l is a finite number depending on j, then we have a *resolution of the identity of m* into lower orders.

Lemma 5.2. *If for some object m of a TLJ category, we have a resolution of its identity id_m into lower orders, then every irrep of the category TLJ is given by a minimal idempotent in $_j TLJ_j$ for some $j < m$.*

Given a TLJ category and two objects $a, c \in \mathrm{TLJ}^0$, there is a subalgebra, denoted by A^a_{cc}, of the algebra $A_{cc} = {_c\mathrm{TLJ}_c}$ consisting all morphisms generated by those factoring through the object a: $f \cdot g, f \in {_c\mathrm{TLJ}_a}, g \in {_a\mathrm{TLJ}_c}$. If e_a is an idempotent of $_a\mathrm{TLJ}_a$, then $A^{e_a}_{cc}$ denotes the subalgebra of A^a_{cc} consisting all morphisms generated by those factoring through e_a, i.e., those of the form $f \cdot e_a \cdot g$.

Lemma 5.3. *Given two objects a, b of a TLJ category, and two minimal idempotents $e_a \in {_a TLJ_a}, e_b \in {_b TLJ_b}$, then*

1): $A^{e_a}_{cc}$ is the simple matrix algebra over the vector space $_c TLJ_a e_a$.

2): If the two representations $e_a TLJ, e_b TLJ$ are isomorphic, then for any $c \in TLJ^0$, which is neither a nor b, the subalgebras $A^{e_a}_{cc}, A^{e_b}_{cc}$ of A_{cc} are equal.

We will use these lemmas to analyze representations of TLJ categories, but first we consider only the low levels.

5.3. **Rectangular Tempeley-Lieb-Jones categories for low levels.** Denote $A_{ij} = {_i\Lambda_j}$. Note that A_{ii} is an algebra, and $A_{ij} = 0$ if $i \neq j$ mod 2. The Markov trace induces an inner product $<,>: A_{ij} \times A_{ij} \to \mathbb{C}$ on all A_{ij} given by $<x, y> = \mathrm{Tr}(\bar{x}y)$.

TABLE 1. Irreps of rectangular level=1

ρ_1	0	1
ρ_0	1	0
	0	1

5.3.1. *Level=1, $d^2 = 1$*. Using $p_2 = 0$, we can "squeeze" a general element $x \in A_{ij}$ so that there are only 0 or 1 arcs cross the mid-level of the rectangle. Such diagrams in Figure 18) in Section 4.1.

The algebra $A_{00} = \mathbb{C}$, and the empty diagram is the generator. The first irrep ρ_0 of $\mathrm{TLJ}^{\mathcal{R}}_{d=\pm 1}$ is given the idempotent p_0, which is just the identity $\mathrm{id}_\emptyset$ on the empty diagram: if j is odd, $\rho_0(j) = 0$; if j is even, $\rho_0(j) = A_{0j} \cong \mathbb{C}$.

The algebra $A_{11} = \mathbb{C}$, generated by a single vertical line. The identity does not factor through the 0-object, so we have a new idempotent p_1 (=idenity on the vertical line). The resulting irrep ρ_1 sends even j to 0, and odd j to $A_{1j} \cong \mathbb{C}$.

Continuing to A_{22}, we see that the identity on two strands does factor through p_0 given by the Jones-Wenzl idempotent p_2. By Lemma 5.2, we have found all the irreps of $\mathrm{TLJ}^{\mathcal{R}}_{\pm 1}$, which are summarized into Table 1.

$\mathrm{TLJ}^{\mathcal{R}}_{d=1}$ does not lead to a TQFT since the resulting S-matrix $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is singular. Although $\mathrm{TLJ}^{\mathcal{R}}_{d=-1}$ does give rise to a TQFT, the resulting theory with S-matrix $= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ is not unitary. The semion theory with S-matrix $= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ can be realized only by the representation category of the quantum group $SU(2)$ at level=1. This subtlety comes from the Frobenius-Schur indicator of the non-trivial label, which is 1 for TLJ and -1 for quantum group.

5.3.2. *Level=2, $d^2 = 2$*. Since p_3 is a resolution of the identity of id_3 into lower orders, it suffices to analyze A_{ii} for $i \leq 2$. The cases of A_{00}, A_{11} are the same as level=1. Since $\dim A_{20}=1$, $\dim A_{21}=0$ and $\dim A_{22}=2$, id_2 does not factor through lower orders, so there is a new idempotent in A_{22}. The 1-dimensional subalgebra A_{22}^0 is generated by e_2, which is the following diagram:

It is easy to check e_2 is the identity of A_{22}^0. Since the identity of A_{22} is the sum of the two central idempotents (the two identities of each 1-dimensional subalgebra), the new idempotent p_2 is $\mathrm{id}_2 - e_2$. The irrep corresponding to p_2 sends each odd j to 0, and each even j to $p_2 A_{2j}$.

ρ_2	0	0	1
ρ_1	0	1	0
ρ_0	1	0	1
	0	1	2

TABLE 2. Irreps of rectangular level=2

Therefore, the irreps of the level=2 $\text{TLJ}^{\mathcal{R}}$ are given by $p_0\text{TLJ}, p_1\text{TLJ}, p_2\text{TLJ}$, which are summarized into Table 2.

5.3.3. *Level=3, $d^2 = 1 + d$ or $d^2 = 1 - d$.* The same analysis for objects $0, 1, 2$ yields three idempotents p_0, p_1, p_2. Direct computation shows $\text{Hom}(3,3) \cong \mathbb{C}^5$, $\text{Hom}(3,0) = \text{Hom}(3,2) = 0$ and $\text{Hom}(3,1) \cong \mathbb{C}^2$. By Lemma 5.3, $A_{33}^{p_1} = A_{33}^1$ is the 4-dimensional algebra of 2×2 matrices over the vector space A_{13}. Let v_1, v_2 be the two vectors of A_{31} represented by diagrams such that $< v_1, v_2 >=< v_2, v_1 >= d^2$, and $< v_1, v_1 >=< v_2, v_2 >= d$. Using Gram-Schmidt on the vectors v_1, v_2, we get an orthonormal basis $e_1 = \frac{v_1}{d}, e_2 = \frac{v_2 - e_1}{d^2 - 1}$ of A_{31}. Hence the identity of the algebra A_{33}^1 is $|e_1 >< e_1| + |e_2 >< e_2|$. Therefore, the remaining idempotent of A_{33} is $\text{id}_3 - |e_1 >< e_1| - |e_2 >< e_2|$, which is just p_3. It follows that the irreps of TLJ are given by $p_0\text{TLJ}, p_1\text{TLJ}, p_2\text{TLJ}, p_3\text{TLJ}$.

5.4. **Annular Temperley-Lieb-Jones theories for low levels.** First we have the following notations for the pictures in the annular morphism sets $A_{00}, A_{11}, A_{02}, A_{22}$, where $\mathbf{1_0}, R, B, \mathbf{1_1}, T_1, \mathbf{1_2}, T_2$ are annular diagrams: $\mathbf{1_0}, \mathbf{1_1}, \mathbf{1_2}$ are identities with $0, 1, 2$ strands, R is the ring, B is the birth, and T_1 is the Dehn twisted curve, and T_2 is the fractional Dehn twisted curve. We also use B' to denote the diagram of RB after Z_2 homology surgery. A diagram with a $^-$ is the one obtained from a reflection through a horizontal line.

5.4.1. *Level=1, $d^2 = 1$.* The Jones-Wenzl idempotent p_2 is a resolution of id_2 into the lower orders, so we need only to find the minimal idempotents of $\text{Hom}(0,0)$ and $\text{Hom}(1,1)$. Since any two parallel lines can be replaced by a turn-back, the algebra A_{00} is generated by the empty picture $\emptyset$ and the ring circle R. Stacking two rings R together and resolving the two parallel lines give $R^2 = 1$, hence A_{00} is the algebra $\mathbb{C}[R]/(R^2 - 1)$. By Lemma B.3, the two minimal idempotents of A_{00} are $e_1 = \frac{\emptyset + R}{2}, e_2 = \frac{\emptyset - R}{2}$. To test which idempotent is of the trivial type, we apply e_1, e_2 to the empty diagram on the disk and obtain $e_1\emptyset = (\frac{d+1}{2})\emptyset, e_2\emptyset = (\frac{1-d}{2})\emptyset$. Hence if $d = 1$, then e_1 is of the trivial type, and if $d = -1$, then e_2 is of the trivial type.

The algebra A_{11} is generated by the straight arc I and the tiwst T. By stacking two rings R together and resolving the two parallel lines, we see that A_{11} is the

algebra $\mathbb{C}[T]/(T^2 - dI)$. By Lemma B.3, for $d = 1$, we have two minimal idempotents $e_{3,1} = \frac{I+T}{2}, e_{4,1} = \frac{I-T}{2}$. For $d = -1$, we have two minimal idempotents $e_{3,-1} = \frac{1-iT}{2}, e_{4,-1} = \frac{1+iT}{2}$. Note that $\text{Hom}(0,1) = \text{Hom}(1,0) = 0$. Therefore, the annular TLJ categories for $d = \pm 1$ have 4 irreps $e_i, i = 1, 2, 3, 4$.

5.4.2. *Level=2, $d^2 = 2$.* For the TLJ categories at level=2, $d^2 = 2$, p_3 is a resolution of the identity of id_3 into lower orders, so we need to analyze the algebras A_{00}, A_{11}, A_{22}. The algebra A_{00} is generated by the empty picture $\emptyset$ and the ring R. Since $R^3 = 2R$, $A_{00} = \mathbb{C}[R]/(R^3 - 2R)$. By Lemma B.3, the three minimal idempotents are $e_1 = \emptyset - \frac{R^2}{2}, e_2 = \frac{R^2+dR}{4}, e_3 = \frac{R^2-dR}{4}$. Testing on the disk, we know that e_2 is of the trivial type.

For A_{11}, we apply the Jones-Wenzl idempotent p_3 to the stacking of two twists T^2. After simplifying, we get $T^4 - dT^2 + 1 = 0$. Again by Lemma B.3, we have 4 minimal idempotents: $\frac{1}{2d}(\alpha^2 I + \alpha T - \alpha^4 T^2 - \alpha^3 T^3)$, where $\alpha^4 - d\alpha^2 + 1 = 0$.

A new phenomenon arises in the algebra A_{22}, which is generated by 8 diagrams: $1_2, \bar{B}B, T_2, \bar{B}'B, \bar{B}B', \bar{B}'B', \bar{B}RB, \bar{B}'RB$. Computing their inner products shows that $A_{22} \cong \mathbb{C}^8$. A_{02} is spanned by B, B', RB, RB'. Using the three minimal idempotents in A_{00}, we see that $e_0 A_{02}$ is spanned by $RB + RB' = f_0$, $e_1 A_{02}$ is spanned by $B - \frac{d}{2}RB' = f_1, B' - \frac{d}{2}RB = f_1'$, and $e_1 A_{02}$ is spanned by $RB - RB' = f_2$. Hence $A_{22}^0 \cong \mathbb{C}^6$ as the direct sum of 2 1×1 matrix algebras generated by f_0, f_2 and a 2×2 algebra generated by f_1, f_1'. Therefore there are two more idempotents in A_{22}. Applying p_3 to the action of the 1/2-Dehn twist F on A_{22}, we get $F^2 = 1$ modulo lower order terms, hence the last two idempotents are of the form $\frac{1}{2}(I_2 \pm iF)$ plus lower order terms in A_{22}^0. Since $A_{22}^0 = BA_{02} + B'A_{20}$, we need to find an x such that $e = -\frac{1}{2}1_2 \pm \frac{i}{2}T_2 + x$ is a projector and $eB = eB' = 0$. Solve the equations, we find

$$e_\pm = \frac{1}{2}1_2 \pm \frac{i}{2}T \mp \frac{i}{2d}\bar{B}'B - \frac{1}{2d}\bar{B}B \mp \frac{i}{2d}BB' - \frac{1}{2d}\bar{B}'B' \pm \frac{i}{2d^2}\bar{B}RB + \frac{1}{2d^2}\bar{B}'RB.$$

5.4.3. *Level=3, $d^2 = 1+d$ or $d^2 = 1-d$.* The algebra A_{00} is the algebra $\mathbb{C}[R]/(R^4 - 3R^2 + 1)$, so we have 4 minimal idempotents.

The algebra A_{11} is generated by the twist T, so A_{11} is the algebra $\mathbb{C}[T]/(T^6 - dT^4 - dT^2 + 1)$, so we have 6 minimal idempotents.

Let F be the fractional Dehn twist on A_{22}, then p_4 results in a dependence among $F^{-2}, F^{-1}, I_2, F, F^2$: $F^4 - dF^2 + 1$ modulo lower order terms. So we have 4 minimal idempotents.

