Springer

Jacobi and the Birth of Lie's Theory of Groups
Author(s): Thomas Hawkins
Source: *Archive for History of Exact Sciences*, Vol. 42, No. 3 (1991), pp. 187-278
Published by: Springer
Stable URL: http://www.jstor.org/stable/41133904
Accessed: 04-09-2016 03:59 UTC

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at http://about.jstor.org/terms

Springer is collaborating with JSTOR to digitize, preserve and extend access to *Archive for History of Exact Sciences*

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

Jacobi and the Birth of Lie's Theory of Groups

THOMAS HAWKINS

Communicated by J. GRAY

Table of Contents

In 1865 when SOPHUS LIE (1842–1899) completed his education at the university in Christiania (now Oslo), he had no idea he was destined to become a mathematician. One of the possibilities for a career he was considering was that of a teacher of mathematics and in this connection he devoted some time, while giving private instruction in mathematics, to speculations on what he called "philosophy of mathematics". Apparently one matter that aroused his interest was the issue of the widespread use of imaginaries in projective geometry, and in this connection he began reading geometrical works by PONCELET, CHASLES and PLÜCKER. Inspired by his reading, he engaged in some original mathematical research on the real representation of imaginary quantities, a portion of which was accepted for publication by CRELLE's journal [LIE 1869b].[1] On the basis of this experience, he decided to devote himself to mathematical research, to become a mathematician. Five years later, during the fall of 1873, he made a second fateful decision: to devote himself to the enormous task of creating a theory of continuous transformations groups. His decision was precipitated by his belief that he could resolve the problem of classifying all finite dimensional continuous groups

[1] A more detailed account was published by the Scientific Society of Christiania [1869a].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

of transformations acting on n-dimensional space, that is all groups of transformations generated by a finite number of infinitesimal transformations of n-dimensional space. It was in the course of working on this problem that he began to introduce the mathematical apparatus that constitutes his theory of finite dimensional transformation groups, the theory to which he devoted his mathematical career.

To understand fully why LIE believed he was in a position, mathematically, to resolve the above problem, and more generally to create a theory of continuous groups, and to understand why he deemed the theory sufficiently important to warrant such a large commitment of his time and energy, it is necessary to examine his mathematical experiences during the first five years of his career: 1869–73. His work during these years can be divided into two periods, each characterized by a different primary source of motivation. During the first period (1869–71) he worked in close contact with FELIX KLEIN and geometry supplied the motivation. During the second period the theory of first order partial differential equations in any number of variables provided the motivation and context within which he developed his ideas. The first period was the main focus of my paper [1989]. Here the focus is on the second period and on the cumulative effect of his work in both periods on his decision to attack the problem of group classification and to become deeply involved with the attendant theory of continuous groups of transformations.

Historians have generally acknowledged that LIE's mathematical activities during 1869–73 are relevant to the creation of his theory of transformation groups. Indeed, LIE himself said as much on many occasions. But to my knowledge no one has attempted to examine the birth of LIE's theory in the light of a careful analysis of his work during those years. This essay together with [1989] constitutes my contribution to such an attempt.[2] What emerges from these essays is, I believe, a much clearer picture of how and why LIE was led to create his theory of groups. The picture is also of historical interest for the manner in which it exhibits the dynamics by means of which the interplay of ideas from diverse mathematical theories — most notably, projective geometry, GALOIS' theory of algebraic equations, the general theory of first order partial differential equations and Hamiltonian mechanics — led, in the hands of LIE, to the creation of an entirely new theory. Except for § 1, which summarizes aspects of LIE's work during 1869–71 essential for an understanding of what follows, the emphasis here is upon LIE's mathematical activity during the period 1871–73, the period in which the primary external source of inspiration came from mathematics that was either due to or inspired by JACOBI.

Given the fact that the JACOBI Identity is fundamental to the theory of LIE groups, JACOBI's influence upon LIE will come as no surprise. But the bald fact that he inherited the Identity from JACOBI fails to convey fully or accurately the historical dimension of the impact of JACOBI's work on partial differential equations. It pervaded LIE's thinking and supplied the context for the development of

[2] Research on this project, much of it done at the Institute for Advanced Study, Princeton, was supported by National Science Foundation grants DIR 8808646 and DMS 8610730. I wish to thank A. BOREL for facilitating my stay at the Institute. I am also grateful to BOB HERMANN and PETER J. OLVER for the many helpful comments they made on a preliminary version of this essay.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

his own ideas, thereby providing him with new research directions in which to test them out. Virtually all the discoveries of LIE discussed in §§ 4–8 are related, directly or indirectly, to JACOBI's work. To understand why this is so, it is necessary to understand something of the development of the theory of first order partial differential equations before JACOBI and the impact of JACOBI's contributions on the course of subsequent developments (§§ 2–3), for LIE's activity in the second period was a part of the wave of research activity created by JACOBI's work, much of which was first published posthumously in 1862–66. It was of considerable interest to mathematicians by virtue of its intrinsic merit and also because of its connections with the HAMILTON-JACOBI approach to dynamics.

What distinguished LIE's reaction to JACOBI's work from that of other mathematicians was the idiosyncratic viewpoint that he brought to bear upon it by virtue of his experiences during his geometrical period, as will be seen in §§ 4–8. Thanks largely to JACOBI's contributions to the theory of first order partial differential equations, LIE found himself confronting a theory that he was able to interpret in terms of the group related concepts of his geometrical period. It would be difficult to overemphasize the historical importance, for LIE's eventual decision to create his theory of transformation groups, of this particular application of group related ideas. During 1869–71 the applications of group related notions made by LIE and KLEIN had been rather specialized: the study of W-curves and W-surfaces and the integration of certain partial differential equations defined in terms of line complexes. The applications all related to their own geometrical research and were not a part of a more broadly based research program. With the application of group related ideas to the general theory of first order partial differential equations, LIE had hit upon an application to an area of mathematics of widespread current interest and with a history to which, in addition to JACOBI, the names of such distinguished mathematicians as LAGRANGE, PFAFF, MONGE, HAMILTON and CAUCHY were associated.

The following essay has two related objectives: to establish the nature and extent of JACOBI's influence upon LIE and to understand how LIE's involvement with the theory of partial differential equations in 1871–73 combined with the experiences of his geometrical period to lead him, in the fall of 1873, to begin working on the creation of a theory of continuous groups. The second objective is complicated by the scant documentary evidence that exists for the year 1872 when, engaged by a plethora of mathematical ideas and discoveries, LIE limited his publications to brief, cryptic research announcements. During this period he reasoned primarily on a conceptualized geometrical level of which few traces are left on paper — if indeed they ever existed in any abundance. By the fall of 1873, he had translated many of his ideas into an analytical form. The important role of the analytical version of his ideas, particularly those relating to infinitesimal transformations (§ 7), for motivating his decision to attempt to create a theory of continuous transformation groups will be evident to the reader by the time § 8 has been read. For a full understanding of the exact progression of LIE's ideas it is thus necessary to know something about the progression of the translation of his synthetic ideas into analytical form. Documentary evidence combined with LIE's recollections provide some idea of the progression, as indicated in § 7, but one significant detail remains uncertain. A central role in the progression was played by the analytical

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

relation that exists between an infinitesimal contact transformation and its characteristic function. The existence of a correspondence between contact transformations and their characteristic functions was established by synthetic means by the spring of 1872 (§ 5). But when did he understand the correspondence in an analytical form such as that of Theorem 7.5? ENGEL suggested that LIE had already achieved this level of understanding by the spring of 1872. I have found ENGEL's arguments unconvincing for reasons given in § 7. In composing this essay I have taken the position that there is no reason to assume that LIE realized something akin to Theorem 7.5 and its implications (such as Theorem 7.6) until the fall of 1873, when Theorem 7.5 was announced in a letter to MAYER. This interpretation of the facts has dictated to some extent the arrangement of the material about LIE, especially in §§ 5–7.

Because many of the terms and notions involved in LIE's work in 1872–74 may not be familiar, a glossary has been included so that readers may refresh their memories about the meaning of a term introduced in *e.g.* § 4 and then referred to in § 6. To facilitate subsequent references, certain mathematical results have been set off as theorems. The standard type of reference notation is employed so that, for example, Theorem 3.2 is the second theorem stated in § 3. Here "theorem" is to be understood more in the literal Greek sense of the word as "that which is beheld or viewed" rather than as a precisely stated proposition capable of a completely rigorous proof. For example, many of the theorems presented are rather vaguely stated and reflect the "local" and "generic" orientation of the reasoning that produced them, reasoning according to which, *e.g.* a differential Ω which satisfies $d\Omega = 0$ is exact ($\Omega = dF$, for some function F) and the relevant inverse and implicit function theorems are always presumed to hold. The theorems in the following sections should be understood as historical statements, rather than as precise mathematical formulations by present day standards. More precise, nongeneric formulations of much of the material in this essay can be found in CARATHÉODORY's book [1935] (for §§ 2–4, 6) and in ENGEL's lectures [1932] (for §§ 4, 6, 8). Readers wishing to see how this material relates to aspects of the contemporary theory of manifolds should begin by consulting the commentaries on LIE's work by HERMANN [1975, 1976] and OLVER's book [1986] on the application of LIE groups to differential equations.

1. Groups and Geometry: 1869–71

LIE's research on the geometry of tetrahedral complexes (1869–70) and on the sphere mapping (1870–71), combined with KLEIN's reaction to this work, had led the two young mathematicians to perceive that there was a continuous analog of the notion of a group of permutations that had interesting and potentially important applications to geometry and to the study of differential equations. These matters have been discussed in detail in [1989]. Drawing upon that work I summarize here the status of LIE's understanding of, and interest in, continuous groups and related questions about the time of the winter of 1871–72, when his interests were shifting to problems more directly related to the theory of first order partial differential equations.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

Neither LIE nor KLEIN gave precise definitions of what they meant by a continuous group of transformations. For them, such a group consisted of a set G of invertible transformations $T: (y_1, \ldots, y_m) \to (y'_1, \ldots, y'_m)$ of an "m-dimensional manifold" of m-tuples. Following RIEMANN, KLEIN and LIE used the term "manifold (*Mannigfaltigkeit*)" to denote a totality of "elements" of the form $y = (y_1, \ldots, y_m)$. The modern conception of a manifold was first introduced in the 20th century by WEYL. Here the term "manifold" will be used in the naive sense in which KLEIN and LIE employed it.[3] The nature and extent of the set D_T of m-tuples for which T is defined was left unclear. Most of the groups they considered in 1869–71 consisted of projective transformations so that for such groups each D_T corresponds to projective space suitably coordinatized. On the other hand, when the T comprising G are contact transformations (defined below), it would seem that the domains of definition D_T vary with the individual T's (as occurs in what are now called pseudogroups). Whatever their understanding of the sets D_T for a particular group G might have been, it was tacitly assumed that any group G possesses the defining property of a permutation group: If T_1 and T_2 belong to G, then so does the composite transformation, $T_1 \circ T_2$. They took it for granted that, as in the case of permutation groups, closure under composition implies closure under inversion. Thus if G had "the group property" of closure under composition, it was assumed that T^{-1} is in G whenever T is. After he had begun to create his theory of transformation groups, LIE realized the assumption could not be taken for granted and attempted to prove it under certain conditions [HAWKINS 1989: note 10].

Closure under composition is what made G a group for KLEIN and LIE. What made G continuous? For many of the groups they considered in 1869–71, the continuity was reflected in the fact that the equations defining the transformations T of G depended upon a fixed number of parameters which, if varied continuously, described the T in G. LIE eventually termed such groups "finite continuous transformation groups." Some of the groups they considered, however, such as the group of all contact transformations of three dimensional space, were not finite in this sense. These "infinite groups" were nonetheless regarded as continuous because, as in the case of finite continuous groups, the transformations of such groups were generated by infinitesimal transformations.

The existence of infinitesimal transformations generating the transformations of a group was regarded as a fundamental characteristic of continuous transformation groups, especially by LIE. An infinitesimal transformation dT sends a point $y = (y_1, \ldots, y_m)$ into a neighboring point $y + dy$, where $dy_i = \eta_i(y)\,dt$, $i = 1, \ldots, m$. LIE and KLEIN identified dT with the system of differential equations

$$\frac{dy_i}{dt} = \eta_i(y), \quad i = 1, \ldots, m. \tag{1.1}$$

By applying dT "continually" to a point y_0 one obtained the points y on the curve with equations $y_i = f_i(y_0, t)$, $i = 1, \ldots, m$, which represented the solution to the system (1.1) satisfying the initial condition $y = y_0$ when $t = t_0$. The one parameter family of transformations $T_t: y \to y'$ defined by $y'_i = f_i(y, t)$, $i =$

[3] On the history of the concept of a manifold see [SCHOLZ, 1980].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

$1, \ldots, m$, defined the noninfinitesimal transformations of the group generated by dT. In this sense dT itself was regarded as belonging to the group.

The notion of a continuous group of transformations with which Lie operated in 1869–71 and throughout 1871–73 was thus extremely vague on specific details. His notion of finite continuous groups evolved into the present day concept of a finite dimensional LIE group, whereas his "infinite groups" are now understood in terms of the concept of a LIE pseudogroup.[4] In this essay, the word "group" will be used in the vague sense of LIE to embrace all these possibilities.

As a consequence of LIE's approach to the study of tetrahedral line complexes, he and KLEIN had come to appreciate the interesting geometrical relations that were associated to configurations that have the property of being left invariant by some finite dimensional continuous group of commuting projective transformations. More generally, as a result of LIE's discovery of his sphere mapping and KLEIN's mathematical reaction to the discovery, they had come to see each and every method of geometrical investigation as determined by a transformation group acting on some manifold. For example, PLÜCKER's line geometry was conceived as the geometrical method determined by the four dimensional quadratic hypersurface $M_4^{(2)} \subset P^5(C)$ of elements $y = (y_1, \ldots, y_6)$ satisfying $\Sigma_{i=1}^6 y_i^2 = 0$ (representing homogeneous line coordinates) and the group of all $T \in PGL(6, C)$ which take $M_4^{(2)}$ into itself (and therefore define projective line transformations). As this example illustrates, most of the geometrical methods of interest to KLEIN and LIE corresponded to noncommutative groups. As a result of the sphere mapping work, they had come to appreciate the importance of the general concept of a continuous transformation group in geometrical investigations. KLEIN appears to have been primarily responsible during this period for emphasizing the broader significance of the group related notions implicit in LIE's early work. In particular, it was KLEIN who articulated the idea of geometrical methods in terms of groups. This first occurred in an essay (no longer extant) on geometrical method written in December of 1871, at the close of the first, or geometrical, period of LIE's research activity. When KLEIN and LIE were together again in the fall of 1872, by which time LIE's research interests were dominated by the theory of partial differential equations, LIE's presence encouraged him to present his ideas on geometrical method as his "Erlangen Program" [1872].

The Erlangen Program was written in October of 1872 while LIE and KLEIN were together in Erlangen, where KLEIN was about to enter the faculty of the University as a Full Professor (*ordentlicher Professor*). KLEIN was the moving force behind the conception of the Program; it was on a level that was more general — "philosophical" was LIE's term — than the specific (albeit far reaching) mathematical problems that engaged LIE's interest at the time. Nonetheless LIE whole heartedly agreed with its viewpoint and in fact pointed out to KLEIN that the research on partial differential equations and contact transformations that had derived from his sphere map work (see below) exemplified KLEIN's viewpoint in ways that were not expected by KLEIN. In particular, there is no doubt that

[4] For the definition of a LIE pseudogroup and a brief discussion of historical developments related to the theory of infinite continuous groups, see [SINGER & STERNBERG 1965].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

LIE concurred with the sentiments KLEIN expressed in the concluding remarks of the Program [1872: 489]. There he called attention to the analogy between the considerations in his Program and GALOIS' theory. When KLEIN composed his Program, the theory and application of permutation groups (or groups of "substitutions" as permutations were called at the time) had already been systematically developed by CAMILLE JORDAN in his monumental *Traité des substitutions* [1870]. KLEIN observed that in works such as JORDAN's, the theory of permutation groups was developed as an independent theory and then applied to the study of algebraic equations (GALOIS' theory). With the paradigm of JORDAN's *Traité* in mind, KLEIN called for an analogous development of an independent theory of continuous transformation groups, a theory which could then be applied to geometry in the sense of his Program.

Although KLEIN and LIE called for the development of a theory of transformation groups in 1872, they did not themselves attempt to contribute to such an enterprise. Their geometrical investigations had led them to appreciate the importance of the *notion* of a transformation group in such research, but it did not provide them with the mathematical tools to develop the *theory* of such groups. This fact is illustrated by LIE's attitude towards a problem suggested by the Erlangen Program, as well as by their earlier work on groups. According to the Erlangen Program, diverse geometrical methods correspond to nonsimilar transformation groups. Roughly stated, two groups of transformations $x' = Tx$ and $y' = Uy$ acting on manifolds of the same dimension are similar if the transformations of the one group go over to those of the other by means of a coordinate change $y = \Phi(x)$. Consequently, it would be of interest to know how many distinct geometrical methods there are besides the ones already known and what sort of "geometries" they determine. The Erlangen Program thus naturally suggests the problem of classifying transformation groups up to similarity. LIE was aware of this problem at the time the Program was written but regarded it then as "absurd and impossible", according to LIE's later recollections in a letter to KLEIN [HAWKINS 1989: note 37]. In other words, in 1872 it appeared ridiculous to attempt to solve such a problem since he lacked the means to do it. This lack reflected the fact that the geometrical work that absorbed his attention in 1869–71 was in spirit far removed from the type of mathematics that would seem necessary to carry out such a classification. Indeed, LIE and KLEIN realized that to carry out a systematic investigation of what they called W-configurations — k-dimensional manifolds in $P^n(C)$ which are the orbits of points under the action of a k-parameter group of projective, commuting transformations — they would have to begin by determining all such groups. Although they did this for $n = 2$ (with KLEIN working out the details) even at $n = 3$ the problem became too complex for KLEIN's patience. For an arbitrary n the problem seemed to KLEIN not only tedious but very difficult. Neither he nor LIE sought to deal with it. Such a problem was not accorded a high priority by either of them.

LIE came away from the geometrical period of research with more than the notion of a continuous group of transformations and an appreciation of its relevance to geometry. As early as 1869 and as a result of KLEIN's suggestive observations, he had come to believe in the value and possibility of developing for the study of differential equations something analogous to GALOIS' theory of algebraic

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

equations. This belief was an *idée fixe* in his mind during the entire period 1869–73 (as well as beyond it) and, as we shall see, played a key role in the events leading to the birth of his theory of groups. It is therefore important to describe more fully the nature of this belief and the extent to which he had translated it into a reality by the end of 1871.

Consider a first order partial differential equation $f(x, y, z, p, q) = 0$, where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. Let us say that this equation *admits* a transformation $T: (x, y, z) \rightarrow (x', y', z')$ if every solution surface $\varphi(x, y, z) = 0$ of the equation is transformed into a solution surface. This is what LIE usually meant when he said that a differential equation, or a system of differential equations, admits (*zulässt, gestattet*) a transformation: the transformation takes solutions into solutions. An equation admits a group of transformations if it admits all the transformations comprising it. LIE's early geometrical work had led him to results which may be summarized as

Theorem 1.1. *Suppose* $r \leqq 3$ *commuting, independent infinitesimal transformations of the variables* x, y, z *are known, which are admitted by* $f(x, y, z, p, q) = 0$. *Then* $f(x, y, z, p, q) = 0$ *can be transformed into a another first order equation in which* r *of the new variables* X, Y, Z *are missing.*

LIE's proof was vague and intuitive, it boiled down to arguing that a suitable change of variables exists so that the r transformations become, in the new variables, translations along r coordinate axes.

If, for example, $r = 3$, Theorem 1.1 asserts that $f(x, y, z, p, q) = 0$ may be transformed into an equation of the form $F(P, Q) = 0$, $P = \frac{\partial Z}{\partial X}$, $Q = \frac{\partial Z}{\partial Y}$, a type of equation studied and integrated by EULER. If $r = 2$ in Theorem 1.1, so that $f(x, y, z, p, q) = 0$ can be transformed into $F(Z, P, Q) = 0$, from known results about differential equations LIE could deduce that the integration of $F(Z, P, Q) = 0$ reduces to a quadrature. In other words, the significance of Theorem 1.1 to LIE was that from the knowledge that the partial differential equation admits a group of known transformations, one is able to obtain information about its integration, about what is involved to effect its integration. LIE saw such results as forming an analog of GALOIS' theory.

The exact extent and accuracy of LIE's knowledge of GALOIS' theory is not known. Although as a student in Norway he had attended SYLOW's lectures on the subject, he claimed he understood very little of what was presented. He may have picked up more information through his contact with KLEIN, who knew a bit more about it. Judging by what KLEIN knew and by LIE's remarks in a letter (quoted below) the following sort of considerations probably reflect LIE's view. Any permutation of the roots of a polynomial equation is admitted by the equation in the sense that it takes a solution (a root) into a solution (some other root). The GALOIS group of a polynomial equation consisting as it does of certain permutations of its roots is thus admitted by the equation. In GALOIS' theory, from

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

knowledge of the GALOIS group of a polynomial equation one is able to obtain information about what is involved in resolving it. That is, assuming a composition series of the group is known one knows that the resolution of the given equation reduces to the successive resolution of the equations corresponding to the factor groups. This is illustrated by a result published by JORDAN in 1869 which was known to KLEIN (and therefore perhaps to LIE through KLEIN). JORDAN had considered the polynomial equation of degree 16 whose roots yield the 16 nodal points on a KUMMER surface. Since he knew the GALOIS group of the equation from the geometry of the situation, he was able to conclude from the composition series of the group that the resolution of the equation of degree 16 reduces to resolving the general equation of degree 6 and several quadratic equations. This is the sort of application of GALOIS' theory LIE seems to have had in mind when he wrote in a letter of February 1874: "Before Galois in the theory of algebraic equations, one only posed the question: Is the equation solvable by radicals and how is it solved? Since Galois' time one also poses ... the question: How is the equation solved *most simply* by radicals? It can be proved e.g. that certain equations of degree six are solvable by means of equations of the second and third degrees and not say, by equations of the second degree" [LIE 1873–4: 586]. In other words, if the composition series of the GALOIS group of a sixth degree equation involves factor groups of orders 2 and 3, then the resolution of the equation will involve adjoining the roots of cubic equation and cannot be reduced to the successive resolution of quadratic equations (as would be the case if the factor groups were all of order 2).

LIE believed that, likewise, from knowledge of transformations admitted by a differential equation one was in a position to say something more about its solution than was otherwise possible, just as, for example, it was possible to say more about the integration of a special type of ordinary differential equation such as the "homogeneous" equation $\frac{dy}{dx} = f\left(\frac{x}{y}\right)$ than about the general first order equation $\frac{dy}{dx} = F(x, y)$. Indeed, for LIE the true ground for this fact was that the homogeneous equation admits the transformations $T_a: (x, y) \rightarrow (ax, ay)$. He proved that if an equation $\frac{dy}{dx} = F(x, y)$ admits any 1-parameter group of transformations generated by a known infinitesimal transformation, then a change of variables is possible such that $\frac{dy}{dx} = F(x, y)$ transforms into a separable equation which can therefore be integrated by quadrature. This theorem and Theorem 1.1 represented the extent to which LIE had pursued this sort of consideration by the end of 1871. The proofs of both are based upon the same sort of reasoning, which LIE had pushed as far as it would go, except for the obvious extension to differential equations in any number of variables.

LIE's belief that some sort on an analog of GALOIS' theory could be developed along the lines indicated above will be referred to in what follows for ease of reference as his *idée fixe*. Up to the end of 1871, his contributions towards the realization of his *idée fixe* had all involved assuming the commutativity of the

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

transformations involved. This was also true of all the group related work stemming from the study of tetrahedral complexes. There is no reason, however, to think that LIE saw his *idée fixe* as limited to commutative groups. If there were to be a complete analogy with GALOIS' theory, one would expect to find results about differential equations admitting noncommutative groups as well. The hypothesis of commutativity, however, had proved to be well suited for translation into information about the integration of a differential equation. The situation must have appeared somewhat similar in GALOIS' theory. In 1869–70, the best known part of the theory involved the polynomials considered by ABEL, which correspond to the case in which the GALOIS group is commutative. This was the only part of the theory considered in SERRET's *Cours d'Algèbre Supérieure* [1866] — the principal text on the theory before the publication of JORDAN's *Traité* in 1870. In the "Abelian case" (and in the case of any solvable group) the composition series of the group yielded a greater amount of information about the equations required to resolve a given equation: they were all of the form $x^p - a = 0$ with p prime (assuming as was usual that all requisite roots of unity are at hand). The emphasis that LIE and KLEIN gave in their early work to the hypothesis of commutativity was thus both understandable from the standpoint of practicability and in keeping with the analogy with algebraic equations.

Presumably LIE hoped that results in the noncommutative case would be forthcoming even though at the time he had no idea how to obtain them. As we shall see in § 5, the JACOBI theory of partial differential equations provided the means for him to develop his *idée fixe* without the hypothesis of commutativity. In this connection it should be noted that, although the work on tetrahedral complexes and the sphere mapping had provided many examples to which Theorem 1.1 applied, the sphere mapping work had also led LIE and KLEIN to turn their attention from commutative to noncommutative groups. In fact, the noncommutative groups associated with the sphere mapping work that particularly interested LIE consisted of contact transformations and were directly related to the study of differential equations. The notion of a contact transformation played a fundamental role in his work during 1871–73, and, in particular, in the development of his *idée fixe* within the context of JABOBI's theory. It will be helpful here to explain this notion as he understood it when he published his work on the sphere mapping in 1871.

Contact transformations had come to LIE from geometry. In geometry they arose as correspondences or "reciprocities" between geometrical objects. For example, an equation of the form $F(x, y, z, X, Y, Z) = 0$ establishes a correspondence between two three dimensional spaces, $\boldsymbol{r}$ and $\boldsymbol{R}$, coordinatized, respectively by x, y, z and X, Y, Z. By virtue of this correspondence various types of geometrical objects in $\boldsymbol{r}$ are associated to types in $\boldsymbol{R}$. For example, if $p = (x, y, z) \in \boldsymbol{r}$ is held fixed, then the equation $F(x, y, z, X, Y, Z) = 0$ associates to the point p a surface in $\boldsymbol{R}$, and so on. When the equation $F = 0$ is $xX + yY + zZ + 1 = 0$, then corresponding to p is a plane; this reciprocity is the well known duality between points and planes in space. PLÜCKER had taken the first step towards a general theory of reciprocities by considering (in the plane) any reciprocity defined by a polynomial equation $F(x, y, X, Y) = 0$. LIE carried PLÜCKER's ideas much further. In three dimensional space he considered reciprocities de-

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

fined by one, two or three equations:

$$\text{(A)}\ F(x,y,z,X,Y,Z)=0, \quad \text{(B)}\ \begin{matrix} F(x,y,z,X,Y,Z)=0, \\ G(x,y,z,X,Y,Z)=0, \end{matrix} \quad \text{(C)}\ \begin{matrix} F(x,y,z,X,Y,Z)=0, \\ G(x,y,z,X,Y,Z)=0, \\ H(x,y,z,X,Y,Z)=0. \end{matrix}$$

Type (A) was the type considered by PLÜCKER, although LIE imposed no restriction on the nature of the function F. Type (C) was also familiar; it consisted of the usual point transformations. That is, if $p=(x,y,z)\in \boldsymbol{r}$ is held fixed, then the equations (C) correspond to the intersection of three surfaces and therefore in general determine a point $P\in \boldsymbol{R}$. Thus (C) defines a transformation $p\to P$ of points. Type (B) was first considered by LIE; his sphere mapping involved a reciprocity of this type.

LIE also realized that all three types of reciprocity determine a correspondence between "surface elements" in spaces $\boldsymbol{r}$ and $\boldsymbol{R}$, and this insight enabled him to see a relevance of these reciprocities to the study of differential equations. A surface element in $\boldsymbol{r}$ was an infinitesimal surface $ds\subset \boldsymbol{r}$, which he regarded as determined by a point $a=(x,y,z)\in ds$ and by the parameters p,q of the normal $\boldsymbol{n}=p\boldsymbol{i}+q\boldsymbol{j}-\boldsymbol{k}$ to ds at a. (Later he formally identified surface elements ds with point-plane pairs a,π, where π is the plane through a with normal $\boldsymbol{n}=p\boldsymbol{i}+q\boldsymbol{j}-\boldsymbol{k}$ — the tangent plane to ds at a.) In the spirit of PLÜCKER's line geometry, where lines are coordinatized, he regarded surface elements in $\boldsymbol{r}$ and $\boldsymbol{R}$ as coordinatized by (x,y,z,p,q) and (X,Y,Z,P,Q). The reciprocities (A)–(C), he realized, define transformations $T:(x,y,z,p,q)\to(X,Y,Z,P,Q)$. For example, for a type (A) reciprocity the transformation T can be defined analytically in the following manner. Fix X,Y,Z and regard $F=0$ as defining z as a function of x and y to obtain, by differentiation of $F=0$, the equations $F_x+F_zp=0$ and $F_y+F_zq=0$, where $F_x=\dfrac{\partial F}{\partial x}$, $p=\dfrac{\partial z}{\partial x}$, *etc.* Similarly with x,y,z held fixed, differentiation of $F=0$ yields $F_X+F_ZP=0$ and $F_Y+F_ZQ=0$. The five equations $F=0$, $F_x+F_zp=0,\dots F_Y+F_ZQ=0$ in the ten unknowns x,y,z,p,q,X,Y,Z,P,Q can be solved for X,Y,Z,P,Q (assuming the implicit function theorem applies) to obtain $X=g_1(x,y,z,p,q)$, $Y=g_2(x,y,z,p,q)$, $Z=g_3(x,y,z,p,q)$, $P=g_4(x,y,z,p,q)$, $Q=g_5(x,y,z,p,q)$. These equations define the contact transformation T associated to the reciprocity $F(x,y,z,X,Y,Z)=0$.

LIE called the transformations $T:(x,y,z,p,q)\to(X,Y,Z,P,Q)$ arising from the reciprocities (A)–(C) *contact transformations*. If a surface s is conceived as composed of elements ds, then a contact transformation T generally takes the surface s so conceived into a surface S, which can be regarded as composed of the elements $dS=T[ds]$. That is, the locus of all points (X,Y,Z) such that $(X,Y,Z,P,Q)=T(x,y,z,p,q)$ for some $ds=(x,y,z,p,q)$ belonging to s, is generally a surface S with surface elements $dS=(X,Y,Z,P,Q)$. (What happens in the exceptional cases will be indicated in § 4). If two surfaces s and s' touch at a point, this means they have a common surface element at that point. A contact transformation, being a transformation of surface elements will there-

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

fore take s, s' into surfaces S, S' which share the transformed surface element and therefore also touch. Contact transformations thus preserve the contact of surfaces and hence LIE's name for them.

Consider now a first order partial differential equation $F(X, Y, Z, P, Q) = 0$, where $P = \frac{\partial Z}{\partial X}, Q = \frac{\partial Z}{\partial Y}$. It was commonplace in the study of such equations to consider a change of variables

$$(1.2) \qquad X = g_1(x, y, z), \quad Y = g_2(x, y, z), \quad Z = g_3(x, y, z).$$

From these equations one obtains expressions for $P = \frac{\partial Z}{\partial X}$ and $Q = \frac{\partial Z}{\partial Y}$ of the form

$$(1.3) \qquad P = g_4(x, y, z, p, q), \quad Q = g_5(x, y, z, p, q),$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.[5] The transformations (1.2)–(1.3) allow the original partial differential equation to be transformed into a new, equivalent one, $f(x, y, z, p, q) = 0$. Occasionally mathematicians such as EULER, LAGRANGE, AMPÈRE and LEGENDRE had used more general transformations such as the transformation

$$(1.4) \qquad X = -q, \quad Y = p, \quad Z = z - px - qy.$$

Since the equations (1.4) imply that $Z = z - Yx + Xy$, and hence that $P = \frac{\partial Z}{\partial X} = y$ and $Q = \frac{\partial Z}{\partial Y} = -x$, they also allow the original equation to be transformed into an equivalent equation, $f(x, y, z, p, q) = 0$.

From LIE's viewpoint, transformations such as (1.4) as well as ithe more customary transformations (1.2)–(1.3) were simply rather special cases of the more general concept of a contact transformation — a contact transformation being conceived as any invertible transformation $T: (x, y, z, p, q) \rightarrow (X, Y, Z, P, Q)$ which is defined by a reciprocity of types (A)–(C). (As will be seen in § 4, he eventually obtained a more satisfactory analytical characterization of those invertible transformations of five variables x, y, z, p, q which represent contact transformations.) The contact transformation defined by (1.4) comes from the equation $Z - z + Yx - Xy = 0$ and is therefore of type (A), whereas (1.2) defines a type (C) point transformation, prolonged to a contact transformation by (1.3). He had consequently hit upon an extensive family of transformations that could be applied to change one partial differential equation into another.

In his work on the line to sphere mapping LIE had discovered interesting classes of first order partial differential equations that could be transformed into one

[5] For the actual form of the expressions (1.3) see [LIE 1896: 577].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

another by the contact transformation defined by the type (B) reciprocity underlying his work. He thus became interested in the following question: given two systems of first order partial differential equations, when can the one system be transformed into the other by means of a contact transformation? If two systems could be so transformed, the integration of the one would be tantamount to the integration of the other, since by means of the contact transformation, the integral surfaces of the one equation would be transformed into the integral surfaces of the other. These considerations raise the question as to what properties of a system of partial differential equations are preserved by all contact transformations. For if one system had some property (P) which is preserved under all contact transformations, then it could only be transformed by such a transformation into systems possessing (P) as well. It was while LIE was thinking about such matters during the winter of 1871–72 that KLEIN communicated to him his ideas on geometrical methods. Once he grasped what KLEIN was getting at, he saw that his ideas on contact transformations and partial differential equations exemplified the viewpoint of KLEIN's essay, provided one allowed the term "geometrical method" a more general scope than that suggested by the examples KLEIN originally had in mind. That is, one can consider the manifold of all surface elements (x, y, z, p, q) and the group $\boldsymbol{B}$ of all contact transformations. According to KLEIN's views, this manifold and group determines a geometrical method, with related methods determined by subgroups.

KLEIN liked to express the essential objective of a geometrical method by borrowing from the language of the algebraic theory of invariants and speaking of it as an "invariant theory" since, given a manifold and a group of transformations of it, the idea was to discover the "invariant relations" with respect to the given group [1871: 321; 1872: 463–464]. LIE realized that his interest in determining properties of partial differential equations preserved by the group of all contact transformations exemplified what KLEIN had in mind. Subsequently he and KLEIN frequently spoke of LIE's "invariant theory" of contact transformations. KLEIN's decision to speak of the "invariant theory" corresponding to a geometrical method, reflected his realization that, in the case of the geometrical method that corresponds to projective geometry, the algebraic theory of invariants of CLEBSCH and GORDAN provided the means to carry out the program of determining the invariant relations of the geometry. In the case of the geometrical method determined by the group $\boldsymbol{B}$ of contact transformations, however, something akin to the algebraic theory of invariants was inappropriate as a means of creating a viable invariant theory. In the spring of 1872, LIE's invariant theory of contact transformations was still largely an unspecified prospective direction for future research rather than a well defined research project in the course of being implemented. As we shall see in § 6, it was in the theory of first order partial differential equations, particularly JACOBI's work, that LIE found the inspiration and mathematical direction for what eventually became, by the spring of 1873, the heart of his invariant theory, his theory of function groups. In turn, the theory of function groups in combination with his growing understanding of relations between the calculus of JACOBI's differential operators and transformation groups (§ 7) provided him with a theoretical basis for taking on the theory and classification of transformation groups (§ 8).

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

2. Jacobi's First Method

The theory of first order partial differential equations was the center of JACOBI's attention during two periods of his career, in 1827 shortly after he had received his doctorate and then in 1836–42, while he was Professor at the University of Königsberg. In 1827 he became interested in the subject as a result of his lectures on the calculus of variations [KOENIGSBERGER, L. 1904: 44]. At that time the major contributions to the general theory of integrating such equations had been made by LAGRANGE and PFAFF. (CAUCHY had also made an important contribution in 1819, but it was not known to JACOBI — or to most mathematicians — until much later. It will be discussed in § 4.) JACOBI evidently made a careful study of their work at this time and produced two papers, one dealing with aspects of LAGRANGE's contributions [1827a] and one discussing PFAFF's theory [1827b]. Although these papers did not advance the theory in fundamental ways, they served to perfect what had been achieved and to present the results in the elegant analytical style that became his hallmark. Since his formulation of the contribitions of LAGRANGE and PFAFF formed the starting point of the later 19th century analytical developments of interest to us here, we shall adhere to it in discussing the theory as he found it in 1827.

