

# A Short Guide to Anyons and Modular Functors

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To the working physicist, anyon theory is meant to describe certain quasi-particle excitations occurring in two dimensional topologically ordered systems. A typical calculation using this theory will involve operations such as  $\otimes$  to combine anyons,  $F_d^{abc}$  to re-associate such combinations, and  $R_c^{ab}$  to commute or braid these anyons. Although there is a powerful string-diagram notation that greatly assists these manipulations, we still appear to be operating on particles arranged on a one-dimensional line, algebraically ordered from left to right. The obvious question is, where is the other dimension? The topological framework for considering these anyons as truly living in a two dimensional space is known as a modular functor, or topological quantum field theory. In this work we show how the apparently one-dimensional algebraic anyon theory is secretly the theory of anyons living in a fully two-dimensional system. The mathematical literature covering this secret is vast, and we try to distill this down into something more manageable.

In this work we describe the theory of two-dimensional topologically ordered systems with anyonic excitations. There are two main approaches to defining these, one being more algebraic and the other more topological.

The algebraic side is known to mathematicians as the study of braided fusion tensor categories, or more specifically, modular tensor categories. This algebraic language appears to be more commonly used in the physics literature, such as the well cited Appendix E of Kitaev [1]. Such algebraic calculations can be interpreted as manipulations of string diagrams [2], or *skeins*. These strings encode connections in the algebra, for example the Einstein summation convention. But they also faithfully encode the twisting that occurs when anyons are re-ordered or braided around each other. And these kinds of spatial relationships are fruitfully studied using topological methods.

Working from the other direction, one starts with a topological space (of low dimension) and attempts to extract a combinatorial or algebraic description of how this space can be built from joining smaller (simpler) pieces together. These topologically rooted constructs are known as modular functors, or the closely related topological quantum field theories (TQFT's).

That these two approaches – algebraic versus topological – meet is one of the great surprises of modern mathematics and physics.

Modular functors can be constructed from skein theory. In the physics literature, this appears as skeins growing out of manifolds [3], or as motivated by renormalization group considerations [4]. A further physical motivation is this: if a skein is supposed to correspond to the  $(2 + 1)$ -dimensional world-lines of particles as in the Schrödinger picture, modular functors would correspond to the algebra of observables, as in a Heisenberg picture.

In the physics literature, modular functors are explicitly used in Refs. [5, 6]. Also, Refs. [7] and [8] use the language of modular functors but they call them TQFT's. This is in fact reasonable because a modular functor can be seen as part of a TQFT, but is quite confusing to the novice who attempts to delve into the mathematical literature.

The definition of a modular functor appears to be well motivated physically. Unfortunately, there are many such definitions in the mathematical literature [9–12]. According to [13] section 1.2 and 1.3, there are several open questions involved in rigorously establishing the connection between these different axiomatizations. In particular, what physicists call anyon theory, and mathematicians call a modular tensor category, has not been established to correspond exactly (bijectively) to any of these modular functor variants. We try not to concern ourselves too much with these details, but merely note these facts as a warning to the reader who may go searching for the “one true formulation” of topological quantum field theory.

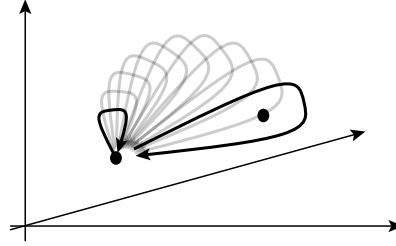
Of the many variants of modular functor found in the literature, one important distinction to be made is the way anyons are labeled. In the mathematical works [10–12] we see that anyons are allowed to have superpositions of charge states. However, in this work we restrict anyons to have definite charge states, as in [6, 7, 9]. This seems to be motivated physically as such configurations would be more stable.

The main goal of the present work is to sketch how a braided fusion tensor category arises from a modular functor. In the mathematical literature, this is covered in [10–12] but as we just noted they use a different formulation for a modular functor.

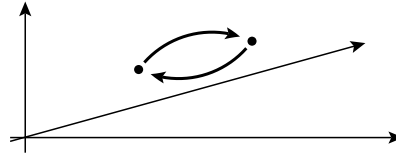
### A. Topological Exchange Statistics

In this section we begin with the familiar question of particle exchange statistics in three dimensions, whose answer is bosons and fermions. We then show how in restricting the particles to two dimensions many more possibilities arise. Our focus will be on the close connection between the algebraic and the topological viewpoints, aiming to motivate the definition of a modular functor given in the next section.

In three spatial dimensions, the process of winding one particle around another, a *monodromy*, is topologically trivial. This is because the path can be deformed back to the identity; there is no obstruction:



The square root of this operation is a swap:



For identical particles this is a *symmetry* of the system. Continuing with this line of thought leads to consideration of the *symmetric group* on  $n$  letters,  $S_n$ . This group is generated by the  $n - 1$  swap operations  $s_1, \dots, s_{n-1}$  that obey the relations

$$\begin{aligned} s_i^2 &= 1, \\ s_i s_j &= s_j s_i \quad \text{for } |i - j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad \text{for } 1 \leq i \leq n - 2. \end{aligned}$$

Writing the Hilbert space of the system as  $V$  we would then expect  $S_n$  to act on this space via unitary transformations  $U(V)$ :

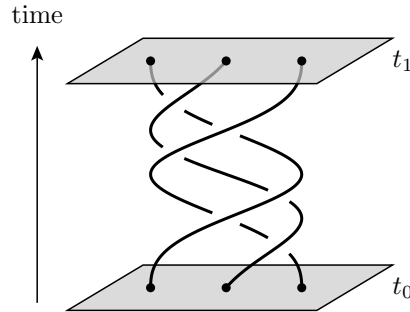
$$\mathcal{H} : S_n \rightarrow U(V).$$

This is the first example of the kind of functor we will be talking about. In this case it is a group representation;  $\mathcal{H}$  is a homomorphism between two groups.

