Random Fibonacci Anyons Simulable via Covariant Action of Mapping Class Group

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Abstract

Technical contribution: (1) discretization of curve diagrams, (2) classification theorem (3) simulation

Introduction

Fibonacci hard

Below bond-percolation threshold we expect to see clusters of charges of size $O(log(n))$ with variance $O(1)$ [2].

Curve diagrams

The disc $D = \{x \in \mathbb{R}^2 \ s.t. \ |x| \leq 1\}$, has boundary $\partial D = \{x \in \mathbb{R}^2 \ s.t. \ |x| = 1\}$.

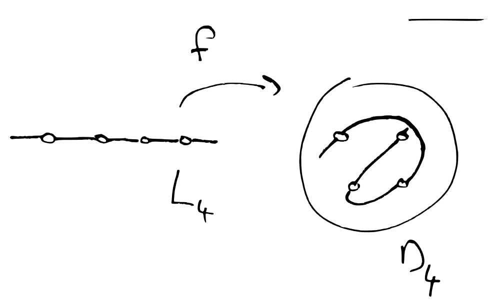
Given a finite set $Q_n \subset D - \partial D$, we denote D_n as the pair (D, Q_n).

We define the mapping class group of D_n, $MCG(D_n)$, as the set of homeomorphisms $D \to D$ that restrict to a permutation on Q_n, modulo isotopy that fixes Q_n. Such an equivelance class of homeomorphisms will be denoted as $\phi : D_n \to D_n$.

We can think of D_n as the unit disc with n holes, and the mapping class group as the set of homeomorphisms on this space, modulo isotopy.

The line $L = \{x \in \mathbb{R} \ s.t. \ |x| \leq 1\}$, and a finite set $R_n \subset L$, $L_n = (L, R_n)$. By a slight abuse of notation, we enumerate the points in R_n in a monotonically increasing order, and understand such notation as $[i, i+1]$ to mean the closed interval in L with (consecutive) endpoints $i, i+1 \in R_n$.

We define a *curve diagram* as an embedding $f : L \to D$ that restricts to a bijection $R_n \to Q_n$, modulo isotopy. This will be denoted as $f : L_n \to D_n$. We can generalize this to $f : L_m \to D_n$ with $m \leq n$ to mean a map $L \to D$ that restricts to an injection $R_m \to Q_n$, modulo isotopy.



XXX $\partial L \to \partial D$ XXX

Given two disjoint curve diagrams $f : L_m \to D_n$ and $f' : L_{m'} \to D_n$ we define their sum $f + f' : L_{m+m'} \to D_n$ ...

Group action

The mapping class group $MCG(D_n)$ acts on curve diagrams $f : L_n \to D_n$ via post-composition (left multiplication). This action is transitive and XXX so we can identify an element of $MCG(D_n)$ by its action on any curve $f : L_n \to D_n$. See [5], chapter 6. XXX $\partial L \to \partial D$ XXX

We single out specific "braid" elements of $MCG(D_n)$. These act on the image of a pair-of-pants embedding: $g : D_2 \to D_n$.

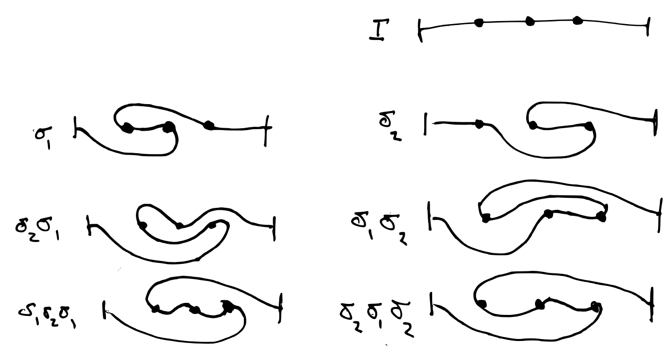
DIAGRAM

The *braid group* on n strands is the group B_n generated by $n-1$ generators $\sigma_1, \sigma_2, ...\sigma_{n-1}$ with the following relations:

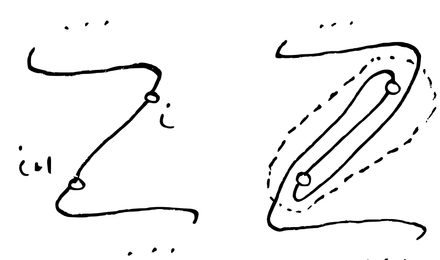
$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i - j| > 2,$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

Here we verify the braid relations directly:



Definition: by a *half-twist* on a curve diagram $f : L_n \to D_n$ at i we mean an element of $MCG(D_n)$ that can be represented by a half-twist that acts on a small neighbourhood of $f([i, i+1])$. We denote a clockwise half-twist by $b(i, f)$:

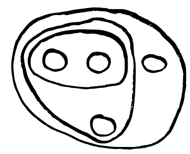


Each half-twist corresponds to a braid action of two neighbouring holes on f. [1]

Anyons

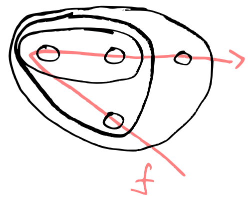
We measure charge contained within a disc. Measurement outcomes belong to a set of charges Λ. Commuting measurements are either disjoint discs or one wholly contained within the other. So (proof?) we get a basis for the Hilbert space of our system from a pair-of-pants (POP) decomposition:

[1]See [7], Section 1.6.2, for more details on half-twists.

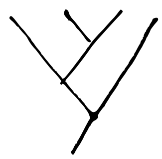


The feet of the POP decomposition form the n holes Q_n in the disc D. [2]

We say that a POP decomposition of D_n *admits* a curve $f : L_n \to D_n$ if (there is a curve in the isotopy class of) the curve performs a depth first traversal of the POP. This means that if the curve visits the trunk of a POP then it visits all the legs of the POP before leaving that POP.

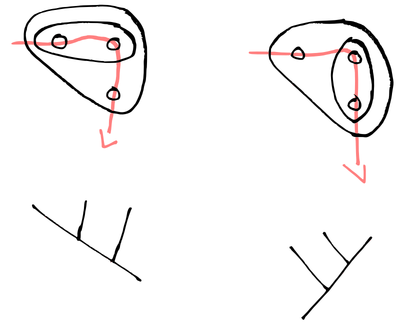


Conversely, given a curve $f : L_n \to D_n$ and a binary rooted tree t, we can recover a unique POP that admits f. The above POP would be represented by the following tree:



(proof: use small neighbourhoods of the curve)

<u>F-moves:</u> given a nested POP and admissable f, the F-move is the change of basis as follows:



XXX F-moves can be nested

<u>R-moves:</u> we examine the action of $MCG(D_n)$ on our state space. Given a single POP and curve $f : L_2 \to D_n$, the half-twist on f acts on ψ as $R\psi$.

DIAGRAM?

This together with f-moves gives all the half-twists on the curve f. Therefore, we may work solely with curve diagrams, forgetting the underlying POP (tree) structure.

We have a (linear) representation of the braid group:

$$\kappa : B_n \to GL(F_n).$$

[2]Every surface that is not a sphere, torus, disc or annulus, has a POP decomposition, see [6], Theorem 2.4.A.

Where F_n is the Hilbert state space of n Fibonacci anyons.
See [9, 8]