Although other 18th century mathematicians such as EULER had studied various special types of first order partial differential equations, LAGRANGE was primarily responsible for initiating the general theory of such equations. One of his contributions involved clarifying and relating the various types of solutions that partial differential equations could have. The integration of specific types of partial differential equations

$$F(x_1, \dots, x_n, z, p_1, \dots, p_n) = 0, \quad p_i = \frac{\partial z}{\partial x_i}, \quad i = 1, \dots, n, \tag{2.1}$$

mostly with $n = 2$ independent variables, had led to solutions

$$z = \varphi(x_1, \dots, x_n; C_1, \dots, C_n) \tag{2.2}$$

involving n arbitrary constants, but it was realized that solutions could exist which were not of this form, some even more general than (2.2) in the sense that they involved an arbitrary function rather than arbitrary constants. LAGRANGE called a solution of the form (2.2) a *complete solution*, and he showed that the other types of solutions, particular solutions and general solutions, which depend on an arbitrary function, could be obtained from (2.2) by appropriately eliminating the constants C_i.[6] In the general theory of first order partial differential equations it was generally considered sufficient to determine a complete solution.

LAGRANGE was also the first to succeed in providing a method of integrating a general first order partial differential equation in two independent variables,

[6] [LAGRANGE 1774, 1806: Ch. 20]. LAGRANGE's reasoning was "generic"; the eliminations he performed, to be justified locally, required in effect that φ should have the property that $\partial^2\varphi/\partial x_i\, \partial C_j$ have a non-vanishing determinant. See [CARATHÉODORY 1935: § 50].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

$F(x, y, z, p, q) = 0$, where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$ [1772, 1806: Ch. 20]. His idea was as follows. Solve the equation for q and express it in the equivalent form

$$(2.3) \qquad q = F_1(x, y, z, p), \quad p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}.$$

To solve (2.3), he considered the differential

$$\Omega = dz - p\,dx - q\,dy = dz - p\,dx - F_1(x, y, z, p)\,dy,$$

where now p does not stand for a partial derivative but for an as yet undetermined function, $p = F_2(x, y, z)$. He observed that if the function F_2 could be chosen so that Ω. multiplied by some factor $M(x, y, z)$, is exact, so that $M\Omega = d\Phi$, then $\Phi(x, y, z) = C$ determines a solution $z = \psi(x, y, C)$ to (2.3). The conclusion follows since, by virtue of the equality $M\Omega = d\Phi$,

$$(2.4) \qquad \frac{\partial z}{\partial x} = -\frac{\Phi_x}{\Phi_z} = F_2, \quad \frac{\partial z}{\partial y} = -\frac{\Phi_y}{\Phi_z} = F_1(x, y, z, F_2).$$

The equations (2.4) mean that $z = \psi(x, y, C)$ is a solution to (2.3). Next he showed that in order to determine F_2 and M so that $M\Omega$ is exact, it is necessary to integrate a linear first order partial differential equation in the four variables x, y, z, p. Since he knew how to integrate linear first order partial differential equations in any number of variables, he regarded the problem of integrating $F(x, y, z, p, q) = 0$ as solved.

LAGRANGE knew how to integrate linear equations in the sense that he knew how to reduce their integration to the problem of integrating a system of first order ordinary differential equations. For him, as for EULER, that was all that was to be expected: "the art of the calculus of partial derivatives is known to consist in nothing more than reducing this calculus to that of ordinary derivatives, and a partial differential equation is regarded as integrated when its integral depends on nothing more than that of one or more ordinary differential equations."[7] LAGRANGE thus set the goal of the theory of integrating first order partial differential equations which was adhered by his successors: to reduce the integration of such equations to that of systems of first order ordinary differential equations.

The reduction of the integration of a linear first order partial differential equation to that of a system of ordinary differential equations is so fundamental to all that is to follow that it needs to be described in some detail. The presentation will be along the lines of JACOBI's elegant paper [1827a], which like all the work to be discussed in this section tended to treat implicitly the "generic" situation. (A more rigorous treatment can be found in CARATHÉODORY's book [1935: 23–25].) Consider first a homogeneous linear first order partial differential equation

$$(2.5) \qquad \xi_1(x_1, \ldots, x_n)\frac{\partial z}{\partial x_1} + \ldots + \xi_n(x_1, \ldots, x_n)\frac{\partial z}{\partial x_n} = 0.$$

[7] [LAGRANGE, J. L. 1781: 625]. Here we have translated "différences" as "derivatives." The same view had been expressed by EULER [1770: 34].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

The integration of this equation was considered equivalent to integrating the system of ordinary differential equations

$$dx_1 : dx_2 : \ldots : dx_n = \xi_1 : \xi_2 : \ldots : \xi_n . \tag{2.6}$$

JACOBI introduced the notation in (2.6) to indicate that any one of the variables x_i could be chosen as independent variable. If x_n is chosen, then (2.6) may be written in the more familiar form

$$\frac{dx_i}{dx_n} = \frac{\xi_i(x_1, \ldots, x_n)}{\xi_n(x_1, \ldots, x_n)}, \quad i = 1, \ldots, n-1. \tag{2.7}$$

Thus (2.6) represents a system of $n-1$ ordinary differential equations, such as the system (2.7). For ease of reference in what follows (2.6) will be referred to as a *system of order* $n-1$. If an auxiliary variable t is introduced into (2.6) by setting the common ratio there equal to dt, then (2.6) may also be written in the form

$$\frac{dx_i}{dt} = \xi_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n. \tag{2.8}$$

To gain an idea of the equivalence of the homogeneous partial differential equation (2.5) and the system of ordinary differential equations (2.6)–(2.8), let $x_k = \phi_k(x_n, C_1, \ldots, C_{n-1})$, $k = 1, \ldots, n-1$, denote the solution to (2.7) corresponding to arbitrary initial conditions $x_k = C_k$ when $x_n = x_n^0$. (In what follows the system (2.6) will be said to have been *completely integrated* when the ϕ_k are known.) It is assumed that the $n-1$ functions ϕ_k are functionally independent. Solving this system of equations for the C_k, we have $C_k = f_k(x_1, \ldots, x_n)$, $k = 1, \ldots, n-1$. Each function $z = f_k(x_1, \ldots, x_n)$ is then a solution to (2.5) since if the equation $f_k = C_k$ is differentiated with respect to x_n, $\sum_{i=1}^{n-1} \left(\frac{\partial f_k}{\partial x_i}\right)\left(\frac{dx_i}{dx_n}\right) + \frac{\partial f_k}{\partial x_n} = 0$, which when combined with (2.7) yields (2.5) with $z = f_k(x_1, \ldots, x_n)$. Hence the functions f_k are $n-1$ functionally independent solutions to (2.5). If $\Phi(u_1, \ldots, u_{n-1})$ is an arbitrary function of $n-1$ variables, then it is readily seen that $z = \Phi(f_1, \ldots, f_{n-1})$ is also a solution to (2.5). It turns out that all solutions are expressible in this form.

The linear partial differential equation that occurred in LAGRANGE's treatment of the general first order partial differential equation (2.3) is actually nonhomogeneous, but it is easy to reduce the nonhomogeneous case to the homogeneous one, as JACOBI observed [1827a: § 4]. Since the reduction establishes a correspondence between nonhomogeneous linear partial differential equations and systems of ordinary differential equations which became the basis for LIE's key discovery of the connection between infinitesimal contact transformations and their characteristic functions (Theorem 5.1), the reduction is worth describing. Consider a nonhomogeneous linear equation of the form

$$\xi_1(x, z)\frac{\partial z}{\partial x_1} + \ldots + \xi_n(x, z)\frac{\partial z}{\partial x_n} = \zeta(x, z), \tag{2.9}$$

where $(x, z) = (x_1, \ldots, x_n, z)$. Let a solution $z = f(x_1, \ldots, x_n)$ to (2.9) be defined implicitly by the equation $\varphi(x_1, \ldots, x_n, z) = 0$. Then since $\frac{\partial \varphi}{\partial x_i} + \frac{\partial \varphi}{\partial z}\frac{\partial z}{\partial x_i} = 0$

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

for $i = 1, \dots, n$, these equations along with (2.9) imply that

$$w = \varphi(x_1, \dots, x_n, z)$$

is a solution to the linear homogeneous equation in $n + 1$ independent variables

$$\xi_1(x, z)\frac{\partial w}{\partial x_1} + \dots + \xi_n(x, z)\frac{\partial w}{\partial x_n} + \zeta(x, z)\frac{\partial w}{\partial z} = 0.$$

Conversly, if $w = \varphi(x_1, \dots, x_n, z)$ is any solution to this homogeneous equation, it follows readily that the equation $\varphi(x_1, \dots, x_n, z) = 0$ defines implicitly a solution $z = f(x_1, \dots, x_n)$ to (2.9). Thus the integration of the nonhomogeneous equation (2.9) is equivalent to the integration of the above homogeneous equation and thus to the associated system of ordinary differential equations, namely

$$(2.10) \qquad \frac{dx_1}{dt} = \xi_1(x, z), \quad \dots, \quad \frac{dx_n}{dt} = \xi(x, z), \quad \frac{dz}{dt} = \zeta(x, z).$$

For JACOBI and his successors the equations (2.5)–(2.10) were regarded as equivalent, and they would switch from one to the other whenever convenient.

Although LAGRANGE had succeeded in integrating the general first order partial differential equation (2.1) in $n = 2$ independent variables, neither he nor his immediate successors were able to extend the ideas behind his method to achieve the integration for $n > 2$ independent variables. The integration of equations with $n > 2$ was first accomplished by JOHANN FRIEDRICH PFAFF (1765–1825), a professor of mathematics at the University of Halle. PFAFF's idea was to consider the more general problem of integrating a total differential equation

$$(2.11) \qquad \Omega_N = A_1(u)\,du_1 + \dots + A_N(u)\,du_N = 0$$

in any number of variables $u = (u_1, \dots, u_N)$ [1815]. At the time he wrote his memoir it was not clear what it meant, in general, to integrate such an expression, even for $N = 3$. PFAFF noted that EULER had expressed the view that it only makes sense to speak of the integration of $\Omega_3 = 0$ when it is "integrable" in the sense that $M\Omega_3$ is exact for some factor M; then, if $M\Omega_3 = d\Phi$, the equation $\Phi = C$ represents an integral surface of $\Omega_3 = 0$. PFAFF pointed out that MONGE had disagreed with EULER and stated that two simultaneous equations $\Phi = C_1$, $\Psi = C_2$, representing a curve, could also be be regarded as an integral of $\Omega_3 = 0$. PFAFF's work supported MONGE's position and showed more generally what it could mean to integrate (2.11). We state the general (generic) theorem implicit in his memoir as follows:

Theorem 2.1 (PFAFF). *A change of variables* $u_i = f_i(v_1, \dots, v_N)$, $i = 1, \dots, N$, *exists such that*

$$(2.12) \qquad \Omega_N = B_1(v)\,dv_1 + \dots + B_m(v)\,dv_m,$$

where $m = N/2$ *if* N *is even and* $m = (N + 1)/2$ *if* N *is odd.*[8]

---

[8] See [PFAFF 1815: § 17] and KOWALEWSKI's notes to the German translation (p. 77 ff.). Theorem 2.1 is "generic" in the sense that (in JACOBI's formulation) it corresponds to the situation when the $N \times N$ skew symmetric matrix of coefficients $a_{ij} = \frac{\partial A_j}{\partial u_i} - \frac{\partial A_i}{\partial u_j}$, introduced by JACOBI [1827b: 23], has full rank N. In the 1870's DARBOUX and FROBE-

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

Theorem 2.1 establishes the existence of solutions to $\Omega_N = 0$. For example, if $v_i = \phi_i(u_1, \ldots, u_N)$, $i = 1, \ldots, N$, and $C_1, \ldots, C_m$ are constants, the m equations

$$\phi_i(u_1, \ldots, u_N) = C_i, \quad i = 1, \ldots, m$$

represent a solution to $\Omega_N = 0$ since these equations imply that $dv_i = 0$ for $i = 1, \ldots, m$ and whence by (2.12) that $\Omega_N = 0$. This solution, which involves m arbitrary constants is analogous to the complete solution of a first order partial differential equation. JACOBI [1827b: 20–21] showed that PFAFF's Theorem leads also to analogs of particular and general solutions.

PFAFF used this theorem to integrate the general first order partial differential equation (2.1) in the following manner. In accordance with LAGRANGE's approach, let (2.1) be solved for p_n to obtain

$$p_n = F_1(x_1, \ldots, x_n, z, p_1, \ldots, p_{n-1}), \tag{2.13}$$

and consider the equation

$$\Omega_{2n} = dz - p_1\,dx_1 - \ldots - p_n\,dx_n = dz - p_1\,dx_1 - \ldots - p_{n-1}\,dx_{n-1} - F_1\,dx_n = 0.$$

This can be regarded as a "Pfaffian" equation (*i.e.* an equation of type (2.11)) in the $N = 2n$ variables $u_1, \ldots, u_N = x_1, \ldots, x_n, z, p_1, \ldots, p_{n-1}$. By PFAFF's theorem Ω_{2n} is expressible in the form (2.12) with $m = n$. Consider, in the notation following Theorem 2.1, the "complete solution" to $\Omega_{2n} = 0$ given by the system of n equations $\phi_i(x_1, \ldots, x_n, z, p_1, \ldots, p_{n-1}) = C_i$, $i = 1, \ldots, n$. Solve this system for the n variables $z, p_1, \ldots, p_{n-1}$ to obtain

$$z = \psi(x_1, \ldots, x_n, C_1, \ldots, C_n), \quad p_i = g_i(x_1, \ldots, x_n, C_1, \ldots, C_n). \tag{2.14}$$

Then $\Omega_{2n} = 0$ when (2.14) holds, or, equivalently, (2.14) implies

$$dz = p_1\,dx_1 + \ldots + p_{n-1}\,dx_{n-1} + F_1\,dx_n.$$

Since, on the other hand, (2.14) implies that $dz = \sum_{i=1}^{n} \left(\frac{\partial\psi}{\partial x_i}\right) dx_i$, comparison of the two expressions for dz shows that $\frac{\partial\psi}{\partial x_i} = p_i$ for $i = 1, \ldots, n-1$ and that $\frac{\partial\psi}{\partial x_n} = F_1(x_1, \ldots, x_n, z, p_1, \ldots, p_{n-1}) = p_n$. Thus ψ is a complete solution to (2.13)/(2.1). In this way PFAFF succeeded in avoiding the impasse that had confronted LAGRANGE, winning thereby the praise of GAUSS, who described PFAFF's result as "a beautiful extension of the integral calculus" [1815: 1026].

PFAFF emphasized that his method of integrating (2.1) also reduced the integration to that of systems of ordinary differential equations. His method rested upon the existence of the change of variables putting Ω_N in the form (2.12). This change of variables was obtained by a succession of $N - m$ changes of variables

---

NIUS independently studied the cases in which the rank is less than N (in which case m can be taken smaller than asserted in PFAFF's proposition, which means the manifolds corresponding to the solutions $\phi_i = C_i$, $i = 1, \ldots, m$, can have a larger dimension than the generic dimension $N - m$.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

the k^{th} of which transformed Ω_N from an expression involving $N-k$ differentials into one involving $N-k-1$ differentials, where $k=0,\ldots,N-(m+1)$. Each such change of variables required completely integrating a system of first order ordinary differential equations (2.6). For example, in the case $N=2n$, which is the relevant case for the integration of the partial differential equation (2.1), the transformation of Ω_{2n} into the form (2.12), and hence the integration of (2.1), required the complete integration of systems (2.6) of respective orders $2n-1$, $2n-3$, $2n-5,\ldots,1$. Thus by PFAFF's method, the integration of (2.1) in the case $n=2$ which LAGRANGE had also resolved involved integrating a system of 3 first order ordinary differential equations in 4 variables and a single first order ordinary differential equation. These results agreed with LAGRANGE's. (To find the function $p=F_2(x,y,z)$ LAGRANGE had to integrate a linear nonhomogeneous partial differential equation in 4 variables — (2.9) with $n=4$ — which is equivalent to integrating a system (2.6) of order 3. The determination of the factor $M(x,y,z)$ involved the integration of an ordinary differential equation, a "system" of order 1.)

For large values of n the number of systems to be integrated in order to integrate a partial differential equation might appear rather excessive, given the fact that in this case $\Omega_{2n}=dz-\sum_{i=1}^{n}p_i\,dx_i$ involves only $n+1$ differentials, one more than the final form (2.13) of PFAFF's Theorem. Nonetheless, if (2.1) is integrated by PFAFF's method, the first step of the Pfaffian process is the "reduction" of Ω_{2n} to an expression involving $2n-1$ differentials. PFAFF's method, as applied to the integration of partial differential equations, might thus seem rather round about. Indeed it is, as JACOBI was to discover in 1836. But in 1827 he expressed no such criticisms. In fact, he showed that the ideas behind LAGRANGE's method could be developed in the case $n>2$ so as to give a new, direct proof of the possibility of the first Pfaffian transformation of $dz-\sum_{i=1}^{n}p_i\,dx_i$ into an expression involving $2n-1$ differentials. Thereby he felt that "the difficulties which stand in the way of extending LAGRANGE's method to [n independent] variables are removed, to the extent that is permitted by the nature of the method" [1827a: 1]. Pushed as far as appeared possible, LAGRANGE's ideas as expounded by JACOBI in 1827 led into PFAFF's method: the further transformation of Ω_{2n} now required, as JACOBI noted, PFAFF's general procedure [1827a: 10]. It was HAMILTON's work in mechanics that caused JACOBI to revise his opinions about the necessity of applying PFAFF's method to integrate partial differential equations and about the limited possibilities of LAGRANGE's approach in the case $n>2$.

JACOBI's second period of interest in the general theory of first order partial differential equations was precipitated by his reaction to two papers by HAMILTON "on a general method of dynamics" [1834, 1835] which he reviewed for DOVE & MOSER's *Repertorium der Physik*. As JACOBI explained to his brother in September of 1836, the study of HAMILTON's papers "led me very deep into the study of the most important mechanical theories, from which a huge manuscript was swelled up."[9] The manuscript was finally completed in December and published in CRELLE's journal [1837], with a French translation appearing the following year in LIOUVILLE's journal. In the above mentioned papers, HAMILTON showed

[9] Letter dated 17 September 1836 and quoted by KOENIGSBERGER [1904: 198].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

that the equations of motion of a mechanical system of m masses could be expressed in the now familiar canonical form

$$(2.16) \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, 3m$$

where q_i, p_i and H denote what are now called, respectively, generalized position and momentum coordinates and the Hamiltonian $H = H(q, p)$ of the system.[10] His main result, however, was that the integration of the equations of motion was equivalent to the integration of two nonlinear first order partial differential equations. The first of these equations,

$$(2.17) \qquad \frac{\partial S}{\partial t} + H(q_1, \dots, q_{3m}, p_1, \dots, p_{3m}) = 0, \quad p_i = \frac{\partial S}{\partial q_i},$$

is now called the HAMILTON-JACOBI equation since JACOBI showed in [1837] that the second equation is superfluous. He also showed that HAMILTON's results could be extended to the case in which the Hamiltonian depends on t, but his most important observation from the viewpoint of this essay had to do with the implications of the equivalence HAMILTON had established.

For HAMILTON the importance of the equivalence lay in the direction of replacing the equations of motion by the two partial differential equations so that thereby the difficulty of determining the motion of a system of masses "is at least transferred from the integration of many equations of one class to the integration of two of another" [1834]. JACOBI realized that, at least in terms of the integration theory of first order partial differential equations, which HAMILTON does not appear to have had in mind, "reduction" of (2.16) to (2.17) is hardly an advance. As JACOBI explained in a letter to the secretary of the mathematics and physics section of the Berlin Academy: "Little would seem to be gained by this reduction to a partial differential equation since according to Pfaff's method ... — and for more than three variables till now nothing further was known about the integration of partial differential equations of the first order — the integration of the one partial differential equation to which the dynamical problem is reduced is much more difficult than integration of the directly given system of ordinary differential equations of motion." He went on to explain, however, that "if Hamilton's investigations are extended to all first order partial differential equations, as can be done without difficulty, it is on the other hand a significant discovery in the theory of first order partial differential equations that they can always be reduced to a single system of ordinary differential equations, which previously according to the Pfaffian method was insufficient" [1837: 50–51].

Here JACOBI was referring to his discovery that the problem of determining a complete solution to the general first order partial differential equation (2.1) reduces to the complete integration of a single system of ordinary differential equations (2.6) of order $2n - 1$, which is in fact the first system that arises in the application of PFAFF's method [1837: 101–102]. In his lectures on dynamics (1842–43), published posthumously in 1866, JACOBI formulated his result in the

[10] For a discussion of HAMILTON's papers from the standpoint of the history of mechanics, see the books by DUGAS [1950] and HANKINS [1980: Ch. 13].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

following manner [1866: 157ff.]. Consider a first order partial differential equation in $n+1$ independent variables $t, x_1, \ldots, x_n$ which, like (2.17), does not involve the dependent variable, z. The consideration of the general equation (2.1) can always be reduced to the case of an equation not explicitly involving the dependent variable, as he realized. (One way to achieve this reduction, which is relevant to LIE's work, is indicated in § 6.) Such an equation is expressible in the form

$$\frac{\partial z}{\partial t} + F(t, x_1, \ldots, x_n, p_1, \ldots, p_n) = 0, \quad p_i = \frac{\partial z}{\partial x_i}, \tag{2.18}$$

and his result may be summarized as:

Theorem 2.2 (JACOBI's First Method). *The problem of integrating* (2.18) *is equivalent to the problem of completely integrating the system of ordinary differential equations*

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial F}{\partial x_i}, \quad i = 1, 2, \ldots, n. \tag{2.19}$$

That is, by means of differentiation and elimination it is possible to pass from a complete solution of (2.18) *to the solutions of* (2.19) *and vice versa.*[11]

When $F = H$, the Hamiltonian of a mechanical system, the equations (2.19) are of course HAMILTON's canonical equations as extended by JACOBI to the case in which H depends on t.

3. Jacobi's New Method

Encouraged by the new mathematical horizons brought into view by HAMILTON's papers, JACOBI pursued his study of mechanics and of the integration of first order partial differential equations far beyond what he published. In 1838 he composed a lengthy manuscript entitled "Nova methodus, aequationes differentiales partiales primi ordinis inter numerum variabilium quemcumque propositas integrandi," which CLEBSCH edited for publication more than a decade after JACOBI's death [1862].

In "Nova methodus" JACOBI reconsidered the viability of LAGRANGE's approach to the integration of (2.1), which will be denoted briefly by $F(x, z, p) = 0$. He restricted his attention to such equations which do not explicitly involve the dependent variable z, *viz.*

$$F(x, p) = F(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0, \quad p_i = \frac{\partial z}{\partial x_i}, \tag{3.1}$$

[11] See [GOURSAT 1921: 259–62] for a discussion of this equivalence that is clearer than JACOBI's.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

since he showed that the general case could be reduced to this one. The underlying idea of LAGRANGE's method can be expressed for any number of variables as follows. The idea is to determine $n-1$ functions $F_i(x, p)$ so that the equations

$$F_i(x_1, \ldots, x_n, p_1, \ldots, p_n) = C_i, \quad i = 1, \ldots, n-1, \tag{3.2}$$

together with (3.1), make a multiple of $\Omega = dz - p_1\, dx_1 - \ldots - p_n\, dx_n$ exact. That is, let the system of n equations (3.1)–(3.2) be solved for the p_i to obtain

$$p_i = G_i(x_1, \ldots, x_n, C_1, \ldots, C_{n-1}), \quad i = 1, \ldots, n. \tag{3.3}$$

Then

$$\Omega = dz - G_1\, dx_1 - \ldots - G_n\, dx_n \tag{3.4}$$

is a differential involving the $n+1$ variables $x_1, \ldots, x_n, z$ as well as the $n-1$ parameters $C_1, \ldots, C_{n-1}$. Suppose that Ω as given by (3.4) has the property of exactness like that of LAGRANGE's original method: a function $\Phi(x_1, \ldots, x_n, z, C_1, \ldots, C_{n-1})$ exists such that $d\Phi = M\Omega$, where $M = M(x_1, \ldots, x_n, z, C_1, \ldots, C_{n-1})$. Then the equation $\Phi = C_n$ defines implicitly a complete solution to (3.1).[12]

The problem of extending LAGRANGE's method to partial differential equations in n variables $F(x, p) = 0$ thus amounts to being able to choose the $n-1$ functions F_i of (3.2) so that $M\Omega = d\Phi$. In terms of the "inverse" functions G_i of (3.3) the necessary and sufficient condition for such a choice to be possible is readily at hand. That is, $M\Omega = d\Phi$ implies that $G_i = \dfrac{\partial\psi}{\partial x_i}$ (see note 12) so that the G_i must satisfy $\dfrac{\partial G_i}{\partial x_k} = \dfrac{\partial G_k}{\partial x_i}$ for all $i, k = 1, \ldots, n$. In terms of the "local-generic" mode of reasoning employed by JACOBI it followed that this condition was not only necessary but also sufficient to yield $M\Omega = d\Phi$.[13] The starting point of JACOBI's new method of integrating $F(x, p) = 0$ was his discovery that this necessary and sufficient condition could be translated into a condition on the functions $F_0(= F), F_1, \ldots, F_{n-1}$ of (3.1) and (3.2) — a condition which could be stated just as succinctly if the bracket notation of POISSON and LAGRANGE is employed [JACOBI 1862: §§ 14–16]:

---

[12] If $\Phi = C_n$ is solved for z to obtain $z = \psi(x_1, \ldots, x_n, C_1, \ldots, C_n)$, then the relation $M\Omega = d\Phi$ together with the relations $\Phi_{x_i} + \Phi_z \dfrac{\partial\psi}{\partial x_i} = 0$ (implied by $\Phi = C_n$) imply the analog of (2.4), namely that $\dfrac{\partial\psi}{\partial x_i} = -\dfrac{\Phi_{x_i}}{\Phi_z} = G_i(x_1, \ldots, x_n, C_1, \ldots, C_{n-1}) = p_i$. Thus $0 = F(x_1, \ldots, x_n, p_1, \ldots, p_n) = F\left(x_1, \ldots, x_n, \dfrac{\partial\psi}{\partial x_1}, \ldots, \dfrac{\partial\psi}{\partial x_n}\right)$, and $z = \psi$ is a complete solution to (3.1).

[13] That is, the condition implies that $\omega = \sum_{i=1}^n G_i\, dx_i$ satisfies $d\omega = 0$ so that $\omega = d\theta$, $\theta = \theta(x_1, \ldots, x_n, C_1, \ldots, C_{n-1})$ and thus $\Omega = d\Phi$, where $\Phi = z - \theta$. This means that the complete solution defined by $\Phi = C_n$ is expressible in the form $z = \theta + C_n$.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

Theorem 3.1. *The equations* (3.1)–(3.2) *make* (3.4) *exact, and hence yield a complete solution to* (3.1), *if and only if they imply the further equations* $(F_i, F_j) = 0$ *for* $i, j = 0, \ldots, n-1$, *where, for any two functions* F, G *of* $2n$ *variables* $x_1, \ldots, x_n, p_1, \ldots, p_n$,

$$(3.5) \qquad (F, G) = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x_i} - \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial p_i} \right).$$

The bracket notation (F, G) had been introduced by POISSON in a memoir on problems in Lagrangian mechanics [1809: 288–289]. JACOBI was actually familiar with POISSON's notation as early as 1827. His reworking of PFAFF's results had led him to an interest in linear systems of equations with skew symmetric coefficients since such a system with coefficients $(\alpha, \beta) = \dfrac{\partial A_\alpha}{\partial u_\beta} - \dfrac{\partial A_\beta}{\partial u_\alpha}$ arose in effecting the first transformation of PFAFF [1827b: 25–9]. At that time he observed that both POISSON and LAGRANGE had already used such a skew symmetric bracket notation to simplify calculations involving linear systems.[14]

Theorem 3.1 showed that a complete solution to the partial differential equation $F(x, p) = 0$ could be obtained (as explained in the discussion preceding Theorem 3.1) if $n - 1$ additional functions $F_i(x, p)$, $i = 1, \ldots, n-1$, could be determined so that $(F_i, F_j) = 0$ for all $i, j = 0, \ldots, n-1$, where $F_0 = F$. (Nowadays the Hamiltonian system defined by $F(x, p)$ is said to be *completely integrable* when the functions $F_1, \ldots, F_{n-1}$ exist [OLVER 1986: 415], a terminology reflecting the conclusions of Theorem 3.1.) JACOBI developed a new method of integration of $F(x, p) = 0$ which showed how to produce the functions $F_1, \ldots, F_{n-1}$. Strictly speaking, it suffices to determine the F_i such that the equations (3.1)–(3.2) imply the further equations $(F_i, F_j) = 0$, but JACOBI sought functions F_i satisfying the stronger condition that the (F_i, F_j) are identically zero. His approach was to find the F_i one at a time. To find F_1 is to find a function f satisfying the equation $(F_0, f) = 0$. The equation $(F_0, f) = 0$ is a first order linear homogeneous partial differential equation in the $2n$ variables x, p since, by (3.5), $(F_0, f) = \sum_{i=1}^{n} a_i \dfrac{\partial f}{\partial x_i} + \sum_{i=1}^{n} b_i \dfrac{\partial f}{\partial p_i}$, where $a_i = \dfrac{\partial F_0}{\partial p_i}$ and $b_i = -\dfrac{\partial F_0}{\partial x_i}$. In the case of $n = 2$ independent variables considered by LAGRANGE, to make Ω exact it was simply necessary to determine F_1, that is to find a solution to the first order linear equation $(F_0, f) = 0$ in 4 variables, in essential agreement with the method of LAGRANGE. For $n > 2$, however, the problem of determining $F_2, \ldots, F_{n-1}$ led JACOBI to the consideration of systems of linear homogeneous partial differential equations. For example F_2 will be a solution to the simultaneous equations $(F_0, f) = 0$, $(F_1, f) = 0$. In general, after $F_1, \ldots, F_{m-1}$ had been determined in this manner, the determination of F_m involved finding a solution $f = F_m$ to the system of equations

$$(3.6) \qquad (F_0, f) = 0, \quad (F_1, f) = 0, \ldots, (F_{m-1}, f) = 0.$$

[14] The account of POISSON's memoir by DUGAS [1950: 368–74] indicates how POISSON used his bracket notation.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

JACOBI thus found it necessary to consider the problem of integrating the linear system (3.6), of reducing the problem of obtaining a solution to (3.6) to the integration of one or more systems of ordinary differential equations.

JACOBI discovered that the system (3.6) has a special property, and to express this property and its implications succinctly he introduced some new notation [1862: § 23]. Linear homogeneous first order partial differential equations in variables $y_1, \ldots, y_m$ were denoted by $A(f) = 0$, $B(f) = 0$, where

$$(3.7)\quad A(f) = a_1(y)\frac{\partial f}{\partial y_1} + \ldots + a_m(y)\frac{\partial f}{\partial y_m}, \quad B(f) = b_1(y)\frac{\partial f}{\partial y_2} + \ldots + b_m(y)\frac{\partial f}{\partial y_m}.$$

He observed that, although the expressions $A(B(f))$ and $B(A(f))$ involve second order derivatives, they all cancel when the one is subtracted from the other so that

$$A(B(f)) - B(A(f))) = c_1(y)\frac{\partial f}{\partial y_1} + \ldots + c_m(y)\frac{\partial f}{\partial y_m} = C(f),$$

where $c_i = A(b_i) - B(a_i)$. Having established this property of the differential operators $A(f)$ and $B(f)$, he turned to the special case of interest to him: $A(f) = (F,f)$, $B(f) = (G,f)$, where F, G, f are functions of the $2n$ variables x, p. Then $A(B(f)) = (F, (G,f))$ and $B(A(f)) = (G, (F,f))$ so that $A(B(f)) - B(A(f)) = (F, (G,f)) - (G, (F,f))$. On the other hand, $A(B(f)) - B(A(f)) = C(f)$, where the coefficients c_i of $C(f)$ are given by $c_i = A(b_i) - B(a_i)$. By calculating the c_i in this manner he observed that $C(f) = ((F, G), f)$. Thus the equality

$$A(B(f)) - B(A(f)) = C(f)$$

yielded the identity $(F, (G,f)) - (G, (F,f)) = ((F, G), f)$. He had thus discovered a new property of the POISSON brackets, which, setting $f = H$ and using the skew symmetry of the brackets, he expressed as follows [1862: § 26, Thm. V]:

Theorem 3.2 (JACOBI's Identity). *If F, G, H are functions of $2n$ variables $x_1, \ldots, x_n$ and $p_1, \ldots, p_n$, then*

$$(3.8)\qquad ((F, G), H) + ((G, H), F) + ((H, F), G) = 0.$$

JACOBI's Identity showed that the system $A_i(f) = (F_i, f) = 0$ of (3.6) makes $A_i(A^j(f)) - A_j(A_i(f)) = ((F_i, F_j), f) = 0$. He therefore considered a system $A(f) = B(f) = 0$ for which $A(B(f)) - B(A(f)) = 0$. From the commutativity relation $A(B(f)) = B(A(f))$ he showed, for example, that $A^iB^j(f) = B^jA^i(f)$ and from this he concluded that if $f = \varphi$ is a solution to $A(f) = 0$, then so are the functions $B^i(\varphi)$, since $A(B^i(\varphi)) = B^i(A(\varphi)) = B^i(0) = 0$. Since, by the theory of linear equations, $A(f) = 0$ possesses exactly $m - 1$ functionally independent solutions (as indicated above in discussing (2.5)–(2.7)), it follows that the l solutions $\varphi, B(\varphi), \ldots, B^{l-1}(\varphi)$ must be functionally dependent for some $l \leqq m$. From this fact, he was able to conclude that a simultaneous solution to $A(f) = B(f) = 0$ may be obtained from φ by integrating a system of ordinary differential equations (2.6) of order at most $m - 1$ (the worst possible case). Since determination of φ itself involved obtaining a solution to the single equation $A(f) = 0$ and hence obtaining a solution to a system of ordinary differential equations of

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

order $m-1$, he had reduced the problem of determining a solution to systems $A(f)=B(f)=0$ satisfying $A(B(f))=B(A(f))$ to the problem of determining one solution to each of two systems of ordinary differential equations, one of order $m-1$ and another of order at most $m-1$. Applied to the systems (2.4), these considerations enabled him to deduce from Theorem 3.1 the following result [1862: § 22]:

Theorem 3.3 (JACOBI's New Method). *The problem of integrating* (3.1) *reduces to the problem of determining* (*in the worst possible case*) *one solution each of the following systems of ordinary differential equations*: *1 system of order* $2n-2$, *2 systems of order* $2n-4$, *3 systems of order* $2n-6, \ldots, n-1$ *systems of order 2.*

JACOBI's first method required the complete integration of one system of order $2n-2$, the equivalent of determining $2n-2$ solutions of a system of order $2n-2$.[15] In his new method, however, some of the systems could be of lower order than indicated by Theorem 3.3, which represents the worst possible case. Both methods were, of course, far more efficient than PFAFF's method. His two methods thus had the effect of refining the goals of the integration theory of first order partial differential equations. No longer was the objective simply to reduce the integration of such equations to the integration of systems of ordinary differential equations. Now the objective was to accomplish this reduction in the most efficient manner possible. As we shall see, LIE's reaction to this goal of the theory is directly related to the developments involved in the birth of his theory of groups.