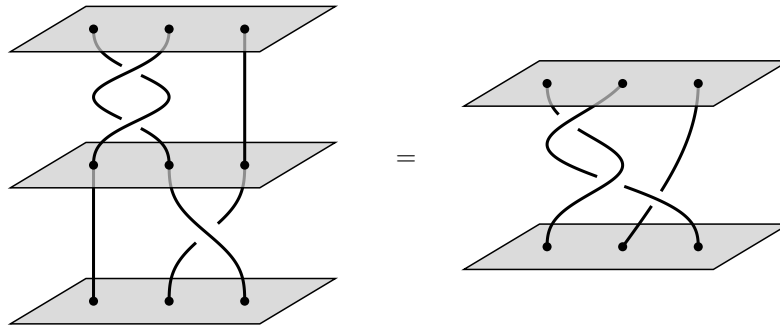
When the above monodromy is constrained to two dimensions we can no longer deform this process to the identity:



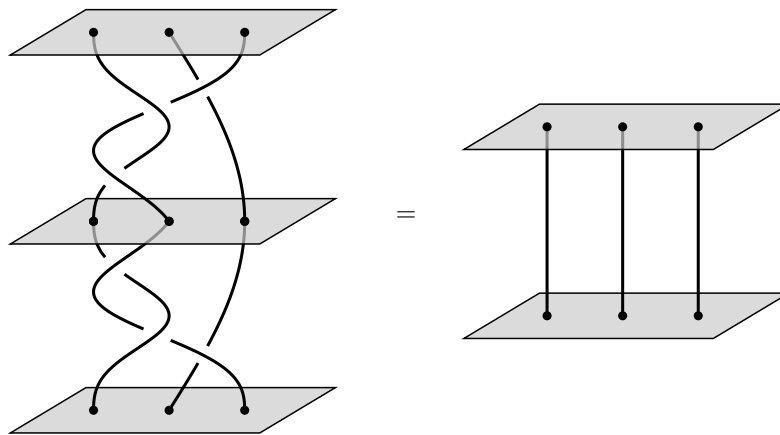
and so in two dimensions we cannot expect a swap to square to the identity. To see this more clearly, we must examine the entire (2+1)-dimensional *world-lines* of these particles. For an example, here we show the world-lines of three particles undergoing an exchange and then returning to their original positions:



Note that if we allow the particles to move in one extra dimension then we can untangle these braided world lines. The question is now, what is the group that acts on the state space from  $t_0$  to  $t_1$ ? Examining the structure of these processes more closely, we see that we can compose them by sequentially performing two such braids, one after the other:



And, by “reversing time”, we can undo the effect of any braid:



This shows that these processes do form a group, known as the *braid group*. For  $n$  particle world-lines we denote this group as  $B_n$ . For identical particles this group acts as symmetries of the state space:

$$\mathcal{H} : B_n \rightarrow U(V).$$

What we have given is a topological description of the group  $B_n$ . More formally, we can describe these braid world lines as paths in the *configuration space* of  $n$  points. This space is defined as the product of a two dimensional space  $M$  for each point, minus the subspace where points overlap:

$$\mathcal{C}_n = \left( \prod_1^n M \right) - \Delta, \quad \Delta = \{(x_1, \dots, x_n) | x_i = x_j \text{ for some } i \neq j\}.$$

Because we are considering identical particles (so far) we use the *unlabelled configuration space*  $\mathcal{UC}_n$  which is the quotient of  $\mathcal{C}_n$  by the natural action of the permutation group  $S_n$ :

$$\mathcal{UC}_n = \mathcal{C}_n / S_n.$$

The *geometric braid group* as we have so far informally described it can now be rigorously defined as the fundamental group:

$$B_n = \pi_1(\mathcal{UC}_n, x)$$

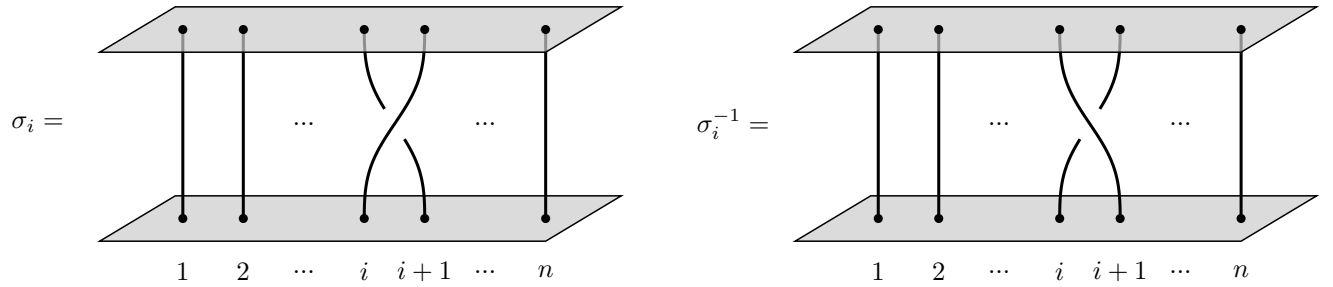
where  $x$  is some reference configuration in  $\mathcal{UC}_n$ . See Ref. [14] for further discussion.

This group also has a purely algebraic description via generators and relations, as was shown by Artin in 1947, [15, 16]. In this description,  $B_n$  is generated by  $n - 1$  elements  $\sigma_1, \dots, \sigma_{n-1}$  that satisfy the following relations:

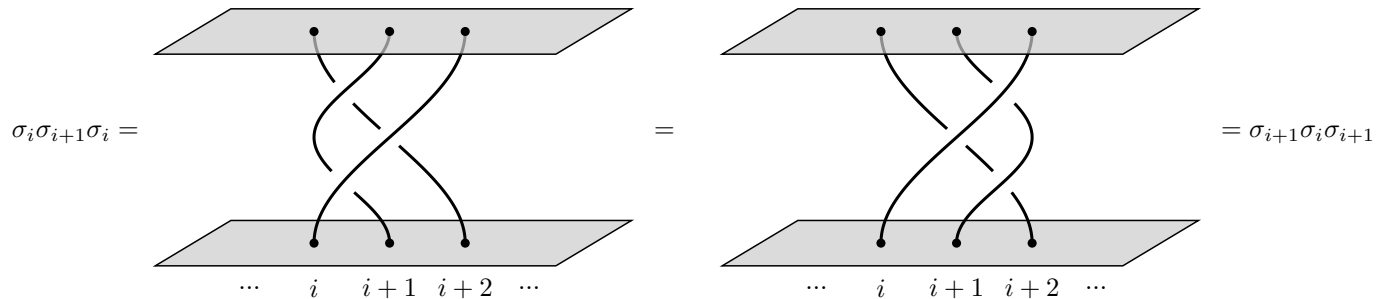
$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{for } |i - j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } 1 \leq i \leq n - 2. \end{aligned}$$

These relations are the same as for  $S_n$  above, except we do not require  $\sigma_i^2 = 1$ .

