ON THE LEGO–TEICHMÜLLER GAME

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Abstract. For a smooth oriented surface Σ, denote by $M(\Sigma)$ the set of all ways to represent Σ as a result of gluing together standard spheres with holes ("the Lego game"). In this paper we give a full set of simple moves and relations which turn $M(\Sigma)$ into a connected and simply-connected 2-complex. Results of this kind were first obtained by Moore and Seiberg, but their paper contains serious gaps. Our proof is based on a different approach and is much more rigorous.

1. Introduction

Let Σ be a smooth oriented surface, possibly with boundary. In many cases—most importantly, for the study of the mapping class group $\Gamma(\Sigma)$ and for the construction of a modular functor—it is convenient to represent Σ as a result of gluing together several simple pieces, which should be surfaces with a boundary. It is easy to show that such a representation (we will call it a *parameterization*) is always possible if we allow these pieces to be spheres with ≤ 3 holes, or, more generally, spheres with n holes. For example, if we want to construct a modular functor, then it suffices to define the vector spaces for each of these "simple pieces", and then, since the behavior of the modular functor under gluing is known, this defines uniquely the vector space which should be assigned to Σ. From the point of view of the mapping class groups, every parameterization defines a homeomorphism of the product of the mapping class groups of the pieces into $\Gamma(\Sigma)$. In particular, in this way one can get a number of elements and relations in $\Gamma(\Sigma)$, and this can be used to get a full set of generators and relations of $\Gamma(\Sigma)$. This approach was first suggested by Grothendieck (see below), who called it "the Lego–Teichmüller game"

In all of these applications, it is important to note that the same surface Σ can have many different parameterizations. Thus, it is natural to ask the following questions. How can one describe different ways of gluing "standard pieces" that give parameterizations of the same surface Σ? Can we define some "simple moves" so that we can pass from a given parameterization to any other by a sequence of these simple moves? And, finally, can one describe all the relations between these simple moves, i.e. describe when a sequence of simple moves applied to a parameterization yields the same parameterization?

These questions were studied in a series of pioneering papers of Moore and Seiberg [MS1, MS2]. These authors used spheres with 3 holes (trinions) as their building blocks, and they gave a complete set of simple moves and relations among them. However, their paper [MS2] has some serious flaws. First of all, they use the language of chiral vertex operators, which is important for applications to conformal field theory, but which is not really relevant for finding the set of simple moves and

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relations, since this question is of purely topological nature. This lead them to miss some "obvious" axioms which are automatically satisfied in any conforaml field theory. What is worse, their proof contains some gaps, the most serious of them being a completely inadequate treatment of the case of surfaces of higher genus with $n > 1$ holes. The reason is that they used an explicit presentation of the mapping class group $\Gamma(\Sigma)$ by generators and relations, found by Wajnryb [Waj], and such a presentation for surfaces of higher genus was known only for surfaces with ≤ 1 holes. (For surfaces with arbitrary number of holes, a presentation of the mapping class groups by generators and relations was recently found in [Ge]; this presentation uses infinite number of generators—all Dehn twists.)

In this paper, we give a reformulation and a rigorous proof of the result of Moore and Seiberg, i.e., we construct a set of simple moves and relations among them, which turn the set of all parameterizations into a connected and simply-connected CW complex. To the best of our knowledge, this has not been done accurately before; the only works in this direction we are aware of are an unfinished and unpublished manuscript by Kevin Walker, and the book [T], in which it is proved that every modular tensor category gives rise to a modular functor.

We mostly use spheres with n holes as our building blocks, which allows us to simplify the arguments; however, we also give version of the main theorem whihc only uses spheres with ≤ 3 holes. We do not use explicit presentation of the mapping class groups $\Gamma_{g,n}$ for $g > 0$ by generators and relations. Instead, we refer to the results of Hatcher and Thurston [HT] and their refinement by Harer [H], who solved a similar problem for the cut systems on Σ. Our exposition is purely topological and requires no knowledge of modular functors.

Our motivation for this work came form the conformal field theory and modular functors. However, this work can also be useful for the study of *Teichmüller tower*, introduced by A. Grothendieck in his famous *Esquisse d'un Programme* [G]. This tower consists of all stable algebraic curves of any genus $g \geq 0$ with any number $n \geq 0$ of marked points (punctures) linked with the operation of "gluing". The fundamental groupoid $T_{g,n}$ of the corresponding moduli space is called the Teichmüller groupoid. The most fascinating thing is that the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the profinite completion $\{\widehat{T}_{g,n}\}$ of the tower of Teichmüller groupoids, and this action is faithful—it is faithful already on $\widehat{T}_{0,4}$ [G] (see also [S, LS] and references therein). Grothendieck states in [G] as a very plausible conjecture that the entire Teichmüller tower can be reconstructed from the first two levels (i.e., the cases when $3g - 3 + n \leq 2$) via the operation of "gluing", level 1 gives a complete system of generators, and level 2 a complete system of relations:

> ... la tour entière se reconstitue à partir des deux premiers étages, en ce sens que via l'opération fondamentale de "recollement", l'étage 1 fournit un système complet de générateurs, et l'étage 2 un système complet de relations.

As Drinfeld says in [D], the above conjecture "has been proved, apparently, in Appendix B of the physics paper [MS2]". As we already discussed, the approach of [MS2] uses heavily the explicit knowledge of the mapping class groups and is not really rigorous. We believe that combining the results of the present paper and of the unpublished manuscript [BFM], one can get a proof of the Grothendieck conjecture. This will be discussed in forthcoming papers.

2. Extended surfaces and parameterizations

2.1. Definition. An *extended surface* (*e-surface*, for short) is a compact oriented smooth 2-dimensional manifold Σ, possibly with a boundary $\partial\Sigma$, with a marked point chosen on each boundary circle of Σ. We will denote the set of boundary components $\pi_0(\partial\Sigma)$ by $A(\Sigma)$; in most cases, the elements of $A(\Sigma)$ will be labeled by Greek letters.

The *genus* $g(\Sigma)$ of an e-surface Σ is defined as the genus of the surface without boundary, obtained from Σ by gluing a disk to every boundary circle.

A *morphism* of extended surfaces is an orientation-preserving homeomorphism $\Sigma \xrightarrow{\sim} \Sigma'$ which maps marked points to marked points. Every such morphism induces a bijection $A(\Sigma) \xrightarrow{\sim} A(\Sigma')$.

An example of an e-surface is shown in Figure 1 below.

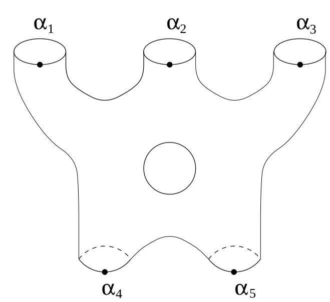


Figure 1. An extended surface.

2.2. Definition. The *mapping class group* $\Gamma(\Sigma)$ of an e-surface Σ is the group of homotopy classes of morphisms $\Sigma \xrightarrow{\sim} \Sigma$. The *pure mapping class group* $\mathrm{P}\Gamma(\Sigma)$ is the subgroup of $\Gamma(\Sigma)$ of those morphisms that act trivially on the set of boundary components.

2.3. **Standard sphere.** For every $n \geq 0$, we define the *standard sphere* $S_{0,n}$ to be the Riemann sphere $\overline{\mathbb{C}}$ with n disks $|z-k| < 1/3$ removed, and with the marked points being $k - \mathrm{i}/3$ ($k = 1, \dots, n$). (Of course, we could have replaced these n disks by any other n non-overlapping disks with centers on the real line and with marked points in the lower half-plane—any two such spheres are homeomorphic, and the homeomorphism can be chosen canonically up to homotopy.) The standard sphere with 4 holes is shown in Figure 2. We will denote by $\Gamma_{0,n} = \Gamma(S_{0,n})$ (respectively, $\mathrm{P}\Gamma_{0,n} = \mathrm{P}\Gamma(S_{0,n})$) the mapping class group (respectively, the pure mapping class group).

Note that the set of boundary components of the standard sphere is naturally indexed by numbers $1, \dots, n$; we will use bold numbers for denoting these boundary components: $A(S_{0,n}) = \{\mathbf{1}, \dots, \mathbf{n}\}$.

Obviously, every connected e-surface of genus zero is homeomorphic to exactly one of the standard spheres $S_{0,n}$, and the set of homotopy classes of such homeomorphisms is a torsor over the mapping class group $\Gamma_{0,n}$.

2.4. Definition. Let Σ be a connected e-surface of genus zero. A *parameterization without cuts* of Σ is a homotopy equivalence class of homeomorphisms $\psi : \Sigma \xrightarrow{\sim} S_{0,n}$.

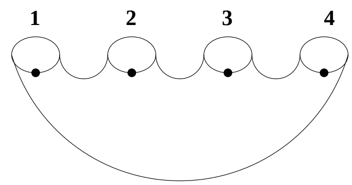


FIGURE 2. A standard sphere (with 4 holes).

The words "without cuts" are added in the definition because later we will consider a more general notion of "parameterization with cuts".

We note that we have natural actions of the mapping class groups $\Gamma(\Sigma)$ and $\Gamma_{0,n}$ on the set of all parameterizations of Σ, given by $\varphi(\psi) = \psi \circ \varphi^{-1}, \varphi \in \Gamma(\Sigma)$ and $\varphi(\psi) = \varphi \circ \psi, \varphi \in \Gamma_{0,n}$. These actions are transitive, so that the set of all parameterizations without cuts can be identified (non canonically!) with $\Gamma_{0,n}$. Note that every parameterization defines an identification $A(\Sigma) \xrightarrow{\sim} \{\mathbf{1}, \ldots, \mathbf{n}\}$ and thus, a natural order on the set of boundary components $A(\Sigma)$.

2.5. Definition. Let Σ be an e-surface. A *cut system* C on Σ is a finite collection of smooth simple closed non-intersecting curves on Σ (called *cuts*) such that each connected component of the complement $\Sigma \setminus C$ is a surface of genus 0. (The cuts are not to be oriented or ordered.) We will denote by $C(\Sigma)$ the set of all cut systems on Σ modulo isotopy.

A cut $c \in C$ is called *removable* if $C \setminus c$ is again a cut system. A cut system is called *minimal* if it contains no removable cuts.[1]

Note that we could have defined a cut to be a simple closed curve with one point marked on it. It is easy to see that this would have given us the same set $C(\Sigma)$: given a cut c and two points p, p' on c, there always exists an isotopy of Σ which is different from identity only in a small neighborhood of c, maps c onto itself and p to p'.

Examples of cut systems and a minimal cut system on an e-surface are shown in Figure 3.

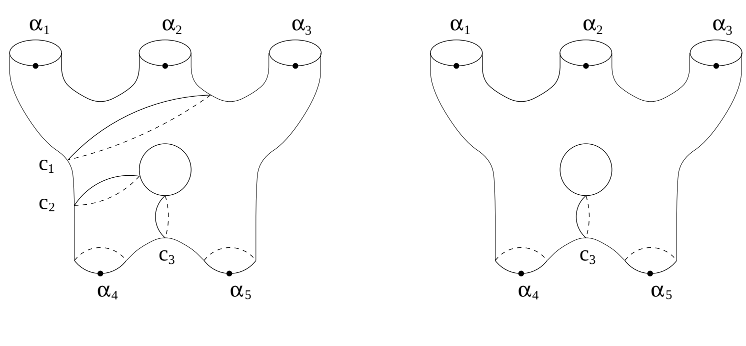


FIGURE 3. Examples of cut systems and a minimal cut system (right).

[1] Our notion of a minimal cut system is exactly what is called a "cut system" in [HT, H].

2.6. Definition. Let Σ be an e-surface. A *parameterization* P of Σ is a collection $(C, \{\psi_a\})$, where C is a cut system on Σ, and ψ_a are parameterizations without cuts of the connected components Σ_a of $\Sigma \setminus C$, i.e. homotopy equivalence classes of homeomorphisms $\psi_a : \Sigma_a \xrightarrow{\sim} S_{0,n_a}$ (see Definition 2.4).

As before, we have a canonical action of $\Gamma(\Sigma)$ on the set of all parameterizations of Σ, given by $\varphi(C, \{\psi_a\}) = (\varphi(C), \{\psi_a \circ \varphi^{-1}\})$.

3. Marked surfaces

In this section, we will introduce some visual language for representing the parameterizations of an e-surface Σ. Let us start with surfaces without cuts.

3.1. Definition. Let $S_{0,n}$ be the standard sphere with n holes (see Subsection 2.3), and let m_0 be the graph on it, shown in Figure 4 (for $n = 4$). This graph has a distinguished edge—the one which connects the vertex $*$ with the boundary component **1**; in the figure, this edge is marked by an arrow. We call m_0 the *standard marking without cuts* of $S_{0,n}$. (For $n = 0$, we let $m_0 = \emptyset$.)

Let Σ be an e-surface of genus zero. A *marking without cuts* of Σ is a graph m on Σ with one marked edge such that $m = \psi^{-1}(m_0)$ for some homeomorphism $\psi \colon \Sigma \xrightarrow{\sim} S_{0,n}$. The graphs are considered up to isotopy of Σ.

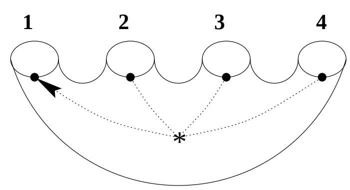


FIGURE 4. Standard marking of the standard sphere (with 4 holes).

Note that the "free ends", i.e., the vertices of the graph m other than $*$, coincide with the marked points on the boundary circles of Σ.

3.2. Proposition. *Let Σ be an e-surface of genus zero. Then there is a bijection between the set of all parameterizations without cuts of Σ and the set of all markings without cuts of Σ, given by $\psi \mapsto \psi^{-1}(m_0)$.*

The proof of this proposition is elementary and is left to the reader. We will denote either of the two sets of the proposition by $M^{\emptyset}(\Sigma)$.