JACOBI's Identity (Theorem 3.2), which he had discovered in the course of developing the idea of LAGRANGE's method into his own new method [1862: § 28], was fundamental to it in the sense that the commutativity of the systems (3.6) was the basis for Theorem 3.3. JACOBI also believed the Identity had implications of fundamental importance in conjunction with his first method. According to that method (Theorem 2.2), the integration of

$$\frac{\partial z}{\partial t}+F(t, x_1, \ldots, x_n, p_1, \ldots, p_n)=0, \qquad p_i=\frac{\partial z}{\partial x_i} \tag{3.9}$$

is equivalent to the complete integration of the system of ordinary differential equations

$$\frac{dx_i}{dt}=\frac{\partial F}{\partial p_i}, \quad \frac{dp_i}{dt}=-\frac{\partial F}{\partial x_i}, \qquad i=1,2,\ldots,n. \tag{3.10}$$

Consider, for the moment, the case in which F does not depend upon t so that $F=F(x,p)$. In this case, the right hand sides of the equations (3.10) do not depend upon t and so (as in the discussion of equations (2.5)–(2.7)) the integration

[15] By Theorem 2.2 (with z and F independent of t) integration of (3.1) requires integration of a system of order $2n-1$ (by the discussion relating (2.6) and (2.8)), but the equation $F(x,p)=0$ allows one of the variables to be eliminated so that a system of order $2n-2$ requires integration. Similar considerations are involved in obtaining the numbers in Theorem 3.3.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

of (3.10) in this case is tantamount to the integration of the linear partial differential equation $(F,f) = 0$. In particular, if $f = \varphi$ is a solution to $(F,f) = 0$, then $\varphi = \text{const.}$ is an integral of (3.10), which means that if $x_i = x_i(t)$, $p_i = p_i(t)$ is any solution to (3.10) then $\varphi(x(t), p(t)) \equiv \text{const.}$ JACOBI observed that his Identity implied that if $f = \varphi$ and $f = \psi$ are solutions of $(F,f) = 0$, then so is $f = (\varphi, \psi)$ since $(F, (\varphi, \psi)) = (\varphi, (F, \psi)) - (\psi, (F, \varphi)) = 0$. In other words, if $\varphi = \text{const.}$ and $\psi = \text{const.}$ are integrals of (3.10), then so is $(\varphi, \psi) = \text{const.}$ As he realized, the case in which $F = F(t, x, p)$ can easily be reduced to the above case by introducing an additional variable [1862: § 25], so that the above conclusion can be extended to this case. These conclusions may be summarized as

Theorem 3.4 (POISSON-JACOBI). *Let $F = F(x, p)$ be any function of $2n$ variables $x_1, \ldots, x_n, p_1, \ldots, p_n$, and consider the linear homogeneous partial differential equation $(F,f) = 0$. If φ and ψ are any two solutions, then (φ, ψ) is also a solution. More generally, if $F = F(t, x, p)$ and if $\varphi(t, x, p) = \text{const.}$, $\psi(t, x, p) = \text{const.}$ are integrals of* (3.10), *then* $(\varphi, \psi) = \text{const.}$ *is also an integral of* (3.10).

At first JACOBI thought Theorem 3.4 was new, but then he discovered that it was implicit in POISSON's memoir [1809: 281–2]. As JACOBI explained at great length in § 28 of "Nova methodus," POISSON proved that if $\varphi(t, x, p) = \text{const.}$ and $\psi(t, x, p) = \text{const.}$ are two integrals of the equations of motion, then (φ, ψ), with $x_i = x_i(t)$, $p_i = p_i(t)$ any solution to these equations, does not depend upon t, a theorem he found "remarkable".[16] This means of course that $(\varphi, \psi) = \text{const.}$ is another integral of these equations. POISSON certainly realized this, but he never mentioned it. Possibly this was because he did not think new integrals would be obtained in this manner. Indeed, in the example he gave to illustrate his theorem, he deduced four integrals of the equations of motion from general physical principles and showed that all pairs of brackets either vanish identically or yield one of the functions defining the four integrals. For POISSON the significance of his theorem was that it yielded a new derivation of a result of LAGRANGE's that certain expressions in a perturbation of the original mechanical system are independent of t. In what follows we shall follow CLEBSCH and LIE and refer to Theorem 3.4 as the *Poisson-Jacobi Theorem.*

From JACOBI's perspective the true importance of POISSON's discovery lay in Theorem 3.4 by virtue of the following considerations, which we present for the case in which F depends upon t: $F = F(t, x, p)$. Suppose $\varphi_1 = \text{const.}$ and $\varphi_2 = \text{const.}$ are two functionally independent integrals of (3.10). Then by Theorem 3.4 $\varphi_3 = (\varphi_1, \varphi_2) = \text{const.}$ is a third integral, an integral, remarkably enough, not obtained by integration but by the differentiations required to form the POISSON bracket in accordance with (3.5). Of course, it could happen that φ_3 is a constant or that it is functionally dependent on φ_1 and φ_2; but, JACOBI reasoned, in general $\varphi_3 = \text{const.}$ would represent a new integral. Likewise, in general, $\varphi_4 = (\varphi_1, \varphi_3) = \text{const.}$ and $\varphi_5 = (\varphi_2, \varphi_3) = \text{const.}$ would be further independent integrals,

[16] POISSON, of course, worked within the context of Lagrangian, not Hamiltonian, mechanics. His proof is expounded in more modern notation by DUGAS [1950: 368–71].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

and so on. In this way $2n$ functionally independent integrals $\varphi_i =$ const. could, in general, be obtained and (3.10) would be completely integrated by the $2n$ equations $\varphi(t, x, p) = C_i$, $i = 1, \ldots, 2n$. (Solved for the $2n$ variables x_i, p_i, these equations yield x_i and p_i as functions of t and the $2n$ arbitrary constants C_i.) For convenience in what follows this conclusion will be referred to as

Jacobi's "Corollary" to Theorem 3.4. *If two functionally independent integrals,* $\varphi =$ const. *and* $\psi =$ const. *to* (3.10) *are known, then "in general" all solutions to this equation may be determined by use of the Poisson bracket.*[17]

By virtue of Theorem 2.2 JACOBI realized his "corollary" had implications for the integration of first order partial differential equations. That is, without loss of generality, it suffices to consider an equation of the form (3.9) so that by Theorem 2.2 its integration may be deduced from that of (3.10). By JACOBI's "corollary," it suffices, in "general" to obtain just two independent integrals of (3.10) in order to obtain, by the differentiation process of POISSON brackets, the $2n$ integrals, required to integrate (3.10) completely and therefore (by Theorem 2.2) to obtain also a complete solution of the partial differential equation (3.9).

Thus in contrast with the lack of any lasting appreciation on the part of POISSON and LAGRANGE for POISSON's discovery that (φ, ψ) is independent of time t, JACOBI declared Theorem 3.4 (including its "corollary") to be "of great significance": "On it as foundation an entirely new theory of integration can be erected for mechanical problems subject to the principle of conservation of living forces[18] and more generally for all problems that can be reduced to a partial differential equation of the first order which, as can be shown, includes the most general isoperimetric problems. Almost all of the present work is based on this foundation .... Nevertheless I believe that what can be obtained from this source for the integration of the dynamical differential equations is far from being exhausted. On the contrary, much of importance still awaits further investigation" [1862: 50–51]. JACOBI's enthusiasm for Theorem 3.4 and its corollary extended as well to the Identity (Theorem 3.2) from which they derived. It was Theorem 3.2 that he described as "a very important theorem" [1862: § 26]. He justified deriving Theorem 3.4 from the Identity because the latter had other uses as well. In stressing the importance of Theorem 3.4 it was clear he was also stressing the importance of the Identity. To the readers of "Nova methodus" the POISSON-JACOBI Theorem and the Identity went hand in hand. They formed the foundation of JACOBI's work in "Nova methodus". The Identity yielded the commutative property of the systems (3.6) that was the key to JACOBI's method for integrating them, the key to his new method (Theorem 3.3). The Identity also yielded, by application of Theorem 3.4, the corollary to that theorem which by virtue of his first method (Theorem 2.2) had implications for the integration of all first order partial differential equations.

Although most of the contents of "Nova methodus" was first revealed to mathematicians through its posthumous publication in 1862, JACOBI called public

[17] When F does not depend on t, $F =$ const. is itself an integral of (3.10). In this case F, φ, ψ are assumed functionally independent.

[18] That is, what is now called the principle of conservation of energy.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

attention to the POISSON-JACOBI Theorem and its "corollary" in 1840. Prompted by receipt of a biographical essay on POISSON, he addressed a letter to the president of the Paris Academy of Sciences regarding "the most profound discovery of M. Poisson," namely that the POISSON bracket of two integrals of the equations of motion is also an integral. In the letter, which was published in the *Comptes rendus* of the Academy and in LIOUVILLE's journal [1840], JACOBI suggested that neither POISSON nor LAGRANGE nor many other illustrious geometers truly grasped the significance of the theorem, "which seems to me to be the most important [theorem] of mechanics and of that part of the integral calculus associated with the integration of systems of ordinary differential equations; nevertheless, it is not found in treatises on the integral calculus or in [LAGRANGE's] *Mécanique analytique*." After stating the gist of the theorem verbally and in the context of a mechanical system of m masses, JACOBI presented his corollary: "By pursuing the same procedure, a fourth, a fifth integral will be found, and, in general, in this manner all the integrals will be deduced from the two given integrals." In his lectures on dynamics during 1842–43, published by CLEBSCH in 1866, he indicated more of the basis for his belief in the corollary [1866: Lecture 34]. Here he considered the case in which F is independent of t so that the integration of $(F,f)=0$ is tantamount to that of (3.10). Since $(F,f)=0$ is a linear homogeneous equation in $2n$ variables x, p, it has $m = 2n-1$ functionally independent solutions $f=\varphi_1, \ldots, \varphi_m$, and $\varphi_{m+1} = \Pi(\varphi_1, \ldots, \varphi_m)$ is the general solution. In the "infinitely vast majority of cases," he concluded, φ_{m+1} together with one of $\varphi_1, \ldots, \varphi_m$, say, φ_1, will suffice to generate by POISSON brackets all the other solutions, "and this is the general case, since $[\varphi_{m+1}]$ set equal to an arbitrary constant represents the most general form of an integral."

In his note of 1840 JACOBI admitted that "in particular cases" the procedure of producing new solutions by POISSON bracketing from two given ones will come to a halt because it produces solutions that are functions of those already obtained. Indeed, as he explained more fully in the above mentioned lecture, in mechanical problems the first solutions obtained usually come from general laws (such as KEPLER's laws), and since they are not peculiar to the problem, they will not furnish all solutions by this procedure. In the note of 1840, he explained that in such cases "the two given integrals enjoy particular properties that are profitable in other ways for the integration of the proposed dynamical equations. This will be seen in a work on which I have been laboring for many years and which perhaps I will soon begin to put into print." In Paris, JACOBI's corollary was received with skepticism by LIOUVILLE, who in a brief note [1840] suggested that JACOBI's "particular case" was far less exceptional than he had suggested. Since by JACOBI's death in 1851, his promised work on these matters had not yet appeared, LIOUVILLE and other French mathematicians, notably BERTRAND and EDMOND BOUR, decided to respond to JACOBI's note. A detailed account of their responses can be found in JESPER LÜTZEN's recent biography of LIOUVILLE [1990: 670ff.]. Here it will suffice to point out that, stimulated by JACOBI's remarks about the particular case, they all showed how the integration of (3.10) could be advanced by integrals $\varphi_i = \text{const.}$ such that $(\varphi_i, \varphi_j) = 0, \pm 1$. In particular, LIOUVILLE rediscovered JACOBI's Theorem 3.1 [1855], and BOUR obtained results on the integration of the canonical equations of motion (2.16) in line with the results of JACOBI's Theorem 3.3

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

[1855a, 1866b, 1856]. The question underlying the work of BERTRAND, LIOUVILLE and BOUR, namely how to use the POISSON-JACOBI Theorem to advance the integration of (3.10), was reconsidered in 1872 by LIE, who perceived therein an avenue for the development of his *idée fixe* (§§ 5–6).

In addition to LIOUVILLE, W. F. DONKIN in England and J. A. WEILER in Germany rediscovered JACOBI's fundamental Theorem 3.1. DONKIN, who was Savilian Professor of Astronomy at Oxford also rediscovered JACOBI's Identity (Theorem 3.2) and used it to prove the POISSON-JACOBI Theorem [1854: 92–3]. Theorem 3.1 arose in WEILER's work as part of his own method of extending LAGRANGE's method to more than two independent variables [1859: 268–84]. Thus by the time "Nova methodus" was published in 1862, some of its results had been rediscovered by others. Nonetheless, both in terms of form and content, "Nova methodus" still contained much that was original, including the calculus of differential operators. Its publication had the effect of intensifying interest in the integration of first order partial differential equations and Pfaffian equations. BOUR devoted many pages to expounding the results of "Nova methodus" in connection with his earlier work [1862a, 1862b]. WEILER was prompted by the appearance of "Nova methodus" to develop his somewhat sketchy ideas more fully in relation to JACOBI's new method and in particular to consider its efficiency in the light of JACOBI's Theorem 3.3. In [1863] he showed that in general his method leads to

Theorem 3.5 (CLEBSCH-WEILER). *The integration of* $F(x, p) = 0$ *can be reduced to obtaining one solution to one system of order* $2n - 2$ *and two systems each of the following orders*: $2n - 4$, $2n - 6$, $2n - 8, \ldots, 2$.

The number of integrations in Theorem 3.5 represented a significant improvement over JACOBI's Theorem 3.3, although WEILER's method lacked the elegance of JACOBI's approach and, in particular, required consideration of numerous special cases. ALFRED CLEBSCH (1833–1873), however, showed [1866: § 5] that WEILER's ideas could be combined with JACOBI's method to yield a natural extension of the latter which yielded the number of integrations specified in Theorem 3.5.

CLEBSCH, whose promising mathematical career was cut short by diphtheria, played an important role in the revival of interest in the integration of first order partial differential equations and Pfaffian systems that was in evidence when LIE began his own mathematical career.[19] He had obtained his doctorate in mathematics in 1854 from Königsberg in the post-JACOBI era. His teachers were FRANZ NEUMANN, RICHELOT and HESSE (who had been JACOBI's student). At Königsberg CLEBSCH received a broad and thorough training in mathematical physics, which included on his part a detailed study of the publications of EULER and JACOBI. He was known personally to the mathematicians in Berlin, where JACOBI had ended his career, since he had taught in various high schools there as well as (briefly) at the University during 1854–58. Apparently C. W. BORCHARDT, as editor of CRELLE's Journal, asked him to edit JACOBI's manuscript for publication in the Journal. He also edited JACOBI's lectures on dynamics [1866].

[19] For details about CLEBSCH's career, see the anonymous memorial essay [1873] and [TOBIES & ROWE 1990: 7ff.].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

The study of "Nova methodus" also prompted CLEBSCH to consider extensions of JACOBI's ideas to Pfaffian equations [1861, 1862, 1863]. In particular, for the Pfaffian systems he considered he showed that their integration could be reduced to the integration of systems of linear homogeneous equations

$$A_i(f) = \sum_{i=1}^{m} a_{ij}(y) \frac{\partial f}{\partial y} = 0, \quad i = 1, \ldots, q \tag{3.11}$$

which are functionally independent. JACOBI had achieved a similar reduction for partial differential equations in "Nova methodus", but there the equations (3.11) had the additional commutativity property $A_i(A_j(f)) - A_j(A_i(f)) = 0$, which was crucial to his reduction of their integration to systems of ordinary differential equations. CLEBSCH termed systems (3.11) with the additional commutativity property *Jacobian systems*. The systems (3.11) occurring in his extension of JACOBI's ideas to Pfaffian equations were not Jacobian systems. Rather, they possessed the more general property

$$A_i(A_j(f)) - A_j(A_i(f)) = \sum_{k=1}^{q} c_{ijk}(y)\, A_k(f), \quad i, j = 1, \ldots, q. \tag{3.12}$$

CLEBSCH called systems with this property *complete systems*. A Jacobian system is thus a special type of complete system.

Using JACOBI's ideas as a starting point, CLEBSCH developed the theory of complete systems in [1866]. In particular he proved the following theorem which generalized the earlier results of LAGRANGE and JACOBI:

Theorem 3.6. (1) *A complete system of* $q \leqq m-1$ *equations* $A_i(f) = 0$, $i = 1, \ldots, q$, *possesses* $m - q$ *functionally independent solutions* $f = \varphi_1, \ldots, \varphi_{m-q}$, *and the general solution is* $f = \Omega(\varphi_1, \ldots, \varphi_{m-q})$, *where* Ω *is an arbitrary function of* $m - q$ *variables.* (2) *The integration of any system of functionally independent equations* $A_i(f) = 0$ *can be reduced to the integration of a complete system.*

As we shall see, the theory of complete systems played an important role in LIE's analytical development of his invariant theory of contact transformations (§ 6) and in the working out of his *idée fixe* in a manner that derermined his decision to create a theory of continuous groups (§ 8). CLEBSCH's work on Pfaffian equations, which like that of his predecessors focused primarily on the generic case, prompted the work of FROBENIUS [1877] on Pfaffian equations which contains the result that has since evolved into FROBENIUS' Theorem on completely integrable systems of vector fields [OLVER 1986: 70].

By the time LIE made CLEBSCH's acquaintance (through KLEIN), CLEBSCH had become a professor of mathematics at the university in Göttingen and his principal research interests had turned to problems in algebraic geometry and the algebraic theory of invariants. His paper [1866] on complete systems, however, became the starting point for ADOLPH MAYER (1839–1908), who, like CLEBSCH but somewhat later, had studied with FRANZ NEUMANN at the University of Königsberg during the post-JACOBI era and was well versed in JACOBI's style of analysis as well as mathe-

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

matical physics.[20] MAYER showed that the number of systems of ordinary differential equations that required solution in order to integrate first order partial differential equations or Pfaffian equations $\Omega_{2n} = 0$ could be reduced even further than CLEBSCH and WEILER had realized. In particular, for first order equations, he proved the following result [1872b]:

Theorem 3.7 (LIE-MAYER). *The integration* $F(x, p) = 0$ *can be reduced to obtaining one solution to one system each of orders* $2n - 2$, $2n - 4$, $2n - 6, \ldots, 2$.

MAYER's result cut almost in half the number of systems of ordinary differential equations requiring solution by the CLEBSCH-WEILER version of JACOBI's new method as stated in Theorem 3.5. Shortly after he submitted his paper containing Theorem 3.7, MAYER learned that LIE had discovered essentially the same result by a completely different (geometrically inspired) method (§ 4). Thus began a long association between LIE and MAYER, who played a role in the birth of LIE's theory of groups by encouranging LIE to translate his ideas into the language of Jacobian analysis.

The intense and widespread interest in the theory of first order partial differential equations precipitated by the appearance of JACOBI's "Nova methodus" is reflected in the publication of V. G. IMSCHENETSKY's monographic essay [1869] of almost 200 pages on the subject, published in GRUNERT's *Archiv der Mathematik und Physik*. Declaring the theory of first order partial differential equations (as presented above) to "form now the most profound and accomplished part of the integral calculus (p. 278)", IMSCHENETSKY explained that "the main object of my study is the 'new method' of Jacobi" (p. 280). IMSCHENETSKY's essay was a remarkably clear treatise on all aspects of the theory of first order partial differential equations relating to JACOBI's work. It seems to have been the primary source from which LIE learned this theory, for, according to KLEIN's recollections, while he and LIE were together in Paris in 1870, LIE studied IMSCHENETSKY's essay with great enthusiasm, to the point of satiety.[21] Thus even before he had developed his ideas on the sphere mapping and its implications (the theory of contact transformations), LIE was eagerly digesting the theory we have been considering. In the following sections, we shall see how he creatively combined this knowledge with the mathematical experiences of his geometrical period of research (1869–71).

4. Lie's Geometrical Approach to Partial Differential Equations

In § 28 of "Nova methodus" JACOBI explained that he had derived the POISSON-JACOBI Theorem (Theorem 3.4) from his Identity (Theorem 3.2) because "it is always useful and not without elegance to reduce all propositions to pure identities." The pleasure that he evidently took in reducing theorems to identities reflected his penchant for elegant formal analysis. JACOBI's successors appreciated

[20] On MAYER's career, see [TOBIES & ROWE 1990: 8ff.].

[21] "Lie studierte in Paris den Imschenetski sehr eifrig, bis zum Ueberdruss" [KLEIN, F. 1916 MS: 10].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

the elegant formalism of the theory and continued to develop and present it in this same spirit. This is true in particular of IMSCHENETSKY's monograph [1869]. LIE, however, did not share the prevalent enthusiasm for analysis. He was a geometer. His geometrically oriented investigations during 1869–71, particularly his experiences investigating the sphere mapping, had encouraged him to aspire to create a geometrical approach to the theory of partial differential equations based upon the geometrical concepts attendant to the concept of a contact transformation. As he learned while in Paris, MONGE had succeeded in reformulating the Lagrangian theory of first order partial differential equations in terms of geometrical constructs such as that of a characteristic curve (discussed below). He hoped to accomplish something analogous relative to the theory of first order partial differential equations as it stood *circa* 1870, the theory that had emerged under the influence of JACOBI's work. Indeed, the emphasis on minimizing the number of integrations of systems of ordinary differential equations that had become a part of the theory as a result of JACOBI's contributions and that is reflected in Theorems 3.5 and 3.7 fit in well with LIE's *idée fixe*. That is, these results surely must have suggested to him the following formulations of his *idée fixe*: *Suppose* $F(x, p) = 0$ *admits known infinitesimal contact transformations* (*which perhaps commute or form a group*). *What does this knowledge imply about the number and order of the systems of ordinary differential equations that require solution in order to integrate* $F(x, p) = 0$? The difficulty facing him was how to relate his geometrical approach – which he tended to refer to as "synthetic" or "conceptual" – to the purely analytical theory of JACOBI and his successors.[22] As we shall see below, this difficulty was epitomized by the POISSON-JACOBI Theorem. The theorem was without question fundamental to JACOBI's theory, but its "true" grounds and significance – that is, its meaning on the synthetic-conceptual level – was a mystery to LIE in 1872.

The purpose of this section is to convey some idea of the "synthetic" notions that LIE developed in 1872 and of the new method of integration which he discovered within this context. He stressed that his method, which yields the same results as MAYER's Theorem 3.7, does not depend upon the POISSON-JACOBI Theorem. However, the method is based upon his notion of a system of partial differential equations in involution, a notion which, in its analytical formulation, was implicit in JACOBI's theory. The notion of systems in involution is of considerable historical importance because in the course of developing it synthetically LIE discovered a fundamental connection with the central "conceptual" notion of his *idée fixe*: that of a differential equation admitting infinitesimal transformations (§ 5). In this manner the crucial link (contained in Theorems 5.1 and 5.2) was forged between his synthetic-conceptual approach to differential equations and

[22] The use of the terms "synthetic" and "conceptual" may have grown out of LIE's association with KLEIN in 1869–70. For example, in 1869 KLEIN referred to himself as a "Synthetiker" surrounded in Berlin by mathematicians working on Weierstrassian function theory [ROWE, D. 1989: 200]. LIE shared many other catch words in common with KLEIN, such as the "invariant theory" of a group (for the study of invariant relations on a manifold determined by a transformation group – see § 1) and the distinction between higher dimensional geometry in the sense of PLÜCKER and in the sense of GRASSMANN [HAWKINS 1989: note 33].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

the analytical Jacobian theory. After some relatively superficial attempts (§ 5) to use this link to develop his *idée fixe*, he was led by it to his theory of function groups (§ 6), a theory which realized both his expectation of an invariant theory of contact transformations along the lines indicated in § 1 and of a development of his *idée fixe* along the lines indicated in the above formulation of it. Ultimately, the link provided him with a conceptual understanding of the POISSON-JACOBI Theorem itself (§ 7), an understanding which was of considerable importance, both mathematically and pyschologically, in encouraging him along the lines that eventuated in the birth of his theory of groups (§ 8).

The central concept in LIE's vision of a Mongean, or geometrically informed, approach to the study of partial differential equations was that of a contact transformation. As explained in § 1, a contact transformation of ordinary 3-space is an invertible transformation of a 5-dimensional manifold $T: (x, y, z, p, q) \to (x', y', z', p', q')$ which is generated by a reciprocity of type (A)–(C). The coordinates (x, y, z, p, q) can be interpreted as determining a surface element ds at $\boldsymbol{a} = (x, y, z)$ with normal $\boldsymbol{n} = p\boldsymbol{i} + q\boldsymbol{j} - \boldsymbol{k}$. Likewise, corresponding to a first order equation in any number of variables,

$$F(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0, \tag{4.1}$$

the $(2n+1)$-tuple $(x, z, p) = (x_1, \ldots, x_n, z, p_1, \ldots, p_n)$ could be regarded as the coordinates of a hypersurface element determined by the point $a = (x, z)$ in $(n+1)$-dimensional space and the hyperplane π through a determined by the p_i, namely the plane through a with equation

$$z^* - z - p_1(x_1^* - x_1) - \ldots - p_n(x_n^* - x_n) = 0. \tag{4.2}$$

The totality of all hypersurface elements (x, z, p) is thus a $(2n+1)$-dimensional manifold, R_{2n+1}, and contact transformations of $(n+1)$-dimensional space (x, z) are transformations $T: R_{2n+1} \to R_{2n+1}$ defined by equations

$$x_i' = X_i(x, z, p), \qquad z' = Z(x, z, p), \qquad p_i' = P_i(x, z, p), \qquad i = 1, \ldots, n, \tag{4.3}$$

which are generated by the "type (A_k) reciprocity" $(1 \leqq k \leqq n+1)$ defined by a system of k equations $F_i(x_1, \ldots, x_n, z, x_1', \ldots, x_n', z') = 0$, $i = 1, \ldots, k$. Transformations $T: (x, z) \to (x', z')$, or "point transformations," correspond to a type (A_{n+1}) reciprocity, the $n+1$ equations defining T being the equations defining the reciprocity and the equations $p_i' = P_i(x, z, p)$ being obtained by prolongation. (Geometrically, the prolongation T^* of T is obtained as follows. Given (x, z, p), let ds be an infinitesimal hypersurface containing $a = (x, z)$ and with normal $\boldsymbol{n} = \langle p_1, \ldots, p_n, -1\rangle$ at a. Then $T^*(x, z, p) = (x', z', p')$, where $(x', z') = T(a)$ and $\boldsymbol{n}' = \langle p_1', \ldots, p_n', -1\rangle$ is the normal to $ds' = T[ds]$ at $a' = T(a)$.) Thus every point transformation T can be regarded as (or prolonged to) a contact transformation T^*.

As explained in § 1, LIE called his transformations contact transformations because they transform a surface, conceived as consisting in the elements (x, y, z, p, q) determined by tangent planes to the surface, into another surface, likewise so conceived. In particular they transform the solution surfaces of one partial differential equation into those of the transformed equation. LIE realized however that exceptional cases exist. For example, if T denotes the EULER-LEGENDRE transformation defined by (1.4), *viz.*, $x' = -q$, $y' = p$, $z' = z - px - qy$,

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

$p' = y$, $q' = -x$, then T takes the surface S defined by $z = 1$, and therefore consisting of the ∞^2 surface elements $(x, y, 1, 0, 0)$, into the "surface" $M_2 = T[S]$ of all elements $(0, 0, 1, y, -x)$. These elements correspond to all planes through the fixed point $(0, 0, 1)$. Thus, within the context of 3-dimensional space (x, y, z), T takes the surface S into a point. This exception corresponds to the fact that the transformation (1.4) takes the partial differential equation $(p')^2 + (q')^2 + (z' - x'p' - y'q' - 1)^2 = 0$ into the "partial differential equation" $x^2 + y^2 + (z - 1)^2 = 0$. LIE's idea was to develop a geometrical theory that would include these exceptions within the same conceptual framework. In a sense, he sought to do something analogous to what had been done in projective geometry through the introduction of points, lines, planes, *etc.* at infinity. Although he never explicitly referred to this analogy, it seems likely that his background in projective geometry encouraged his approach to the study of partial differential equations. Other examples of analogies between his ideas and those of projective geometry will be seen in §§ 5–6.

Within the context of the manifold R_5 of surface elements, both S and M_2 are two dimensional manifolds. In fact they are both examples of what LIE later termed an "element manifold" [1874a: 155]. Within the context of R_{2n+1}, he defined an *element manifold* to be any solution of the Pfaffian equation

$$\Omega_{2n+1} = dz - p_1\, dx_1 - \ldots - p_n\, dx_n = 0$$

in the $2n + 1$ variables x, z, p [1871e: 20]. As indicated in § 2, since the time of LAGRANGE, the idea of regarding solutions to partial differential equations as solutions to $\Omega_{2n+1} = 0$ had been central, as LIE certainly realized. However, in the traditional analytical theory $\Omega_{2n+1} = 0$ was reduced to a form in $2n$ variables, $\Omega_{2n} = 0$, by using the given partial differential equation $F(x, z, p) = 0$ to eliminate one of the p_i. For LIE, however, this was not necessarily possible since he wished to include "exceptional" cases in which $F = 0$ does not involve any of the p_i. Thus it was essential to work with the equation $\Omega_{2n+1} = 0$. Any hypersurface $\varphi(x, z) = 0$ is an n-dimensional element manifold since it corresponds to the solution to $\Omega_{2n+1} = 0$ defined by the $n + 1$ equations, $\varphi = 0$, $p_i = -\varphi_{x_i}/\varphi_z$, $i = 1, \ldots, n$. Also $M_2 = T[S]$, where T is the EULER-LEGENDRE transformation (1.4), is a 2-dimensional element manifold since it satisfies $z - p\,dx - q\,dy = 0$. Element manifolds have the geometrical property that "infinitesimally close" points, (x, z, p) and $(x + dx,\ z + dz,\ p + dp)$, of such manifolds lie "united" or "joined" (*vereinigt*). That is, the point $(x + dx, z + dz)$ in $(n + 1)$-dimensional space lies on the hyperplane π through $a = (x, z)$ associated to the element (x, z, p) in the sense that for $(x^*, z^*) = (x + dx, z + dz)$ the equation (4.2) defining π takes the form $\Omega_{2n+1} = 0$.

For LIE any equation $F(x, z, p) = 0$ was to be regarded as a first order partial differential equation, even e.g. $F(x, y, z, p, q) = x^2 + y^2 + (z - 1)^2 = 0$. In general, $F(x, z, p) = 0$ defines a $2n$-dimensional manifold $M_{2n} \subset R_{2n+1}$. Solutions of this equation are by definition n-dimensional element manifolds $M_n \subset M_{2n}$. Thus, for example, the manifold $M_2 = T[S]$, where T denotes the EULER-LEGENDRE transformation (1.4), is a solution of $F = x^2 + y^2 + (z - 1)^2 = 0$. The problem of integrating $F(x, z, p) = 0$ is to determine all solutions, a problem which may be reduced to determining a family of ∞^n element manifolds

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

$M_n(c_1, \ldots, c_n) \subset M_{2n}$ [1872e: 21]. These correspond to the complete solutions (2.2) of the analytical theory.

Since contact transformations are transformations of R_{2n+1} which take solutions of partial differential equations into the same, they must take element unions into element unions and thus must leave the Pfaffian equation $\Omega_{2n+1} = 0$ invariant [1872e: 20]. LIE discovered that an invertible transformation $T: R_{2n+1} \to R_{2n+1}$ is a contact transformation if and only if it leaves the equation $\Omega_{2n+1} = 0$ invariant. Equivalently, the equations (4.3) define a contact transformation provided that

$$(4.4)\quad dz' - p_1' \, dx_1' - \ldots - p_n' \, dx_n' = \varrho(x, z, p)\,(dz - p_1 \, dx_1 - \ldots - p_n \, dx_n),$$

where ϱ does not vanish. This eventually became his preferred definition of a contact transformation [1873b].

By the standards of today, LIE's formulation of his ideas is unacceptably vague. Whether his variables are real or complex is never made precise. Just what differentiability the functions X_i, Z, P_i defining the contact transformation (4.3) were assumed to possess is never made clear. It is unclear whether he assumed such transformations had to be defined and invertible on all of R_{2n+1} or (as examples suggest) only on some subset. And so on. Nonetheless it is possible to relate his ideas to notions from the present day theory of manifolds, a theory to which in a sense his fertile geometrical imagination aspired. For example, contact transformations which fix z turn out to be particularly important in LIE's theory (§ 6). They may be regarded as transformations $T: (x, p) \to (x', p')$ of R_{2n} which (by LIE's Theorem 6.6 below) leave POISSON brackets invariant:

$$(F \circ T, G \circ T)_{(x,p)} = (F, G)_{(x',p')} \circ T.$$

These contact transformations (at least when all variables are real) can thus be identified with the canonical transformations of mechanics, and are examples of the more general notion of a POISSON map [OLVER 1986: 390]. In the case of general contact transformations (4.3), LIE's manifold R_{2n+1} could be interpreted as a contact manifold, contact transformations as contact diffeomorphisms, and n-dimensional element manifolds as LEGENDRE manifolds in the sense in which these terms are used by ARNOLD [1989: Appendix 4]. Likewise, the group of all contact transformations on R_{2n+1} could be interpreted as a LIE pseudogroup. And so on.[23] No such means of interpreting his ideas was of course available to mathematicians in the 1870's.

Although LIE's notions of element unions and contact transformations were presented fairly clearly — KLEIN helped him articulate them in the form given above[24] — to most of his contemporaries these notions were too radically differ-

[23] R. HERMANN has considered in detail the meaning and significance of LIE's ideas within the framework of the current theory of manifolds [1900: v. 2, Ch. 2; 1973: Ch. 3; 1975; 1976]. See also [CECIL & CHERN 1987] for information on the current revival of interest in generalizations of LIE's sphere geometry where *e.g.* LEGENDRE manifolds figure prominently.

[24] As he had on other occasions, KLEIN wrote up for publication LIE's paper [1872e], which first contained these concepts. He also presented them in his Erlangen Program [1872: § 9].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

ent from the manner in which they were accustomed to thinking about mathematics. And LIE himself made matters worse because his geometrical reasoning with these notions did not consistently display a clarity comparable to that found in the above definitions. Frequently he reasoned with older, more obscure formulations of these notions.[25] As a result, even such mathematicians as KLEIN, CLEBSCH and MAYER found it difficult, if not impossible, to follow his mathematical arguments. For example, in the fall of 1872 he found himself in Göttingen in the company of these three mathematicians. Seizing the opportunity, he presented the geometrical reasoning behind his recent discoveries in the theory of partial differential equations. According to his recollections, none of them was able to understand him. MAYER, in particular, could not at all grasp his "synthetic" presentation.[26]

The discoveries which LIE sought in vain to explain and justify in Göttingen had led him to a new method of integrating first order equations. The method grew out of his research related to the sphere mapping and led him to generalize many attendant notions to n-dimensional space [1871a, 1871b]. One such notion, which became fundamental to his geometrical-conceptual approach to partial differential equations, was that of a characteristic curve. These curves had already been related to LAGRANGE's theory of partial differential equations by MONGE. Consider a complete solution $z = \varphi(x_1, \ldots, x_n, c_1, \ldots, c_n)$ to $F(x, z, p) = 0$, as in (2.2). LAGRANGE had showed that φ may be used to generate what was called the general solution, a solution depending on an arbitrary function σ, rather than on arbitrary constants. Let σ denote any function of $n-1$ variables, and consider the family of solutions

$$z = \varphi_c = \varphi(x_1, \ldots, x_n, c_1, \ldots, c_{n-1}, \sigma(c_1, \ldots, c_{n-1})), \tag{4.5}$$

which depends on the $n-1$ parameters $c = (c_1, \ldots, c_{n-1})$. Differentiation of $z - \varphi_c = 0$ with respect to c_i, for $i = 1, \ldots, n-1$ yields the system of $n-1$ equations

$$\frac{\partial \varphi}{\partial c_i} + \frac{\partial \varphi}{\partial c_n}\frac{\partial \sigma}{\partial c_i} = 0, \quad i = 1, \ldots, n-1 \tag{4.6}$$

which may be used to eliminate the c_i from (4.5). That is, assuming the equations (4.6) can be solved for $c_1, \ldots, c_{n-1}$, let $c_i = g_i(x)$. Substitution in (4.5) yields the equation of the general solution

$$z = \Phi_\sigma(x) = \varphi(x_1, \ldots, x_n, g_1, \ldots, g_{n-1}, \sigma(g_1, \ldots, g_{n-1})).$$

That Φ_σ is a solution to $F(x, z, p) = 0$ follows from the fact that the chain rule combined with (4.6) implies that

$$\frac{\partial \Phi_\sigma}{\partial x_i}(x) = \frac{\partial \varphi}{\partial x_i}(x_1, \ldots, x_n, g_1, \ldots, g_{n-1}, \sigma(g_1, \ldots, g_{n-1})). \tag{4.7}$$

[25] See *e.g.* LIE's proofs (*circa* late 1872) that a single partial differential equation possesses no invariants (*Ges. Abh.* 7, p. 112ff.) and that every infinitesimal contact transformation determines a first order partial differential equation whose integral manifolds and "characteristic strips" (defined below) are left invariant by the transformation (*Ges. Abh.* 7, p. 47).

[26] LIE, *Ges. Abh.* 7, p. 194 (written *circa* 1881).

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

The surface $z = \Phi_\sigma$ is the enveloping surface of the family of surfaces $z = \varphi_c$ defined in (4.5). Equation (4.7) says that $z = \Phi_\sigma$ touches a specific member $z = \varphi_{c^0}$ of this family determined by setting $c = c^0 = (c_1^0, \ldots, c_{n-1}^0)$ at all points (x, z) which satisfy the n equations $g_i(x) = c_i^0$, $i = 1, \ldots, n-1$, $z = \Phi_\sigma(x)$. These n equations define a curve in $(n+1)$-dimensional space (x, z), which thus lies on both surfaces and along which the enveloping surface touches a particular member of the enveloped family.