It is easy to see that the geometric braid group satisfies these relations. Here we show the braids corresponding to the generators  $\sigma_i$ :



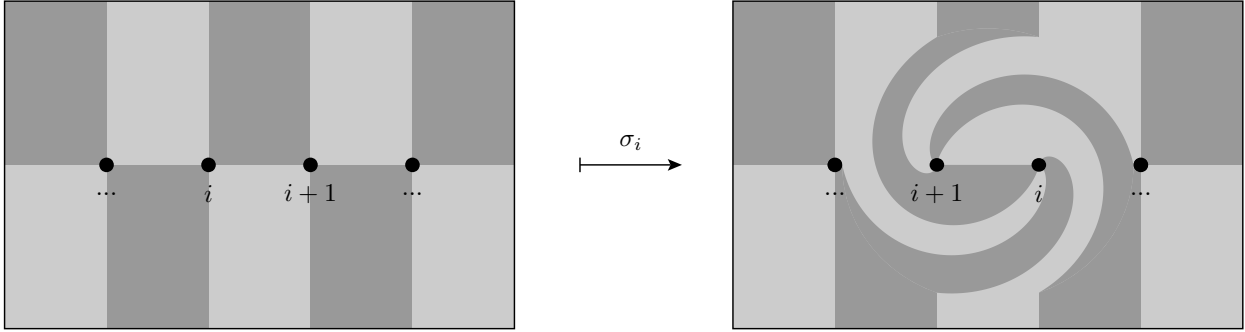
So that  $\sigma_i \sigma_j = \sigma_j \sigma_i$  for  $|i - j| > 1$  because the two braids are operating on disjoint world-lines. The second relation is also easy to see:



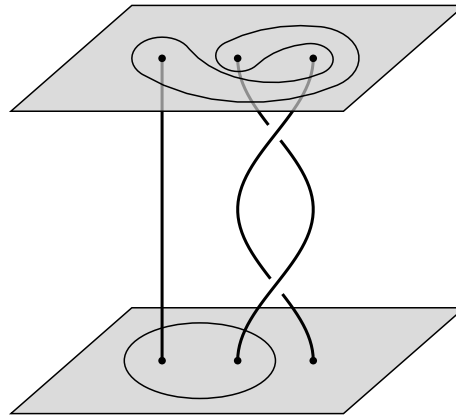
There is another important geometric representation of the braid group which is purely two-dimensional. We pick an  $n$ -point finite subset  $Q_n \subset M$  and consider diffeomorphisms  $f : M \rightarrow M$  that map  $Q_n$  to itself. The *mapping class group* of  $M$  relative to  $Q_n$  is the set of all such diffeomorphisms up to an equivalence relation  $\sim_{\text{iso}}$ :

$$MCG(M, Q_n) = \{f : M \rightarrow M \text{ such that } f(Q_n) = Q_n\} / \sim_{\text{iso}}$$

The equivalence relation  $\sim_{\text{iso}}$  is called *isotopy* which allows for any continuous deformation of  $f : M \rightarrow M$  that fixes each point in  $Q_n$ . Each generator  $\sigma_i$  of the braid group is found in  $MCG(M, Q_n)$  as a *half-twist* that swaps two points  $i, i+1 \in Q_n$ :



In order to show the action of this half-twist we have decorated the manifold with a checkerboard pattern, but there is a more important object that lives on the manifold itself. An *observable* is a simple closed curve in  $M$  that does not intersect  $Q_n$ . Such a closed curve is called an observable because these will be associated to measurements of the total anyonic charge on the interior of the curve. The importance of understanding the braid group as identical to the mapping class group is now manifest: whereas geometric braids act on states as in a Schrödinger picture, elements of the mapping class group act on the observables as in a Heisenberg picture.



This diagram should give the reader some idea as to why these two definitions of the braid group are equivalent, but the actual proof of this is somewhat involved. We cite Ref. [17] for an excellent contemporary account that fills in these gaps.

The definition given above for the geometric braid group and the mapping class group make sense for any two dimensional manifold  $M$  but for concreteness we consider  $M$  to be a flat disc. The corresponding algebraic definition of the braid group will in general be altered depending on the underlying manifold  $M$ .

So far we have been studying the exchange statistics for  $n$  identical particles. Without this restriction, one needs to constrain the allowed exchange processes so as to preserve particle type. For example, if all

particles are different we would use the *pure braid group*  $PB_n$ . The geometric description of this group is as the fundamental group of the labelled configuration space

$$PB_n = \pi_1(\mathcal{C}_n, x).$$

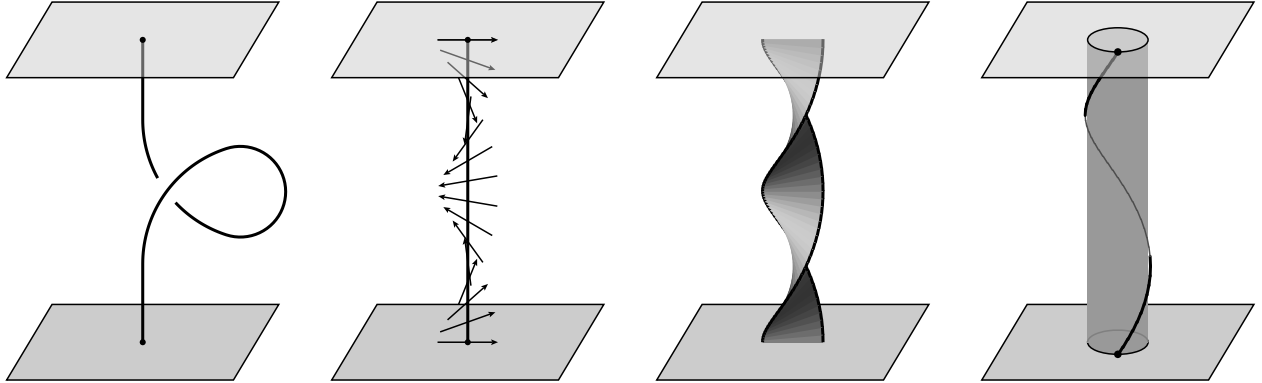
Loops in  $\mathcal{C}_n$  correspond to braids where each world-line returns to the point it started from. The algebraic description of this group is somewhat complicated and we omit this. In terms of the mapping class group, we can describe  $PB_n$  as the *pure mapping class group*:

$$PMCG(M, Q_n) = \{f : M \rightarrow M \text{ such that } f(x) = x \text{ for } x \in Q_n\} / \sim_{\text{iso}}.$$

One further complication arises when particles have a rotational degree of freedom: ie., they can be rotated by  $2\pi$  in-place and this effects the state of the system. To capture this action, we use the *framed braid group*  $FB_n$ . This group can be presented algebraically using the same generators and relations as for the braid group  $B_n$ , along with “twist” generators  $\theta_i$  for  $i = 1, \dots, n$ . These must satisfy the further relations

$$\begin{aligned} \theta_i \theta_j &= \theta_j \theta_i \\ \theta_i \sigma_j &= \sigma_j \theta_i \text{ if } i < j \text{ or } i \geq j + 2 \\ \theta_{i+1} \sigma_i &= \sigma_i \theta_i \\ \theta_i \sigma_i &= \sigma_i \theta_{i+1}. \end{aligned}$$

Here we show four geometric approaches to representing a twist. Any person that has struggled to untangle their headphone cable will immediately see what is going on here.



