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Quadratic Integers and Coxeter Groups

Dedicated to H. S. M. Coxeter, mentor and friend

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Abstract. Matrices whose entries belong to certain rings of algebraic integers can be associated with discrete groups of transformations of inversive n-space or hyperbolic $(n+1)$-space H^{n+1}. For small n, these may be Coxeter groups, generated by reflections, or certain subgroups whose generators include direct isometries of H^{n+1}. We show how linear fractional transformations over rings of rational and (real or imaginary) quadratic integers are related to the symmetry groups of regular tilings of the hyperbolic plane or 3-space. New light is shed on the properties of the rational modular group $\mathrm{PSL}_2(\mathbb{Z})$, the Gaussian modular (Picard) group $\mathrm{PSL}_2(\mathbb{Z}[i])$, and the Eisenstein modular group $\mathrm{PSL}_2(\mathbb{Z}[\omega])$.

1 Introduction

Each of the classical spaces of constant curvature has a continuous group of isometries that (for some n) is a subgroup of the general linear group $\mathrm{GL}_n(\mathbb{R})$ of $n \times n$ invertible matrices over $\mathbb{R}$ or its central quotient group, the projective general linear group $\mathrm{PGL}_n(\mathbb{R})$. The *orthogonal* group O_n of real $n \times n$ matrices A such that $AA^\vee = I$ (the inverted circumflex denoting the transpose) is the group of isometries of the $(n-1)$-sphere S^{n-1}, and the *projective orthogonal* group $\mathrm{PO}_n \cong O_n/\langle -I\rangle$ is the group of isometries of elliptic $(n-1)$-space $\tilde{\mathrm{P}}^{n-1}$. If T^n is the additive (translation) group of $\mathbb{R}^n$, the *Euclidean* group $E_n \cong T^n \rtimes O_n$ of isometries of Euclidean n-space E^n can be represented by "transorthogonal" matrices of order $n+1$. Real $(n+1)\times(n+1)$ matrices A such that $AHA^\vee = H$, where H is the diagonal matrix $\backslash 1, \dots, 1, -1\backslash$, form the *pseudo-orthogonal* (or "Lorentzian") group $O_{n,1}$, and the *projective pseudo-orthogonal* group $\mathrm{PO}_{n,1} \cong O_{n,1}/\langle -I\rangle$ is the group of isometries of hyperbolic n-space H^n (see [14, pp. 444–447], [27, pp. 58–60, 67–68]).

Here we shall be primarily concerned with discrete groups of isometries, many of which are related to the symmetry groups of regular polytopes or regular honeycombs (tilings) of Euclidean or non-Euclidean space. Of particular interest will be representations of hyperblic isometries, which are in one-to-one correspondence with the circle-preserving transformations of inversive geometry.

Certain groups of transformations of inversive n-space I^n or hyperbolic $(n+1)$-space $\mathrm{H}^{n+1} (n \le 4)$ can be represented by 2×2 invertible matrices whose entries belong to the ring $\mathbb{Z}$ of rational integers or to a suitable ring of *quadratic integers*, *i.e.*, real numbers, complex numbers, or quaternions that are zeros of a monic polynomial of degree 2 with coefficients in $\mathbb{Z}$. When discrete, such groups are subgroups of groups generated by reflections, or *Coxeter groups*, frequently the symmetry groups of regular honeycombs of H^{n+1}.

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When $n = 1$, the ring of integers may be either $\mathbb{Z}$ itself or a real quadratic integral domain. When $n = 2$, it is one of the complex quadratic integral domains $\mathbb{G}$ or $\mathbb{E}$ of Gaussian or Eisenstein integers. Groups for $n = 3$ and $n = 4$ are related to quaternionic integral skew-domains. While some of these connections have long been known, others have been discovered only recently as more has been learned about discrete groups of hyperbolic isometries and about the algebraic systems themselves.

Felix Klein [16, pp. 120–121] proved that $\mathrm{PSL}_2(\mathbb{Z})$ (the "modular group") is isomorphic to the group of rotations of the regular hyperbolic tessellation $\{3, \infty\}$. Émile Picard [26] considered the analogous group $\mathrm{PSL}_2(\mathbb{G})$ (the "Picard group"). Luigi Bianchi [2], [3] showed that if D is an imaginary quadratic integral domain, the group $\mathrm{PSL}_2(D)$ acts discontinuously on hyperbolic 3-space. Fricke & Klein [10, pp. 76–93] identified $\mathrm{PSL}_2(\mathbb{G})$ with a subgroup of the rotation group of the regular honeycomb $\{3, 4, 4\}$.

Graham Higman, Bernhard Neumann, and Hanna Neumann [11] showed how to construct infinite groups in which any two elements, apart from the identity, are conjugate. Both the modular group $\mathrm{PSL}_2(\mathbb{Z})$ and the Picard group $\mathrm{PSL}_2(\mathbb{G})$ contain normal subgroups that are free products with amalgation of such HNN *groups*, a fact that gives considerable insight into their structure. However, the corresponding group $\mathrm{PSL}_2(\mathbb{E})$ over the Eisenstein integers does not have this property [1, pp. 2935–2936].

Wilhelm Magnus [20, pp. 107–122] gave geometric descriptions of $\mathrm{PSL}_2(\mathbb{Z})$ and some of its subgroups and quotient groups. Benjamin Fine [8, chap. 5] undertook a thorough algebraic treatment of $\mathrm{PSL}_2(\mathbb{G})$. Fine & Newman [9] investigated normal subgroups of $\mathrm{PSL}_2(\mathbb{G})$, and Roger Alperin [1] did likewise for $\mathrm{PSL}_2(\mathbb{E})$. Schulte & Weiss [29, p. 246] showed that $\mathrm{PSL}_2(\mathbb{E})$ can be identified with a subgroup of the rotation group of $\{3, 3, 6\}$.

Spherical and Euclidean reflection groups were completely classified by H. S. M. Coxeter [4]. Hyperbolic reflection groups with simplicial fundamental regions, which exist in H^n only for $n \leq 9$, were enumerated by Folke Lannér [18], Coxeter & Whitrow [7], and Jean-Louis Koszul [17]. Groups with nonsimplicial fundamental regions have been described by Ernest Vinberg and others. All regular honeycombs of H^n were determined by Klein [16], Schlegel [28], and Coxeter [5]; these exist only for $n \leq 5$.

It is our purpose here to show how the properties of Coxeter groups and their subgroups provide a basis for a unified theory of linear fractional transformations as represented by 2×2 matrices over rings of real, complex, or quaternionic integers. Such transformations may be taken as projectivities on a projective line, homographies of real inversive space, or direct isometries of a real hyperbolic space. For each system of integers the corresponding group of linear fractional transformations is isomorphic to a subgroup of some hyperbolic Coxeter group.

In discussing groups over different rings R that are algebraic extensions of the real field $\mathbb{R}$ or the ring $\mathbb{Z}$ of rational integers, we find it convenient to adopt a uniform notation for certain standard cases. We identify R^n with the left linear space (or lattice) of rows $(x) = (x_1, \ldots, x_n)$. The one-dimensional subspaces of linear space R^n spanned by nonzero rows are the elements $\langle (x) \rangle$ of a projective linear space $\mathrm{P}R^n$.