Let F be the fractional Dehn twist on A_{33}, then p_4 results in a relation between F^{-1}, I_3, F. So we have 2 minimal idempotents.

We leave the exact formula for the idempotents to interested readers. Note that the number of irreps of the annular TLJ categories is the square of the corresponding TLJ rectangular categories.

5.5. **Temperley-Lieb-Jones categories for primitive $4r$th roots of unity.**
Let A be a primitive $4r$-th root of unity, and $d = -A^2 - A^{-2}$. $\text{TLJ}^{\mathcal{R},k,A}$ is just
the TL_d modulo its annihilator p_{r-1}. We found that it has minimal idempotents
$p_0, p_1, p_2, \cdots, p_k$, $k = r - 2$ and with image (p_{k+1}) being the annihilator of the
Hermitian paring $\langle\,,\,\rangle$.

The case A a primitive $2r$-th or rth root of unity, r odd, e.g. $A = e^{2\pi i/6}, k =
1, d = 1$; is identical as far as the rectangle categories go, but for the annular
categories is more complicated; it is analyzed in the next section.

Theorem 5.4. Rectangle diagrams with $p_i, 0 \le i \le k$, near the bottom and object
t at top span spaces $\{W_{A,i}^t\} := W_{A,i}$ on which $\Lambda := \Lambda^{\mathcal{R},k,A}$ acts from above. The
families $\{W_{A,i}\}$ (as i varies) are the $k + 1$ (isomorphism classes of) irreducible
representations of Λ. The involution $\hat{}$ is the identity.

Proof. Most of the argument is by now familiar. Resolving the identity shows that
any representation is a direct sum of $\{W_{A,i}\}, 0 \le i \le k$.

For the first time $\dim(W_{A,i}^t)$ may be > 1 and there will not be invertible mor-
phism $t \to t'$ but irreducibility can still be proved as follows: for all $m = p_i \cdot m_0$
and $m' = p_i \cdot m_0'$ one may construct morphism x and y so that $m' = mx$ and
$m = m'y$, where $p_i \in {}_i\Lambda_i, m, m_0 \in {}_i\Lambda_a, m', m_0' \in {}_i\Lambda_b, x \in {}_a\Lambda_b, y \in {}_a\Lambda_b$.   □

It is a bit harder to find the irreps of $\Lambda^{\mathcal{A},k,A} := \Lambda$, but we will do this now.
Similar irreps for $\text{TL}^{\mathcal{A}}$ categories were previously found by Graham-Lehner [GL],
in a different context.

We do not know how to proceed in a purely combinatorial fashion but must
invoke the action of the doubled theory on the undoubled. Topologically this
amounts to the action on pictures in the solid cylinder $(B^2 \times I, B^2 \times \partial I)$ under
the addition of additional strands in a shell $(B_2^2 \backslash B_1^2 \times I; B_2^2 \backslash B_1^2 \times \partial I)$. Logically
our calculation should be done until we have established[1] the undoubled TQFT
based on $\Lambda^{\mathcal{R},k,A}$ where the hypothesis A a primitive $4r$th root is used. This can be
done in Section 3 already or from here by going directly to Section 7 which does
not depend on this section. Therefore, we will freely invoke this material.

In the low level cases we found that $\#\text{irreps}\Lambda^{\mathcal{A}} = (\#\text{irreps}\Lambda^{\mathcal{R}})^2$. This is not an
accident but comes from identifying $\Lambda^{\mathcal{A}}$ with $\text{End}(\Lambda^{\mathcal{R}})$. $\Lambda^{\mathcal{A},k,A}$ is too complicated
to "guess" the irreps so we compute them from the endomorphism view point.

Recall from Section 3.4 the projectors $\omega_a = \sum\limits_{c=0}^{k} \frac{\Delta_{(a+1)(c+1)}[c]}{D}$ onto the a-label, and
$D^2 = \sum\limits_{c=0}^{k} \Delta_{c+1}^2$.

---

[1]It has been shown in Lemma 3.6 that the Temperley-Lieb-Jones theories $\Lambda^{\mathcal{R},k,A}$ violate an
important TQFT-axiom when $A^{2r} = 1$. The S-matrix is singular, half the expected rank, so the
action of the mapping class group is not completely defined. In this case irreps of $\Lambda^{\mathcal{A},k,A}$ have a
more complicated structure.

Also recall from Section 7 if $Y = \partial X$ and $\gamma \subset$ interior$X$ is a family of sccs labelled by $\tilde{\omega}_a$ and γ cobounds a family of imbedded annuli $\mathcal{A} \subset X$ with $\gamma' \subset Y_{,,}$ i.e. $\partial \mathcal{A} = \gamma \cup \gamma'$, then $Z(X, \gamma_{\omega_a}) \in V(Y\backslash\gamma; a, \hat{a}) \subset \bigoplus_{\text{admissible}} V(Y\backslash\gamma; l, \hat{l}) = V(Y)$.

Consider the 4$-$component formal tangle in annulus cross interval, $-A \times I$, where $h = |i - j|$:
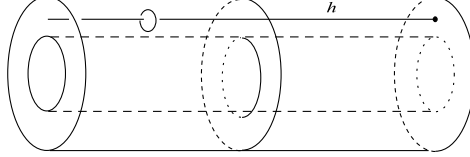

FIGURE 27. 4-component formal tangle

Let X be the 3-manifold made by removing small tubular neighborhoods of the h-labeled arc, and write $X \cong Y \times I$, where Y is the annulus with a new puncture, and $\partial X = DY$ the double of Y. Let (Y, l_0) be Y with ∂Y labeled as follows: outer boundary $\to j$, inner boundary $\to i$, new boundary $\to h$. From Lemma 3.16 we know $V(Y, l_0) \cong V_{i,j,h} \cong \mathbb{C}$.

Another useful decomposition of ∂X results from expanding the inner and outer boundary components of Y to annuli, A_i and A_0: $\partial X = -Y \cup +Y \cup -A_i \cup +A_0 \cup A_h$. Applying V we have: $V(\partial X) =$

$$\bigoplus_{\substack{\text{admissible labels}}} V^*(Y, l) \bigotimes V(Y, l) \bigotimes V^*(A_i, l) \bigotimes V(A_0, l) \bigotimes V(A_h, l). (*)$$

Let us restrict to label: l_0. By lemma, $\dim V(Y, l_0) = 1$ and let x be the unit normalized vector $\kappa \in V(Y, l_0)$,

The Jones-Wenzl projectors p_i, p_j and p_h are inserted as shown.
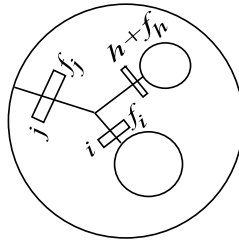

FIGURE 28. Insert Jones-Wenzl

The arc diagram should be pushed into a ball B^+ bounding the 2-sphere S^2 made by capping ∂Y, to define an element of $V(Y, l_0)$. The root $\theta-$symbol normalizes $||x'||^2$ to the invariant of $D(B^3) = S^3$ and the $s_{00}^{\frac{1}{2}}$ kills this factor so as defined $||x'||^2 = 1$.

Let $V_{l_0}(\partial X)$ denote the l_0 summand of the rhs (*). Fixing x', and therefore its dual $\hat{x}'$, give an isomorphism $\epsilon'_{ij} : V_{l_0} \to \operatorname{Hom}(V(A_i; i, \hat{i}), V(A_0; j, \hat{j}) =: \operatorname{Hom}(V_i, V_j)$.

Consider the partition function $Z(X, L)$ of (X, L), where L is the 3-component link in X labelled by ω_i, ω_j and ω_h in FIG??, and $Z(X, L) \subset V_{l_0}(X)$.

We now check that $\epsilon'_{i,j}(Z(X, L))$ is a non-zero vector in $\operatorname{Hom}(V_i, V_j)$ whose definition is independent of phase(X).

The pairing axiom can be used to analyze the result of gluing X to the genus two handlebody $(H, \overline{\theta}_{i,j,h})$ which is a thickening of the i, j, h labelled θ-graph (with the graph inside), we get:

$$S_{00}\theta_{ijh} = S_{0i}^2 S_{0j}^2 S_{0h} \langle x, \hat{x} \rangle \langle \beta_{ii}^*, \beta_{ii} \rangle \langle \beta_{jj}, \beta_{jj}^* \rangle. \qquad (**)$$

The two factors of S_{0i} and S_{0j} come from gluing along the seems separating off the inner and outer annuli (respectively); the factor S_{0h} derives from gluing across the "new component" of ∂Y, S_{00} is the 3-sphere normalization constant, making the lhs (**) a "spherical θ-symbol". Previously we arranged $\langle x', \hat{x}' \rangle = 1$ and $\langle \beta_{aa}^*, \beta_{a0} \rangle = S_{0,a}^{-1}$ so if we define $x = \frac{\sqrt{S_{0i}} \sqrt{S_{0j}} \sqrt{S_{0h}}}{\sqrt{S_{00}} \sqrt{\theta_{ijk}}} \cdot x' = \frac{\sqrt{S_{0i}} \sqrt{S_{0j}} \sqrt{S_{0h}}}{S_{00}} \kappa$, and redefine ϵ'_{ij} to ϵ_{ij} by replacing x' with x in its definition, we obtain:

$$\epsilon_{ij}(Z(X, L)) = \beta_{ii}^* \beta_{jj},$$

i.e. the canonical element of $\operatorname{Hom}(V_i, V_j)$.

Now attach a 2-handle to X along the "new" component of ∂Y to reverse our original construction: $X \cup 2\text{-handle} = A \times I$. The co-core of the 2-handle should now be labeled by h and the ω_h-labeled component can be dispensed with (it is now irrelevant). Call this new idempotent 3$-$component formal tangle $\bar{L}_{ij}$. Fix x, as above, a map $\overline{\epsilon_{ij}}$ closely related to ϵ_{ij} is now defined: $\bar{\epsilon}_{ij} : V(A \times I, \bar{L}) \to \operatorname{Hom}(V_i, V_j)$ and as before we have

Lemma 5.5. $\bar{\epsilon}_{ij}(Z(A \times I, \bar{L})) = \beta_{ii}^* \beta_{jj}.$

Using the geometric interpretation of links in a product as operators, and using the product structure from the middle factor in $A \times I = S^1 \times I \times I$, we see $\beta_{i,i}^* \beta_{jj}$ realized by a formal knot projection $(\overline{L}) \subset S^1 \times \frac{1}{2} \times I$. This projection can be Kauffman-resolved to a formal 1$-$submanifold $=: L_{ij} \subset S^1 \times I$ (ignoring the constant $\frac{1}{2}$). This is the minimal, in fact 1-dimensional idempotent of $\Lambda^{A,k,A}$. In fact by counting we see that we have achieved a complete resolution of the identity, and in the annular algebras $_i\Lambda_i^A$ a complete list of isomorphisms classes of irreducible representations of the full annular category $\Lambda^{A,k,A}$.

Our assumption in Section 7 is that A is a primitive $4r$-th root of unity, $r = k+2$. Section 7 constructs a TQFT with {labels} = {irreps$\Lambda^{R,k,A}$}. Using the s-matrix of this TQFT, we have just constructed a basis $\{\beta_{ii}^* \beta_{jj}\}$ of $(k + 1)^2$ operators for

$\text{Hom}\left(\bigoplus_{i=0}^{k} V_i, \bigoplus_{j=0}^{k} V_j\right)$ which are geometrically represented as formal submanifolds $\{L_{ij} \subset A\}$, also an idempotent in $_h\Lambda_h^{A,k,A}$.

The counting argument below holds for A a primitive $4r$th or $2r$th root of unity r odd or rth root of unity r odd, and so applies in the next section as well.

By a direct count of classical (not formal) pictures up to the projector relation $p_{k+1} = p_{r-1}$ we find:

$$\dim\left(_h\Lambda_h^{A,k,A}\right) \leq \begin{cases} k, & \text{for} \quad i = 0 \\ 2k + 2 - 2h, & \text{for} \quad 1 \leq h \leq k \end{cases} \quad (***).$$

Summing over h, $\dim\left(\bigoplus_{h=0}^{k} {}_h\Lambda_h^{A,k,A}\right) \leq (k+1)^2$.

Since $\{L_{ij}, 0 \leq i, j \leq k\}$ represent as $(k+1)^2$ linearly independent operators, the above inequalities must, in fact, be equalities.

$_0\Lambda_0^{A,k,A}$ is spanned by the empty picture $\emptyset$, the ring circle R, and its powers up to R^k. The projector decomposes R^{k+1} into a linear combination of lower terms.