Note that any marking without cuts of the surface Σ defines a bijection $A(\Sigma) \xrightarrow{\sim} \{\mathbf{1}, \dots, \mathbf{n}\}$. In particular, it defines an order on $A(\Sigma)$ and a *distinguished boundary component*, corresponding to **1**.

Thus, these graphs provide a nice pictorial way of describing parameterizations of e-surfaces. Similar to the constructions in the previous section, we now define a more general notion of a marking with cuts.

3.3. Definition. Let Σ be an e-surface. A *marking* M of Σ is a pair (C, m), where C is a cut system on Σ and m is a graph on Σ with some distinguished edges such that it gives a marking without cuts of each connected component of $\Sigma \setminus C$. We will denote the set of all markings of a surface Σ modulo isotopy by $M(\Sigma)$. A *marked surface* (*m-surface*) is an e-surface Σ together with a marking M on it.

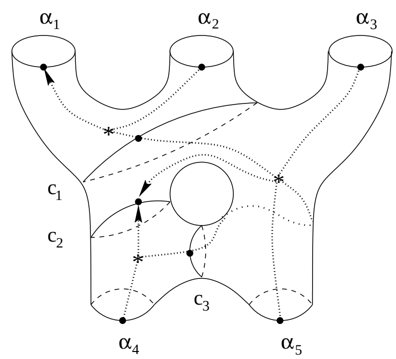


FIGURE 5. A marked surface.

An example of an m-surface is shown in Figure 5.

Note that, by definition, we have a canonical "forgetting map" $\pi\colon M(\Sigma) \to C(\Sigma)$ (recall that $C(\Sigma)$ is the set of all cut systems on Σ, see Definition 2.5).

As before, the main reason for defining these markings is the following result, which immediately follows from Proposition 3.2.

3.4. Proposition. *Let Σ be an e-surface. Then there is a bijection between the set of all parameterizations of Σ and the set of all markings of Σ.*

3.5. **Operations on markings.** Rewriting the action of the mapping class group $\Gamma(\Sigma)$ on the set of all parameterizations of Σ, defined in Subsection 2.4, in terms of markings, we see that $\Gamma(\Sigma)$ acts on $M(\Sigma)$ by $\varphi(C,m) = (\varphi(C), \varphi(m))$. We also have the following obvious operations:

Disjoint union: $\amalg\colon M(\Sigma_1) \times M(\Sigma_2) \to M(\Sigma_1 \amalg \Sigma_2)$.

Gluing: If Σ is an e-surface and $\alpha, \beta \in A(\Sigma)$ is an unordered pair with $\alpha \neq \beta$, then we have a map

$$\amalg_{\alpha,\beta}\colon M(\Sigma) \to M(\Sigma'),$$

where $\Sigma' := \amalg_{\alpha,\beta}(\Sigma)$ is obtained from Σ by identifying the boundary components α, β so that the marked points are glued to each other (this defines Σ' uniquely up to homotopy). The image of α and β is a cut on Σ'; the marking on Σ' is shown in Figure 6. If either of the edges ending at α, β was marked by an arrow, then we keep the arrow after the gluing.

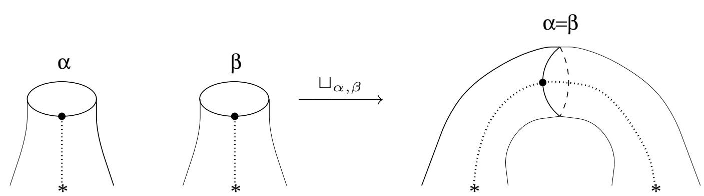


FIGURE 6. Gluing of m-surfaces.

The above two operations satisfy natural associativity properties, which we do not list here (compare with Subsection 4.2). Note also that if Σ_1, Σ_2 are two

e-surfaces, and $\alpha \in A(\Sigma_1)$, $\beta \in A(\Sigma_2)$, then we can define gluing

$$\amalg_{\alpha,\beta}\colon M(\Sigma_1) \times M(\Sigma_2) \to M(\Sigma_1 \amalg_{\alpha,\beta} \Sigma_2)$$

as the composition $M(\Sigma_1) \times M(\Sigma_2) \to M(\Sigma_1 \amalg \Sigma_2) \to M\big(\amalg_{\alpha,\beta}(\Sigma_1 \amalg \Sigma_2)\big) =: M(\Sigma_1 \amalg_{\alpha,\beta} \Sigma_2)$.

3.6. **Marking graphs.** Finally, note that for any m-surface Σ and a marking $(C, m) \in M(\Sigma)$, the graph m has some additional structure. Namely, the orientation of Σ gives a natural (counterclockwise) cyclic order on the set of germs of edges starting at a given vertex. Also, we have a distinguished set of 1-valent vertices (called "free ends"), corresponding to the boundary components; these vertices are in bijection with the set $A(\Sigma)$. It is also easy to see that all "internal edges"—i.e., edges that do not have a free end—are in bijection with the cuts of C. We will always draw such graphs on the plane so that the cyclic order on edges coincides with the counterclockwise order; as before, we will mark the distinguished edges by arrows. For example, the surface in Figure 5 gives the graph shown in Figure 7.

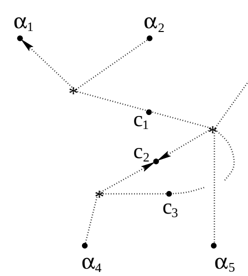


FIGURE 7. A marking graph.

We also note that Proposition 3.4 easily implies the following "rigidity theorem".

3.7. Theorem. *Let Σ be a connected e-surface, $\alpha \in A(\Sigma)$, and $M = (C, m)$ — a marking of Σ. If $\varphi \in \Gamma(\Sigma)$ is such that $\varphi(M) = M, \varphi(\alpha) = \alpha$, then $\varphi = \mathrm{id}$.*

4. THE COMPLEX $\mathcal{M}(\Sigma)$

Let Σ be an extended surface (see Definition 2.1). In Subsections 4.1, 4.2, 4.7, 4.19, 4.22 below, we will define a 2-dimensional CW complex $\mathcal{M}(\Sigma)$, which has the set $M(\Sigma)$ of all markings of Σ as the set of vertices. The edges of $\mathcal{M}(\Sigma)$ will be directed; we call them *moves*. It is convenient to look at $\mathcal{M}(\Sigma)$ as a groupoid with objects—all vertices and morphisms between two vertices—the set of homotopy classes of paths on the edges of $\mathcal{M}(\Sigma)$ from the first vertex to the second one (going along an edge in the direction opposite to its orientation is allowed). We will use group notation writing a path composed of edges $E_1, E_2, \dots$ as a product $E_1 E_2 \cdots$, and we will write E^{-1} if the edge E is traveled in the opposite direction. Then the 2-cells are interpreted as relations among the moves: we will write $E_1 \cdots E_k = \mathrm{id}$ if the closed loop formed by the edges $E_1, \dots, E_k$ is contractible in $\mathcal{M}(\Sigma)$; if we want to specify the base point for the loop, we will write $E_1 \cdots E_k(M) = \mathrm{id}(M)$. We will write $E : M \rightsquigarrow M'$ if the edge E goes from M to M'.

Our Main Theorems 4.9 and 4.24 state that the complex $\mathcal{M}(\Sigma)$ is connected and simply-connected. Up to Subsection 4.19, the e-surface Σ will be of genus 0.

4.1. **Genus** 0 **moves.** We define the following *simple moves*:

Z-move: Let Σ be a connected e-surface of genus 0 and $M = (\emptyset, m)$ a marking without cuts on Σ. Then we define the Z-move $Z : (\emptyset, m) \rightsquigarrow (\emptyset, m')$, where m' is the same graph as m but with a different distinguished edge, see Figure 8.

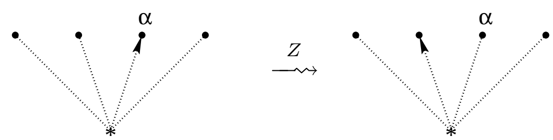


FIGURE 8. Z-move ("rotation").

F-move: Let Σ be a connected e-surface of genus 0 and $(\{c\}, m) \in M(\Sigma)$ be a marking with only one cut such that the edges ending at this cut is the distinguished ("first") edge for one of the components and the "last" edge for the other, as illustrated in Figure 9 below. Then we define the F-move ("fusion") $F_c : (\{c\}, m) \rightsquigarrow (\emptyset, m')$, where the graph m' is obtained from m by contracting the edges ending at c, see Figure 9.

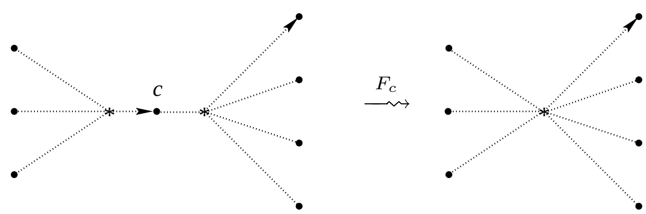


FIGURE 9. F-move ("fusion" or "cut removal").

B-move: Let $S_{0,3}$ be the standard sphere with three holes (a trinion), with no cuts and with the standard marking m_0, shown in the left hand side of Figure 10. We define the "braiding" move $B_{\alpha,\beta}$ by Figure 10.

More generally, let Σ be an e-surface and φ be a homeomorphism $\varphi\colon \Sigma \xrightarrow{\sim} S_{0,3}$. Then we define the move $B_{\alpha,\beta}\colon \big(\emptyset, \varphi^{-1}(m_0)\big) \rightsquigarrow \big(\emptyset, \varphi^{-1}(B_{\alpha,\beta}(m_0))\big)$ in $\mathcal{M}(\Sigma)$.

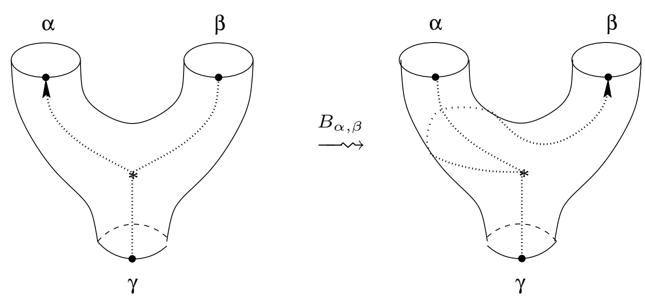


FIGURE 10. B-move ("braiding").

It easily follows from the definition that the set of moves is invariant under the action of the mapping class groupoid: for every edge $E: M_1 \rightsquigarrow M_2, M_1, M_2 \in M(\Sigma)$ and $\varphi: \Sigma \to \Sigma'$, we also have an edge $\varphi(E): \varphi(M_1) \rightsquigarrow \varphi(M_2)$.

4.2. **Propagation of moves.** We define the edges of the complex $\mathcal{M}(\Sigma)$ to be all that can be obtained from the above defined simple moves by taking disjoint unions and gluings modulo obvious equivalence relations. More precisely, we define the set of edges of $\mathcal{M}(\Sigma)$ to be all edges that can be obtained from the simple moves Z, B, F by applying the two operations listed below, modulo the equivalence relations below.

Operations:

Disjoint union: If $\Sigma = \Sigma_1 \sqcup \Sigma_2$, then for every edge $E: M_1 \rightsquigarrow M_1'$ in $\mathcal{M}(\Sigma_1)$, and every marking $M_2 \in M(\Sigma_2)$, we add an edge $E \sqcup \mathrm{id}_{M_2}: M_1 \sqcup M_2 \rightsquigarrow M_1' \sqcup M_2$ in $\mathcal{M}(\Sigma)$.

Gluing: If $\Sigma_1 = \sqcup_{\alpha,\beta}\Sigma$, then for every edge $E: M \rightsquigarrow M'$ in $\mathcal{M}(\Sigma)$ we add an edge $\sqcup_{\alpha,\beta}E: M_1 \rightsquigarrow M_1'$ in $\mathcal{M}(\Sigma_1)$, where $M_1 = \sqcup_{\alpha,\beta}M, M_1' = \sqcup_{\alpha,\beta}M'$, (cf. Figure 6).

Equivalence relations:

Functoriality: If E, E' are edges in $\mathcal{M}(\Sigma_1)$, and EE' is defined, then

$$(E \sqcup \mathrm{id}_{M_2})(E' \sqcup \mathrm{id}_{M_2}) = (EE' \sqcup \mathrm{id}_{M_2}).$$

Similarly, we have

$$\sqcup_{\alpha,\beta}(EE') = (\sqcup_{\alpha,\beta}E)(\sqcup_{\alpha,\beta}E').$$

Associativity 1: For every edge E in $\mathcal{M}(\Sigma_1)$ and markings $M_2 \in M(\Sigma_2)$, $M_3 \in M(\Sigma_3)$, we have

$$(E \sqcup \mathrm{id}_{M_2}) \sqcup \mathrm{id}_{M_3} = E \sqcup \mathrm{id}_{M_2 \sqcup M_3}.$$

Associativity 2: If $\alpha, \beta, \gamma, \delta \in A(\Sigma)$ are four different boundary components of Σ and $E: M \rightsquigarrow M'$ is an edge, then

$$\sqcup_{\alpha,\beta}(\sqcup_{\gamma,\delta}E) = \sqcup_{\gamma,\delta}(\sqcup_{\alpha,\beta}E).$$

Associativity 3: If $\Sigma = \Sigma_1 \sqcup \Sigma_2, \alpha, \beta \in A(\Sigma_1)$, and $E: M \rightsquigarrow M'$ is an edge in $\mathcal{M}(\Sigma_1)$, then

$$\sqcup_{\alpha,\beta}(E \sqcup \mathrm{id}) = \sqcup_{\alpha,\beta}(E) \sqcup \mathrm{id}.$$

4.3. *Remark.* Note that an edge $E: M \rightsquigarrow M'$ just means that we connect the points corresponding to M and M' in $\mathcal{M}(\Sigma)$. It is **not** a homeomorphism of surfaces. The relation between the groupoid $\mathcal{M}(\Sigma)$ and the mapping class group $\Gamma(\Sigma)$ can be described as follows: if $\varphi \in \Gamma(\Sigma)$ and $M \in M(\Sigma)$ then we can ask if it is possible to connect M with $\varphi(M)$ by a path in $\mathcal{M}(\Sigma)$. For example, the braiding move B connects M with $b^{-1}(M)$ for a certain $b \in \Gamma(\Sigma)$ ("braiding"). One of the main results of the next sections will be that for every φ, there is a path in $\mathcal{M}(\Sigma)$, connecting M with $\varphi(M)$, and this path is unique up to homotopy. However, as the example of the F-move shows, we also have edges $E: M \rightsquigarrow M'$, where M' cannot be obtained from M by the action of $\Gamma(\Sigma)$.