On the left we have a loop; it is not a braid because it travels backwards in time. If we pull on this loop to make it straight, we introduce a twist. This is shown in the next figure, where we show a *framing* which is a non-degenerate vector field along the world-lines of a braid. The initial and final vectors in the vector field must be the same. By non-degenerate we mean that the vector field is everywhere non-zero and non-tangent to the world-line. In the next figure we show a *ribbon*: instead of point particles we have short one-dimensional curves in  $M$ . On the right the particle is represented as a boundary component (a hole) of  $M$  with a distinguished point. In this picture the world-line looks like a tube.

All these representations of twists carry essentially the same information. In the sequel we will stick to thinking of particles as boundary components because this fits well with the way we are formulating observables as simple closed curves in  $M$ .

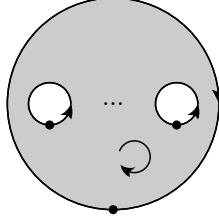
At this point in the narrative we are close to our next destination. All of these considerations, of framed or unframed, labelled or not, with possibly different underlying manifolds, together with observables, is meant to be captured by the formalism of a modular functor which we turn to next.

## B. Modular Functors

We list the axioms for a *2-dimensional unitary topological modular functor*.

For our purposes a *surface* will be a compact oriented 2-dimensional differentiable manifold with boundary. We will not require surfaces to be connected. By a *hole* of  $M$  we mean a connected component of its boundary. Each hole will inherit the manifold orientation, contain a distinguished *base point*, and be labeled with an element from a fixed finite set  $\mathcal{A}$ . This is the set of “anyon labels”, and comes equipped with a vacuum element  $\mathbb{I}$  and an involution  $\hat{\phantom{x}}$  such that  $\hat{\mathbb{I}} = \mathbb{I}$ . The involution maps an anyon label to the “antiparticle” label.

We will mostly be concerned with planar such surfaces, that is, a *disc* with holes removed from the interior. Such surfaces will be given a clockwise orientation, which induces a counterclockwise orientation on any *interior* hole and a clockwise orientation on the *exterior* hole.



Although it helps to draw such a surface as a disc with holes, we stress that there is no real distinction to be made between interior holes and the exterior hole. That is, a disc with  $n$  interior holes is (equivalent to) a sphere with  $n + 1$  holes. We merely take advantage of the fact that a sphere with at least one hole can be flattened onto the page by “choosing” one of the holes to serve as the exterior hole.

By a *map of surfaces*  $f : M \rightarrow N$  we mean a diffeomorphism that preserves manifold orientation, hole labels, and base points. Note that we also deal with maps of various other objects (vector spaces, sets, etc.) but a map of surfaces will have these specific requirements.

Two maps of surfaces  $f : M \rightarrow N$  and  $f' : M \rightarrow N$  will be called *isotopic* when one is a continuous deformation of the other. In detail, we have a continuously parametrized family of maps  $f_t : M \rightarrow N$  for  $t \in [0, 1]$  such that  $f_0 = f$ ,  $f_1 = f'$  and the restriction of  $f_t$  to the set of marked points  $X \subset \partial M$  is constant:  $f_t|_X = f|_X$  for  $t \in [0, 1]$ . Such a family  $\{f_t\}_{t \in [0, 1]}$  is called an *isotopy* of  $f$ . This is an equivalence relation on maps  $M \rightarrow N$ , and the equivalence class of  $f$  under isotopy is called the *isotopy class* of  $f$ . The weaker notion of homotopy of maps will not be used here, but for the maps we use it turns out that homotopy is equivalent to isotopy. Furthermore, we can weaken the requirement that maps be differentiable, because every continuous map  $f : M \rightarrow N$  is (continuously) isotopic to a differentiable map [18].

A *modular functor*  $\mathcal{H}$  associates to every surface  $M$  a finite dimensional complex vector space  $\mathcal{H}(M)$ , called the *fusion space* of  $M$ . For each map  $f : M \rightarrow N$  the modular functor associates a unitary transformation  $\mathcal{H}(f) : \mathcal{H}(M) \rightarrow \mathcal{H}(N)$  that only depends on the isotopy class of  $f$ . Functoriality requires that  $\mathcal{H}$  respect composition of maps.

We have the following axioms for  $\mathcal{H}$ .

**Unit axioms.** The fusion space of an empty surface is one dimensional,  $\mathcal{H}(\emptyset) \cong \mathbb{C}$ . For  $M_a$  a disc with boundary label  $a$  we have  $\mathcal{H}(M_{\mathbb{I}}) \cong \mathbb{C}$  and  $\mathcal{H}(M_a) \cong 0$  for  $a \neq \mathbb{I}$ . For an annulus  $M_{a,b}$  with boundary labels  $a, b$  we have  $\mathcal{H}(M_{a,\hat{a}}) \cong \mathbb{C}$  and  $\mathcal{H}(M_{a,b}) \cong 0$  for  $a \neq \hat{b}$ .

**Monoidal axiom.** The disjoint union of two surfaces  $M$  and  $N$  is associated with the tensor product of fusion spaces:

$$\mathcal{H}(M \amalg N) \cong \mathcal{H}(M) \otimes \mathcal{H}(N).$$

This is natural from the point of view of quantum physics, where the Hilbert space of two disjoint systems is the tensor product of the space for each system.

**Gluing axiom.** Denote a surface  $M$  with (at least) two holes labeled  $a, b$  as  $M_{a,b}$ . If we constrain  $b = \hat{a}$  then we may *glue* these two holes together to form a new surface  $N$ . To construct  $N$  we choose a diffeomorphism from one hole to the other that maps base point to base point and reverses orientation. Identifying the two holes along this diffeomorphism gives the glued surface  $N$ . (There is a slight technicality in ensuring that  $N$  is then differentiable, but we will gloss over this detail.) The image of these holes in  $N$  we call a *seam*. The fusion space of  $M_{a,\hat{a}}$  then embeds unitarily in the fusion space of  $N$ . Moreover, there is an isomorphism called a *gluing map*:

$$\bigoplus_{a \in \mathcal{A}} \mathcal{H}(M_{a,\hat{a}}) \xrightarrow{\cong} \mathcal{H}(N)$$

and this isomorphism depends only on the isotopy class of the seam in  $N$ .

**Unitarity axiom.** Reversing the orientation of the surface  $M$  to form  $\overline{M}$  we get the dual of the fusion space:  $\mathcal{H}(\overline{M}) = \mathcal{H}(M)^*$ .