Such curves (in three dimensional space) were called *characteristics* by MONGE, who showed that a geometrically informed theory of integration of $F(x, y, z, p, q) = 0$ could be based on the consideration of these curves (by contrast with LAGRANGE's analytical theory discussed in § 2).[27] MONGE showed that the characteristics satisfy a system of ordinary differential equations in x, y, z, p, q and that the integration of these equations yields the integration of $F(x, y, z, p, q) = 0$. In extending MONGE's theory to a space (x, z) of $n + 1$ dimensions LIE found that he had in effect rediscovered a method of integration introduced by CAUCHY in [1819]. CAUCHY had showed that the integration of $F(x, z, p) = 0$ could be reduced to the complete integration of the system of ordinary differential equations

$$(4.8) \quad F = 0, \quad \frac{dx_i}{dt} = \frac{\partial F}{\partial p_i}, \quad \frac{dz}{dt} = \sum_{i=1}^{n} p_i \frac{\partial F}{\partial p_i}, \quad \frac{dp_i}{dt} = -\left(\frac{\partial F}{\partial x_i} + p_i \frac{\partial F}{\partial z}\right).$$

CAUCHY presented his method in an obscure journal and never published a promised sequel comparing his method with PFAFF's. It was not until after the appearance of JACOBI's paper on his first method [1837] that CAUCHY sought to attract attention to his own method. For partial differential equations of the form $F(x, p) = 0$, CAUCHY's method agrees in essence with JACOBI's first method.

Although LIE never read any of CAUCHY's papers on his method, IMSCHENETSKY included a brief discussion of it in the final chapter of his essay [1869] which is probably where he learned about it. He saw that CAUCHY's equations (4.8) were the $(n+1)$-dimensional analog of MONGE's equations for the characteristics. CAUCHY's method thus agreed with the conclusions of MONGE's theory of characteristics in the case $n = 2$ and with the conclusions of LIE's generalization of MONGE's theory when $n > 2$. A solution to (4.8) represented, for LIE, a one dimensional manifold $t \to (x(t), z(t), p(t))$ in the space R_{2n+1} of contact elements (x, z, p). In terms of the $(n+1)$-dimensional space (x, z) a solution to (4.8) defined a curve $t \to (x(t), z(t))$ — a characteristic curve — with a hyperplane (determined by the $p_i(t)$) through each point. LIE therefore referred to $t \to (x(t), z(t), p(t))$ as a *characteristic strip*.[28]

CAUCHY's theory was, to LIE, simply an analytical version of the geometrical theory of characteristics, and for this reason he found CAUCHY's method "transpa-

[27] MONGE's theory was first communicated in memoirs written in the time period of LAGRANGE's work but was summed up in his book *Application de l'analyse à la géométrie*. LIE probably read about MONGE's theory in the fifth and final edition [1850: 421 ff.].

[28] For a rigorous introduction to the theory of characteristics, see [CARATHÉODORY 1935: Ch. 3].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

rent" (*durchsichtig*). He could see through it to the underlying geometrical concepts which were, for him, its true basis. By contrast, he had reservations about JACOBI's new method, which, he noted, mathematicians tended to regard as the "last word" on the integration of first order equations:

> On the one hand the method is not at all transparent. This is probably due essentially to the Poisson-Jacobi Theorem, the inner essence of which has not yet been investigated .... On the other hand, Jacobi's method requires an extraordinarily large number of integrations, significantly more than Cauchy's. Regarding this last point, certainly the modification suggested by Herr Weiler [Theorem 3.5] now places Jacobi's method in a significantly more favorable light. There are nonetheless other circumstances which raise doubts as to whether Jacobi's method is so perfect as mathematicians tend to maintain.

These words were written in an unfinished essay on first order equations which LIE appears to have written in the spring of 1872.[29] They reflect his interest in the conceptual basis of the POISSON-JACOBI Theorem and his acceptance of the research program implicit in JACOBI's work — to reduce as much as possible the number of integrations required to integrate a first order equation.

LIE's critical attitude towards JACOBI's method was perhaps partly encouraged by the fact that he himself had discovered a new and quite different method for integrating first order equations, one which, as he emphasized, did not depend upon the POISSON-JACOBI Theorem and in addition required fewer integrations than even the CLEBSCH-WEILER modification of JACOBI's new method (Theorem 3.5). He had discovered his new method geometrically, through his further generalization of MONGE's theory of characteristics to systems of first order equations. Although it did not involve the POISSON-JACOBI Theorem, it did conceptualize other simple extensions of the ideas in JACOBI's "Nova methodus" that had been made by others an that were contained in IMSCHENETSKY's essay [1869]. Since these extensions became a basic part of LIE's thinking regarding his *idée fixe*, they and their "Jacobian" roots will be indicated.

In "Nova methodus" JACOBI had not considered the problem of integrating systems of first order equations, $F_i(x, p) = 0$, $i = 1, \ldots, q$, but, as BOUR noted [1862b], his ideas apply directly to this problem because, in order for such a system to have a common solution it is necessary that those equations imply the further equations $(F_i, F_j) = 0$ for all $i, j = 1, \ldots, q$. Once these conditions are fulfilled, however, the reasoning behind JACOBI's Theorem 3.1 applies so that a complete solution may be obtained by determining $n - q$ additional functions $F_{q+1}, \ldots, F_n$ such that $(F_i, F_j) = 0$ for all $i, j = 1, \ldots, n$. Furthermore, the additional F_i may be found by JACOBI's method, according to which, *e.g.* F_{q+1} is obtained as a solution to the Jacobian system of linear equations $(F_i, f) = 0$,

[29] Excerpts of this essay are given in *Ges. Abh.* 7. The above quotation is on p. 56. The dating of the essay is based on the fact that in referring to his new method of integrating first order equations (discovered in the spring of 1872) he does not refer to MAYER's similar result, about which he learned (through CLEBSCH) in early June. See CLEBSCH's note to [LIE 1872f: 12]. A further discussion of the essay is included in § 5 below.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

$i = 1, \dots, q$. As IMSCHENETSKY pointed out in his presentation of these matters [1869: 400], the problem of integrating a system of equations $F_i(x, p) = 0$, $i = 1, \dots, q$, is in principle easier than integrating one such equation since one need find only $n - q$ F_i rather than $n - 1$. Once the F_i have been found the complete solution to the system may be obtained by the procedure underlying LAGRANGE's method as described by way of introduction to Theorem 3.1, the only difference being that the complete solution will involve $n - q + 1$ arbitrary constants rather than n. (This is because the functions G_i of (3.3) in this case are obtained by inverting the equations $F_1 = 0, \dots, F_q = 0, F_{q+1} = C_1, \dots, F_n = C_{n-q}$.) Thus JACOBI's new method led readily to the realization that the q equations $F_i(x, p) = 0$ imply the equations $(F_i, F_j) = 0$ if and only if (generically speaking) the q equations $F_i = 0$ have a common complete solution.

By reducing the problem of integrating $F(x, z, p) = 0$ to the problem for equations of the form $F(x, p) = 0$, JACOBI had been able to take advantage of the POISSON brackets, but it is also possible to generalize the brackets to apply to functions involving z [IMSCHENETSKY 1869: 393 ff.]. Given $F = F(x, z, p)$ and $G = G(x, z, p)$, let

$$(4.9) \qquad [F, G] = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{DG}{Dx_i} - \frac{DF}{Dx_i} \frac{\partial G}{\partial p_i} \right), \quad \frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial z}.$$

The operator D/Dx_i and the brackets $[\,,\,]$ arise readily from the fact that if $V = F(x, z, p)$ and $z = \varphi(x)$, then by the chain rule $\dfrac{\partial V}{\partial x_i} = DF/Dx_i$. The generalized POISSON brackets possess the same fundamental properties as the original. In particular, Theorems 3.1 and 3.2 (JACOBI Identity) extend to generalized brackets. It should also be noted that $[F, f] = 0$ is a linear partial difference equation in the $2n + 1$ variables x, z, p, namely

$$[F, f] = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \right) \frac{\partial f}{\partial x_i} + \left(\sum_{i=i}^{n} p_i \frac{\partial F}{\partial p_i} \right) \frac{\partial f}{\partial z} + \sum_{i=1}^{n} - \left(\frac{\partial F}{\partial x_i} + p_i \frac{\partial F}{\partial z} \right) \frac{\partial f}{\partial p_i} = 0.$$

By virtue of the correspondence between linear homogeneous partial differential equations and systems of ordinary differential equations established by LAGRANGE — see (2.5)–(2.10) — LIE realized that the integration of the linear homogeneous equation $[F, f] = 0$ is tantamount to integrating the system (4.8) of CAUCHY's method.

It follows readily from the above extensions of JACOBI's ideas that a system of equations which involve the variable z. viz. $F_i(x, z, p) = 0$, $i = 1, \dots, q$, will have a common complete solution involving $n - q + 1$ arbitrary constants if and only if these equations imply the further equations $[F_i, F_j] = 0$, $i, j = 1, \dots, q$. This generic result became the starting point for LIE's concept of an involutive system of equations. That is: equations $F_i(x, z, p) = 0$, $i = 1, \dots, q$ are said to be *in involution* if $[F_i, F_j] = 0$ for all $i, j = 1, \dots, q$ and for all (x, z, p) for which $F_1 = F_2 = \dots = F_q = 0$ [1872a: 1; 1872b: 6]. The stronger condition that $[F_i, F_j] \equiv 0$ for all i, j he later described by saying that in this case the *functions* F_i (rather than the equations $F_i = 0$) are in involution. As will be seen in § 5, corresponding to an infinitesimal contact transformation $d\omega$ is its characteristic function

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

$W(x, z, p)$. If $d\omega$ is admitted by an equation $F(x, z, p) = 0$, and hence leaves the equation $F = 0$ invariant, then the equations $F = W = 0$ are in involution. If $d\omega$ leaves the function F invariant, then the functions F, W are in involution. LIE was primarily interested in equations in involution.

LIE translated the notion of involutive equations into geometrical terms and in that form developed characteristic properties of such systems involving, among other things, his generalization of the notion of a characteristic strip. (See, for example, *Satz 2* of [1872b].) The resultant theory will be referred to as his synthetic theory of involutive systems. The details of this theory as he developed it in 1872 are not known, although ENGEL has attempted to reconstruct them. (See pp. 601 ff. of LIE, *Ges. Abh.* **3**.) Here the emphasis will be upon the analytic form of certain synthetically derived results which are relevant to his creation of his theory of transformation groups.

LIE's first major disovery by means of his synthetic theory of involutive systems was a new method for integrating a first order partial differential equation which yielded the same conclusion as MAYER's Theorem 3.7 [LIE 1872b]. This discovery brought him face-to-face with the fact that his synthetic mode of thought was unacceptable, indeed unintelligible to other mathematicians. In analytical terms the *gist* of his method is easy enough to understand [1872b: 9–10]. Given the equation $F(x, z, p) = 0$, determine a solution $f = \Phi$ to the linear equation $[F, f] = 0$ associated with Cauchy's method. Hence the equations $F = 0$, $\Phi - c_n = 0$ are in involution since $[F, \Phi - c_n] = [F, \Phi] = 0$. Solving $\Phi - c_n = 0$ for p_n to obtain

$$p_n = \psi(x_1, \ldots, x_n, z, p_1, \ldots, p_{n-1}, c_n), \tag{4.10}$$

eliminate p_n from $F = 0$ by making the substitution (4.10) in $F = 0$ to obtain the equation $G(x, z, p_1, \ldots, p_{n-1}, c_n) = 0$, where $G = F(x, z, p_1, \ldots, p_{n-1}, \psi)$. Regard $G = 0$ as a partial differential equation in $x_1, \ldots, x_{n-1}$, and consider a complete solution, $z = \varphi$. It will be a function of $x_1, \ldots, x_{n-1}$ and involve $n - 1$ arbitrary constants $c_1, \ldots, c_{n-1}$ but it will also depend upon x_n and c_n, so $z = \varphi(x_1, \ldots, x_n, c_1, \ldots, c_n)$. According to LIE, from this complete solution a complete solution to $F(x, z, p) = 0$ could be determined "by means of differentiation," although he did not explain exactly how this was to be done.

In LIE's method, the original problem of determining a complete solution to $F = 0$ is reduced to the same problem for an equation $G = 0$ in one less variable, and repeated application of this reduction leads to Theorem 3.7. MAYER had submitted his paper [1872b] containing Theorem 3.7 to *Mathematische Annalen* in February of 1872, but in May when LIE published his method in Norway and sent something by agency of KLEIN to CLEBSCH for publication in the *Göttinger Nachrichten*, CLEBSCH, unaware of MAYER's result, declined to submit LIE's communication because he could not follow the argument and doubted the correctness of the low number of integrations, slightly over half of what was required by the CLEBSCH-WEILER modification of JACOBI's method (Theorem 3.5). It was only after he learned that MAYER had arrived at essentially the same result that he decided to have both LIE and MAYER announce their work in the *Nachrichten*. Nonetheless, neither CLEBSCH nor MAYER was completely convinced of the correctness of LIE's method — especially that "differentiations" performed

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

on φ led to a complete solution to $F = 0$. Despite LIE's efforts to explain his ideas in geometrical terms to them in Göttingen in the fall of 1872, they remained unconvinced. MAYER, who by LIE's recollections understood his geometrical arguments least of all, took up the task of providing a completely analytical justification of LIE's method. Using some notes provided by LIE, MAYER succeeded quickly [1872a, 1873].

Because of MAYER's expertise in differential equations, especially as related to JACOBI's work, and by virtue of the serious interest he showed in providing respectable analytical formulations of LIE's theories, LIE cultivated his association with MAYER and began to correspond with him regularly, sharing his new ideas and results as they came to him.[30] Indeed, with the death of CLEBSCH at the end of 1872, MAYER had effectively become the key "Jacobian analyst" for LIE to convince regarding the validity of his synthetically derived results. LIE was fortunate that MAYER was interested in the analytic implications of his synthetically inspired ideas. It was with MAYER's encouragement that he began to develop his invariant theory of contact transformations in analytical form (§ 6) and was thereby led to the discovery of relations between partial differential equations, the calculus of differential operators and the composition of transformation groups (§ 7) that played a vital role in the events of the fall of 1873 (§ 8). Before discussing these matters, however, we need to consider how, by the middle of 1872, he had discovered synthetically the crucial link between involutive systems and partial differential equations admitting infinitesimal transformations, the link that motivated and guided the developments discussed in §§ 6–8.

5. Involutive Systems and Lie's *Idée Fixe*

Although during 1872 LIE ended up presenting his ideas on the integration of first order partial differential equations in brief, cryptic research announcements, documents he did not publish reveal that he had attempted to incorporate them into major essays. Representative portions of these essays were edited by ENGEL and are included in the seventh volume of LIE's *Abhandlungen* as items III, IIIa, IV and IVa. These essays document the fact that during 1872 his *idée fixe* remained just that — an idea that was always in the back of his mind. Indeed, the projected essay was first conceived as being devoted entirely to the integration of first order partial differential equations admitting known infinitesimal transformations (IIIa, III). Here he pointed out that the integration of differential equations admitting known infinitesimal transformations was a subject that he had considered in several of the papers that came out of his geometrical period of research. In particular, "stimulated by Herr Klein who pointed out to me that there was an analogy between the above methods of integration [of equations with known transformations] and the theory of substitutions in the discipline of algebraic equations, I pursued these considerations further and succeeded in obtaining some

[30] Excerpts from this correspondence, mostly LIE's letters to MAYER, are scattered throughout ENGEL's editorial notes to volumes 3–6 of LIE's collected works. The entire letters however were never published, and the present disposition of the correspondence is not known.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

general results which appear noteworthy" (p. 27). Eventually he decided to incorporate this essay into one devoted to all aspects of the theory of first order partial differential equations (IVa, IV). Here he expressed the view that such a theory had two basic goals: the development of new general methods (such as his own new method of integration) and the development of "particular methods" which "resolve or at least simplify the integration problem in special cases, that is, when the equation possesses a particular form or generally possesses certain known properties" (p. 58). The property that fascinated him, of course, was that of admitting infinitesimal transformations.

Lie believed that the second goal of the theory was an important one and that the study of the integration of equations known to admit infinitesimal transformations was destined to occupy an important place within the theory of differential equations. He expressed this conviction in a draft of one of his research announcements [1872d]. Pointing out that "in recent, largely unpublished works Herr Klein and I have concerned ourselves with groups of continuous transformations of a manifold $(x_1, \dots, x_n)$," he referred to his joint work with Klein on W-curves and his own work on the sphere mapping, in both of which the integration of equations admitting infinitesimal transformations is considered (essentially by means of the ideas involved in Theorem 1.1). He then continued by expressing his "conviction that these theories will some day find a place in science. Up till now efforts have been concentrated on finding general methods of integration and on settling certain problems, such as the three body problem. While not in any way denying the significance of these considerations, I would nonetheless venture the opinion that it is important for the advancement of these theories to investigate in general what advantage for integration can be drawn from the fact that certain properties of the problem are known a priori."[31]

Although Lie's essays contain the generalization of Theorem 1.1 to any number of variables, they also contain new manifestations of his *idée fixe* that are closely linked to both his new method of integration and to Jacobi's (IIIa, p. 47ff.). The results themselves are less important than the means by which they were established, namely Theorems 5.1 and 5.2 below. Through these theorems he established a link between involutive systems and infinitesimal transformations admitted by a partial differential equation, a link that led to his theory of function groups, a theory which, on one level at least, was a working out of his *idée fixe*. More generally this link played a fundamental role in the considerations that led him to attempt to create a theory of transformation groups.

The discovery communicated by Lie in his essay and referred to above as Theorem 5.1 may be stated as follows (IIIa, Nr. 10):

Theorem 5.1. *Associated to an infinitesimal contact transformation $d\omega$ is a unique partial differential equation $W(x, z, p) = 0$ whose characteristic strips are transformed into themselves by $d\omega$. The characteristic strip through a point (x_0, z_0, p_0) of the manifold $W(x, z, p) = 0$ is obtained by applying $d\omega$ continually to (x_0, z_0, p_0).*

[31] *Ges. Abh.* 7, p. 89. This passage was omitted from the published version which was submitted in November 1872 and announced further results in this direction, which according to Engel (Lie, *Ges. Abh.* 3, p. 636) Lie published in [1874c] (discussed here in § 8 below).

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

As explained in § 4, the characteristic strips associated to $W(x, z, p) = 0$ are solution curves $t \to (x(t), z(t), p(t))$ to the system of equations (4.8):

$$(5.1) \quad W = 0, \quad \frac{dx_i}{dt} = \frac{\partial W}{\partial p_i}, \quad \frac{dz}{dt} = \sum_{i=1}^{n} p_i \frac{\partial W}{\partial p_i}, \quad \frac{dp_i}{dt} = -\left(\frac{\partial W}{\partial x_i} + p_i \frac{\partial W}{\partial z}\right).$$

Lie's justification of Theorem 5.1 suggests how he probably discovered it. He began by considering first the special case in which $d\omega$ is a point transformation $(x, z) \to (x + dx, z + dz)$, where

$$(5.2) \qquad \frac{dx_i}{dt} = \xi_i(x, z), \quad i = 1, \dots, n, \quad \frac{dz}{dt} = \zeta(x, z).$$

He realized, as did anyone familiar with the Lagrange-Jacobi theory of partial differential equations, that solving the system of ordinary differential equations (5.2) was equivalent to solving the linear partial differential equation $W(x, z, p) = 0$, where

$$(5.3) \qquad W(x, z, p) = \sum_{i=1}^{n} \xi_i(x, z) \frac{\partial z}{\partial x_i} - \zeta(x, z) = \sum_{i=1}^{n} \xi_i p_i - \zeta.$$

(See the discussion of the equivalence of (2.9) and (2.10), which is similar to that found in Imschenetsky's essay [1869: 295–6] where Lie probably learned it.) The correspondence between the system of ordinary differential equations (5.2) and the partial differential equation $W = 0$ defined by (5.3), was for Lie, however, also a correspondence between the infinitesimal point transformation defined by (5.2) and the partial differential equation $W = 0$. It follows readily that the function W of (5.3) is precisely the one whose existence is asserted in Theorem 5.1. To see this, recall that if $t \to (x(t), z(t), p(t))$ is a characteristic strip of $W = 0$ then $t \to (x(t), z(t))$ is the associated characteristic curve. Since for the special function W given by (5.3), $\frac{dx_i}{dt} = \frac{\partial W}{\partial p_i} = \xi_i$, and, on the manifold $W = 0$, $\frac{dz}{dt} = \sum p_i \frac{\partial W}{\partial p_i} = \sum \xi_i p_i = W + \zeta = \zeta$, the equations for the characteristic curves coincide with (5.2), the equations defining $d\omega$. This means, in the language of § 1, that $d\omega$ applied infinitely often to a point of the manifold $W = 0$ generates a characteristic curve through that point, and consequently $d\omega$ takes these curves into themselves. From this fact, Lie could conclude that $d\omega$ — or, more precisely, its prolongation to a contact transformation

$$(x, z, p) \to (x + dx, z + dz, p + dp)$$

— takes characteristic strips into themselves and so satisfies the conclusions of Theorem 5.1.

In the special case of an infinitesimal point transformation, Theorem 5.1 was thus in essence simply a restatement of a well known fact from the Lagrange-Jacobi theory of partial differential equations. To obtain Theorem 5.1 in its full generality, Lie transformed the above special result by means of a contact transformation. That is, if $d\omega : (x, z, p) \to (x + dx, z + dz, p + dp)$ is now assumed to be any contact transformation, then drawing upon generalizations of ideas in his sphere mapping work, he argued synthetically that a non-infinitesimal contact transformation $\beta : (x, z, p) \to (X, Z, P)$ could be determined so that

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

$d\omega_1 = \beta\, d\omega\, \beta^{-1}$ is a point transformation $(X, Z) \to (X + dX, Z + dZ)$. If $W_1(X, Z, P) = 0$ is the partial differential equation associated to this point transformation by the above reasoning, then $W(x, z, p) = 0$, where $W(x, z, p) = \beta^{-1} W_1 \beta$, is the partial differential equation that corresponds to $d\omega = \beta^{-1}\, d\omega_1\, \beta$.

This type of reasoning according to which a simple fact is transformed by means of a contact transformation into a more general one, was used on many occasions by LIE. In this case the simple fact was provided by the theory of first order partial differential equations. It shows just how closely the seminal ideas behind his creation of a theory of continuous groups were linked to the theory of first order partial differential equations. Of course what is interesting to observe is how basic ideas from that theory took on new dimensions when combined by LIE with the notions he had acquired during his geometrical period — in this case the notions of infinitesimal and contact transformations. More generally, the technique of establishing general results by applying an "arbitrary" contact transformation to a special case is reminiscent of the technique of projective geometry, whereby results in special cases (*e.g.* properties of a special type of conic such as a circle) are transformed by projective transformations into more general theorems. The viewpoint of projective geometry was of course second nature to LIE. That the technique as applied within the context of the geometrical method determined by the group of contact transformations was basic to his synthetic approach to partial differential equations is also suggested by other documents, such as that in LIE, *Ges. Abh.* 7, IX (p. 113).

Theorem 5.1 establishes a correspondence $d\omega \to W$ between infinitesimal contact transformations $d\omega$ and partial differential equations $W = 0$. LIE undoubtedly realized that if any equation $W(x, z, p) = 0$ is given, then there is an infinitesimal transformation $d\omega$ such that $W = 0$ is the equation which corresponds to $d\omega$ by Theorem 5.1. I base this conclusion upon the fact that by the time he published Theorem 5.5 in May of 1872 [1872a] he had convinced himself that the following theorem holds: *Any two first order partial differential equations may be transformed into one another by means of a contact transformation.*[32] The correspondence $W \to d\omega$ can be deduced from this theorem by a line of reasoning that is essentially that used by LIE to prove Theorem 5.1. Given $W(x, z, p) = 0$, it follows from the theorem that a contact transformation $\beta : (x, z, p) \to (X, Z, P)$ exists which transforms $W = 0$ into a linear equation, $W_1(X, Z, P) = \beta W \beta^{-1} = \sum_{i=1}^{n} \xi_i(X, Z)\, P_i - \zeta(X, Z) = 0$. Then corresponding to $W_1 = 0$ is the infinitesimal point transformation $d\omega_1$ given by $dX_i/dt = \xi_i$, $dZ/dt = \zeta$, and it follows from LIE's proof of Theorem 5.1 that $d\omega = \beta^{-1}\, d\omega_1\, \beta \to W$. Thus the correspondence between infinitesimal contact transformations and partial differential equations is actually two way and will be denoted by $d\omega \leftrightarrow W$. LIE eventually named the function W in $d\omega \leftrightarrow W$ the *characteristic function* of $d\omega$. For convenience of reference this appellation will be used in what follows.

---

[32] Although this theorem is not mentioned in [1872a], it is contained in a manuscript which was evidently a draft of parts of that paper. (The manuscript is item IX of LIE, *Ges. Abh.* 7, pp. 112–114. Compare especially Nr. 4 of the manuscript with Nr. 4 of [1872a].) The theorem was first stated in the paper which KLEIN wrote up for LIE in October 1872 [LIE 1872e: 19].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

The significance of the correspondence $d\omega \leftrightarrow W$ for LIE came by virtue of the following fundamental discovery (IIIa, Nr. 13):

Theorem 5.2. *If* $F(x, z, p) = 0$ *admits an infinitesimal contact transformation* $d\omega$, *and if* $W(x, z, p)$ *is its characteristic function, then* $F = 0$ *and* $W = 0$ *are in involution. Thus the equations* $F = 0$, $W = 0$ *imply the equation* $[F, W] = 0$.

His sketchy justification of Theorem 5.2 was almost entirely verbal and drew upon his synthetic theory of involutive systems. Theorems 5.1 and 5.2 were known to him by the spring of 1872. They provided the key by means of which he was able to connect his *idée fixe* with the theory of first order partial differential equations. The general problem underlying his *idée fixe* may be stated as follows: suppose an equation $F(x, z, p) = 0$ is known to admit infinitesimal contact transformations $d\omega_1, \ldots, d\omega_q$ which perhaps either commute or form a group. What does this information say about the integration of $F = 0$? By virtue of the correspondence $d\omega \leftrightarrow W$ and Theorem 5.2, he knew that the existence of the $d\omega_i$ meant that q functions $W_i(x, z, p)$ exist such that each equation $W_i(x, z, p) = 0$ is in involution with $F(x, z, p) = 0$: $[F_i, W_i] = 0$ when $F = W_i = 0$. This was the sort of information that had a definite relevance to theories of integration of first order equations that had emerged from JACOBI's work, and in the manuscript (IIIa) under consideration he began to explore the possibilities. These early efforts are of historical interest as the prelude to the development of his *idée fixe* into what he called his theory of function groups. They also illustrate how he conceived of the composition of infinitesimal transformations.

LIE began by considering two infinitesimal transformations, $d\omega_1$ and $d\omega_2$ which are admitted by $F(x, z, p) = 0$. Then: "By composition of these, ∞^1 infinitesimal transformations are obtained, each of which arises by first applying the transformation $[d\omega_1]$ n times and then $[d\omega_2]$ p times. For different ratios of n and p we obtain different infinitesimal transformations."[33] In other words and in the notation of (1.1), the transformation $d\omega_c$ corresponding to the ratio $n:p = c$ is given by the system of equations $dy_i/dt = n\eta_i^{(1)} + p\eta_i^{(2)}$, $i = 1, \ldots, m$, where the system defining $d\omega_j$, $j = 1, 2$ is $dy_i/dt = \eta_i^{(j)}$. In particular, despite LIE's manner of expressing himself, n and p need not be integers. Thus for each c there is by Theorems 5.1–5.3 an equation $W(x, z, p, c) = 0$ associated to $d\omega_c$ and in involution with $F = 0$. If $W = 0$ is solved for c to obtain $c = \Phi(x, z, p)$ then the conclusion may be stated as

Theorem 5.3. *If* $F(x, z, p) = 0$ *admits two infinitesimal contact transformations, then there exists an equation* $\Phi - c = 0$ *in involution with* $F = 0$.

Theorem 5.3 relates directly to LIE's new method of integration (§ 4) and shows that when $F = 0$ admits two infinitesimal contact transformations, its integration can be reduced to that of an equation with one less variable. This reduction should not be confused with that of his earlier Theorem 1.1. Here the reduced equation involves one less independent variable and therefore one less partial derivative as well.

[33] IIIa, p. 48. LIE used the notation do_1, do_2 for the infinitesimal transformations.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

LIE's proof of Theorem 5.3, sketched above, shows that he conceived of the composition of infinitesimal transformations in the customary way. That is, if $d\mu$ and $d\nu$ are infinitesimal transformations defined, respectively by the systems of equations $dy_i/dt = \xi_i(y)$ and $dy_i/dt = \eta_i(y)$, $i = 1, \ldots, m$, then the composite $d\mu \circ d\nu$, corresponds to the system $dy/dt = \xi_i(y) + \eta_i(y)$, $i = 1, \ldots, m$. This means of course that $d\nu \circ d\mu$ is indistinguishable from $d\mu \circ d\nu$, whereas the non-infinitesimal transformations generated, respectively, by $d\mu$ and $d\nu$ need not commute. In § 7 we shall see how LIE dealt, at the infinitesimal level, with the possible noncommutativity of the noninfinitesimal transformations, guided once again by JACOBI's theory.

The ideas behind LIE's method of integration implied the following extension: if q equations $\Phi_i - c_i = 0$, $i = 1, \ldots, q$, exist so that the *system* of $q + 1$ equations $F = 0$, $\Phi_1 - c_1 = 0, \ldots, \Phi_q - c_q = 0$ is in involution, then the integration of $F = 0$ can be reduced to that of a partial differential equation in q fewer independent variables (IIIa, pp. 52–53). Theorem 5.3 shows that if $F = 0$ admits $q + 1$ infinitesimal contact transformations $d\omega_1, \ldots, d\omega_{q+1}$, then each pair $d\omega_i, d\omega_{q+1}$ determines the ∞^1 infinitesimal transformations $d\omega_{c_i} = (d\omega_i)^n (d\omega_{q+1})^p$ with associated equation $\Phi_i + c_i = 0$ in involution with $F = 0$. Thus each pair $F = 0$, $\Phi - c_i = 0$ is in involution. The entire system of $q + 1$ equations would be in involution if each pair $\Phi_i - c_i = 0$, $\Phi_j - c_j = 0$ were also known to be in involution. LIE realized that this would be the case provided the known infinitesimal transformations commuted, for he deduced from his synthetic theory of involutive systems

Theorem 5.4. *If $d\omega_1$ and $d\omega_2$ are commuting infinitesimal transformations, then the associated equations $W_1(x, z, p) = 0$, $W_2(x, z, p) = 0$, are in involution.*[34]

Theorem 5.4 shows that if the $q + 1$ $d\omega_i$ commute, so that the q $d\omega_{c_i}$ likewise commute, then the system of equations $F = 0$, $\Phi_1 - c_1 = 0, \ldots, \Phi_q - c_q = 0$ is in involution. His extended method of integration could thus be articulated as

Theorem 5.5. *If $F(x, z, p) = 0$ admits $q + 1$ commuting infinitesimal contact transformations, then there exist q equations $\Phi_i - c_i = 0$ which are in involution with each other and with $F = 0$* [IIIa, Nr. 22]. *Hence the integration of $F = 0$ can be reduced to that of a partial differential equation with q fewer variables.*

Theorem 5.5, like the earlier Theorem 1.1, was based upon the existence of commuting transformations admitted by the equation. That LIE aspired to similar results for non-commuting transformations is indicated in the passage following Theorem 5.5: "Finally I indicate briefly how it can be useful to know nonpermutable infinitesimal contact transformations [admitted by a partial differential equation]. I suspect, by the way, that a more general theory can be built upon this point than what I now sketch. If first of all the equation has only three variables:

[34] ENGEL's conjectured reconstruction of how LIE might have proved Theorem 5.4 in 1872 (LIE, *Ges. Abh.* **3**, p. 616) is consistent with LIE's brief proof in IIIa. (It is unclear whether ENGEL was familiar with IIIa when he published his reconstruction.)

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

$F(z, x_1, x_2, p_1, p_2) = 0$, and if infinitesimal transformations do_1, do_2 [admitted by $F = 0$] are known which are not permutable, form the infinitesimal transformation $d\tilde{\omega} = do_1 \cdot do_2$ and seek the associated partial equation $\Omega = C$ which lies in involution with $F = 0$" [IIIa, p. 53]. Unfortunately he did not develop his thoughts any further in the manuscript. By the end of April of 1872 when he announced Theorem 5.5 in print, his consideration of noncommuting transformations had led to the following result: "If $F = 0$ admits three known infinitesimal contact transformations which are not permutable, then in general the integration of $F = 0$ can be reduced to the determination of an integrability factor. In this connection, the Poisson-Jacobi Theorem is applied" [1872a: 2].

Because LIE did not elaborate further on this statement, its exact meaning is not certain. However, as ENGEL observed, the reference to the POISSON-JACOBI Theorem (Theorem 3.4) and the phrase "in general" which figures in JACOBI's "corollary" to the theorem, suggest that he had something in mind along the following lines. Consider first of all the context of the POISSON-JACOBI Theorem: functions of (x, p) rather than (x, z, p). CAUCHY's method, applied to an equation of the form $F(x, p) = 0$, implies that if independent functions $W_1, \ldots, W_{2n-2}$ of (x, p) have been determined such that $(F, W_i) = 0$ when $F = 0$, then the integration of $F = 0$ reduces to that of a first order ordinary differential equation, an equation which may be solved by obtaining an integrating factor.[35] If JACOBI's "corollary" to the POISSON-JACOBI Theorem is accepted, it implies that "in general" it would suffice to know just two such W_i, say W_1 and W_2 in order to obtain, by applying the POISSON-JACOBI Theorem, the requisite $2n - 1$ functionally independent solutions $F, W_1, \ldots, W_{2n-2}$. By virtue of Theorems 5.1–5.2, if $F(x, p) = 0$ admits two infinitesimal transformations, $d\omega_1$ and $d\omega_2$, then the corresponding characteristic functions W_1, W_2 are precisely what is needed "in general" to reduce the integration of $F = 0$ to an integrability factor.

So far in these considerations the context has been functions of (x, p) rather than of (x, z, p). Within the context of variables (x, z, p), $F(x, p) = 0$ actually admits a third infinitesimal transformation, namely $d\omega_3 : (x, z, p) \to (x, z + dt, p)$. ENGEL has suggested that the result of LIE's quoted above was obtained by transforming the special result into a more general result by means of a non-infinitesimal contact transformation, a technique he employed on other occasions as we have seen. (See ENGEL's detailed discussion on pp. 616–618 of LIE, *Ges. Abh.* **3**.) Whatever the exact content of LIE's result might have been, it clearly reflects his interest in trying to relate the POISSON-JACOBI Theorem to his *idée fixe*. These efforts eventually bore fruit within the context of his theory of function groups.

---

[35] When $F = F(x, p)$, equation (4.8) of CAUCHY's method splits up into (1) the "Hamiltonian" system of ordinary differential equations equivalent to finding all Φ such that $(F, \Phi) = 0$ when $F = 0$ and (2) the equation $\dfrac{dz}{dt} = \sum_{i=1}^{n} p_i \dfrac{\partial F}{\partial p_i}$. The complete integration of (1) requires finding $W_1, \ldots, W_{2n-2}$ which are independent in the sense that $F, W_1, \ldots, W_{2n-2}$ are functionally independent. Assuming that (1) has been completely integrated, the x_i and p_i may be expressed as functions of t and the expression for $\dfrac{dz}{dt}$ in (2) reduces to a function of the two variables z, t.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

The attempt described above by LIE to develop his *idée fixe* for non-commuting infinitesimal contact transformations involved utilizing what amounts to JACOBI's "corollary" to the POISSON-JACOBI Theorem, the only difference being that LIE considered solutions to $F = 0$, $(F, \Phi) = 0$. That is, in the terminology explained in § 4, he considered pairs of equations in involution, $F = 0$, $\Phi = 0$, rather than pairs of functions in involution.) As we have seen, however, many mathematicians questioned the validity of the ' corollary" even as a generic theorem. LIE, too, seems to have had this doubs about it; or at least he also explored the situation in which the "corollary" is not assumed to hold. In that case, if one starts with independent solutions $\Phi = W_1, W_2$ to $(F, \Phi) = 0$, then, after repeatedly applying the POISSON-JACOBI Theorem to obtain further solutions, one will end up with $r < 2n - 1$ solutions $\Phi = u_i$, $i = 1, \ldots, r$, to $(F, \Phi) = 0$ (among which F, W_1 and W_2 are included) such that for all i, j the solution $\Phi = (u_i, u_j)$ yields nothing new in the sense that (u_i, u_j) is functionally dependent upon $u_1, \ldots, u_r$. That is, a function Ω_{ij} of r variables exists such that

$$(u_i, u_j) = \Omega_{ij}(u_1, \ldots, u)_r. \tag{5.4}$$

This suggests

Problem 5.1. *Given a partial differential equation* $F(x, p) = 0$, *suppose* r *functionally independent solutions* $\Phi = u_i(x, p)$, $i = 1, \ldots, r$, *to* $(F, \Phi) = 0$ *are known which satisfy the relations* (5.4). *What can be said about the integration of* $F = 0$?