To avoid confusion, we will denote elements of $\Gamma(\Sigma)$ by lowercase letters $(b, s, t, z, \dots)$ and moves by uppercase letters $(B, S, T, Z, \dots)$. For the same reason, we use a different style of arrows for edges.

4.4. *Remark.* When describing paths in the complex $\mathcal{M}(\Sigma)$, it is useful to note that for every marking $M \in M(\Sigma)$ and any $\alpha, \beta \in A(\Sigma)$, there exists at most one edge of the form $B_{\alpha,\beta}$ originating from M: if we write $B_{\alpha,\beta} : M \rightsquigarrow M'$, then M' is uniquely defined. The same applies to Z and F_c. Thus, when describing a path in $\mathcal{M}(\Sigma)$ it suffices to give the initial marking M and a sequence of moves $B^{\pm 1}, F^{\pm 1}, Z^{\pm 1}$. This will define all the subsequent markings. However, to assist the reader, in many cases we will make pictures of the intermediate markings or at least of the corresponding graphs m.

4.5. Example (Generalized F-move). Let Σ and $(\{c\}, m) \in M(\Sigma)$ be as in the definition of the F-move, but with possibly different distinguished edges for m. Let us fix an order of the connected components of $\Sigma \setminus c$, so that $\Sigma = \Sigma_1 \sqcup_c \Sigma_2$. We will call any composition of the form $Z^a F_c(Z_1^k \sqcup \mathrm{id})(\mathrm{id} \sqcup Z_2^l)$ a *generalized F-move*; for brevity, we will frequently denote it just by F_c. The Rotation axiom formulated below implies that up to homotopy, such a composition is uniquely determined by the marking M and by the choice of the distinguished edge for the resulting marking $F_c(M)$. Moreover, the Symmetry of F axiom along with the commutativity of disjoint union, also formulated below, imply that if we switch the roles of Σ_1 and Σ_2, then we get (up to homotopy) the same generalized F-move. Thus, the homotopy class of the generalized F-move is completely determined by the marking M and by the choice of the distinguished edge for the resulting marking $F_c(M)$.

4.6. Example (Generalized braiding). Let Σ be a surface of genus zero, and let m be a marking without cuts of Σ. As discussed before, this defines an order on the set of boundary components of Σ. Let us assume that we have a presentation of $A(\Sigma)$ as a disjoint union, $A(\Sigma) = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$, where the order is given by $I_1 < I_2 < I_3 < I_4$ (some of the I_k may be empty). Then we define the *generalized braiding move* B_{I_2,I_3} to be the product of simple moves shown in Figure 11 below (note that we are using generalized F-moves, see above).

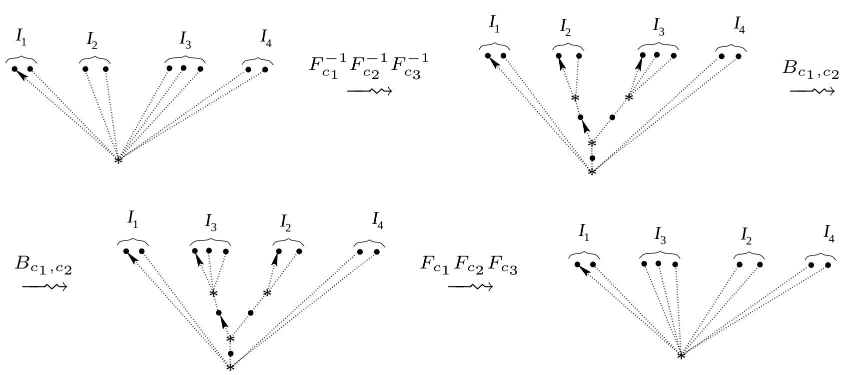


FIGURE 11. Generalized braiding move.

It follows from the Cylinder axiom (4.3) below that when Σ is a three-holed sphere as in the definition of the B-move, then the B-move $B_{\alpha,\beta}$ is homotopic in $\mathcal{M}(\Sigma)$ to the generalized braiding move $B_{\{\alpha\},\{\beta\}}$.

We will use generalized moves to simplify our formulas.

4.7. **Genus** 0 **relations.** Let us impose the following relations among the moves:

Rotation axiom: If Σ, M are as in the definition of the Z-move, then $Z^n = \mathrm{id}$, where n is the number of boundary components of Σ.

Commutativity of disjoint union: If $\Sigma = \Sigma_1 \sqcup \Sigma_2$ and E_i is an edge in $\mathcal{M}(\Sigma_i)$ $(i = 1, 2)$, then in $\mathcal{M}(\Sigma)$

$$(E_1 \sqcup \mathrm{id})(\mathrm{id} \sqcup E_2) = (\mathrm{id} \sqcup E_2)(E_1 \sqcup \mathrm{id}). \tag{4.1}$$

We will denote either of these two products by $E_1 \sqcup E_2$.

Symmetry of F-move: Let Σ, M be as in the definition of the F-move. Then $Z^{n_1 - 1} F_c = F_c(Z^{-1} \sqcup Z)$, where $n_1 = |A(\Sigma_1)|$.

Associativity of cuts: If Σ is a connected surface of genus zero, and $M = (C, m) \in M(\Sigma)$ is a marking with two cuts: $C = \{c_1, c_2\}$, then

$$F_{c_1} F_{c_2}(M) = F_{c_2} F_{c_1}(M) \tag{4.2}$$

(here F denotes generalized F-moves).

Cylinder axiom: Let $S_{0,2}$ be a cylinder with boundary components α_0, α_1 and with the standard marking $M_0 = (\emptyset, m_0)$. Let Σ be an e-surface, $M = (C, m) \in M(\Sigma)$ be a marking, and $\alpha \in A(\Sigma)$ be a boundary component of Σ. Then, for every move $E : M \rightsquigarrow M'$ in $\mathcal{M}(\Sigma)$ we require that the following square be contractible in $\mathcal{M}(\Sigma \sqcup_{\alpha,\alpha_1} S_{0,2})$:

$$\begin{tikzcd} M \sqcup_{\alpha,\alpha_1} M_0 \arrow[r, "E \sqcup_{\alpha,\alpha_1} \mathrm{id}"] \arrow[d, "F_\alpha"'] & M' \sqcup_{\alpha,\alpha_1} M_0 \arrow[d, "F_\alpha"] \\ M \arrow[r, "E"'] & M' \end{tikzcd}, \tag{4.3}$$

where in the last line, we used the homeomorphism $\varphi : \Sigma \sqcup_{\alpha,\alpha_1} S_{0,2} \xrightarrow{\sim} \Sigma$, which is equal to identity ouside of a neghborhood of $S_{0,2}$ and which maps $F_\alpha(M \sqcup_{\alpha,\alpha_1} M_0)$ to M (see Figure 12).

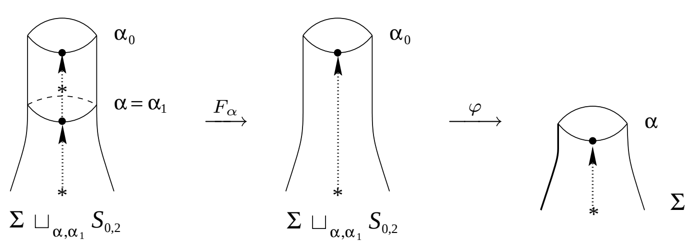


FIGURE 12. Cylinder Axiom.

Braiding axiom: Let Σ be an m-surface isomorphic to the sphere with 4 holes, indexed by α, β, γ, δ, and M be a marking such that the graph m is as shown in the left hand side of Figure 13. Then

$$B_{\alpha,\beta\gamma}(M) = B_{\alpha,\gamma} B_{\alpha,\beta}(M), \tag{4.4}$$

$$B_{\alpha\beta,\gamma}(M) = B_{\alpha,\gamma} B_{\beta,\gamma}(M). \tag{4.5}$$

For an illustration of Eq. (4.4), see Figure 13. Note that all braidings involved are generalized braidings, cf. Example 4.6.

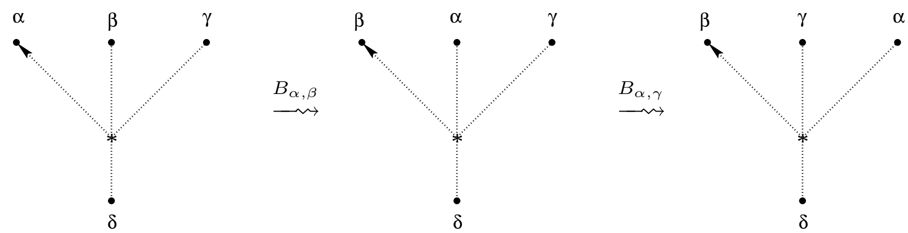


FIGURE 13. Triangle axiom (4.4).

Dehn twist axiom: Let Σ be a sphere with 2 holes, indexed by α, β, and let $M = (\emptyset, m)$ be a marking without cuts with the distinguished vertex α. Then

$$ZB_{\alpha,\beta}(M) = B_{\beta,\alpha}Z(M) \tag{4.6}$$

(generalized braidings). This axiom is equivalent to the identity $T_\alpha = T_\beta$, where T_α is the Dehn twist defined in Example 4.15 below (see Figure 15).

4.8. Proposition. *All of the relations above make sense, i.e. they describe closed paths in $\mathcal{M}(\Sigma)$.*

This proposition can be immediately verified explicitly.

We also add all relations that can be obtained from the above by taking disjoint unions and results of gluing of relations:

Propagation rules: For every relation $E = E'$ in $\mathcal{M}(\Sigma_1)$ we add relations $E \sqcup \mathrm{id} = E' \sqcup \mathrm{id}$ in $\mathcal{M}(\Sigma_1 \sqcup \Sigma_2)$ and $\sqcup_{\alpha,\beta}(E) = \sqcup_{\alpha,\beta}(E')$ in $\mathcal{M}(\sqcup_{\alpha,\beta}(\Sigma_1))$; compare with Subsection 4.2.

This completes the definition of the complex $\mathcal{M}(\Sigma)$. Note that, by definition, this complex is invariant under the action of the mapping class groupoid: for every edge $E : M_1 \rightsquigarrow M_2$ and $\varphi : \Sigma \to \Sigma'$ we also have an edge $\varphi(E) : \varphi(M_1) \rightsquigarrow \varphi(M_2)$. Similarly, for every relation $E_1 \cdots E_n = \mathrm{id}$ we also have a relation $\varphi(E_1) \cdots \varphi(E_n) = \mathrm{id}$.

Now we can formulate our main result for genus 0 surfaces.

4.9. Main Theorem ($g = 0$)**.** *Let Σ be an e-surface of genus 0. Then the above defined complex $\mathcal{M}(\Sigma)$ is connected and simply-connected.*

This theorem will be proved in Section 6. Here we give several examples, which will play an important role later.

4.10. Example (Associativity of cuts). If Σ is a surface of genus zero, $M = (C, m) \in M(\Sigma)$ and $c_1, c_2 \in C$ are two of the cuts, then

$$F_{c_1}F_{c_2}(M) = F_{c_2}F_{c_1}(M). \tag{4.7}$$

Indeed, let us consider the connected components of $\Sigma \setminus (C \setminus \{c_1, c_2\})$. If c_1, c_2 are in the same connected component, then (4.7) follows from the Associativity axiom (4.2) and the Propagation rules. If c_1 and c_2 are in different connected components, then (4.7) follows from the Commutativity of disjoint union (4.1).

4.11. Example. Let Σ be an e-surface of genus zero and let M be a marking with one cut and with the marking graph shown in Figure 14. Then

$$F_c B_{\{\alpha_1,\dots,c,\dots,\alpha_k\},J} = B_{\{\alpha_1,\dots,I,\dots,\alpha_k\},J} F_c. \tag{4.8}$$

Indeed, this easily follows from the definition of the generalized braiding and (4.7).

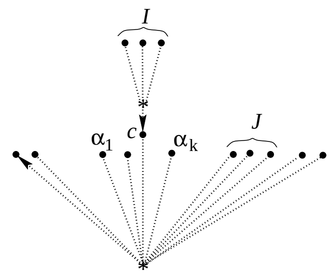


FIGURE 14.

4.12. Example. If we glue a disk to the hole β of the four-holed sphere considered in the Braiding axiom, we get $B_{\alpha,\gamma} = B_{\alpha,\gamma} B_{\alpha,\emptyset}$, which proves that $B_{\alpha,\emptyset} = \mathrm{id}$. We leave it to the reader to write accurately the formal deduction of this from the above axioms (this requires the use of the Associativity axiom and the Commutativity of disjoint union). Similarly, one can prove that $B_{\emptyset,\alpha} = \mathrm{id}$, $B_{\emptyset,\emptyset} = \mathrm{id}$.

4.13. Example (Generalized braiding). Let $\alpha_1 < \cdots < \alpha_k < \beta_1 < \cdots < \beta_l$ be boundary components of an m-surface Σ of genus zero with no cuts. Then, applying repeatedly the Braiding axiom, we get

$$B_{\{\alpha_1,\dots,\alpha_k\},\{\beta_1,\dots,\beta_l\}} = (B_{\alpha_1,\beta_l} \cdots B_{\alpha_k,\beta_l}) \cdots (B_{\alpha_1,\beta_1} \cdots B_{\alpha_k,\beta_1}). \tag{4.9}$$

(This argument also uses implicitly the Cylinder axiom and the Propagation rules.)