Half-twist factorization of $MCG(D_n)$

Definition: by a *sequence of half-twists* on a curve diagram $f : L_n \to D_n$ we mean a sequence of curve diagrams $f_k : L_n \to D_n$, and a sequence of half-twists:

$$b(i_1, f_1),\ b(i_2, f_2),\ \dots\ b(i_N, f_N)$$

such that $f_1 = f$, and

$$f_{k+1} = b(i_k, f_k) f_k, \quad 1 \le k < N.$$

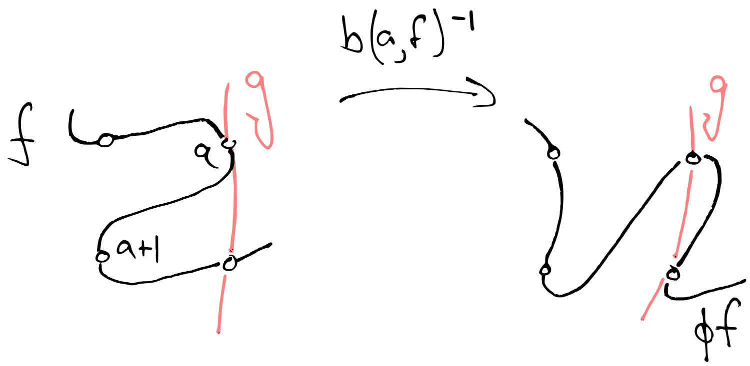
Problem: Given curve diagrams $f : L_n \to D_n$ and $g : L_2 \to D_n$ find $\phi \in MCG(D_n)$ as a product of a sequence of half-twists such that $g = \phi f i$, where i is an inclusion of L_2 into L_n :

$$\begin{tikzcd} L_2 \arrow[rr, "g"] \arrow[dr, "i"'] & & D_n \\ & L_n \arrow[ur, "\phi f"'] & \end{tikzcd}$$

The idea is that we would like to apply half-twists to a curve diagram until the two holes indicated by g become neighbours.

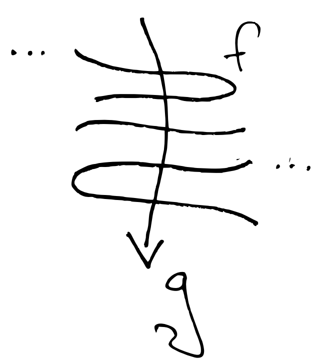
Examples:

In this example we use an inverse half-twist about the segment $[a, a+1]$:

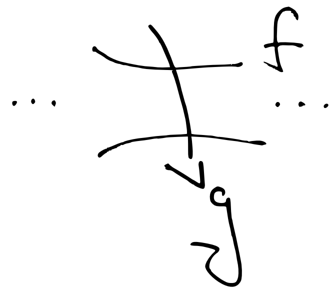


Solution:

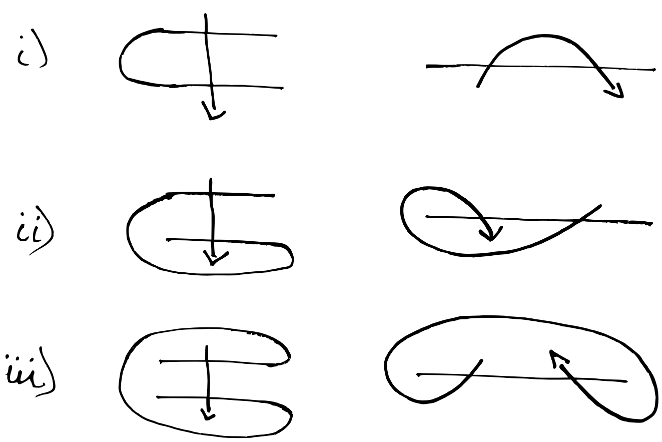
1) decompose f along (finitely many) intersections with g.



2) on each component k:



let $h_k = g_k + f_k$. This is a simple closed curve in D which splits D into two regions, the inside (simply connected) and the outside. We find three basic cases (there are more including reflections and rotations of these), which we can distinguish between using the winding number and whether the end points of f are on the inside or outside of h_k:



XXX

Planar diagrams

We aim to discretize the problem for software implementation. Our first task is to characterize what happens to a curve if we "forget" any braiding inside a sub-disc. We adapt the Abramsky combinatorial description of the Temperly-Lieb algebra to our needs (see [1], section 6.)