**Compatibility axioms.** Loosely put, we require that the above operations play nicely together, and commute with maps of surfaces. For example, a sequence of gluing operations applied to a surface can be performed regardless of the order (gluing is associative) and we require the various gluing maps for these operations to similarly agree. For another example, we require  $\mathcal{H}$  to respect that gluing commutes with disjoint union.

**Observables.** The seam along which gluing occurs can be associated with an observable as follows. We take a gluing map  $g$  and projectors  $P_a$  onto the summands in the above direct sum:

$$P_a : \bigoplus_{b \in \mathcal{A}} \mathcal{H}(M_{b,\hat{b}}) \rightarrow \mathcal{H}(M_{a,\hat{a}}).$$

then the observable will be the set of operators  $\{P_a g^{-1}\}_{a \in \mathcal{A}}$ . For each  $a \in \mathcal{A}$  we call the image of  $P_a$  a *charge sector* for that observable. Note that in the glued surface  $N$  the seam has no preferred orientation (or base point). If we choose an orientation for the seam this corresponds to choosing one of the two boundary components in the original surface  $M_{a,\hat{a}}$ . If these boundary components come from disconnected components of  $M_{a,\hat{a}}$  the seam cuts  $N$  into two pieces and the orientation chooses an *interior*: following the orientation around the seam presents the interior to the left. We intentionally confuse the distinction between an observable as a set of operators, and the associated seam along which gluing occurs.

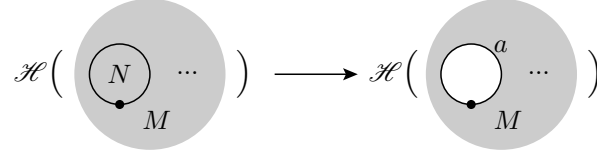
**Consequences of axioms.** A common operation is to glue two separate surfaces. We can do this by first taking disjoint union (tensoring the fusion spaces) and then gluing. Here we show this process applied to two surfaces  $M$  and  $N$ .

We display the surface  $N$  with the  $\hat{a}$  boundary on the outside, to show more clearly how  $N$  fits into  $M$ . In the glued surface we indicate the placement of  $M$  and  $N$  and the seam along which gluing occurred, as well as the identification of base points.

We note two other consequences of the axioms. A hole of  $M$  labeled with  $\mathbb{I}$  can be replaced with a disk (by gluing) and this does not change the fusion space of  $M$ . That is, a hole that carries no charge can be “filled-in”. And, the dimensionality of the fusion space of a torus is the cardinality of  $\mathcal{A}$ . This can be seen by gluing one end of an annulus (cylinder) to the other.

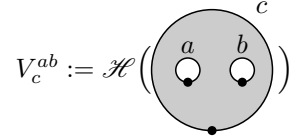
**Fusion.** When a surface can be presented as the gluing of two separate surfaces, we have projectors onto the fusion space of either glued surface:



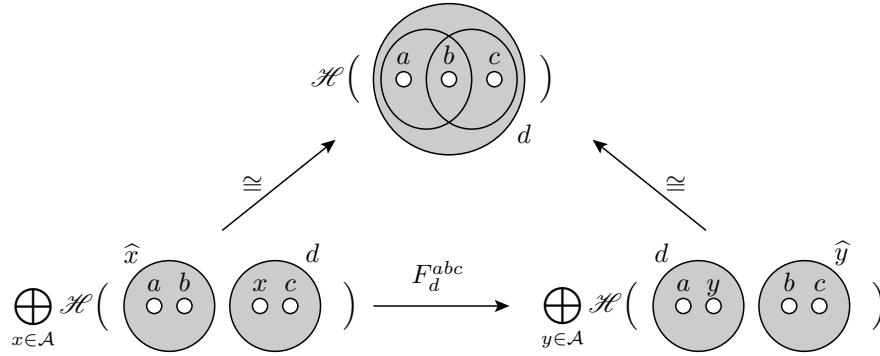


In this case, we define the operation of *fusion* to replace the interior of an observable by a single hole. This is an operation on the manifold itself, and we will only do this when the interior piece is a disc with zero or more holes.

**F-move.** The fusion space of the disc and annulus are specified by the axioms, and we define the fusion space of the disc with two holes, or *pair-of-pants* as:



The *F-move* is constructed from two applications of a gluing map (one in reverse) as the following commutative diagram shows:

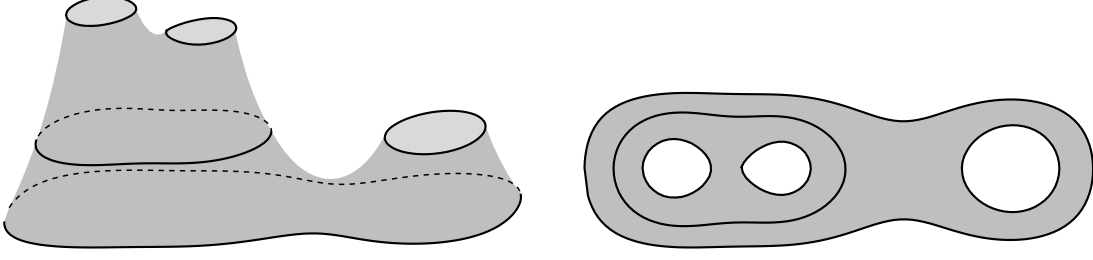


Here we have a surface  $M_d^{abc}$  with four labeled boundary components, as well as two separate ways of gluing pairs-of-pants to get  $M_d^{abc}$ . We can also write this out in terms of the fusion spaces of pair-of-pants:

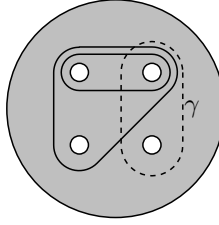
$$F_d^{abc} : \bigoplus_{x \in \mathcal{A}} V_{\hat{x}}^{ab} \otimes V_d^{xc} \rightarrow \bigoplus_{y \in \mathcal{A}} V_d^{ay} \otimes V_{\hat{y}}^{bc}.$$

By using gluing, and summing over charge sectors, we can extend this operation to apply where any of the boundary components  $a, b, c$  or  $d$  are merely seams in a larger manifold.