Hence, any path B_{I_2,I_3} can be written as a product of braiding moves of the form $B_{\alpha,\beta}$ ($\alpha \in I_2, \beta \in I_3$). Note, however, that these $B_{\alpha,\beta}$ are again generalized braidings.

4.14. Example (Braid relations). In the setup of the Braiding axiom, one has the following relation:

$$B_{\alpha,\beta} B_{\alpha,\gamma} B_{\beta,\gamma} = B_{\beta,\gamma} B_{\alpha,\gamma} B_{\alpha,\beta}. \tag{4.10}$$

Indeed, by (4.4, 4.5) this is equivalent to

$$B_{\alpha,\beta} B_{\alpha\beta,\gamma} = B_{\beta\alpha,\gamma} B_{\alpha,\beta},$$

which follows from the commutativity of disjoint union (4.1). By the Propagation rules, it follows that (4.10) holds for a sphere with n holes if we choose as the basepoint a marking without cuts such that $\alpha < \beta < \gamma$.

4.15. Example (Dehn twist around a boundary component). Let α be a boundary component of an m-surface Σ. Recall that the *Dehn twist* t_α around α is the element of the mapping class group $\Gamma(\Sigma)$ which twists the boundary component α by 360 degrees counterclockwise. For any marking (C, m), we construct a path $T_\alpha : (C, m) \rightsquigarrow (C, t_\alpha^{-1}(m))$ on the edges of $\mathcal{M}(\Sigma)$ as follows.

First, we define T_α for the standard cylinder $S_{0,2}$ with boundary components α, β and the standard marking $(\emptyset, m_0)$ with a distinguished vertex α. Then let $T_\alpha := B^{-1}_{\alpha,\beta} Z$, where $B_{\alpha,\beta} = B_{\{\alpha\},\{\beta\}}$ is the generalized braiding from Example 4.6 (see Figure 15).

For an arbitrary marked surface Σ and $\alpha \in A(\Sigma)$, define the Dehn twist as the following composition: $Z^{-k} F_c T_\alpha F_c^{-1} Z^k$, where c is a small circle encompassing α and k is such that $Z^k(M)$ has α as the first boundary component. By the Cylinder axiom, for Σ being a cylinder this coincides with the previously defined.

The Dehn twist axiom (4.6) is equivalent to the identity $T_\alpha = T_\beta$ for a cylinder with boundary components α, β.

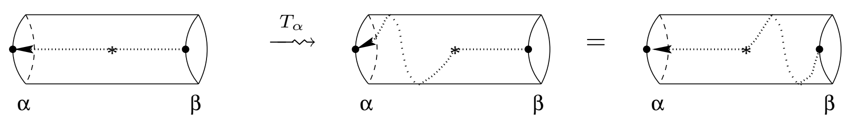


FIGURE 15. Dehn twist ($T_\alpha = T_\beta$).

Similarly, if Σ is a connected surface of genus 0, $\alpha \in A(\Sigma)$, and m is a marking without cuts such that $A(\Sigma) = \alpha \sqcup I, \alpha < I$, then

$$T_\alpha^{-1} = Z B_{\alpha,I} = B_{I,\alpha} Z^{-1}. \tag{4.11}$$

This follows from the definitions and the Dehn twist axiom.

By the commutativity of disjoint union, we have

$$T_\alpha T_\beta = T_\beta T_\alpha \tag{4.12}$$

for any $\alpha, \beta \in A(\Sigma)$. Also note that, by Example 4.12, we have $T_\alpha = \mathrm{id}$ for a sphere with one hole α, i.e. a disk.

4.16. *Remark.* One might ask why we chose T_α to connect m with $t_\alpha^{-1}(m)$ rather then $t_\alpha(m)$. The reason is that if we recall that markings m correspond to the homeomoprhisms $\psi : \Sigma \to S_{0,n}$ by $m = \psi^{-1}(m_0)$, then the marking $t_\alpha^{-1}(m)$ corresponds to the homeomorphism $t_i \circ \psi$, where i is the index of the hole in $S_{0,n}$ corresponding to α. Thus, the edge T_α connects ψ with $t_i \circ \psi$. Similarly, the edge $B_{\alpha_i,\alpha_{i+1}}$ connects a homeomorphism ψ with $b_i \circ \psi$, where $b_i \in \Gamma(S_{0,n})$ is the braiding of i-th and $(i+1)$-st holes (cf. Proposition 6.6).

4.17. Example (Dehn twist around a cut). Let Σ be a surface, and M be a marking containing a cut c. Define the moves T'_c, T''_c by doing the same construction as above on either of the sides of the cut c. Then $T'_c = T''_c$. Indeed, it suffices to prove this when Σ is a cylinder, in which case it follows from the Cylinder axiom and the Dehn twist axiom that $T'_c = T_\alpha$, $T''_c = T_\beta$, $T_\alpha = T_\beta$. We will use the notation T_c for both T'_c, T''_c.

4.18. Example. Let Σ be a sphere with 3 holes, labeled by α, β, γ, and let M be a marking without cuts such that it defines the order $\alpha < \beta < \gamma$. Then we claim that

$$T_\alpha(M) = T_\beta T_\gamma B_{\gamma,\beta} B_{\beta,\gamma}(M). \tag{4.13}$$

Indeed, by (4.11) and the Braiding axiom, we have

$$T_\alpha^{-1} = B_{\beta\gamma,\alpha} = B_{\beta,\alpha}B_{\gamma,\alpha}, \quad T_\beta^{-1} = B_{\beta,\alpha}B_{\beta,\gamma}, \quad T_\gamma^{-1} = B_{\gamma,\beta}B_{\gamma,\alpha},$$

which implies $T_\gamma^{-1}T_\alpha T_\beta^{-1} = B_{\gamma,\beta}B_{\beta,\gamma}$. Now (4.13) follows from the commutativity (4.12).

4.19. **Higher genus moves.** Now let us consider e-surfaces Σ of positive genus. In this case, we need to add to the complex $\mathcal{M}(\Sigma)$ one more simple move and several more relations.

S-move: Let $S_{1,1}$ be a "standard" torus with one boundary component and one cut, and with the marking M shown on the left hand side of Figure 16. Then we add the edge $S\colon M \rightsquigarrow M'$ where the marking M' is shown on the right hand side of Figure 16.

More generally, let Σ be an e-surface and ψ be a homeomorphism $\psi\colon \Sigma \xrightarrow{\sim} S_{1,1}$. Then we add the move $S\colon \psi^{-1}(M) \rightsquigarrow \psi^{-1}(M')$.

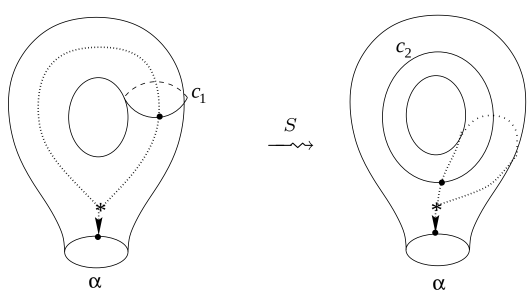


FIGURE 16. S-move.

Of course, we also add all moves that can be obtained from the above Z-, F-, B- and S-moves by taking disjoint unions and gluing, as in Subsection 4.2.

4.20. *Remark.* If Σ is a surface of genus one with one hole, we can identify the set of all markings with one cut on Σ with the set of all homeomorphisms $\psi\colon \Sigma \xrightarrow{\sim} S_{1,1}$ (see Theorem 3.7). Then the S-move connects the marking ψ with $s \circ \psi$, where $s \in \mathrm{P}\Gamma(S_{1,1})$ acts on $\mathrm{H}_1(\overline{S_{1,1}}) = \mathbb{Z}c_1 \oplus \mathbb{Z}c_2 \simeq \mathbb{Z}^2$ by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. (Here $\overline{S_{1,1}}$ is the closed torus obtained by gluing a disk to the boundary component of $S_{1,1}$.)

4.21. Example (Generalized S-move). Let Σ be a torus with n holes, and let $M = (\{c_1\}, m)$ be a marking with one cut c_1, such that the graph m is shown on the left hand side of Figure 17. Then we define the *generalized S-move S* as the composition of moves shown in Figure 17.

It can be shown that the cut c, and thus, the S-move, is uniquely defined by c_1 and m for $n \geq 1$. As before, it follows from the Cylinder axiom that for $n = 1$, this generalized S-move coincides with the one defined in Subsection 4.19. For $n = 0$ there are two possible choices for the cut c, and thus, there are two generalized S-moves $S^{(1)}, S^{(2)} : M \rightsquigarrow M'$. It will follow from the relation (4.14) below that $S^{(1)} = S^{(2)}$.

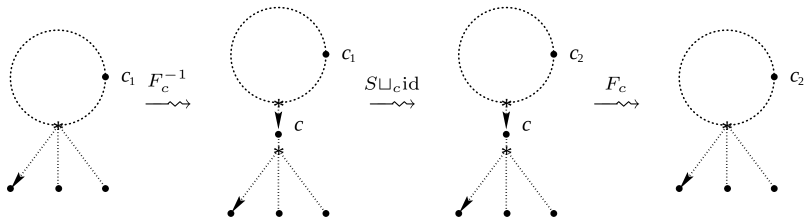


FIGURE 17. Generalized S-move (for $n = 3$).

4.22. **Higher genus relations.** Besides those from Subsection 4.7, we have the following additional relations:

Relations for $g = 1, n = 1$: Let Σ be a marked torus with one hole α, isomorphic to the one shown on the left hand side of Figure 18. For any marking $M = (\{c\}, m)$ with one cut, we let T act on M as the Dehn twist T_c around c (see Example 4.17). Then we impose the following relations:

$$S^2 = Z^{-1} B_{\alpha,c_1}, \tag{4.14}$$

$$(ST)^3 = S^2. \tag{4.15}$$

The left hand side of relation (4.14) is shown in Figure 18. For an illustration of (4.15), see Appendix A.

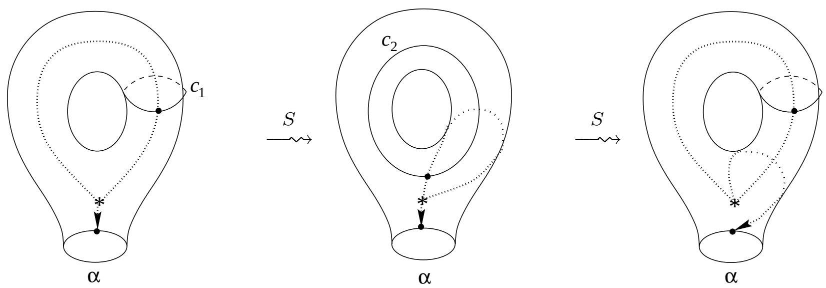


FIGURE 18. The relation $S^2 = Z^{-1} B_{\alpha,c_1}$.

Relation for $g = 1, n = 2$: Let Σ be a marked torus with two holes α, β, isomorphic to the one shown in Figure 19. As before, for any marking $M = (\{c\}, m)$ of Σ with one cut, we let T act on M as the Dehn twist T_c (see Example 4.17). Similarly, let $\widetilde{T} := T_{c+\beta} = T_c T_\beta B_{\beta,c} B_{c,\beta}$ (cf. Examples 4.17, 4.18). Then we have (note that we use generalized S-moves):

$$B_{\alpha,\beta} F_{c_1} F_{c_2}^{-1} = S^{-1} \widetilde{T}^{-1} T S. \tag{4.16}$$

The basepoint of the path (4.16) is the marking shown in Figure 19 below. A detailed picture of the whole path is presented in Appendix B.

Note that, by their construction, the above relations are invariant under the action of the mapping class group.

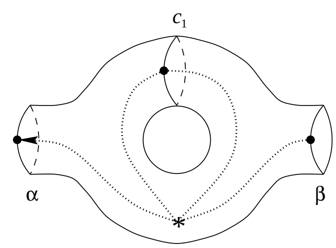


FIGURE 19. A marked torus with two holes.

It is not trivial that the relations (4.15, 4.16) make sense, i.e. that they are indeed closed paths in $\mathcal{M}(\Sigma)$. This is equivalent to checking that the corresponding identities hold in the mapping class group $\Gamma(\Sigma)$. This is indeed so (see, e.g., [Bir, MS2]). Another way of verifying that the relations (4.14–4.16) make sense is to draw explicitly all the marked surfaces and moves and to show that this is indeed a closed loop. We give such pictures in Figure 18 and in Appendices A and B.

4.23. Example. Let Σ be a marked torus with one cut c_1 and one hole α (see the left hand side of Figure 16). Then we have:

$$(ST)^3 = S^2, \tag{4.17}$$

$$S^2T = TS^2, \tag{4.18}$$

$$S^4 = T_\alpha^{-1}. \tag{4.19}$$

Indeed, (4.17) is exactly (4.15). Equation (4.18) follows from (4.14), the Cylinder axiom, and the commutativity of disjoint union, and (4.19) easily follows from (4.14) and (4.11).

In particular, this implies that the elements $t, s \in \Gamma_{1,1}$ (cf. Remark 4.3) satisfy the relations (4.17–4.19). In fact, it is known that these are the defining relations of the group $\Gamma_{1,1}$ (see [Bir]).

Now we can formulate our main result for arbitrary genus.

4.24. Main Theorem ($g \geq 0$)**.** *Let Σ be an e-surface. Let $\mathcal{M}(\Sigma)$ be the above defined complex with vertices: all markings of Σ, edges obtained from the Z-, F-, B-, and S-moves by disjoint unions and gluing, and 2-cells given by the relations in Subsections* 4.7, 4.22*. Then $\mathcal{M}(\Sigma)$ is connected and simply-connected.*

This theorem will be proved in Section 7.

