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The Category of Representations of the General Linear Groups over a Finite Field

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This paper shows that the complex representations of the general linear groupoid over a fixed finite field form a braided monoidal category which is furthermore describable in terms of generators and relations.

INTRODUCTION

Fix a field $\mathbf{F}$ with q elements. This paper provides a structural approach to the study of complex representations of the finite general linear groups $GL(n,q) = GL(n,\mathbf{F})$ for $n \geq 0$. We show that the totality of these representations forms a *braided tensor category* which, we furthermore show, can be completely described in terms of generators and relations. More explicitly, let $\mathscr{R}GL(n,q)$ be the category of finite dimensional complex representations of $GL(n,q)$ and let $XGL(n,q)$ be its character group. There is a classical *external* product of representations defined by using induction on parabolic subgroups. It is a functor of two variables

$$\mathscr{R}GL(n,q) \times \mathscr{R}GL(m,q) \to \mathscr{R}GL(m+n,q),$$

which is associative up to coherent isomorphism [ML]. We shall call it the external *tensor* product and use the notation $M \otimes N$ for the external

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tensor product of M with N. It leads to a corresponding (external) product of characters

$$XGL(n,q) \times XGL(m,q) \to XGL(m+n,q)$$

determining a graded ring structure on the direct sum

$$XGL(q) = \bigoplus_{n \geq 0} XGL(n,q).$$

Green [Gn] proved that the ring $XGL(q)$ is actually commutative. This means that the exterior products $M \otimes N$ and $N \otimes M$ are isomorphic, without exhibiting an isomorphism. However, an explicit isomorphism was defined by Harish-Chandra for cuspidal representations M and N. A fuller explanation of the commutativity of $XGL(q)$ would be achieved if we could find a natural isomorphism

$$c_{M,N}: M \otimes N \to N \otimes M,$$

satisfying the coherence properties of a *symmetry* for the tensor product [ML]. Such a symmetry does not seem to exist. We shall exhibit a natural isomorphism satisfying the coherence conditions of a *braiding* [JS1, JS2]. Assembling together the categories $\mathscr{R}GL(n,q)$, we obtain a braided tensor category $\mathscr{R}\mathscr{G}\mathscr{L}(q)$ which is semisimple abelian. We show that it can be described purely in terms of generators and relations, or equivalently, in terms of representations of generalised Hecke algebras [HL].

1. THE GENERAL LINEAR GROUPOID

The general linear groupoid over $\mathbf{F}$ is the category $\mathscr{G}\mathscr{L}(q)$ whose objects are the finite vector spaces over $\mathbf{F}$ and whose arrows are the linear *isomorphisms*. For objects V, W of $\mathscr{G}\mathscr{L}(q)$, we write $Iso(V,W)$ for the set of linear isomorphisms

$$\rho: V \xrightarrow{\sim} W.$$

The general linear group $GL(V) = Iso(V,V)$ acts on $Iso(V,W)$ by composition on the right, and $GL(W)$ acts on the left. In particular, the group $GL(\mathbf{F}^n) = GL(n,q)$ acts on the right of $Iso(\mathbf{F}^n, V)$. The set $Iso(\mathbf{F}^n, V)$ is a torsor (that is, a principal homogeneous set) over $GL(n,q)$ when $\dim V = n$; otherwise, it is empty.

Let $\mathscr{V}ect$ denote the category of finite-dimensional complex vector spaces, and *all* linear functions. A (finite-dimensional complex) *representation* of $\mathscr{G}\mathscr{L}(q)$ is a functor

$$M: \mathscr{G}\mathscr{L}(q) \to \mathscr{V}ect.$$

An *arrow* between two representations is a natural transformation. We shall denote the category of representations by $[\mathcal{GL}(q), \mathcal{V}ect]$. There is obviously a close relationship between the representations of the groupoid $\mathcal{GL}(q)$ and the representations of the groups $GL(n, q)$. The fact is that the groupoid $\mathcal{GL}(q)$ is *equivalent* to the disjoint union of the groups $GL(n, q)$. It follows that we have an equivalence of categories

$$[\mathcal{GL}(q), \mathcal{V}ect] \xrightarrow{\sim} \Pi_{n \geq 0} \mathcal{R}GL(n, q),$$

where $\Pi_{n \geq 0} \mathcal{R}GL(n, q)$ is the *product* of the categories $\mathcal{R}GL(n, q)$. Recall that an *object* of the product category is a sequence of representations

$$GL(n, q) \times M[n] \to M[n], \qquad n \geq 0,$$

and an *arrow*

$$f: (M[n] \| n \geq 0) \to (N[n] \| n \geq 0)$$

is a sequence $f = (f_n | n \geq 0)$ of homomorphisms f_n: $M[n] \to N[n]$. More explicitly, the equivalence is defined by restricting each representation M of $\mathcal{GL}(q)$ to the groups $Iso(\mathbf{F}^n, \mathbf{F}^n) = GL(n, q)$, $n \geq 0$, from which a sequence $M[n] = M(\mathbf{F}^n)$, $n \geq 0$, can be obtained. Conversely, from any such sequence $(M[n] \| n \geq 0)$, we can define a functor $\mathbf{M}$: $\mathcal{GL}(q) \to \mathcal{V}ect$ as follows: for any V of dimension n in $\mathcal{GL}(q)$, take $M(V)$ to be the quotient of the complex vector space $Iso(\mathbf{F}^n, V) \times M[n]$ by the relations

$$(\rho g, e) \sim (\rho, ge) \qquad \text{for } \rho \in Iso(\mathbf{F}^n, V),\ g \in GL(n, q),\ e \in M[n].$$

There is a canonical isomorphism $M(\mathbf{F}^n) \cong M[n]$.

Clearly $[\mathcal{GL}(q), \mathcal{V}ect]$ is an abelian category. However, the noetherian condition fails in an infinite product of categories. This motivates the introduction of the full subcategory $\mathcal{RGL}(q)$ of $[\mathcal{GL}(q), \mathcal{V}ect]$. We define an object M of $[\mathcal{GL}(q), \mathcal{V}ect]$ to belong to $\mathcal{RGL}(q)$ if and only if $M[n] = 0$ for all but a finite number of n. In this case, we can write M as a (finite) direct sum

$$M = \bigoplus_{n \geq 0} M_n,$$

where $M_n(V)$ is equal to 0, except when $\dim V = n$ in which case it is equal to $M(V)$. The components M_n and $M[n]$ obviously determine each other, and we shall often identify them. We have an equivalence

$$\mathcal{RGL}(q) \xrightarrow{\sim} \Pi'_{n \geq 0} \mathcal{R}GL(n, q)$$

between $\mathcal{RGL}(q)$ and the *weak product* of the categories $\mathcal{R}GL(n, q)$. The category $\mathcal{RGL}(q)$ is thus semisimple since each of the categories $\mathcal{R}GL(n, q)$ is. Each simple object of $\mathcal{RGL}(q)$ belongs to exactly one of the subcategories $\mathcal{R}GL(n, q)$ for some $n \geq 0$; that is, we have

$$\mathcal{GL}(q)^{\vee} = \sum_{n \geq 0} GL(n, q)^{\vee},$$

where $\mathcal{GL}(q)^{\vee}$ denotes the set of isomorphism class of simple objects of $\mathcal{RGL}(q)$, and $GL(n, q)^{\vee}$ denotes the set of isomorphism classes of irreducible representations of $GL(n, q)$.

There are three ways to conjugate a functor M: $\mathcal{GL}(q) \to \mathcal{V}ect$. Recall that the *complex conjugate* $\overline{E}$ of a vector space E is obtained by taking a copy $\overline{E}$ of E and defining

$$\lambda \bar{x} = \overline{\bar{\lambda} x}$$

for all $\lambda \in \mathbf{C}$ and $x \in E$. The *complex conjugate* $\overline{M}$ is obtained by composing M with the complex conjugation functor $(\overline{\ })$: $\mathcal{V}ect \to \mathcal{V}ect$. To obtain the *conjugate* M^*, we precompose and postcompose M with the contravariant duality functors $(-)^*$: $\mathcal{GL}(q) \to \mathcal{GL}(q)$ and $(-)^*$: $\mathcal{V}ect \to \mathcal{V}ect$, respectively. To obtain the *contragredient representation* $M^{\vee}$, we postcompose M with the contravariant duality functor $(-)^*$: $\mathcal{V}ect \to \mathcal{V}ect$ and precompose it with the contravariant inversion functor $(-)^{-1}$: $\mathcal{GL}(q) \to \mathcal{GL}(q)$ (which is the identity on objects and inverts the arrows). Both constructions M^* and $M^{\vee}$ are contravariant functors in M. The construction $M^{*\vee}$ can be obtained by precomposing M with the (mutually commuting contravariant) functors $(-)^*$ and $(-)^{-1}$. We have canonical isomorphisms $M^{*\vee} \cong M^{\vee *}$, $M^{**} \cong M$, and $M^{\vee\vee} \cong M$.

Many examples of linear representations are obtained by linearising permutation representations. A *permutation* representation of $\mathcal{GL}(q)$ is a functor

$$E\colon \mathcal{GL}(q) \to \mathcal{S}et$$

taking its values in finite sets. Much of what we have said so far about linear representations of $\mathcal{GL}(q)$ can be repeated for permutation representations. For example, a permutation representation E is entirely determined by the sequence of set-theoretic actions

$$GL(n, q) \times E[n] \to E[n], \qquad n \geq 0,$$

where $E(n) = E(\mathbf{F}^n)$. Equivalently, E decomposes as a disjoint union

$$E = \sum_{n \geq 0} E_n,$$

where $E_n(V)$ is empty unless $\dim V = n$.

Most of the set-valued functors on $\mathcal{GL}(q)$ are conceptually obtained by considering certain *species of structures* on vector spaces [J1]. We provide some examples. The *n-grassmanian* G_n is defined to be the functor whose value at V is the set $G_n(V)$ of all subspaces of V of dimension n. There is the *n-flag functor* F_n; an *n-flag* A *on* V (or *V-flag of length* n) is an n-chain of subspaces

$$A: 0 = A_0 \leq A_1 \leq A_2 \leq \cdots \leq A_{n-1} \leq A_n = V.$$

The *frame* functor R has its value at V equal to the set $R(V)$ of bases of V. The functor R decomposes as a disjoint union $R = \Sigma_n R_n$ where R_n is the *n-frame* functor whose value at V is the set of bases of V of size n. There is an obvious isomorphism $R_n \cong Iso(\mathbf{F}^n, -)$ (so R_n is a "representable functor" [ML]).

Given a functor E: $\mathcal{GL}(q) \to \mathcal{S}et$, it is often convenient to use the terminology that an element $s \in E(V)$ is a *structure of species E on V*. If $E(\rho)(s) = t$ for ρ: $V \xrightarrow{\sim} W$, we say that ρ is an *isomorphism* between $s \in E(V)$ and $t \in E(W)$. If moreover $s = t$, we say that ρ is an *automorphism* of s. A functor E is *connected* when there is exactly one isomorphism class of structure of species E. When E is connected and $s \in E[n]$, we have a canonical isomorphism

$$E \xrightarrow{\sim} Iso(\mathbf{F}^n, -)/H,$$

where $H \leq GL(n,q)$ is the subgroup of automorphisms of s. In other words, $E[n]$ is isomorphic to the permutation representation $GL(n,q)/H$.

For any set S, we denote by $\mathbf{C}S$ the complex vector space with basis S. Any action of a group on S extends linearly to a linear representation of the group on $\mathbf{C}S$. Similarly, a functor E: $\mathcal{GL}(q) \to \mathcal{S}et$ with values in finite sets can be extended to a linear representation $\mathbf{C}E$: $\mathcal{GL}(q) \to \mathcal{V}ect$. For example, the representation $\mathbf{C}R_n$ is the *regular representation* of $GL(n,q)$.

2. THE EXTERNAL TENSOR PRODUCT

The external *tensor product* $M \otimes N$ of representations, mentioned in the Introduction, takes on a particularly simple expression in terms of objects

M, N in $[\mathscr{GL}(q), \mathscr{V}ect]$. It can be defined by the formula

$$(M \otimes N)(V) = \bigoplus_{A \leq V} M(A) \otimes N(V/A),$$

where the direct sum runs over the set of all subspaces A of V, where the tensor product on the right-hand side is the usual one for complex vector spaces, and where V/A is the quotient space. Each linear isomorphism $\rho: V \to W$ induces isomorphisms $A \to B$, $V/A \to V/B$ (where $B = \rho(A) \leq W$), and hence a linear isomorphism

$$(M \otimes N)(\rho) : (M \otimes N)(V) \to (M \otimes N)(W).$$

This defines the external tensor product functor

$$\otimes : [\mathscr{GL}(q), \mathscr{V}ect] \times [\mathscr{GL}(q), \mathscr{V}ect] \to [\mathscr{GL}(q), \mathscr{V}ect].$$

The unit I for this tensor product is given by

$$I(V) = \mathbf{C} \text{ for } V = 0, \qquad \text{and} \qquad I(V) = 0 \text{ otherwise}.$$

The associativity of the external tensor product will be obvious from the consideration of n-fold external tensor products. By definition, the n-fold exterior tensor product of $M_1, M_2, \ldots, M_n \in [\mathscr{GL}(q), \mathscr{V}ect]$ is given by the formula

$$\begin{aligned}(M_1 \otimes M_2 \otimes \cdots \otimes M_n)(V) \\ = \bigoplus_{A \in F_n(V)} M_1(A_1) \otimes M_2(A_2/A_1) \otimes \cdots \otimes M_n(V/A_{n-1}),\end{aligned}$$

where the direct sum runs over the n-flags $A \in F_n(V)$. Clearly there is a canonical isomorphism between $(M \otimes N) \otimes P$ and $M \otimes N \otimes P$ and also between $M \otimes N \otimes P$ and $M \otimes (N \otimes P)$. Composing these isomorphisms, we obtain associativity constraints

$$(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$$

which satisfy the pentagonal condition, so that $[\mathscr{GL}(q), \mathscr{V}ect]$ becomes a monoidal category (also called "tensor category") in the usual sense [ML, p. 158].