**POP decomposition.** Given a manifold  $M$  which is a disc with two or more holes, we show how to present  $M$  as the gluing of various pair-of-pants. Such an arrangement will be termed a *POP decomposition*. (We refer to Ref. [19] for more details on this construction, and Ref. [14] for a leisurely description of Morse theory.) This will yield a decomposition of  $\mathcal{H}(M)$  into a direct sum of fusion spaces of pair-of-pants. The key idea is to choose a “height” function  $h : M \rightarrow \mathbb{R}$  with some specific properties that allow us to cut the manifold up along level sets of  $h$ . First, we need that critical points of  $h$  are isolated. This is the defining condition for  $h$  to be a *Morse function*. Also, we need that  $h$  is constant on  $\partial M$ , and the values of  $h$  at different critical points are distinct. Now choose a sequence of non-critical values  $a_1 < a_2 < \dots < a_n$  in  $\mathbb{R}$  such that every interval  $[a_{i-1}, a_i]$  contains exactly one critical value of  $h$  and the image of  $h$  lies within  $[a_1, a_n]$ . Each component of  $h^{-1}([a_{i-1}, a_i])$  is then either an annulus, a disc, or a pair-of-pants depending on the index of (any) critical point it contains. We then re-glue any annuli or discs until there are no more of these and we have only pair-of-pants.

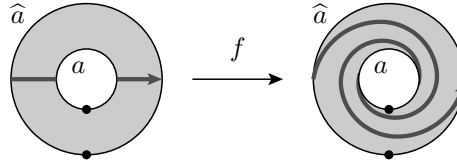


Clearly such a POP decomposition is not unique, and the goal here is to understand how to switch between decompositions, and in particular, given an observable  $\gamma$ , and a given POP decomposition, find a sequence of “moves” such that  $\gamma$  is in the resulting POP decomposition:



One way to achieve this is via Cerf theory, which is the theory of how one may deform Morse functions into other Morse functions and the kind of transitions involved in their critical point structure. This was the approach used in Ref. [6]. In this work we use a simpler method, which is essentially the same as skein theory. This is the refactoring theorem that we describe below.

**Dehn twist.** Consider a surface  $M_{a,\hat{a}}$  with two boundary components,  $a$  and  $\hat{a}$ . Let  $f$  be a map  $M_{a,\hat{a}} \rightarrow M_{a,\hat{a}}$  which performs a clockwise  $2\pi$  full-twist or *Dehn twist*. Here we show the action of  $f$  by highlighting the equator of the annulus:



We define the induced map on fusion spaces as  $\theta_a := \mathcal{H}(f)$ . Because  $\mathcal{H}(M_{a,\hat{a}})$  is one-dimensional this will be multiplication by a complex number which we also write as  $\theta_a$ .

If we now take  $M$  to be an arbitrary surface, and  $\gamma$  an observable on  $M$ , we can consider a neighbourhood of  $\gamma$  which will be an annulus, and perform a Dehn twist there, which we denote as  $f_\gamma : M \rightarrow M$ . Writing  $M$  as a gluing along  $\gamma$  of another manifold  $N_{a,\hat{a}}$ , the action of  $\mathcal{H}(f_\gamma)$  will decompose as a direct sum over charge sectors:

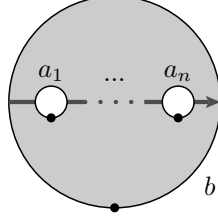
$$\bigoplus_{a \in \mathcal{A}} \theta_a \mathcal{H}(N_{a,\hat{a}}).$$

**Standard surfaces.** For each  $n = 0, 1, \dots$  and every ordered sequence of any labels  $a_1, \dots, a_n, b$  we choose a *standard surface*. This is a surface with  $n + 1$  boundary components labeled  $a_1, \dots, a_n, b$  which we denote  $M_b^{a_1 \dots a_n}$ .

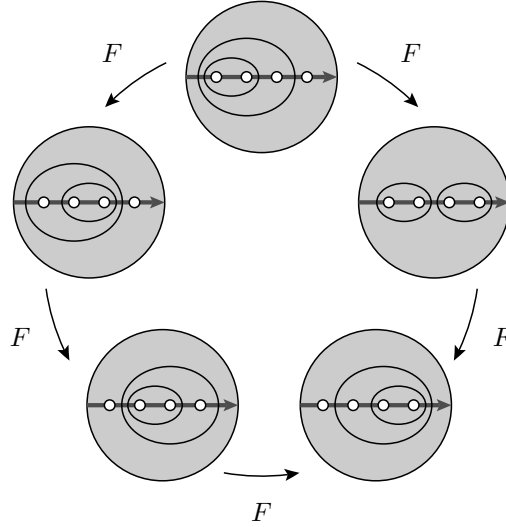
For concreteness we define this surface using the following expression for a closed disc less  $n$  open discs:

$$\left\{ (x, y) \in \mathbb{R}^2 \text{ st. } \left| (x, y) - \left( \frac{1}{2}, 0 \right) \right| \leq \frac{1}{2} \right\} - \left\{ (x, y) \in \mathbb{R}^2 \text{ st. } \left| (x, y) - \left( \frac{2i-1}{2n}, 0 \right) \right| < \frac{1}{4n} \right\}_{i=1, \dots, n}.$$

We then label the interior holes  $a_1, \dots, a_n$  in order of increasing  $x$  coordinate, and the exterior hole is labeled  $b$ . The base points are placed in the direction of negative  $y$  coordinate. As a notational convenience we highlight the *equator* of the surface which is the intersection of the  $x$  axis with the surface:



Each standard surface comes with a collection of *standard POP decompositions*: these will be POP decompositions where we require each observable to cross the equator twice and have counterclockwise orientation. Up to isotopy, a given standard surface will have only finitely many of these. On a standard surface with three interior holes there are two standard POP decompositions, and one  $F$ -move that relates these. On a standard surface with four interior holes there are five standard POP decompositions and five  $F$ -moves that relate these. In this case the  $F$ -moves themselves satisfy an equation that is an immediate consequence of the way we have defined  $F$ -moves. This is known as the pentagon equation, which we depict as the following commutative diagram:



We are now starting to confuse the notation for the topological space  $M$  and the fusion space  $\mathcal{H}(M)$ . Also, each of these  $F$ -moves is referring to an isomorphism that is block decomposed according to the charge sectors of the indicated observables.

We define the vector spaces  $V_b^{a_1 \dots a_n} := \mathcal{H}(M_b^{a_1 \dots a_n})$ . We now choose a basis for each of the  $V_b^{a_1 a_2}$  to be  $\{v_{b,\mu}^{a_1 a_2}\}_\mu$ . For every standard POP decomposition of  $M_b^{a_1 \dots a_n}$ , we get a decomposition of  $V_b^{a_1 \dots a_n}$  into direct sums of various  $V_{b'}^{a'_1 a'_2}$ . This then gives a *standard basis* of  $V_b^{a_1 \dots a_n}$  relative to this standard POP decomposition using the corresponding  $\{v_{b',\mu}^{a'_1 a'_2}\}_\mu$  for each of the  $V_{b'}^{a'_1 a'_2}$ .