As usual, we denote by $\mathrm{GL}_n(R)$ the *general linear* group of $n \times n$ invertible matrices over R and by $\mathrm{SL}_n(R)$ the *special linear* group of $n \times n$ matrices of determinant 1. For R an extension of $\mathbb{R}$ or $\mathbb{Z}$, $\mathrm{SL}_n(R)$ is the commutator subgroup of $\mathrm{GL}_n(R)$. (This is true of matrix groups over arbitary rings except for 2×2 matrices over the finite fields $\mathbb{F}_2$ and $\mathbb{F}_3$.) We define the *unit linear* group $\bar{\mathrm{S}}\mathrm{L}_n(R)$ to be the group of $n \times n$ matrices over R whose determinant has an absolute value of 1. Over a ring of integers, $\bar{\mathrm{S}}\mathrm{L}_n(R)$ is the same as $\mathrm{GL}_n(R)$; in any case, $\bar{\mathrm{S}}\mathrm{L}_n(R)$ contains $\mathrm{SL}_n(R)$ as a normal subgroup.

If multiplication in R is commutative, the centre of $\mathrm{GL}_n(R)$ is the *general scalar* group $\mathrm{GZ}(R)$ of nonzero matrices λI, the centre of $\bar{\mathrm{S}}\mathrm{L}_n(R)$ is the *unit scalar* group $\bar{\mathrm{S}}\mathrm{Z}(R)$ of matrices λI with $|\lambda| = 1$, and the centre of $\mathrm{SL}_n(R)$ is the *special scalar* group $\mathrm{SZ}_n(R)$ of $n \times n$ matrices λI with $\lambda^n = 1$. The respective central quotient groups are the *projective general linear* group $\mathrm{PGL}_n(R) \cong \mathrm{GL}_n(R)/\,\mathrm{GZ}(R)$, the *projective unit linear* group $\mathrm{P}\bar{\mathrm{S}}\mathrm{L}_n(R) \cong \bar{\mathrm{S}}\mathrm{L}_n(R)/\bar{\mathrm{S}}\mathrm{Z}(R)$, and the *projective special linear* group $\mathrm{PSL}_n(R) \cong \mathrm{SL}_n(R)/\,\mathrm{SZ}(R)$. Depending on the ring R and the value of n, these three groups may or may not be distinct.

For D a ring of rational or quadratic integers, our methods lead to matrix representations of groups related to the special linear group $\mathrm{SL}_2(D)$, as well as generators and relations for the projective special linear group $\mathrm{PSL}_2(D)$ and other groups of interest. Coxeter groups can be used to show how each such projective group is realized as a discrete group of isometries in hyperbolic space of dimension 2, 3, 4, or 5. Here we obtain representations in H^2 and H^3 of projective linear groups over real and complex integers—especially the groups $\mathrm{PSL}_2(\mathbb{Z})$, $\mathrm{PSL}_2(\mathbb{G})$, and $\mathrm{PSL}_2(\mathbb{E})$. Elsewhere [15] we extend the application of the theory to groups over quaternionic integers with realizations in H^4 and H^5.

2 Reflection Groups and Their Subgroups

A Coxeter group P is generated by reflections $\rho_0, \rho_1, \ldots, \rho_n$ in the facets of a polytope P each of whose dihedral angles is a submultiple of π. If the angle between the i-th and j-th facets is π/p_{ij}, the product of reflections ρ_i and ρ_j is a rotation of period p_{ij}. A Coxeter group is thus defined by the relations

$$(1) \qquad (\rho_i\rho_j)^{p_{ij}} = 1 \quad (0 \le i \le j \le n,\ p_{ii} = 1).$$

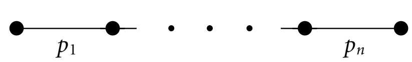
If two facets are parallel, as in the case of an asymptotic triangle in the hyperbolic plane, the corresponding relation with $p_{ij} = \infty$ may be omitted.

The polytope P—generally a simplex—whose closure forms the fundamental region for a Coxeter group P (or the group itself) is conveniently denoted by its *Coxeter diagram*, a graph whose nodes represent the facets of P (or the generators of P). Nodes i and j are joined by a branch marked 'p_{ij}' if the period of the product of the i-th and j-th generators is p_{ij}, except that when $p_{ij} = 3$, the mark is customarily omitted, and when $p_{ij} = 2$, the nodes are not joined; in the latter case, the corresponding generators commute.

When P is an *orthoscheme*, a simplex whose facets may be ordered so that any two that are not consecutive are orthogonal, the relations for P take the form

$$(2) \qquad (\rho_i\rho_j)^{p_{ij}} = 1 \quad (0 \le i \le j \le n, p_{ii} = 1, p_{ij} = 2 \text{ for } j - i > 1),$$

and it is convenient to abbreviate $p_{j-1,j}$ as p_j. Such a group corresponds to the "string diagram"



and is denoted by the *Coxeter symbol*

$$[p_1, \ldots, p_n].$$

When $p_j > 2$ for each j ($1 \le j \le n$), this is the symmetry group of a regular honeycomb of spherical, Euclidean, or hyperbolic n-space or an isomorphic regular $(n+1)$-polytope whose *Schläfli symbol* is

$$\{p_1, \dots, p_n\}.$$

This is a regular polygon $\{p\}$ or a regular apeirogon $\{\infty\}$ if $n = 1$ and for higher n is the regular polytope or honeycomb whose *facet* or *cell* polytopes are $\{p_1, \dots, p_{n-1}\}$'s and whose *vertex figures* are $\{p_2, \dots, p_n\}$'s.

A group generated by reflections in the facets of a polytope that is not an orthoscheme may likewise be given a Coxeter symbol that suggests the form of its Coxeter diagram. For instance, the group whose fundamental region is the closure of a triangle $(p\ q\ r)$ with acute (or zero) angles π/p, π/q, π/r is denoted by the symbol $[(p, q, r)]$ or, if $p = q = r$, simply by $[p^{[3]}]$.

A Coxeter group is said to be *spherical, Euclidean*, or *hyperbolic* according as it is generated by reflections in the facets of a convex polytope in spherical, Euclidean, or hyperbolic space. It is *finitary* if this polytope has finite content ("n-volume"). If in addition each subgroup generated by all but one of the reflections is spherical, the group is *compact*. If each such subgroup is either spherical or Euclidean, including at least one of the latter, it is *paracompact*. If at least one such subgroup is hyperbolic, it is *hypercompact*. A Coxeter group (1) is *crystallographic* if it leaves invariant some $(n+1)$-dimensional lattice [13, pp. 135–137].

A compact Coxeter group may be spherical, Euclidean, or hyperbolic, and its fundamental region is the closure of an ordinary simplex. A paracompact Coxeter group is either Euclidean or hyperbolic, the fundamental polytope being respectively a prism or an asymptotic *Koszul simplex*. A hypercompact group can only be hyperbolic, and the fundamental *Vinberg polytope* is not a simplex. In a crystallographic group the periods of the products of distinct generators are restricted to the values 2, 3, 4, 6, and ∞.

Each element of a Coxeter group P is an isometry of the underlying space. Those elements that are products of an even number of reflections constitute the *direct subgroup* P^+, of index 2. The product of two reflections is a rotation, a pararotation, or a translation according as the mirrors intersect, are parallel, or have a common perpendicular. Other important subgroups occur when the product of certain pairs of generators is of even or infinite period.