An *external* product $E \underline{\times} F$ of set-valued functors E, F in $[\mathscr{GL}(q), \mathscr{S}et]$ is defined similarly. We put

$$(E \underline{\times} F)(V) = \sum_{A \leq V} E(A) \times F(V/A)$$

in which the product $\times$ on the right-hand side is the cartesian product of sets. If we think of $E(V)$ (respectively, $F(V)$) as a set of structures of species E (respectively, F) on V, the definition means that *a structure of species* $E \underline{\times} F$ *on* V is a triple (A, s, t) where A is a subspace of V, where s is a structure of species E on A, and t is a structure of species E on V/A [J1, J2]. This description makes it clear that $E \underline{\times} F$ is connected if both E and F are. Also, it follows that we have an isomorphism $F_n \underline{\times} F_m \cong F_{n+m}$ since the giving of an $n + m$ flag on V is equivalent to the giving of a triple (A, s, t) where s is an n-flag on a subspace $A \leq V$ and t is an m-flag on V/A. Clearly, we have a general isomorphism

$$\mathbf{C}(E \underline{\times} F) \cong \mathbf{C}E \otimes \mathbf{C}F$$

from which it follows that we have $\mathbf{C}F_n \otimes \mathbf{C}F_m \cong \mathbf{C}F_{n+m}$. This shows that $\mathbf{C}F_n$ is the n-fold external tensor product of the *trivial* representation $\mathbf{C}F_1$ of $GL(1, q)$. With the n-frame functor R_n for $n \geq 0$, we see that $\mathbf{C}R_n$ is the regular representation of $GL(n, q)$.

The external tensor product $S \otimes T$ of representations S of $GL(s, q)$ and T of $GL(t, q)$ is a representation of $GL(s + t, q)$ which can be reducible even if S and T are both irreducible. Let us say that $S \otimes T$ is a *non-trivial* product if $s, t > 0$. We now describe the elementary theory of cuspidal representations needed in this paper.

DEFINITION 2.1 (Harish-Chandra). An irreducible representation X of $GL(n, q)$, $n \neq 0$, is cuspidal if it does not occur as a direct factor of a non-trivial external product $S \otimes T$ with S in $\mathscr{R}GL(s, q)$ and T in $\mathscr{R}GL(t, q)$ where $n = s + t$.

By decomposing the representations S and T into irreducible constituents, we see that we can suppose in the definition that S and T are irreducible. Clearly every irreducible representation of $GL(1, q)$ is cuspidal.

PROPOSITION 2.2. *Every irreducible representation is a direct factor of an r-fold* ($r \geq 0$) *external tensor product* $M_1 \otimes M_2 \otimes \cdots \otimes M_r$ *of cuspidal representations.*

Proof. Let J be an irreducible representation of $GL(n, q)$. We argue by induction on n. If $n = 0$, the result is true with $r = 0$. Assume $n > 0$. If J is cuspidal then the result is true with $r = 1$. Otherwise, J is an irreducible constituent of a non-trivial external product $S \otimes T$ with S an irreducible representation of $\mathscr{R}GL(s, q)$ and T an irreducible representation of $\mathscr{R}GL(t, q)$. We have $s < n$ and $t < n$ so that, by the induction hypothesis, S, T appear as direct factors of products $X_1 \otimes \cdots \otimes X_h$, $Y_1 \otimes \cdots \otimes Y_k$, respectively, with $X_1, X_2, \ldots, X_h, Y_1, Y_2, \ldots, Y_k$ all cuspidal. It follows that J is a direct factor of the external product $(X_1 \otimes \cdots \otimes X_h) \otimes (Y_1 \otimes \cdots \otimes Y_k)$. Q.E.D.

We need to introduce a few concepts. The *parabolic subgroup* $P(A)$ of $GL(V)$ associated with a flag $A \in F_m(V)$ consists of the automorphisms of A; that is,

$$P(A) = \{\rho \in GL(V) | \rho A_i = A_i \text{ for } i = 1, \ldots, m\}.$$

The kernel of the canonical map

$$p\colon P(A) \to GL(A) = \prod_{i=1}^{m} GL(A_i/A_{i-1})$$

is the *unipotent radical* $U(A)$ of $P(A)$; it consists of those ρ which induce the identity map on the consecutive quotients A_i/A_{i-1}.

LEMMA 2.3. *The external tensor product* $\mathbf{C}R_n \otimes \mathbf{C}R_m$ *is isomorphic to the representation* $\mathbf{C}GL(n+m,q)/H$ *where* H *is the unipotent radical* $U(A)$ *associated with the flag*

$$A\colon 0 \leq \mathbf{F}^n \leq \mathbf{F}^{n+m}.$$

Proof. We have an isomorphism $\mathbf{C}R_n \otimes \mathbf{C}R_m \cong \mathbf{C}(R_n \underline{\times} R_m)$, and the functor $R_n \underline{\times} R_m$ is connected since both of the functors R_n and R_m are. An element of $(R_n \underline{\times} R_m)(V)$ is a triple (A, s, t) where $A \leq V$, where s is an n-frame on A, and t is an m-frame on V/A. Clearly, the group of automorphisms of (A, s, t) is the unipotent radical for the flag $0 \leq A \leq V$. This shows that we have

$$R_n \underline{\times} R_m \cong GL(n+m,q)/H. \quad \text{Q.E.D.}$$

The following result provides a simple characterisation of cuspidal representations. A flag A on V is said to be *proper* when it contains at least one subspace other than 0 and V.

PROPOSITION 2.4. *An irreducible representation* M *of* $GL(n,q)$ *is cuspidal if and only if it contains no non-zero* $U(A)$*-invariant vector for any proper flag* A *on* V. *Moreover, for that, it suffices to take the standard flags* A: $0 < \mathbf{F}^r < \mathbf{F}^n$.

Proof. Let us remark first that an irreducible representation M of $GL(n,q)$ is cuspidal if and only if it is not a direct factor of a representation $\mathbf{C}R_r \otimes \mathbf{C}R_{n-r}$ where $0 < r < n$. This is because any irreducible representation is a direct factor of a regular representation. Equivalently,

M is a cuspidal if and only if there is no nonzero map $\mathbf{C}R_r \otimes \mathbf{C}R_{n-r} \to M$ for all $0 < r < n$. But, according to Lemma 2.3, we have an isomorphism

$$\mathbf{C}R_r \otimes \mathbf{C}R_{n-r} \cong \mathbf{C}GL(n,q)/H,$$

where $H = U(A)$ is the unipotent radical associated with the standard flag $0 < \mathbf{F}^r < \mathbf{F}^n$. It follows that the set of maps $\mathbf{C}R_r \otimes \mathbf{C}R_{n-r} \to M$ is in bijection with the set M^H of H-invariant vectors of $M[n]$. The proposition is proved with the set of standard flags of length 2. The rest follows from the facts that every flag is isomorphic to a standard flag and that, for any proper flag A with $0 < A_i < V$, we have $U(A) \leq U(A')$ where A': $0 < A_i < V$ is a flag of length 2. Q.E.D.

The external tensor product behaves well with respect to the duality functors.

PROPOSITION 2.5. *There are canonical isomorphisms*

$$(M \otimes N)^* \cong N^* \otimes M^*, \qquad (M \otimes N)^\vee \cong M^\vee \otimes N^\vee.$$

Proof. We shall describe only the first. From any short exact sequence

$$A \to V \to V/A,$$

we obtain, by duality, a short exact sequence

$$(V/A)^* \to V^* \to A^*$$

or equivalently, a short exact sequence

$$A^\circ \to V^* \to V/A^\circ,$$

where A° is the orthogonal complement of A in V^*. It follows that

$$\begin{aligned}
&(M \otimes N)^*(V) \\
&\quad = (M \otimes N)(V^*) = \sum_{A \leq V} M(A^\circ) \otimes N(V^*/A^\circ) \\
&\quad = \sum_{A \leq V} M((V/A)^*) \otimes N(A^*) = \sum_{A \leq V} M^*(V/A) \otimes N^*(A) \\
&\quad \cong \sum_{A \leq V} N^*(A) \otimes M^*(V/A) = (N^* \otimes M^*)(V). \quad \text{Q.E.D.}
\end{aligned}$$

It follows from these identities that M^* and $M^\vee$ are cuspidal representations whenever M is.

3. CHARACTER SERIES

At this point it is instructive to consider the dimension series and the character series associated to a functor $M\colon \mathcal{GL}(q) \to \mathcal{V}ect$. By definition, the *dimension series* is the power series

$$\dim M = \sum_{n \geq 0} \dim M[n] \frac{x^n}{\phi_n(q)},$$

where $\phi_n(q) = (q^n - 1)(q^{n-1} - 1) \cdots (q - 1)$ for every $n \geq 0$.

Obviously $\dim(M \oplus N) = \dim M + \dim N$. To see that we have the identity

$$\dim(M \otimes N) = \dim M \times \dim N,$$

we need to prove that

$$\frac{\dim(M \otimes N)[n]}{\phi_n(q)} = \sum_{k+r=n} \frac{\dim M[k]}{\phi_k(q)} \times \frac{\dim N[r]}{\phi_r(q)};$$

or equivalently,

$$\dim(M \otimes N)[n] = \sum_{k+r=n} [n/k]_q \dim M[k] \times \dim N[r],$$

where $[n/k]_q = \phi_n(q)/\phi_k(q)\phi_{n-k}(q)$ is the number of subspaces of $\mathbf{F}^n$ of dimension k. This follows from the formula

$$(M \otimes N)(V) = \bigoplus_{A \leq V} M(A) \otimes N(V/A)$$

since we have $\dim M(A) = \dim M(B)$ and $\dim M(V/A) = \dim M(V/B)$ for subspaces A, B of V of the same dimension.

A *class function* is a map $f\colon \Sigma_n GL(n, q) \to \mathbf{C}$ which is constant on conjugacy classes. We can identify a conjugacy class c of $GL(n, q)$ with the class function taking the value 1 on the members of c, and 0 elsewhere. A class function f can be written as an infinite sum

$$f = \sum_c f(c)c$$

in which the summation index runs over the disjoint union of the set of conjugacy classes of the groups $GL(n, q)$ for $n \geq 0$. Let $\mathcal{A}ut(q)$ be the category whose objects are pairs (V, σ) where V is the $\mathcal{GL}(q)$ and σ is an automorphism of V, and whose arrows $\rho\colon (V, \sigma) \to (W, \tau)$ are the linear isomorphisms $\rho\colon V \to W$ such that $\rho\sigma = \tau\rho$. The isomorphism classes of objects of $\mathcal{A}ut(q)$ are in bijection with the disjoint union of the set of conjugacy classes of the groups $GL(n, q)$ for $n \geq 0$. So, a class function is exactly a map $f: \mathcal{A}ut(q) \to \mathbf{C}$ which is constant on isomorphism classes of $\mathcal{A}ut(q)$ (actually, f is a functor if we provide the set $\mathbf{C}$ of complex numbers with the structure of discrete category). The *product* of the class functions f and g is defined by the formula

$$(fg)(V, \sigma) = \sum_{\sigma(A)=A} f(A, \sigma|A) \times g(V/A, \sigma/A),$$

where the summation runs over the subspaces $A \leq V$ such that $\sigma(A) = A$, where $\sigma|A$ is the restriction of σ to A and where σ/A is the automorphism of V/A induced by σ. It is easy to see that this product is associative by considering 3-fold products of class series.

PROPOSITION 3.1. *The class function ring is commutative.*

Proof. Define the *conjugate* f^* of a class function by the formula $f^*(\sigma) = f(\sigma')$ where σ' is the transpose of σ. From a short exact sequence

$$A \to V \to V/A$$

with $\sigma(A) = A$, by duality we obtain a short exact sequence

$$A^\circ \to V^* \to V/A^\circ$$

with $\sigma'(A^\circ) = A^\circ$ where A° is the orthogonal complement of A in V^*. We have

$$(fg)^*(V, \sigma) = (fg)(V^*, \sigma') = \sum_{\sigma(A)=A} f(A^\circ, \sigma'|A^\circ) \times g(V^*/A^\circ, \sigma/A^\circ)$$

$$= \sum_{\sigma(A)=A} f^*(V/A, \sigma/A) \times g^*(A, \sigma|A) = (g^*f^*)(V, \sigma).$$

This shows that we have $(fg)^* = g^*f^*$ for class functions f and g. But the transpose σ' of a matrix σ in $GL(n, q)$ is conjugate to σ and therefore $f(\sigma') = f(\sigma)$ for every σ. The relation $f^* = f$ for every f implies that $fg = (fg)^* = g^*f^* = gf$. Q.E.D.

The *character series* χ_M of M is defined to be the class function whose value at (V, σ) is the trace of $M(\sigma)$. This can be written as an infinite sum

$$\chi_M = \sum_c \operatorname{Tr} M(c)c,$$

where $\operatorname{Tr} M(c)$ is defined to be $\operatorname{Tr} M(\sigma)$ for any σ in the class c. Obviously, $\chi_{M \oplus N} = \chi_M + \chi_N$. We also have the identity

$$\chi_{M \otimes N} = \chi_M \chi_N .$$

To check this we need to prove that, for every automorphism σ of V,

$$\operatorname{Tr}(M \otimes N)(\sigma) = \sum_{\sigma(A)=A} \operatorname{Tr} M(\sigma | A) \times \operatorname{Tr} N(\sigma/A).$$

But the endomorphism $(M \otimes N)(\sigma)$ has a block decomposition corresponding to the decomposition of $(M \otimes N)(V)$ as the direct sum of subspaces $M(A) \otimes N(V/A)$. The trace of $(M \otimes N)(\sigma)$ is equal to the sum of the traces of the diagonal blocks. Therefore, it is the sum of the traces of the $M(\sigma | A) \otimes N(\sigma/A)$ for subspaces $A \leq V$ such that $\sigma(A) = A$.