For $h = 1$, let I denote the straight arc picture and T the counter clockwise Dehn twist. The pictures: $\bar{T}^{(k-1)}, \bar{T}^{k-2)}, \cdots, I, T, \cdots T^{k-1}$ appear (and are) independent but there is an obvious dependency if the list is expanded to $T^{-k} \cdots, T^k$. This dependency leads quickly to the claimed bound for $h = 1$.

For $h > 1$, the argument is similar to the above, except a fractional Dehn twist F replaces T.

Lemma 5.6. $\{L_{ij}, 0 \leq i, j \leq k\}$ is a complete set of minimal idempotents for $\{_h\Lambda_h^{A,k,A}\}, 0 \leq h \leq k$.

Proof: From Fig. 27, each $\bar{L}$ is a minimal idempotent and L_{ij} represents the same operator.

Fixing $h > 0$ now consider the action of "fractional Dehn twist", F on L_{ij}.

Lemma 5.7. For $i < j$, $F(L_{ij}) = -A^{i+j+2}L_{ij}$, and for $i > j$, $F(L_{ij}) = -A^{i+j+2}L_{ij}$.

Proof: Use the Kauffman relation to resolve the diagram below, noting the left kink is equal to a factor of $-A^3$ and that only the resolution indicated by arrows gives a term not killed by the projectors; its coefficient is A^{i+j-1}.

For $i > j$ one considers the mirror image of the above, interchanging A and A^{-1}.

If $h = 0, i = j$, then $L_{ij} = \tilde{\omega}_i$ and we may consider the action R of ring addition to ω_i. Since ω_i is the projector to the i-th label, we have $R(\tilde{\omega}_i) = -(A^{2i+2} + A^{-2i-2})\tilde{\omega}_i$.

This establishes:

Lemma 5.8. $R(L_{ii}) = -(A^{2i+2} + A^{-2i-2})L_{ii}$.

Let V_{ij} be the vector space spanned by formal 1-submanifolds in the annulus which near the inner boundary agree with L_{ij}.
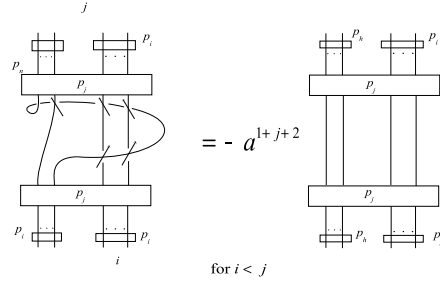
FIGURE 29. Annular idempotent

Essentially the same argument employed for the rectangle categories: resolution of the identities but now for id $\in {}_i\Lambda_i^{A,k,A}, 0 \le i \le k$ shows:

Theorem 5.9. *The spaces $\{V_{ij}\}$ form a complete set of irreps for $Rep(\Lambda^{A,k,A})$. Direct sum decompositions into these irreducibles are unique.*

The irreps V_{ij} have another "diagonal" indexing by $(h = |i-j|, \text{eigenvalue}(i,j))$. We think of h as the "crude label" it specifies the boundary condition (object); it is refined into a true label by the additional information of eigenvalue under fractional Dehn twist.

Note that for ring addition: eigenvalue(i,j) $= -A^{i+j+2}$ for $i \ne j$.

5.6. **Temperley-Lieb-Jones categories for primitive $2r$th root or rth root of unity, r odd.** In Lemma 3.6 we compute the S matrix associated to the rectangle category $\Lambda^{\mathcal{R},k,A}$, A a primitive $2r^{\text{th}}$ root or rth root of unity, r odd and find it is singular (Note that the last theorem of Ch XI [Tu] holds only for even r because the S matrix is singular for odd r). We find there is an involution on the label set $^-\colon \{0, \cdots, k\} \to \{0, \cdots, k\}$ defined by $\bar{a} = k - a$ so that $S = S_{\text{even}} \otimes \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right)$.) We use the notation i_{even} or just $i_e, 0 \le i \le k$, to denote the even number i or $\bar{i}$. (Note that $^-$ is not the usual duality $\hat{\ }$ on labels which is trivial in the TLJ theory. Also note that since k is odd exactly one of i and $\bar{i}$ is even,) The $\frac{k+1}{2}$ by $\frac{k+1}{2}$ matrix S_{even} is nonsingular and defines an $SU(2)_k^{\text{even}}$-TQFT[2] on the even labels at level k, explicitly:

$$S_{i_e j_e} = \sqrt{\frac{2}{r}}(-1)^{i+j}\Big([i+1][j+1]\Big). \qquad (5c.1)$$

The formal $1-$submanifolds $L_{i_e j_e}, 0 \le i,j \le k$ can be defined just as in last section. As operators on the $SU(2)_k^{\text{even}}$ TQFT they are $\beta_{i_e i_e}^* \beta_{j_e j_e}$. Also each $L_{i,j}$

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[2] These TQFTs are called $SO(3)$-TQFTs by many authors. As noted in [RSW], there is some mystery about those TQFTs as $SO(3)$-Witten-Chern-Simons TQFTs. Since they are the same TQFTs as $SU(2)$-Witten-Reshetikhin-Turaev TQFTs restricted to integral spins, therefore we adopt this notation. Their corresponding MTCs are denoted by $(A_1, k)_{\frac{1}{2}}$ in [RSW].

has an interpretation as a formal 1-manifold in the category $\Lambda^{\mathcal{A},k,e^{2\pi i/6}}$. (This is the "$d = 1$" category ("$\mathbb{Z}_2$"-gauge theory) that we have been developing as a simple example.)

Letting i_0 denote the odd index, i or $\bar{i}$, the tensor decomposition of the S-matrix implies $\omega_{i_e} = \omega_{i_0}$. It follows that $L_{i_e,j_e} = L_{i_0,j_0}$ and $L_{i_e,j_0} = L_{i_0,j_e}$ so we have found only half of the expected number of minimal idempotents. Let R_{i_e,j_e} be "reverse"(L_{i_e,j_e}), L_{i_e,j_e} with certain (-1) phase factors. That is, if $L_{i_e,j_e} = \sum_n a_n \alpha_n$ where α_i is a classical tangle then $R_{i_e,j_e} = \sum (-1)^{k_n} a_n \alpha_n$, where $k_n \overset{\text{def}}{=} k(\alpha_n))$ is the transverse intersection number with a radial segment, $s \times I \subset S^1 \times I = \mathcal{A}$ in the annulus. Similarly define R_{i_e,j_0}.

Recall the four irreps of $\mathbb{Z}_2$-gauge theory, $0 = \emptyset + R$, $e = \emptyset - R$, $m = I + T$, and $em = I - T$, and consider the following bijection:

$$\{L_{i_e,j_e}, R_{i_e,j_e}, L_{i_e,j_0}, R_{i_e,j_0}\} \overset{\beta}{\to}$$
$$\{\beta^*_{i_e i_e} \bigotimes \beta_{j_e j_e} \bigotimes 0, \beta^*_{i_e i_e} \bigotimes \beta_{j_e j_e} \bigotimes e, \beta^*_{i_e i_e} \bigotimes \beta_{j_e j_e} \bigotimes m, \beta^*_{i_e i_e}$$
$$\bigotimes \beta_{j_e j_e} \bigotimes em\} \hspace{4cm} (5c.2)$$

Theorem 5.10. β *is a bijection between the minimal idempotents of two graded algebras:*

$$_h\Lambda_h^{\mathcal{A},k,A} \overset{\beta}{\to} {}_h\Lambda_h^{\mathcal{A},k,e^{2\pi i/6}} \bigotimes End\left(\bigoplus_{j=even} {}_j\Lambda_j^{\mathcal{R},k,A}\right),$$

where A is a primitive $2r^{th}$ root or rth root of unity, r odd. The bijection β induces a bijection between the isomorphism classes of irreps of categories:

$$irreps.\left(\Lambda^{\mathcal{A},k,A}\right) \overset{\bar{\beta}}{\to} irreps.\left(\Lambda^{\mathcal{A},k,e^{2\pi i/6}} \bigotimes End\left(\Lambda_{even}^{\mathcal{R},k,A}\right)\right).$$

Proof: The second statement is by now the familiar consequence of the first and a "resolution of the identity."

The dimension count (upper bound) of last section applies equally for primitive $2r^{th}$ roots or rth roots, so it suffices to check that R_{i_e,j_e} and R_{i_e,j_o} are idempotents. Writing either as reverse$(L) = \sum_n (-1)^{k_n} a_n \alpha_n$ we square:

$$(\text{reverse}(L))^2 = (\sum_n (-1)^{k_n} a_n \alpha_n)^2 = \sum_{n,m} (-1)^{k_n+k_m} a_n a_m \alpha_n \alpha_m =$$

$$\text{reverse}\left(\sum_{n,m} a_n a_m \alpha_n \alpha_m\right) = \text{reverse}(L^2) = \text{reverse}(L). \quad (5c.3)$$

In the third equality holds since intersection number with a product ray in $\mathcal{A}$ is additive under stacking annuli:

$$k_n + k_m = k(\alpha_n) + k(\alpha_m) = k(\alpha_n \alpha_m) = k_{n,m}.$$

6. The definition of a TQFT

There are two subtle ingredients in the definition of a TQFT: the framing anomaly and the Frobenius-Schur (FS) indicator. For the TQFTs in this paper, the diagram and black-white TQFTs have neither anomaly nor non-trivial FS indicators, therefore, they are the easiest in this sense. The Jones-Kauffman TQFTs have anomaly, but no non-trivial FS indicators. Our version of the Turaev-Viro $SU(2)$-TQFTs have non-trivial FS indicators, but no anomaly; while the WRT TQFTs have both anomaly, and non-trivial FS indicators.

Our treatment essentially follows [Wal1] with two variations: first the axioms in [Wal1] apply only to TQFTs with trivial FS indicators, so we extend the label set to cover the non-trivial FS indicators; secondly we choose to resolve the anomaly for 3-manifolds only half way in the sense that we endow every 3-manifold with its canonical extension, so the modular functors lead to only projective representations of the mapping class groups. One reason for our choices is to minimize the topological prerequisite, and the other is that for application to quantum physics projective representations are adequate.

6.1. **Refined labels for TQFTs.** A TQFT assigns a vector space $V(Y)$ to a surface Y. If Y has boundaries, then certain conditions for ∂Y have to be specified for the vector space $V(Y)$ to satisfy desired properties for a TQFT. In Section 4, we see that crude boundary conditions need to be refined to the irreps of the picture categories, which are the labels. But for more complicated theories such as Witten-Reshetikhin-Turaev TQFTs, labels are not sufficient to encode the FS indicators. Therefore, we will introduce a boundary condition category to formalize boundary conditions. More precisely, boundary conditions are for small annular neighborhoods of the boundary circles. Our boundary condition category will be a strict weak fusion category $\mathcal{C}$, which enables us to encode the FS indicator for a label by marking boundaries with $\pm U$, where $U \in \mathcal{C}^0$. In our examples, the strict weak fusion categories are the representation categories of the TLJ categories. Then the labels are irreps of TLJ categories. In anyonic theory, labels are called superselection sectors, topological charges, or anyon types, etc. Boundary conditions which are labels are preferred because anyonic systems with such boundary conditions are more stable, while general boundary conditions such as superpositions of labels are difficult to maintain.

A fusion category is a finitely dominated semi-simple rigid linear monoidal category with finite dimensional morphism spaces and simple unit. A weak fusion category is like a fusion category except that rigidity is relaxed to weak rigidity as follows. A monoidal category $\mathcal{C}$ is weakly rigid if every object U has a weak dual: an object U^* such that $Hom(1, U \otimes W) \cong Hom(U^*, W)$ for any object W of $\mathcal{C}$.

A refined label set for a TQFT with a boundary condition category $\mathcal{C}$ is a finite set $L^e = \{\pm V_i\}_{i \in I}$, where the label set $L = \{V_i\}_{i \in I}$ is a set of representatives of isomorphism classes of simple objects of $\mathcal{C}$, and I a finite index set with a

distinguished element 0 and $V_0 = 1$. An involution $\hat{\ } : L^e \longrightarrow L^e$ is defined on refined labels $l = \pm V_i$ by $\hat{l} = -l$ formally. There is also an involution on the index set I of the label set: $\hat{i} = j$ if $V_j \cong V_i^*$. A label $V_i \in L$ is self-dual if $\hat{i} = i$, and a refined label is self dual if the corresponding label is self dual. A (refined) label set is self-dual if every (refined) label is self-dual. Each label V_i has an FS indicator ν_l: 0 if not self-dual, and ± 1 if self-dual. A self-dual label V_i is symmetrically self-dual or real in conformal field theory language if $V_i^* = V_i$ in $\mathcal{C}$, then we say $\nu_i = 1$, and anti-symmetrically self-dual or pseudo-real if otherwise, then we say $\nu_i = -1$, i.e., V_i^* is not the same object V_i in $\mathcal{C}$, though they are isomorphic. Secretly $-V_i$ is V_i^*, and we will identify the label $-V_i$ with V_i if the label is symmetrically self-dual; but we cannot do so if the label is anti-symmetrically self-dual, e.g., in the Witten-Reshetikhin-Turaev $SU(2)$ TQFTs. Frobenius-Schur indicators are determined by the modular S and T matrices [RSW]. Note that the trivial label 1 is always symmetrically self-dual.