Let the generators $\rho_0, \rho_1, \dots, \rho_n$ of a Coxeter group P (relabeled if necessary) be partitioned into sets of $k+1$ and $n-k$, where $0 \le k \le n$, so that for each pair of generators ρ_j and ρ_l with $0 \le j \le k < l \le n$ the period p_{jl} of the product $\rho_j\rho_l$ is even (or infinite), and let Q be the distinguished subgroup of P generated by reflections ρ_0 through ρ_k (if $k = n$, then $Q = P$). The group Q has a direct subgroup Q^+ generated, if $k \ge 1$, by even transformations (rotations, pararotations, or translations) $\tau_{ij} = \rho_i\rho_j$ ($0 \le i < j \le k$). The transformations actually needed to generate Q^+ usually include all the τ's of period greater than 2. When the Coxeter diagram for Q is connected, these suffice; otherwise, certain "linking" half-turns are also required.

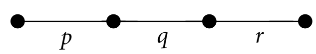
The group P then has a subgroup of index 2 generated by the even transformations τ_{ij} ($0 \le i < j \le k$), the reflections ρ_l ($k < l \le n$), and the conjugate reflections $\rho_{jlj} = \rho_j\rho_l\rho_j$ ($0 \le j \le k < l \le n$). (Some generators may turn out to be superfluous.) This is a *halving subgroup* if $k = 0$, a *semidirect subgroup* if $0 < k < n$, or the direct subgroup P^+ if $k = n$.

Such a subgroup is denoted by affixing a superscript plus sign to the Coxeter symbol for P so that the resulting symbol contains the symbol for the subgroup Q^+, minus the enclosing brackets. In the Coxeter diagram nodes corresponding to omitted reflections are replaced by rings and detached from any branches joining them to nodes corresponding to retained reflections.

For example, the group $[p, q, r]$, generated by reflections $\rho_0, \rho_1, \rho_2, \rho_3$, satisfying the relations

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_2\rho_3)^r = 1, \tag{3}$$
$$\rho_0 \rightleftharpoons \rho_2, \quad \rho_0 \rightleftharpoons \rho_3, \quad \rho_1 \rightleftharpoons \rho_3$$

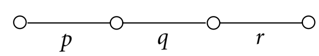
as indicated in the Coxeter diagram



has a direct subgroup $[p, q, r]^+$, generated by the rotations $\sigma_1 = \tau_{01} = \rho_0\rho_1$, $\sigma_2 = \tau_{12} = \rho_1\rho_2$, and $\sigma_3 = \tau_{23} = \rho_2\rho_3$, with the defining relations

$$\sigma_1^p = \sigma_2^q = \sigma_3^r = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1. \tag{4}$$

The group $[p, q, r]^+$ has the Coxeter diagram



If r is even, the semidirect subgroup $[(p, q)^+, r]$, generated by the rotations σ_1 and σ_2 and the reflection ρ_3, is defined by the relations

$$\sigma_1^p = \sigma_2^q = \rho_3^2 = (\sigma_1\sigma_2)^2 = (\sigma_2^{-1}\rho_3\sigma_2\rho_3)^{r/2} = 1, \quad \sigma_1 \rightleftharpoons \rho_3. \tag{5}$$

Likewise, if q is even, the semidirect subgroup $[p^+, q, r]$, generated by the rotation σ_1 and the reflections ρ_2 and ρ_3, has the defining relations

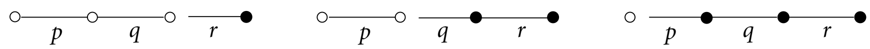
$$\sigma_1^p = \rho_2^2 = \rho_3^2 = (\sigma_1^{-1}\rho_2\sigma_1\rho_2)^{q/2} = (\rho_2\rho_3)^r = 1, \quad \sigma_1 \rightleftharpoons \rho_3. \tag{6}$$

Also, if p is even, the halving subgroup $[1^+, p, q, r]$ is generated by the reflections ρ_1, ρ_2, ρ_3, and $\rho_{010} = \rho_0\rho_1\rho_0$, with the defining relations

$$\rho_{010}^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = (\rho_{010}\rho_1)^{p/2} = (\rho_1\rho_2)^q = (\rho_2\rho_3)^r = 1, \quad \rho_1 \rightleftharpoons \rho_3. \tag{7}$$

These relations imply that $(\rho_{010}\rho_2)^q = 1$ and $\rho_{010} \rightleftharpoons \rho_3$, so that $[1^+, p, q, r]$ is itself a Coxeter group, which may be denoted by $[r, q^{1,1}]$ if $p = 4$ or by $[r, 3^{[3]}]$ if $p = 6$ and $q = 3$ (*cf.* [21, pp. 9–11]).

The groups $[(p, q)^+, r]$, $[p^+, q, r]$, and $[1^+, p, q, r]$ have the respective Coxeter diagrams



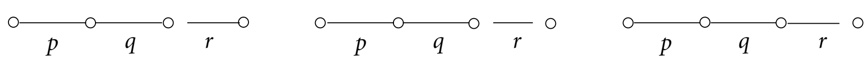
When the Coxeter diagram for P has one or more branches marked with even numbers (or ∞), the generators can be combined to yield further subgroups whose symbols include two or more superscript plus signs, each '+' doubling the index. For example, if r is even, the group $[p, q, r]$ and the three subgroups $[(p, q)^+, r]$, $[p, q, r]^+$, and $[p, q, r, 1^+]$ have a common subgroup

$$[(p, q)^+, r]^+ \cong [(p, q)^+, r, 1^+] \cong [p, q, r, 1^+]^+,$$

of index 4 in $[p, q, r]$ and of index 2 in the other three, generated by the rotations σ_1, σ_2, and $\sigma_{33} = \sigma_3^2 = (\rho_2\rho_3)^2$, with the defining relations

$$\sigma_1^p = \sigma_2^q = \sigma_{33}^{r/2} = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_{33})^q = (\sigma_1\sigma_2\sigma_{33})^2 = 1. \tag{8}$$

The Coxeter diagram for this group may take any of the equivalent forms



Any such subgroup of a Coxeter group P, with a symbol containing one or more plus signs, will be called an *ionic subgroup*, by analogy with an atom that has lost one or more electrons. The subgroup obtained by introducing the maximum number of plus signs into the symbol for P, so that no further partitioning of the generators is possible, is the commutator subgroup P^{+c}, of index 2^c, "where c is the number of pieces into which the graph falls when any branches that have even marks are removed" [6, p. 126].