5. The complex $\mathcal{M}^{\max}(\Sigma)$

In this section, we formulate and prove a version of the Main Theorem 4.24 in which one uses only spheres with ≤ 3 holes. [2]

Clearly, any surface Σ can be cut into a union of spheres with ≤ 3 holes, i.e. trinions, cylinders, disks, and spheres. Let us call such a cut system on Σ *maximal*. We will define a 2-dimensional CW complex $\mathcal{M}^{\max}(\Sigma)$ with vertices: all markings

[2]This is referred to as “a small lego box” by Grothendieck [G].

$M = (C, m) \in M(\Sigma)$ such that the cut system C is maximal (we call such M maximal). The edges and the 2-cells of $\mathcal{M}^{\max}(\Sigma)$ are described below.

EDGES:

Z-move: defined in the same way as in Subsection 4.1, but for spheres with ≤ 3 holes.

F-move: $F_c : M \rightsquigarrow M'$ – defined as in Subsection 4.1, but for $M, M' \in \mathcal{M}^{\max}(\Sigma)$. It is easy to see that this happens iff $\Sigma = \Sigma_1 \sqcup_c \Sigma_2$ with only one cut c and one of Σ_1, Σ_2 is either a cylinder or a disk. If $\Sigma = \Sigma_1 \sqcup_c \Sigma_2$ and both Σ_1 and Σ_2 are trinions, then F_c is not defined in $\mathcal{M}^{\max}(\Sigma)$, because the result would be a sphere with 4 holes without cuts.

A-move: if Σ is a sphere with 4 holes, and M is a marking with one cut shown in the left-hand side of Figure 20 below, then we define the A-move $A : M \rightsquigarrow M'$ as in Figure 20.

B-move: defined in the same way as in Subsection 4.1.

S-move: defined in the same way as in Subsection 4.19.

As before, we also add all edges which can be obtained from these ones by disjoint unions and gluing modulo the equivalence relations of Subsection 4.2.

Thus, the edges of the complex $\mathcal{M}^{\max}(\Sigma)$ are those edges of $\mathcal{M}(\Sigma)$ which have both endpoints in $M^{\max}(\Sigma)$, plus the new A-moves. Note that for every A-move $A_{c',c} : M \rightsquigarrow M'$, the same markings can be connected in $\mathcal{M}(\Sigma)$ by the path $F_{c'}^{-1} F_c(\Sigma)$.

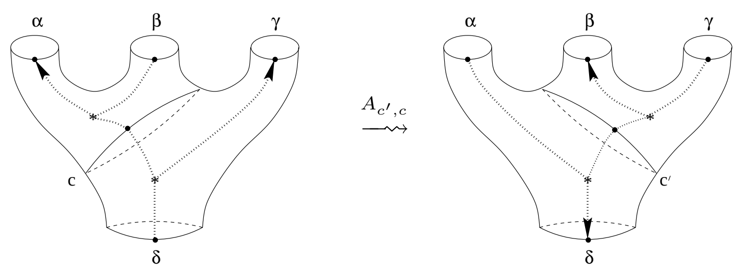


FIGURE 20. A-move ("associativity constraint").

Note that the generalized braiding moves can not be defined in $\mathcal{M}^{\max}(\Sigma)$, with the exception of the Dehn twist T_c: its definition can be repeated in $\mathcal{M}^{\max}$.

RELATIONS:

Weak associativity of cuts: if Σ is a surface of genus zero, and $M \in \mathcal{M}^{\max}(\Sigma)$ is a marking with two cuts c_1, c_2 such that $\Sigma = \Sigma_1 \sqcup_{c_1} \Sigma_2 \sqcup_{c_2} \Sigma_3$, and Σ_2 is a cylinder, then $F_{c_1} = F_{c_2}$.

Symmetry of F: same as in Subsection 4.7, but one of Σ_1, Σ_2 has at most 2 holes, and the other at most 3.

Rotation axiom: same as in Subsection 4.7, but for $n \leq 3$.

Commutativity of disjoint union: same as in Subsection 4.7.

Cylinder axiom: same as in Subsection 4.7.

Pentagon axiom: (see Appendix C).

Two Hexagon axioms: (see Appendix C).

Self-duality of associativity: if Σ, M are as in definition of the A-move, then $A^2(M) = \mathrm{id}(M)$.

Triangle axiom: (see Appendix C).
Dehn twist axiom: same as in Subsection 4.7.
Relations for $g = 1, n = 1$: same as in Subsection 4.22.
Relation for $g = 1, n = 2$: (see Appendix B).

As before, we also add all the relations that can be obtained from these ones by disjoint unions and gluing. This completes the definition of the complex $\mathcal{M}^{\max}(\Sigma)$.

Note that the Hexagon axioms are essentially the Braiding axioms (4.4), (4.5), only rewritten so that they start at a maximal marking and instead of products of the form $F_{c'}^{-1}F_c$ we used the A-moves. Indeed, by the definition of generalized braiding (Example 4.6), the left hand side of (4.4) is $F_{c_2}B_{\alpha,c_2}F_{c_2}^{-1}$, while the right hand side is $F_{c_3}B_{\alpha,\gamma}F_{c_3}^{-1}F_{c_1}B_{\alpha,\beta}F_{c_1}^{-1}$. Therefore, (4.4) can be rewritten as

$$B_{\alpha,c_2}A_{c_2,c_1} = A_{c_2,c_3}B_{\alpha,\gamma}A_{c_3,c_1}B_{\alpha,\beta}, \tag{5.1}$$

which is the Hexagon relation.

The same can be said about the relation for $g = 1, n = 2$, see Appendix B. Thus, the only new relations are the Pentagon and Triangle axioms.

5.1. Theorem. *For any e-surface Σ, the complex $\mathcal{M}^{\max}(\Sigma)$ defined above is connected and simply-connected.*

Proof. It suffices to show that the complexes $\mathcal{M}^{\max}(\Sigma)$ and $\mathcal{M}(\Sigma)$ are homotopically equivalent, after which the result follows from Theorem 4.24. To show the equivalence, we introduce the following notion. Let $M, M' \in M(\Sigma)$. We say that M' is a subdivision of M if M can be obtained by applying to M' a sequence of F-moves (not F^{-1}!) and Z-moves. We will write $M \subset M'$.

Now, for a given $M \in M(\Sigma)$, denote

$$\mathrm{Sub}(M) = \{M' \in M^{\max}(\Sigma) | M \subset M'\}.$$

5.2. Lemma. 1. *Every two markings $M', M'' \in \mathrm{Sub}(M)$ can be connected by a path in $\mathrm{Sub}(M)$ consisting of a sequence of the F-, Z-, and A-moves and their inverses.*

2. *Every loop in $\mathrm{Sub}(M)$, composed of F-, Z-, and A-moves and their inverses, is contractible in $\mathcal{M}^{\max}(\Sigma)$.*

The proof of this lemma is left to the reader. Obviously, it suffices to consider the case $\Sigma = S_{0,n}, M = (\emptyset, m_0)$, in which case it is essentially a version of MacLane's coherence theorem.

Now, let us choose for every $M \in M(\Sigma)$ one element $\tau(M) \in \mathrm{Sub}(M)$ ("maximal subdivision") in such a way that we do not add any new cuts to components which already are spheres with ≤ 3 holes. Then one easily sees that the map $\tau : M(\Sigma) \to M^{\max}(\Sigma)$ can be extended to a map of CW complexes. Indeed, it is obvious how this map is defined on the edges of B and S type. As for the F-edge, let us define for $F_c : M_1 \sqcup_c M_2 \rightsquigarrow M$ its image $\tau(F_c)$ as any path $\tau(M_1) \sqcup_c \tau(M_2) \rightsquigarrow \tau(M)$, composed of A-, Z-, and F-moves in $\mathcal{M}^{\max}$; by the lemma above, such a path is unique up to homotopy. Similarly we define $\tau(Z)$. It is immediate to see that τ respects all the relations in $\mathcal{M}$.

Conversely, we have an obvious embedding $\mathcal{M}^{\max} \subset \mathcal{M}$. It is immediately verified that the composition of this embedding with τ is an auto-equivalence of $\mathcal{M}^{\max}$. Thus, every loop in $\mathcal{M}^{\max}$ is homotopic to a loop of the form $\tau(l)$ for some closed loop in $\mathcal{M}$. But every loop l in $\mathcal{M}$ is contractible; thus, the same holds for $\tau(l)$. □

6. Proof of the main theorem for genus 0

In this section we will prove the Main Theorem 4.9 for genus 0: that for any extended surface Σ of genus 0 the complex $\mathcal{M}(\Sigma)$, defined in Subsections 4.1, 4.2, 4.7, is connected and simply-connected.

6.1. **Outline of the proof.** Let us start by slightly modifying the complex $\mathcal{M}(\Sigma)$. Namely, let us add to it edges corresponding to each of the paths B_{I_1,I_2} defined in Example 4.6 and the Dehn twists T_α (see examples 4.15, 4.17).

For each of these new moves, we add the expression for it as a product of simple moves Z, F, B as a new relation. As before, we also add all edges and relations that can be obtained from these ones by disjoint union and gluing. Let us call the complex obtained in this way $\widetilde{\mathcal{M}}(\Sigma)$. Obviously, if $\widetilde{\mathcal{M}}(\Sigma)$ is connected and simply-connected, then so is $\mathcal{M}(\Sigma)$.

The proof is based on extending the forgetting map $\pi : M(\Sigma) \to C(\Sigma)$ (see Subsection 3.3) to a map of CW complexes $\pi : \widetilde{\mathcal{M}}(\Sigma) \to \mathcal{C}(\Sigma)$, and showing that both the base and any fiber are connected and simply-connected. More precisely, we will use the following proposition, whose easy proof is left to the reader. For future use, we formulate it in a slightly more general form than we need now.

6.2. Proposition. *Let $\mathcal{M}, \mathcal{C}$ be 2-dimensional CW complexes (with directed edges), and let $\pi\colon \mathcal{M}^{[1]} \to \mathcal{C}^{[1]}$ be a map of their 1-skeletons, which is surjective both on vertices and on edges. Suppose that the following conditions are satisfied:*

1. *$\mathcal{C}$ is connected and simply-connected.*
2. *For every vertex $C \in \mathcal{C}$, $\pi^{-1}(C)$ is connected and simply-connected in $\mathcal{M}$ (that is, every closed loop l which lies completely in $\pi^{-1}(C)$ is contractible in $\mathcal{M}$).*
3. *Let $C_1 \overset{e}{\rightsquigarrow} C_2$ be an edge in $\mathcal{C}$, and let $M_1' \overset{e'}{\rightsquigarrow} M_2'$ and $M_1'' \overset{e''}{\rightsquigarrow} M_2''$ be two its liftings to $\mathcal{M}$. Then one can choose paths $M_1' \overset{e_1}{\rightsquigarrow} M_1''$ in $\pi^{-1}(C_1)$ and $M_2' \overset{e_2}{\rightsquigarrow} M_2''$ in $\pi^{-1}(C_2)$ such that the square*
$$\begin{tikzcd} M_1' \arrow[r, "e'"] \arrow[d, "e_1"'] & M_2' \arrow[d, "e_2"] \\ M_1'' \arrow[r, "e''"'] & M_2'' \end{tikzcd}$$
is contractible in $\mathcal{M}$.
4. *For every 2-cell X in $\mathcal{C}$, its boundary ∂X can be lifted to a contractible loop in $\mathcal{M}$.*

Then the complex $\mathcal{M}$ is connected and simply-connected.

6.3. **The complex $\mathcal{C}(\Sigma)$.** The set of vertices of $\mathcal{C}(\Sigma)$ is the set $C(\Sigma)$ of all cut systems on Σ. The (directed) edges of $\mathcal{C}(\Sigma)$ will correspond to the following

$\bar{\mathrm{F}}$-move: Let Σ be an e-surface of genus zero, and let $C \in C(\Sigma)$ be a cut system on Σ, consisting of a single cut: $C = \{c\}$. Then we define the $\bar{\mathrm{F}}$-move $\bar{F}_c\colon C \rightsquigarrow \emptyset$, which removes c.

As before, we also add all the moves that can obtained from the $\bar{\mathrm{F}}$-move above by disjoint unions and gluing subject to the obvious associativity relations as in Subsection 4.2. In particular, for any $C \in C(\Sigma)$ (Σ not necessarily of genus zero) and a removable cut $c \in C$, we have a move $\bar{F}_c : C \rightsquigarrow C \setminus \{c\}$.

Let us impose the following relations for these moves:

Associativity of cut removal: Let $c_1, c_2 \in C$, $c_1 \neq c_2$. Then

$$\bar{F}_{c_1}\bar{F}_{c_2}(C) = \bar{F}_{c_1}\bar{F}_{c_2}(C).$$

We add these relations, as well as all relations obtained by taking disjoint unions and gluing (cf. Subsection 4.8), as 2-cells of the complex $\mathcal{C}(\Sigma)$.

By construction, there is a canonical map of CW complexes $\pi\colon \widetilde{\mathcal{M}}(\Sigma) \to \mathcal{C}(\Sigma)$, which extends the forgetting map $\pi\colon M(\Sigma) \to C(\Sigma) : (C, m) \mapsto C$. Namely, we define π on edges by $\pi(F) = \bar{F}$, $\pi(B) = \mathrm{id}$, $\pi(Z) = \mathrm{id}$.