Note that for any standard surface  $M_b^{a_1 \dots a_n}$  and any choice of  $k$  contiguous holes  $a_j, \dots, a_{j+k-1}$  we can find an observable that encloses exactly these holes in at least one of the standard POP decompositions of  $M_b^{a_1 \dots a_n}$ .

We next show how to glue the exterior hole of a standard surface  $M$  to an interior hole of another standard surface  $N$ . To do this within  $\mathbb{R}^2$  we rescale and translate the two surfaces so that the exterior hole of  $M$  coincides with the interior hole of  $N$ . At this point the union is not in general going to produce

another standard surface, and so we remedy this by applying any isotopy within  $\mathbb{R}^2$  that fixes the  $x$ -axis while moving the surface to a standard surface.

**Curve diagram.** We next study maps from standard surfaces to arbitrary surfaces. The reader should think of this as akin to choosing a basis for a vector space. The set of all maps from  $M_b^{a_1 \dots a_n}$  to a surface  $N$  will be denoted as  $\text{Hom}(M_b^{a_1 \dots a_n}, N)$ . We call each such map a *curve diagram*, or more specifically, a curve diagram on  $N$ . The reason for this terminology is that we can reconstruct (up to isotopy) any map  $f : M_b^{a_1 \dots a_n} \rightarrow N$  from the restriction of  $f$  to the equator of  $M_b^{a_1 \dots a_n}$ . In other words, we can uniquely specify any map  $f : M_b^{a_1 \dots a_n} \rightarrow N$  by indicating the action of  $f$  on the equator. (This reconstruction works because a simple closed curve on a sphere cuts the sphere into two discs.) We take full advantage of this fact in our notation: any figure of a surface with a “green line” drawn therein is actually notating a diffeomorphism, not a surface!

Any curve diagram will act on a standard POP decomposition of  $M_b^{a_1 \dots a_n}$  sending it to a POP decomposition of  $N$ . Surprisingly, the converse of this statement also holds: any POP decomposition of  $N$  (a disc with holes) comes from some curve diagram acting on a standard POP decomposition. The proof of this is constructive, and we call this the refactoring theorem below.

The operation of gluing of surfaces can be extended to gluing of curve diagrams as long as we are careful with the way we identify along the seam: the identification map needs to respect the equator of the curve diagram. (Keep in mind that a curve diagram is really a map of surfaces, and so gluing two such maps involves two separate gluing operations.)

We note in passing two connections to the mathematical literature. Such curve diagrams have been used in the study of braid groups [20], and this is where the name comes from, although our curve diagrams respect the base points and so could be further qualified as “framed” curve diagrams. And, we note the similarity of curve diagrams and associated modular functor to the definition of a planar algebra [21], the main difference being that planar algebras allow for not just two but any even number of curve intersections at each hole.

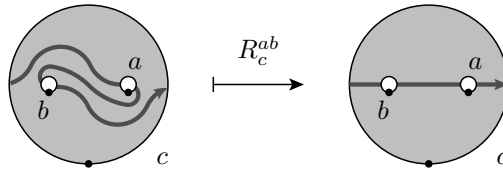
**Z-move.** We let  $z$  be a diffeomorphism of standard surfaces  $z : M_b^{a_1 \dots a_n} \rightarrow M_{a_1}^{a_2 \dots a_n b}$  that preserves the equator. (Considering the standard surface as a sphere with holes placed uniformly around a great circle,  $z$  is seen to be a “rotation”.) This acts by pre-composition to send a curve diagram  $f \in \text{Hom}(M_b^{a_1 \dots a_n}, N)$  to  $fz \in \text{Hom}(M_{a_1}^{a_2 \dots a_n b}, N)$ . This we call a  $Z$ -move of  $f$ .

In this way, any curve diagram can be seen as another curve diagram that has a cyclic permutation of the labels of the underlying standard surface.

**R-move.** Given anyon labels  $a, b$  and  $c$ , and arbitrary surface  $N$ , we now define the following map of curve diagrams on  $N$  :

$$R_c^{ab} : \text{Hom}(M_c^{ab}, N) \rightarrow \text{Hom}(M_c^{ba}, N).$$

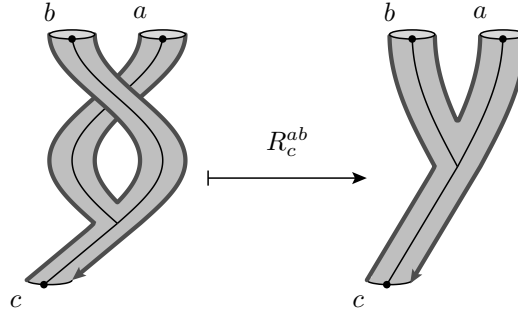
This map works by taking a curve diagram  $f : M_c^{ab} \rightarrow N$  to the composition  $f\sigma$  where  $\sigma : M_c^{ba} \rightarrow M_c^{ab}$  is a counterclockwise “half-twist” map that exchanges the  $a$  and  $b$  holes. Here we show the action of  $R_c^{ab}$  on one particular curve diagram:



Such an application of  $R_c^{ab}$  to a particular curve diagram we call an  $R$ -move. As noted above, a curve diagram serves to pick out a basis for the fusion space, and the point of this  $R$ -move is to switch between different curve diagrams for the same surface. This is highlighted to draw the readers attention to the fact that the  $R$ -move does not swap the labels on the holes: the surface itself stays the same.

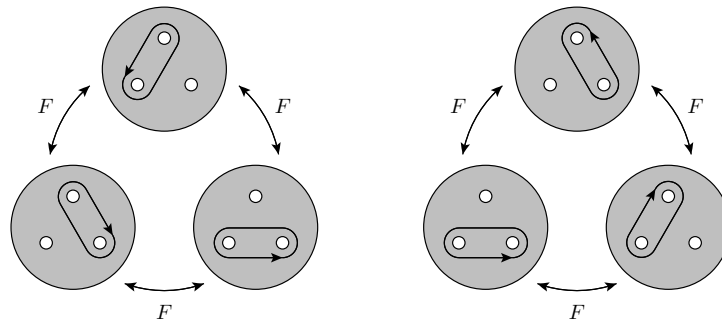
As we extended Dehn twists under gluing, and  $F$ -moves under gluing, we also do this for  $R$ -moves.