The finite Coxeter group $[3, 3] \cong S_4$ has a direct subgroup $[3, 3]^+ \cong A_4$, and the latter has a normal subgroup $[3, 3]^{\triangle} \cong D_2$ of index 3. Analogously, when r is even, the group $[3, 3, r]$ has a *trionic subgroup* $[(3, 3)^{\triangle}, r]$, of index 6 in $[3, 3, r]$ and of index 3 in its semidirect subgroup $[(3, 3)^+, r]$, generated by the half-turns $\sigma_{12} = \sigma_1\sigma_2$ and $\sigma_{21} = \sigma_2\sigma_1$ and the reflection ρ_3, satisfying the relations

$$\sigma_{12}^2 = \sigma_{21}^2 = \rho_3^2 = (\sigma_{12}\sigma_{21})^2 = (\sigma_{12}\rho_3)^r = (\sigma_{21}\rho_3)^r = (\sigma_{12}\sigma_{21}\rho_3)^r = 1. \tag{9}$$

The trionic subgroup $[(3, 3)^{\triangle}, r]$ and the ionic subgroup $[(3, 3)^+, r, 1^+]$ have a common subgroup $[(3, 3)^{\triangle}, r, 1^+]$, of index 2 in $[(3, 3)^{\triangle}, r]$, of index 3 in $[(3, 3)^+, r, 1^+]$, of index 6 in $[(3, 3)^+, r]$ and $[3, 3, r, 1^+]$, and of index 12 in $[3, 3, r]$. This group, generated by the half-turns σ_{12} and σ_{21} and their conjugates $\bar{\sigma}_{12} = \sigma_1\sigma_2\sigma_{33} = \rho_3\sigma_{12}\rho_3$ and $\bar{\sigma}_{21} = \sigma_2\sigma_{33}\sigma_1 = \rho_3\sigma_{21}\rho_3$, is the commutator subgroup of $[(3, 3)^+, r, 1^+] \cong [3, 3, r]^{+2}$, which is itself the commutator subgroup of $[3, 3, r]$.

As above, given a Coxeter group $P \cong \langle \rho_0, \rho_1, \ldots, \rho_n \rangle$, let Q be a distinguished subgroup generated by reflections ρ_0 through ρ_k (for some $k < n$), such that for each pair of generators ρ_j and ρ_l with $0 \leq j \leq k < l \leq n$ the period p_{jl} of the product $\rho_j\rho_l$ is even (or infinite). Then P has a *radical subgroup* $P(Q^*)$, generated by the reflections ρ_l $(k < l \leq n)$ and their conjugates by all the elements of Q. When Q is finite, $P(Q^*)$ is a finitely generated reflection group, *i.e.*, another Coxeter group. When $k = 0$, it is the halving subgroup described above. Moreover, at least for the cases to be considered here, the quotient group $P/P(Q^*)$ is isomorphic to Q. A radical subgroup is denoted by affixing a superscript asterisk to the portion of the Coxeter symbol for P corresponding to the subgroup Q.

For example, when q is even, the group $[p, q]$, generated by reflections ρ_0, ρ_1, ρ_2 has a radical subgroup $[p^*, q]$ of index $2p$ generated by the reflection ρ_2 and its conjugates $\rho_{121} = \rho_1\rho_2\rho_1$, $\rho_{01210} = \rho_0\rho_{121}\rho_0$, *etc.*, the number of distinct conjugates depending on the value of p. In particular, we have

$$[3^*, 4] \cong [2, 2], \quad [4^*, 4] \cong [\infty, 2, \infty], \quad [3^*, 6] \cong [3^{[3]}].$$

Other instances of radical subgroups will be noted as they arise.

3 The Rational Modular Group

Each point X of the real projective line P^1 may be given real homogeneous coordinates $(x) = (x_1, x_2)$. Equivalently, the point X may be associated with the single number x_1/x_2 if $x_2 \neq 0$ or with the extended value ∞ if $x_2 = 0$. If $M = [(a, b), (c, d)]$ is an invertible matrix over $\mathbb{R}$, a projectivity $\mathrm{P}^1 \to \mathrm{P}^1$ is then induced either by a projective linear transformation $\langle\cdot M\rangle : \mathrm{P}\mathbb{R}^2 \to \mathrm{P}\mathbb{R}^2$, with $\langle(x)\rangle \longmapsto \langle(x_1a + x_2c, x_1b + x_2d)\rangle$, or by a *linear fractional transformation*

$$(10) \qquad \cdot\langle M\rangle : \mathbb{R} \cup \{\infty\} \longrightarrow \mathbb{R} \cup \{\infty\}, \quad \text{with } x \longmapsto \frac{xa + c}{xb + d}, ad - bc \neq 0,$$

where $x \longmapsto \infty$ if $xb + d = 0$, $\infty \longmapsto a/b$ if $b \neq 0$, and $\infty \longmapsto \infty$ if $b = 0$. The class of scalar multiples of (x) or M is denoted by $\langle(x)\rangle$ or $\langle M\rangle$. The projectivity is direct or opposite according as $ad - bc$ is positive or negative.

The product of two projectivities induced by linear fractional transformation $\cdot\langle M\rangle$ and $\cdot\langle N\rangle$, in that order, is the projectivity induced by the transformation $\cdot\langle MN\rangle$. The group of all direct projectivities of P^1 is the projective general linear group $\mathrm{PGL}_2(\mathbb{R})$. The group of all direct projectivities of P^1 is the projective special linear group $\mathrm{PSL}_2(\mathbb{R})$, a subgroup of index 2 in $\mathrm{PGL}_2(\mathbb{R})$.

The (integral) *special linear* group $\mathrm{SL}_2(\mathbb{Z})$ of 2×2 integer matrices of determinant 1 is generated by the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The *unit linear* group $\bar{\mathrm{S}}\mathrm{L}_2(\mathbb{Z})$ of 2×2 integer matrices of determinant ± 1, which is the same as the general linear group $\mathrm{GL}_2(\mathbb{Z})$ of invertible matrices with integer entries, is generated by the matrices A, B, and $L = \backslash -1, 1\backslash$. (Such matrices and groups are sometimes called "modular" or "unimodular", but the usage of these terms is not consistent in the literature.)

Various other representations of both groups are possible (see [6, pp. 83–88], [20, pp. 107–111]; [12, pp. 365–371]). In particular, if we let $R_0 = AL$, $R_1 = LB$, and $R_2 = L$, then $\bar{\mathrm{S}}\mathrm{L}_2(\mathbb{Z})$ is generated by the matrices

$$R_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Likewise, with $S_1 = R_0R_1 = AB$ and $S_2 = R_1R_2 = B^{-1}$, $\mathrm{SL}_2(\mathbb{Z})$ is generated by

$$S_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The two groups have a common commutator subgroup $\mathrm{SL}'_2(\mathbb{Z})$, of index 2 in $\mathrm{SL}_2(\mathbb{Z})$ and of index 4 in $\bar{\mathrm{S}}\mathrm{L}_2(\mathbb{Z})$, generated by the matrices $S = S_1$ and $W = S_2^{-1}S_1S_2$. Each of these matrix groups may be regarded as a group of linear transformations of the lattice $\mathbb{Z}^2$ of points with integer coordinates (x_1, x_2).

The centre of both $\bar{\mathrm{S}}\mathrm{L}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z})$ is the *special scalar* group $\mathrm{SZ}_2(\mathbb{Z})$ of scalar matrices of determinant 1, *i.e.*, the matrices $\pm I$. Denoting the projective linear transformations determined by the above matrices by corresponding Greek letters, we see that the generators ρ_0, ρ_1, ρ_2 of the (rational) *extended modular* group $\mathrm{P}\bar{\mathrm{S}}\mathrm{L}_2(\mathbb{Z}) \cong \bar{\mathrm{S}}\mathrm{L}_2(\mathbb{Z})/\,\mathrm{SZ}_2(\mathbb{Z})$ satisfy the relations

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^3 = (\rho_0\rho_2)^2 = 1, \tag{11}$$

while the generators $\sigma_1 = \rho_0\rho_1$ and $\sigma_2 = \rho_1\rho_2$ of the (rational) *modular* group $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z})/\,\mathrm{SZ}_2(\mathbb{Z})$ satisfy

$$\sigma_1^3 = (\sigma_1\sigma_2)^2 = 1. \tag{12}$$

If we set $\sigma = \sigma_1$ and $\tau = \sigma_1\sigma_2$, then $\mathrm{PSL}_2(\mathbb{Z})$ has the simpler presentation