COROLLARY 3.2. *For any M and N in $[\mathscr{GL}(q), \mathscr{V}ect]$, the external tensor products $M \otimes N$ and $N \otimes M$ are isomorphic.*

Proof. We have $\chi_{M \otimes N} = \chi_M \chi_N = \chi_N \chi_M = \chi_{N \otimes M}$; and objects of $[\mathscr{GL}(q), \mathscr{V}ect]$ with the same characteristic series are isomorphic. Q.E.D.

This result leads to the question of finding a *natural* isomorphism between $M \otimes N$ and $N \otimes M$. The concept of naturality should be taken in the categorical sense as a natural transformation $c_{M,N}\colon M \otimes N \to N \otimes M$ which is invertible for all M, N. We shall see that this question has a positive answer. A similar question arises of finding a natural isomorphism between M and its *conjugate* M^* where $M^*(V) = M(V^*)^*$. The conjugate M^* is always isomorphic to M since we have $\chi_{M^*} = (\chi_M)^* = \chi_M$. But there is no natural isomorphism between M and M^* for a simple reason: the group $GL(0, q)$ is trivial and $M^*[0]$ is merely the dual of the complex vector space $M[0]$; and it is well known that there is no natural isomorphism between a vector space and its dual.

4. THE BRAIDING

To describe the *braiding*

$$c = c_{M,N}\colon M \otimes N \to N \otimes M,$$

we shall define a linear map

$$\theta = c_{M,N}(V)\colon (M \otimes N)(V) \to (N \otimes M)(V)$$

for each V and prove in Section 6 that it is an isomorphism. For each pair A, B of *complementary* subspaces of V, let

$$r_{A,B}\colon A \xrightarrow{\sim} V/B, \qquad s_{A,B}\colon V/A \xrightarrow{\sim} B$$

be the canonical isomorphisms, and take $\theta_{A,B}$ to be the composite

$$M(A) \otimes N(V/A) \xrightarrow{\text{switch}} N(V/A) \otimes M(A) \xrightarrow{N(s_{A,B}) \otimes M(r_{A,B})} N(B) \otimes M(V/B)$$

(where "switch" is the usual symmetry for complex vector spaces). When A, B are *not* complementary, put $\theta_{A,B} = 0$. These $\theta_{A,B}$ are the entries of the matrix $(\theta_{A,B})$ defining the map

$$\theta\colon \bigoplus_{A \leq V} M(A) \otimes N(V/A) \to \bigoplus_{B \leq V} N(B) \otimes M(V/B).$$

PROPOSITION 4.1. *The arrows* $c_{M,N}\colon M \otimes N \to N \otimes M$ *are natural in* M, N *and render the following triangles commutative.*

$$\begin{tikzcd} M \otimes N \otimes P \arrow[rr, "c_{M,N\otimes P}"] \arrow[dr, "c_{M,N}\otimes 1_P"'] & & N \otimes P \otimes M \\ & N \otimes M \otimes P \arrow[ur, "1_N \otimes c_{M,P}"'] & \end{tikzcd} \qquad \begin{tikzcd} M \otimes N \otimes P \arrow[rr, "c_{M\otimes N,P}"] \arrow[dr, "1_M \otimes c_{N,P}"'] & & P \otimes M \otimes N \\ & M \otimes P \otimes N \arrow[ur, "c_{M,P}\otimes 1_N"'] & \end{tikzcd}$$

Proof. Naturality is clear. To check commutativity of the first triangle (the second is similar), note that the switch symmetry of the tensor product of complex vector spaces allows us to translate the value of the triangle at V to the triangle

$$\begin{tikzcd} \bigoplus_{A \leq B \leq V} M(A) \otimes N(B/A) \otimes P(V/B) \arrow[rr, "\gamma"] \arrow[dr, "\alpha"'] & & \bigoplus_{C \leq D \leq V} M(V/D) \otimes N(C) \otimes P(D/C) \\ & \bigoplus_{C' \leq B' \leq V} M(B'/C') \otimes N(C') \otimes P(V/B') \arrow[ur, "\beta"'] & \end{tikzcd}$$

where

(i) the matrix γ has component $\gamma_{A,B,C,D}$ zero unless A, D are complementary subspaces of V and the canonical isomorphism $V/A \cong D$ sends B/A to C, in which case $\gamma_{A,B,C,D}$ is induced by the isomorphisms

$$A \cong V/D, \qquad B/A \cong C, \qquad V/B \cong D/C;$$

(ii) the matrix α has component $\alpha_{A,B,C',B'}$ zero unless $B = B'$ and A, C' are complementary subspaces of B, in which case $\alpha_{A,B,C',B'}$ is induced by the isomorphisms

$$A \cong B/C', \qquad B/A \cong C', \qquad V/B \cong V/B';$$

(iii) the matrix β has component $\beta_{C',B',C,D}$ zero unless $C = C'$ and B'/C, D/C are complementary subspaces of V/C, in which case $\beta_{C',B',C,D}$ is induced by the isomorphisms

$$B'/C \cong V/D, \qquad C' \cong C, \qquad V/B' \cong D/C.$$

The desired result $\gamma = \beta\alpha$ follows from the equation

$$\gamma_{A,B,C,D} = \beta_{C,B,C,D}\,\alpha_{A,B,C,B}$$

which holds when either side is non-zero. Q.E.D.

Clearly, to prove that $c_{M,N}$ is an isomorphism, we can suppose that both M and N belong to $\mathcal{RGL}(q)$. Using naturality, we can suppose that M and N are irreducible. In fact, we can suppose more, as in the next lemma.

LEMMA 4.2. *If the map $c_{M,N}\colon M \otimes N \to N \otimes M$ is invertible for all cuspidal representations M and N then it is invertible for every M and N.*

Proof. It follows from Proposition 4.1 that, for a given M, the collection of those N for which $c_{M,N}$ is invertible is closed under external tensor product. By naturality of $c_{M,N}$ it is also closed under direct factors and sums. If the hypothesis of the lemma is true, these two properties entail that $c_{M,N}$ is invertible for every N when M is cuspidal. Reversing the roles of M and N a similar argument shows that $c_{M,N}$ is invertible for every M and every N. Q.E.D.

Remark 4.3. The exterior tensor product and the braiding for representations can be expressed in terms of structure on the groupoid $\mathcal{GL}(q)$. We have a *protensor-product* functor

$$[\![-, -, -]\!]\colon \mathcal{GL}(q)^{op} \times \mathcal{GL}(q)^{op} \times \mathcal{GL}(q) \to \mathcal{S}et,$$

into the category $\mathcal{S}et$ of sets, whose value $[\![A, B, V]\!]$ at any object (A, B, V) is the set

$$\Big\{(f, g) \,|\, 0 \to A \xrightarrow{f} V \xrightarrow{g} B \to 0 \text{ is a short exact}$$
$$\text{sequence of F-linear functions}\Big\};$$

and, for all $\alpha \in Iso(A', A)$, $\beta \in Iso(B', B)$, $\rho \in Iso(V, V')$,

$$[\![\alpha, \beta, \rho]\!](f, g) = (\rho f \alpha, \beta^{-1} g \rho^{-1}) \in [\![A', B', V']\!].$$

We see easily (in the notation of Lemma 2.3) that there is a canonical isomorphism of species

$$[\![\mathbf{F}^m, \mathbf{F}^n, -]\!] \xrightarrow{\sim} R_m \underline{\times} R_n.$$

It follows from the general theory of promonoidal categories [D] that the external tensor product of representations can be described in terms of the protensor product. Furthermore, the braiding on the external tensor product can be expressed in terms of the protensor product as follows. We define a linear function

$$b\colon \mathbf{C}[\![A, B, V]\!] \to \mathbf{C}[\![B, A, V]\!]$$

represented by a matrix μ whose entries are all either 0 or 1. The entry $\mu_{(f,g),(h,k)}$ is $1 \in \mathbf{C}$ if and only if the diagram

$$A \underset{k}{\overset{f}{\rightleftarrows}} V \underset{h}{\overset{g}{\rightleftarrows}} B$$

is a direct sum situation ($k \circ f = 1_A, g \circ h = 1_B, f \circ k + h \circ g = 1_V$). It is readily verified that this linear function b transports to the braiding c: $\mathbf{C}R_m \otimes \mathbf{C}R_n \to \mathbf{C}R_n \otimes \mathbf{C}R_m$ under the canonical isomorphisms $\mathbf{C}[\![\mathbf{F}^m, \mathbf{F}^n, -]\!] \cong \mathbf{C}R_m \otimes \mathbf{C}R_n$, $\mathbf{C}[\![\mathbf{F}^n, \mathbf{F}^m, -]\!] \cong \mathbf{C}R_n \otimes \mathbf{C}R_m$. It follows that the braiding $c_{M,N}$: $M \otimes N \to N \otimes M$ is invertible for all representations M, N if and only if the matrix μ is invertible for all $A, B, V \in \mathcal{GL}(q)$. However, this is not the strategy we use to prove the invertibility.

5. MAPS BETWEEN MULTIPLE TENSOR PRODUCTS

There is a natural decomposition of maps between multiple tensor products. It is described in the following theorem whose proof will occupy this whole section.

THEOREM 5.1. *For any objects $M_1, \ldots, M_m, N_1, \ldots, N_n$ of $[\mathcal{GL}(q), \mathcal{V}ect]$, there is a natural decomposition*

$$\begin{aligned} &\operatorname{Hom}(M_1 \otimes \cdots \otimes M_m, N_1 \otimes \cdots \otimes N_n) \\ &\quad = \prod_{\mathbf{a}} \operatorname{Hom}_{\mathbf{a}}(M_1 \otimes \cdots \otimes M_m, N_1 \otimes \cdots \otimes N_n), \end{aligned}$$

where the indices $\mathbf{a}$ of the product run over the set of all $m \times n$ matrices of *natural numbers. When $M_1, \dots, M_m, N_1, \dots, N_n$ belong to $\mathscr{RGL}(q)$, the product can be replaced by a direct sum. Moreover, if M_i belongs to $\mathscr{R}GL(e_i, q)$ and N_j to $\mathscr{R}GL(f_j, q)$, then the indexing set consists of the matrices* **a** *whose ith row has sum e_i and jth column has sum f_j.*

Let us begin by analysing an arrow

$$f\colon M_1 \otimes M_2 \otimes \cdots \otimes M_m \to N_1 \otimes N_2 \otimes \cdots \otimes N_n.$$

By definition, f_V is a linear map between direct sums as

$$\bigoplus_{A \in F_m(V)} M_1(A_1) \otimes M_2(A_2/A_1) \otimes \cdots \otimes M_m(V/A_{m-1})$$
$$\to \bigoplus_{B \in F_n(V)} N_1(B_1) \otimes N_2(B_2/B_1) \otimes \cdots \otimes N_n(V/B_{n-1})$$

and so can be expressed as a matrix

$$f_V = (f_{A,B})_{(A,B) \in F_m(V) \times F_n(V)},$$

where each arrow

$$M_1(A_1) \otimes M_2(A_2/A_1) \otimes \cdots \otimes M_m(V/A_{m-1})$$
$$\xrightarrow{f_{A,B}} N_1(B_1) \otimes N_2(B_2/B_1) \otimes \cdots \otimes N_n(V/B_{n-1})$$

is a linear function. The condition of naturality on f_V translates into a naturality condition for the components $f_{A,B}$: that is, for all linear isomorphisms $\rho\colon V \to W$, if C, D are the image W-flags of A, B under ρ, and if

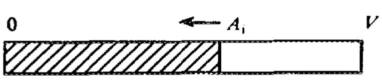
$$\sigma_i\colon A_i/A_{i-1} \to C_i/C_{i-1}, \qquad \tau_j\colon B_j/B_{j-1} \to D_j/D_{j-1}$$

are the isomorphisms induced by ρ, then the following square commutes.

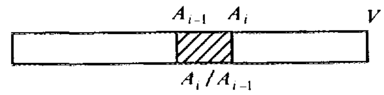
$$\begin{tikzcd}
M_1(A_1) \otimes \cdots \otimes M_m(V/A_{m-1}) \arrow[r, "f_{A,B}"] \arrow[d, "M_1(\sigma_1) \otimes \cdots \otimes M_m(\sigma_m)"'] & N_1(B_1) \otimes \cdots \otimes N_n(V/B_{n-1}) \arrow[d, "N_1(\tau_1) \otimes \cdots \otimes N_n(\tau_n)"] \\
M_1(C_1) \otimes \cdots \otimes M_m(W/C_{m-1}) \arrow[r, "f_{C,D}"'] & N_1(D_1) \otimes \cdots \otimes N_n(W/D_{n-1})
\end{tikzcd} \tag{5.2}$$

The equality $(\rho(A), \rho(B)) = (C, D)$ really means that $\rho\colon V \to W$ is an isomorphism between the pair of flags (A, B) and (C, D), which suggests examining isomorphism classes of pairs of flags in some detail.