This was a natural analytical question to ask in response to JACOBI's "corollary" and several mathematicians (LIOUVILLE, BERTRAND, BOUR) had already asked it in the 1850's (§ 3), although they limited their attention to what can be said when $(u_i, u_j) = 0, \pm 1$. LIE was the first to consider the general problem stated above.

For LIE this problem had an extra dimension which made it especially attractive to him. It could be understood as an expression of his *idée fixe* in the case in which the given infinitesimal transformations do not necessarily commute but rather "form a group" in the following sense. Since the functions $u_1, \ldots, u_r$ are solutions to $(F, \Phi) = 0$, it follows that any function of the u_i, say $u = \Omega(u_1, \ldots, u_r)$, is also a solution because the chain rule implies that $(F, u) = \sum_{k=1}^{n} \frac{\partial \Omega}{\partial u_k} (F, u) = 0$. If $\mathscr{F}$ denotes the totality of all functions of $u_1, \ldots, u_r$, then all these solutions are in effect given when $u_1, \ldots, u_r$ are given. Now each $u \in \mathscr{F}$ is the characteristic function of an infinitesimal contact transformation $d\mu$ of (x, p). If we let $g_{\mathscr{F}}$ denote the totality of all such $d\mu$, that is, if

$$g_{\mathscr{F}} = \{d\mu : d\mu \leftrightarrow u \;\&\; u \in \mathscr{F}\}, \tag{5.5}$$

then $g_{\mathscr{F}}$ has the property that it is closed under composition. (If $d\mu \leftrightarrow u$ and $dv \leftrightarrow v$ are in $g_{\mathscr{F}}$, then so is $d\mu \circ dv \leftrightarrow u + v$ since $u + v \in \mathscr{F}$.) This property will be referred to as *the* group property because, as indicated in § 1, during this period LIE took it for granted that a continuous family of transformations with this property formed a group. Not only does $g_{\mathscr{F}}$ have the group property, it turns out that the equation $F = 0$ admits all the transformations of $g_{\mathscr{F}}$. Thus Problem 5.1 can also be stated as

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

Problem 5.2. *Given a partial differential equation* $F(x, p) = 0$, *suppose it admits the group* $g_{\mathscr{F}}$ *of infinitesimal contact transformations. What can be said about the integration of* $F = 0$?

The purely analytical Problem 5.1 was thus for LIE equivalent to an expression of his *idée fixe*. That is why, in his purely analytical publications dealing with Problem 5.1, although $g_{\mathscr{F}}$ itself is not introduced, he called $\mathscr{F}$ a group and the theory he developed to resolve it his "theory of groups." The fact that $F = 0$ admits $g_{\mathscr{F}}$ follows immediately from LIE's discoveries involving the analytical calculus of infinitesimal transformations as will be seen in § 7. It is doubtful, however, that he had made all these discoveries — specifically Theorems 7.5–7.6 — by the time he chose the name "group" for $\mathscr{F}$. He probably first perceived this property of $g_{\mathscr{F}}$ through synthetic considerations.

The fact that $g_{\mathscr{F}}$ is admitted by $F = 0$ is a reflection of the fact that for infinitesimal contact transformations $d\mu$ of (x, p) the converse of Theorem 5.2 holds as well so that in the correspondence $d\mu \leftrightarrow u$ an equation $F(x, p) = 0$ admits $d\mu$ if and only if $F = 0$ and $u = 0$ are in involution, that is, if and only if $F = 0$ implies $(F, u) = 0$. As will be seen more fully in §§ 7–8, this link between LIE's *idée fixe* and involutive systems — a fundamental concept in the theories of first order equations inspired by JACOBI's work — and more generally the link between LIE's conceptual world of continuous groups and their infinitesimal transformations and the analytical properties of POISSON brackets, played a key role in his decision to attempt to create a theory of continuous groups. Part of the reason for this came from the discoveries he made as he worked on Problem 5.1, his *idée fixe*. They transcended in significance the resolution of Problem 5.1 and constituted his hoped for "invariant theory of contact transformations." These discoveries will be briefly described in § 6. Their full significance for group theory and history will come out in §§ 7–8.

6. Lie's Invariant Theory of Contact Transformations

The earliest documentary evidence of LIE's interest in Problems 5.1–5.2 is contained in a letter received by MAYER on December 15, 1872. There he wrote: "Soon I will send you a little note on various new matters. These days I am even working on a new theory which seems very noteworthy. In a sense it is a combination of many older theories. The following is an indication: Let

$$F(x_1, \ldots, x_n, p_1, \ldots, p_n) = a$$

be given. I know two functions φ such that $(F\varphi_1) = (F\varphi_2) = 0$. Here I assume that $(\varphi_1\varphi_2) = \text{const.}$ yields nothing new."[36] He then went on to draw conclusions about the integration of $F = 0$ which, a week later, he explained to MAYER were not quite correct. "Nonetheless it is certain that I have made some fairly important discoveries. Here is what is involved: when a number of equations $\varphi_1 = \text{const.}, \ldots, \varphi_q = \text{const.}$ have been found which are in involution with a given equation $F(x_1, \ldots, p_n) = 0$, integrate $F = 0$ in the simplest manner

[36] Quoted by ENGEL on p. 636 of LIE, *Ges. Abh.* **3**. My translation.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

possible."[37] The theory which he had begun to develop to solve this problem was what he was to call his "theory of groups." His terminology reflected the synthetic-conceptual understanding of the matter as presented in Problem 5.2. After he had begun to develop his theory of finite dimensional transformation groups, he occasionally spoke instead of his theory of function groups to distinguish the two theories (*Ges. Abh.* **3**, pp. 30–31)). To avoid confusion in what follows the theory emerging from Problems 5.1–5.2 will be called the theory of function groups. Recently it has been recognized that the theory of function groups represents a significant precursor of the theory of POISSON structures.[38]

Although LIE's theory of function groups was for him a development of his *idée fixe*, and although he initially developed the theory synthetically, in his publications he kept the discussion on the analytical level of Problem 5.1 — the level on which it would be understood and appreciated by mathematicians such as MAYER. In a note submitted to the Christiania Scientific Society on December 20 [1872c] he began to announce some of his results. The following March, he published two fairly extensive papers in which he presented and justified his results analytically [1873a, 1873c]. These papers in fact represented his earliest efforts at a systematic analytical derivation of some of his discoveries in the theory partial differential equations. According to LIE, it was as a result of his correspondence with MAYER and MAYER's urgings that he undertook to develop "an algebraic presentation" of his synthetically conceived results [1874b: 33]. The experience of attempting to translate his synthetic ideas into an "algebraic form" took time and often was painful and frustrating, but he seems to have benefitted from it in ways he could not appreciate but which are directly related to the creation of his theory of groups.

Functionally independent functions $u_1, \ldots, u_r$ of the variables (x, p) which satisfy (5.4), *viz.*

$$(u_i, u_j) = \Omega_{ij}(u_1, \ldots, u_r), \tag{6.1}$$

were said by LIE to form a "closed group" in [1872c] and simply a "group" or an "r-term group" thereafter. The functions $u_1, \ldots, u_r$ associated to a partial differential equation $F(x, p) = 0$ as in Problem 5.1 are thus an example of a function group, although partial differential equations do not figure in the general definition. A function $v = v(x, p)$ was said to belong to the function group $u_1, \ldots, u_r$ if v is functionally dependent on $u_1, \ldots, u_r$, i.e., $v = \Omega(u_1, \ldots, u_r)$ for some function Ω of r variables. LIE realized that if v and w belong to $u_1, \ldots, u_r$, so that $v = \varphi(u_1, \ldots, u_r)$, $w = \psi(u_1, \ldots, u_r)$, then so does (v, w) since by the chain rule

$$(v, w) = \sum_{i,j=1}^{r} \frac{\partial\varphi}{\partial u_i}\frac{\partial\psi}{\partial u_j}(u_i, u_j) = \sum_{i,j=1}^{r} \frac{\partial\varphi}{\partial u_i}\frac{\partial\psi}{\partial u_j}\,\Omega_{ij}(u_1, \ldots, u_r), \tag{6.2}$$

[37] Letter to MAYER received on December 22, 1872, quoted in part by ENGEL (LIE, *Ges. Abh.* **3**, pp. 636–637).

[38] See [OLVER 1986] for the definition of a POISSON structure (p. 380) and the historical connections with LIE's theory of function groups (pp. 417–18). See also [HERMANN 1976: 146–7; WEINSTEIN 1983].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

and hence (v, w) is a function of the u_i. If $s < r$ independent functions $v_1, \ldots, v_s$ belong to the group defined by $u_1, \ldots, u_r$ and if they form a group in their own right so that (v_i, v_j) is a function of $v_1, \ldots, v_s$, then they were said to form an s-term subgroup of $(u_1, \ldots, u_r)$.

In his definitions LIE was following the nineteenth century practice of speaking in terms of representative bases for sets of objects rather than intrinsically in terms of the sets themselves. However, it should be clear that, in effect, the function group $u_1, \ldots, u_r$ is indistinguishable from the totality $\mathscr{F}$ of all functions belonging to it, *i.e.*, functionally dependent on the u_i. To distinguish the r "basis" functions $u_1, \ldots, u_r$ from the function group they define, LIE himself frequently denoted the latter by $(u_1, \ldots, u_r)$. Thus v belongs to $(u_1, \ldots, u_r)$ if and only if $v \in \mathscr{F}$. If this identification is made, then by virtue of (6.2) a function group $\mathscr{F}$, endowed with the POISSON bracket, is a LIE algebra (of infinite dimension): $v, w \in \mathscr{F}$ implies $\alpha v + \beta w \in \mathscr{F}$ and $(v, w) \in \mathscr{F}$. As we shall see in § 7, for LIE these properties of $\mathscr{F}$ reflected the fact that the corresponding set of infinitesimal transformations, $g_{\mathscr{F}}$, defined by (5.5), has the group property. In particular, it should be noted that if $\mathscr{G} = (v_1, \ldots, v_s)$ is a subgroup of $\mathscr{F} = (u_1, \ldots, u_r)$, then g is a subgroup of $g_{\mathscr{F}}$. Except for the telltale terminology, however, LIE suppressed his conceptual understanding of the theory in terms of $g_{\mathscr{F}}$. In fact, upon giving the definition of a function group, he simply remarked that "the group concept belongs in essence to Jacobi; it is useful to have a name for this fundamental concept" [1873c: 34 n. 2].[39]

An especially important subgroup of a given function group for LIE's theory consists of the "distinguished functions" [1873c: 35]. These are the functions v belonging to $(u_1, \ldots, u_r)$ such that $(v, u_i) = 0$ for $i = 1, \ldots, r$ (and hence $(v, u) = 0$ for all $u \in \mathscr{F}$). Evidently they form an m-term subgroup of $(u_1, \ldots, u_r)$, where m denotes the number of functionally independent distinguished functions belonging to $(u_1, \ldots, u_r)$. Distinguished functions have a direct bearing on the integration of partial differential equations. Consider, for example, a single equation $F(x, p) = 0$. Suppose that r solutions $\Phi = u_i$ to $(F, \Phi) = 0$ are known and form a function group $(u_1, \ldots, u_r)$. (This is the context of Problem 5.1.) Let $(F_1, \ldots, F_m)$ denote the subgroup of distinguished functions, where $F_1 = F$ (since F is evidently distinguished). Then the system of equations

$$(6.3) \qquad F_1 = 0, \quad F_2 = C_1, \quad \ldots, \quad F_m = C_{m-1}$$

is in involution since $(F_i, F_j) = 0$ for all i, j. To LIE it was evident that if this system could be completely integrated, then the complete integration of $F = 0$ was also achieved. (On a heuristic level this can be seen as follows. Solutions to the system (6.3) will depend upon the constants $C_1, \ldots, C_{m-1}$. A complete solution $z = \psi$ to (6.3). depends, in addition, upon $n - m + 1$ constants, $C_m, \ldots, C_n$. Thus $z = \psi(x_1, \ldots, x_n, C_1, \ldots, C_n)$ is a complete solution to $F_1(x, p) = 0$.)

[39] However, in an earlier brief note about his results LIE let slip the cryptic remark that the theory "is intimately related to my theory of those differential equations which admit infinitesimal transformations" [1872c: 30]. Many years later he explained that he had chosen the term "group" with the group of infinitesimal transformations $g_{\mathscr{F}}$ expressly in mind [1884b: 112 n. 1].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

LIE had also extended CAUCHY's method to systems in involution [LIE 1872e: 25]. From this extension it followed that the integration of the system (6.3) reduced to the complete integration of the system of linear partial differential equations $(F_i, \Phi) = 0$, $i = 1, \ldots, m$. Since this is a complete system (indeed a Jacobian system), to integrate completely the system $(F_i, \Phi) = 0$ required the determination of $2n - m$ independent solutions. The r functions $\Phi = u_i$ comprising the function group are solutions to this system since the F_i are distinguished. If $r = 2n - m$, then the integration of $F = 0$ is achieved. Of course in general it cannot be expected that $r = 2n - m$. By developing the further properties of function groups, however, LIE was able to show how to replace the given function group with another for which $r = 2n - m$.

LIE had discovered these properties synthetically, as a byproduct of his attempts to understand the true essence of the POISSON-JAACOBI Theorem.[40] In developing the theory analytically, he made extensive, ingenious use of the CLEBSCH-JACOBI theory of complete systems, the theory of which Theorem 3.6 is a part. According to LIE, the "algebraic starting point" of the analytical theory was the fact that if $(u_1, \ldots, u_r)$ is a function group then the system of equations $A_i(f) = (u_i, f) = 0$, $i = 1, \ldots, r$ is a complete system. This follows from JACOBI's Identity (3.8), which, in conjunction with (6.1), shows that

$$A_i(A_j(f)) - A_j(A_i(f)) = ((u_i, u_j), f) = (\Omega_{ij}(u_1, \ldots, u_r), f). \tag{6.4}$$

The completeness property (3.12), then follows by expansion of the POISSON bracket (Ω_{ij}, f) by means of the chain rule.

Using the properties of function groups, LIE showed how, by obtaining solutions to certain systems of ordinary differential equations, the function group $(u_1, \ldots, u_r)$ of Problem 5.1 could be replaced by another, $(u_1^*, \ldots, u_{r^*}^*)$ with the property that $r^* = 2n - m^*$ so that the complete integration of $F = 0$ followed. In this manner he established

Theorem 6.1. *Suppose $F(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0$ is such that r solutions to $(F, \Phi) = 0$ are known which form a function group $\mathscr{F} = (u_1, \ldots, u_r)$. Then the integration of $F = 0$ can be achieved by obtaining one solution each to systems of ordinary differential equations of orders $m - 1, m - 2, \ldots, 3, 2, 1$, and $2n - 2q - 2m - 2, 2n - 2q - 2m - 4, \ldots, 6, 4, 2$, where m denotes the number of distinguished functions of $\mathscr{F}$, and $q = \frac{1}{2}(r - m)$.*

Implicit in the statement of Theorem 6.1 is the fact that for any function group $r - m$ is always even so that q is an integer.

LIE worked out many cases of Theorem 6.1 to show how his results compared with what could be achieved by the best previously known methods. For example, a partial differential equation $F(x, p) = 0$ with $n = 10$ variables

[40] See LIE's remarks [1873c: 36 n. 2, 1874b: 39 n. 1]. His interest in the synthetic essence of the POISSON-JACOBI Theorem is documented in the manuscripts quoted and discussed in § 4 following equation (4.8). ENGEL attempted to reconstruct LIE's synthetic theory of function groups (LIE, *Ges. Abh.* 3, p. 639ff.).

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

$x_1, \ldots, x_{10}$ required according to the best result of the time (Theorem 3.7) one solution each of systems of ordinary differential equations of orders 18, 16, 14, ..., 4, 2. Knowing 8 functionally independent solutions $f = u_1, \ldots, u_8$ to the associated linear equation $(F, f) = 0$ reduceed the orders to: 10, 9, 8, ..., 3, 2, 1. Now suppose $F(x, p) = 0$ admits 8-term function group. Since $m = 8 - 2q$ and $m \geqq 1$, the possibilities are $m = 2, 4, 6, 8$. For $m = 4$, Theorem 6.1 shows that the orders become 4, 3, 2, 1, and 6, 4, 2.

There can be no doubt that LIE regarded Theorem 6.1 in terms of transformation groups, as a development of his *idée fixe*. When solutions to $(F, \Phi) = 0$ are known, as happens for example in problems of celestial mechanics, then the POISSON-JACOBI Theorem shows that without any loss of generality the solutions can be assumed to form a function group $\mathscr{F}$. On the synthetic level this means that $F = 0$ admits known infinitesimal contact transformations $g_{\mathscr{F}}$ with the group property. When the GALOIS group of a polynomial equation $f(x) = 0$ is known, it follows from the composition series of the group just what degree polynomial equations must be resolved in order to resolve $f(x) = 0$. Here when $g_{\mathscr{F}}$ is known it follows from the values of r and m by Theorem 6.1 precisely what orders of integration are required in order to integrate $F = 0$. Admittedly LIE never made this analogy as explicit as this in his publications in 1873, but they were written to be intelligible and acceptable to analysts. It is known that he did conceive of GALOIS' theory as performing the type of service described above. In a letter received by MAYER on 3 February 1874, LIE explained the significance of GALOIS' work by writing: "Before Galois, in the theory of algebraic equations, one only posed the question: Is the equation solvable by radicals and how is it solved? Since Galois, one also poses ... the question: How is the equation solved *most simply* by radicals?" It can be proved *e.g.* that certain equations of degree six are solvable by means of equations of the second and third degrees and not, say, by equations of the second degree.[41] He believed that that by means of Theorem 6.1, he had probably achieved the simplest way to integrate a partial differential equation $F(x, p) = 0$.[42]

Theorem 6.1 represented LIE's solution to Problems 5.1–5.2, but his theory of function groups transcended the problem which originally motivated it and became the core of his invariant theory of contact transformations. Theorem 6.1 was not the only part of this theory that is relevant to his creation of the theory of transformation groups. As indicated in § 1, LIE was interested in

Problem 6.1. *Given two systems of equations* $F_i(x, z, p) = 0, \quad i = 1, \ldots, r,$ *and* $G_i(x', z', p') = 0$, $i = 1, \ldots, r$, *when does a contact transformation* $T\colon (x, z, p) \to (x', z', p')$ *exist which transforms the one system into the other in the sense that for every* i, $G_i = F_i \circ T^{-1}$?

For $r = 1$, LIE had discovered the answer is "always" by virtue of the theorem that any two first order partial differential equations are transformable into one

[41] Letter quoted by ENGEL in LIE, *Ges. Abh.* **5**, p. 586. This quotation and its remaning were discussed in § 1.

[42] See [LIE 1874b: 3]. *Cf.* [LIE, S. 1876a: 163, 232ff.].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

another by means of a contact transformation (§ 5). Consequently a single equation has no special properties that are invariant under contact transformations. For $r > 1$, invariant properties exist, such as the property of being in involution.

A special case of Problem 6.1 arises when the functions F_i and G_i define r-term function groups. Since in this case the F_i and G_i do not involve the dependent variables z, z', the requisite contact transformations will have equations of the form

$$(6.5) \quad x_i' = X_i'(x, p), \quad z' = Z(x, z, p), \quad p_i' = P_i(x, p), \quad i = 1, \ldots, n.$$

A contact transformation T with equations of the form (6.5) takes the function group $\mathscr{F} = (F_1, \ldots, F_r)$ into another function group

$$\mathscr{F}' = \{G = F \circ T^{-1} : F \in \mathscr{F}\}.$$

If m' denotes the number of independent distinguished functions of $\mathscr{F}'$, then $m' = m$. For LIE the equality $m' = m$ was an immediate consequence of the invariance of involution under contact transformations. This means that $(v, u)_{(x,p)} = 0$ if and only if $(v', u')_{(x',p')} = 0$. Thus $v \in \mathscr{F}$ is distinguished if and only if the corresponding $v' \in \mathscr{F}'$ is distinguished and so $m' = m$. In this sense, the number m is an invariant of the r-term function group $\mathscr{F}$. In [1873c] he established the importance of this invariant by proving

Theorem 6.2. *Let $\mathscr{F}$ and $\mathscr{F}'$ denote r-term function groups in variables (x, p) and (x', p'), respectively. If m, m' denote the number of functionally independent distinguished functions of $\mathscr{F}$, $\mathscr{F}'$, respectively, then a contact transformation $T: (x, p) \to (x', p')$ exists which takes $\mathscr{F}$ into $\mathscr{F}'$ if and only if $m = m'$.*

He regarded Theorem 6.2 "extremely important", the most important result of his paper on function groups [1873c: 36, 49–50].

Applied to Problem 6.1, Theorem 6.2 shows that for systems of equations forming r-term function groups, a necessary condition for the desired transformation to exist is that $m = m'$. If $m = m'$, however, Theorem 6.2 simply establishes the existence of a contact transformation T which transforms the totality $\mathscr{F}$ into $\mathscr{F}'$; T need not take the individual functions F_i into the given functions G_i as required by Problem 6.1. In March 1873, when LIE submitted his papers on function groups, he had not yet solved Problem 6.1. It was not until the following November that he was able to announce a solution to MAYER.[43] Although he sketched the solution (for $n = 2$) in his letter, due to the interruption caused by his effort to create the theory of continuous groups during of the winter of 1873–74, it was not until July of 1874 that he presented it for publication [1874b: 60–63]. Judging by that presentation, it would seem that the key to the solution was provided by his experiences developing the theory of contact transformations analytically.

The fruits of LIE's efforts to create an analytical theory of contact transformations were submitted for publication in June 1873 [1873b]. Concerning this work, he wrote to MAYER: "the analytical edition of my work on contact transforma-

---

[43] Letter received by MAYER on 10 November 1873. Quoted in part by ENGEL on p. 513 of LIE, *Ges. Abh.* **4**.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

tions has been terribly difficult. To be sure, I have thereby made new discoveries."[44] One of the matters he explored in detail was the type of contact transformations that occur in the theory of function groups, that is, contact transformations with equations of the form (6.5) [1873b: 104–116], which, following LIE, will be referred to as *contact transformations of* (x, p). His main results may be summarized as follows:

Theorem 6.3. (1) *Contact transformations of* (x, p) *are all of the form* T_A: $(x, z, p) \to (x', z', p')$ *where* $A \neq 0$ *is constant and*

$$x'_i = X_i(x, p), \quad z' = Az + \Pi(x, p), \quad p'_i = P_i(x, p). \tag{6.6}$$

(2) *Functions* $X_i = X_i(x, p)$, $P_i = P_i(x, p)$ *determine a contact transformation* T_A *of* (x, p) *with equations of the form* (6.6) *if and only if for all* $i, j = 1, \ldots, n$:

$$(X_i, X_j) = 0, \quad (X_i, P_j) = 0, \quad (P_i, P_j) = 0, \quad (P_i, X_i) = A. \tag{6.7}$$

An immediate consequence of the relations (6.7) and the chain rule is that if $F_i = G_i \circ T_A$, then $(F_i, F_j)_{(x,p)} = A(G_i, G_j)_{(x',p')}$ Thus the contact transformations T_A with $A = 1$ leave the POISSON bracket invariant: $(F_i, F_j)_{(x,p)} = (G_i, G_j)_{(x',p')}$, or, equivalently, $(G_i \circ T, G_j \circ T)_{(x,p)} = (G_i, G_j)_{(x',p')}$. LIE realized that, as transformations $(x, p) \to (x', p')$, they are the canonical transformations of mechanics. Indeed, by 1872 he had learned that consideration of the canonical transformations of Hamiltonian mechanics had led Jacobi, in the pages of "Nova methodus". to introduce the equivalent of the notion of a contact transformation, albeit in a different, exclusively analytical form]1862: § 57 ff]. In considering Problem 6.1 for function groups, he restricted his attention to the transformations T_A with $A = 1$. These transformations will be referred to as *canonical transformations of* (x, p), or simply as canonical transformations.

With these results from the analytical theory of contact transformations in mind, consider the question of when two systems $F_i(x, p) = 0$, $u = 1, \ldots, r$, and $G_i(x', p') = 0$, $i = 1, \ldots, r$, such that $(F_1, \ldots, F_r)$, $(G_1, \ldots, G_r)$ are function groups are equivalent under some canonical transformation of (x, p). Let $(F_i, F_j) = \Omega_{ij}(F_1, \ldots, F_r)$, and $(G_i, G_j) = \Omega'_{i,j}(G_1, \ldots, G_r)$. If a canonical transformation T exists such that $G_i \circ T = F_i$, then, in terms of functions of the variables (x, p):

$$\Omega_{ij}(F_1, \ldots, F_r) = (F_i, F_j) = (G_i, G_j) \circ T = \Omega'_{i,j}(G_1, \ldots, G_r) \circ T = \Omega'_{ij}(F_1, \ldots, F_r),$$

or, in terms of (x', p'), $\Omega_{ij}(G_1, \ldots, G_r) = \Omega'_{ij}(G_1, \ldots, G_r)$. As LIE put it: (G_i, G_j) is the same function of $G_1, \ldots, G_r$ as (F_i, F_j) is of $F_1, \ldots, F_r$. This is a necessary condition for one system of equations forming a function group to be transformable by a canonical transformation T into another such system. Using results from the theory of function groups as developed in [1873c], he was able to show that it is also sufficient:

[44] Letter quoted in part by ENGEL on pp. 650–51 of LIE, *Ges. Abh.* 3.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

Theorem 6.4. *Two function groups $(F_1, \ldots, F_r)$ in (x, p) and $(G_1, \ldots, G_r)$ in (x', p') are transformable into one another by a canonical transformation T in the sense that $F_i = G_i \circ T$ for all i if and only if for all $i, j (G_i, G_j)_{(x',p')}$ is the same function of $G_1, \ldots, G_r$ as $(F_i, F_j)_{(x,p)}$ is of $F_1, \ldots, F_r$.*

Theorem 6.4 enabled LIE to establish necessary and sufficient conditions such that any system of functions of (x, p) be transformable by a canonical transformation T into any other system of functions of (x', p').[45] He had thus solved a problem which he regarded as "of the utmost importance in my theory of invariants."[46] The solution made it possible to determine when two systems of partial differential equations not containing the dependent variable can be transformed into one another by a canonical transformation of (x, p).

Not long after LIE communicated Theorem 6.4 to MAYER in November 1873 — and possibly even by the time of communication — he realized that the theorem is also relevant to the problem of determining all finite dimensional groups of transformations in n variables, the problem that launched him into his investigations on the theory of continuous groups. Theorem 6.4 played a key role in the considerations that encouraged him to attack the group classification problem for transformations in any number of variables. The remainder of this section is devoted to indicating the mathematical context within which Theorem 6.4 played this role, namely the theory of homogeneous function groups.

LIE's theory of function groups was applicable to the problem of integrating differential equations of the form $F(x, p) = 0$, that is, equations that do not explicitly involve the dependent variable z. This had been the context of the work of JACOBI and his successors. JACOBI had justified the restriction by showing that the integration of $F(x, z, p) = 0$ can be reduced to that of an equation of the form $F(x, p) = 0$, but his demonstration was shown to be flawed by BERTRAND. In his essay, IMSCHENETSKY presented the following considerations to justify the reduction [1869: § 14]. Given $F(x, z, p) = 0$, express any solution in the form $\gamma(x_1, \ldots, x_n, z) = 0$. Differentiation of this equation with respect to x_i yields

$$\frac{\partial\gamma}{\partial x_i} + \frac{\partial\gamma}{\partial z} p_i = 0 \quad \text{or} \quad p_i = -\frac{\dfrac{\partial\gamma}{\partial x_i}}{\dfrac{\partial\gamma}{\partial z}}$$

and therefore

$$F(x, z, p) = F\left((x_1, \ldots, x_n, z, -\frac{\dfrac{\partial\gamma}{\partial x_1}}{\dfrac{\partial\gamma}{\partial z}}, \ldots, -\frac{\dfrac{\partial\gamma}{\partial x_n}}{\dfrac{\partial\gamma}{\partial z}}\right)$$

$$\stackrel{\text{def}}{=} G\left(x_1, \ldots, x_n, z, \frac{\partial\gamma}{\partial x_1}, \ldots, \frac{\partial\gamma}{\partial x_n}, \frac{\partial\gamma}{\partial z}\right) = 0.$$

[45] See Theorem XIV [LIE 1874b: 62–3].

[46] Letter to MAYER (received 10 November 1873), quoted by ENGEL in LIE, *Ges. Abh.* **4**, p. 513.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

The equation $G = 0$ is a first order partial differential equation in the $n + 1$ independent variables $x_1, \ldots, x_n, z$ which does not explicitly involve the dependent variable y. Thus JACOBI's new method applies to $G = 0$ and yields a complete solution, which (as indicated in § 3, note 13) has the special form $y = \varphi(x_1, \ldots, x_n, z, C_1, \ldots, C_n) + C_{n+1}$. A straightforward calculation shows that the equation $\varphi(x_1, \ldots, x_n, z, C_1, \ldots, C_n) = 0$ implicitly defines a complete solution to the original equation $F(x, z, p) = 0$. Thus, as JACOBI had claimed, it suffices to develop a theory of integration for equations of the form $F(x, p) = 0$.

LIE was familiar with these considerations by IMSCHENETSKY.[47] They implied that he was justified in developing his theory of function groups, which relates exclusively to partial differential equations of the form $F(x, p) = 0$. But to LIE, who approached differential equations with the background and inclinations of a geometer, the above considerations had another implication, which in fact motivated him to develop another theory of function groups. IMSCHENETSKY's reduction procedure involves transforming the original partial differential equation $F(x, z, p) = 0$ by means of the change of variable

$$(6.8) \quad x_i = y_i \quad (i = 1, \ldots, n), \qquad z = y_{n+1}, \qquad p_i = \frac{-q_i}{q_{n+1}} \quad (i = 1, \ldots, n),$$

where $q_i = \dfrac{\partial y}{\partial x_i}$, $q_{n+1} = \dfrac{\partial y}{\partial z}$. By means of (6.8) $F(x, z, p) = 0$ is transformed into $G(y, q) = 0$. The change of variables (6.8) became the starting point for LIE's theory of homogeneous function groups [1873a], created largely in February 1873.[48]

The change of variables (6.8) formally resembles the change of variables $p_i = \dfrac{q_i}{q_{n+1}}$ from Cartesian coordinates $p_1, \ldots, p_n$ to homogeneous coordinates $q_1, \ldots, q_{n+1}$. The change to homogeneous coordinates was of course fundamental to projective geometry. It makes any polynomial in the p_i correspond to a homogeneous polynomial in the q_i, and it makes any projective transformation of the $p_i \left(p_i' = \dfrac{a_{i1} p_1 + \ldots + a_{in} p_n + a_{in+1}}{a_{n+11} p_1 + \ldots + a_{n+1n} p_n + a_{n+1n+1}} \right)$ correspond to a linear homogeneous transformation of the q_i ($q_i' = a_{i1} q_1 + \ldots + a_{in+1} q_{n+1}$). By virtue of these correspondences, the algebraic aspects of projective geometry can be developed with greater facility and elegance in terms of homogeneous coordinates. Given LIE's background in projective geometry and his expressed interest in applying the ideas of geometry to advance the theory of differential equations, it is likely that the analogy of (6.8) with the change to homogeneous coordinates encouraged him to see if (6.8) enables the analytic aspects of the theory of contact transformations and function groups to be developed with greater facility and elegance. Certainly this is in effect what he discovered, and the fact that he referred to the resultant theory as his theory of homogeneous contact transformations and function groups suggests that he appreciated the analogy.

---

[47] In presenting his theory of homogeneous function groups, LIE expressly referred to § 14 of "das wertvolle Werk von Imschenetsky" [1973a: 64 n. 1].

[48] See ENGEL's remarks on pp. 647–8 of LIE, *Ges. Abh.* 3.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

Consider, for example, the function G into which F is transformed by (6.8). It turns out that G is a homogeneous function of degree zero in the variables q_i, provided the notion of a homogeneous function is interpreted in a manner suitable to the consideration of differential rather than polynomial equations. Accordingly a function $G(y, q)$ is said to be *homogeneous of degree h in the* q_i if h is a nonnegative integer such that G satisfies the EULER condition $\sum_{i=1}^{n+1} q_i \, \partial G/\partial q_i = hG$. It is easily verified that (6.8) transforms any partial differential equation $F(x, z, p) = 0$ into a partial differential equation $G(y, q) = 0$ for which G is homogeneous in the q_i of degree 0 in this generalized sense. Thus there is an analogy with the change to homogeneous coordinates in projective geometry.

LIE discovered the analogy goes further. Just as projective transformations are the transformations which define projective geometry in the sense of KLEIN's Erlangen Program, contact transformations define LIE's approach to partial differential equations through invariant theory. Projective transformations correspond in homogeneous coordinates to linear homogeneous transformations, which take homogeneous polynomials of a given degree into the same and are generally more amenable to algebraic treatment. LIE considered what type of transformations correspond, through use of (6.8), to contact transformations. If

$$x_i' = X_i(x, z, p), \quad z' = Z(x, z, p), \quad p_i' = P_i(x, z, p) \tag{6.9}$$

denotes the general contact transformation of the $2n + 1$ variables (x, z, p) then under the variable change (6.8), (6.9) takes the form

$$y_i' = Y_i(y, p), \quad q_i' = Q_i(y, q), \quad i = 1, \dots, n + 1, \tag{6.10}$$

where Y_i and Q_i are of homogeneous of degrees 0 and 1, respectively, in the variables q_i. Moreover, the defining condition that the transformation (6.9) be a contact transformation, namely

$$dz' - (p_1' \, dx_1' + \dots + p_n' \, dx_n') = \varrho(x, z, p) \, [dz - (p_1 \, dx_1 + \dots + p_n \, dx_n)],$$

becomes

$$q_1' \, dy_1' + \dots + q_{n+1}' \, dy_{n+1}' = q_1 \, dy_1 + \dots + q_{n+1} \, dy_{n+1}. \tag{6.11}$$

Equation (6.11) says that the transformation (6.10) is also a contact transformation. That is, if w is introduced as dependent variable and if the equation $w' = w$ is added to those of (6.10), then since (6.11) implies

$$dw' - (q_1' \, dy_1' + \dots + q_{n+1}' \, dy_{n+1}) = dw - (q_1 \, dy_1 + \dots + q_{n+1} \, dy_{n+1}),$$

we have a contact transformation $(y, w, q) \to (y', w', q')$ which, in LIE's parlance, is a transformation of (y, q). Moreover it is evident that, in the notation of Theorem 6.3, $A = 1$ for these transformations, so that they are what have been termed canonical transformations of (y, q). The transformation (6.10) is somewhat special since the functions Y_i and Q_i defining it are homogeneous in the q_i of degrees 0 and 1, respectively; as a result q-homogeneous functions of (y, q) of degree h are taken into q'-homogeneous functions of (y', q') of degree h. The analogy with homogeneous polynomials and linear transformations is thus clear. It is not surprising that LIE termed the transformations (6.10) *homogeneous contact transformations*. It turns out that the homogeneity conditions on the Y_i and Q_i

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

actually follow from the fact that $w' = w$. That is, the homogeneous contact transformations are precisely those canonical transformations which fix w.[49] Every contact transformation in $2n + 1$ variables (x, z, p) thus corresponds to a homogeneous contact transformation in $2n + 2$ variables (y, q). Since, as LIE showed, every homogeneous contact transformation in $2n + 2$ variables corresponds in similar fashion to a contact transformation in $2n + 1$ variables, he regarded the theory of homogeneous contact transformations as equivalent to the general theory of contact transformations and consequently as especially important.[50]

Expressed in terms of the (x, p) variable notation, the above considerations imply that the theory of partial differential equations $F(x, p) = 0$, where F is homogeneous of degree 0 in the p_i, and of their transformation under homogeneous contact transformations of (x, p) is worthy of special attention because of its relevance to the integration of first order partial differential equations involving the dependent variable. In this connection LIE developed his theory of *homogeneous function groups* [1873a]. These are function groups $\mathscr{F}$ with the property that there exist p-homogeneous functions $h_1, \ldots, h_r \in \mathscr{F}$ such that $\mathscr{F} = (h_1, \ldots, h_r)$. (Of course, not every function which belongs to a homogeneous group $\mathscr{F}$ need be homogeneous.) The theory of homogeneous function groups was the core of his invariant theory of homogeneous contact transformations. Except for the fact that these function groups possess an additional invariant, their theory is analogous to the general theory of function groups.[51] In particular, the analog of Theorem 6.4 holds:

Theorem 6.5. *Let* $\mathscr{F} = (F_1, \ldots, F_r)$ *and* $\mathscr{G} = (G_1, \ldots, G_r)$ *denote homogeneous function groups, where the* F_i *are p-homogeneous functions of* (x, p) *and the* G_i *are p'-homogeneous functions of* (x', p'). *They are transformable into one another by a homogeneous contact transformation T in the sense that* $F_i = G_i \circ T$ *for all i if and only if for all i, j,* $(G_i, G_j)_{(x',p')}$ *is the same function of* $G_1, \ldots, G_r$ *as* $(F_i, F_j)_{(x,p)}$ *is of* $F_1, \ldots, F_r$.