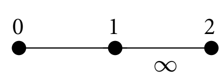
$$\sigma^3 = \tau^2 = 1. \tag{13}$$

The generators $\sigma = \sigma_1$ and $\omega = \sigma_2^{-1}\sigma_1\sigma_2$ of the commutator subgroup $\mathrm{PSL}'_2(\mathbb{Z}) \cong \mathrm{SL}'_2(\mathbb{Z})/\,\mathrm{SZ}_2(\mathbb{Z})$ [6, p. 86] satisfy

$$\sigma^3 = \omega^3 = 1. \tag{14}$$

The relations (11) define the paracompact Coxeter group $[3, \infty]$, the symmetry group of the regular hyperbolic tessellation $\{3, \infty\}$ of triangles with three absolute vertices [6, p. 87], [20, pp. 111, 174], [12, pp. 354–355], [8, pp. 41–45]. The subgroup $[3, \infty]^+$, which is the rotation group of $\{3, \infty\}$, is defined by the relations (12).

The group $[3, \infty]$, generated by reflections ρ_0, ρ_1, ρ_2 in the sides of a Koszul triangle with one absolute vertex and finite angles $\pi/3$ and $\pi/2$, is represented by the Coxeter diagram



where each node has been marked with the subscript of the corresponding generator. It has three ionic subgroups of index 2 and one of index 4.

The direct subgroup $[3, \infty]^+ \cong \mathrm{PSL}_2(\mathbb{Z})$ is generated either by the rotation $\sigma_1 = \rho_0\rho_1$ and the pararotation $\sigma_2 = \rho_1\rho_2$, satisfying (12), or by the rotation $\sigma = \sigma_1$ and the half-turn

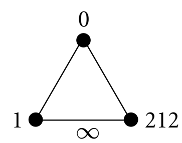
$\tau = \sigma_1\sigma_2 = \rho_0\rho_2$, satisfying (13). The semidirect subgroup $[3^+, \infty]$ is generated by the rotation σ_1 and the reflection ρ_2, satisfying

$$\sigma_1^3 = \rho_2^2 = 1. \tag{15}$$

The halving subgroup $[3, \infty, 1^+] \cong [(3, 3, \infty)]$ is itself a Coxeter group, generated by the reflections ρ_0, ρ_1, and $\rho_{212} = \rho_2\rho_1\rho_2$, satisfying the relations

$$\rho_0^2 = \rho_1^2 = \rho_{212}^2 = (\rho_0\rho_1)^3 = (\rho_0\rho_{212})^3 = 1, \tag{16}$$

as indicated in the diagram

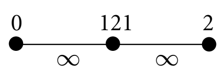


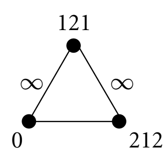
These three groups have a common subgroup, of index 2 in each and of index 4 in $[3, \infty]$: the commutator subgroup $[(3, 3, \infty)]^+ \cong [3^+, \infty, 1^+] \cong [3, \infty]^{+2} \cong \mathrm{PSL}_2'(\mathbb{Z})$, generated by the rotations $\sigma = \sigma_1$ and $\omega = \sigma_2^{-1}\sigma_1\sigma_2 = \rho_2\sigma_1^{-1}\rho_2 = \rho_{212}\rho_0$, satisfying (14). This is the rotation group of the "half regular" tessellation $\mathrm{h}\{\infty, 3\}$, each of whose vertices is surrounded by three triangles and three apeirogons.

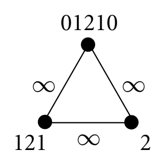
The group $[3, \infty]$ has three other subgroups

$$[\infty, \infty], \quad [(3, \infty, \infty)], \quad [\infty^{[3]}],$$

of indices 3, 4, 6, that are themselves Coxeter groups, with diagrams







Nodes marked '121', '212', and '01210' correspond to generating reflections $\rho_{121} = \rho_1\rho_2\rho_1$, $\rho_{212} = \rho_2\rho_1\rho_2$, and $\rho_{01210} = \rho_0\rho_{121}\rho_0$. The group $[\infty^{[3]}]$ is the radical subgroup $[3^*, \infty]$. The direct subgroup $[\infty^{[3]}]^+ \cong [3^*, \infty, 1^+]$, generated by the pararotations $\sigma\omega = \rho_{01210}\rho_2$ and $\omega\sigma = \rho_2\rho_{121}$ is the free group with two generators.

4 Semiquadratic Modular Groups

For any square-free integer $d \neq 1$, the *quadratic field* $\mathbb{Q}(\sqrt{d})$ has elements $r + s\sqrt{d}$, where r and s belong to the rational field $\mathbb{Q}$. A *quadratic integer* is a root of a monic quadratic equation with integer coefficients. For $d \equiv 2$ or $d \equiv 3 \pmod 4$, $r + s\sqrt{d}$ is a quadratic integer if and only if r and s are both integers; for $d \equiv 1 \pmod 4$, r and s may be both integers or both halves of odd integers. The quadratic integers of $\mathbb{Q}(\sqrt{d})$ form an integral domain, a two-dimensional algebra $\mathbb{Z}^2(d)$ over $\mathbb{Z}$, whose invertible elements, or *units*, have *norm* $r^2 - s^2d = \pm 1$ [19, pp. 187–189].

When d is negative, the invertible elements of $\mathbb{Z}^2(d)$ are complex numbers of modulus 1, and there are only finitely many units: four if $d = -1$, six if $d = -3$, and two in all other cases. When d is positive, $\mathbb{Z}^2(d)$ has an infinite number of units, each expressible as ± 1 times an integral power of a certain *fundamental unit* [12, p. 441]. It may be noted that $\mathbb{Z}^2(d)$ is just the ring $\mathbb{Z}[\delta]$ of numbers of the form $m + n\delta$ (m and n in $\mathbb{Z}$), where $\delta = \sqrt{d}$ if $d \equiv 2$ or $d \equiv 3 \pmod 4$ and $\delta = -\frac{1}{2} + \frac{1}{2}\sqrt{d}$ if $d \equiv 1 \pmod 4$; in the latter case $\mathbb{Z}^2(d)$ contains $\mathbb{Z}[\sqrt{d}]$ as a proper subdomain.

The group $\bar{\mathrm{S}}\mathrm{L}_2\big(\mathbb{Z}^2(d)\big)$ of 2×2 invertible matrices over $\mathbb{Z}^2(d)$ has two discrete subgroups analogous to the groups $\bar{\mathrm{S}}\mathrm{L}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z})$ discussed in the last section. In each instance the four entries of a matrix are partitioned, with rational integers on one diagonal and integral multiples of $\sqrt{d}$ on the other; entries of the form $r + s\sqrt{d}$ with $rs \neq 0$ do not occur. The *semiquadratic unit linear* group $\bar{\mathrm{S}}\mathrm{L}_{1+1}(\mathbb{Z}[\sqrt{d}])$ is generated by the matrices

$$R_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_1 = \begin{pmatrix} -1 & 0 \\ \sqrt{d} & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the *semiquadratic special linear* group $\mathrm{SL}_{1+1}(\mathbb{Z}[\sqrt{d}])$ is generated by

$$S_1 = R_0R_1 = \begin{pmatrix} \sqrt{d} & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad S_2 = R_1R_2 = \begin{pmatrix} 1 & 0 \\ -\sqrt{d} & 1 \end{pmatrix}.$$

Replacing each matrix A in either of these groups by the equivalence class α of matrices $\pm A$ (and $\pm iA$ in the first case if $d = -1$), we obtain the *semiquadratic extended modular* group $\mathrm{P}\bar{\mathrm{S}}\mathrm{L}_{1+1}(\mathbb{Z}[\sqrt{d}])$, with generators ρ_0, ρ_1, ρ_2, and the *semiquadratic modular* group $\mathrm{PSL}_{1+1}(\mathbb{Z}[\sqrt{d}])$, with generators $\sigma_1 = \rho_0\rho_1$ and $\sigma_2 = \rho_1\rho_2$.