6.2. **Anomaly of TQFTs and extended manifolds.** In diagram TQFTs in Section 3, we see that $Z(X_1 \cup_{Y_2} X_2) = Z(X_1) \cdot Z(X_2)$ as composition of linear maps. For general TQFTs, this identity only holds up to a phase factor depending on X_1, X_2 and the gluing map. Moreover, for general TQFTs, the vector spaces $V(Y)$ for oriented surface Y are not defined canonically, but depend on extra structures under the names of 2-framing, Lagrange subspace, or p_1 structure, etc. A Lagrangian subspace of a surface Y is a maximal isotropic subspace of $H_1(Y; \mathbb{R})$ with respect to the intersection pairing of $H_1(Y; \mathbb{R})$. We choose to work with Lagrange subspaces to resolve the anomaly of a TQFT.

An extended surface Y is a pair (Y, λ), where λ is a Lagrangian subspace of $H_1(Y; \mathbb{R})$. Note that if $\partial X = Y$, then Y has a canonical Lagrange subspace $\lambda_X = ker(H_1(Y; \mathbb{R}) \longrightarrow H_1(X; \mathbb{R}))$. In the following, the boundary Y of a 3-manifold X is always extended by the canonical Lagrangian subspace λ_X unless stated otherwise. For any planar surface Y, $H_1(Y; \mathbb{R}) = 0$, so the extension is unique. Therefore, extended planar surfaces are just regular surfaces.

To resolve the anomaly for surfaces, we define a category of labeled extended surfaces. Given a boundary condition category $\mathcal{C}$, and a surface Y, a labeled extended surface is a triple $(Y; \lambda, l)$, where λ is a Lagrnagian subspace of $H_1(Y; \mathbb{R})$, and l is an assignment of a signed object $\pm U \in \mathcal{C}^0$ to each boundary circle. Moreover each boundary circle is oriented by the induced orientation from Y, and parameterized by an orientation preserving map from the standard circle S^1 in the plane.

Given two labeled extended surfaces $(Y_i; \lambda_i, l_i), i = 1, 2$, their disjoint union is the labeled extended surface $(Y_1 \coprod Y_2; \lambda_1 \oplus \lambda_2, l_1 \cup l_2)$. Gluing of surfaces has to be carefully defined to be compatible with the boundary structures and Lagrangian subspaces. Given two components γ_1 and γ_2 of ∂Y parameterized by ϕ_i and labeled by signed objects $\pm U$, and let gl be a diffeomorphism $\phi_2 \cdot r \cdot \phi_1^{-1}$, where r is the

standard involution of the circle S^1. Then the glued surface Y_{gl} is the quotient space of $q : Y \to Y_{gl}$ given by $x \sim x'$ if $\mathrm{gl}(x) = \mathrm{gl}(x')$. If Y is extended by λ, then Y_{gl} is extended by $q_*(\lambda)$. The boundary surface ∂M_f of the mapping cylinder M_f of a diffeomorphism $f : Y \to Y$ of an extended surface $(Y; \lambda)$ has a canonical extension by the inclusions of λ.

Labeled diffeomorphisms between two labeled extended surfaces are orientation, boundary parameterization, and label preserving diffeomorphisms between the underlying surfaces. Note that we do not require the diffeomorphisms to preserve the Lagrangian subspaces.

6.3. **Axioms for TQFTs.** The category $\mathfrak{X}^{2,e,l}$ of labeled extended surfaces is the category whose objects are labeled extended surfaces, and the morphism set of two labeled extended surfaces (Y_1, λ_1, l_1) and (Y_2, λ_2, l_2) are labeled diffeomorphisms.

The anomaly of a TQFT is a root of unity κ, and to match physical convention, we write $\kappa = e^{\pi i c/4}$, and $c \in \mathbb{Q}$ is well-defined mod 8, and called the central charge of a TQFT. Therefore, a TQFT is anomaly free if and only if the central charge c is 0 mod 8.

Definition 6.1. A $(2+1)$-TQFT with a boundary condition category $\mathcal{C}$, a refined label set L^e, and anomaly κ consists of a pair (V, Z), where V is a functor from the category $\mathfrak{X}^{2,e,l}$ of oriented labeled extended surfaces to the category $\mathcal{V}$ of finitely dimensional vector spaces and linear isomorphisms composed up to powers of κ, and Z is an assignment for each oriented 3-manifold X with extended boundary, $Z(X, \lambda) \in V(\partial X; \lambda)$, where ∂X is extended by a Lagrangian subspace λ. We will use the notation $Z(X), V(\partial X)$ if ∂X is extended by the canonical Lagrangian subspace λ_X. V is called a modular functor. Z is the partition function if X is closed in physical language, and we will call Z the partition function even when X is not closed.

Furthermore, V and Z satisfy the following axioms.

Axioms for V:

(1) Empty surface axiom:

$$V(\emptyset) = \mathbb{C}$$

(2) Disk axiom:

$$V(B^2; l) \cong \begin{cases} \mathbb{C} & \text{if } l \text{ is the trivial label} \\ 0 & \text{otherwise} \end{cases}, \text{ where } B^2 \text{ is a 2-disk.}$$

(3) Annular axiom:

$$V(\mathcal{A}; a, b) \cong \begin{cases} \mathbb{C} & \text{if } a = \hat{b} \\ 0 & \text{otherwise} \end{cases}, \text{ where } \mathcal{A} \text{ is an annulus, and } a, b \in L^e \text{ are}$$

refined labels.

(4) Disjoint union axiom:

$V(Y_1 \amalg Y_2; \lambda_1 \oplus \lambda_2, l_1 \amalg l_2) \cong V(Y_1; \lambda_1, l_1) \otimes V(Y_2; \lambda_2, l_2)$. The isomorphisms are associative, and compatible with the mapping class group actions.

(5) Duality axiom:

$$V(-Y; l) \cong V(Y; \hat{l})^*.$$

The isomorphisms are compatible with mapping class group actions, with orientation reversal and disjoint union axiom as follows:

a) The isomorphisms $V(Y) \to V(-Y)^*$ and $V(-Y) \to V(Y)^*$ are mutually adjoint.

b) Given $f : (Y_1; l_1) \to (Y_2; l_2)$ and let $\bar{f} : (-Y_1; \hat{l}_1) \to (-Y_2; \hat{l}_2)$, then $< x, y >=< V(f)x, V(\bar{f})y >$, where $x \in V(Y_1; l_1), y \in V(-Y_1; \hat{l}_1)$.

c) Let $\alpha_1 \otimes \alpha_2 \in V(Y_1 \amalg Y_2) = V(Y_1) \otimes V(Y_2)$, and $\beta_1 \otimes \beta_2 \in V(-Y_1 \amalg -Y_2) = V(-Y_1) \otimes V(-Y_2)$, then

$$< \alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2 >=< \alpha_1, \beta_1 >< \alpha_2, \beta_2 > .$$

(6) Gluing Axiom:

Let Y_{gl} be the surface obtained from gluing two boundary components of Y, then $V(Y_{gl}) \cong \oplus_{l \in L} V(Y; (l, \hat{l}))$, where $l, \hat{l}$ label the two glued boundary components. The isomorphism is associative and compatible with mapping class group actions.

Moreover, the isomorphism is compatible with duality as follows: let $\oplus_{i \in L} \alpha_i \in V(Y_{gl}; l) = \oplus_{i \in L} V(Y; l, (i, \hat{i}))$ and $\oplus_i \beta_i \in V(-Y_{gl}; \hat{l}) = \oplus_{i \in L} V(-Y; \hat{l}, (i, \hat{i}))$, then there are non-zero real numbers s_i for each label V_i such that

$$< \oplus_i \alpha_i, \oplus_i \beta_i >= \sum_i s_i < \alpha_i, \beta_i > .$$

Axioms for Z:

(1) Disjoint axiom:

If $X = X_1 \amalg X_2$, then $Z(X) = Z(X_1) \otimes Z(X_2)$.

(2) Naturality axiom:

If $f : (X_1, (\partial X_1, \lambda_1)) \longrightarrow (X_2, (\partial X_2, \lambda_2))$ is a diffeomorphism, then $V(f) : V(\partial X_1) \longrightarrow V(\partial X_2)$ sends $Z(X_1, \lambda_1)$ to $Z(X_2, \lambda_2)$.

(3) Gluing axiom:

If $\partial X_1 = -Y_1 \amalg Y_2, \partial X_2 = -Y_2 \amalg Y_3$, then $Z(X_1 \cup_{Y_2} X_2) = \kappa^n Z(X_1) Z(X_2)$, where $n = \mu((\lambda_- X_1), \lambda_2, (\lambda_+ X_2))$ is the Maslov index (see Appendix C).

More generally, if X is an oriented 3-manifold and let $Y_i, i = 1, 2$ be disjoint surfaces in ∂X, extended by $\lambda_i \subset \lambda_X, i = 1, 2$, and $f : Y_1 \rightarrow Y_2$ be an orientation reversing dffeomorphism sending λ_1 to λ_2.

Then $V(\partial X)$ is isomorphic to $\sum_{l_1, l_2} V(Y_1; l_1) \otimes V(Y_2; l_2) \otimes V(\partial X \backslash (Y_1 \cup Y_2); (\hat{l}_1, \hat{l}_2))$ by multiplying κ^m, where l_i runs through all labelings of Y_i, and $m = \mu(K, \lambda_1 \oplus \lambda_2, \Delta)$ (see Appendix C). Hence $Z(X) = \oplus_{l_1, l_2} \kappa^m \sum_j \alpha_{l_1}^j \otimes \beta_{l_2}^j \otimes \gamma_{\hat{l}_1, \hat{l}_2}^j$.

If gluing Y_1 to Y_2 by f results in the manifold X_f, then

$$Z(X_f) = \kappa^m \sum_{j,l} < V(f) \alpha_l^j, \beta_l^j > \gamma_{\hat{l}, l}^j.$$

(4) Mapping cylinder axiom:

If Y is closed and extended by λ, and $Y \times I$ is extended canonically by $\lambda \oplus (-\lambda)$. Then $Z(Y \times I, \lambda \oplus (-\lambda)) = \mathrm{id}_{V(Y)}$.

More generally, let I_{id} be the mapping cylinder of $\mathrm{id} : Y \rightarrow Y$, and id_l be the identity in $V(Y; l) \otimes V(Y; l)^*$, then

$$Z(\mathrm{I}_{\mathrm{id}}, \lambda \oplus (-\lambda)) = \oplus_{l \in L(Y)} \mathrm{id}_l.$$

$\square$

First we derive some easy consequences of the axioms:

Proposition 6.2.   (1) $V(S^2) \cong \mathbb{C}$
(2) $Z(X_1 \sharp X_2) = \frac{Z(X_1) \otimes Z(X_2)}{Z(S^3)}$.
(3) *Trace formula: Let X be a bordism from closed surfaces Y, extended by λ, to itself, and X_f be the closed 3-manifold obtained by gluing Y to itself with a diffeomorphism f.*
    Then $Z(X_f) = \kappa^m Tr_{V(Y)}(V(f))$, where $m = \mu(\lambda(f), \lambda_Y \oplus f_*(\lambda), \Delta_Y)$ and $\lambda(f)$ is the graph of f_*, Δ_Y is the diagonal of $H_1(-Y; \mathbb{R}) \oplus H_1(Y; \mathbb{R})$. In particular, $Z(Y \times S^1) = dim(V(Y))$.
(4) *The dimension of $V(T^2)$ is the number of particle types.*

For a TQFT with anomaly, the representations of the mapping class groups are projective in a very special way. From the axioms, we deduce:

Proposition 6.3. *The representations of the mapping class groups are given by the mapping cylinder construction: given a diffeomorphsim $f : Y \longrightarrow Y$ and Y extended by λ, the mapping cylinder Y_f induces a map $V(f) = Z(Y_f) : V(Y) \longrightarrow V(Y)$. We have $V(fg) = \kappa^{\mu(g_*(\lambda), \lambda, f_*^{-1}(\lambda))} V(f) V(g)$.*

It follows from this proposition that the anomaly can be incorporated by an extension of the bordisms X, in particular, modular functors yield linear representations of certain central extensions of the mapping class groups.