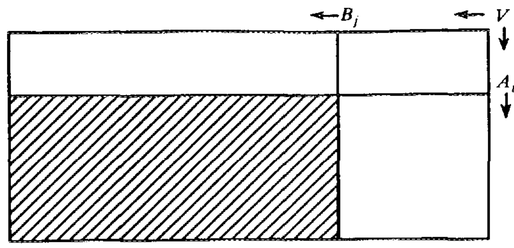
It is helpful to illustrate a V-flag A of length m by a rectangle



where the subspace A_i is represented by the shaded section, and the consecutive quotient A_i/A_{i-1} is represented by a small brick:



Suppose A, B are V-flags of length m, n. This can be illustrated by a rectangle

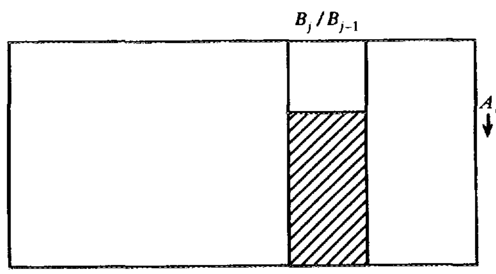


in which the shaded rectangle represents the intersection $A_i \cap B_j$. We see that the flag A induces a flag

$$A \cap (B_j/B_{j-1}) : 0 \leq (A_1 \cap B_j + B_{j-1})/B_{j-1}$$
$$\leq \cdots \leq (A_m \cap B_j + B_{j-1})/B_{j-1}$$

of length m on B_j/B_{j-1}. The shaded region in the next picture represents the **F**-vector space

$$A_i \cap B_j/A_i \cap B_{j-1} \qquad (\cong (A_i \cap B_j + B_{j-1})/B_{j-1}).$$



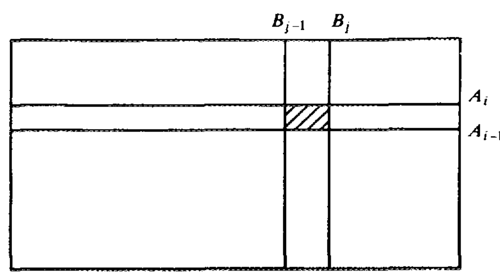
From the following 3×3 diagram of short exact sequences

$$\begin{tikzcd}
A_{i-1} \cap B_{j-1} \arrow[r, tail] \arrow[d, tail] & A_{i-1} \cap B_j \arrow[r, two heads] \arrow[d, tail] & A_{i-1} \cap B_j / A_{i-1} \cap B_{j-1} \arrow[d, tail] \\
A_i \cap B_{j-1} \arrow[r, tail] \arrow[d, two heads] & A_i \cap B_j \arrow[r, two heads] \arrow[d, two heads] & A_i \cap B_j / A_i \cap B_{j-1} \arrow[d, two heads] \\
A_i \cap B_{j-1} / A_{i-1} \cap B_{j-1} \arrow[r, tail] & A_i \cap B_j / A_{i-1} \cap B_j \arrow[r, two heads] & A_i \cap B_j / (A_i \cap B_{j-1} + A_{i-1} \cap B_j)
\end{tikzcd} \tag{5.3}$$

we see that the consecutive quotients for the flags $A \cap (B_j/B_{j-1})$, $0 \le j \le n$, and the consecutive quotients for the flags $(A_i/A_{i-1}) \cap B$, $0 \le i \le m$, lead to the same spaces

$$\langle A, B \rangle_{ij} = A_i \cap B_j/(A_i \cap B_{j-1} + A_{i-1} \cap B_j)$$

which are depicted by small bricks as shaded below.



The dimension of the space corresponding to a shaded region is a measure of the region. Every shaded region is a disjoint union of bricks so its measure is the sum of the measures of the constituent bricks. Putting $d_{ij}(A, B) = \dim\langle A, B \rangle_{ij} \in \mathbb{N}$, we define an $m \times n$ matrix $\mathbf{d}(A, B) = (d_{ij}(A, B))$ of natural numbers which suffices for calculating the dimension of any element in the lattice of subspaces generated by the A_i's and the B_j's. For example, the dimensions of the consecutive quotients can be obtained from the matrix by adding entries in rows, or in columns:

$$\dim(A_i/A_{i-1}) = \sum_{j=1}^{n} d_{ij}(A, B), \qquad \dim(B_j/B_{j-1}) = \sum_{i=1}^{m} d_{ij}(A, B).$$

Also,

$$d_{ij}(A, B) = \dim(A_i \cap B_j) - \dim(A_{i-1} \cap B_j) \\ - \dim(A_i \cap B_{j-1}) + \dim(A_{i-1} \cap B_{j-1}).$$

The matrix $\mathbf{d}(A, B)$ is a complete invariant for the isomorphism class of the pair (A, B); that is:

LEMMA 5.4. *Two pairs of flags $(A, B) \in F_m(V) \times F_n(V)$ and $(C, D) \in F_m(W) \times F_n(W)$ are isomorphic if and only if there is an equality of $m \times n$ matrices*

$$\mathbf{d}(A, B) = \mathbf{d}(C, D).$$

Proof. Suppose $\mathbf{d}(A, B) = \mathbf{d}(C, D)$. For $1 \le i \le m$ and $1 \le j \le n$, let S_{ij} be a subspace of $A_i \cap B_j$ complementary to $(A_i \cap B_{j-1}) + (A_{i-1} \cap B_j)$, and let T_{ij} be a subspace of $C_i \cap D_j$ complementary to $(C_i \cap D_{j-1}) + (C_{i-1} \cap D_j)$. We have the direct sum decompositions

$$V = \bigoplus_{i,j} S_{ij} \qquad \text{and} \qquad W = \bigoplus_{i,j} T_{ij}.$$

By hypothesis, we have $\dim S_{ij} = \dim T_{ij}$ for all $1 \le i \le m$ and $1 \le j \le n$, and therefore we can choose linear isomorphisms

$$\rho_{ij}: S_{ij} \to T_{ij}.$$

Taking the direct sum of these isomorphisms, we obtain an isomorphism $\rho: V \to W$ such that $\rho(A) = C$ and $\rho(B) = D$. Q.E.D.

Returning now to our map $f: M_1 \otimes M_2 \otimes \cdots \otimes M_m \to N_1 \otimes N_2 \otimes \cdots \otimes N_n$ in the category $[\mathscr{GL}(q), \mathscr{V}ect]$, we can decompose it into "blocks" $f_{\mathbf{a}}$, where $\mathbf{a}$ runs over the set of $m \times n$ matrices of natural numbers, by putting

$$(f_{\mathbf{a}})_{A,B} = \begin{cases} f_{A,B} & \text{when } \mathbf{a} = \mathbf{d}(A, B) \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, for every $\mathbf{a}$, the family $(f_{\mathbf{a}})_{A,B}$ satisfies the naturality condition (5.2) so that $f_{\mathbf{a}}$ is indeed a map $M_1 \otimes \cdots \otimes M_m \to N_1 \otimes \cdots \otimes N_n$ in $[\mathscr{GL}(q), \mathscr{V}ect]$. For every $x \in (M_1 \otimes M_2 \otimes \cdots \otimes M_m)(V)$, we have

$$f(x) = \sum_{\mathbf{a}} f_{\mathbf{a}}(x).$$

This sum is actually finite because the non-zero terms are among those indexed by matrices having their entries summing to $\dim V$. The assignments $f \mapsto f_{\mathbf{a}}$ are projections for the required product decomposition

$$\operatorname{Hom}(M_1 \otimes \cdots \otimes M_m, N_1 \otimes \cdots \otimes N_n) = \prod_{\mathbf{a}} \operatorname{Hom}_{\mathbf{a}}(M_1 \otimes \cdots \otimes M_m, N_1 \otimes \cdots \otimes N_n).$$

6. GENERAL PERMUTATION MAPS

The goal of this section is to define a natural map

$$c_\phi : M_1 \otimes M_2 \otimes \cdots \otimes M_n \to M_{\phi(1)} \otimes M_{\phi(2)} \otimes \cdots \otimes M_{\phi(n)}$$

for any permutation ϕ of $[n] = \{1, 2, \ldots, n\}$. This will reduce to

$$c_{M,N} : M \otimes N \to N \otimes M$$

when $n = 2$ and $\phi : [2] \to [2]$ is the switch map.

We now distinguish a special kind of pair (A, B) of flags on V which can be obtained from a permutation and a direct sum decomposition of V.

DEFINITION 6.1. Let ϕ be a permutation of $[n]$. Two flags $A, B \in F_n(V)$ are ϕ-related when there exists a direct sum decomposition

$$V = \bigoplus_{i=0}^{n} D_i$$

such that, for all $0 \leq j \leq n$,

$$A_j = \bigoplus_{i=0}^{j} D_i \quad \text{and} \quad B_j = \bigoplus_{i=0}^{j} D_{\phi(i)}.$$

We shall also say that (A, B) is a *ϕ-related pair* when A and B are ϕ-related flags. Obviously, if (A, B) is a ϕ-related pair then (B, A) is a ϕ^{-1}-related pair. Two flags are ϕ-related for the identity permutation ϕ if and only if they are equal. This shows that the factors D_i, $0 \leq i \leq n$, in the direct sum decomposition of V, are generally not determined uniquely by the triple (A, B, ϕ).

EXAMPLE 6.2. Let σ be the switch permutation of $1, 2$. Flags $0 \leq A \leq V$, $0 \leq B \leq V$ are σ-related if and only if $A \oplus B = V$. More generally, for $0 < i < n$, let s_i be the simple transposition switching $i, i+1$. Two flags $0 = A_0 \leq A_1 \leq A_2 \leq \cdots \leq A_{n-1} \leq A_n = V$ and $0 = B_0 \leq B_1 \leq B_2$

$\leq \cdots \leq B_{n-1} \leq B_n = V$ are s_i-related if and only if $A_j = B_j$ for every $j \neq i$ and A_i and B_i are complementary subspaces of A_{i+1} relative to A_{i-1} (that is, $A_i + B_i = A_{i+1}$ and $A_i \cap B_i = A_{i-1}$).

Let A and $B \in F_n(V)$ be two ϕ-related flags. We can define a map

$$\theta_{AB}: M_1(A_1) \otimes \cdots \otimes M_n(V/A_{n-1}) \to M_{\phi(1)}(B_1) \otimes \cdots \otimes M_{\phi(n)}(V/B_{n-1})$$

as follows. First choose a direct sum decomposition as in Definition 6.1. Then we have the isomorphisms

$$A_i/A_{i-1} \cong D_j \qquad \text{and} \qquad D_{\phi(i)} \cong B_i/B_{i-1}.$$

The map θ_{AB} is defined as the composite of the three isomorphisms

$$M_1(A_1) \otimes \cdots \otimes M_n(V/A_{n-1}) \xrightarrow{\sim} M_1(D_1) \otimes \cdots \otimes M_n(D_n)$$

$$M_1(D_1) \otimes \cdots \otimes M_n(D_n) \xrightarrow{\sim} M_{\phi(1)}(D_{\phi(1)}) \otimes \cdots \otimes M_{\phi(n)}(D_{\phi(n)})$$

$$M_{\phi(1)}(D_{\phi(1)}) \otimes \cdots \otimes M_{\phi(n)}(D_{\phi(n)}) \xrightarrow{\sim} M_{\phi(1)}(B_1) \otimes \cdots \otimes M_{\phi(n)}(V/B_{n-1})$$

in which the second arrow is the map $x_1 \otimes \cdots \otimes x_n \mapsto x_{\phi(1)} \otimes \cdots \otimes x_{\phi(n)}$. That this definition of θ_{AB} does not depend on the choice of the direct sum decomposition D is an easy consequence of the following lemma.

LEMMA 6.3. *Let $(A, B) \in F_n(V) \times F_n(V)$ and let ϕ be a permutation of $[n]$. The pair (A, B) is ϕ-related if and only if the matrix $\mathbf{d}(A, B)$ is such that $d_{ij}(A, B) = 0$ unless $i = \phi(j)$. Moreover, if V has a direct sum decomposition D as in Definition* 6.1, *then the composite of the isomorphisms*

$$A_{\phi(j)}/A_{\phi(j)-1} \cong D_{\phi(j)} \cong B_j/B_{j-1}$$

is independent of the choice of decomposition.

Proof. The first part of the statement is left to the reader; it involves proving that, for $i = \phi(j)$, the maps $A_i/A_{i-1} \to \langle A, B\rangle_{ij}$ and $B_j/B_{j-1} \to \langle A, B\rangle_{ij}$ are isomorphisms, where (as in Section 4) $\langle A, B\rangle_{ij} = A_i \cap B_j/(A_i \cap B_{j-1} + A_{i-1} \cap B_j)$. For the second part, it suffices to check that these two isomorphisms combine to give the composite $A_i/A_{i-1} \cong D_i \cong B_j/B_{j-1}$. Q.E.D.

Now we are ready to define the map

$$c_\phi: M_1 \otimes M_2 \otimes \cdots \otimes M_n \to M_{\phi(1)} \otimes M_{\phi(2)} \otimes \cdots \otimes M_{\phi(n)}.$$

Its component at V is given by the matrix

$$\theta = (\theta_{AB})_{A,B \in F_n(V)},$$

where θ_{AB} is the map defined above when A, B are ϕ-related, and $\theta_{AB} = 0$ otherwise.

EXAMPLE 6.4. If σ is the switch permutation of $1, 2$ then c_σ is the map

$$c_{M_1, M_2} \colon M_1 \otimes M_2 \to M_2 \otimes M_1$$

defined in Section 4.