The period of the matrix S_1 is finite when d has one of the values 0, 1, 2, or 3, with $S_1^p = -I$ for p equal to 2, 3, 4, or 6, respectively. Thus the generators of the groups $\mathrm{P}\bar{\mathrm{S}}\mathrm{L}_{1+1}(\mathbb{Z}[\sqrt{2}])$ and $\mathrm{PSL}_{1+1}(\mathbb{Z}[\sqrt{2}])$ satisfy the respective relations

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^4 = (\rho_0\rho_2)^2 = 1, \tag{17}$$

$$\sigma_1^4 = (\sigma_1\sigma_2)^2 = 1, \tag{18}$$

while the generators of $\mathrm{P}\bar{\mathrm{S}}\mathrm{L}_{1+1}(\mathbb{Z}[\sqrt{3}])$ and $\mathrm{PSL}_{1+1}(\mathbb{Z}[\sqrt{3}])$ respectively satisfy

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^6 = (\rho_0\rho_2)^2 = 1, \tag{19}$$

$$\sigma_1^6 = (\sigma_1\sigma_2)^2 = 1. \tag{20}$$

We observe that (17) and (19) define the Coxeter groups $[4, \infty]$ and $[6, \infty]$, which are the symmetry groups of the regular hyperbolic tessellations $\{4, \infty\}$ and $\{6, \infty\}$, with the rotation groups $[4, \infty]^+$ and $[6, \infty]^+$ being defined by (18) and (20).

The group $[4, \infty]$ contains two other Coxeter groups as halving subgroups, namely, $[4, \infty, 1^+] \cong [(4, 4, \infty)]$, generated by the reflections ρ_0, ρ_1, and $\rho_{212} = \rho_2\rho_1\rho_2$ and satisfying the relations

$$\rho_0^2 = \rho_1^2 = \rho_{212}^2 = (\rho_0\rho_1)^4 = (\rho_0\rho_{212})^4 = 1, \tag{21}$$

and $[1^+, 4, \infty] \cong [\infty, \infty]$, generated by reflections ρ_1, ρ_2 and $\rho_{010} = \rho_0\rho_1\rho_0$ and satisfying

$$\rho_{010}^2 = \rho_1^2 = \rho_2^2 = (\rho_{010}\rho_1)^2 = 1. \tag{22}$$

The latter group has a subgroup $[\infty^{[3]}]$, of index 4 in $[4, \infty]$, generated by the reflections ρ_1, ρ_2, and $\rho_{01210} = \rho_{010}\rho_2\rho_{010}$ and satisfying

$$\rho_{01210}^2 = \rho_1^2 = \rho_2^2 = 1. \tag{23}$$

The commutator subgroup of $[4, \infty]$ and $[4, \infty]^+$, of index 8 in the former and of index 4 in the latter, is $[1^+, 4, 1^+, \infty, 1^+] \cong [4, \infty]^{+3}$, generated by the half-turns σ_1^2 and $\sigma_2^{-1}\sigma_1^2\sigma_2$ and the pararotation σ_2^2.

Likewise, $[6, \infty]$ contains the halving subgroups $[6, \infty, 1^+] \cong [(6, 6, \infty)]$, generated by the reflections ρ_0, ρ_1, and $\rho_{212} = \rho_2\rho_1\rho_2$ and satisfying

$$\rho_0^2 = \rho_1^2 = \rho_{212}^2 = (\rho_0\rho_1)^6 = (\rho_0\rho_{212})^6 = 1, \tag{24}$$

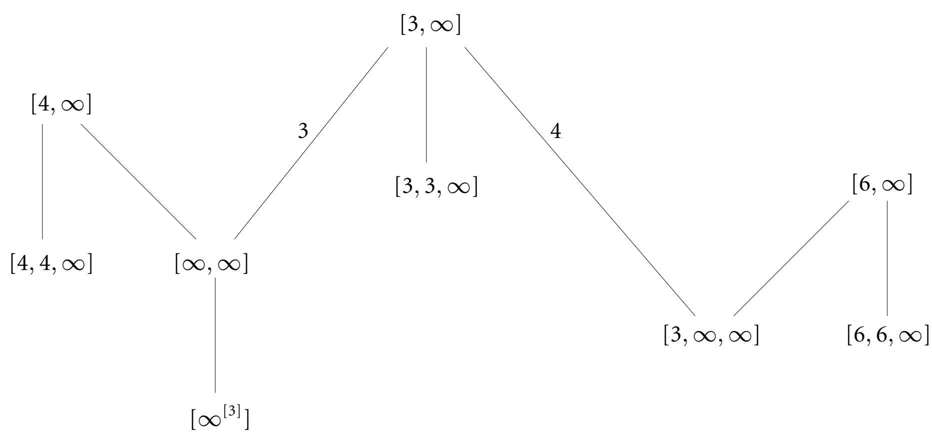
and $[1^+, 6, \infty] \cong [(3, \infty, \infty)]$, with generators ρ_1, ρ_2, and $\rho_{010} = \rho_0\rho_1\rho_0$, satisfying

$$\rho_{010}^2 = \rho_1^2 = \rho_2^2 = (\rho_{010}\rho_1)^3 = 1. \tag{25}$$

The commutator subgroup of $[6, \infty]$ and $[6, \infty]^+$, of index 8 in the former and of index 4 in the latter, is $[1^+, 6, 1^+, \infty, 1^+] \cong [6, \infty]^{+3}$, generated by the rotations σ_1^2 and $\sigma_2^{-1}\sigma_1^2\sigma_2$, both of period 3, and the pararotation σ_2^2.

In addition to its representation as $\mathrm{P\bar{S}L}_{1+1}(\mathbb{Z}[\sqrt{2}])$, the group $[4, \infty]$ is isomorphic to the (integral) *projective pseudo-orthogonal* group $\mathrm{PO}_{2,1}(\mathbb{Z})$, the central quotient group of the group $O_{2,1}(\mathbb{Z})$ of 3×3 pseudo-orthogonal matrices with integer entries [7, pp. 423–424] (*cf.* [27, pp. 299–300]).

Connections between these groups and the subgroups of $[3, \infty]$ discussed in the last section may be seen in the following diagram. When two groups are joined by a line, the lower is subgroup of the upper, of index 2 unless otherwise indicated.



If d contiguous replicas of the fundamental region for a Coxeter group P can be amalgamated to form the fundamental region for another group Q, then Q is a subgroup of index d in P. Thus the dissection of the Koszul triangle $(\infty\,\infty\,\infty)$ into two triangles $(2\,\infty\,\infty)$, four triangles $(2\,4\,\infty)$, or six triangles $(2\,3\,\infty)$ shows that $[\infty^{[3]}]$ is a subgroup of index 2 in $[\infty, \infty]$, of index 4 in $[4, \infty]$, and of index 6 in $[3, \infty]$.