6.4. **More consequences of the axioms.** For refined labels a, b, c, we have vector spaces $V_a = V(B^2; a), V_{a,b} = V(\mathcal{A}_{ab}), V_{a,b,c} = V(P_{abc})$, where P is a pair of pants or three-punctured sphere. Denote the standard orientation reversing maps on $B^2, \mathcal{A}_{ab}, P_{abc}$ by ψ. Then $\psi^2 = id$, therefore ψ induces identifications $V_{abc} = V^*_{\hat{a}\hat{b}\hat{c}}$, $V_{a\hat{a}} = V^*_{a\hat{a}}$, and $V_1 = V_1^*$. Choose basis $\beta_1 \in V_1, \beta_{a\hat{a}} \in V_{a\hat{a}}$ such that $< \beta_a, \beta_a > = \frac{1}{d_a}$.

Proposition 6.4.   (1) $Z(B^2 \times I) = \beta_1 \otimes \beta_1$
   (2) $Z(S^1 \times B^2) = \beta_{11}$
   (3) $Z(X \backslash B^3) = \frac{1}{D} Z(X) \otimes \beta_1 \otimes \beta_1$.

Proof. Let B^3 be a 3-ball regarded as the mapping cylinder as the identity map $id : B^2 \longrightarrow B^2$. By the mapping cylinder axiom, $Z(B^3) = \beta_1 \otimes \beta_1$. Gluing two copies of B^3 together yields S^3. By the gluing axiom $Z(S^3) = s_{00} = \frac{1}{D}$. It follows that $Z(X \backslash B^3) = \frac{1}{D} Z(X) \otimes \beta_1 \otimes \beta_1$.

$\square$

Proposition 6.5. *The action of the left-handed Dehn twist along a boundary component labeled by a of $B^2, \mathcal{A}_{ab}, P_{abc}$ on $V_1, V_{a,\hat{a}}$ or V_{abc} is a multiplication by a scalar θ_a. Furthermore, $\theta_1 = 1, \theta_a = \theta_{\hat{a}}$, and θ_a is a root of unity for each refined label a.*

6.5. **Framed link invariants and modular representation.** Let K be a framed link in a 3-manifold X. The framing of K determines a decomposition of the boundary tori of the link compliment $X \backslash \mathrm{nbd}(K)$ into annuli. With respect to this decomposition,

$$Z(X \backslash \mathrm{nbd}(K)) = \oplus_l J(K; l) \beta_{a_1 \hat{a_1}} \otimes \cdots \otimes \beta_{a_n \hat{a_n}},$$

where $J(k; l) \in \mathbb{C}$ and $l = (a_1, \cdots, a_n)$ ranges over all labelings of the components of K. $J(K; l)$ is an invariant of the framed, labeled link $(K; l)$. When (V, Z) is a Jones-Kauffman or WRT TQFT, and $X = S^3$, the resulting link invariant is a version of the celebrated colored Jones polynomial evaluated at a root of unity. This invariant can be extended to an invariant of labeled, framed graphs.

A framed link K represents a 3-manifold $\chi(K)$ via surgery. Using the gluing formula for Z, we can express $Z(\chi(K))$ as a linear combination of $J(K;l)$:

$$Z(\chi(K)) = \sum_l c_l J(K;l).$$

Consider the Hopf link H_{ij} labeled by $i, j \in L$. Let $\tilde{s}_{ij}$ be the link invariant of H_{ij}. Note that when a component is labeled by the trivial label, then we may drop the component from the link when we compute link invariant. Therefore, the first row of $\tilde{s}$ consists of invariants of the unknot labeled by $i \in L$. Denote $\tilde{s}_{i0}$ as d_i, and d_i is called the quantum dimension of label i. In Prop. 6.5, each label is associated with a root of unity θ_i, which will be called the twist of label i. Define $D = \sqrt{\sum_{i \in L} d_i^2}$, and $S = \frac{1}{D}\tilde{s}, T = (\delta_{ij}\theta_i)$, then S, T give rise to a representation of $SL(2,\mathbb{Z})$, the mapping class group of T^2.

6.6. **Verlinde algebras and Verlinde formulas.** Let $T^2 = S^1 \times S^1 = \partial D^2 \times S^1$ be the standard torus. Define the meridian to be the curve $\mu = S^1 \times 1$ and the longitude to be the curve $\lambda = 1 \times S^1$.

Let (V, Z) be a TQFT, then the Verlinde algebra of (V, Z) is the vector space $V(T^2)$ with a multiplication defined as follows: consider the two decompositions of T^2 into annuli by splitting along μ and λ, respectively. These two decompositions determine two bases of $V(T^2)$ denoted as $m_a = \beta_{a\hat{a}}$, and $l_a = \beta_{\hat{a}a}$. These two bases are related by the modular S-matrix as follows:

$$(6.6) \qquad l_a = \sum_b s_{ab} m_b, \, m_a = \sum_b s_{\hat{a}b} l_b.$$

Define $N_{abc} = \dim V(P_{abc})$, then we have

$$(6.7) \qquad m_b m_c = \sum_a N_{a\hat{b}\hat{c}} m_a.$$

The multiplication makes $V(T^2)$ into an algebra, which is called the Verlinde algebra of (V, Z).

In the longitude bases l_a, the multiplication becomes

$$(6.8) \qquad l_a l_b = \delta_{ab} s_{0a}^{-1} l_a.$$

This multiplication also has an intrinsic topological definition: $Z(P \times S^1)$ gives rise to a linear map from $V(T^2) \times V(T^2) \to V(T^2)$ by regarding $P \times S^1$ as a bordism from $T^2 \coprod T^2$ to T^2.

The fusion coefficient N_{abc} can be expressed in terms of s_{ab}, we have

$$(6.9) \qquad N_{abc} = \sum_{x \in L} \frac{s_{ax} s_{bx} s_{cx}}{s_{0x}}.$$

More generally, for a genus=g surface Y with m boundaries labeled by $l = (a_1 \cdots a_m)$,

$$(6.10) \qquad\qquad \mathrm{dim}V(Y) = \sum_{x \in L} s_{0x}^{2-2g-n}(\prod s_{a_i x}).$$

7. Diagram and Jones-Kauffman TQFTs

For the remaining part of the paper, we will construct picture TQFTs and verify the axioms for those TQFTs. Our approach is as follows: start with a local relation and a skein relation, we first define a picture category Λ whose objects are points with decorations in a 1-manifold X which is either an interval I or a circle S^1, and morphisms are unoriented sub-1-manifolds in $X \times I$ with certain structures connecting objects (=points in $X \times \{0\}$ or $X \times \{1\}$). More generally, the morphisms can be labeled trivalent graphs with coupons. Those picture categories serve as crude boundary conditions for defining picture spaces for surfaces with boundaries. Secondly, we find the representation category $\mathcal{C}$ of Λ, which is a spherical tensor category. The irreps will be the labels. In the cases that we are interested, the resulting spherical categories are all ribbon tensor categories. Thirdly, we define colored framed link invariants with the resulting ribbon tensor category in the second step. Invariants of the colored Hopf links with labels form the so-called modular S-matrix. Each row of the S-matrix can be used to define a projector ω_i which projects out the i-th label if a labeled strand goes through a trivial circle labeled by ω_i.

Figure 30. Projectors

The projector ω_0 is used to construct the resulting 3-manifold invariant. Finally, we define the partition function Z for a bordism X using a handle decomposition. This construction will yield a TQFT if the S-matrix is nonsingular, which is always true for the annular TLJ cases. If the S-matrix is singular, we still have a 3-manifold invariant, but we cannot define the representations of the mapping class groups for high genus surfaces, though representations of the braid groups are still well defined.

7.1. **Diagram TQFTs.** In this section, we outline the proof that for some $r \geq 3$, A a primitive $4r$th root of unity, or a primitive $2r$th root of unity and r odd, or a primitive rth root of unity and r odd, the diagram theories $\mathrm{Pic}^A(Y), Z_D$ defined in Section 3 indeed satisfy the axioms of TQFTs.

The diagram TQFTs are constructed based on the TLJ annular categories. The boundary condition categories $\mathcal{C}$ are the representation categories of the TLJ annular categories Λ. A nice feature of those TQFTs is that we can identify the objects of the TLJ annular categories Λ as boundary conditions using Theorem B.8: each object in TLJ gives rise to a representation of Λ and therefore becomes an object of $\mathcal{C}$, which is in general not simple, i.e., not a label. Hence picture vector spaces are naturally vector spaces for the diagram TQFTs.

For the diagram TQFTs, all labels are self-dual with trivial FS indicators. Therefore, it suffices to use only the label set. The label sets of the diagram TQFTs are given by the idempotents $L = \{\omega_{i,j,h}\}$ in Fig. 12. Given a surface Y with $\partial Y = \gamma_1, \cdots, \gamma_m$, and each boundary circle γ_i labeled by an idempotent $e_i \in L$. Then the picture space $\mathrm{Pic}_D^A(Y; e_1, \cdots, e_m)$ consists of all formal pictures that agree with e_i inside a small annular neighborhood $\mathcal{A}_i$ of the boundary γ_i modulo the Jones-Wenzl projector p_{r-1} outside all $\mathcal{A}_i$'s in Y. Given a bordism X from Y_1 to Y_2, the partition function $Z_D(X)$ is defined in Section 3.6. Now we verify that (Pic^A, Z_D) is indeed a TQFT.

For the axioms for modular functor V:

(1) is obvious.

(2) Since Jones-Wenzl projectors kill any turn-backs, then $\mathrm{Pic}^A(B^2; \omega_{i,j,h}) = 0$ unless $h = 0$. For $h = 0$, all pictures are multiples of the empty diagram.

(3) Since $\mathrm{Hom}(p_i, p_j) = 0$ unless $i = j$, so $\mathrm{Pic}^A(\mathcal{A}; \omega_{i',j',h'}, \omega_{i,j,h}) = 0$ unless $h = h'$. If $h = h'$, then we have $\omega_i \cdot \omega_{i'}$ and $\omega_j \cdot \omega_{j'}$, respectively in the annulus. Recall that $\omega_a \omega_b = \delta_{ab} \omega_a$, it follows that unless $i = i', j = j'$, $\mathrm{Pic}^A(\mathcal{A}; \omega_{i',j',h'}, \omega_{i,j,h}) = 0$.

(4) Obvious

(5) $\mathrm{Pic}^A(-Y) = \mathrm{Pic}^A(Y)$, hence duality is obvious.

(6) Gluing follows from Morita equivalence.

The axioms of partition function Z follow from handle-body theory and properties of the S matrix.

The action of the mapping class groups is easy to see: a diffeomorphism maps one multicurve to another. Since a diffeomorphism preserves the local relation and skein relation, this action sends skein classes to skein classes. The compatibility of the action with the axioms for vectors spaces is easy to check.

7.2. **Jones-Kauffman TQFTs.** In this section, we outline the proof that for $r \geq 3$, A a primitive $4r$th root of unity, the Jones-Kauffman skein theories $V_{JK}^A(Y), Z_{JK}$ defined in Section 3 indeed satisfy the axioms of TQFTs.

The boundary condition category for a Jones-Kauffman TQFT is the representation category of a TLJ rectangular category. The label set is $L = \{p_i\}_{i \in I}$, and $I = \{0, 1, \cdots, r-2\}$. Same reason as for the diagram TQFTs, we need only the label set.

The new feature of the Jones-Kauffman TQFTs is the framing anomaly. If A and r as in Lemma 3.4, then the central charge is $\frac{3(r-2)}{r}$.

Given an extended surface $(Y; \lambda)$, the modular functor $V(Y; \lambda)$ is defined in Section 3.14. If $\partial X = Y$, then we define $Z(X)$ as the skein class in $K_A(\partial X)$ represented by the empty skein. TQFT axioms for V and Z follow from theorems in Section 3.14. The non-trivial part is the mapping class group action. This is explained at the end of Section 3.14.

8. WRT and Turaev-Viro $SU(2)$-TQFTs

The pictorial approach to the Witten-Reshetikhin-Turaev $SU(2)$ TQFTs was based on [KM]. The paper [KM] finished with 3-manifold invariants, just as [KL] for the Jones-Kauffman theories. The paper [BHMV] took the picture approach in [KL] one step further to TQFTs, but the same for WRT TQFTs has not been done using a pictorial approach. The reasons might be either people believe that this has been done by [BHMV] or realize that the Frobennius-Scur indicators make a picture approach more involved. It is also widely believed that the two approaches resulted in the same theories. But they are different. The spin $1/2$ representation of quantum group $SU(2)_q$ for $q = e^{\pm 2\pi i/r}$ has a Frobenius-Schur indicator$=-1$, whereas the corresponding label 1 in Temperley-Lieb-Jones theories has Frobenius-Schur indicator $= 1$. The Frobenus-Schur indicators -1 in the Witten-Reshetikhin-Turaev theories introduce some -1's into the S-matrix, hence for the odd levels k, these -1's change the S-matrix from singular in the Jones-Kauffman theories when $A = \pm ie^{\pm \frac{2\pi i}{4r}}$ to non-singular. For even levels k, the S-matrices are the same as those of the Jones-Kauffman TQFTs, even though the TQFTs are different theories (see [RSW] for the level$=2$ case).