By November of 1873 LIE knew Theorems 6.4 and 6.5. By that time he had discovered that the development of the analytical theory of infinitesimal contact transformations is also facilitated by consideration of homogeneous transformations. In particular, the correspondence between infinitesimal contact transformations and their characteristic functions established by Theorem 5.1 takens on a particularly simple analytical form in the homogeneous case (Theorem 7.5). It is the analytical form that makes Theorem 6.5 relevant to the problem of classifying finite dimensional transformation groups.

[49] See [LIE, S. 1890: 134–137]. In [LIE 1873b: §4, 1874b: §5] a somewhat more general definition of a homogeneous contact transformations was given (so that $w' = Aw + B$), but in 1873–74 it was the above type that were of primary interest to him.

[50] This opinion, which is implicit in his publications on homogeneous function groups, is made explicit in his paper on the analytical theory of contact transformations [1873b: 116], although he nowhere explained it fully. See also [LIE 1874b: 24].

[51] The number m' of independent distinguished functions which are homogeneous in the p_i of degree 0 is also an invariant. Either $m' = m$ or $m' = m - 1$ [1873a: 92]. Corresponding to the two possibilities for m' are two canonical forms for a homogeneous function group.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

7. The Calculus of Infinitesimal Transformations

When LIE had occasion to think about infinitesimal transformations analytically, as for example, in his joint work with KLEIN on W-curves and surfaces (1870), he identified such a transformation $dT: y \to y + dy$, where $dy_i = \eta_i(y)\, dt$, $i = 1, \ldots, m$, with the system of ordinary differential equations, $\frac{dy_i}{dt} = \eta_i(y)$, $i = 1, \ldots, m$ as explained in connection with equation (1.1). For anyone immersed in the theory of first order partial differential equations, as LIE was in 1872–73, it was natural to identify this system with the partial differential equation $X(f) = \sum_{i=1}^{m} \eta_i(y) \frac{\partial f}{\partial y_i} = 0$ whose integration was tantamount to that of the system (as explained in § 2). Indeed, this identification had inspired his fundamental Theorem 5.1, which he knew by April 1872. By the fall of 1873, he had decided to denote dT by $X(f)$. Part of the motivation for this notational innovation came from observations he had already made by the end 1872. By that time he had discovered several analytical properties of infinitesimal transformations that are succinctly expressible in terms of the JACOBI operator notation $X(f)$, as he eventually observed.

For example, in 1872 LIE considered an equation $F(y_1, \ldots, y_m) = 0$ which admits an infinitesimal transformation dT.[52] He interpreted this to mean that if y is a solution to $F = 0$ then so is $dT(y) = y + dy$, which means that $F(y) = 0$ implies $F(y + dy) = 0$. Since $F(y + dy) = F(y) + \sum_{i=1}^{m} \frac{\partial F}{\partial y_i} dy_i + \ldots = F(y) + \sum_{i=1}^{m} \frac{\partial F}{\partial y_i} \eta_i(y)\, dt + \ldots$, he concluded that $\sum_{i=1}^{m} \frac{\partial F}{\partial y_i} \eta_i(y) = 0$. For brevity he denoted the last equality by $\frac{dF}{dt} = 0$. Thus he could conclude that $F(y) = 0$ admits dT when $F = 0$ implies $\frac{dF}{dt} = 0$. Expressed in differential operator notation, the conclusion is that $F(y) = 0$ implies $X(F) = 0$. Analogous reasoning applies to a linear homogeneous partial differential equation $A(f) = \sum_{i=1}^{m} \alpha_i(y) \frac{\partial f}{\partial y_i} = 0$ which admits the infinitesimal transformation dT. If $f(y)$ is a solution to $A(f) = 0$, then so is $f \circ dT = f(y + dy) = f(y) + \sum_{i=1}^{m} \frac{\partial f}{\partial y_i} \eta_i(y)\, dt + \ldots$. Thus $0 = A(f(y + dy)) = A(f) + A\left(\sum_{i=1}^{m}\left(\frac{\partial f}{\partial y_i}\eta_i(y)\right)\right) dt + \ldots = A\left(\sum_{i=1}^{m}\left(\frac{\partial f}{\partial y_i}\eta_i(y)\right)\right) dt + \ldots$ Expressed in differential operator notation, the equation is $A(X(f)) = 0$ so that $A(f) = 0$ implies $A(X(f)) = 0$. $A(f) = 0$ also implies that $X(A(f)) = X(0) = 0$ and therefore that $A(X(f)) - X(A(f)) = 0$. He had realized the last differential operator relation by 1872 (see below), although he had not

[52] See document V of LIE, *Ges. Abh.* 7 (p. 90). Regarding the dating of this document see p. 93.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

yet decided to employ differential operator notation systematically to express analytic results pertaining to infinitesimal transformations. For future reference, however, the above results will be summarized in the differential operator notation as

Theorem 7.1. (i) $F(y) = 0$ *admits the infinitesimal transformation* dT *given by* $dy_i = \eta_i(y)\,dt$ *if and only if* $F(y) = 0$ *implies* $X(F) = 0$, *where* $X(f) = \sum_{i=1}^{m} \eta_i(y) \frac{\partial f}{\partial y_i}$. (ii) *If* $A(f) = \sum_{i=1}^{m} \alpha_i(y) \frac{\partial f}{\partial y_i} = 0$ *admits* dT, *then* $A(f) = 0$ *implies* $A(X(f)) - X(A(f)) = 0$.

There is documentary evidence that by the end of 1872 LIE had also discovered the equivalent of

Theorem 7.2. *If* dU *is a second infinitesimal transformation defined by* $dy_i = \mu_i(y)\,dt$, *then the non-infinitesimal transformations generated respectively by* dT *and* dU *commute when the operators* $X(f)$ *and* $Y(f) = \sum_{i=1}^{m} \mu_i(y) \frac{\partial f}{\partial y_i}$ *which correspond to* dT *and* dU, *respectively, satisfy* $X(Y(f)) - Y(X(f)) = 0$.

Although there are no documents from 1872 that indicate how he might have discovered this theorem, he probably proceeded by a line of reasoning similar to what he indicated to MAYER in a letter of April 1874 [LIE 1873–4: 600–601] and which reappears in later publications in connection with this and related theorems [1874:e 3, 1876c: 51–2, 1878a: 85, 1884b: 114–5]. The reasoning proceeds as follows. If the infinitesimal transformation dT is applied continually to a point y, then it generates the ∞^1 "finite transformations" $y' = T_t(y)$ which represent the solution $y'_i = f_i(y, t)$, $i = 1, \ldots, m$, to the system of ordinary differential equations $\frac{dy'_i}{dt} = \eta_i(y')$ satisfying the initial condition: $y' = y$, when $t = t_0$. Consider the TAYLOR expansion of $y'_i = f_i(y, t)$ with respect to the variable t:

$$y'_i = f_i(y, t) = f_i(y, 0) + \frac{\partial f_i}{\partial t}(y, 0)\, t + \frac{\partial^2 f_i}{\partial t^2}(y, 0)\, t^2 + \ldots .$$

Since calculation shows that $\frac{\partial f_i}{\partial t}(y, 0) = \eta_i(y)$ and $\frac{\partial f_i}{\partial t}(y, 0) = \sum_{k=1}^{m} \frac{\partial \eta_i}{\partial y_k}(y)\eta_k(y)$,[53] the series expansion may be written as

$$y'_i = y_i + \eta_i(y)\, t + \left[\sum_{k=1}^{m} \eta_k(y) \frac{\partial \eta_i}{\partial y_k}\right] \frac{t^2}{2} + \ldots . \tag{7.1}$$

The infinitesimal transformation dT is given by the expansion (7.1) with t infinitely small, say $t = dt$, so that, neglecting higher order infinitesimals, $y'_i = y_i + \eta_i(y)\,dt$ or $dy_i = y'_i - y_i = \eta_i(y)\,dt$.

[53] $\frac{\partial f_i}{\partial t}(y, 0) = \left[\frac{dy'_i}{dt}\right]_{t=0} = [\eta_i(y')]_{t=0} = \eta_i(y)$. Hence $\frac{\partial^2 f_i}{\partial t^2}(y, 0) = \left[\frac{\partial}{\partial t}\left(\frac{\partial f_i}{\partial t}(y, t)\right)\right]_{t=0}$ $= \left[\frac{\partial \eta_i}{\partial t}(y')\right]_{t=0} = \left[\sum_{k=1}^{m} \frac{\partial \eta_i}{\partial y_k}(y') \frac{dy'_k}{dt}\right]_{t=0} = \left[\sum_{k=1}^{m} \frac{\partial \eta_i}{\partial y_k}(y')\, \eta_k(y')\right]_{t=0} = \frac{\partial \eta_i}{\partial y_k}(y)\, \eta_k(y).$

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

Consider now the ∞^1 transformations $y'' = U_s(y')$ generated by dU with analogous series expansion

$$(7.2)\qquad y_i'' = y_i' + \mu_i(y')\,s + \left[\sum_{k=1}^{m} \mu_k(y') \frac{\partial \mu_i}{\partial y_k}(y')\right] \frac{s^2}{2} + \dots.$$

Then the expansion for $y'' = U_s(y') = U_s(T(y))$ is obtained by substituting (7.1) in (7.2) and expanding terms such as $\mu_k(y') = \mu_k(y + \eta_k(y)\,t + \dots)$ in TAYLOR expansions about y. After simplification, the result up to terms of the second order, is

(7.3)

$$y_i'' = y_i + \eta_i t + \mu_i s + \left[\sum_{k=1}^{m} \eta_k \frac{\partial \eta_i}{\partial y_k}\right] \frac{t^2}{2} + \left[\sum_{k=1}^{m} \mu_k \frac{\partial \mu_i}{\partial y_k}\right] \frac{s^2}{2} + \left[\sum_{k=1}^{m} \frac{\partial \eta_i}{\partial y_k} \mu_k\right] ts + \dots,$$

where the functions η_j, μ_j and their derivatives are all evaluated at y. Similarly, the series expansion for $y^{**} = T_t(U_s(y))$ is

(7.4)

$$y_i^{**} = y_i + \eta_i t + \mu_i s + \left[\sum_{k=1}^{m} \eta_k \frac{\partial \eta_i}{\partial y_k}\right] \frac{t^2}{2} + \left[\sum_{k=1}^{m} \mu_k \frac{\partial \mu_i}{\partial y_k}\right] \frac{s^2}{2} + \left[\sum_{k=1}^{m} \frac{\partial \mu_i}{\partial y_k} \eta_k\right] ts + \dots.$$

Comparison of (7.3) and (7.4) shows that these expressions for y_i'' and y_i^{**} differ in the ts term. If T_t and U_s commute so that $y'' = U_s(T_t(y)) = T_t(U_s(y)) = y^{**}$ for all y, then the two ts terms must agree for all y and for all i, which means that

$$(7.5)\qquad \sum_{k=1}^{m} \left(\eta_k \frac{\partial \mu_i}{\partial y_k} - \mu_k \frac{\partial \eta_i}{\partial y_k}\right) = 0, \quad i = 1, \dots, m.$$

For anyone (such as LIE) familiar with JACOBI's calculus of differential operators (7.5) states that $X(\mu_i) - Y(\eta_i) = 0$, where $X(f) = \Sigma_{i=1}^{m} \eta_i(y) \dfrac{\partial f}{\partial y_i}$, $Y(f) = \Sigma_{i=1}^{m} \mu_i(y) \dfrac{\partial f}{\partial y_i}$. Since, as JACOBI had shown in "Nova methodus",

$$(7.6)\qquad X(Y(f)) - Y(X(f)) = \sum_{i=1}^{m} (X(\mu_i) - Y(\eta_i)) \frac{\partial f}{\partial y_i},$$

it follows that (7.5) means that $X(Y(f)) - Y(X(f)) \equiv 0$. By this sort of reasoning LIE probably arrived at Theorem 7.2.

According to LIE's recollections [1884b: 114–15], he also realized in 1872 the following further consequence of the above reasoning:

Theorem 7.3. *Let dT and dU be infinitesimal transformations belonging to a group G in the sense that the non-infinitesimal transformations they generate belong to G. Let dT be defined by $dy_i = \eta_i(y)\,dt$ and dU by $dy_i = \mu_i(y)\,dt$, and let the corresponding differential operators be, respectively, $X(f) = \sum_{i=1}^{m} \eta_i(y) \dfrac{\partial f}{\partial y_i}$, and $Y(f) = \sum_{i=1}^{m} \mu_i(y) \dfrac{\partial f}{\partial y_i}$. Then the infinitesimal transformation defined by the system*

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

of ordinary differential equations corresponding to $Z(f) = X(Y(f)) - Y(X(f))$ *also belongs to* G.

Since Theorem 7.3 follows readily from the considerations outlined in connection with Theorem 7.2, and since there is documentary evidence that LIE knew Theorem 7.2 in 1872, his recollection that he knew Theorem 7.3 as well in 1872 seems reliable. Indeed, at the time of these recollections, he proved Theorem 7.3 in a manner that fits in well with the above considerations. He wrote down the equivalent of equations (7.3) and (7.4) and concluded that "it is now clear that the infinitesimal transformation which brings the system of values $[y_i'']$ into $[y_i^{**}]$ belongs to our group" [1884b: 114–5]. Since $y^{**} = T_t U_s [U_s T_t]^{-1} (y'')$, the infinitesimal transformation to which he was referring is the above commutator with $t = dt$ and $s = ds$. As he observed, it follows from (7.3) and (7.4) that

$$(7.7)\quad y_i^{**} = y_i'' + \left[\sum_{k=1}^{m} \eta_k \frac{\partial \mu_i}{\partial y_k} - \mu_k \frac{\partial \eta_i}{\partial y_k}\right] dt\, ds = y_i'' + [X(\mu_i) - Y(\eta_i)]\, dt\, ds.$$

The infinitesimal transformation (7.7), which may be denoted by

$$dy_i = [X(\mu_i) - Y(\eta_i)]\, d\tau,$$

corresponds to the differential operator $X(Y(f)) - Y(X(f))$ by virtue of (7.6), and so Theorem 7.3 follows.

Another theorem about infinitesimal transformations which was certainly familiar to LIE during his geometrical period (1869–71) is

Theorem 7.4. *If infinitesimal transformations* $dy_i = \eta_i(y)\, dt$ *and* $dy_i = \mu_i(y)\, ds$ *belong to a group, then so do all infinitesimal transformations* $dy_i = (a\eta_i + b\mu_i)\, du$, *that is, all transformations corresponding to the differential operators* $aX(f) + bY(f)$.

As noted in § 5, LIE regarded $dy_i = (a\eta_i + b\mu_i)\, du$ as representing the "composite" $(dT)^a \circ (dU)^b$. In terms of (7.3), the infinitesimal transformation corresponding to $y' = U_t(T_s(y))$, with $t = dt$ and $s = ds$ infinitely small is $y' = y + (\eta\, dt + \mu\, ds)$, which may be written as $dy = (a\eta + b\mu)\, du$, where $dt : ds = a : b$. Here higher order infinitesimals have been ignored, as was the customary practice. To LIE, Theorem 7.4 was thus an immediate consequence of the group closure property. All mathematicians who had occasion to consider infinitesimal transformations (or motions) regarded $dy_i = (\eta + \mu)\, du$ as corresponding to the composite of $dy_i = \eta_i\, dt$ and $dy_i = \mu\, ds_i$. In this regard LIE was no exception, and since for him dT and dU belonged to the group G it "followed" that the composite, *viz.* $dy_i = (a\eta_i + b\mu_i)\, du$ would also belong to G. What distinguished LIE from most other mathematicians was his observation of Theorem 7.3, which reflects at the infinitesimal level that the composition of the non-infinitesimal transformations of G might not commute.[55]

The analytical properties of infinitesimal transformations as expressed in the above theorems appear to have been developed in connection with LIE's interest in working out his *idée fixe*. For example, in a document written in 1872, he stated

[54] Independently of LIE, WILHELM KILLING also discovered the equivalent of Theorem 7.3. See [HAWKINS, T. 1980].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

and proved Part (i) of Theorem 7.1 and apparently took for granted Part (ii) as well as Theorem 7.2. This occurred in his consideration of a partial differential equation $A(f) = \sum_{i=1}^{m} \alpha_i(y) \frac{\partial f}{\partial y_i} = 0$ assumed to admit q commuting infinitesimal transformations corresponding to differential operators $X_k(f) = \sum_{i=1}^{m} \xi_i(y) \frac{\partial f}{\partial y_i}$, $k = 1, \ldots, q$.[55] From these assumptions he concluded at once that the system of $q+1$ equations $A(f) = 0$, $X_1(f) = 0, \ldots, X_q(f) = 0$ is Jacobian, and then applied results on Jacobian systems by himself and MAYER [MAYER 1872b: 465–6]. Recall that the equations $A(f) = 0$, $X_1(f) = 0, \ldots, X_q(f) = 0$ form a Jacobian system when they all commute: (1) $A(X_k(f)) - X_k(A(f)) \equiv 0$, $k = 1, \ldots, q$ and (2) $X_k(X_l(f)) - X_l(X_k(f)) \equiv 0$ for all k, l. Now (2) follows from Theorem 7.2. As for (1), it "almost" follows from Part (ii) of Theorem 7.1, which implies that $A(X_k(f)) - X_k(A(f)) = 0$ provided $A(f) = 0$. It would seem that he was invoking this theorem, although it is unclear whether his conclusions about the integration of $A(f) = 0$ follow given that the system is not Jacobian in the strict sense of that word. In his subsequent study of complete systems of equations $A_i(f) = 0$, he made a more consequential application of Part (ii) of Theorem 7.1, as will be seen in § 8. As for Theorem 7.3, the first significant use he made of it appears to have occurred in the fall of 1873, when he discovered an analytical counterpart to his Theorem 5.1 associating a partial differential equation $W = 0$ to an infinitesimal contact transformation $d\omega$. This discovery, which he communicated to MAYER in November 1873, and its immediate analytical consequences enabled him to perceive the "conceptual" basis (or basis in group theory) for the POISSON-JACOBI Theorem, and seems to have encouraged his belief that the entire theory of first order partial differential equations could be regarded, with profit, from this conceptual standpoint.

In a letter received by MAYER on November 9 1873, LIE announced: "I have managed to discover an extremely interesting analytical form for an infinitesimal contact transformation of x, p."[56] As indicated in Theorem 6.3, a contact transformation of (x, p) involves a constant A, where $z' = Az + \Pi(x, p)$. LIE's discovery pertained to such transformations with $A = 1$ — the canonical transformations of (x, p) as they were called in § 6. It may be stated in the following form:

Theorem 7.5. *If* $d\omega : (x, z, p) \to (x + dx, z + dz, p + dp)$ *is an infinitesimal canonical transformation, where* $dx_i = \xi_i(x, p)\, dt$, $dz = \zeta(x, p)\, dt$, $dp_i = \pi_i(x, p)\, dt$, *then a function* $W(x, p)$ *exists such that* $\xi_i = \frac{\partial W}{\partial p_i}$, $\pi_i = -\frac{\partial W}{\partial x_i}$, *and* $\zeta = \sum_{k=1}^{n} p_k \frac{\partial W}{\partial p_k} - W$. *Conversely if* $W(x, p)$ *is any function, the equations*

$$dx_i = \left(\frac{\partial W}{\partial p_i}\right) dt, \quad dz = \left(\sum_{k=1}^{n} p_k \frac{\partial W}{\partial p_k} - W\right) dt, \quad dp_i = -\left(\frac{\partial W}{\partial x_i}\right) dt$$

define an infinitesimal canonical transformation $d\omega$.

[55] This is document V, LIE, *Ges. Abh.* **7**, pp. 89–95.

[56] Letter quoted by ENGEL on pp. 612–13 of LIE, *Ges. Abh.* **3**.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

The proof of Theorem 7.5 which LIE sent to MAYER used results in his paper on the analytical theory of contact transformations [1873b]. By Theorem 6.3 a canonical transformation has equations of the form $x_i' = X_i(x, p), z' = z + \Pi(x, p), p'_i = P_i(x, p)$, where

$$(7.8) \qquad (X_i, X_j) = 0, \quad (X_i, P_j) = 0, \quad (P_i, P_j) = 0, \quad (P_i, X_i) = 1.$$

Since $d\omega$ is infinitesimal, LIE wrote $X_i = x + \xi_i\varepsilon$, $\Pi = \zeta\varepsilon$ and $P_i = p_i + \pi_i\varepsilon$ and applied the relations (7.8) to obtain (ignoring ε^2 terms): $\frac{\partial\xi_i}{\partial p_j} = \frac{\partial\xi_j}{\partial p_i}, \frac{\partial\pi_i}{\partial p_i} = -\frac{\partial\xi_i}{\partial x_i}, \frac{\partial\pi_i}{\partial x_j} = \frac{\partial\pi_j}{\partial x_i}$. These equations imply that $\Omega = \sum_{i=1}^{n} \xi_i\, dp_i - \sum_{i=1}^{n} \pi_i\, dx_i$ satisfies $d\Omega = 0$ and so he could conclude that a W exists such that $dW = \Omega$, that is, $\frac{\partial W}{\partial p_i} = \xi_i$ and $\frac{\partial W}{\partial x_i} = -\pi_i$. He omitted the proof that $\zeta = \sum_{k=1}^{n} p_k \frac{\partial W}{\partial p_k} - W$, but it follows in similar fashion from his theorem that any contact transformation $x'_i = X_i(x, z, p)$, $z' = Z(x, z, p)$, $p'_i = P_i(x, z, p)$ satisfies $[Z, X_i] = 0$ and $[Z, P_i] = P_i[X_i, P_i]$ [1873b: 116]. The converse part of Theorem 7.5 also follows readily by considerations similar to those given above.

Theorem 7.5 may be regarded as an analytic version of Theorem 5.1 for canonical transformations $d\omega$ of (x, p). Theorem 7.5 asserts that the system of ordinary differential equations defining $d\omega$ has the following form for points on the manifold

$$W = 0: \ \frac{dx_i}{dt} = \frac{\partial W}{\partial p_i}, \ \frac{dz}{dt} = \sum_{i=1}^{n} p_i \frac{\partial W}{\partial p_i}, \ \frac{dp_i}{dt} = -\frac{\partial W}{\partial x_i}.$$

These are precisely the equations (5.1) for the characteristic strips of $W = 0$ when W is independent of z. Thus, in the language of § 5, by continually applying $d\omega$ to a point of $W = 0$, the characteristic strips of the partial differential equation $W = 0$ are generated. In the notation of § 5, $d\omega \leftrightarrow W$. For this reason, LIE eventually decided to call W the *characteristic function* associated to $d\omega$. He apparently first used this term in 1888 [1888b: 275]. Theorem 7.5 was first published in [1874b: 26–27]. There is an analogous theorem for any infinitesimal contact transformation [1890: 252]. ENGEL has suggested that LIE recognized this more general theorem even so early as the spring of 1872, and that in his letter to MAYER (quoted above) he meant to say that he had simply discovered a new and interesting *derivation* of the previously known analytical form of an infinitesimal contact transformation of (x, p). (See LIE, *Ges. Abh.* **3**, p. 613). His main reason for this conclusion was that LIE's remarks in [1872a] referring to the POISSON-JACOBI Theorem in connection with his *idée fixe* are otherwise unintelligible. However, the discussion in § 5, based on documents possibly unfamiliar to ENGEL when he drew the above conclusion, shows this is not at all the case, that Theorems 5.1–5.2 suffice to make LIE's remarks intelligible.

As indicated at the end of § 6, homogeneous contact transformations may be regarded as canonical transformations of (x, p) which leave z invariant: $z' = z$. Theorem 7.5 thus applies in particular to infinitesimal homogeneous contact transformations. In fact, if $d\omega \leftrightarrow W$ is a canonical transformation of (x, p) which fixes

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

z so that $dz = \zeta\, dt = 0$, then $\sum_{k=1}^{n} p_k \frac{\partial W}{\partial p_k} = W$, which means that W is homogeneous in the p_i of degree 1. From this it follows by straightforward calculation that the functions ξ_i and π_i are p-homogeneous of degrees 0 and 1, respectively, so that $d\omega$ is a homogeneous contact transformation. In other words Theorem 7.5 makes it clear that homogeneous transformations $d\omega$ and canonical transformations $d\omega$ which fix z are one and the same. LIE tended to think in terms of homogeneous contact transformations since their theory embraces that of all contact transformations (as indicated in § 6) and because in general analytical relations take on an especially simple form for them. During the fall of 1873 he drew some historically significant conclusions from Theorem 7.5 as applied to infinitesimal homogeneous contact transformations (§ 8). Although they are first manifest in his letters to MAYER in the winter of 1873–74 it would seem that he most certainly realized these conclusions soon after the discovery of Theorem 7.5 since, as will now be shown, they follow readily from that theorem in conjunction with analytic properties of infinitesimal transformations known already to him in 1872, namely Theorems 7.1–7.4.

Let $d\omega$ denote a homogeneous contact transformation. Since $dz = 0$, Theorem 7.5 applied to $d\omega$ says that $d\omega$ corresponds to the differential operator $X(f) = \sum_{i=1}^{n} \frac{\partial W}{\partial p_i}\frac{\partial f}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial W}{\partial x_i}\frac{\partial f}{\partial p_i}$, which is expressible in terms of a POISSON bracket: $X(f) = (W, f)$. Suppose that $dv \leftrightarrow V \leftrightarrow Y(f)$ is a second such transformation. Then by the reasoning behind Theorem 7.3 the commutator infinitesimal transformation $d\omega \circ dv \circ [dv \circ d\omega]^{-1}$ corresponds to $X(Y(f)) - Y(X(f))$, which by the above bracket representation equals, $(W, V, f)) - (V, (W, f)) = ((W, V), f)$, by virtue of JACOBI's Identity. Summing up, Theorem 7.5 implies

Theorem 7.6. *If $d\omega$ is a homogeneous contact transformation then $X(f) = (W, f)$ is the corresponding differential operator, where $d\omega \leftrightarrow W$ in accordance with Theorem 7.5. If $dv \leftrightarrow V \leftrightarrow Y(f)$ is a second such transformation, then the commutator transformation $d\omega \circ dv \circ [dv \circ d\omega]^{-1}$ is given by*

$$X(Y(f)) - Y(X(f)) = ((W, V), f). \tag{7.9}$$

Hence $d\omega \circ dv \circ [d\circ \circ d\omega]^{-1} \leftrightarrow (W, V)$.

The link between infinitesimal transformations and the POISSON bracket provided by Theorem 7.6 yields simple analytical proofs of many of LIE's theorems relating infinitesimal transformations to the integration of first order partial differential equations such as those in § 5. Consider, for example, the fundamental Theorem 5.2 which for homogeneous $d\omega$ states that if $F(x, p) = 0$ admits $d\omega$, then $(W, F) = 0$ where $d\omega \leftrightarrow W$. The conclusion follows immediately from Theorems 7.1 and 7.5–7.6 which imply that $(W, F) = X(F) = 0$. This simple analytical demonstration of Theorem 5.2 apparently prompted LIE's remark, in a letter to MAYER received the day before the letter containing Theorem 7.5, that he had "generally succeeded in conceiving of the theory of differential equations as well as Pfaff's

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

Problem as a theory of infinitesimal transformations. Such an interpretation is of value because, among other things, it gives a new viewpoint for research."[57]

Theorem 7.6 also reveals what LIE regarded as the "conceptual" grounds for the truth of the POISSON-JACOBI Theorem. Actually there were two versions of this theorem for LIE, although he did not always clearly distinguish them. One version (JACOBI's Theorem 3.4) says that if $\Phi = W$ and $\Phi = V$ are two solutions to $(F, \Phi) = 0$, then $\Phi = (W, V)$ is also a solution. The second version says that if $\Phi = W$ and $\Phi = V$ are two solutions to $F = 0 \Rightarrow (F, \Phi) = 0$, then $\Phi = (W, V)$ is also a solution. The second version is considered here, since this is the one LIE tended to favor. Let us therefore suppose that $\Phi = W(x, p)$ and $\Phi = V(x, p)$ are solutions to $F = 0 \Rightarrow (F, \Phi) = 0$. On LIE's conceptual level, this means that the equation $F = 0$ admits the canonical transformations, $d\omega$ and dv, with characteristic functions W and V. Now the totality g of all infinitesimal canonical transformations of (x, p) admitted by $F = 0$ is evidently closed under composition. For LIE during this period this meant that these transformations form a group. Since $d\omega$ and dv belong to g, it followed that the "comutator" infinitesimal transformation corresponding to (7.9) also belongs to g and is therefore likewise admitted by $F = 0$. Since by Theorem 7.6 the differential operator associated to the commutator transformation is $Z(f) = ((W, V), f)$, being admitted by $F = 0$ means that $F = 0 \Rightarrow Z(F) = ((W, V), F) = 0$ so that $\Phi = (W, V)$ is also a solution. The proof of the first version of the theorem is entirely analogous.[58]

In this manner the POISSON-JACOBI Theorem is proved and seen, on the conceptual level, to be a consequence of the properties of the group of infinitesimal contact transformations admitted by $F = 0$ (or by F). JACOBI's proof of the POISSON-JACOBI Theorem was of course much simpler since for JACOBI the Theorem follows immediately from his Identity. The above proof also requires the Identity (to obtain (7.9)) as well as additional reasoning. Undoubtedly JACOBI would have regarded the additional reasoning as extraneous, as obscuring the formal simplicity of the proof. From LIE's perspective, however, the additional reasoning revealed the true, underlying basis for the theorem. From his point of view the POISSON-JACOBI Theorem stated the evident fact that the totality of infinitesimal canonical transformations of (x, p) admitted by $F = 0$ has the closure property characteristic of a group of transformations.[59]

Theorem 7.5 (through Theorem 7.6) also brings out clearly the connections

---

[57] Quoted by ENGEL on pp. 582–3 of LIE, *Ges. Abh.* **3**. LIE made a similar remark in [LIE 1873 MSb: 3] after stating what amounts to Theorems 7.5 and 5.2.

[58] It follows from the reasoning behind Theorem 7.1, that $d\omega \leftrightarrow X(f)$ admits a *function* F (*i.e.* leaves F invariant) if and only if $X(F) \equiv 0$. The first version then follows by applying the above reasoning to the group of all infinitesimal transformations which admit $F(x, p)$.

[59] In a letter to an unidentified "Herr Professor", probably written in the spring or summer of 1876, LIE put the matter this way: "the interpretation of the solutions Φ to the equation $(F\Phi) = 0$ as infinitesimal transformations which leave the equation $F = a$ invariant makes the POISSON-JACOBI Theorem evident. It says that the succession of two such transformations supplies another such transformation." (Quoted by ENGEL on pp. 696–7 of LIE, *Ges. Abh.* **3**. The dating of the letter is also ENGEL's.)

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

between the function group $(u_1, \ldots, u_r) = \mathscr{F}$ and the group $g_{\mathscr{F}}$ of infinitesimal transformations $d\omega$ with characteristic functions W which belong to $\mathscr{F}$. Recall that $W = W(x, p) \in \mathscr{F}$ means that W is expressible as a function of $u_1, \ldots, u_r$. As noted in § 6, if $W, V \in \mathscr{F}$ then $aW + bV \in \mathscr{F}$ and $(W, V) \in \mathscr{F}$ so that $\mathscr{F}$ has the defining properties of what is now called a LIE algebra. These properties of $\mathscr{F}$ correspond to properties of $g_{\mathscr{F}}$. If $W \leftrightarrow d\omega \leftrightarrow X(f)$ and $V \leftrightarrow dv \leftrightarrow Y(f)$, then $aX(f) + bY(f) = (aW + bV, f)$ so the "composite" $[d\omega]^a \circ [dv]^b \leftrightarrow aW + bV \in \mathscr{F}$, which means that $[d\omega]^a \circ [dv]^b \in g_{\mathscr{F}}$. Likewise $d\omega \circ dv \circ [dv \circ d\omega]^{-1} \leftrightarrow (W, V) \in \mathscr{F}$, which means that $d\omega \circ dv \circ [dv \circ d\omega]^{-1} \in g_{\mathscr{F}}$. In other words, $g_{\mathscr{F}}$ has the properties of infinitesimal transformations belonging to a group that are singled out in Theorems 7.3 and 7.4.

According to LIE's recollections [1884b: 112 n. 1], when he chose the name "group" in December 1872 in his first note on function groups, "On the Invariant Theory of Contact Transformations" [1872c], he realized that the infinitesimal transformations $g_{\mathscr{F}}$ determine a group of contact transformations. Whatever "synthetic" grounds he originally may have had for this understanding of $g_{\mathscr{F}}$, Theorem 7.5 and its consequences provided further analytical support by November 1873. From LIE's point of view, the entire theory of function groups could be interpreted in terms of group related concepts. For example, it follows readily from Theorem 7.6 that the distinguished functions of function group $\mathscr{F}$ are precisely the functions $v \in \mathscr{F}$ which are invariants of the group $G_{\mathscr{F}}$ generated by $g_{\mathscr{F}}$ since $v \circ d\omega = v + (W, v)\, dt = v$. Likewise, from LIE's viewpoint, the fundamental relation (6.4) which formed the starting point of the analytical theory of function groups was simply an expression of the fact that $G_{\mathscr{F}}$ is a group.

On a strictly mathematical level, the connection between function groups and transformation groups was not particularly significant. That is, the theory of function groups could be, and indeed was, developed and applied by LIE in 1873–74 without explicitly considering either $g_{\mathscr{F}}$ or $G_{\mathscr{F}}$. Even in *Theorie der Transformationsgruppen*, which includes the theory of function groups in the second volume [1890], the existence of $G_{\mathscr{F}}$ is established more or less in passing and never used. Nevertheless in 1873, the connection between function groups and transformation groups was important for psychological reasons. The perception that function groups correspond to transformation groups, made a great impression upon LIE, for it showed that continuous groups had a relevance to 19th century mathematics that in scope was far broader than the rather specialized applications of continuous groups he had made in his geometrical period. As he himself explained a decade later: "In my investigations of line and sphere geometry in the years 1870–71 many similar [group related] integration methods were applied with some success to special geometrical problems. Soon however these investigations took on an entirely different significance. I discovered that the integration theory of first order partial differential equations founded by Lagrange, Pfaff, Cauchy, Jacobi and their successors could be conceived in a natural way as a transformation theory of these equations … . Every known group of [contact] transformations which took a first order partial differential equation $[F = 0]$ into itself supplied, corresponding to its infinitesimal transformations, a certain number of integrals $u_1, \ldots, u_r$ of the associated simultaneous system $[(F, \Phi) = 0]$ which satisfied pairwise relations of the form $(u_i, u_k) = f_{ik}(u_1, \ldots, u_r)$" [1884a: 140–41].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

The correspondence between function groups and transformation groups enabled LIE to regard the theory of function groups, and particularly results such as Theorem 6.1, as an analog of GALOIS' theory for partial differential equations of the first order and hence as a realization of his *idée fixe*. His letters to MAYER confirm that he did attach this sort of significance to the analogy. For example, that the primary significance of Theorem 6.1 was seen to arise from group theory rather than simply as a another theorem in a chain of theorems going back to PFAFF and JACOBI (*e.g.* Theorems 2.2, 3.3, 3.5, 3.7), is evident from a letter he wrote to MAYER in April 1873. Referring to his publications [1873a, 1873c] on function groups he explained: "To judge my investigations correctly, it must be kept in mind that simplifying the integration [of partial differential equations] is only of secondary interest to me. The main concern is my invariant theory."[60] As explained in § 1, he had adopted the phrase "invariant theory of contact transformations" from KLEIN's characterization of geometrical methods by group theory to refer to all possible aspects of the study and classification of partial differential equations in terms of the group of all contact transformations and its subgroups. Theorem 6.1 showed what could be said about the integration of a partial differential equation when one knew it admitted the infinitesimal transformations $g_{\mathscr{F}}$ of the subgroup $G_{\mathscr{F}}$. In the euphoria surrounding his initial efforts to create a theory of continuous groups, he wrote to MAYER: "the invariant theory of contact transformations reigns over the integration theory of partial differential equations of the first order in the same sense that the theory of substitutions [permutation groups] rules over the theory of algebraic equations!!!"[61] The belief that KLEIN and LIE shared, as early as 1869, that a continuous analog of GALOIS' theory of algebraic equations should exist for differential equations was now perceived as becoming a reality. Of course the analogy between Theorem 6.1 and GALOIS' theory is superficial, but this fact never appears to have concerned LIE, whose knowledge of GALOIS' theory was admittedly limited.