6.4. Theorem ($g = 0$)**.** *The above complex $\mathcal{C}(\Sigma)$ is connected and simply-connected.*

Proof. It is easy to see from the Associativity axiom that every product $\bar{F}_{c_1}\bar{F}_{c_2}^{-1}$ can be replaced by either $\bar{F}_{c_2}^{-1}\bar{F}_{c_1}$ or by identity. Thus, every loop can be deformed to one of the form $\bar{F}^{-1}\cdots\bar{F}^{-1}\bar{F}\cdots\bar{F}$. On the other hand, every cut system can be connected to the empty one using $\bar{\mathrm{F}}$-moves (this is where we need that Σ is of genus zero!). Thus, it suffices to consider only loops starting at the empty cut system. But every loop of the form $\bar{F}^{-1}\cdots\bar{F}^{-1}\bar{F}\cdots\bar{F}$ starting at the empty cut system must be homotopic to identity. □

6.5. **Simply-connectedness of the fiber.** Let C be a vertex of $\mathcal{C}(\Sigma)$, i.e. a cut system on Σ. Denote by $\{\Sigma_a\}$ the set of connected components of $\Sigma \setminus C$. Then $\pi^{-1}(C) \subset M(\Sigma)$ can be canonically identified with the product $\prod_a M^{\emptyset}(\Sigma_a)$, where $M^{\emptyset}(\Sigma_a)$ is the set of all markings without cuts of Σ_a, (cf. Definition 3.1). Thus, to check assumption 2 of Proposition 6.2, it is enough to check that every $\widetilde{\mathcal{M}}^{\emptyset}(\Sigma)$, where Σ is a sphere with n holes, is connected and simply-connected. (Here $\widetilde{\mathcal{M}}^{\emptyset}(\Sigma)$ is the subcomplex of $\widetilde{\mathcal{M}}(\Sigma)$ with vertices $M^{\emptyset}(\Sigma)$, and edges given by Z-moves and the generalized B-moves.)

By Proposition 3.2, the set $M^{\emptyset}(\Sigma)$ is in bijection with the mapping class group $\Gamma_{0,n} = \Gamma(S_{0,n})$. Let us consider the following elements of $\Gamma_{0,n}$:

$t_i, i = 1, \dots, n$	: Dehn twist around i-th puncture
$b_i, i = 1, \dots, n-1$	: Braiding of i-th, $(i+1)$-st punctures
z	: Rotation, i.e. a homeomorphism which acts on the set of boundary components by $\mathbf{i} \mapsto \mathbf{i+1}, \mathbf{n} \mapsto \mathbf{1}$ and preserves the real axis.

6.6. Proposition. *The group $\Gamma_{0,n}$ is generated by elements b_i, $i = 1, \dots, n-1$, t_i, $i = 1, \dots, n$, and z with the following defining relations*

$$b_i b_j = b_j b_i, \qquad |i-j| > 1, \tag{6.1}$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \tag{6.2}$$

$$b_i t_j = t_j b_i, \qquad |i-j| > 1,\ i = j+1, \tag{6.3}$$

$$b_i^{\pm 1} t_i = t_{i+1} b_i^{\pm 1}, \tag{6.4}$$

$$t_i t_j = t_j t_i, \tag{6.5}$$

$$z^n = 1, \tag{6.6}$$

$$b_1 \dots b_{n-1} t_n = z, \tag{6.7}$$

$$zt_n = t_1 z. \tag{6.8}$$

This proposition is known (see, e.g., [MS2], where it is formulated in a somewhat different form), so we skip the proof.

6.7. *Remark.* Denote the boundary components of Σ by $\alpha_1, \dots, \alpha_n$. Let ψ be a homeomorphism $\Sigma \xrightarrow{\sim} S_{0,n}$ which induces the order $\alpha_1 < \dots < \alpha_n$. As was noted before, such a homeomorphims can be viewed as an element of $M^\emptyset(\Sigma)$. Then in $\widetilde{\mathcal{M}}^\emptyset(\Sigma)$ we have the edges

$$\begin{aligned} B_{\alpha_i,\alpha_{i+1}} : \psi &\rightsquigarrow b_i \circ \psi, && i = 1, \dots, n-1, \\ T_{\alpha_i} : \psi &\rightsquigarrow t_i \circ \psi, && i = 1, \dots, n, \\ Z : \psi &\rightsquigarrow z \circ \psi, \end{aligned} \tag{6.9}$$

compare with Remark 4.16.

6.8. Now we can prove that the complex $\widetilde{\mathcal{M}}^\emptyset(\Sigma)$ is connected and simply-connected. To prove that it is connected, it suffices to check that the homeomorphisms $\psi, g \circ \psi$, where g is one of the generators of the group $\Gamma_{0,n}$, can be connected by a path in $\widetilde{\mathcal{M}}^\emptyset(\Sigma)$. This is obvious because $\widetilde{\mathcal{M}}^\emptyset(\Sigma)$ contains the edges (6.9).

To prove that $\widetilde{\mathcal{M}}^\emptyset(\Sigma)$ is simply-connected, note first that it follows from Example 4.13 that every path can be deformed to a path that only uses $B_{\alpha,\beta}$ for neighboring boundary components $\alpha < \beta$. Therefore, any path can be contracted to a sequence of moves of the form (6.9). Thus, it remains to show that any closed loop composed of the moves (6.9) is contractible. Since these moves correspond to the generators of the group $\Gamma_{0,n}$, used in Proposition 6.6, this reduces to checking that the loops corresponding to the relations (6.1–6.8) are contractible. This is straightforward. The braid relation (6.2) has already been established in Example 4.14. Using (4.9), we can show that it suffices to check the relations (6.7), (6.8) for $n = 2$, in which case they immediately follow from the Dehn twist axiom. The other relations follow from the commutativity of disjoint union (4.1) and the Cylinder axiom. For example, both (6.3) and (6.4) correspond to the identity $B_{\alpha_i,\alpha_{i+1}} T_{\alpha_j} = T_{\alpha_j} B_{\alpha_i,\alpha_{i+1}}$.

This proves that $\widetilde{\mathcal{M}}^\emptyset(\Sigma)$ is simply-connected, and thus establishes assumption 2 of Proposition 6.2.

6.9. **Finishing the proof.** So far, we have defined the map $\pi : \widetilde{\mathcal{M}}(\Sigma) \to \mathcal{C}(\Sigma)$ and proved that both the base and the fiber are connected and simply-connected, thus establishing assumptions 1 and 2 of Proposition 6.2. Assumption 4 is quite obvious, since the only 2-cells in $\mathcal{C}(\Sigma)$ are those obtained from the associativity axiom, and they can be lifted to the 2-cells in $\widetilde{\mathcal{M}}(\Sigma)$ also given by the associativity axiom. Thus, the only thing that remains to be checked is the assumption 3.

It is easy to see from the results of Subsection 6.8 that any two markings with the same cut system can be connected by a product of the moves $Z, B_{\alpha,\beta}$ ($\alpha, \beta \in A(\Sigma_a)$ where Σ_a is a connected component of $\Sigma \setminus C$), cf. Example 4.13.

Thus, we only need to consider assumption 3 with e_1 being either Z or B. For Z, the statement immediately follows from the symmetry of F axiom.

Hence, it suffices to check that for $\Sigma = S_{0,n} \sqcup_c S_{0,k}$ and $\alpha, \beta \in A(S_{0,n})$, there exists a path e_2 such that the following square is contractible in $\widetilde{\mathcal{M}}(\Sigma)$:

$$\begin{tikzcd} M_1' \arrow[r, "F_c"] \arrow[d, "B_{\alpha,\beta}"'] & M_2' \arrow[d, "e_2"] \\ M_1'' \arrow[r, "F_c"'] & M_2'' \end{tikzcd} .$$

This can be easily proved explicitly, using the axioms and Example 4.13. Indeed, if both α and β are distinct from c, we can take $e_2 = B_{\alpha,\beta}$. If $\beta = c$, then we can take $e_2 = B_{\alpha,I}$ where $I = A(S_{0,k}) \setminus c$.

Thus, we see that the map $\pi\colon \widetilde{\mathcal{M}}(\Sigma) \to \mathcal{C}(\Sigma)$ satisfies all assumptions of Proposition 6.2 and thus, $\widetilde{\mathcal{M}}(\Sigma)$ is connected and simply-connected. This concludes the proof of Theorem 4.9.

7. Proof of the main theorem for higher genus

In this section we will prove the Main Theorem 4.24 for higher genus: that for any extended surface Σ the complex $\mathcal{M}(\Sigma)$, defined in Subsect. 4.1, 4.7, 4.19, 4.22, is connected and simply-connected. The strategy of the proof is similar to the one used in the genus 0 case.

First, we extend the complex $\mathcal{M}(\Sigma)$ by adding all disjoint unions and gluings of generalized braidings and generalized S-moves as new edges, and adding their definitions as new 2-cells. We denote this new complex by $\widetilde{\mathcal{M}}(\Sigma)$; again, $\mathcal{M}(\Sigma)$ is connected and simply-connected iff $\widetilde{\mathcal{M}}(\Sigma)$ is connected and simply-connected. Second, we define a complex $\widetilde{\mathcal{C}}(\Sigma)$ with vertices the set $C(\Sigma)$ of all cut systems of Σ. Then we apply Proposition 6.2 to the canonical projection $\pi : \widetilde{\mathcal{M}}(\Sigma) \to \widetilde{\mathcal{C}}(\Sigma)$. The most difficult part of the proof is checking that the complex $\widetilde{\mathcal{C}}(\Sigma)$ is simply-connected, which is based on the results of [HT] and [H].

7.1. **The complex $\mathcal{C}(\Sigma)$.** The definition of the complex $\widetilde{\mathcal{C}}(\Sigma)$ is parallel to the definition of $\widetilde{\mathcal{M}}(\Sigma)$. First, we define a complex $\mathcal{C}(\Sigma)$ with vertices the set $C(\Sigma)$ of all cut systems of Σ (see Definition 2.5). The (directed) edges of $\mathcal{C}(\Sigma)$ are the following moves:

$\bar{\text{F}}$-move: Let Σ be an e-surface of genus zero, and let $C \in C(\Sigma)$ be a cut system on Σ, consisting of a single cut: $C = \{c\}$. Then we define the $\bar{\text{F}}$-move $\bar{F}_c\colon C \rightsquigarrow \emptyset$, which removes c.

$\bar{\text{S}}$-move: Let Σ be an e-surface of genus one with one boundary component, and let C be a cut system on Σ, consisting of a single cut: $C = \{c\}$. Let c' be a simple closed curve on Σ such that c' intersects c transversally at exactly one point (see Figure 21). Then we add an edge $\bar{S}_{c,c'} : \{c\} \rightsquigarrow \{c'\}$.

As before, we also add all the edges which can be obtained from the $\bar{\text{F}}$-, $\bar{\text{S}}$-edges above by applying the operations of disjoint union and gluing as in Subsection 4.2. This implies that for every removable cut $c \in C, C \in C(\Sigma)$ (Σ not necessarily of genus zero), we have an edge $\bar{F}_c : C \rightsquigarrow C \setminus \{c\}$.

7.2. Example (Generalized $\bar{\text{S}}$-move). Let Σ be a torus with n holes, and let c, c_1 be cuts on Σ as in Example 4.21. Then we define the *generalized $\bar{S}$-move* as the composition of moves shown in Figure 17 with F, S replaced by $\bar{F}, \bar{S}$. Again, it

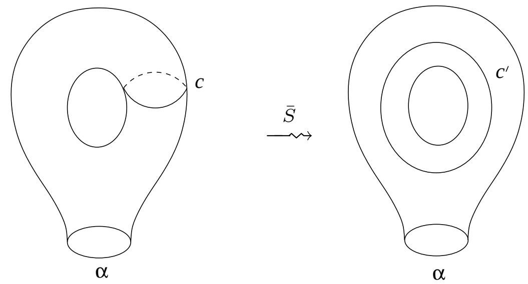


FIGURE 21. $\bar{\mathsf{S}}$-move.

can be shown that the cut c is uniquely determined by c_1, c_2. Thus, we will denote this generalized S̄-move by $\bar{S}_{c_1,c_2}$.

It is easy to see that if $C = \{c_1, \dots, c_k\} \in C(\Sigma)$, and c_1' is a simple closed curve on Σ which intersects c_1 transversally at exactly one point, and does not intersect any other cuts in C, then the connected component Σ_1 of $\Sigma \setminus \{c_2, \dots, c_k\}$ which contains c_1, c_1' has genus one, and thus we have a generalized S̄-move $\bar{S}_{c_1,c_1'}$: $\{c_1, c_2, \dots, c_k\} \rightsquigarrow \{c_1', c_2, \dots, c_k\}$ obtained by gluing the generalized S̄-move on Σ_1 with the identity on other components.

7.3. **Relations in $\mathcal{C}(\Sigma)$.** Let us impose the following relations for the F̄- and S̄-moves:

Associativity of cut removal: If $c_1, c_2 \in C$ are two cuts on Σ such that $\bar{F}_{c_1}\bar{F}_{c_2}(C)$ is defined, then $\bar{F}_{c_2}\bar{F}_{c_1}(C)$ is defined and

$$\bar{F}_{c_1}\bar{F}_{c_2}(C) = \bar{F}_{c_2}\bar{F}_{c_1}(C). \tag{7.1}$$

Inverse for S̄: Let Σ be a surface of genus one with one hole, and c, c' be as in the definition of S̄-move, cf. Figure 21. Then

$$\bar{S}_{c',c}\bar{S}_{c,c'}(\{c\}) = \mathrm{id}(\{c\}). \tag{7.2}$$

Relation between S̄ and F̄: Let Σ be a surface of genus one with two holes, and c_1, c_2, c_3 be three cuts as shown in Figure 22 below. Then

$$\bar{F}_{c_1}\bar{F}_{c_2}^{-1}(\{c_1\}) = \bar{S}_{c_3,c_2}\bar{S}_{c_1,c_3}(\{c_1\}). \tag{7.3}$$

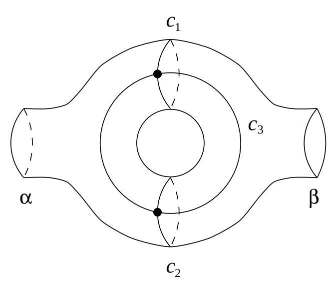


FIGURE 22. Relation between $\bar{\mathsf{S}}$ and $\bar{\mathsf{F}}$.