In the pictorial TLJ approach to TQFTs, there is no room to encode the Frobenius-Schur indicators -1. In this section, we introduce "flag" decorations on each component of a framed multicurve which can point to either side of the component. These flags allow us to encode the FS indicator -1, hence reproduce the Witten-Reshetikhin-Turaev $SU(2)$ TQFTs exactly. The doubled theories of WRT TQFTs are not the diagram TQFTs, and will be called the Turaev-Viro $SU(2)$-TQFTs. They are direct products of WRT theories with their mirror theories.

8.1. **Flagged TLJ categories.** In flagged TLJ categories, the local relation is still the Jones-Wenzl projectors, but the skein relation is not the Kauffman bracket exactly, but a slight variation discovered by R. Kirby and P. Melvin in [KM].

The skein relation for resolving a crossing p is given in [KM] is as follows: if the two strands of the crossing belongs to two different components of the link, then the resolution is the Kauffman bracket in Figure 1; but if the two strands of the crossing p are from the same component, then a sign $\epsilon(p) = \pm 1$ is well-defined, and the skein relation is:

The flagged TLJ categories have objects signed points in the interval and morphism flagged multicurves as follows: given an oriented surface Y, and a multicurve

FIGURE 31. Kirby-Melvin skein relation

γ in the interior of Y, and no critical points of γ are within small neighborhoods of ∂Y. Let $\gamma \times [-\epsilon, \epsilon]$ be a small annulus neighborhood of γ. A flag of γ at $p \in \gamma$ is an arc $p \times [0, \epsilon]$ or $p \times [-\epsilon, 0]$. A flag is admissible if p is not a critical point of γ. A multicurve γ is flagged if all flags on γ are admissible and the number of flags has the same parity as the number of critical points of γ. An admissible flag on γ can be parallel transported on γ so that when the flag passes through a critical point, it flips to the other side. In the plane, this is the same as parallel transport by keeping the flag parallel at all times in the plane. A multicurve is flagged if all its components are flagged.

Given a surface Y with signed points on the boundary. Each signed point is flagged so that if the sign is $+$, the flag agrees with the induced orientation of the boundary; if the sign is $-$, the flag is opposite to the induced orientation. Let $\mathbb{C}[\mathbb{S}]$ be the space of all formal flagged multicurves in $\mathcal{R}$ with signed points at the bottom and top, then the morphism set between the bottom signed point and the top signed point of $\mathrm{TL}^{\mathrm{flag}}$ is the quotient space of $\mathbb{C}[\mathbb{S}]$ such that

   (1) Flags can be parallel transported
   (2) Flipping a flag to the other side results in a minus sign
   (3) Two neighboring flags can be cancelled if there are no critical points between them and they are on the opposite sides.
   (4) Apply Jones-Wenzl projector to any part of an multicurve with no flags.

Then all discussions for TL apply to $\mathrm{TL}^{\mathrm{flag}}$. The representation category is similarly given by the same Jones-Wenzl projectors. The biggest difference from TL is the resulting framed link invariant.

Lemma 8.1. *Given a framed link diagram D, then the WRT invariant $< D >_{KM}$ of D using Kirby-Melvin skein relation and the Jones-Kauffman invariant $< D >_K$ using Kauffman bracket is related by:*

$$(8.2) \qquad\qquad < D >_{KM} (A) = (-i)^{D \cdot D} < D >_K (iA).$$

8.2. **Turaev-Viro Unitary TQFTs.** Fix $A = \pm e^{\pm \frac{2\pi i}{4r}}$ for some $r \geq 3$.

The label set is the same as that of the corresponding diagram TQFT, but for the first time we need to work with the refined label set.

Given a surface Y with boundaries labeled by refined labels $\epsilon_i V_i$. If $\epsilon_i = 1$, we flag the point to the orientation of ∂Y; if $\epsilon_i = -1$, we flag the points opposite

to the orientation of ∂Y. Then define the modular functor space analogous to the skein space replacing multicurves with flagged multicurves. The theories are similar enough so we will leave the details to interested readers. The difference is that when the level=k is odd, our version of the Turaev-Viro theory is a direct product-a trivial quantum double, while the corresponding diagram TQFT is a non-trivial quantum double.

8.3. **WRT Unitary TQFTs.** Fix $A = \pm e^{\pm \frac{2\pi i}{4r}}$ for some $r \geq 3$.

The label set of a WRT TQFT is the same as that of the corresponding Jones-Kauffman TQFT, but it needs to be extended to the refined label set. The central charge of a level=k theory is $\frac{3k}{k+2}$. The discussion together with the Turaev-Viro theories is completely parallel to the Jones-Kauffman TQFTs with diagram TQFTs.

9. BLACK-WHITE TQFTS

Interesting variations of the TLJ categories can also be obtained by 2-colorings: the black-white annular categories TLJ_d^{BW}. The objects of the category are the objects of the corresponding annular TLJ category enhanced by two colorings of the complements of the points. In particular there are two circles: black and white. Morphisms between two objects are enhanced by colorings of the regions. A priori there are two enhancements of each Jones-Wenzl idempotent, but it has been proved in [Fn] that the two versions are equivalent.

9.1. **Black-white TLJ categories.** Fix some $r \geq 3$ and A, where A is a primitve $4r$th root of unity, or a primitive $2r$th root of unity and r odd, or a primitive rth root of unity and r odd

The objects of black-white TLJ categories are points in the interval or S^1 with a particular 2-coloring of the complementary intervals so that adjacent intervals having different colors. Given two objects, morphisms are multicurves from the bottom to top whose complement regions have black-white colors that are compatible with the objects, and any two neighboring regions receive different colors. The local relation is the 2-color enhanced Jones-Wenzl projector and the skein relation is the 2-color enhancement of the Kauffman bracket. The representation theories of the black-white categories are considerably harder to analyze.

The object with no points in the circle has two versions $0_B, 0_W$, which might be isomorphic. Indeed sometimes they are isomorphic and sometimes not. Therefore a skeleton of a black-white TLJ category can be identified with $\{0_B, 0_W, 2, 4, \cdots\}$ with the possibility that $0_B = 0_W$. We will draw the black object 0_b as a bold solid circle, and the white object as a dotted circle. Interface circles between black and white regions will be drawn as regular solid circles. Morphisms will be drawn inside annuli, directed and composed from inside-out. There are two color changing morphisms $r_{bw} \in \text{Hom}(0_b, 0_w), r_{wb} \in \text{Hom}(0_w, 0_b)$.

ρ_3	0	0	1
ρ_2	1	0	1
ρ_1	0	1	1
ρ_0	1	1	1
	0_b	0_w	2

TABLE 3. Irreps of black-white level=2

ρ_3	0	0	1
ρ_2	0	0	1
ρ_1	1	1	2
ρ_0	1	1	1
	0_b	0_w	2

TABLE 4. Irreps of black-white level=3

Let us denote the two compositions $r_{bw} \cdot r_{wb} = x_b \in \mathrm{Hom}(0_b, 0_b)$, $r_{wb} \cdot r_{wb} = x_w \in \mathrm{Hom}(0_w, 0_w)$, which are just rings in the annulus.

Given an oriented closed surface Y, a 2-colored multicurve in Y is a pair (γ, c), where γ is a multicurve, and c is an assignment of black or white to all regions of $Y \backslash \gamma$ so that any two neighboring regions have opposite colors. Let $\mathbb{C}[\mathcal{S}]$ be the vector space of formal 2-colored multicurves, and $\mathrm{Pic}^{\mathrm{BW}}(Y)$ be the quotient space of $\mathbb{C}[\mathcal{S}]$ modulo the BW-enhancement of JW projectors.

9.2. **Labels for black-white theories.** Recall in Section 5.1, we define the element $q_{2m} \in C_{2m}(x)$.

Lemma 9.1. *The element q_{2m} is a minimal idempotent of $C_{2m}(x)$.*

Proposition 9.2. *(1): If r is even, then x_b, x_w are not invertible, hence 0_b is not isomorphic to 0_w.*
   (2): If r is odd, then x_b, x_w are invertible, hence 0_b and 0_w are isomorphic.
   (3): The color swap involution is the identity on the TQFT vector spaces.

9.2.1. *Level=2, $d^2 = 2$.* The algebra $A_{0_b 0_b} \cong \mathbb{C}^2$, and so is the algebra $A_{0_w 0_w}$. Both are generated by x, so $\cong \mathbb{C}[x]/(x^2 = 2x)$.
   $\mathrm{Hom}(0_b, 0_w) \cong \mathbb{C}$ is generated by r_{bw}. Similarly, $\mathrm{Hom}(0_w, 0_b) \cong \mathbb{C}$ is generated by r_{wb}.
   $A_{2,2} \cong \mathbb{C}^4$. Following the same analysis as in Section 5, we get the irreps denoted by the following:

9.2.2. *Level=3.* The algebra $A_{0_B, 0_B} \cong \mathbb{C}^2$ is generated by x so $\cong \mathbb{C}[x]/(x^2 - 3x + 1)$.
   The algebra $A_{22} \cong \mathbb{C}^7$. Similar analysis as above leads to:

9.3. **BW TQFTs.**

Theorem 9.3. *(1): If $r \geq 3$, and A a primitive $4r$th root of unity, or a primitive $2r$th root of unity and r odd, or a primitive rth root of unity and r odd, then (V_{BW}^A, Z_{BW}^A) is a TQFT.*
(2): If r odd, then (V_{BW}^A, Z_{BW}^A) is isomorphic to the doubled even TLJ subcategory TQFT, i.e., the TQFT from the quantum double of the even TLJ subcategory at the corresponding A.

The proof of this theorem and the irreps for all r are left to a future publication.

We have not be able to identify the BW TQFTs with known ones when r is even, and A is a primitive $4r$th root of unity. If $r = 4$, then (V_{BW}^A, Z_{BW}^A) is isomorphic to the toric code TQFT. We conjecture Theorem 9.3 (2) still holds for these cases. Furthermore, each (V_{BW}^A, Z_{BW}^A) decomposes into a direct product of the toric code TQFT with another TQFT.

10. Classification and Unitarity

In this section, we classify all TQFTs based on Jones-Wenzl projectors and Kauffman brackets. Then we decide when the resulting TQFT is unitary. In literature A has been chosen to be either as a primitive $4r$-th root of unity or as a primitive $2r$th root of unity. We notice that for r odd, when A is a primitive rth root of unity, the resulting TLJ rectangular categories give rise to ribbon tensor categories with singular S-matrices, but their annular versions lead to TQFTs which are potentially new. Also when A is a primitive $4r$th root of unity and r even, the BW TQFTs seem to contain new theories.

10.1. **Classification of diagram local relations.** By d generic we mean that d is not a root of some Chebyshev polynomial $\triangle_i$. Equivalently $d \neq B + \bar{B}$ for some B such that $B^\ell = 1$.

Let us consider d-isotopy classes of multicurves on a closed surface Y. Call this vector space $\mathrm{TL}_d(Y)$. This vector space has the subtle structure of gluing formula associated to cutting into subsurfaces (and then regluing); there is a product analogous to both times and tensor products in TLJ_d. Also for special values of d $\mathrm{TL}_d(Y)$ has a natural singular Hermitian structure.

Theorem 10.1. *If d has the form: $d = -A^2 - A^{-2}$, A a root of unity. Then there is a (single) local relation $R(d)$ so that $\mathrm{TL}_d(Y)$ modulo $R(d)$, denoted by $V_d(Y)$, have finite nonzero dimension. If d is not of the above form then $V_d(Y) = 0$ or $= \mathrm{TL}_d(Y)$ for any given $R(d)$. Furthermore the quotient space $V_d(Y)$ of $\mathrm{TL}_d(Y)$ when it is neither $\{0\}$ nor $\mathrm{TL}_d(Y)$ is uniquely determined, and when A is a primitive $4r$th root of unity, then $V_d(Y)$ is the "Drinfeld double" of a Jones-Kauffman TQFT at level $k = r - 2$.*
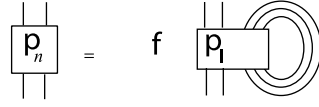
FIGURE 32. Partial trace

Proof. Consider a local relation $R_0(d)$ of smallest degree, say $2n$, which holds in TL_d (i.e. is a consequence of $R(d)$). Arbitrarily draw $R_0(d)$ in a rectangle with n endpoints assigned to the top and n endpoints assigned to the bottom, to place $R_0(d)$ in the algebra $\mathrm{TL}_n(d)$. Adding any cup or cap to R_0 gives a consequent relation of degree $= 2n - 2$; this relation, by minimality, must be zero. This implies that $e_i R_0(d) = R_0(d) e_i = 0$ for $1 \leq i \leq n - 1$. So by Proposition 2.15, $R_0(d) = c p_{n,d}$, c a nonzero scalar.