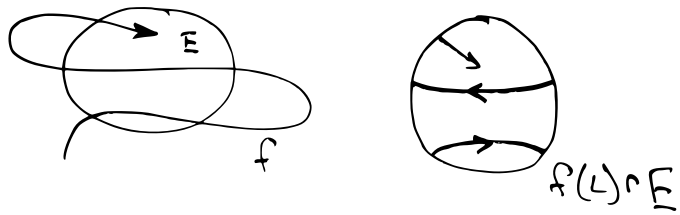
The set of *planar diagrams* is defined as:

$$P = \{f(L) \cap E/\sim \text{ s.t. } f : L \to D\}$$

where $f : L \to D$ is a continuous function whose image $f(L)$ intersects a disc $E \subset D$ in a finite number of components, and the equivalence relation $\sim$ consists of isotopies that fix ∂E. Each component of $f(L) \cap E$ we call a *strand.*

Elements of P have a simple combinatorial description as a finite sequence (a word) consisting of the four labels **(,), H, T** such that the parentheses are balanced. To construct such a word, begin at a point on ∂E, and proceeding clockwise: visiting a strand for the first time produces a **(** label, and visiting a strand for the second time produces a **)** label. The two labels **H** and **T** specify the head and tail endpoints of f, each of which appears at most once in the word.

For example, the following planar diagram can be represented by the word **(())H**:

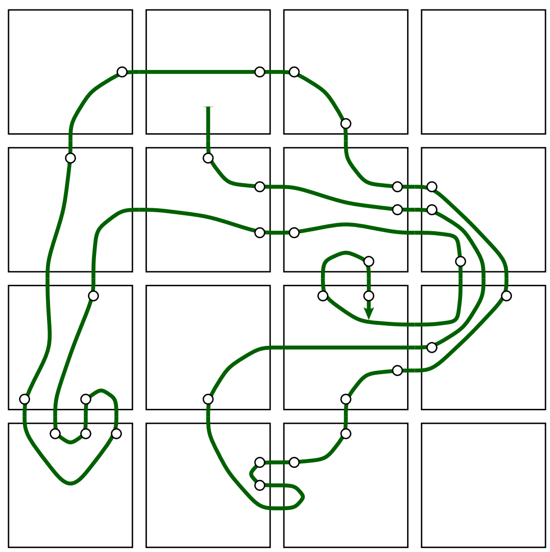


Tiling

We now consider a cellulation of a disc. Concretely, we choose a regular l by l square tiling of a square region. The holes Q_n will always be contained within the interior of the tiles.

Definition: the discrete curve space $C_n^{(l)}$ consists of curves $L_n \to D_n$ modulo isotopy that fixes the boundary of every tile.

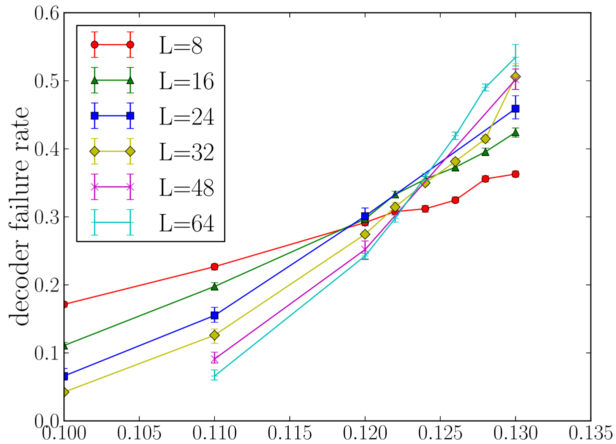
Combinatorialy, we store an element of $C_n^{(l)}$ by patching together words for planar diagrams in a compatible way: neighbouring tiles must agree on the strands that they share.



Simulation and noise model

Poisson process, pair creation t_{sim}
PICTURE
Extraction of syndrome
PICTURE
Multi-round Clustering of charges, [4] [3]
Fail on big braid

Numerical results



Threshold appears at $t_{\text{sim}} \simeq 0.124 \pm 0.004$
Plot of computational power...

Conclusion/Discussion

Acknowledgements

Yo to my bro's and homies

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