Triangle relation for $\bar{\mathrm{S}}$: Let Σ be a torus with one hole, and c_1, c_2, c_3 be three cuts as shown in Figure 23 below. Then

$$\bar{S}_{c_3,c_1}\bar{S}_{c_2,c_3}\bar{S}_{c_1,c_2}(\{c_1\}) = \mathrm{id}(\{c_1\}). \tag{7.4}$$

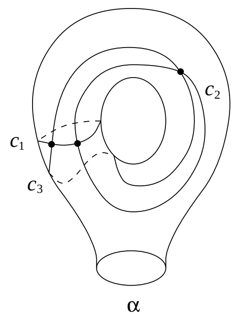


FIGURE 23. Triangle relation for $\bar{\mathrm{S}}$.

Commutativity of disjoint union: If $\Sigma = \Sigma_1 \sqcup \Sigma_2$ and E_i is an edge in $\mathcal{C}(\Sigma_i)$ $(i = 1, 2)$, then in $\mathcal{C}(\Sigma)$

$$(E_1 \sqcup \mathrm{id})(\mathrm{id} \sqcup E_2) = (\mathrm{id} \sqcup E_2)(E_1 \sqcup \mathrm{id}). \tag{7.5}$$

Note that we also add the relations obtained from the above under the action of the mapping class group; for example, in Eq. (7.3), c_1, c_2, c_3 may be any three cuts such that c_1, c_3 and c_2, c_3 intersect at exactly one point and there are no other intersections.

Again, we add the propagation rules, i.e. we add all relations obtained by taking disjoint unions and gluing, cf. Subsection 4.7. Note that when Σ is of genus 0, the complex $\mathcal{C}(\Sigma)$ is the same as the one defined in Subsection 6.3.

Finally, as before, let us replace the complex $\mathcal{C}(\Sigma)$ by the equivalent complex $\widetilde{\mathcal{C}}(\Sigma)$, obtained by adding the generalized $\bar{\mathrm{S}}$-moves as new edges (rather than considering them as composition of moves), and adding the definition of these moves as new relations.

7.4. **The projection.** We define the map of CW complexes $\pi\colon \widetilde{\mathcal{M}}(\Sigma) \to \widetilde{\mathcal{C}}(\Sigma)$, such that on the vertices it is given by the canonical forgetting map $\pi\colon M(\Sigma) \to C(\Sigma)$, and $\pi(Z) = \mathrm{id}$, $\pi(F) = \bar{F}$, $\pi(B) = \mathrm{id}$, $\pi(S) = \bar{S}$.

Our goal is to prove that the projection map π satisfies all the assumptions of Proposition 6.2. Obviously, as soon as we prove this, we get a proof of the Main Theorem 4.24.

First of all, we need to check that the map π is surjective on edges, i.e. that every move in $\widetilde{\mathcal{C}}(\Sigma)$ can be obtained by a projection of a move in $\widetilde{\mathcal{M}}(\Sigma)$. This is obvious for the $\bar{\mathrm{F}}$-move, and almost obvious for the generalized $\bar{\mathrm{S}}$-move.

7.5. **Checking assumption 2.** Let us check that assumption 2 of Proposition 6.2 holds for the projection map defined in Subsection 7.4. Clearly, for every cut system C, $\pi^{-1}(C) \subset M(\Sigma) = \prod_a M^{\emptyset}(\Sigma_a)$ (cf. Subsection 6.5), and a path l which lies in $\pi^{-1}(C)$ must be composed of Z-, B-moves only (in particular, it cannot include an S-move). Thus, the same proof as in the genus zero case (see Subsect. 6.5–6.8) applies here.

7.6. **Checking assumption 3.** Let us check that assumption 3 of Proposition 6.2 holds for the projection map defined in Subsection 7.4, i.e. that for every edge $e : C_1 \rightsquigarrow C_2$ and two its liftings e', e'' to $\widetilde{\mathcal{M}}$, they can be included in a commutative square. If the edge e is of the $\bar{\mathrm{F}}$-type, then the same proof as in the genus 0 case (see Subsection 6.9) applies.

Thus, we have to consider the case when the edge e is of $\bar{\mathrm{S}}$-type. This reduces to asking what different liftings a given generalized $\bar{\mathrm{S}}$-move has. This is answered by the following lemma.

7.7. Lemma. *Let Σ be a torus with n holes $\alpha_1, \dots, \alpha_n$, and let c be a cut on it as shown in Figure 24. Let $M = (\{c\}, m)$ be a marking on Σ such that $S(M) = (\{c'\}, m')$, with the cut c' shown in Figure 24 (recall that the generalized S-move is uniquely defined by M). Then any such M can be connected by a sequence of moves $B_{\alpha_i,\alpha_{i+1}}$ $(i = 1, \dots, n-1)$ and T_{α_i} $(i = 1, \dots, n)$ and their inverses with one of the two standard markings shown in Figure 24.*

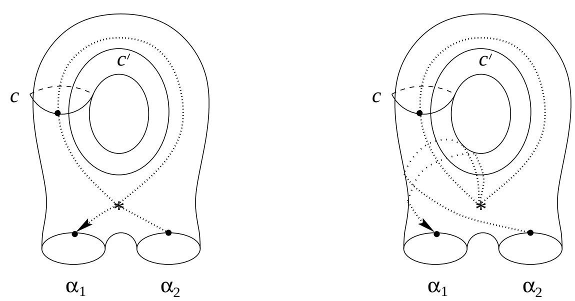


FIGURE 24. Two standard markings of a torus with n holes (for $n = 2$).

Proof. Denote the two markings in Figure 24 by M', M''. Now, let M be the marking satisfying the conditions of the theorem. Then there exists a homeomorphism $\varphi' : \Sigma \to \Sigma$ such that $\varphi'(M) = M'$ and $\varphi'(c) = c$. Moreover, it is easy to see that we must also have $\varphi'(c') = c'$.

Presenting the torus as a rectangle with identified opposite sides, we see that such a φ' is the same as a homeomorphism of a rectangle with n holes onto itself which maps vertical sides to vertical and horizontal to horizontal. If φ' preserves each of the sides, then without loss of generality we may assume that it acts as identity on the boundary of the rectangle. But it is well-known that the group of such homeomorphisms is generated by the elements b_i $(i = 1, \dots, n-1)$ and t_j $(j = 1, \dots, n)$, cf. [Bir, Theorem 1.10]. Thus, in this case M and M' can be connected by a sequence of B, T moves as in the theorem.

If the homeomorphism φ' interchanges the opposite sides of rectangle (i.e., interchanges the sides of the cuts c, c'), then we need to repeat the same argument for M''; it is easy to check that in this case the homeomorphism φ'' will preserve each of the sides of the rectangle. □

Arguing as in Subsection 6.9, we see that it suffices to check that we can find a path e_2 which does not change the cut system and such that $Se_1 = e_2S$, with e_1 being $B_{\alpha_i,\alpha_{i+1}}$ $(i = 1, \dots, n-1)$ or T_{α_i} $(i = 1, \dots, n)$. This is obvious, since we

can take $e_2 = e_1$, and the equality would follow from the commutativity of disjoint union. And, finally, it remains to show that we can find e_1, e_2 such that the square

$$\begin{array}{ccc} M' & \xrightarrow{S} & M_2' \\ {\scriptstyle e_1}\downarrow & & \downarrow{\scriptstyle e_2} \\ M'' & \xrightarrow{S} & M_2'' \end{array}$$

is commutative, where M_1', M_1'' are the standard markings in Figure 24. But this can be easily achieved by letting $e_1 = Z^{-1}B_{\{\alpha_1,\dots,\alpha_n\},c}$, $e_2 = Z^{-1}B_{\{\alpha_1,\dots,\alpha_n\},c'}$. Indeed, it follows from (4.14) that $e_1 = S^2, e_2 = S^2$ (cf. Figure 18), and thus the square above is obviously commutative.

7.8. **Checking assumption 4.** Let us check that assumption 4 of Proposition 6.2 holds for the projection map defined in Subsection 7.4, i.e. that for every every 2-cell X in $\widetilde{\mathcal{C}}(\Sigma)$, its boundary can be lifted to a contractible loop in $\widetilde{\mathcal{M}}(\Sigma)$. In other words, we need to check that every relation in $\widetilde{\mathcal{C}}(\Sigma)$ can be obtained by projecting some relation in $\widetilde{\mathcal{M}}(\Sigma)$. Clearly, it suffices to check this for the basic relations (7.1–7.4).

For the associativity axiom (7.1), this is obvious: it can be obtained by projecting the associativity axiom (4.7). The inverse axiom (7.2) can be easily obtained from the relation $S^2 = Z^{-1}B$ (see (4.14) and Figure 18). The relation (7.3) between $\bar{S}$ and $\bar{F}$ is nothing else but the projection of the defining relation (4.16), see Appendix B. Similarly, the triangle relation (7.4) is exactly the projection of the relation $(ST)^3 = S^2$ (see (4.15) and Appendix A).

7.9. Theorem ($g \geq 0$)**.** *The complex $\widetilde{\mathcal{C}}(\Sigma)$ is connected and simply-connected.*

This theorem is proved in Subsections 7.10, 7.11 below.

Without loss of generality we may assume that Σ is connected. Recall that a cut system is called minimal if it contains no removable cuts; this is exactly what is called a "cut system" in [HT, H]. Let $\widetilde{\mathcal{C}}^{\min}(\Sigma)$ be the complex with vertices: all minimal cut systems, edges: all generalized $\bar{\text{S}}$-moves, and the relations induced by the relations in $\widetilde{\mathcal{C}}(\Sigma)$ (i.e., a path in $\widetilde{\mathcal{C}}^{\min}(\Sigma)$ is contractible if it is contractible as a path in $\widetilde{\mathcal{C}}(\Sigma)$).

7.10. Proposition. *The subcomplex $\widetilde{\mathcal{C}}^{\min}(\Sigma)$ is connected and simply-connected.*

Proof. The proof is based on the results of [H, Section 2], where a certain 2-dimensional CW complex Y_2 is introduced, which has the same vertices and edges as $\widetilde{\mathcal{C}}^{\min}(\Sigma)$, but different 2-cells. Since Y_2 is connected and simply-connected [H, Theorem 2.2], it suffices to show that all the relations of Harer follow from the relations in $\widetilde{\mathcal{C}}(\Sigma)$.

The first relation [H, Eq. (R_1)] has the form

$$\bar{S}_{c_3,c_1}\bar{S}_{c_2,c_3}\bar{S}_{c_1,c_2}(\{c_1\}) = \mathrm{id}(\{c_1\}), \tag{7.6}$$

where c_1, c_2, c_3 are some cycles on a surface Σ. There are many different choices, displayed in [H, Figure 4]. However, it is easy to see that, by making one additional cut, they all reduce to the configuration shown in Figure 25.

To prove (7.6) for the cuts shown in Figure 25, redraw Figure 25 as shown in Figure 26, and add one more cut c_4.

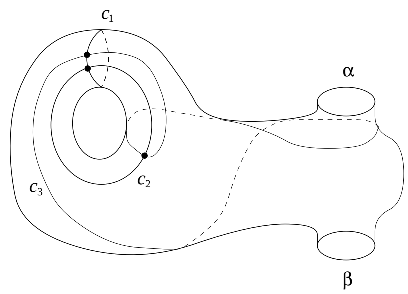


FIGURE 25. Harer's first relation.

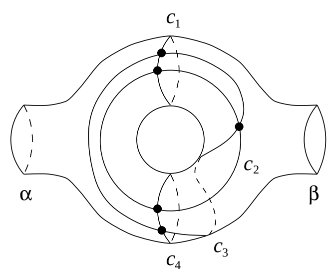


FIGURE 26. Proof of the Harer's first relation.

Then consider the following diagram:

$$\begin{tikzcd}
& \{1\} \arrow[d] \arrow[ddr, "\bar{S}"] & \\
& \{4\} \arrow[dl, "\bar{S}"'] \arrow[dr, "\bar{S}"] & \\
\{3\} \arrow[uur, "\bar{S}"] & & \{2\} \arrow[ll, "\bar{S}"]
\end{tikzcd}$$

where $\{1\}$ stands for the cut system consisting of one cut c_1, etc., and the vertical arrow $\{1\} \to \{4\}$ is given by $\bar{F}_{c_1}\bar{F}_{c_4}^{-1}$. The outer triangle is exactly the left-hand side of the relation (7.6). On the other hand, the two top small triangles are contractible by (7.3), and the bottom triangle is contractible because it can be obtained from the Triangle relation (7.4) by gluing a three-punctured sphere to the hole. This completes the proof of the first Harer's relation (7.6).

The second relation [H, Eq. (R_2)] states that if $c_1, \dots, c_4$ are 4 cycles such that c_1 intersects c_2 at one point, c_3 intersects c_4 at one point, and there are no other intersections (this is illustrated by the diagram in Figure 27), then

$$\bar{S}_{12}\bar{S}_{34} = \bar{S}_{34}\bar{S}_{12},$$

where $\bar{S}_{ij} = \bar{S}_{c_i,c_j}$. This follows from the commutativity of disjoint union (7.5).

The third relation [H, Eq. (R_3)] is

$$\bar{S}_{42}\bar{S}_{54}\bar{S}_{61}\bar{S}_{26}\bar{S}_{35}\bar{S}_{13}(\{c_1, c_2\}) = \mathrm{id}(\{c_1, c_2\}),$$

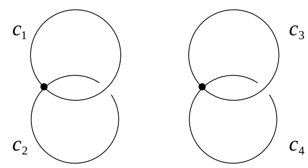


FIGURE 27. Harer's second relation.

where $c_1, \ldots, c_6$ are cuts on an e-surface Σ of genus two with one hole, displayed in Figure 28 (cf. [H, Figure 3]).

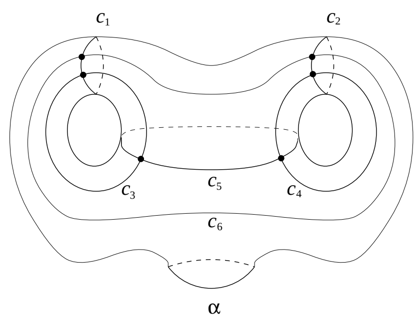


FIGURE 28. Harer's third relation.