The trace, $\mathrm{tr}(p_{n,d}) \in \mathbb{C}$ is a degree $= 0$ consequence of $p_{n,d}$ so unless d is a root of $\triangle_n$, $\mathrm{tr}(p_{n,d}) \neq 0$ and so generates all relations: $p_{n,d}(Y) = 0$.

Now suppose d is the root of two Chebyshev polynomials $\triangle_m$ and $\triangle_\ell$, $m < \ell$. This happens exactly when $(m + 1)$ divides $(\ell + 1)$. In fact to understand the roots of $\triangle_n(d)$ introduces a change of variables $d = B + B^{-1}$, then: $\triangle_n(d) = (B^{n+1} - B^{-n-1})/(B - B^{-1})$. The r.h.s. vanishes (simply) when (and only when) B is a $2n + 2-$ root of unity $\neq \pm 1$. In particular if d is a root of $\triangle_m$ and $\triangle_\ell$ then $p_{m,d}$ is a consequence of $p_{m,\ell}$ by "partial trace" as shown in Figure 32.

Trace both sides to verify the coefficient $f = \frac{\triangle_m}{\triangle_\ell}$ and note that since both numerator and denominator have simple roots at d the coefficient at d is well defined and nonzero.

Thus for a diagram local relation (or set thereof) to yield a nontrivial set of quotient space $\neq 0$, d must be a root of lowest degree $\triangle_n$ and the relation(s) are equivalent to the single relation $p_{n,d}$.

Geometrically, $0 = p_n = 1 + U$ implies $1 = -U$ means that the multicurves whose multiplicity is less than n along the $1-$ cells, $SK^1(Y)$, ("bonds" in physical language) of any fixed triangulation of Y determine $p_n(Y)$. Here multiplicity $\leq n$ for a multicurve γ means that γ runs near $SK^1(Y)$ and with fewer than n parallel copies of a $1-$ cell (bond) of $SK^1(Y)$. Finite dimensionality of $p_n(Y)$ is an immediate consequence. (SK^1 stands for "1$-$skeleton.")

The quotient space $\neq 0$, for $Y = S^2$ this follows from the nonvanishing of certain θ-symbols; for Y of higher genus the Verlinde formulas. $\qquad\square$

10.2. **Unitary TQFTs.** A unitary modular functor is a modular functor such that each $V(Y)$ is endowed with a non-degenerate Hermitian pairing:

$$<,>: \overline{V(Y)} \times V(Y) \longrightarrow \mathbb{C},$$

and each morphism is unitary. The Hermitian structures are required to satisfy compatibility conditions as in the naturality axiom of a modular functor. In particular,

$$< \oplus_i v_i, \oplus_j w_j >= \sum_i s_{i0} < v_i, w_j > .$$

Note that this implies that all quantum dimensions of particles are positive reals. It might be true that any theory with all quantum dimensions positive is actually unitary. Moreover, the following diagram commutes for all Y:

$$
\begin{array}{ccc}
V(Y) & \xrightarrow{\cong} & V(-Y)^* \\
\cong \downarrow & & \downarrow \cong \\
\overline{V(Y)^*} & \xrightarrow{\cong} & \overline{V(-Y)}
\end{array}
$$

A unitary TQFT is a TQFT whose modular functor is unitary and whose partition function satisfies $Z(-M) = \overline{Z(M)}$.

10.3. **Classification and unitarity.** There are two kinds of TQFTs that we studied in this paper: undoubled and doubled, which are indexed by the Kauffman variable A.

When A is a primitive $4r$th root of unity for $r \geq 3$, we have the Jones-Kauffman TQFTs. The even sub-categories of TLJs yield TQFTs for r odd, but have singular S-matrix if r even. If r is even, and $A = \pm i e^{\pm \frac{2\pi i}{4r}}$, then the Jones-Kauffman TQFTs are unitary.

We also have the WRT $SU(2)$-TQFTs for $q = e^{\pm \frac{2\pi i}{r}}$, which are unitary. WRT TQFTs were believed to be the same as the Jones-Kauffman TQFTs with $q = A^{\pm 4}$, but they are actually different. Jones-Kauffman TQFTs and WRT TQFTs are related by a version of Schur-Weyl duality as alluded in Section 2 for the braid group representations.

All above theories can be doubled to get picture TQFTs: the doubled Jones-Kauffman TQFTs are the diagram TQFTs, while the doubled WRT TQFTs are the Turaev-Viro TQFTs [TV]. The doubles of even sub-categories for r odd form part of the Black-White TQFTs in Theorem 9.3, while for r even this is still a conjecture.

When A is a primitive $2r$th or rth root of unity and r odd, the TLJ categories do not yield TQFTs. But the restrictions to the even labels lead to TQFTs. When $A = \pm i e^{\pm \frac{2\pi i}{4r}}$, the resulting TQFTs are unitary. Those unitary TQFTs are the same as those obtained from the restrictions of WRT TQFTs to integral spins. All can be doubled to picture TQFTs. Note that for these cases when A is a primitive rth root of unity, then $-A$ is a primitive $2r$th root of unity. For the even sub-categories, they lead to the same TQFTs, which form part of the Black-White TQFTs in Theorem 9.3.

<div align="right">□</div>

Appendix A. Topological phases of matter

Fractional quantum Hall liquids are new phases of matter exhibiting topological orders, and Chern-Simons theories are proposed as effective theories to describe the universal properties of such quantum liquids. Quantum Chern-Simons theories are (2+1)-dimensional topological quantum field theories (TQFTs), so we define a topological phase of matter as a quantum system with a TQFT effective theory.

A.1. **Ground states manifolds as modular functors.** While in real experiments, we will prefer to work with quantum systems in the plane, it is useful in theory to consider quantum systems on 2-dimensional surfaces such as the torus. Given a quantum system on a 2-dimensional oriented closed surface Σ, there associates a Hilbert space $\mathbb{H}$ consisting of all states of the system. The lowest energy states form the ground states manifold $V(\Sigma)$, which is a subspace of $\mathbb{H}$. For a given theory, the local physics of the quantum systems on different surfaces are the same, so there are relations among the ground states manifolds $V(\Sigma)$ for different Σ's dictated by the local physics. In a topological quantum system, the ground states manifolds form a modular functor—the 2-dimensional part of a TQFT. In particular, $V(\Sigma)$ depends only on the topological type of Σ.

A.2. **Elementary excitations as particles.** A topological quantum system has many salient features including an energy gap in the thermodynamic limit, ground states degeneracy and the lack of continuous evolutions for the ground states manifolds. The energy gap implies that elementary excitations are particle-like and particle statistics is well-defined. These quasi-particles are anyons, whose statistics are described by representations of the braid groups rather than representations of the permutation groups.

The mathematical model for an anyonic system is a ribbon category. In this model an anyon is pictured as a framed point in the plane: a small interval. Given a collection of n anyons in the plane, we will arbitrarily order them and place them onto the real axis, so we can represent them by intervals $[i - \epsilon, i + \epsilon], i = 1, 2, \cdots, n$ on the real axis for some small ϵ. The worldlines of any n anyons from time $t = 0$ to the same set of n anyons at time $t = 1$ form a *framed* braid in $\mathbb{R}^2 \times [0, 1]$. We will represent worldlines of anyons by diagrams of ribbons in the plane which are projections from $\mathbb{R}^2 \times [0, 1]$ to the real axis $\times [0, 1]$ with crossings. (Technically we need to perturb the worldlines in order to avoid more singular projections.) A further convention is the so-called blackboard framing: we will draw only single lines to represent ribbons with the understanding the ribbon is the parallel thickening of the lines in the plane.

Suppose n elementary excitations of a topological quantum system on a surface Σ are localized at points $p_1, p_2, \cdots, p_n$, by excising the particles from Σ, we have a topological quantum system on a punctured surface Σ' obtained from Σ by deleting a small disk around each point p_i. Then the ground states manifold of

the quantum system on Σ' form a Hilbert space $V(\Sigma')$. The resulting Hilbert space should depend only on the topological properties of the particles—particle types that will be referred to also as labels. In this way we assign Hilbert spaces $V(\Sigma, a_1, a_2, \cdots, a_n)$ to surfaces with boundary components $\{1, 2, \cdots, n\}$ labelled by $\{a_1, a_2, \cdots, a_n\}$.

A.3. **Braid statistics.** The energy gap protects the ground states manifold, and when two particles are exchanged adiabatically within the ground states manifolds, the wavefunctions are changed by a unitary transformation. Hence particle statistics can be defined as the resulting unitary representations of the braid groups.

APPENDIX B. REPRESENTATION OF LINEAR CATEGORY

Category theory is one of the most abstract branch of mathematics. It is extremely convenient to use category language to describe topological phases of matters. It remains to be seen whether or not this attempt will lead to useful physics. But tensor category theory might prove to be the right generalization of group theory for physics. On a superficial level, the two layers of structures in a category fit well with physics: objects in a category represent states, and morphisms between objects possible "physical processes" from one state to another. For quantum physics, the category will be linear so each morphism set is a vector space. Functors might be useful for the description of topological phase transitions and condensations of particles or string-nets. For more detailed introduction, consult the book [Ma].

A *category* $\mathcal{C}$ consists of a collection of objects, denoted by $a, b, c, \cdots$, and a morphism set $_a\mathcal{C}_b$ (also denoted by $\mathrm{Mor}(a, b)$) for each ordered pair (a, b) of objects which satisfy the following axioms:

Given $f \in {_a\mathcal{C}_b}$ and $g \in {_b\mathcal{C}_c}$, then there is a morphism $f \cdot g \in {_a\mathcal{C}_c}$ such that
1)(Associativity):

If $f \in {_a\mathcal{C}_b}, g \in {_b\mathcal{C}_c}, h \in {_c\mathcal{C}_d}$, then $(f \cdot g) \cdot h = f \cdot (g \cdot h)$.
2)(Identity):

For each object a, there is a morphism $\mathrm{id}_a \in {_a\mathcal{C}_a}$ such that for any $f \in {_a\mathcal{C}_b}$ and $g \in {_c\mathcal{C}_a}$, $\mathrm{id}_a \cdot f = f$ and $g \cdot \mathrm{id}_a = g$.

We denote the objects of $\mathcal{C}$ by $\mathcal{C}^0$ and write $a \in \mathcal{C}^0$ for an object of $\mathcal{C}$. We use $\mathcal{C}^1$ to denote the disjoint union of all the sets $_a\mathcal{C}_b$. The morphism $f \cdot g \in {_a\mathcal{C}_c}$ is usually called the *composition* of $f \in {_a\mathcal{C}_b}$ and $g \in {_b\mathcal{C}_c}$, but our notation $f \cdot g$ is different from the usual convention $g \cdot f$ as we imagine the composition as the join of two consecutive arrows rather than the composition of two functions. This convention is convenient when the composition in $_a\mathcal{C}_a$ of a linear category is regarded as a multiplication to turn $_a\mathcal{C}_a$ into an algebra.

A category $\mathcal{C}$ is a *linear category* if each morphism set $_a\mathcal{C}_b$ is a finitely dimensional vector space, and the composition of morphisms is a bilinear map of vector spaces.

It follows that for each object a, $_a\mathcal{C}_a$ is a finitely dimensional unital algebra. It follows that a finitely dimensional unital algebra can be regarded as a linear category with a single object. Another important linear category is the category of finitely dimensional vectors spaces $\mathcal{V}$. An object of $\mathcal{V}$ is a finitely dimensional vector space V. The morphism set $\text{Mor}(V, W)$ between two objects V, W is $\text{Hom}(V, W)$. More generally, given any finite set I, consider the linear category $\mathcal{V}[I]$ of I-graded vector spaces, which is a categorification of the group algebra $\mathbb{C}[G]$ if I is a finite group G. An object of $\mathcal{V}[I]$ is a collect of finitely dimensional vector spaces $\{V_i\}_{i \in I}$ labelled by elements of I, and the morphism set $\text{Mor}(\{V_i\}_{i \in I}, \{W_j\}_{j \in I})$ is the (graded) vector space of linear maps $\oplus_{i \in I} \text{Hom}(V_i, W_i)$. In the following all categories will be linear categories, and we will see that any semisimple linear category with finitely many irreducible representations is isomorphic to a category of a finite set graded vector spaces.