Given objects $M_1, \dots, M_n, N_1, \dots, N_n$ of $[\mathscr{GL}(q), \mathscr{V}ect]$, a permutation $\phi \colon [n] \cong [n]$ and, for all $1 \le i \le n$, a morphism $u_i \colon M_{\phi(i)} \to N_i$, we can define a morphism

$$M_1 \otimes M_2 \otimes \cdots \otimes M_n \to N_1 \otimes N_2 \otimes \cdots \otimes N_n$$

by composing

$$c_\phi \colon M_1 \otimes M_2 \otimes \cdots \otimes M_n \to M_{\phi(1)} \otimes M_{\phi(2)} \otimes \cdots \otimes M_{\phi(n)}$$

with

$$u_1 \otimes u_2 \otimes \cdots \otimes u_n \colon M_{\phi(1)} \otimes M_{\phi(2)} \otimes \cdots \otimes M_{\phi(n)} \to N_1 \otimes N_2 \otimes \cdots \otimes N_n.$$

We obtain a map

$$d_\phi \colon \bigotimes_{j=1}^{n} \operatorname{Hom}(M_{\phi(j)}, N_j) \to \operatorname{Hom}\left(\bigotimes_{i=1}^{n} M_i, \bigotimes_{j=1}^{n} N_j \right).$$

Collecting these maps together, we obtain a map

$$d \colon \bigoplus_{\phi} \bigotimes_{j=1}^{n} \operatorname{Hom}(M_{\phi(j)}, N_j) \to \operatorname{Hom}\left(\bigotimes_{i=1}^{n} M_i, \bigotimes_{j=1}^{n} N_j \right).$$

We now state our version of a result of Harish-Chandra [H, Chap. 1, Theorem 4.1].

THEOREM 6.5. *For any cuspidal representations* $M_1, \dots, M_m, N_1, \dots, N_n \in \mathscr{RGL}(q)$,

$$\operatorname{Hom}\left(\bigotimes_{i=1}^{m} M_i, \bigotimes_{j=1}^{n} N_j \right) = 0$$

unless $m = n$, *in which case the map* d *is invertible.*

Proof. Notice first that, when m = n, the map d_ϕ sends

$$P_\phi = \bigotimes_{j=1}^{n} \operatorname{Hom}(M_{\phi(j)}, N_j)$$

into the **a** component of its codomain, where **a** is the permutation matrix $(i = \phi(j))$. Moreover, from the irreducibility of the representations $M_1, \ldots, M_m, N_1, \ldots, N_n$, it follows that $P_\phi = 0$ unless $M_{\phi(j)} \cong N_j$ for every j, in which case it is of dimension 1 by Schur's Lemma. In this case the map d_ϕ is non-zero since c_ϕ is non-zero, $(u_1 \otimes \cdots \otimes u_n)$ can be chosen to be invertible, and $d_\phi(u_1 \otimes \cdots \otimes u_n) = (u_1 \otimes \cdots \otimes u_n)c_\phi$. It follows that, when $M_1, \ldots, M_m, N_1, \ldots, N_n$ are irreducible, d is injective since each map d_ϕ is. Let us now suppose that $f\colon M_1 \otimes \cdots \otimes M_m \to N_1 \otimes \cdots \otimes N_n$ is a non-zero map in $\mathscr{RGL}(q)$. To prove $m = n$ we can suppose that f is homogeneous; that is, $f = f_{\mathbf{a}}$ for some matrix **a**. We shall see that in this case **a** is a permutation matrix $(i = \phi(j))$ and moreover that there are isomorphisms $u_j\colon M_{\phi(j)} \cong N_j$ such that $(u_1 \otimes \cdots \otimes u_n) = c_\phi = f_{\mathbf{a}}$. This will show that d is surjective and finish the proof. So consider flags A, B on V for which $\mathbf{d}(A,B) = \mathbf{a}$ and the map

$$\begin{aligned} f_{A,B}\colon M_1(A_1) \otimes M_2(A_2/A_1) \otimes \cdots \otimes M_m(V/A_{m-1}) \\ \to N_1(B_1) \otimes N_2(B_2/B_1) \otimes \cdots \otimes N_n(V/B_{n-1}) \end{aligned}$$

is non-zero. Now the parabolic subgroups $P(A)$, $P(B)$ act on the domain, codomain (respectively) of $f_{A,B}$; if $\rho \in P(A) \cap P(B)$ then the commutativity of the square (5.2) shows that $f_{A,B}$ intertwines the action of $P(A) \cap P(B)$ on its domain and codomain. So $P(A) \cap P(B)$ acts on the image $\operatorname{Im}(f_{A,B})$. The action of $U(A)$ on the domain of $f_{A,B}$ is trivial. So the action of $U(A) \cap P(B)$ on $\operatorname{Im}(f_{A,B})$ is trivial. We need the following lemmas.

LEMMA 6.6. *For any flag A, the following short exact sequence splits*:

$$1 \to U(A) \to P(A) \to GL(A) \to 1.$$

Proof. To see this, for all $1 \le i \le n$, choose a subspace $D_i \le A_i$ complementary to A_{i-1}. If H denotes the subgroup of elements $\rho \in GL(V)$ such that $\rho(D_i) = D_i$ for all $i = 1, \ldots, m$, then $H \le P(A)$ and the projection $p\colon P(A) \to GL(A)$ induces an isomorphism $H \xrightarrow{\sim} GL(A)$. Q.E.D.

LEMMA 6.7. *For $(A, B) \in F_m(V) \times F_n(V)$, the canonical group homomorphism*

$$\zeta\colon P(A) \cap P(B) \to \prod_{j=1}^{n} P\big(A \cap (B_j/B_{j-1})\big)$$

is surjective. The restriction of ζ to $U(A) \cap P(B)$ is also a surjection

$$\zeta': U(A) \cap P(B) \to \prod_{j=1}^{n} U\big(A \cap (B_j/B_{j-1})\big)$$

whose kernel contains $U(A) \cap U(B)$.

Proof. The map ζ is actually a split surjection. In order to see this, we shall find a subgroup $H \leq P(A) \cap P(B)$ such that the restriction $\zeta|H$ is an isomorphism. As in the proof of Lemma 5.4, choose a direct sum decomposition

$$V = \bigoplus_{i,j} S_{ij} \qquad \text{such that } A_i = \bigoplus_{k \leq i} \bigoplus_{j} S_{kj} \text{ and } B_j = \bigoplus_{i} \bigoplus_{r \leq j} S_{ir};$$

and put

$$T_{ij} = \bigoplus_{k \leq i} S_{kj}.$$

Then H is the set of elements $\rho \in GL(V)$ such that $\rho(T_{ij}) = T_{ij}$ for all $0 \leq i \leq m$, $0 \leq j \leq n$. The second statement is proved similarly by looking at the subgroup K of H consisting of those elements ρ which induce the identity map on the consecutive quotients T_{ij}/T_{i-1j}. Q.E.D.

Returning to the proof of the theorem, we see that Lemma 6.7 implies that the action of

$$U(A \cap B_1) \times \cdots \times U(A \cap (V/B_{n-1}))$$

on $\mathrm{Im}(f_{A,B})$ is trivial. Let $\lambda_1, \ldots, \lambda_n$ be linear forms on $N_1(B_1), \ldots, N_n(V/B_{n-1})$, respectively, such that $\lambda_1 \otimes \cdots \otimes \lambda_n$ takes a non-zero value on the subspace $\mathrm{Im}(f_{A,B})$ of $N_1(B_1) \otimes \cdots \otimes N_n(V/B_{n-1})$. For all $1 \leq j \leq n$, let

$$p_j: N_1(B_1) \otimes \cdots \otimes N_n(V/B_{n-1}) \to N_j(B_j/B_{j-1})$$

be the map $\lambda_1 \otimes \cdots \otimes \lambda_{j-1} \otimes \mathrm{id} \otimes \lambda_{j+1} \otimes \cdots \otimes \lambda_n$. Then $\mathrm{Im}(p_j \circ f_{A,B})$ is non-zero and invariant under $U(A \cap (B_j/B_{j-1}))$. But the representation $N_j(B_j/B_{j-1})$ of $GL(B_j/B_{j-1})$ is cuspidal since it is non-zero and N_j is cuspidal. It follows from Proposition 2.4 that the flag $A \cap (B_j/B_{j-1})$ is improper for each $1 \leq j \leq n$. This means that, for all $1 \leq j \leq n$, there exists precisely one $1 \leq i \leq m$ such that $d_{ij}(A, B)$ is non-zero. Put $i =$

$\phi(j)$. To see that ϕ is a bijection, we use the fact that the category $\mathcal{RGL}(q)$ is *self-dual*. We have the equivalence of categories

$$(-)^*\colon \mathcal{RGL}(q)^{op} \xrightarrow{\sim} \mathcal{RGL}(q)$$

given by $L^*(W) = L(W)^*$, $L^*(\rho) = L(\rho^t)^t$, for all objects W and arrows ρ of $\mathcal{GL}(q)$. Moreover, the functor $(-)^*$ is tensor reversing so that we have

$$f^*\colon N_n^* \otimes \cdots \otimes N_1^* \to M_m^* \otimes \cdots \otimes M_1^*.$$

Thus we can apply the argument of the last paragraph to the non-zero map f^* and use the fact that the representations M_i^* are cuspidal. It follows that, for all $1 \le i \le m$, there exists precisely one $1 \le j \le n$ such that $d_{ij}(A, B)$ is non-zero. In other words, the matrix $\mathbf{a}$ is a permutation matrix ($i = \phi(j)$) and $m = n$. This means that the flags A and B are ϕ-related; so we have a direct sum decomposition D of V as in Definition 6.1. Using the isomorphisms $A_i/A_{i-1} \cong D_i$ and $B_i/B_{i-1} \cong D_{\phi(i)}$, we obtain a map g as the composite of the four maps

$$M_1(D_1) \otimes \cdots \otimes M_n(D_n) \to M_1(A_1) \otimes \cdots \otimes M_n(V/A_{n-1})$$

$$f_{A,B}\colon M_1(A_1) \otimes \cdots \otimes M_n(V/A_{n-1}) \to N_1(B_1) \otimes \cdots \otimes N_n(V/B_{n-1})$$

$$N_1(B_1) \otimes \cdots \otimes N_n(V/B_{n-1}) \to N_1(D_{\phi(1)}) \otimes \cdots \otimes N_n(D_{\phi(n)})$$

$$N_1(D_{\phi(1)}) \otimes \cdots \otimes N_n(D_{\phi(n)}) \to N_{\psi(1)}(D_1) \otimes \cdots \otimes M_{\psi(n)}(D_n),$$

where the last arrow is the map $x_1 \otimes \cdots \otimes x_n \mapsto x_{\psi(1)} \otimes \cdots \otimes x_{\psi(n)}$ and ψ is the inverse of ϕ. The map g is a non-zero homomorphism between irreducible representations of the product group $GL(D)$ since the representations $M_i(D_i)$ and $N_{\psi(i)}(D_i)$ of $GL(D_i)$ are irreducible. It follows that g is of the form $(v_1 \otimes \cdots \otimes v_n)$ for some isomorphism of representations $v_i\colon M_i(D_i) \cong N_{\psi(i)}(D_i)$. Let $w_i\colon M_i \cong N_{\psi(i)}$ be the unique isomorphism of functors extending v_i and let $u_i = w_{\phi(i)}$. We shall prove that $(u_1 \otimes \cdots \otimes u_n) \circ c_\phi = f$. But the left-hand side is homogeneous of degree $\mathbf{a}$ where $\mathbf{a}$ is the permutation matrix ($i = \phi(j)$), so it suffices to verify that $(u_1 \otimes \cdots \otimes u_n) \circ \theta_{A,B} = f_{A,B}$. To compute the left-hand side we use the commutative squares

$$\begin{array}{ccc} M_{\phi(i)}(D_{\phi(i)}) & \longrightarrow & M_{\phi(i)}(B_i/B_{i-1}) \\ {\scriptstyle v_{\phi(i)}}\big\downarrow & & \big\downarrow{\scriptstyle u_i} \\ N_i(D_{\phi(i)}) & \longrightarrow & N_i(B_i/B_{i-1}) \end{array}$$

(which follow from the naturality of the transformations u_i) to see that the upper path around the following square is equal to the lower leg which, by the definition of g, is equal to $f = f_{A,B}$.

$$\begin{tikzcd}
M_1(A_1) \otimes \cdots \otimes M_n(V/A_{n-1}) \arrow[r] \arrow[d] & M_1(D_1) \otimes \cdots \otimes M_n(D_n) \arrow[r] & M_{\phi(1)}(D_{\phi(1)}) \otimes \cdots \otimes M_{\phi(n)}(D_{\phi(n)}) \arrow[d] \\
M_1(D_1) \otimes \cdots \otimes M_n(D_n) \arrow[d, "g = v_1 \otimes \cdots \otimes v_n"] & & M_{\phi(1)}(B_1) \otimes \cdots \otimes M_{\phi(n)}(V/B_{n-1}) \arrow[d, "u_1 \otimes \cdots \otimes u_n"] \\
N_{\psi(1)}(D_1) \otimes \cdots \otimes N_{\psi(n)}(D_n) \arrow[r] & N_1(D_{\phi(1)}) \otimes \cdots \otimes N_n(D_{\phi(n)}) \arrow[r] & N_1(B_1) \otimes \cdots \otimes N_n(V/B_{n-1})
\end{tikzcd}$$

We have thus shown that each map d_ϕ is surjective. So d is bijective. Q.E.D.

The referee has pointed out that [Jm, Theorem 4.10] treats the special case of Theorem 6.5 for the endomorphism algebra of $M \otimes M \otimes \cdots \otimes M$ for a single cuspidal representation M.

COROLLARY 6.8. *For any cuspidal representation M, the set $\{c_\phi \mid \phi \in \mathfrak{S}_n\}$ is a basis for the algebra*

$$\operatorname{End}(M^{\otimes n}) = \operatorname{Hom}(M^{\otimes n}, M^{\otimes n}).$$

COROLLARY 6.9. *Let $M_1, \ldots, M_n$ be a sequence of pairwise non-isomorphic cuspidal representations. Then $M_1 \otimes \cdots \otimes M_n$ is irreducible.*

Proof. According to the theorem, the vector space $\operatorname{End}(M_1 \otimes \cdots \otimes M_n)$ is generated by $c_1 = \mathrm{id}$. It follows that $M_1 \otimes \cdots \otimes M_n$ is irreducible. Q.E.D.