This relation follows from (7.3):

$$\begin{tikzcd}
 & \{1,2\} \arrow[rr, "\bar{S}"] \arrow[dr, shift left, "\bar{F}^{-1}"] & & \{2,3\} \arrow[dr, "\bar{S}"] & \\
\{1,4\} \arrow[ur, "\bar{S}"] & & \{1,2,5\} \arrow[ul, shift left, "\bar{F}"] \arrow[rr, shift left, "\bar{F}"] \arrow[dl, shift left, "\bar{F}"] & & \{2,5\} \arrow[ll, shift left, "\bar{F}^{-1}"] \arrow[dl, "\bar{S}"] \\
 & \{1,5\} \arrow[ul, "\bar{S}"] \arrow[ur, shift left, "\bar{F}^{-1}"] & & \{5,6\} \arrow[ll, "\bar{S}"] &
\end{tikzcd} .$$

This completes the proof that $\widetilde{\mathcal{C}}^{\min}(\Sigma)$ is simply-connected. $\square$

7.11. **Proof of Theorem 7.9.** Since any cut system can be joined to a minimal one by erasing cuts, and $\widetilde{\mathcal{C}}^{\min}(\Sigma)$ is connected, it follows that $\widetilde{\mathcal{C}}(\Sigma)$ is connected.

To prove that $\widetilde{\mathcal{C}}(\Sigma)$ is simply-connected, we first note that every path $\bar{F}_{c_1}\bar{F}_{c_2}^{-1}$ is homotopic to either id, or $\bar{F}_{c_2}^{-1}\bar{F}_{c_1}$, or $\bar{S}_{c_3,c_2}\bar{S}_{c_1,c_3}$ for certain c_3. Indeed, if both $\bar{F}_{c_1}\bar{F}_{c_2}^{-1}(C)$ and $\bar{F}_{c_2}^{-1}\bar{F}_{c_1}(C)$ are defined, then they are equal. Suppose that the first one is defined but the second one is not. Then c_1 and c_2 do not intersect and $\Sigma \setminus C$ becomes of positive genus if we remove them from the cut system C. Hence, there is a cut c_3 which intersects both of them, and we can apply (7.3).

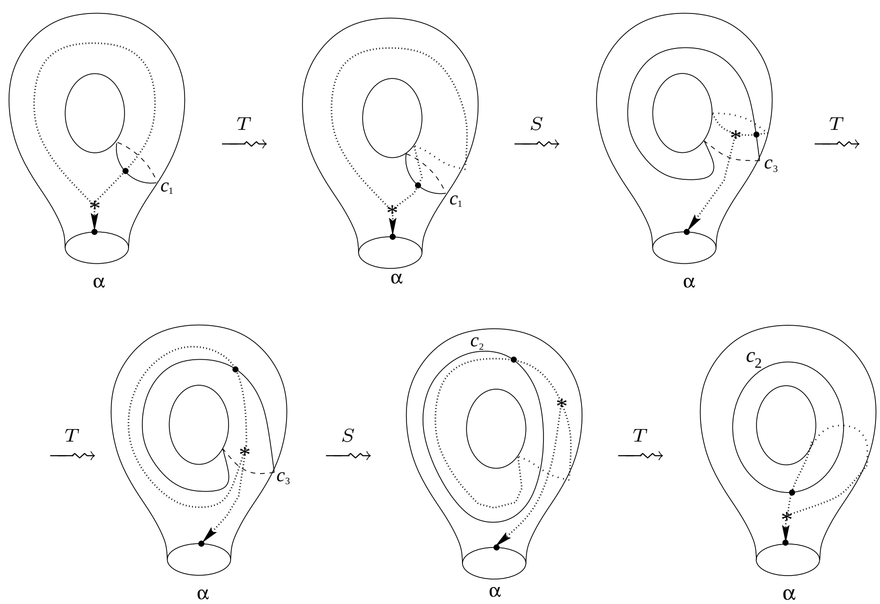
Similarly, note that every path $\bar{S}_{c_2,c_3}\bar{F}_{c_1}^{-1}$ is homotopic to $\bar{F}_{c_1}^{-1}\bar{S}_{c_2,c_3}$. Indeed, $\bar{S}_{c_2,c_3}\bar{F}_{c_1}^{-1}(C)$ being defined implies that c_1 intersects neither c_2 nor c_3. Then by the commutativity of disjoint union, $\bar{F}_{c_1}^{-1}\bar{S}_{c_2,c_3}(C)$ is also defined and they are equal.

Now take any closed loop l in $\widetilde{\mathcal{C}}(\Sigma)$. Without loss of generality, we may assume that it has a minimal cut system as the basepoint. Using the above two remarks, we can deform l into a loop composed only of $\bar{\mathrm{S}}$-moves. Indeed, we can move any $\bar{\mathrm{F}}^{-1}$-move to the left until it meets an $\bar{\mathrm{F}}$-move and either cancels out or creates a pair of $\bar{\mathrm{S}}$-moves. Repeating this procedure, we will get a loop composed only of $\bar{\mathrm{F}}$- and $\bar{\mathrm{S}}$-moves. But since the number of cuts in the initial cut system should be the same, it is actually composed only of $\bar{\mathrm{S}}$-moves.

If we start with a minimal cut system and apply to it a sequence of $\bar{\mathrm{S}}$-moves, we again get a minimal cut system. Since the subcomplex $\widetilde{\mathcal{C}}^{\min}(\Sigma)$ of $\widetilde{\mathcal{C}}(\Sigma)$ is simply-connected by Proposition 7.10, it follows that l is contractible.

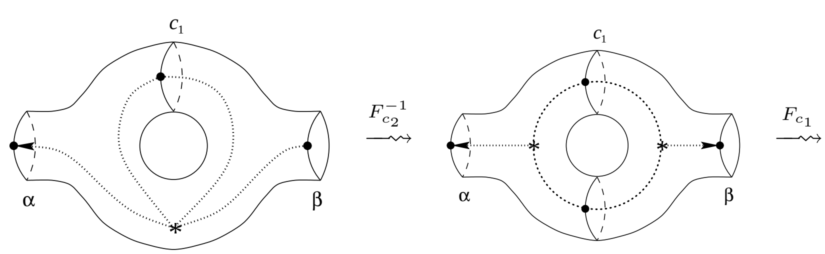
This completes the proof that $\widetilde{\mathcal{C}}(\Sigma)$ is simply-connected. Therefore, we have checked all the assumptions of Proposition 6.2, and thus, we have proved that the complex $\mathcal{M}(\Sigma)$ is connected and simply-connected.

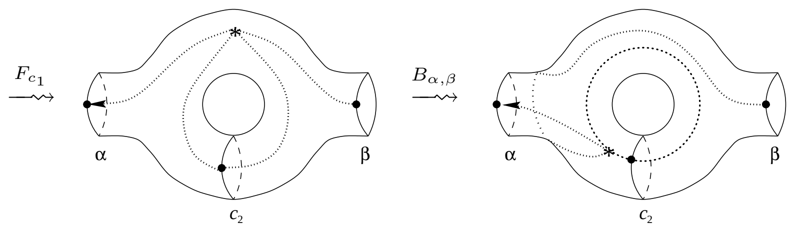
Appendix A. The relation (4.15): $TSTST = S$



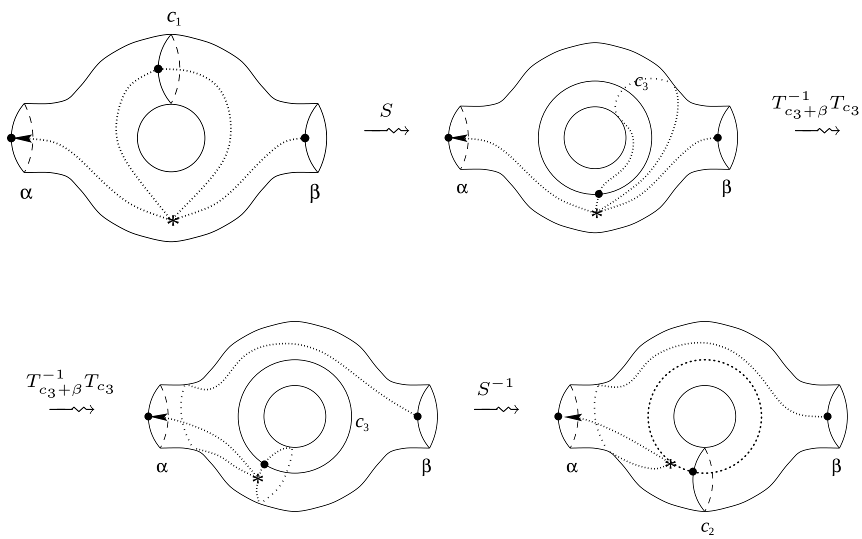
Appendix B. The relation (4.16): $B_{\alpha,\beta}F_{c_1}F_{c_2}^{-1} = S^{-1}\widetilde{T}^{-1}TS$

The left hand side of (4.16) is:

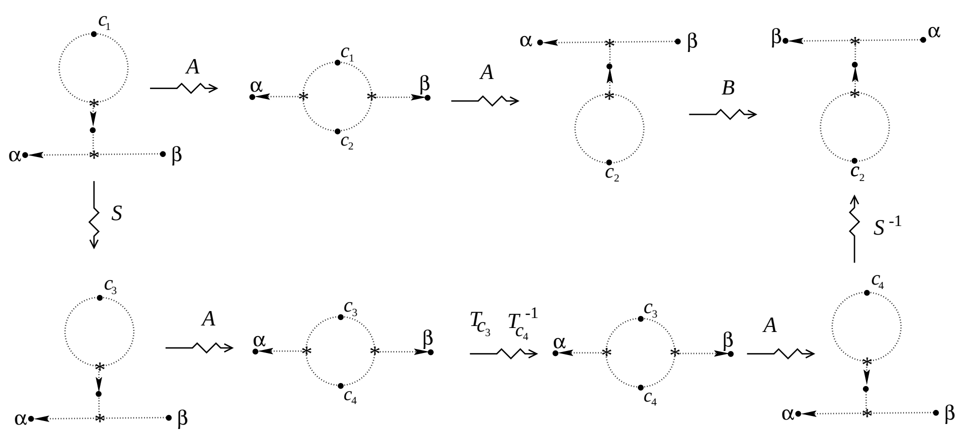




The right hand side of (4.16) is:



Below is a version of the same relation which makes sense in the complex $\mathcal{M}^{\max}$, i.e. which only uses spheres with ≤ 3 holes—for brevity, we just wrote the corresponding marking graphs.



Appendix C. Triangle, Pentagon, and Hexagon relations

In this appendix, we formulate the Triangle, Pentagon, and Hexagon axioms, which were used in Section 5. For brevity, we only give pictures of the corresponding marking graphs; as was mentioned before, this is sufficient to uniquely reconstruct the moves. All unmarked edges in these diagrams are compositions of the form $(Z^*\sqcup Z^*)A(Z^*\sqcup Z^*)$; the powers of Z are uniquely determined by the distinguished edges in the diagram and by the requirement that this composition is well-defined.

The Triangle axiom requires that the diagram in Figure 29 below be commutative.

The Pentagon relation is shown in Figure 30.

Finally, there are two Hexagon axioms. One of them claims the commutativity of the diagram in Figure 31; the other is obtained by replacing all occurrences of B by B^{-1}, so that $B_{\alpha\beta}$ is replaced by $B^{-1}_{\beta\alpha}$, etc.

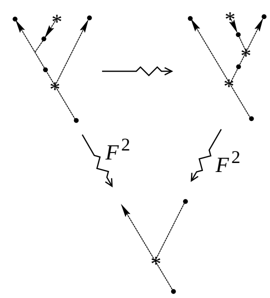


Figure 29. Triangle relation.

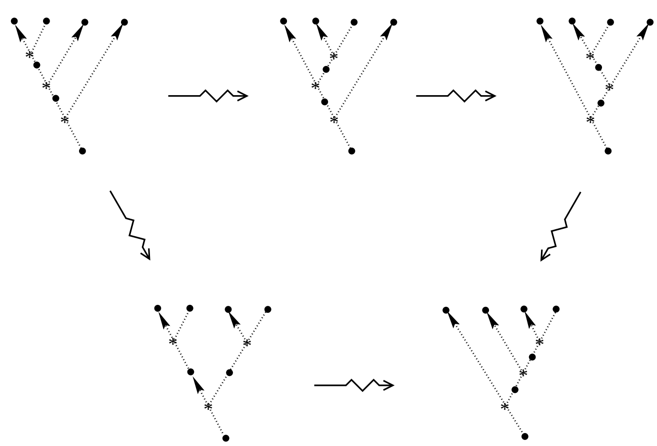


Figure 30. Pentagon relation.

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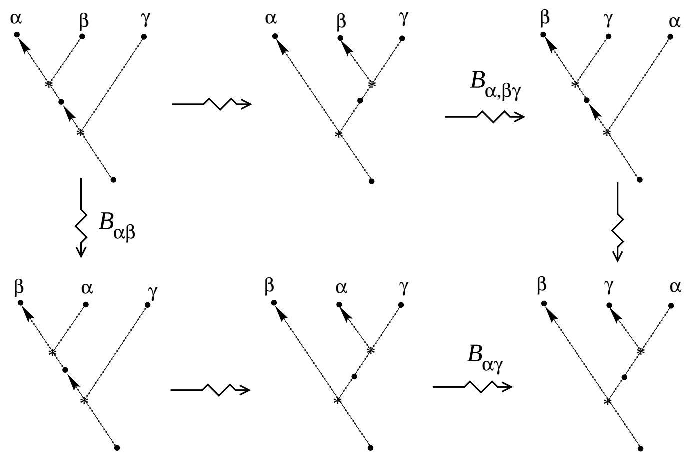

FIGURE 31. Hexagon relation.

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