B.1. **General representation theory.**

Definition B.1. A (right) representation of a linear category $\mathcal{C}$ is a functor $\rho : \mathcal{C} \to \mathcal{V}$, where $\mathcal{V}$ is the category of finitely dimensional vector spaces. The action is written on the right: $\rho(a) = V_a$ and given an $f \in {}_a\mathcal{C}_b, v.\rho(f) = v.f = v \cdot \rho(f) : V_a \to V_b$ for any $v \in V_a$.

The 0-representation of a category is the representation which sends every object to the 0-vector space. Fix an object $a \in \mathcal{C}^0$, we have a representation of the category $\mathcal{C}$, denoted by $a\mathcal{C}$: the representation sends a to the vector space $_a\mathcal{C}_a$, and any other $b \in \mathcal{C}^0$ to $_a\mathcal{C}_b$. An important construct which gives rise to all the representations of a semi-simple linear category is as follows: fix an object $a \in \mathcal{C}^0$ and a right ideal J_a of the algebra $_a\mathcal{C}_a$, then the map which sends each object $b \in \mathcal{C}^0$ to $J_a \cdot {}_a\mathcal{C}_b \subseteq {}_a\mathcal{C}_b$ affords $\mathcal{C}$ a representation, where $J_a \cdot {}_a\mathcal{C}_b$ is the subspace of $_a\mathcal{C}_b$ generated by all elements $f \cdot g, f \in J_a \subseteq {}_a\mathcal{C}_a, g \in {}_a\mathcal{C}_b$. If the right ideal J_a is generated by an element $p_a \in {}_a\mathcal{C}_a$, then the resulting representation of $\mathcal{C}$ will be denoted by $p_a\mathcal{C}$. In particular if $J_a = {}_a\mathcal{C}_a$, we will have the regular representation $a\mathcal{C}$.

The technical part of the paper will be the analysis of the representations of certain picture categories. In order to do this, we first recall the representation theory for an algebra—a linear category with a single object.

Definition B.2. Let A be an algebra, an element $e \in A$ is an idempotent if $e^2 = e \neq 0$. Two idempotents e_1, e_2 are orthogonal if $e_1 e_2 = e_2 e_1 = 0$. An idempotent is minimal if it is not the sum of two orthogonal idempotents.

Given an idempotent e of a finitely dimensional semi-simple algebra A, the right ideal eA is an irreducible representation of A if and only if the idempotent e is minimal. Since every irreducible right representation of A is isomorphic to a right ideal eA for some idempotent of A, the representations of A are completely known

once we find a collection of pairwise orthogonal minimal idempotents e_i of A such that $1 = \oplus_i e_i$. It follows that $A = \oplus_{i=1}^n e_i A$.

Let $p(x)$ be a polynomial of degree=n with n distinct roots $a_1, a_2, \cdots, a_n$ and A be the quotient algebra $\mathbb{C}[x]/(p(x))$ of the polynomial algebra $\mathbb{C}[x]$. Let $u_j = \prod_{i=1,i\neq j}^n (x - a_j), \lambda_j = \prod_{i=1,i\neq j}^n (\lambda_j - \lambda_i)$ and $e_j = \frac{u_j}{\lambda_j}$. Then we have the following lemma.

Lemma B.3. *The idempotents $\{e_j\}_{j=1}^n$ of A are pair-wise orthogonal and $\oplus_{j=1}^n e_j = 1$. It follows that A is semi-simple and a direct sums of $\mathbb{C}$'s. Note that e_j is an eigenvector of the element $x \in A$ associated with the eigenvalue a_j.*

Proof. Since $u_j(x - a_j) = p(x) = 0$, so $u_j \cdot x = u_j \cdot a_j = a_j u_j$. It follows that $u_j^2 = u_j \prod_{i=1,i\neq j}^n (x - a_i) = \lambda_j u_j$, therefore $e_j^2 = e_j \neq 0$. Now consider $u_i \cdot u_j$, in u_j there is the factor $(x - a_i)$ if $i \neq j$, but $u_i(x - a_i) = p(x) = 0$, hence $u_i u_j = 0$.

The polynomial $g(x) = (\sum_{j=1}^n e_j) - 1$ is a polynomial of degree $n - 1$, but it has n distinct roots $a_1, a_2, \cdots, a_n$, so $g(x)$ is identically 0. $\qquad\square$

A representation ρ of $\mathcal{C}$ is *reducible* if ρ is the direct sum of two non-zero representations of $\mathcal{C}$. Otherwise ρ is *irreducible*. A linear category $\mathcal{C}$ is *semi-simple* if every representation $\{\rho, \mathcal{V}\}$ of $\mathcal{C}$ is a direct sum of irreducible representations.

Definition B.4. Λ has a positive definite Hermitian inner product (pdhi) iff each morphism set ${}_a\Lambda_b$ has a finite dimensional pdhi and composition ${}_a\Lambda_b \otimes {}_b\Lambda_c \xrightarrow{P} {}_a\Lambda_c$ satisfies the compatibility $< P({}_a m_b \otimes {}_b m_c), {}_c m_d > = < {}_a m_b, P({}_b m_c \otimes {}_c m_d) >, {}_i m_j \in {}_i\Lambda_j$, and for all $i, j \in \Lambda^0, {}_i\Lambda_j$ is identified with $\overline{{}_j\Lambda_i}$.

Lemma B.5. *Suppose Λ has positive definite Hermitian inner product, then A is semi-simple.*

Proof. If Λ has a pdhi, then any (finite dimensional) representation $\{\rho, V\}$ of Λ may also be given a pdhi structure. This means that the V_i are individually pdhi-spaces and that for all morphisms $m, (\rho(m))^\dagger = \rho(\bar{m})$. One may check that any collections of pdhi-structures on $\{V\}$ which are averaged under the invertible morphisms (and therefore invariant) satisfies this condition. $\qquad\square$

Most of the usual machinery of linear algebra, including Schur's lemma, holds for $\mathbb{C}$-linear categories.

Lemma B.6. (Schur's Lemma for $\mathbb{C}$-linear categories) *Suppose $\{\rho_m, V_i\}$ and $\{\chi_m, W_i\}, i \in obj(\Lambda), m \in Morph(i, j)$, are irreducible representations of a $\mathbb{C}$-linear category Λ (called an algebroid by some authors e.g. [BHMV]). Irreducibility means no ρ_m invariant class of proper subspaces $V_i' \subset V_i$ exists. Suppose that $\phi : \{V\} \rightarrow \{W\}$ is a Λ-map commuting with the action Λ. That is for $m \in Morph(i, j)$ and $v_i \in V_i$ we have $\chi_m(\phi_i(v_i)) = \phi_j \cdot \rho_m(v_i)$. Then either ϕ is identically zero for all $i, \phi_i : V_i \rightarrow W_i$, or ϕ is an isomorphism. If $\{V\} = \{W\}$ then $\phi = \lambda \cdot id$ for some $\lambda \in \mathbb{C}$.*

Proof. As in the algebra case $\ker(\phi)$ (and $\text{image}(\phi)$) are both invariant families of subspaces (indexed by $i \in \text{obj}(\Lambda)$.) So if either is a nontrivial proper subspace for any i irreducibility of $\{V\}$ (or $\{W\}$) fails. For the second assertion, since $\mathbb{C}$ is algebraically closed for any $i \in \text{obj}(\Lambda)$, the characteristic equation $\det(\phi_i - x\mathrm{I}) = 0$, has roots, call one λ_i. The Λ-map $\phi - \lambda(I)$ has non-zero kernel (at least at object i) so by part one, $\phi - \lambda I = 0$ identically or $\phi = \lambda I$. □

Corollary B.7. *Suppose the representation $\{\rho, V\}$ of Λ has decomposition $\{V\} = V_{a_1} \otimes \{V_1\} \oplus \cdots \oplus V_{a_k} \otimes \{V_k\}$, where the V_l are distinct (up to isomorphism) irreducible representations, l is finite index, $1 \le l \le k$, and the V_{a_l} are ordinary $\mathbb{C}$-vector spaces with no Λ-action, (Dimension $(V_{a_l}) =: d(a_l)$ is the multiplicity of V_l.) The decomposition is unique up to permutation and of course isomorphism of V_{a_l} and scalars acting on $\{V_l\}$.*

Proof. Suppose $\{V\} = \bigoplus_m W_{a_m} \otimes \{W_j\}$. Apply Schur's lemma to compositions:

$$v_{a_l} \bigotimes \{V_l\} \to \{V\} \to w_{a_m} \bigotimes \{W_m\}.$$

for all $v_{a_l} \in V_{a_l}$ and $w_{a_l} \in W_{a_l}$ to conclude that given l, $V_l \cong W_m$ for some m and $d(a_l) = d(a_m)$. This established uniqueness. □

Now we state a structure theorem from [Wal2] for the representation theory of semisimple linear categories. Both the statement and the proof are analogous to those for the semi-simple algebras. A *right ideal* of $\mathcal{C}$ is a subset J of $\mathcal{C}^1$ such that for each object $a \in \mathcal{C}^0$, $J \cap {}_a\mathcal{C}_a$ is a right ideal of ${}_a\mathcal{C}_a$. Note that each right ideal of $\mathcal{C}$ affords $\mathcal{C}$ a representation.

Theorem B.8. *1): Let $\mathcal{C}$ be a semi-simple linear category, and $\{X_i\}_{i \in I}$ be a complete set of representatives for the simple right ideals of $\mathcal{C}$. Then $\mathcal{C}$ is naturally isomorphic to the category of the finite set I-graded vector spaces with each object $a \in \mathcal{C}^0$ corresponding to the graded vector space X_{ia}, where X_{ia} is $X_i \cap {}_a\mathcal{C}_a$.*

2): Each irreducible representation ρ of $\mathcal{C}$ is given by a right ideal of the form $e_a\mathcal{C}$ for some object $a \in \mathcal{C}^0$, where e_a is a minimal idempotent of ${}_a\mathcal{C}_a$. If for some $b \in \mathcal{C}^0$, which may be a, and e_b is a minimal idempotent of ${}_b\mathcal{C}_b$, then the irrep $e_b\mathcal{C}$ of $\mathcal{C}$ is isomorphic to $e_a\mathcal{C}$ if and only if there exist $f \in {}_a\mathcal{C}_b$ and $g \in {}_b\mathcal{C}_a$ such that $f \cdot g = e_a$, $g \cdot f = e_b$.

Appendix C. Gluing and Maslov index

C.1. **Gluing.** Gluing of 3-manifolds needs to be addressed carefully due to anomaly. The basic problem is when X is a bordism, the canonical Lagrangian subspace $\lambda_X \in H_1(\partial X; \mathbb{R})$ is in general not a direct sum. λ_X is determined by the intrinsic topology as it is the kernel of the inclusion homomorphism: $H_1(\partial X; \mathbb{R}) \to H_1(X; \mathbb{R})$. But the anomaly is related to the parameterizations of the bordisms, which are extrinsic.

Suppose $X_i, i = 1, 2$ are bordisms from $-Y_i$ to Y_{i+1} extended by $\lambda_j, j = 1, 2, 3$. The canonical Lagrangian subspace λ_{X_i} defines a Lagrangian subspace of $H_1(Y_2; \mathbb{R})$ as follows: let $\lambda_- X_1 = \{b \in H_1(Y_2; \mathbb{R}) | \text{for some} \quad a \in \lambda_1, (a, b) \in \lambda_{X_1}\}$, and $\lambda_+ X_2 = \{c \in H_1(Y_2; \mathbb{R}) | \text{for some} \quad d \in \lambda_3, (c, d) \in \lambda_{X_2}\}$. Then we have three Lagrangian subspaces in $H_1(Y_2; \mathbb{R})$ together with λ_2.

More generally, let (Y_i, λ_i) be extended sub-surfaces of $(\partial X; \lambda_X)$, and $f : (Y_1; \lambda_1) \to (-Y_2; \lambda_2)$ be a gluing map. Then we have three Lagrangian subspaces in $H_1(Y_1; \mathbb{R}) \oplus H_1(Y_2; \mathbb{R})$: the direct sum $\lambda_1 \oplus \lambda_2$, the anti-diagonal $\Delta = \{(x \oplus -f_*(x)\}$, and K— the complement of λ_i in λ_X mapped here.

C.2. **Maslov index.** Given three isotropic subspaces $\lambda_i, i = 1, 2, 3$ of a symplectic vector space (H, ω), we can define a symmetric bilinear form $<, >$ on $(\lambda_1 + \lambda_2) \cap \lambda_3$ as follows: for any $v, w \in (\lambda_1 + \lambda_2) \cap \lambda_3$, write $v = v_1 + v_2, v_i \in \lambda_i$, then set $< v, w >= \omega(v_2, w)$. The Maslov index $\mu(\lambda_1, \lambda_2, \lambda_3)$ is the signature of the symmetric bilinear form $<, >$ on $(\lambda_1 + \lambda_2) \cap \lambda_3$.

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