5 The Complex Projective Line

Each point Z of the complex projective line $\mathbb{C}\mathrm{P}^1$ may be given complex homogeneous coordinates $(z) = (z_1, z_2)$ or associated with the single number z_1/z_2 if $z_2 \neq 0$ or with the extended value ∞ if $z_2 = 0$. Any three distinct points U, V, W lie on a unique chain $\mathbb{R}(UVW)$, consisting of all points Z for which the cross ratio $\{UV, WZ\}$ is real or infinite (*cf.* [30, p. 165], [31, p. 222]). Points that lie on the same chain are said to be *concatenate*. A chain-preserving transformation of $\mathbb{C}\mathrm{P}^1$ is a *concatenation*, and every concatenation is either a projectivity or an antiprojectivity, according as cross ratios are preserved or replaced by their complex conjugates.

If $M = [(a, b), (c, d)]$ is an invertible matrix over $\mathbb{C}$, a projectivity $\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ is induced by the *linear fractional transformation*

$$\cdot\langle M\rangle\colon \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}, \quad \text{with } z \longmapsto \frac{za+c}{zb+d}, ad-bc \neq 0, \tag{26}$$

where $z \longmapsto \infty$ if $zb + d = 0$, $\infty \longmapsto a/b$ if $b \neq 0$, and $\infty \longmapsto \infty$ if $b = 0$. Likewise, an antiprojectivity $\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ is induced by the *antilinear fractional transformation*

$$\bar{\cdot}\langle M\rangle\colon \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}, \quad \text{with } z \longmapsto \frac{\bar{z}a+c}{\bar{z}b+d}, ad-bc \neq 0. \tag{27}$$

Since a projectivity preserves all cross ratios and an antiprojectivity preserves real cross ratios, each of these transformations takes chains into chains; they are in fact the only transformations of $\mathbb{C}\mathrm{P}^1$ that do so [31, p. 252].

The real inversive sphere I^2, regarded as a one-point compactification of the Euclidean plane E^2, can be taken as a conformal model (the *Riemann sphere*) for the complex projective line $\mathbb{C}\mathrm{P}^1$ [31, pp. 250–252], [24, chap. 17]. The chains of $\mathbb{C}\mathrm{P}^1$ are the real circles of I^2. Concatenations of $\mathbb{C}\mathrm{P}^1$ become circle-preserving transformations, or *circularities* of I^2, a projectivity (26) corresponding to a direct circularity, or *homography*, and an antiprojectivity (27) to an opposite circularity, or *antihomography*. Homographies are also called *Möbius transformations*.

A hyperbolic antiinvolution of $\mathbb{C}\mathrm{P}^1$, *i.e.*, an involutory antiprojectivity leaving all the points of a chain invariant, is an *inversion* in the corresponding circle of I^2. The product of an even number of inversions is a homography; the product of an odd number is an antihomography. If M and N are any two invertible matrices, we have the following rules:

$$\begin{aligned} \cdot\langle M\rangle \cdot \langle N\rangle &= \cdot\langle MN\rangle, \quad & \cdot\langle M\rangle \,\bar{\cdot}\, \langle N\rangle &= \bar{\cdot}\langle \bar{M}N\rangle, \\ \bar{\cdot}\langle M\rangle \cdot \langle N\rangle &= \bar{\cdot}\langle MN\rangle, \quad & \bar{\cdot}\langle M\rangle \,\bar{\cdot}\, \langle N\rangle &= \cdot\langle \bar{M}N\rangle. \end{aligned} \tag{28}$$

Monson & Weiss [22, p. 188], [23, p. 103] give analogous rules for multiplying linear and antilinear transformations.

The set of all 2×2 invertible matrices over $\mathbb{C}$ forms the (complex) *general linear* group $\mathrm{GL}_2(\mathbb{C})$, whose centre $\mathrm{GZ}(\mathbb{C})$ is the group of nonzero scalar matrices. The central quotient group $\mathrm{PGL}_2(\mathbb{C}) \cong \mathrm{GL}_2(\mathbb{C})/\mathrm{GZ}(\mathbb{C})$ is the *projective general linear* group. This is the group of all projectivities of $\mathbb{C}\mathrm{P}^1$, which is isomorphic to the "Möbius group" of all homographies of I^2. The group of all concatenations of $\mathbb{C}\mathrm{P}^1$, *i.e.*, the group of all projectivities and antiprojectivities, is the *complemented projective general linear* group $\bar{P}\,\mathrm{GL}_2(\mathbb{C})$, containing $\mathrm{PGL}_2(\mathbb{C})$ as a subgroup of index 2. The group $\bar{\mathrm{P}}\mathrm{GL}_2(\mathbb{C})$ is isomorphic to the "inversive group" of all circularities of I^2, i.e., the group of all homographies and antihomographies. Since I^2 is the absolute sphere of hyperbolic 3-space H^3, $\bar{\mathrm{P}}\mathrm{GL}_2(\mathbb{C})$ is the group of all isometries of H^3, and $\mathrm{PGL}_2(\mathbb{C})$ is the subgroup of direct isometries.

Complex matrices of determinant 1 form the *special linear* group $\mathrm{SL}_2(\mathbb{C})$, whose centre $\mathrm{SZ}_2(\mathbb{C})$ consists of the two matrices $\pm I$. Because every invertible matrix over $\mathbb{C}$ is a scalar multiple of some matrix of determinant 1, the *projective special linear* group $\mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C})/\,\mathrm{SZ}_2(\mathbb{C})$ is isomorphic to $\mathrm{PGL}_2(\mathbb{C})$. Discrete subgroups of $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PSL}_2(\mathbb{C})$ are obtained by restricting matrix entries to a ring $D = \mathbb{Z}^2(d)$ of quadratic integers in a field $\mathbb{Q}(\sqrt{d})$, where d is a square-free negative integer. The group $\mathrm{PSL}_2(D)$ is a *Bianchi group*; when d has one of the values -1, -2, -3, -7, or -11, there is a Euclidean algorithm on the norm of D [12, p. 448], [8, pp. 71–72]. When d is -1 or -3, $\mathrm{PSL}_2(D)$ is a subgroup of the symmetry group of a regular honeycomb of H^3.

6 The Gaussian Modular Group

The integral domain $\mathbb{G} = \mathbb{Z}[i] = \mathbb{Z}^2(-1)$ of *Gaussian integers* comprises the complex numbers $g = g_0 + g_1 i$, where $(g_0, g_1) \in \mathbb{Z}^2$ and $i = \sqrt{-1}$ is a primitive fourth root of unity, so that $i^2 + 1 = 0$. This system was first described by Carl Friedrich Gauss circa 1830. Each Gaussian integer g has a norm $N(g) = |g|^2 = g_0^2 + g_1^2$. The units of $\mathbb{G}$ are the four numbers with norm 1, namely ± 1 and $\pm i$, which form the Gaussian *unit scalar* group $\bar{\mathrm{S}}\mathrm{Z}(\mathbb{G}) \cong C_4 \cong \langle i \rangle$, with the proper subgroup $S_2Z(\mathbb{G}) \cong C_2 \cong \langle -1 \rangle$.