Theorems 7.1–7.6 had thus put LIE in a position to interpret, by purely analytical means, the theory of first order partial differential equations in terms of group theory. His "conceptual" interpretation of the POISSON-JACOBI Theorem, in particular, seemed to epitomize his belief that the entire theory of such equations could be pursued profitably on the level of group theory. The view that such a state of mind on LIE's part was an important factor in the events leading to his decision to create a theory of continuous groups is also supported by his own recollections:

> In the course of investigations on first order partial differential equations, I observed that the formulas which occur in this discipline become amenable to a remarkable conceptual interpretation by means of the concept of an infinitesimal transformation. In particular, what is called the Poisson-Jacobi Theorem is closely connected with the composition of infinitesimal transformations. By following up on this observation I arrived at the surprising result that all

[60] Letter to MAYER written ca. 7–10 April 1873 and quoted in part by ENGEL in LIE, *Ges. Abh.* **3**, p. 648. My translation.

[61] Letter of 5 July 1874, quoted by ENGEL on pp. 582–3 of LIE, *Ges. Abh.* **4**.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

transformation groups of a simply extended manifold can be reduced to the linear form by a suitable choice of variables, and also that *the determination of all groups of an n-fold extended manifold can be achieved by the integration of ordinary differential equations*. This discovery ... became the starting point of my many years of research on transformation groups. (LIE [1879b: 93]; the emphasis on the penultimate sentence is LIE's.)

This passage leaves no doubt about the important role played by the conceptual interpretation facilitated by Theorems 7.1–7.6. How it led to the decisive involvement with the determination of transformation groups still requires explanation.

8. The Birth of Lie's Theory of Groups

According to LIE, it was during the month of October 1873 that the beginnings of his theory of groups emerged.[62] The documentary evidence at hand supports his recollection and suggests what may have occurred at that time. For example, in a letter received by MAYER on November 8, 1873, he announced that the had "generally succeeded in conceiving of the theory of differential equations as well as Pfaff's Problem as a theory of infinitesimal transformations. Such an interpretation is of value because, among other things, it gives a new viewpoint for research."[63] It was in the letter received by MAYER the following day (November 9) that he stated and proved Theorem 7.5, which we have seen to be fundamental to the interpretation of the theory of partial differential equations in terms of groups of infinitesimal transformations.[64] That he was reporting on discoveries made during October is suggested by a document [LIE 1873 MSa] which is a draft of a letter received by MAYER on November 12. In the draft, he reported on some of his recent research: "I studied higher order equations very intensely; to a large extent it all ended in illusions. This is in particular the way October passed. (Incidentally, during this period I obtained results about infinitesimal transformations which are to me of the greatest interest.) In September I worked on the Pfaffian Problem. I succeeded in finding an algebraic representation of my method."[65] In the actual letter sent to MAYER, he rephrased his parenthetical remark and revealed another aspect of the work with infinitesimal transformations. After mentioning his unsuccessful "illusory" research efforts, he wrote: "Only my work on infinitesimal transformations and various related theories of multiplicators and integrability factors is of interest to me."[66]

These "related theories" were published by LIE a year later [1874c, 1874d]. In [1874d], he returned to a problem he had considered already in his joint paper with KLEIN [1871: § 7]: the integration of $\frac{dy}{dx} = f(x, y)$ when it admits an in-

[62] According to ENGEL. See his remarks on p. 583 of LIE, *Ges. Abh.* **5**.

[63] LIE, *Ges. Abh.* **3**, 582–3.

[64] LIE, *Ges. Abh.* **3**, 612–13.

[65] A transcription of a portion of [LIE 1873 MSa], including the part translated here, was published by GULDBERG [1913: 9].

[66] LIE, *Ges. Abh.* **4**, 470.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

finitesimal transformation. The reasoning in the 1871 treatment had been of the vague synthetic sort typical of his work then, but in [1874d] it is thoroughly analytical and reflects the greater emphasis upon the analytical formulation of his ideas that is first evident in his publications on his invariant theory of contact transformations. He showed that the existence of an infinitesimal transformation admitted by the equation, now expressed in the form $M\,dx + N\,dy = 0$, is equivalent to the existence of an integrability factor.

It was in connection with JACOBI's generalization of the notion of an integrability factor — his theory of multiplicators — that LIE developed another aspect of his *idée fixe* in [1874c], which deals with

Problem 8.1. *Suppose a system of ordinary differential equations or, equivalently, a system of independent linear, homogeneous partial differential equations*

$$(8.1) \qquad A_i(f) = \sum_{j=1}^{n} a_{ij}(y) \frac{\partial f}{\partial y_j} = 0, \quad i = 1, \dots, m \quad (m \leqq n)$$

is known to admit infinitesimal transformations (which perhaps commute or form a group). What does this information imply about the integration of the system?

This was a natural enough question to pose since the integration of first order and Pfaffian equations was known to reduce to the integration of complete systems. The work on Problem 8.1 along the lines of LIE's paper [1874c] appears to have provided the fillip that precipitated the considerations that gave birth to his theory of groups.[67]

According to LIE [1884b: 105; 1884a: 141] his interest in Problem 8.1 went back to the fall of 1872, and some of his discoveries were communicated in a brief research announcement submitted in November 1872. The relevant passage begins: "I have succeeded in broadening my work on partial [differential] equations with infinitesimal transformations in various directions and in particular have also extended it to the Pfaff Problem and to systems of ordinary differential equations. I consider both commutative transformations as well as those which form a group" [1872d: 27]. The remainder of the passage vaguely describes results which support his own recollections that in 1872 he considered Problem 8.1 under the assumption that (8.1) admits a known group of transformations G of the type that had predominated during his geometrical period: unlike the groups $G_{\mathscr{F}}$ associated to function groups $\mathscr{F}$, G is assumed to be generated by a finite number of infinitesimal transformations, and its non-infinitesimal transformations thus depend continuously on a finite number of parameters. LIE subsequently termed such groups finite continuous transformation groups; here they will be referred to as finite dimensional groups. In his study of Problem 8.1 in 1872 not only the infinitesimal transformations of G but also the non-infinitesimal transformations were assumed known, and his announced results pertained exlusively to simplifications that arise by virtue of knowing the non-infinitesimal transformations. Considera-

[67] According to ENGEL [1899: xxxvii, xlff.], who presents this interpretation as fact not conjecture, presumably having learned it from LIE. See also his remarks in LIE, *Ges. Abh.* 6, p. xiiiff.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

tions of this sort were not presented in [1874c]; they were first presented in a much more developed form in [1884a: § 10].

Whatever state his work on Problem 8.1 had attained by November 1872, it did not inspire him to begin creating a theory of continuous groups. Probably he shunted the work aside during the spring of 1873, when he concentrated on the analytical formulation of his invariant theory of contact transformations. A byproduct of the analytical focus was Theorem 7.5 and its implications. By October 1873, when he was pursuing his research on infinitesimal transformations and multiplicators, he had at his disposal Theorems 7.1–7.6 and realized that resusults about partial differential equations, such as the POISSON-JACOBI Theorem, could be understood in terms of group-related concepts. It would seem that it was during this period of renewed interest in Problem 8.1 that he arrived at a greater appreciation of the role of finite dimensional groups in the study of differential equations, for he discovered that Problem 8.1 under the more general assumption that (8.1) simply admits a finite number of infinitesimal transformations can always be reduced to the case in which (8.1) admits a finite dimensional group of transformations.

In discussing LIE's work on Problem 8.1 and its role in the birth of his theory of groups, the presentation as it appears in [1874c] has been followed since earlier versions are not extant. Although [1874c] was submitted in November 1874 and thus well after he had launched his theory of groups, it appears to be a polished form of what he had discovered by the fall of 1873 rather than a work that benefited from his deliberations on transformation groups during 1874. The only reference in [1874c] to groups is a passing one in the introductory remarks: "Strictly speaking the theory of transformation groups which I recently outlined [1874e] plays a fundamental role [in the theory presented in [1874c]]; to be sure, this is not at all apparent in this preliminary work which in no way assumes the above cited [1874e] as known." Here he seems to be saying that, although the contents of [1874c] have important implications for group theory, the presentation in [1874c], being preliminary, does not attempt to develop them. He was so occupied with his theory of transformation groups in 1874 and thereafter that some of these implications were first worked out in [1884a]. A presentation of his involvement with Problem 8.1 as it might have stood in the fall of 1873 based on what is found in [1874c], consequently seems reasonable and justified under the circumstances. The state of his knowledge in the fall of 1873, which included Theorems 7.1–7.6, certainly put him in a position to make the discoveries discussed below.

Let $dT_j : y \to y + dy$, $j = 1, \ldots, q$, denote the infinitesimal transformations admitted by the system $A_i(f) = 0$ in (8.1), where the equations for dT_j are

$$dy_k = \eta_{jk}(y)\, dt, \qquad k = 1, \ldots, n. \tag{8.2}$$

It follows from Theorem 7.1 that if f is any solution to the system $A_i(f) = 0$, then $A_i((X_j)f) - X_j(A_i(f)) = 0$ for all i, j, where $X_j(f) = \sum_{k=1}^{n} \eta_{jk}(y) \dfrac{\partial f}{\partial y_k}$ is the differential operator corresponding to dT_j. In other words, the system of equations (8.1) and the larger system obtained by adding to (8.1) the equations $A_i(X_j(f)) - X_j(A_i(f)) = 0$ have the same solutions. Because the system (8.1) is complete,

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

it follows from CLEBSCH's theory of these systems that each additional equation $A_i(X_j(f)) - X_j(A_i(f)) = 0$ must be linearly dependent on the original equations $A_i(f) = 0$.[68] Therefore,

(8.3) $$A_i(X_j(f)) - X_j(A_i(f)) = \sum_{k=1}^{m} c_{ijk}(y) A_k(f).$$

In the ensuing discussion of LIE's work, some additional notation for differential operators $B_i(f) = \sum_{k=1}^{n} b_{ik}(y) \frac{\partial f}{\partial y_k}$ will be used. First, let

(8.4) $$(B_i, B_j) \stackrel{\text{def}}{=} B_i(B_j(f)) - B_j(B_i(f)).$$

LIE introduced this bracket notation (without comment) in his correspondence with MAYER during the winter of 1873–74.[69] That POISSON brackets were intimately related to expressions of the form $B_i(B_j(f)) - B_j(B_i(f))$ was implicit in JACOBI's proof of his Identity (Theorem 3.2). In [1871: 94] MAYER had added the observation that if corresponding to $B_i(f)$ one sets $f_i(x, p) = \sum_{k=1}^{n} b_{ik}(x) p_k$, then JACOBI's expression $B_i(B_j(f)) - B_j(B_i(f))$ can be identified with the POISSON bracket (f_i, f_j). The notational convention in (8.4) was consequently a natural one. For LIE the adoption of the bracket notation was all the more natural since if the $B_i(f)$ correspond to infinitesimal homogeneous contact transformations with characteristic functions W_i, then by Theorem 7.6 $B_i(B_j(f)) - B_j(B_i(f))$ equals the POISSON bracket $((W_i, W_j), f)$ and (W_i, W_j) is the characteristic function of the infinitesimal transformation $B_i(B_j(f)) - B_j(B_i(f))$.

In addition to the bracket notation, the following convention will be adopted: *If operators $B(f)$, $C(f)$ are such that $B(f) - C(f)$ is linearly dependent on the $A_i(f)$ defining the system* (8.1) *so that $B(f) - C(f) = \sum_{i=1}^{m} \lambda_i(y) A_i(f)$, write $B(f) \equiv C(f)$ or $B(f) - C(f) \equiv 0$.*

With this convention, the result of (8.3) may be expressed in the form

(8.5) $$(A_i, X_j) \equiv 0, \quad i = 1, \ldots, m; \; j = 1, \ldots, q.$$

Equation (8.5) was a consequence of the fact the system (8.1) admits the dT_j. LIE realized that the converse also holds so that (8.5) is both necessary and sufficient that the system $A_i(f) = 0$ admit the infinitesimal transformations $X_j(f)$ [1874c: Theorem I].

LIE's approach to Problem 8.1 was along the following lines. By definition the m equations (8.1) are linearly independent. If the number of infinitesimal transfor-

[68] Since the augmented system has the same solutions as the original complete system, by Theorem 3.6 it has exactly q independent solutions. If the augmented system had $q^* > q$ independent equations, then by Theorem 3.6 it would be equivalent to a complete system with $q^{**} \geqq q^*$ equations. Thus the augmented system would have $q^{**} > q$ independent solutions.

[69] See LIE, *Ges. Abh.* 5, p. 589. There is also occasional use of bracket notation in [LIE 1873 MS b]. The first use of the notation in a publication appears to be in [LIE 1874c: 191].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

mations $X_i(f)$ is sufficiently large, it is to be expected that the $m+q$ equations $A_1(f)=0,\ldots,X_q(f)=0$ will be linearly dependent. This is necessarily the case, for example, when $m+q>n$. Now suppose that the equations $A_i(f)=0$, $X_j(f)=0$ are linearly dependent and that the notation has been chosen so that the equations $X_{l+1}(f)=0,\ldots,X_q(f)=0$ are superfluous and that $A_1(f)=0,\ldots,X_l(f)=0$ are linearly independent. Thus

$$(8.6) \qquad X_{l+k}(f) \equiv \sum_{j=1}^{l} \beta_{kj}(y)\, X_j(f), \quad k=1,\ldots,q-l.$$

LIE discovered that the $X_{l+k}(f)$ are a source of potentially new solutions to the original system by proving

Theorem 8.1. *The functions* $\beta_{kj}(y)$ *defined by equation* (8.6) *are solutions to the system* $A_i(f)=0$. [1874c: Theorem II].

The proof follows readily from (8.5) and properties of differential operators.

Since the integration of (8.1) is facilitated by knowledge of solutions — the more independent solutions the better — Theorem 8.1 indicates that it is desirable to obtain as many infinitesimal transformations as possible which are admitted by the system (8.1) since they are a potential source of new independent solutions. In this connection, LIE observed that if the system (8.1) admits $X_1(f),\ldots,X_q(f)$, then it also admits all infinitesimal transformations (X_j, X_k). This observation was an obvious one for him given the reasoning behind his conceptual understanding of the POISSON-JACOBI Theorem (as discussed in § 7 following Theorem 7.6). That is, according to that sort of reasoning, consider the totality of all infinitesimal transformations admitted by (8.1). It evidently has the group property (closure under composition) and so is a group. The commutator infinitesimal transformations, which correspond by Theorem 7.3 to the (X_i, X_j), belong to this group and hence are admitted by the system (8.1). By repeated applications of this consequence of his conceptualization of the POISSON-JACOBI Theorem, LIE therefore sought to generate by bracketing more and more infinitesimal transformations which are admitted by the system (8.1). Here he was applying the type of reasoning used by CLEBSCH in proving the second part of Theorem 3.6 — a line of reasoning that is also analogous to what is involved in the considerations on which the notion of a function group are based.

Once infinitesimal transformations are generated by bracketing, Theorem 8.1 is applied to obtain solutions to (8.1). If in this way p functionally independent solutions to (8.1) are obtained, and if $p=n-m$, then the integration of (8.1) is accomplished since by Theorem 3.6 the general solution is an arbitrary function of these $n-m$ solutions. If $p<n-m$, then these p solutions can be used to introduce new variables and reduce the integration of the original system to that of a new complete system in $n-p$ variables admitting known infinitesimal transformations. To this new system, the same reasoning can be applied and so on until the process stops. The final complete system is a system admitting infinitesimal transformations $X_1(f),\ldots,X_r(f)$ which is immune to the procedure

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

described above. In particular, no new infinitesimal transformations are generated by bracketing which implies that

$$(8.7) \qquad (X_i, X_j) = \sum_{k=1}^{r} c_{ijk} X_k(f), \quad c_{ijk} = \text{const.}$$

The study of the integration of complete systems admitting infinitesimal transformations could thus be reduced to the case of systems in which the infinitesimal transformations satisfy the additional relations in (8.7).[70]

Given LIE's penchant for imbuing analytical relations with a conceptual meaning, it is easy to imagine how the relations (8.7) would have prompted him to seek to interpret (8.7) in terms of groups. Indeed, the differential operator (X_i, X_j) on the left hand side he knew corresponded to the commutator infinitesimal transformation $dT_i \circ dT_j \circ (dT_j \circ dT_i)^{-1}$ (Theorem 7.3). Assuming that a continuous group G exists whose infinitesimal transformations are all of the form $dT = \sum_{i=1}^{r} a_i dT_i$, then (8.7) simply reaffirms Theorem 7.3 for the group G: the commutator of infinitesimal transformations belonging to G also belongs to G, and since this commutator must be a linear combination of the dT_i (8.7) follows. If G always exists, then the reasoning leading to (8.7) shows that the integration of complete systems admitting known infinitesimal transformations can always be reduced to the case of complete systems admitting a finite dimensional group G, a type of group he had frequently encountered during his geometrical period. To LIE this would have meant that finite dimensional groups are also fundamental to the working out of his *idée fixe* within the context of the 19th century theory of first order and Pfaffian equations.

The above sort of considerations apparently motivated him to seek to use the relations (8.7) actually to determine the groups G in the special case of transformations of a single variable, y. Perhaps he wanted to test his intuition that (8.7) is sufficient to guarantee the existence of a group G with infinitesimal transformations satisfying (8.7). In the case of one variable, the infinitesimal transformation dT_i of (8.2) is simply $dy = \eta_i(y)\, dt$ and so is identifiable with the function $\eta_i(y)$. The relations (8.7) in this case state that the functions η_i must satisfy relations of the form $\eta_i \dfrac{d\eta_j}{dy} - \eta_j \dfrac{d\eta_i}{dy} = \sum_{k=1}^{n} c_{ijk}\eta_k$. By means of elementary considerations he discovered that the number r of η_i must satisfy $r \leqq 3$ and that, up to variable and parameter changes, the distinct systems of infinitesimal transformations in one variable satisfying (8.7) correspond to the finite dimensional groups $y' = y + b$ $(r = 1)$, $y' = ay + b\,(r = 2)$, and $y' = \dfrac{ay + b}{y + c}$ $(r = 3)$.[71] By the third

[70] In many equations in [1874c], including (8.7), LIE wrote "=" although (as ENGEL's editorial additions indicate) the explicit reasoning supported only the weaker "≡." Later LIE showed explicitly that in the final, reduced complete system equation (8.7) can be assumed to hold in the sense of "=" [1884a: 150]. It is unclear whether in 1873–74 he carelessly glossed over the distinctions between the two senses of equality or whether he believed that strict equality could be shown to hold but did not bother to go into such details.

[71] LIE announced these results in a letter answered by MAYER on December 2 1873 and sent MAYER proofs in a letter received April 19 1874 [LIE 1873–4: 584–85, 599–604].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

group, he meant the group of all "linear" (*i.e.* projective) transformations of the line $y' = \frac{ay+b}{cy+d}$, $a\,d - bc \neq 0$, which depends on three independent parameters. He had therefore determined all non-similar groups of transformations in one variable, and he had discovered that they can all be represented as projective transformations.

Judging by LIE's letters to MAYER in the winter of 1873–74, these discoveries set off a chain reaction in his mind, causing the experiences with groups and with the implementation of his *idée fixe* which had been accumulating during 1869–73, and especially in 1872–73, now to become the focal point of his attention, the basis for a long term research program on groups. Initially he concentrated on finite dimensional groups because, not only could they now be seen to be relevant to the 19th century theory of partial differential equations, they also seemed to be particularly tractable by virtue of the relations (8.7). In his first letter to MAYER announcing the above discoveries and the new focus of his research interests, he began: "Dear Mayer! My heartfelt thanks for both your letters [of 3 and 14 November 1873] which however I will not answer today. Instead I will explain why I cannot write longer letters these days. The reason is that I am concentrating all my intellectual powers on a mathematical investigation of quite extraordinary interest to me. I have obtained highly interesting results and expect many, many more. It concerns an idea which originated in the earlier works of Klein and me, namely that of introducing the concepts of the theory of substitutions [= permutation groups] in the theory of differential equations" [LIE 1873–4: 584]. In this undated letter (answered by MAYER on December 2 1873), LIE conveyed the above discoveries to MAYER in the form of two theorems. The first stated that groups of ∞^r transformations of one variable $x' = f(x, a_1, \ldots, a_r)$ exist only for $r \leqq 3$ and the second that all such groups were similar to subgroups of the group of projective transformations of the line. "For n-fold extended manifolds corresponding theorems exist," he continued, "but here I am not at all finished." By the time of his first letter to MAYER, he was evidently already dealing with groups of transformations in any number n of variables and was confident that he could match his success with the case $n = 1$. He even seemed to suggest that analogs of the two stated theorems exist for groups in $n > 1$ variables, although his work on the n variable case was still in progress.

LIE's optimism that he could create a theory of groups of transformations in n variables for any n does not appear to have been based solely, or even primarily, on his success with the case $n = 1$. As soon as $n = 2$, the number of non-similar groups becomes infinite. Exactly when he realized this fact is uncertain, although his letters to MAYER suggest that he realized it, or suspected as much, from the outset. Among all his admittedly provisional conjectures about the n-variable theory, he never spoke of the possibility of a finite number of similar groups, but rather of the possibility of a classification scheme involving a finite number of distinct "types" of similarity classes. Thus there is reason to believe he quickly realized the group classification problem was far more difficult for $n > 1$. He knew by past experience that the simplicity of the matter for a low dimension did not ensure that the same would be true for manifolds of higher or arbitrary dimension. During 1869–71 he and KLEIN had been led by their geometrical work on

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

W-curves and surfaces to a group classification problem which they were able to resolve completely for $n = 2$ variables but not for $n = 3$ and certainly not for n arbitrary. It is not surprising that in 1872 LIE had dismissed the general group classification problem suggested by KLEIN's Erlangen Program as "absurd and impossible". By December 1873, however, we find him confident that he was now in a position to deal with the theory and classification of transformation groups in any number of variables. The next letter to MAYER, received by him on February 3, 1874, suggests that the sanguine expectations regarding a general theory of finite dimensional transformation groups conveyed in his previous letter had been due in large part to the links he had discovered between finite dimensional transformation groups and his invariant theory of contact transformations, particularly his theory of function groups.

In a mathematical essay accompanying the letter of February 3 [1873–74: 586, 595], LIE began, after giving some preliminary definitions, by stating without any indication of a proof, the following fundamental proposition:

Theorem 8.2. *Infinitesimal transformations* $X_i(f) = \sum_{j=1}^{n} \eta_{ij}(y) \frac{\partial f}{\partial y}$, $i = 1, \ldots, r$ *determine an r-dimensional group of transformations of* $y = (y_1, \ldots, y_n)$ *if and only if they satisfy* (8.7), *viz.* $(X_i, X_j) = \Sigma_{k=1}^{r} c_{ijk} X_k(f)$.

His computations for groups in one variable had confirmed Theorem 8.2 in this special case, and judging by his essay he did not regard the general theorem as questionable or its proof as problematic.[72] Indeed, he explained to MAYER that, by virtue of Theorem 8.2, the problem of determining all r-dimensional groups of transformations in n variables was reduced to

Problem 8.2. *Determine all linearly independent infinitesimal transformations* $X_i(f) = \sum_{j=1}^{n} \eta_{ij}(y) \frac{\partial f}{\partial y_j}$, $i = 1, \ldots, r$, *which satisfy relations of the form* $(X_i, X_j) = \Sigma_{k=1}^{r} c_{ijk} X_k(f)$.

By "determine all ..." he always meant "determine up to similarity all ..." Today the problem would be formulated as the problem of determining all finite dimensional LIE algebras of vector fields up to equivalence under diffeomorphisms. In 1873–74 it represented for LIE the main problem facing his prospective theory of groups.

To deal with Problem 8.2, LIE turned to his theory of homogeneous contact transformations. For transformations in $n > 1$ variables $(x, z) = (x_1, \ldots, x_{n-1}, z)$, in addition to point transformations $(x, z) \to (x', z')$, there are contact transformations $(x, z, p) \to (x', z', p')$ to consider. He realized that, without loss of generality, his study of finite dimensional groups of point and contact transformations in $n > 1$ variables can be restricted to groups of homogeneous contact transformations of $(x, p) = (x_1, \ldots, x_n, p_1, \ldots, p_n)$, since every point transfor-

[72] His first published proof [1876b: 58–63] was admittedly incomplete since it considered only the generic case, but LIE felt the other cases presented no problems.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

mation can be prolonged to a contact transformation (§ 4) and every contact transformation can be identified with a homogeneous one (§ 6). Suppose that $X_1(f), \ldots, X_r(f)$, $f = f(x,p)$, denote linearly independent infinitesimal homogeneous contact transformations of (x,p) which satisfy the fundamental relations (8.7). By Theorem 8.2, corresponding to the $X_i(f)$ is a finite dimensional group G of homogeneous contact transformations. Now by virtue of Theorem 7.5, corresponding to each infinitesimal transformation $X_i(f)$ is its characteristic function $H_i(x,p)$, which is homogeneous in p of degree 1 and satisfies $X_i(f) = (H_i, f)$ for all $f = f(x,p)$. Theorem 7.6 combined with (8.7) shows that for all f

$$((H_i, H_j), f) = (X_i, X_j) = \sum_{k=1}^{r} c_{ijk} X_k(f) = \sum_{k=1}^{r} c_{ijk} (H_k, f) \tag{8.8}$$

so that the functions $H_i(x,p)$ satisfy

$$(H_i, H_k) = \sum_{k=1}^{r} c_{ijk} H_k, \tag{8.9}$$

an exact analog of (8.7).[73]

Equation (8.9) was especially important to LIE because it implied that the functions H_i determine a homogeneous function group. The r functions H_i are linearly independent but not necessarily functionally independent. Let the notation be chosen so that $H_1, \ldots, H_\varrho$ are functionally independent and that the remaining H_i are functions of these. Then functions K_l exist satisfying

$$H_{\varrho+l} = K_l(H_1, \ldots, H_\varrho), \quad l = 1, \ldots, r - \varrho, \tag{8.10}$$

and (8.9) may be expressed in the form

$$(H_i, H_j) = \sum_{k=1}^{\varrho} c_{ijk} H_k + \sum_{k=\varrho+1}^{r} c_{ijk} K_k(H_1, \ldots, H_\varrho) \overset{\text{def}}{=} \Omega_{ij}(H_1, \ldots, H_\varrho). \tag{8.11}$$

Equation (8.11), restricted by $1 \leqq i, j \leqq \varrho$, states that $H_1, \ldots, H_\varrho$ form a function group $\mathscr{F}$. Thus the r-dimensional transformation group G determined by infinitesimal transformations with characteristic functions $H_1, \ldots, H_r$ corresponds to a ϱ-term function group $\mathscr{F} = (H_1, \ldots, H_\varrho)$. LIE discovered in this correspondence a way to use his theory of function groups to develop the theory of transformation groups. The groups G fall into three categories determined by the properties of the associated function group $\mathscr{F} = (H_1, \ldots, H_\varrho)$. Category I consists of all G for which $\varrho = r$. When $\varrho < r$ there are two cases to consider. According to the theory of function groups $\varrho = 2q + m$, where m is the number of functionally independent distinguished functions of $\mathscr{F}$. Category II consists of all groups G for which $m = 0$, and category III contains those with $m > 0$.

---

[73] Equation (8.8) shows that $((H_i, H_j) - \sum_{k=1}^{r} c_{ijk} H_k, f) = 0$ for all $f = f(x,p)$, which implies that $(H_i, H_j) - \sum_{k=1}^{r} c_{ijk} H_k = \text{const.}$ Since the H_i are p-homogeneous of degree $d = 1$, direct calculation shows that (H_i, H_j) is also p-homogeneous of degree 1, so that the same is true of $K(x,p) = (H_i, H_j) - \sum_{k=1}^{r} c_{ijk} H_k$. Equation (8.9) then follows since, the p-homogeneity of $K = \text{const.}$ means $K = \sum_{i=1}^{n} p_i \frac{\partial K}{\partial p_i} = 0$.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

The groups G of category I are the simplest to consider. In this case, $\mathcal{F} = (H_1, \ldots, H_r)$. Suppose that G^* is another such group, with infinitesimal transformations $X_i^*(f)$ and associated characteristic functions $H_i^*(x^*, p^*)$, $i = 1, \ldots, r$. Suppose also that the two groups have the same constants c_{ijk} in the sense that suitable linear combinations of the $X_i^*(f)$ may be chosen so that $(X_i^*, X_j^*) = \sum_{k=1}^r c_{ijk} X_k^*(f)$. (Stated in more familiar terms, the assumption is that G and G^* have isomorphic LIE algebras.) Then the corresponding H_i^* satisfy $(H_i^*, H_j^*) = \sum_{k=1}^r c_{ijk} H_k^*$, and (H_i^*, H_j^*) is the same function of $H_1^*, \ldots, H_r^*$ as (H_i, H_j) is of $H_1, \ldots, H_r$. Theorem 6.5 thus asserts that a homogeneous contact transformation $T: (x, p) \to (x^*, p^*)$ exists which changes each H_i^* into the corresponding H_i. This means that T also changes the $X_i^*(f)$ into the $X_i(f)$ and therefore that G and G^* are similar. Whereas similar groups always have the same constants c_{ijk} (in the sense explained above), the converse is not generally true. For groups of category I, however, the above reasoning shows that all groups which have the same constants c_{ijk} are actually similar. This discovery prompted LIE to conclude that "herewith the existence of a limited number of types of transformation group of the special kind considered is proved or at least suggested" [1873–4: 592].

LIE's conclusion seems to derive from the following line of thought. He realized that the constants c_{ijk} in (8.7) and (8.9) satisfy many relations that follow from the properties of POISSON brackets, notably from the JACOBI Identity $((H_i, H_k), H_j) + ((H_k, H_j), H_i) + ((H_j, H_i), H_k) = 0$ [1873–74: 591] and of course from $(H_i, H_j) + (H_j, H_i) = 0$. Although he did not write them down for MAYER, the following relation follow readily by direct computation using (8.9):

$$c_{ijk} + c_{jik} = 0, \qquad i, j, k = 1, \ldots, r,$$

$$\sum_{s=1}^{r} \{c_{iks} c_{sjt} + c_{kjs} c_{sit} + c_{jis} c_{skt}\} = 0, \quad i, j, k, t = 1, \ldots, r. \tag{8.12}$$

His results about groups of category I showed that the non-similar groups of this type have non-isomorphic LIE algebras and thus correspond to "non-isomorphic" sets of constants c_{ijk}. (LIE of course would have expressed these facts differently.) For a fixed value of r, there are at most as many non-similar groups of category I as there are non-isomorphic sets c_{ijk} satisfying the many relations in (8.12). He probably conjectured, that the number of such sets would be significantly limited by all the conditions imposed by (8.12). The algebraic problem of actually classifying up to isomorphism the c_{ijk} satisfying (8.12), and consequently defining an abstract LIE algebra, was first studied fifteen years later by KILLING, who resolved it for semi-simple algebras. In 1873–74 LIE turned aside from what turned out to be an extremely difficult problem.

In the case of groups of categories II and III, equation (8.11) indicates that Theorem 6.5 will have an analogous application if the functions K_l are the same for the two groups. LIE observed that for $1 \leqq i \leqq \varrho$ and $j = \varrho + l$, by (8.11), $\Omega_{i\varrho+l} = (H_i, H_{\varrho+l}) = (H_i, K_l(H_1, \ldots, H_r))$. Expansion of the POISSON bracket $(H_i, K_l(H_1, \ldots, H_r))$ by the standard technique of the chain rule turns the above equalities into

$$\Omega_{i\varrho+l} = \sum_{j=1}^{\varrho} \frac{\partial K_l}{\partial H_j} (H_i, H_j), \quad i = 1, \ldots, \varrho. \tag{8.13}$$

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

Equation (8.13) can be intepreted as a system of ϱ linear equations in the unknowns $\frac{\partial K_l}{\partial H_j}$ with matrix of coefficients (H_i, H_j), $1 \leqq i, j \leqq \varrho$. This matrix was familiar to LIE from his theory of function groups [1873c: 41], its rank equals $\varrho - m$ and so determines m. For category II groups $m = 0$, and so the system (8.13) is non-singular and may be solved for the $\frac{\partial K_l}{\partial H_j}$ to obtain

$$(8.14) \qquad \frac{\partial K_l}{\partial H_j} = f_{lj}(H_1, \ldots, H_\varrho, K_1, \ldots, K_{r-\varrho}), \quad 1 \leqq j \leqq \varrho, 1 \leqq l \leqq r - \varrho.$$

Equation (8.14) repesents a system of $\varrho \times (r - \varrho)$ first order partial differential equations in $\varrho = 2q$ independent variables H_i and $r - \varrho$ dependent variables K_l. The functions K_l of (8.10) are therefore solutions to this system. LIE concluded that the K_l are functions of the H_i and a certain number of constants of integration $C_1, \ldots, C_e$:

$$(8.15) \qquad K_l = K_l(H_1, \ldots, H_\varrho, C_1, \ldots, C_e).$$

Substitution of (8.15) into (8.11) shows that two category II groups G, G^* such that $\varrho = \varrho^*$, $c_{ijk} = c^*_{ijk}$, and $C_i = C^*_i$, so that $\Omega_{ij}(H_1, \ldots, H_\varrho) = \Omega^*_{ij}(H^*_1, \ldots, H^*_\varrho)$ for all $1 \leqq i, j \leqq \varrho$, will be similar by Theorem 6.5. "Thus the existence of a limited number of types of transformation groups is easily recognised" [1873–74: 593]. Each type (for groups of fixed dimension r) is determined by the integer ϱ and by the isomorphism class of the constants c_{ijk}. Two groups G, G^* within the same type are similar when the H_i and H^*_i can be chosen so that the corresponding structure constants c_{ijk} and constants of integration C_i are identical. LIE showed that the same reasoning extends, with minor modifications, to groups in category III and yields an analogous division into a finite number of types.

Immediately after presenting these results on the classification into types LIE added: "I suspect that these propositions will form the foundations for what I would almost like to call a new discipline. This transformation theory would occupy the same position within the theory of continuous magnitudes that is occupied within the theory of discrete magnitudes by the theory of substitutions [= permutation groups] ... I have posited many general propositions on transformation groups, but they do not have the import of the preceding" [1873–74: 594]. These words support the view that the link between function groups and transformation groups established by Theorems 7.5–7.6 and 6.5, and the results that flowed from it, were primarily responsible for convincing LIE that this work on partial differential equations and particularly his invariant theory of contact transformations, had put him in a position to create an entirely new theory — a theory of continuous groups.

LIE's letters to MAYER contain further evidence supporting this view. For example, it was also on the basis of the link between transformation groups and function groups that he told MAYER, in his first letter on transformation groups, that the two theorems he had stated for groups in one variable had analogs for groups in any number of variables. In his second letter he conjectured such theorems for category II groups. Thus corresponding to the one variable theorem that

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

$r \leqq 3$, he now conjectured that for r-dimensional category II groups in any number of variables, $r \leqq R$, where R depends only on ϱ [1873–4: 594]. And corresponding to the one variable theorem that every finite dimensional group is similar to a subgroup of the group of "linear" transformations $y' = \frac{ay + b}{cy + d}$, he conjectured the theorem that groups of category II in any number of variables are similar to a linear group.[74] Although he never pursued these conjectures any further,[75] the fact that he formulated them is a further indication of his confidence that he could deal with the theory of transformation groups in any number of variables by applying his theory of homogeneous function groups.