COROLLARY 6.10. *Suppose M, N are cuspidal representations of $\mathscr{RGL}(q)$ with $M[r] \neq 0$, $N[s] \neq 0$.*

(i) *If M is not isomorphic to N then*

$$c_{NM} c_{MN} = q^{rs} 1.$$

(ii) *If $M = N$ then there exists a complex number γ such that*

$$c_{MM} c_{MM} = \gamma c_{MM} + q^{rr} 1.$$

Proof. Let V be a vector space of dimension $r + s$ in $\mathscr{GL}(q)$. We have $c_{MN} = (\theta_{AB})$ where $\theta_{AB} = 0$ unless the subspaces A and B are complementary and $\dim A = r$, $\dim B = s$. Similarly, we have $c_{NM} = (\theta'_{BC})$ where $\theta'_{BC} = 0$ unless the subspaces B and C are complementary and $\dim B = s$,

$\dim C = r$. Let us compute the components ρ_{AA} of the product $\rho_{AC} = \sum_B \theta'_{BC}\theta_{AB}$. But we have

$$\theta_{AB}\colon M(A) \otimes N(V/A) \to M(V/B) \otimes N(B) \to N(B) \otimes M(V/B)$$

and

$$\theta'_{BA}\colon N(B) \otimes M(V/B) \to N(V/A) \otimes M(A) \to M(A) \otimes N(V/A)$$

and therefore

$$\theta'_{BA}\theta_{AB} = \mathrm{id}$$

for every complementary subspace B of A. This shows that $\rho_{AA} = q^{rs}\mathrm{id}$ since the number of complementary subspaces of A is q^{rs} (each such complement amounts to a splitting of the canonical projection $V \to V/A$, which in turn amounts, after choosing one such splitting, to a linear map $V/A \to A$). The result then follows from Corollaries 6.8 and 6.9. Q.E.D.

COROLLARY 6.11. *The braiding map $c_{M,N}\colon M \otimes N \to M \otimes N$ is invertible.*

Proof. By Lemma 4.2, it suffices to observe invertibility for M, N cuspidal. But, by Corollary 6.10, in this case $c_{M,N}$ "satisfies a quadratic equation with invertible constant term," and so is invertible. Q.E.D.

Write $|\phi|$ for the *length* of the permutation ϕ; it is the number of inversions involved in ϕ. We shall prove the following result generalising Proposition 4.1.

THEOREM 6.12. *If ϕ, ψ are permutations of $1, 2, \ldots, n$ such that $|\phi\psi| = |\phi| + |\psi|$ then*

$$c_\psi c_\phi = c_{\phi\psi}.$$

The proof uses the following lemma:

LEMMA 6.13. *If ϕ and $\psi\colon [n] \to [n]$ are permutations such that $|\phi\psi| = |\phi| + |\psi|$, then two flags A and C are $\phi\psi$-related if and only if there exists a flag B such that the pair (A, B) is ϕ-related and the pair (B, C) is ψ-related. Furthermore, in this case, B is unique.*

Proof. If A and C are $\phi\psi$-related we have $A_j = \oplus_{0 \le i \le j} D_i$ and $C_j = \oplus_{0 \le i \le j} D_{\phi\psi(i)}$ for some direct sum decomposition $V = \oplus_{0 \le i \le n} D_i$. Let B be the flag defined by $B_j = \oplus_{0 \le i \le j} D_{\phi(i)}$. Then the pair (A, B) is ϕ-related and the decomposition $V = \oplus_{0 \le i \le n} D_{\phi(i)}$ shows that (B, C) is a ψ-related pair. Conversely, suppose that (A, B) is ϕ-related and (B, C) is ψ-related. Let us assume first that ψ is the simple transposition $s_k =$

$(k, k+1)$. The assumption that (A, B) is ϕ-related means that for some decomposition $V = \oplus_{0 \le i \le n} D_i$ we have $A_j = \oplus_{0 \le i \le j} D_i$ and $B_j = \oplus_{0 \le i \le j} D_{\phi(i)}$ for all $0 \le j \le n$. In particular, we have the equality

$$B_{k-1} \oplus D_{\phi(k)} \oplus D_{\phi(k+1)} = B_{k+1}. \qquad (*)$$

The assumption that (B, C) is s_k-related means (Example 6.2) that $B_k \cap C_k = B_{k-1}$, $B_k + C_k = B_{k+1}$, and that $B_i = C_i$ for $i \ne k$. Using $B_{k-1} \le C_k \le B_{k+1}$, we see from $(*)$ that there is a unique subspace E of $D_{\phi(k)} + D_{\phi(k+1)}$ such that $B_{k-1} + E = C_k$. We have

$$D_{\phi(k)} + E = D_{\phi(k)} + D_{\phi(k+1)}$$

since

$$\begin{aligned} B_{k-1} + D_{\phi(k)} + E &= B_{k-1} + E + B_{k-1} + D_{\phi(k)} \\ &= C_k + B_k = B_{k+1} = B_{k-1} + D_{\phi(k)} + D_{\phi(k+1)} \end{aligned}$$

and the direct summand B_{k-1} can be cancelled from the first and last terms of this string of equalities. Putting $E_i = D_i$ for $i \ne \phi(k+1)$ and $E_i = E$ for $i = \phi(k+1)$, we obtain a direct sum decomposition $V = \oplus_{0 \le i \le n} E_i$ such that $C_j = \oplus_{0 \le i < j} E_{\phi\psi(j)}$ for all $0 \le j \le n$. The hypothesis on the length of $\phi\psi$ means that $\phi(k) < \phi(k+1)$, so $A_j = \oplus_{0 \le i \le j} E_i$ for all $0 \le j \le n$. This shows that (A, C) is $\phi\psi$-related. The uniqueness of B is a consequence of the equality

$$B_k = C_{k-1} + \left(A_{\phi(k)} \cap C_{k+1}\right)$$

which can be rewritten as $B_k = B_{k-1} + (A_{\phi(k)} \cap B_{k+1})$, and this follows from (A, B) ϕ-related and $\phi(k) < \phi(k+1)$. Finally, the general case is obtained by induction on the length of a minimal decomposition of ψ into a product of simple transitions. Q.E.D.

Proof of Theorem 6.12. The map c_ϕ is given as a matrix $c_\phi = (\theta_{AB})$ in which the indices run over the pairs (A, B) of ϕ-related flags (since the coefficients of the other pairs are all zero), and similarly for $c_\psi = (\theta_{BC})$ and $c_{\phi\psi} = (\theta_{AC})$. We must prove the relation $\theta_{AC} = \Sigma_B \theta_{BC} \theta_{AB}$. According to the lemma, both sides of this equality are zero when the flags A and C are not $\phi\psi$-related. On the other hand, when A and C are $\phi\psi$-related, there is exactly one flag B such that the pair (A, B) is ϕ-related and the pair (B, C) is ψ-related; and we only need to prove the relation $\theta_{AC} = \theta_{BC}\theta_{AB}$. For this, we can assume that we have a direct sum decomposition $V = \oplus_{0 \le i \le n} D_i$ such that $A_j = \oplus_{0 \le i \le j} D_i$, $B_j = \oplus_{0 \le i \le j} D_{\phi(i)}$, and $C_j =$

$\oplus_{0 \leq i \leq j} D_{\phi\psi(i)}$ for all $0 \leq j \leq n$. In this case, the relation follows directly from the definition of the maps θ_{AC}, θ_{BC}, and θ_{AB}. Q.E.D.

COROLLARY 6.14. *The maps c_ϕ are invertible.*

7. THE HECKE ALGEBROID

This section provides an abstract description of $\mathscr{RGL}(q)$ as a tensor category generated by object and arrow symbols satisfying certain relations. More precisely, we shall see that it is generated by the cuspidal representations together with a Yang-Baxter operator satisfying a certain quadratic relation.

Let $\mathscr{A}$ be a tensor category (also called monoidal category [ML]). A *Yang-Baxter operator* [JS1] on a family $(A(s)|s \in S)$ of objects of $\mathscr{A}$ is a family $y = (y_{st}|(s,t) \in S \times S)$ of isomorphisms y_{st}: $A(s) \otimes A(t) \to A(t) \otimes A(s)$ such that

$$\begin{aligned}(A(u) \otimes y_{st}) \circ (y_{su} \otimes A(t)) \circ (A(s) \otimes y_{tu})\\ = (y_{tu} \otimes A(s)) \circ (A(t) \otimes y_{su}) \circ (y_{st} \otimes A(u)),\end{aligned}$$

or, more simply, such that

$$(1 \otimes y_{st}) \circ (y_{su} \otimes 1) \circ (1 \otimes y_{tu}) = (y_{tu} \otimes 1) \circ (1 \otimes y_{su}) \circ (y_{st} \otimes 1),$$

for all $s, t, u \in S$.

When $\mathscr{A}$ is braided, the braiding $c_{A(s), A(t)}$: $A(s) \otimes A(t) \to A(t) \otimes A(s)$ provides an example of a Yang-Baxter operator on $(A(s)|s \in S)$.

Let Q denote the set of (isomorphism classes of) cuspidal representations in $\mathscr{RGL}(q)$. The family $(c_{u,v}|(u,v) \in Q \times Q)$ is a Yang-Baxter operator on the set of cuspidal representations. It satisfies a quadratic relation that we now compute. For this we define the *parity* $\varepsilon_M \in \{-1, 1\}$ of an irreducible representation M in $\mathscr{RGL}(q)$ as follows. Each non-zero element $a \in \mathbf{F}$ determines a homothety a_V on each vector space $V \in \mathscr{GL}(q)$ and therefore a map $M(a_V)$: $M(V) \to M(V)$. The maps $M(a_V)$ actually constitute a morphism $M(a)$: $M \to M$ since $fa_V = a_W f$ for any f: $V \to W$ in $\mathscr{GL}(q)$. Since M is irreducible, Schur's Lemma implies there exists a complex number a_M with $M(a) = a_M \mathrm{id}$. The map $a \mapsto a_M$ is *the central character* of M; it is a representation of the multiplicative group of $\mathbf{F}$ in the complex numbers. Putting $\varepsilon_M = (-1)_M$, we have $\varepsilon_M \in \{-1, 1\}$ since $(\varepsilon_M)^2 = 1$. For any u in Q, let $d(u)$ be the degree of u (that is, $d(u) = n$ when u is a representation of $GL(n, q)$).

PROPOSITION 7.1. *The family* $(c_{u,v}\colon (u,v) \in Q \times Q)$ *is a Yang-Baxter operator on the set of cuspidal representations. It satisfies the equations*

$$c_{v,u}c_{u,v} = q^{d(u)d(v)}1_{u\otimes v} \qquad \textit{for } u \neq v,$$

and

$$c_{u,u}c_{u,u} = \varepsilon_u\left(q^{d(u)(d(u)+1)/2} - q^{d(u)(d(u)-1)/2}\right)c_{u,u} + q^{d(u)d(u)}.$$

Proof. The first equation was proved in Corollary 6.10. It remains to find the coefficient γ in the second equation $c_{M,M} \circ c_{M,M} = \gamma c_{M,M} + q^{rr}1$ of Corollary 6.10. Let $\rho_{AC} = \Sigma_B \theta'_{BC}\theta_{AB}$ be the matrix representing the map $c_{M,M} \circ c_{M,M}$ at $V \in \mathcal{GL}(q)$. To find the coefficient γ we look only at those A, C with $\dim A = \dim C = r$ and which are σ-related where σ is the switch permutation on [2], that is, those A, C of dimension r which are complementary subspaces of V. From the definition of $c_{M,M}$ and matrix multiplication, we see that ρ_{AC} is the sum, over all simultaneous complements B of both A and C, of the composites

$$M(A) \otimes M(V/A) \xrightarrow{M(r)\otimes M(s)} M(V/B)$$
$$\otimes M(B) \xrightarrow{M(s')\otimes M(r')} M(C) \otimes M(V/C),$$

where r, s, r', s' are the obvious canonical arrows in $\mathcal{GL}(q)$. But the simultaneous complements B are in bijection with linear isomorphisms u: $A \to C$ according to the prescription

$$u = s' \circ r \qquad \text{and} \qquad B = \{a - u(a) \in A \oplus C \mid a \in A\}.$$

Note that $s_{C,A} \circ r' \circ s \circ r_{C,A} = -u^{-1}$ where

$$r_{C,A}\colon C \xrightarrow{\sim} V/A, \qquad s_{C,A}\colon V/C \xrightarrow{\sim} A.$$

Hence we are led to calculate the sum

$$S = \sum_{u \in Iso(A,C)} M(u) \otimes M(-u^{-1})\colon M(A) \otimes M(C) \to M(C) \otimes M(A).$$

This we do by computing the trace of its composite with the switch map $\langle\sigma\rangle$: $M(C) \otimes M(A) \to M(A) \otimes M(C)$ (compare [H, p. 10]). Notice that, if $\mathbf{a}$ is a diagonal matrix with eigenvalues λ_i, then multiplication by the matrix $\langle\sigma\rangle\mathbf{a} \otimes \mathbf{a}^{-1}$ takes basis elements $e_i \otimes e_j$ to $\lambda_i\lambda_j^{-1}e_j \otimes e_i$ which is a linear map whose trace is equal to the size of the matrix $\mathbf{a}$. It follows that each $\langle\sigma\rangle M(u)\otimes M(u^{-1})$ has trace equal to $\dim M(A)$. So, using $M(-u^{-1}) = \varepsilon_M M(u^{-1})$, we have

$$\mathrm{Tr}(\langle\sigma\rangle S) = \varepsilon_M \dim M(A)\#Iso(A,C) = \dim M(A)\#GL(A).$$

But $M(A)\otimes M(C)$ is an irreducible representation of $GL(A)\times GL(C)$; so, by Schur's Lemma, $\varepsilon_M\langle\sigma\rangle S$ is a scalar multiple of the identity map. Comparing traces, we get

$$\varepsilon_M\langle\sigma\rangle S = \frac{\#GL(A)}{\dim M(A)} 1_{M(A)\otimes M(C)}.$$

It follows that

$$\begin{aligned}\rho_{AC} &= (1\otimes M(r_{A,C}))\circ S\circ(1\otimes M(s_{A,C}))\\ &= \gamma M(s_{A,C})\otimes M(r_{A,C})\circ\langle\sigma\rangle = \gamma\theta_{A,C},\end{aligned}$$

where $\gamma = \varepsilon_M \#GL(A)/\dim M(A)$. But according to the work of [GG, G1] (see [H, Appendix 3]), we have $\dim M(A) = (q^{r-1}-1)\cdots(q-1)$, and therefore

$$\#GL(A)/\dim M(A) = q^{r(r-1)/2}(q^r-1). \quad \text{Q.E.D.}$$

Recall that a category $\mathscr{A}$ is **C**-*linear* if it is enriched over the category of complex vector spaces. That is, for each pair of objects A, B in $\mathscr{A}$, the set $\mathscr{A}(A,B)$ of arrows from A to B has the structure of a complex vector space and composition of arrows is bilinear. The concept of **C**-*linear functor* between **C**-linear categories and the concept of **C**-linear *tensor* category are defined in the obvious way.