**Skeins.** The previous figure can be seen as a “top-down” view of the following three dimensional arrangement:

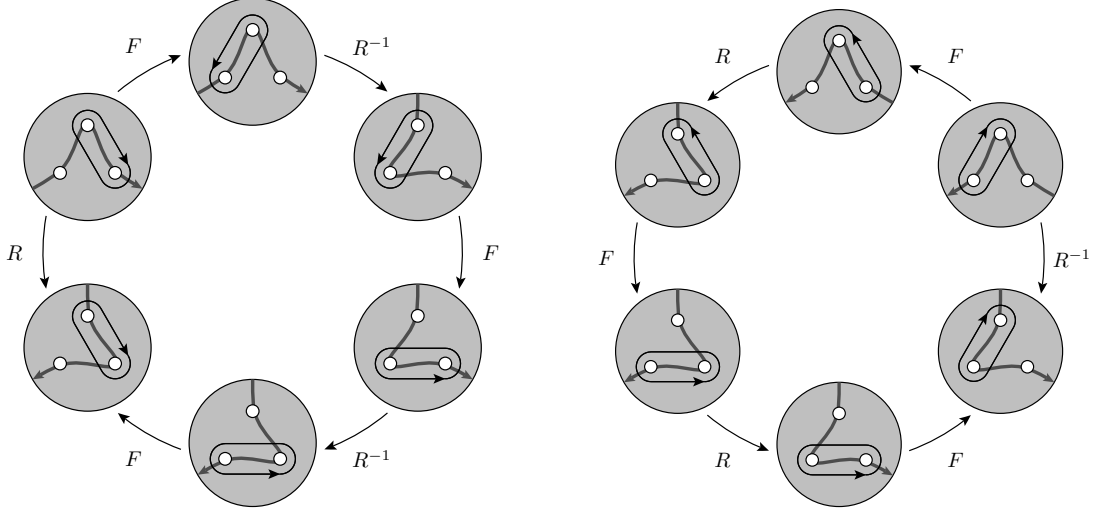


This figure is intended to be topologically the same as the previous flat figure, with the addition of a third dimension, and the  $c$  boundary has been shrunk. Also note thin black lines connecting the base points. The black lines do not add any extra structure, they can be seen as a part of the hexagon cut out by the green line and boundary components. But notice this: the black lines are “framed” by the green lines. These are the ribbons used in skein theory! Note that all the holes are created equal: there is no distinction between “input” holes and “output” holes (as there is with cobordisms or string diagrams.)

**Hexagon equation.** For a surface with three interior holes there are infinitely many POP decompositions (up to isotopy). These are all made by choosing an observable that encloses two holes. The  $F$ -moves allow us to switch between these POP decompositions, and the axioms for the modular functor make these consistent. Here we show two such triangle consistency requirements (they are reflections of each other):



Given a POP decomposition of a surface  $N$  there are various curve diagrams on  $N$  that produce this decomposition from a standard POP decomposition, and there are certain  $R$ -moves that will map between these. Once again, these moves must be consistent, and here we note the two “hexagon equations” corresponding to the above two triangles:



Note that we have neglected to indicate the base points here. Given a curve diagram, we can agree that base points occur “to the right” of the image of the equator. But in notating diagrams such as these, there is still an ambiguity: the position of the base points should be the same in each surface in the diagram. If we try to correct for this post-hoc by rotating individual holes, we will then be correct only up to possible Dehn twist(s) around each hole. We can certainly track these twists if we wanted to, but in the interests of simplicity we do not. This introduces a global phase ambiguity into the calculations.

The reason why we mention the pentagon and hexagon equations is that these become important in an algebraic description of the theory. Because we have defined everything in terms of a modular functor we get these equations “for free”.

### C. Refactoring theorem

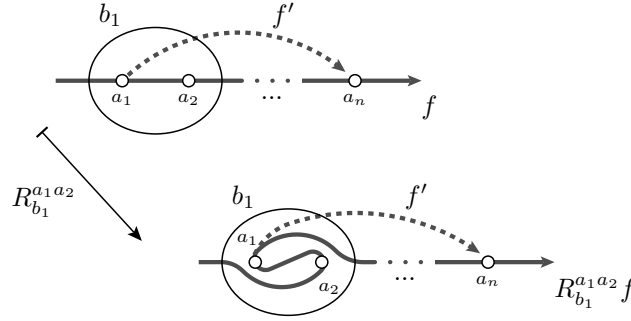
In this section we specialize to considering planar surfaces only. The theorem we are building towards shows that the  $R$ -moves act transitively on curve diagrams. By transitive we mean that given two curve diagrams  $f$  and  $f'$  on  $N$  we can find a sequence of  $R$ -moves that transform  $f$  to  $f'$ . The key idea is to consider the (directed) image of the equator under these curve diagrams. As mentioned previously, this image is sufficient to define the entire diffeomorphism (up to isotopy) and so we are free to work with just this image, or *curve*, and we confuse the distinction between a curve diagram (a diffeomorphism) and its curve.

The construction proceeds by considering adjacent pairs of holes along  $f'$  and then acting on the portion of  $f$  between these same two holes so that they then become adjacent on the resulting curve. Continuing this process for each two adjacent holes of  $f'$  will then show a sequence of  $R$ -moves that sends  $f$  to  $f'$ .

To this end, consider a sequence of holes  $a_1, \dots, a_n$  appearing sequentially along  $f$  such that  $a_1$  and  $a_n$  appear sequentially on  $f'$ . We may need to apply some  $Z$ -moves to  $f$  to ensure that  $a_1$  appears sequentially before  $a_n$ . Now consider the case such that along  $f'$  between these two holes there is no intersection with  $f$ . We form a closed path  $\xi$  by following the  $f'$  curve between  $a_1$  and  $a_n$  and then following the  $f$  curve in reverse from  $a_n$  back to  $a_1$ . (Note that to be completely rigorous here we would need to include segments of path contained within boundary components.) The resulting closed path bounds a disc. If  $\xi$  has clockwise orientation we apply the following sequence of  $R$ -moves to  $f$ :

$$R_{b_1}^{a_1 a_2}, R_{b_2}^{a_1 a_3}, \dots, R_{b_{n-1}}^{a_1 a_{n-1}}.$$

Depicted here is the first such move:



After each of these  $R$ -moves the closed path formed by following the  $f'$  curve between  $a_1$  and  $a_n$  and then following the  $R$ -moved  $f$  curve back to  $a_1$  will traverse one less hole, and still bound a disc. After all of the  $R$ -moves this path will only touch  $a_1$  and  $a_n$ , and bound a disc. Therefore, we have acted on the  $f$  curve so that the resulting curve has  $a_1$  and  $a_n$  adjacent and the bounded disc gives an isotopy for that segment of the curve.

When the closed curve  $\xi$  has anti-clockwise orientation we use the same sequence of  $R$ -moves but with  $R$  replaced by  $R^{-1}$ .

Generalizing further, when the  $f'$  curve between  $a_1$  and  $a_n$  has (transverse) intersections with  $f$  we use every such intersection to indicate a switch between using  $R$  and  $R^{-1}$ .

We continue in this way moving backwards (from head to tail) sequentially applying this procedure.

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