By the time of his first letter to MAYER on transformation groups (2 December 1873), LIE was evidently already committed to his new theory, brimming with confidence that he was in a position to make important contributions. This was certainly due in part to the explicit resolution of Problem 8.2 for groups in one variable, which he communicated in that letter. Judging by the essay in his second letter on transformations groups (3 February 1874), it would seem that his confidence was based primarily on the connection he had discovered between function groups and transformation groups, a connection which provided a foothold for dealing with groups in any number of variables. In other words, it would seem that this connection and some of its implications had already been discovered by the time of his first letter. Such an interpretation is plausible given the fact that by November 1873 he had discovered the key theorems from which the connection follows (Theorems 6.5 and 7.5). It is also supported by an undated manuscript [1873 MSb] which LIE later estimated to have been written in December 1873. It is a brief introduction to infinitesimal transformations and transformation groups which in many respects resembles a rough first draft of the essay he included in the letter of February 3. Half of the document is devoted to groups of contact transformations, and the connection with function groups is made through (8.9). The presentation is very sketchy but indicates that, at least for groups of category I, he already recognized the classification into a "limited number" of types.[76]

Indirect evidence also suggests the historical importance of the connection between function and transformation groups and places LIE's theory of similarity types in a different light. At the conclusion of § 7 LIE's recollections from 1879 on the birth of his theory of groups were quoted at length. Judging by that passage, and the portion of it chosen for emphasis by him, his concern with the theory

[74] Exactly what LIE had in mind as justification of this conjecture is unknown, but it may have been linked to his discovery of what would now be called the adjoint representation of a LIE algebra, for that representation is included in his first publication developing the above ideas [1876b: 73–74]. (He used it to conclude erroneously that any finite-dimensional group of homogeneous contact transformations is isomorphic to a linear group. It was only later that he observed that the existence of non-trivial center meant that the representation was not faithful.)

[75] According to ENGEL (LIE, *Ges. Abh.* **5**, p. 595).

[76] Sei nun $\phi_1, \ldots, \phi_r$ im alten Sinne eine Gruppe; als dann lässt sich leicht beweisen, dass es nur eine begrenzte Zahl *typische* Formen giebt" [1873 MSb: 3].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

of transformation groups in the years 1874–79 derived especially from his discovery "that *the determination of all groups of a n-fold extended manifold can be achieved by the integration of ordinary differential equations*." Here he seems to be claiming that he had discovered how to resolve Problem 8.2 for groups in any number of variables, in the sense that he had reduced the problem to integrating systems of ordinary differential equations, precisely the same sort of reduction that had been the goal of the general theory of first order partial differential equations since the work of LAGRANGE. Although such a resolution is not mentioned explicitly in his letters to MAYER (or elsewhere in documents from 1873–74), it follows from the considerations leading to the classification of groups into similarity types by reasoning that was familiar to him from his theory of function groups. The discovery that helped launch LIE's lifelong investigation of transformation groups was, according to this interpretation, an aspect of his classification by types.

I base this interpretation on the presentation of the classification in one of LIE's earliest publications containing details of his new theory of groups [1876b]. In [1876b: 68–73] he presented the classification by types more or less as he had in his letter to MAYER, but he added a new interpretation which he explained for category II groups as follows: "all transformation groups $H_1, \ldots, H_r$ with composition given by the equation $(H_i, H_j) = \sum_s c_{ijs} H_s$ can be determined in the following manner. Choose an arbitrary even number no greater than r, say $[2q]$, and make the first hypothesis that the related function group contains $[2q]$ terms. Then from among r quantities $H_1, \ldots, H_r$ choose arbitrarily $[2q]$, say H_a, $H_b, \ldots, H_g$, and make the second hypothesis that they are independent functions. Then by the theory developed above the remaining H are determined as functions of the chosen H ... [and constants]. Finally, if the constants are given specific values all transformation groups hereby determined are similar" [1876b: 70]. To clarify these remarks, let $H_a, H_b, \ldots, H_g$ be denoted by $H_1, \ldots, H_\varrho$, where $\varrho = 2q$, and the remaining H by $H_{\varrho+l} = K_l$, in accordance with the notation employed above. Since the c_{ijk} are assumed to be given, the expressions (H_i, H_j) and $\Omega_{i\varrho+l}$ in (8.13) are known functions of the variables $H_1, \ldots, H_\varrho, K_1, \ldots, K_{r-\varrho}$. By formally solving the linear system of equations (8.13) (using determinants), equations of the form (8.14) are obtained in which the f_{lj} are known functions of the variables H_i and K_l. Thus (8.14) represents a known system of first order equations, and its integrals yield the remaining H, in the sense that now

$$H_{\varrho+l} = K_l = K_l(H_1, \ldots, H_\varrho, C_1, \ldots, C_e).$$

At this point, in characteristic fashion, he ended his explanation, leaving it for the reader to fill in the remaining details.

At this point, the H's are still symbols and not functions of $(x_1, \ldots, x_n, p_1, \ldots, p_n)$ for some n. The above considerations indicate that it suffices to show that the symbols $H_1, \ldots, H_\varrho$ correspond to an actual ϱ-term function group in (x, p). The problem is to show that ϱ functions $H_i(x, p)$ exist which satisfy $(H_i, H_j) = \Omega_{ij}(H_1, \ldots, H_\varrho)$, where the functions Ω_{ij} are given: they are determined by the constants c_{ijk} and by the functions $K_l(H_1, \ldots, H_\varrho)$. In LIE's *Theorie der Transformationsgruppen*, where the details of many of his discoveries were first worked out, this type of problem is resolved [1890: 233ff.] and then applied to Problem 8.2 [1890: 336ff.]. The idea (specialized to the present context) is to show that $\varrho = 2q$

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

independent functions $X_1, \ldots, X_q, P_1, \ldots, P_q$ of the H_i exist such that $\{P_i, X_j\} = \delta_{ij}$, and $\{X_i, X_j\} = \{P_i, P_j\} = 0$, where for any two functions of variables $H_1, \ldots, H_\varrho$, $\{F, G\} = \sum_{i,j=1}^{\varrho} \Omega_{ij}(H_1, \ldots, H_\varrho) \frac{\partial F}{\partial H_i} \frac{\partial G}{\partial H_j}$. (The definition of $\{F, G\}$ is motivated by the fact that when the H_i are functions of (x, p), so that F and G are also, the chain rule shows that $(F, G) = \{F, G\}$.) Then the system of $\varrho = 2q$ equations $x_i = X_i(H_1, \ldots, H_\varrho)$, $p_i = P_i(H_1, \ldots, H_\varrho)$, $i = 1, \ldots, q$, may be inverted to yield the H_i as functions of (x, p) which satisfy $(H_i, H_j) = \{H_i, H_j\} = \Omega_{ij}$. Since this idea had already been used by LIE to obtain canonical forms for function groups [1873c: § 3], it is most likely what he had in mind when he composed [1876b].

The existence of the functions X_i, P_i is established by successively integrating linear systems of first order partial differential equations. Consequently the determination of $H_1, \ldots, H_\varrho$ as functions of (x, p), and hence the determination of the group G they define, is reduced to the integration of first order equations: the system (8.14) and the above mentioned linear equations. Since the integration of first order equations reduces in turn to that of systems of ordinary differential equations, LIE was in a position to see that the determination of all finite dimensional groups of homogeneous contact transformations reduces to the integration of systems of ordinary differential equations.[77] His recollections of 1879 suggest that he had realized this by the winter of 1873–74. Seeing he could solve Problem 8.2 in the same sense that *e.g.* JACOBI's new method solves the problem of integrating first order equations, he believed he had laid the foundations for an entirely new theory, a theory based upon the fundamental relations (8.7) of Theorem 8.2.

During the critical period when LIE was deciding to commit himself to the task of creating a theory of finite dimensional transformation groups, the link between finite dimensional groups and his invariant theory of contact transformations, played a vital role in convincing him that he was prepared for, perhaps even destined for, this task. The creation of a theory of continuous groups would build upon and be a continuation of his previous work on contact transformations and function groups — his invariant theory of contact transformations. At the beginning of his second letter to MAYER, he expressed such a view in the following words: "You will be interested ... to see that I have found beautiful interpretations of the symbols $A(f), A_iA_k - A_kA_i$, (H_i, H_k) etc. If I am not mistaken the so-called operational calculus gains herewith an unexpected conceptual content. It is worth noting that my investigations on [function] groups, homogeneous [function] groups, contact transformations, as well as my older works, lay ready at hand, so to speak, in order to found the new theory of transformation groups" [1873–4: 586]. This passage also confirms the emphasis in this essay on the historical importance of his discovery of the conceptual intrepretation of JACOBI's calculus of differential operators that Theorems 7.1–7.6 facilitated. In particular Theorems 7.5–7.6 forged the link with his theory of function groups. As he explained in a paper submitted July 6, 1874, the theorem that every infinitesimal homogeneous contact transformation is of the form $X(f) = (H, f)$ (Theorem 7.6) "has become the starting point of my investigations on transformation groups" [1874b: 26].

[77] Although this conclusion is not explicitly drawn in [1876b], LIE later cited [1876b] as its source [1890: 345].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

In analyzing the birth of LIE's theory of groups, I have divided his work during 1869–73 into two periods, a geometrically oriented period (1869–71) and the period (1871–73) covered in this essay. From this essay it should be clear that the influence of JACOBI on LIE, both directly through his work and indirectly through the work of his successors, was pervasive. Without the work of JACOBI it is difficult to imagine the birth of LIE's theory of groups occurring in the fall of 1873. But this essay also brings out the fundamental importance of LIE's work during his geometrical period. Although JACOBI's influence upon LIE was indeed pervasive, it was always subservient to the concepts and research programs which had emerged during his geometrical period and which continued to guide his research thereafter. During 1869–71 his work was dominated by two successive research projects: the study of the geometry of tetrahedral complexes (1869–70) and the study of the sphere mapping (1870–71). From the first research project came a fundamental concept and a fundamental idea: the concept of a continuous group of transformations and the idea of a continuous analog of GALOIS' theory of algebraic equations — his *idée fixe*. Likewise a fundamental concept and a related idea originated in the sphere mapping work: the concept of a contact transformation and the idea of an invariant theory of contact transformations. During the period 1871–73, these concepts and ideas were developed within the remarkably fertile context of the theory of first order partial differential equations in the form given to it by JACOBI and his successors. As a result the ideas were realized. LIE was able to create a substantial invariant theory of contact transformations and to work out his *idée fixe* both within that context (the theory of function groups) and within the context of complete systems. In the process, he discovered his interpretive proof of the POISSON-JACOBI Theorem, which epitomizes his realization that group related concepts are fundamental to the general theory of partial differential equations. In LIE's mind, this realization must have increased the significance of the challenge issued at the end of KLEIN's Erlangen Program. Speaking also for LIE (who was with him when he composed it), KLEIN had called for the development of a theory of continuous transformation groups to parallel the theory of permutation groups. As was the case in JORDAN's *Traité des substitutions*, the theory was to be developed independently of its applications, which would then follow. In October 1872, when KLEIN wrote this Program, LIE felt that such an enterprise was too far fetched to be taken seriously. By the end of 1873, his pessimism had been replaced by exuberant optimism. He had discovered in the mathematics that he had created in developing his invariant theory and his *idée fixe* the approach and requisite mathematical tools for laying the foundations of a theory of finite dimensional continuous groups. During the winter of 1873–74 he began to take up the challenge of the Erlangen Program, to develop the theory of continuous groups. It became his life work.

Epilogue. The resolution of Problem 8.2 which LIE had obtained by applying the theory of function groups was theoretical. It implied the existence of a finite number of types of similarity classes for r-dimensional groups. The construction of groups within a given type was reduced to the integration of systems of ordinary differential equations. But such a construction did not yield an explicit solution such as he had obtained for groups in one variable since the integrations required

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

were not practicable. Assured by the general theory that the number of types was limited, LIE posed the problem of explicitly determining them [LIE 1873–74: 603–4]. He envisioned the solution of this problem as having important applications to other areas of mathematics such as differential equations. During the winter and spring of 1874 he concentrated on groups acting on the complex plane by means of point or contact transformations. By June 1874 he had solved Problem 8.2 explicitly for these groups, but the calculations were so involved that he did not publish the classification until he discovered a simpler derivation [1878a].[78] During 1876–77 he resolved Problem 8.2 explicitly for groups of point transformations acting on complex 3-dimensional space, but because of the extensive nature of the calculations his solution was never completely published.[79] Although a theoretical solution to Problem 8.2 by means of the theory of function groups remained in LIE's definitive presentation of his theory of groups [1890: 336ff.], the classification into types based on integrating (8.13) was abandoned in favor of a more rigorous approach.

9. References

Anonymus, 1873. "Zum Andenken an Rudolf Friedrich Alfred Clebsch," *Math. Ann.* **6** (1873), 197–202.

ARNOLD, V. I., 1989. *Mathematical Methods of Classical Mechanics*, New York, Springer-Verlag, 2nd edn., K. VOGTMANN & A. WEINSTEIN, transl., 1989.

BOUR, E., 1855a. "Sur l'intégration des équations différentielles de la mécanique analytique," *Comptes Rendus, Acad. Sci. Paris* **40** (1855), 524–526.

BOUR, E., 1855b. "Sur l'intégration des équations différentielles de la mécanique analytique," *Jl. des math. pures et appl.*, **20** (1855), 185–208.

BOUR, E., 1856. "Mémoire sur l'intégration des équations différentielles de la mécanique analytique," *Mémoires présentés par divers savants à l'Institut de France, Paris* **14** (1856), 792–812.

BOUR, E., 1862a. "Sur l'intégration des équations différentielles partielles du premier et du second ordre," *Comptes Rendus, Acad. Sci. Paris* **54** (1862), 439–444; 549–554; 588–593; 645–655.

BOUR, E., 1862b. "Sur l'intégration des équations différentielles partielles du première et du second ordre," *Journal de l'Ecole Polytechnique*, **Cahier 39** (1862), 149–191.

BOURBAKI, N., 1972. "Note historique," *Elements de mathématiques. Fasc. XXXVII. Groupes et Algèbres de Lie*, Paris, 1972, pp. 286–308.

CARATHÉODORY, C., 1935. *Variationsrechnung und partielle Differentialgleichungen erster Ordnung*, Leipzig, Teubner, 1935. Translated and edited by R. DEAN & J. BRANDSTATTER, as *Calculus of Variations and Partial Differential Equations of the First Order*, 2 vols., San Francisco, 1965 & 1967; 2nd edn., 1 vol., New York (Chelsea), 1982.

CAUCHY, A., 1819. "Note sur l'intégration des équations aux différences partielles du premiere ordre à un nombre quelconque de variables," *Bull. de la Société Philomatique* (1819), 10–21. Reprinted in *Oeuvres* (2) **2**, 238–252. Translated into German and

[78] For recent work on this problem, see [GONZÁLEZ-LOPEZ, 1990a, 1990b].

[79] See LIE's remarks on pp. 493–94 of *Ges. Abh.* **5**. A partial solution was presented in [1893: 122–178].

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

annotated by G. KOWALEWSKI in *Ostwald's Klassiker der exakten Wissenschaften, Nr. 113* (Leipzig, 1900).

CECIL, T., & S. CHERN, 1987. "Tautness and Lie Sphere Geometry," *Math. Ann.* **278** (1987), 381–399.

CLEBSCH, A., 1861. "Ueber Jacobi's Methode, die partiellen Differentialgleichungen erster Ordnung zu integriren, und ihre Ausdehnung auf das Pfaff'sche Problem, Auszug aus einem Schreiben an den Herausgeber," *Jl. für die reine u. angew. Math.* **59** (1861), 190–192.

CLEBSCH, A., 1862. „Ueber das Pfaffsche Problem," *Jl. für die reine u. angew. Math.* **60** (1862), 193–251.

CLEBSCH, A., 1863. „Ueber das Pfaffsche Problem. Zweite Abhandlung," *Jl. für die reine u. angew. Math.* **61** (1983), 146–179.

CLEBSCH, A., 1866. „Ueber die simultane Integration linearer partieller Differentialgleichungen," *Jl. für die reine u. angew. Math.* **65** (1866), 257–268.

DONKIN, W. F., 1854. "On a Class of Differential Equations, including those which occur in Dynamical Problems, Part I," *Philosophical Transactions* **144** (1854), 71–113.

DUGAS, R., 1950. *Histoire de la mécanique*, Paris and Neuchâtel, 1950.

ENGEL, F., 1899. "Sophus Lie," *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Math.-phys. Cl.* (1899), xi–lxi.

ENGEL, F., 1932. *Die Liesche Theorie der Partiellen Differentialgleichungen erster Ordnung*, Leipzig, Teubner, 1932.

EULER, L., 1770. *Institutiones calculi integralis. Volumen tertium*, St. Petersburg, 1770.

FROBENIUS, G., 1877. „Über das Pfaffische Probleme," *Jl. für die reine u. angew. Math.* **82** (1877), 230–315.

GAUSS, C., 1815. Review of [PFAFF 1815], *Göttingische gelehrte Anzeigen* 1815, 1025–1038.

GONZÁLEZ-LOPEZ, A., N. KAMRAN & P. OLVER, 1990a. "Lie Algebras of Differential Operators in Two Complex Variables," *Institute for Mathematics and Its Applications. Preprint Series # 617* (1990).

GONZÁLEZ-LOPEZ, A., N. KAMRAN & P. OLVER, 1990b. "Lie Algebras of Vector Fields in the Real Plane," preprint (1990).

GOURSAT, E., 1921. *Leçons sur l-intégration des équations aux dérivées partielles du premier ordre*, Paris, 2nd edn., 1921.

GULDBERG, A., 1913. „Verzeichnis über den wissenschaftlichen Nachlass von Sophus Lie. Zweite Mitteilung," *Videnskapsselskapet Skrifter* 1. *Mat.-Naturw. Klasse* 1913, *No.* 5.

HAMILTON, W. R., 1834. "On a General Method in Dynamics by which the Study of the Motions of all Free Systems of Attracting or Repelling Points is Reduced to the Search and Differentiation of One Central Relation, or Characteristic Function," *Phil. Trans. R. Soc. London, 1834* (*part II*), (1834), 247–308. Reprinted in *Papers* **2**, 103–161.

HAMILTON, W. R., 1835. "Second Essay on a General Method in Dynamics," *Phil. Trans. R. Soc. London, 1835* (*part I*), (1835), 95–144. Reprinted in *Papers* **2**, 162–211.

HANKINS, T. L., 1980. *Sir William Rowan Hamilton*, Baltimore, Johns Hopkins Press, 1980.

HAWKINS, T., 1980. "Non-Euclidean Geometry and Weierstrassian Mathematics: The Background to Killing's Work on Lie Algebras," *Historia Mathematica* **7** (1980), 289–342.

HAWKINS, T., 1982. "Wilhelm Killing and the Structure of Lie Algebras," *Archive for Historyof Exact Sciences* **26** (1982), 127–192.

HAWKINS, T., 1989. "Line Geometry, Differential Equations and the Birth of Lie's Theory

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

of Groups," *The History of Modern Mathematics*, Vol. 1, J. McCleary & D. Rowe, ed., New York, Academic Press, 1989, pp. 275–327.

Hermann, R., 1970. *Vector Bundles in Mathematical Physics*, vol. 2, New York 1970.

Hermann, R., 1973. *Geometry, Physics, and Systems*, New York 1973.

Hermann, R., 1975. *Sophus Lie's 1880 Transformation Group Paper. Translated by M. Ackerman. Comments by Robert Hermann*, Brookline, Mass. (Math Sci Press), 1975.

Hermann, R., 1976. *Sophus Lie's 1884 Differential Invariant Paper. Translated by M. Ackerman. Comments and Additional Material by Robert Hermann*, Brookline, Mass. (Math Sci Press), 1976.

Imschenetsky, W., 1869. "Sur l'intégration des équations aux dérivées partielles du première ordre," *Archiv für Math. u. Physik* **50** (1869), 278–474.

Jacobi, C., 1827a. „Über die Integration der partiellen Differentialgleichungen erster Ordnung," *Jl. für die reine u. angew. Math.* **2** (1827), 317–329. Reprinted in *Werke* **4**, 3–15.

Jacobi, C., 1827b. „Über die Pfaffsche Methode, eine gewöhnliche lineare Differentialgleichung zwischen $2n$ Variabeln durch ein System von n Gleichungen zu integriren," *Jl. für die reine u. angew. Math.* **2** (1827), 347–357. Reprinted in *Werke* **4**, 19–29.

Jacobi, C., 1837. „Zur Theorie der Variations-Rechnung und der Differential-Gleichungen," *Jl. für die reine u. angew. Math.* **17** (1837), 68–82. Reprinted in *Werke* **4**, 41–55. [29 November 1836.]

Jacobi, C., 1837. „Über die Reduction der Integration der partiellen Differentialgleichungen erster Ordnung zwischen irgend einer Zahl Variablen auf die Integration eines einzigen Systems gewöhnlicher Differentialgleichungen," *Jl. für die reine u. angew. Math.* **17** (1837), 97–162. Reprinted in *Werke* **4**, 59–127.

Jacobi, C., 1840. "Sur un théorème de Poisson," *Comptes Rendus, Acad. Sci. Paris*, **11** (1840), 529–30. Reprinted in *Werke* **4**, 145–146.

Jacobi, C., 1862. "Nova methodus, aequationes differentiales partiales primi ordinis inter numerum variabilium quemcumque propositas integrandi," *Jl. für die reine u. angew. Math.* **60** (1862), 1–181. Reprinted in *Werke* **4**, 3–189. [A. Clebsch, ed.] Translated into German and annotated by G. Kowalewski as *Ostwald's Klassiker der exakten Wissenschaften, Nr.* 156 (Leipzig, 1906).

Jacobi, C., 1866. *Vorlesungen über Dynamik*, Berlin, A. Clebsch, ed., 1866.

Jordan, C., 1870. *Traité des substitutions et des équations algébriques*, Paris, 1870.

Klein, F., 1871. „Über die sogenannte Nicht-Euklidische Geometrie (Zweiter Aufsatz)," *Math. Ann.* **6** (1873), 311–343. Reprinted in *Abhandlungen* **1**, 311–343. [Göttingen 8 June 1872.]

Klein, F., 1872. *Vergleichende Betrachtungen über neuere geometrische Forschungen. Programm zum Eintritt in die philosophische Fakultät und den Senat der k. Friedrich-Alexanders-Universität zu Erlangen*, Erlangen, 1872. Reprinted with additional notes in *Math. Ann.* **43** (1893) = *Abhandlungen* **1**, 460–497.

Klein, F., 1916 MS. Manuscript entitled „Mitteilungen von Felix Klein über seine Beziehungen zu Lie. (Nach Aufzeichnungen, die Engel im Sept. 1916 in Göttingen gemacht hat, wohin zu kommen ihn Klein aufgefordert hatte.)" 25 pages. Manuscript in the hand of Friedrich Engel. In possession of Professor Ernst Hölder, Mainz, Germany.

Klein, F., & S. Lie, 1871. „Über diejenigen ebenen Kurven, welche durch ein geschlossenes System von einfach unendlich vielen vertauschbaren linearen Transformationen in sich übergehen," *Math. Ann.* **4** (1871), 50–84. Reprinted in Klein, *Abhandlungen* **1**, 424–459 and Lie, *Abhandlungen* **1**, 229–285. [March 1871.]

Koenigsberger, L., 1904. *Carl Gustav Jacob Jacobi*, Leipzig, 1904.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

LAGRANGE, J. L., 1772. "Sur l'intégration des équations à differences partielles du premier ordre," *Nouv. Mém. Acad. des Sci. et Belles-Lettres Berlin, 1772*, (1774), 353–372. Reprinted in *Oeuvres* **3**, 549–575. Translated into German and annotated by G. KOWALEWSKI in *Ostwald's Klassiker der exakten Wissenschaften, Nr. 113* (Leipzig, 1900).

LAGRANGE, J. L., 1774. "Sur les intégrales particulières des équations différentielles," *Nouv. Mém. Acad. des Sci. et Belles-Lettres Berlin, 1774*. Reprinted in *Oeuvres* **4**, 5–108.

LAGRANGE, J. L., 1781. "Sur différentes questions d'analyse relatives à la théorie des integrales particulières," *Nouv. Mém. Acad. des Sci. et Belles-Lettres Berlin*, 1779, (1781), 121–160. Reprinted in *Oeuvres* **4**, 585–634.

LAGRANGE, J. L., 1806. *Leçons sur le calcul des fonctions. Novelle édition*, Paris, 1806.

LIE, S., 1869a. „Repräsentation der Imaginären der Plangeometrie," *Forhandlinger Christiania 1869*, (1870), 16–38, 107–146. Reprinted in *Abhandlungen* **1**, 16–66. [August 1869.]

LIE, S., 1869b. „Über eine Darstellung des Imaginären in der Geometrie," *Jl. für die reine u. angew. Math.* **70** (1869), 346–353. Reprinted in *Abhandlungen* **1**, 1–11. [March 1869.]

LIE, S., 1871a. „Zur Theorie eines Raumes von *n* Dimensionen," *Göttingen Nachrichten 1871*, (1871), 535–557. Reprinted in *Abhandlungen* **1**, 271–285. [15 November 1871.]

LIE, S., 1871b. „Zur Theorie eines Raumes von *n* Dimensionen II," *Abhandlungen* **7** (1960), 1–10. [3 Dezember 1871.]

LIE, S., 1872a. „Kurzes Résumé mehrerer neuer Theorien," *Forhandlinger Christiania 1872*, (1873), 24–27. Reprinted in *Abhandlungen* **3**, 1–4. [30 April 1872.]

LIE, S., 1872b. „Neue Integrationsmethode partieller Gleichungen erster Ordnung zwischen *n* Variabeln," *Forhandlinger Christiania 1872*, (1873), 28–34. Reprinted in *Abhandlungen* **3**, 5–11. [8 May 1872.]

LIE, S., 1872c. „Zur Invariantentheorie der Berührungstransformationen," *Forhandlinger Christiania 1872*, (1873), 133–135. Reprinted in *Abhandlugen* **3**, 29–30. [20 December 1872.]

LIE, S., 1872d. „Zur Theorie der Differentialprobleme," *Forhandlinger Christiania 1872*, (1873), 132–133. Reprinted in *Abhandlungen* **3**, 27–28. [14 November 1872.]

LIE, S., 1872e. „Zur Theorie partieller Differentialgleichungen erster Ordnung, insbesondere über eine Klassifikation derselben," *Göttingen Nachrichten 1872*, (1872), 473–489. Reprinted in *Abhandlungen* **3**, 16–26. [Erlangen 11 October 1872.]

LIE, S., 1872f. „Über eine neue Integrationsmethode partieller Differentialgleichungen erster Ordnung," *Göttingen Nachrichten 1872*, (1872), 321–326. Reprinted in *Abhandlungen* **3**, 12–15. [Christiania, 4 June 1872.]

LIE, S.. 1873a. „Partielle Differentialgleichungen erster Ordnung, in denen die unbekannte Funktion explizite vorkommt," *Forhandlinger Christiania 1873*, (1874), 52–85. Reprinted in *Abhandlungen* **3**, 64–95. [21 March 1873.]

LIE, S., 1873b. „Zur analytischen Theorie der Berührungstransformationen," *Forhandlinger Christiania 1873*, (1874), 237–262. Reprinted in *Abhandlungen* **3**, 96–119. [August 1873.]

LIE, S., 1873c. „Über partielle Differentialgleichungen erster Ordnung," *Forhandlinger Christiania 1873*, (1874), 16–51. Reprinted in *Abhandlungen* **3**, 32–63. [21 March 1873.]

LIE, S., 1873 MSa. Draft of a letter received by MAYER on November 12, 1873. Four folio pages. Royal University Library, Oslo, Ms. fol. 3839 XL: 14.

LIE, S., 1873 MSb. Manuscript with headings, „Begriff einer infinitesimalen Transformation," „Transformations-Gruppen", and „Gruppen von Berührungs-Transformationen." 4 folio pages, undated but bearing a later annotation in LIE's hand „Sehr alt, etwa Decbr. 1873," Royal University Library, Oslo, Ms. fol. 3839 XL: 11.

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

LIE, S., 1874a. „Allgemeine Theorie partieller Differentialgleichungen erster Ordnung,“ *Forhandlinger Christiania 1874*, (1875), 198–226. Reprinted in *Abhandlungen* **3**, 149–174. [20 November 1874.]

LIE, S., 1874b. „Begründung einer Invariantentheorie der Berührungstransformationen,“ *Math. Ann.* **8** (1874), 215–303. Reprinted in *Abhandlungen* **4**, 1–96. [Christiania 5 July 1874.]

LIE, S., 1874c. „Verallgemeinerung und neue Verwertung der Jacobischen Multiplikatortheorie,“ *Forhandlinger Christiania 1874*, (1875), 255–274. Reprinted in *Abhandlungen* **3**, 188–205. [November 1874.]

LIE, S., 1874d. „Zur Theorie des Integrabilitätsfaktors,“ *Forhandlinger Christiania 1874*, (1875), 242–254. Reprinted in *Abhandlungen* **3**, 176–186.

LIE, S., 1874e. „Über Gruppen von Transformationen,“ *Göttingen Nachrichten 1874*, (1874), 529–542. Reprinted in *Abhandlungen* **5**, 1–8. [3 December 1874.]

LIE, S., 1875. „Allgemeine Theorie der partiellen Differentialgleichungen erster Ordnung,“ *Math. Ann.* **9** (1875–76), 245–296. Reprinted in *Abhandlungen* **4**, 97–151. [Christiania June 1875.]

LIE, S., 1876a. „Allgemeine Theorie der partiellen Differentialgleichungen erster Ordnung. Zweite Abhandlung,“ *Math. Ann.* **11** (1877), 464–557. Reprinted in *Abhandlungen* **4**, 163–262. [Christiania 15 October 1876.]

LIE, S., 1876b. „Theorie der Transformationsgruppen. Abhandlung II,“ *Archiv for Mathematik* **1** (1876), 152–193. Reprinted in *Abhandlungen* **5**, 42–75. [Kristiania May 1876.]

LIE, S., 1876c. „Theorie der Transformationsgruppen. Erster Abhandlung,“ *Archiv for Mathematik* **1** (1876), 19–57. Reprinted in *Abhandlungen* **5**, 9–41.

LIE, S., 1878a. „Theorie der Transformationsgruppen III. Bestimmung aller Gruppen einer zweifach ausgedehnten Punktmannigfaltigkeit,“ *Archiv for Mathematik* **3** (1878), 93–128. Reprinted in *Abhandlungen* **5**, 78–133.

LIE, S., 1878b. „Theorie der Transformationsgruppen. Abhandlung IV,“ *Archiv for Mathematik* **3** (1878), 375–460. Reprinted in *Abhandlungen* **5**, 136–197. [Christiania 15 October 1878.]

LIE, S., 1879a. „Theorie der Transformationsgruppen V,“ *Archiv for Mathematik* **4** (1879), 232–261. Reprinted in *Abhandlungen* **5**, 199–222.

LIE, S., 1879b. „Theorie der Transformationsgruppen I,“ *Math. Ann.* **16** (1880), 441–528. Reprinted in *Abhandlungen* **6**, 1–94. [Christiania, December 1879.]

LIE, S., 1884a. „Allgemeine Untersuchungen über Differentialgleichungen, die eine kontinuierliche, endliche Gruppe gestatten,“ *Math. Ann.* **25** (1885), 71–151. Reprinted in *Abhandlungen* **6**, 139–223. [Christiania, 5 July 1884.]

LIE, S., 1884b. „Über Differentialinvarianten," *Math. Ann.* **24** (1884), 537–578. Reprinted in *Abhandlungen* **6**, 95–138. [Christiania, 29 May 1884.]

LIE, S., 1888a, 1890, 1893. *Theorie der Transformationsgruppen. Unter Mitwirkung von ... Friedrich Engel*, 3 vols., Leipzig, 1888–1893.

LIE, S., 1888b. „Zur Theorie der Berührungstransformationen,“ *Abhandlungen k. sächs. Ges. Wiss. Leipzig Math. phys. Kl.* **14** (1888), 537–562. Reprinted in *Abhandlungen* **4**, 265–290. [7 July 1888.]

LIE, S., 1889. „Über irreduzible Berührungstransformationsgruppen,“ *Berichte über d. Verh. d. Sächsischen Gesell. der Wiss., math.-phys. Klasse, 1889*, (1890), 320–327. Reprinted in *Abhandlungen* **6**, 260–266. [1 July 1889.]

LIE, S., 1896. *Geometrie der Berührungstransformationen. Dargestellt von Sohus Lie und Georg Scheffers*, Leipzig, 1896.

LIOUVILLE, J., 1840. "Note sur un des théorèms de Jacobi," *Jl. des math. pures et appl.* **5** (1840), 351–355.

LIOUVILLE, J., 1855. "Sur l'intégration des équations différentielles de la Dynamique,

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

présentée au Bureau des Longitudes le 29 juin 1853," *Jl. des math. pures et appl.* **20** (1855), 137–138.

LÜTZEN, J., 1990. *Joseph Liouville 1809–1882: Master of Pure and Applied Mathematics*, New York, Springer-Verlag, 1990.

MAYER, A., 1871. "„Ueber die Integration simultaner partieller Differentialgleichungen der ersten Ordnung mit derselben unbekannten Function," *Math. Ann.* **4** (1871), 88–94.

MAYER, A., 1872a. „Die Lie'sche Integrationsmethode der partiellen Differentialgleichungen erster Ordnung," *Göttingen Nachrichten 1872*, (1872), 467–472. [9 October 1872.]

MAYER, A., 1872b. „Ueber unbeschränkt integrable Systeme von linearen totalen Differentialgleichungen und die simultane Integration linearer partieller Differentialgleichungen," *Math. Ann.* **5** (1872), 448–470. [Leipzig, February 1872.]

MAYER, A., 1873. „Die Lie'sche Integrationsmethode der partiellen Differentialgleichungen erster Ordnung," *Math. Ann.* **6**, 162–196. [October 1872.]

MONGE, G., 1850. *Application de l'analyse a la géométrie. Cinquième édition, revue, corrigée et annotée par M. Liouville*, Paris, Bachelier, 1850.

OLVER, P. J., 1986. *Applications of Lie Groups to Differential Equations*, New York, Springer-Verlag, 1986.

PFAFF, J. F., 1815. "Methodus generalis aequationes differentiarum partialium, nec non aequationes differentiales vulgares, utrasque primi ordinis, inter quotcunque variabiles, complete integrandi," *Abhandlungen d. Akad. der Wiss. zu Berlin 1814–15*, (1818), 76–136. [15 May 1815.] Translated into German and annotated by G. KOWALEWSKI as *Ostwald's Klassiker der exakten Wissenschaften, Nr. 128* (Leipzig, 1902).

POISSON, S., 1809. "Mémoire sur la variation des constantes arbitraires dans les questions de mécanique," *Journal de l'Ecole Polytechnique*, Cahier 15 (1809), 266–344.

ROWE, D., 1989. "The Early Geometrical Works of Felix Klein and Sophus Lie," *The History of Modern Mathematics*, Vol. 1, J. MCCLEARY & D. ROWE, ed., New York, Academic Press, 1989, pp. 200–273.

SCHOLZ, E., 1980. *Geschichte des Mannigfaltigkeitsbegriffes von Riemann bis Poincaré*, Boston, Birkhäuser, 1980.

SERRET, J., 1866. *Cours d'Algèbre Supérieure*, 3rd edn., Paris, 1866.

SINGER, I., & S. STERNBERG, 1965. "The Infinite Groups of Lie and Cartan, Part I (The Transitive Groups)," *Journal d'analyse mathématique* **15** (1965), 1–114.

TOBIES, R., & D. ROWE, 1990. *Korrespondenz Felix Klein-Adolph Mayer. Auswahl aus den Jahren 1871–1907 Herausgegeben, eingeleitet und kommentiert von R. Tobies und D. Rowe*, Leipzig (B. G. Teubner).

WEILER, J. A., 1859. „Integration der partiellen Differentialgleichungen erster und zweiten Ordnung," *Archiv der Mathematik und Physik* **33** (1859), 171–329.

WEILER, J. A., 1863. „Integration der partiellen Differentialgleichungen 1. O. mit $n + 1$ Veränderlichen," *Ztschr. für Math. u. Physik* **8** (1963), 264–292.

WEINSTEIN, A., 1983. "Sophus Lie and symplectic geometry," *Expositiones Mathematicae*, **1** (1983), 95–96.

10. Glossary

admitting transformations: see above Theorem 1.1.
canonical (contact) transformation: see following Theorem 6.3.
CAUCHY's method: see (4.8).
characteristic curve: see following (4.5).

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms

Department of Mathematics
Boston University

(Received October 27, 1990)

This content downloaded from 142.66.3.42 on Sun, 04 Sep 2016 03:59:22 UTC
All use subject to http://about.jstor.org/terms