Let S be a set and let $\gamma\colon S\times S\to \mathbf{C}\setminus\{0\}$, $\delta\colon S\to\mathbf{C}$ be prescribed functions with γ symmetric. Let $\mathscr{A}$ be a **C**-linear tensor category. We shall say that a Yang-Baxter operator y on the family $(A(s)|s\in S)$ of objects of $\mathscr{A}$ *satisfies equations* $E(\gamma,\delta)$ if we have

$$y_{ts}y_{st} = \gamma(s,t)\mathrm{id} \qquad \text{for all } s\neq t,$$

and

$$y_{ss}y_{ss} = \delta(s)y_{ss} + \gamma(s,s)\mathrm{id} \qquad \text{for all } s.$$

DEFINITION 7.2. The *Hecke algebroid* $\mathscr{H}(\gamma,\delta)$ is the **C**-linear (strict) tensor category universally generated by a Yang-Baxter operator satisfying equations $E(\gamma,\delta)$.

Recall that a tensor category is *strict* if its tensor product is strictly associative. We assume strictness only for simplicity. Here is a more explicit presentation of $\mathscr{H}(\gamma, \delta)$:

the generating objects are the elements s of S;

the generating arrows are symbols

$$y_{s,t}: s \otimes t \to t \otimes s,$$

subject to the relations

$$\begin{aligned}(u \otimes y_{s,t})(y_{s,u} \otimes t)(s \otimes y_{t,u}) &= (y_{t,u} \otimes s)(t \otimes y_{s,u})(y_{s,t} \otimes u),\\ y_{t,s}y_{s,t} &= \gamma(s,t)1_{s\otimes t} \qquad \text{if } s \neq t,\\ y_{s,s}y_{s,s} &= \delta(s)y_{s,s} + \gamma(s,s)1_{s\otimes s}.\end{aligned}$$

When the set S contains a single element, the functions γ and δ are complex numbers $\alpha \neq 0$ and β. In this case, $\mathscr{H}(\alpha, \beta)$ has a simple description in terms of the *Hecke algebras* $H_n(\alpha, \beta)$. Recall that $H_n(\alpha, \beta)$ is generated by elements $g_1, \dots, g_{n-1}$ subject to the relations

$$\begin{aligned} g_i g_j &= g_j g_i \qquad \text{for } |i-j| > 1\\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \qquad \text{for } 1 \le i \le n-2\\ g_i g_i &= \beta g_i + \alpha 1 \qquad \text{for } 1 \le i \le n-1.\end{aligned}$$

Now $H_n(\alpha, \beta)$ has a basis $(g_\phi | \phi \in \mathfrak{S}_n)$ indexed by the permutations of $[n] = \{1, \dots, n\}$. More precisely, if ϕ is the transposition $(i, i+1)$ then $g_\phi = g_i$, and $g_\psi g_\phi = g_{\psi\phi}$ when $|\psi\phi| = |\psi| + |\phi|$. The algebras $H_n(\alpha, \beta)$ can be assembled to form the C-linear category $\mathscr{H}(\alpha, \beta)$. The objects of this category are the natural numbers. The vector space $\mathrm{Hom}(m, n)$ is zero unless $m = n$ in which case it is equal to $H_n(\alpha, \beta)$. Composition in $\mathscr{H}(\alpha, \beta)$ is the product in the algebras $H_n(\alpha, \beta)$. The category $\mathscr{H}(\alpha, \beta)$ has a tensor product defined as $m \otimes n = m + n$, $g_i \otimes g_j = g_i g_{m+j}$ for $g_i \in H_m(\alpha, \beta)$, and $g_j \in H_n(\alpha, \beta)$. The element $g_1 \in H_2(\alpha, \beta)$ is a Yang-Baxter operator $y : 1 + 1 \to 1 + 1$ satisfying the relation $y^2 = \beta y + \alpha 1$. The tensor category $\mathscr{H}(\alpha, \beta)$ is actually equipped with a *braiding* $c_{m,n}$: $m + n \to n + m$. We have $c_{m,n} = g_\phi$ where ϕ is the permutation such that $\phi(i) = i + n$ for $1 \le i \le m$ and $\phi(i) = i - m$ for $m + 1 \le i \le m + n$.

Here is a concrete description of the general $\mathscr{H}(\gamma, \delta)$.

PROPOSITION 7.3. *The objects of* $\mathscr{H}(\gamma, \delta)$ *are finite sequences* $u_1 \otimes u_2 \otimes \cdots \otimes u_m$ *of elements of* S. *The vector space* $\mathrm{Hom}(u_1 \otimes \cdots \otimes u_m, v_1$

$\otimes \cdots \otimes v_n$) has a basis consisting of elements g_ϕ indexed by the bijections ϕ: $[m] \to [n]$ such that $u_i = v_{\phi(i)}$ for all $1 \le i \le n$. These basis elements s_ϕ are characterised by the following properties: $g_{id} = \mathrm{id}$, $g_\psi g_\phi = g_{\psi\phi}$ when $|\psi\phi| = |\psi| + |\phi|$; $s_\psi \otimes s_\phi = g_{\psi+\phi}$ where $\psi + \phi$ is the disjoint sum of the permutations ψ and ϕ; and $g_\sigma = y_{u,v}$: $u \otimes v \to v \otimes u$ where σ is the transposition $(1,2)$. The tensor category $\mathscr{H}(\gamma, \delta)$ is braided.

Proof. For any permutation $\phi \in \mathfrak{S}_n$, we can define an element $g_\phi \in \mathrm{Hom}(u_1 \otimes \cdots \otimes u_n, v_{\phi(1)} \otimes \cdots \otimes v_{\phi(n)})$ by induction on the length of ϕ. If $\sigma \in \mathfrak{S}_n$ is the transposition $(i, i+1)$ then $g_\sigma = g_i$ is the tensor product

$$\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes c_{u_i, u_j} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}$$

(in which g_1 occupies position i). For a general permutation $\phi \in \mathfrak{S}_n$, we put

$$g_\phi = g_{\sigma_1} \cdots g_{\sigma_r},$$

where $\phi = \sigma_1 \cdots \sigma_r$ is a minimal decomposition of ϕ as a product of simple transpositions. That this definition of g_ϕ is independent of the choice of the decomposition follows from the relations $g_i g_j = g_j g_i$ for $|i-j| > 1$ and $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$. If $\sigma = (i, i+1)$, we have, by associativity of composition, that $g_\sigma g_\phi = g_{\sigma\phi}$ when $|\sigma\phi| = |\sigma| + |\phi|$, and

$$g_\sigma g_\phi = \gamma(u_i, u_{i+1}) g_{\sigma\phi} \qquad \text{if } u_i \neq u_{i+1}$$

and

$$g_\sigma g_\phi = g(u_i) g_\phi + \gamma(u_i, u_i) g_{\sigma\phi} \qquad \text{if } u_i = u_{i+1}$$

when $|\sigma\phi| < |\sigma| + |\phi|$. This proves that the linear span of the g_ϕ, $\phi \in \mathfrak{S}_n$, is closed under composition. It is also closed under tensor product since we have $g_\psi \otimes g_\phi = g_{\psi+\phi}$. It remains to prove that the g_ϕ's are linearly independent. For this, we shall construct a category $\mathscr{H}_1$ and a functor F: $\mathscr{H}(\gamma, \delta) \to \mathscr{H}_1$ such that the $F(g_\phi)$'s are independent. For each $s \in S$, let $\mathscr{H}(s)$ be the Hecke category $\mathscr{H}(\gamma(s), \delta(s))$ generated by one object s. Let $\mathscr{H}_1$ be the *tensor product* of the categories $\mathscr{H}(s)$:

$$\mathscr{H}_1 = \bigotimes_{s \in S} \mathscr{H}(s).$$

An *object* of $\mathscr{H}_1$ is an abelian word in the alphabet S. Such an object can be represented as a formal sum

$$\sum_{s \in S} n(s) s,$$

where $n(s) = 0$ except for a finite number of elements $s \in S$. An *arrow*

$$\sum_{s \in S} m(s)s \to \sum_{s \in S} n(s)s$$

is an element of the tensor product

$$\bigotimes_{s \in S} \mathrm{Hom}_s(m(s), n(s)),$$

where $\mathrm{Hom}_s(m, n)$ is the vector space of arrows from m to n in $\mathscr{H}(s)$. Notice that this tensor product makes sense even when S is infinite since $\mathrm{Hom}_s(m(s), n(s)) = \mathrm{Hom}_s(0,0) = \mathbf{C}$ for all but a finite number of $s \in S$. Composition of arrows is defined in the obvious manner (as in a tensor product of algebras). The category $\mathscr{H}_1$ is equipped with a tensor product defined in the obvious way from the tensor products on each of the categories $\mathscr{H}(s)$. To define the functor $F\colon \mathscr{H}(\gamma, \delta) \to \mathscr{H}_1$ we shall use the universal property of $\mathscr{H}(\gamma, \delta)$. Choose a function $\rho\colon S \times S \setminus \Delta(S) \to \mathbf{C} \setminus \{0\}$ such that $\rho(s,t)\rho(ts) = \gamma(s,t)$ for all $s \neq t$ in S. There are many possible choices for ρ but we do not need to indicate a particular one. Using ρ we define an arrow $y_{s,t}\colon s + t \to t + s$ in $\mathscr{H}_1$ for $(s,t) \in S \times S$ as follows. If $s \neq t$, we put $y_{s,t} = \rho(s,t)1_{s+t}$ (remember that $s + t = t + s$ in $\mathscr{H}_1$). If $s = t$, we put $y_{t,t} = s_1 \in \mathrm{Hom}_t(2,2) = \mathrm{Hom}_t(2t, 2t)$. It is straightforward to verify that $y = (y_{s,t} | (s,t) \in S \times S)$ is a Yang-Baxter operator satisfying the equations $E(\gamma, \delta)$. We obtain a functor $F\colon \mathscr{H}(\gamma, \delta) \to \mathscr{H}_1$. We now verify that it is an equivalence of categories. For this, choose a linear order on S. Let $\mathscr{H}_0 \hookrightarrow \mathscr{H}(\gamma, \delta)$ be the full subcategory whose objects are the products $a_1^{n(1)} \otimes \cdots \otimes a_k^{n(k)}$ where $a_1 < \cdots < a_r$. For $a_1^{n(1)} \otimes \cdots \otimes a_k^{n(k)}$ and $b_1^{m(1)} \otimes \cdots \otimes b_r^{m(r)}$ in $\mathscr{H}_0$, consider the map

$$\mathrm{Hom}\left(a_1^{n(1)} \otimes \cdots \otimes a_k^{n(k)}, b_1^{m(1)} \otimes \cdots \otimes b_r^{m(r)}\right) \to \mathrm{Hom}\left(\sum_i n(i)a_i, \sum_j m(j)b_j\right)$$

induced by F. Note that both of these spaces are zero unless $a_1^{n(1)} \otimes \cdots \otimes a_k^{n(k)} = b_1^{m(1)} \otimes \cdots \otimes b_r^{m(r)}$ in which case both domain and codomain have generators indexed by the product

$$\prod_i \mathfrak{S}_{n(i)}$$

and F is the identity matrix on these generators. This proves that the generators of the domain are linearly independent since those of the codomain are. It follows that F is an equivalence of categories since every object of $\mathscr{H}(\gamma, \delta)$ is isomorphic to an object in $\mathscr{H}_0$. It follows that F defines a bijection on every hom space and therefore that the g_ϕ's are independent. The braiding on $\mathscr{H}(\gamma, \delta)$ is defined exactly as on the tensor categories $\mathscr{H}(\alpha, \beta)$. Q.E.D.

We shall say that a **C**-linear category is *projectively complete* if it is closed under finite direct sums and idempotents split. Any **C**-linear category $\mathscr{A}$ has a *projective completion* $\mathscr{A}'$ obtained in two steps:

Step 1. The *matrix completion* $M(\mathscr{A})$ is obtained by adjoining finite direct sums. An *object* of $M(\mathscr{A})$ is a finite family of objects of $\mathscr{A}$. An *arrow* f: $(A_i | i \in I) \to (B_j | j \in J)$ is a family $f = (f_{ji} | i \in I, j \in J)$ where f_{ji}: $A_i \to B_j$. If also $g: (B_j | j \in J) \to (C_k | k \in K)$, the composite gf is $(h_{ki} | i \in I, k \in K)$ where $h_{ki} = \sum_j g_{kj} f_{ji}$.

Step 2. The *Karoubi completion* $K(\mathscr{A})$ is obtained by formally splitting the idempotents of $\mathscr{A}$. An *object* of $K(\mathscr{A})$ is a pair (A, e) where A is an object of $\mathscr{A}$ and e: $A \to A$ is an idempotent (that is, $ee = e$). An *arrow* f: $(A, e) \to (B, p)$ is an arrow f: $A \to B$ such that $pfe = f$. Composition is as in $\mathscr{A}$.

The projective completion $\mathscr{A}'$ is equal to $K(M(\mathscr{A}))$. If $\mathscr{A}$ is a **C**-algebra R then $\mathscr{A}'$ is equivalent to the category of finitely generated projective R-modules. Any tensor product on $\mathscr{A}$ can be extended to a tensor product on $\mathscr{A}'$.

A **C**-linear category $\mathscr{A}$, with all hom vector spaces $\mathscr{A}(A, B)$ finite dimensional, is *semisimple* when the projective completion $\mathscr{A}'$ is an abelian category in which all short-exact sequences split. When $\mathscr{A}$ is semisimple, every object of $\mathscr{A}'$ decomposes into a finite direct sum of simple objects.

PROPOSITION 7.4. *The Hecke algebroid* $\mathscr{H}(\gamma, \delta)$ *is semisimple if* $\delta(s)^2/\gamma(s)$ *does not lie in the real closed interval* $[-4, 0]$ *for any* $s \in S$.

Proof. It is a fact [B, pp. 54–56] that the usual Hecke algebra $H_n(q, q-1)$ is semisimple iff the "q-factorial" of n is non-zero; this holds iff $q = 1$ or q is not an n!th root of unity; and this, in turn, is certainly guaranteed, for example, by $|q| \neq 1$. Changing the generators of the algebra by a common scalar multiple, we obtain an isomorphism between $H_n(\alpha, \beta)$ and $H_n(q, q-1)$ for q satisfying $q^2 - (2 + \beta^2/\alpha)q + 1 = 0$. We see that $q = e^{i\theta}$ iff $\cos\theta = 1 + \beta^2/2\alpha$. This means $|q| \neq 1$ iff β^2/α does not lie in the closed interval $[-4, 0]$ on the real axis; so $H_n(\alpha, \beta)$ is semisimple in this case. Hence, under the conditions of the proposition, the Hecke algebras $H_n(\gamma(s), \delta(s))$ are semisimple, and so, assemble to form a semisimple Hecke algebraoid $\mathscr{H}(\gamma(s), \delta(s))$. In the proof of Proposition 7.3 we showed that $\mathscr{H}(\gamma, \delta)$ is equivalent to the tensor product of the Hecke algebroids $\mathscr{H}(\gamma(s), \delta(s))$ for $s \in S$; as such, $\mathscr{H}(\gamma, \delta)$ is also semisimple. Q.E.D.

Let Q be the set of cuspidal representations. Consider the Hecke algebroid $\mathscr{H}(\gamma_0, \delta_0)$ where

$$\gamma_0(u, v) = q^{d(u)d(v)} \quad \text{and} \quad \delta_0(u) = \varepsilon_u\left(q^{d(u)(d(u)+1)/2} - q^{d(u)(d(u)-1)/2}\right).$$

We have shown that the family $(c_{u,v}|(u, v) \in Q \times Q)$ is a Yang-Baxter operator satisfying the equations $E(\gamma_0, \delta_0)$. Using the universality of $\mathscr{H}(\gamma_0, \delta_0)$, we obtain a tensor functor $F: \mathscr{H}(\gamma_0, \delta_0) \to \mathscr{RGL}(q)$ taking $y_{u,v}$ to $c_{u,v}$. The functor F can be extended to a functor F': $\mathscr{H}(\gamma_0, \delta_0)' \to \mathscr{RGL}(q)$ since $\mathscr{RGL}(q)$ is projectively complete.

THEOREM 7.5. *The functor F' is an equivalence of categories*

$$F': \mathscr{H}(\gamma_0, \delta_0)' \xrightarrow{\sim} \mathscr{RGL}(q).$$

Proof. First we prove that the functor F: $\mathscr{H}(\gamma_0, \delta_0) \to \mathscr{RGL}(q)$ is full and faithful. For this we need that the map

$$\begin{aligned}\mathrm{Hom}(&u_1 \otimes \cdots \otimes u_m, v_1 \otimes \cdots \otimes v_n)\\ &\to \mathrm{Hom}(Fu_1 \otimes \cdots \otimes Fu_m, Fv_1 \otimes \cdots \otimes Fv_n)\end{aligned}$$

is a bijection for any $u_1 \otimes \cdots \otimes u_m$ and $v_1 \otimes \cdots \otimes v_n$ in $\mathscr{H}(\gamma_0, \delta_0)$. It is easy to see by induction on the length of ϕ that we have $F(g_\phi) = c_{\phi'}$ where ϕ' is the inverse of the permutation ϕ and the c_ϕ's are the permutation maps defined in Section 6. The result follows from Theorem 6.5 and Proposition 7.3. But the full faithfulness of F clearly implies the full faithfulness of F'. Also, F' is essentially surjective since any irreducible representation is a factor of an external product of cuspidals (Proposition 2.2). Q.E.D.

Henceforth, we take P to be the set of irreducible monic polynomials $p \in \mathbf{F}[x]$ with nonzero constant term (that is, the polynomial x is excluded). Let $d(p)$ denote the degree of p and let ε: $P \to \{1, -1\}$ be the function with $\varepsilon(p) = 1$ if and only if the constant term $p(0)$ is a square in $\mathbf{F}$.

PROPOSITION 7.6. *There is a bijection π: $Q \to P$ such that $d(\pi(u)) = d(u)$ and $\varepsilon(\pi(u)) = \varepsilon_u$ for all $u \in Q$.*

Proof. Let $\mathbf{F}_n$ be the field extension of degree n of $\mathbf{F}$ and $\mathbf{F}_n^\times$ its multiplicative group of non-zero elements. Let $\mathbf{D}_n = \mathrm{Hom}(\mathbf{F}_n^\times, \mathbf{T})$ be the character group of $\mathbf{F}_n^\times$. The Frobenius automorphism $F(x) = x^q$ acts on both $\mathbf{F}_n^\times$ and $\mathbf{D}_n$. For all $n \geq 1$, let Q_n be the set of (isomorphism classes of) cuspidal representations of $GL(n, q)$. There is a bijection between Q_n and the set of orbits of size n for the action of F on $\mathbf{D}_n$ [Md]. The groups $\mathbf{D}_n$ and $\mathbf{F}_n^\times$ are abstractly isomorphic since they are both cyclic groups of order $(q^n - 1)$.

If we choose an isomorphism $i\colon \mathbf{F}_n^\times \to \mathbf{D}_n$, we obtain a bijection between Q_n and the set of orbits of size n for the action of F on $\mathbf{F}_n^\times$. These orbits are exactly the roots of irreducible polynomials with coefficients in $\mathbf{F}$. When q is even, this finishes the proof. Suppose that q is odd. For any $\xi \in \mathbf{D}_n$ whose orbit is of size n, let us write $M(\xi)$ for the cuspidal representation corresponding to this orbit. It is shown in [Md, p. 153, Example 2] that the central character of $M(\xi)$ is obtained by restricting ξ to $\mathbf{F}_1^\times$ (the restriction depending only on the orbit of ξ). In particular, we have $\varepsilon_{M(\xi)} = \xi(-1)$. Consider the exact sequence

$$\{1, -1\} \to \mathbf{F}_n^\times \to \mathbf{F}_n^\times,$$

where the first map is the inclusion and the second is the squaring operation. By duality, we obtain an exact sequence

$$\mathbf{D}_n \to \mathbf{D}_n \to \{1, -1\},$$

where the first map is the squaring operation and the second map is evaluation at -1. This shows that $\xi(-1) = 1$ if and only if ξ is a square in $\mathbf{D}_n$. But $\mathbf{D}_n$ is a cyclic group of order $(q^n - 1)$, so ξ is a square if and only if $\xi^r = 1$ where $r = (q^n - 1)/2$. Consider the subgroup $\mathbf{D}_n^F$ of F-invariant elements of $\mathbf{D}_n$. It is a cyclic group of order $(q - 1)$, so an element $\alpha \in \mathbf{D}_n^F$ is a square in $\mathbf{D}_n^F$ if and only if $\xi^s = 1$ where $s = (q - 1)/2$. We have the norm map

$$N\colon \mathbf{D}_n \to \mathbf{D}_n^F$$

given by

$$N(\alpha) = \prod_{0 \leq k \leq n-1} F^k(\alpha) = \alpha^{(q^n-1)/(q-1)}.$$

If we write $(q^n - 1)/2 = ((q^n - 1)/(q - 1))(q - 1)/2$, we see that $\xi^r = N(\xi)^s$ and therefore that ξ is a square in $\mathbf{D}_n$ if and only if $N(\xi)$ is a square in $\mathbf{D}_n^F$. Using the chosen isomorphism $i\colon \mathbf{F}_n^\times \to \mathbf{D}_n$, we obtain that, if $p(x) = \prod_j (x + z_j)$ is an irreducible polynomial of degree n with coefficients in $\mathbf{F}$, then $i(z_1)(-1) = 1$ if and only if $N(z_1) = p(0)$ is a square in $\mathbf{F}$. Q.E.D.

COROLLARY 7.7. *Let P be the set of irreducible monic polynomials in $\mathbf{F}[x]$, with non-zero constant term (that is, excluding the polynomial x). Let d_u*

denote the degree of $u \in P$ and let ε: $P \to \{1, -1\}$ be the function with $\varepsilon_u = 1$ if and only if the constant term $u(0)$ is a square in **F**. *Let $\mathscr{H}(P, d, \varepsilon)$ be the strict-monoidal* **C**-*linear category presented as follows*:

the generating objects are the elements u of **P**;

the generating arrows are symbols

$$c_{u,v}: u \otimes v \to v \otimes u,$$

subject to the relations

$$(v \otimes c_{u,w})(c_{u,v} \otimes w)(u \otimes c_{v,w}) = (c_{v,w} \otimes u)(v \otimes c_{u,w})(c_{u,v} \otimes w),$$

$$c_{v,u} \circ c_{u,v} = q^{d_u d_v} \qquad \text{for } u \neq v,$$

and

$$\left(c_{u,u} - \varepsilon_u q^{(1/2)d_u(d_u+1)}\right) \circ \left(c_{u,u} + \varepsilon_u q^{(1/2)d_u(d_u-1)}\right) = 0.$$

The tensor category $\mathscr{H}(P, d, \varepsilon)$ is braided and semisimple. Its projective completion is equivalent to the braided tensor category $\mathscr{RGL}(q)$.

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REFERENCES

[B] N. Bourbaki, "Groupes et Algèbres de Lie, IV, V, VI," Masson, Paris, 1982.

[D] B. J. Day, On closed categories of functors, *in* "Reports of the Midwest Category Seminar IV," pp. 1–38, Lecture Notes in Math., Springer-Verlag, Berlin, 1970.

[EK] S. Eilenberg and G. M. Kelly, Closed categories, *in* "Proc. Conf. Categorical Algebra at La Jolla, 1965," pp. 421–562, Springer-Verlag, New York/Berlin, 1966.

[F] P. J. Freyd, "Abelian Categories," Harper & Row, New York, 1964.

[GG] I. Gelfand and M. Graev, The construction of irreducible representations of simple algebraic groups over a finite field, *Dokl. Akad. Nauk USSR* **147** (1962), 529–532.

[GI] S. Gelfand, Representations of the general linear group over a finite field, *Mat. Sb.* **83** (1970), 15–41.

[Gn] J. A. Green, The characters of the finite general linear groups, *Trans. Amer. Math. Soc.* **80** (1955), 402–447.

[HC] Harish-Chandra, Eisenstein series over finite fields, *in* "Functional Analysis and Related Fields" (F. Browder, Ed.), pp. 76–88, Springer-Verlag, Berlin/New York, 1969.

[H] R. Howe, Harish-Chandra homomorphisms for p-adic groups, *in* "Regional Conference Series in Math.," Vol. 59, Amer. Math. Soc., Providence, 1985.

[HL] R. B. Howlett and G. I. Lehrer, Induced cuspidal representations and generalised Hecke rings, *Invent. Math.* **58**, (1980), 37–64.

[Jm] G. James, The irreducible representations of the finite general linear groups, *Proc. London Math. Soc.* (*3*) **52** (1986), 236–268.

[J1] A. Joyal, Une théorie combinatoire des séries formelles, *Adv. Math.* **42** (1981), 1–82.

[J2] A. Joyal, Foncteurs analytiques et espèces de structures, *in* "Colloque de combinatoire énumérative" (G. Labelle and P. Leroux, Eds.), pp. 126–159, Lecture Notes in Math., Vol. 1234, Springer-Verlag, Berlin, 1985.

[JS1] A. Joyal and R. Street, Braided tensor categories, *Adv. Math.* **102** (1993), 20–78.

[JS2] A. Joyal and R. Street, The geometry of tensor calculus, I, *Adv. Math.* **88** (1991), 55–113.

[Md] I. G. Macdonald, "Symmetric Functions and Hall Polynomials," Clarendon, Oxford, 1979.

[ML] S. Mac Lane, Categories for the working mathematician, *in* "Graduate Texts in Math.," Vol. 5, Springer-Verlag, Berlin, 1971.

[Z] A. V. Zelevinsky, Representations of finite classical groups, *in* "Lecture Notes in Math.," Vol. 869, Springer-Verlag, Berlin, 1981.