The *special linear* group $\mathrm{SL}_2(\mathbb{G})$ of 2×2 Gaussian integer matrices of determinant 1 is generated by the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$$

[2, p. 314]. The *semispecial linear* group $S_2L_2(\mathbb{G})$ of 2×2 matrices S over $\mathbb{G}$ with $(\det S)^2 = 1$ is generated by A, B, C, and $L = \backslash -1, 1\backslash$. The *unit linear* group $\bar{\mathrm{S}}\mathrm{L}_2(\mathbb{G})$ of matrices U with $|\det U| = 1$ is generated by A, B, and $M = \backslash -i, 1\backslash$ [29, pp. 230–231]. Note that $C = MBM^{-1}$ and $L = M^2$. The centre of $\mathrm{SL}_2(\mathbb{G})$ is the *special scalar* group $\mathrm{SZ}_2(\mathbb{G}) \cong \langle -I \rangle$, and the centre of both $\bar{\mathrm{S}}\mathrm{L}_2(\mathbb{G})$ and $S_2L_2(\mathbb{G})$ is the *unit scalar* group $\bar{\mathrm{S}}\mathrm{Z}(\mathbb{G}) \cong \langle iI \rangle$.

The *Gaussian modular* group

$$\mathrm{PSL}_2(\mathbb{G}) \cong \mathrm{SL}_2(\mathbb{G})/\,\mathrm{SZ}_2(\mathbb{G}) \cong S_2L_2(\mathbb{G})/\bar{\mathrm{S}}\mathrm{Z}(\mathbb{G})$$

(the "Picard group") is generated in H^3 by the half-turn $\alpha = \cdot\langle A \rangle$ and the pararotations $\beta = \cdot\langle B \rangle$ and $\gamma = \cdot\langle C \rangle$. The *Gaussian extended modular* group

$$\mathrm{P}\bar{\mathrm{S}}\mathrm{L}_2(\mathbb{G}) \cong \bar{\mathrm{S}}\mathrm{L}_2(\mathbb{G})/\bar{\mathrm{S}}\mathrm{Z}_2(\mathbb{G})$$

is likewise generated by the half-turn α, the pararotation β, and the quarter-turn $\mu = \cdot\langle M \rangle$.

When the complex field $\mathbb{C}$ is regarded as a two-dimensional vector space over $\mathbb{R}$, the Gaussian integers constitute a two-dimensional lattice I_2. The points of I_2 are the vertices of a regular tessellation $\{4,4\}$ of the Euclidean plane E^2, whose symmetry group $[4,4]$ is generated by three reflections ρ_1, ρ_2, ρ_3, satisfying the relations

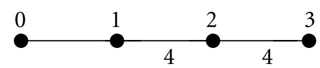
$$\rho_1^2 = \rho_2^2 = \rho_3^2 = (\rho_1\rho_2)^4 = (\rho_1\rho_3)^2 = (\rho_2\rho_3)^4 = 1. \tag{29}$$

The tessellation $\{4,4\}$ is the vertex figure of a regular honeycomb $\{3,4,4\}$ of hyperbolic 3-space H^3, the cell polyhedra of which are regular octahedra $\{3,4\}$ whose vertices all lie on the absolute sphere.

The symmetry group $[3,4,4]$ of the honeycomb $\{3,4,4\}$ is generated by four reflections ρ_0, ρ_1, ρ_2, ρ_3, satisfying (29) as well as

$$\rho_0^2 = (\rho_0\rho_1)^3 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = 1. \tag{30}$$

The combined relations (29) and (30) are indicated in the Coxeter diagram



The generators ρ_0, ρ_1, ρ_2, ρ_3 can be represented by antilinear fractional transformations $\bar{\cdot}\langle R_0\rangle$, $\bar{\cdot}\langle R_1\rangle$, $\bar{\cdot}\langle R_2\rangle$, $\bar{\cdot}\langle R_3\rangle$, determined by the matrices

$$R_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The direct subgroup $[3,4,4]^+$ is generated by three rotations $\sigma_1 = \rho_0\rho_1$, $\sigma_2 = \rho_1\rho_2$, $\sigma_3 = \rho_2\rho_3$, with the defining relations

$$\sigma_1^3 = \sigma_2^4 = \sigma_3^4 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1. \tag{31}$$

The generators σ_1, σ_2, σ_3 can be represented by linear fractional transformations $\cdot\langle S_1\rangle$, $\cdot\langle S_2\rangle$, $\cdot\langle S_3\rangle$, corresponding to the unit matrices

$$S_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} -i & 0 \\ i & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix},$$

with entries in $\mathbb{G} = \mathbb{Z}[i]$ and determinants in $\bar{\mathrm{S}}\mathrm{Z}(\mathbb{G}) \cong \langle i\rangle$. Our presentation of these groups follows that of [29, pp. 234–235], except that the order of the generators has been reversed and, in accordance with the convention followed here that transformations are multiplied from left to right, all matrices have been transposed.

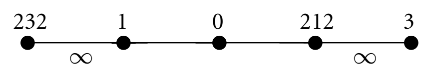
The matrices S_1, S_2, S_3 belong to and generate the unit linear group $\bar{\mathrm{S}}\mathrm{L}_2(\mathbb{G}) \cong \langle A, B, M\rangle$, since

$$S_1S_2S_3^{-1} = A, \quad S_3S_2^{-1} = B, \quad \text{and} \quad S_3 = M.$$

Thus the group $[3,4,4]^+$, generated by σ_1, σ_2, and σ_3 is the Gaussian extended modular group $\mathrm{P}\bar{\mathrm{S}}\mathrm{L}_2(\mathbb{G}) \cong \langle \alpha, \beta, \mu\rangle$.

Generators and relations for the Gaussian modular (Picard) group $\mathrm{PSL}_2(\mathbb{G})$ as a Euclidean Bianchi group are given by Fine [8, pp. 74–84]. Schulte & Weiss [29, pp. 235–236] have shown that $\mathrm{PSL}_2(\mathbb{G})$ is a subgroup of index 2 in $[3, 4, 4]^+$, and Monson & Weiss [22, pp. 188–189] have exhibited it as a subgroup of index 2 in the hypercompact Coxeter group $[\infty, 3, 3, \infty]$. In discussing the Picard group, Magnus [20, pp. 152–153] gives correct generators but incorrect relations for the parent group $[3, 4, 4]$. Here we obtain an explicit geometric presentation for $\mathrm{PSL}_2(\mathbb{G})$ as an ionic subgroup of each of the groups just mentioned.

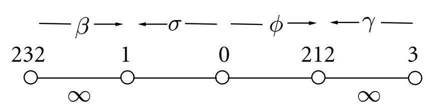
The group $[3, 4, 4]$ has a halving subgroup $[3, 4, 1^+, 4] \cong [\infty, 3, 3, \infty]$, generated by the reflections $\rho_0, \rho_1, \rho_3, \rho_{212} = \rho_2\rho_1\rho_2$, and $\rho_{232} = \rho_2\rho_3\rho_2$, satisfying the relations indicated in the diagram



The five mirrors are the bounding planes of a quadrangular pyramid whose apex lies on the absolute sphere. This group has an involutory automorphism, conjugation by ρ_2, interchanging generators ρ_1 and ρ_{212}, ρ_3 and ρ_{232}. The two groups $[3, 4, 4]^+$ and $[3, 4, 1^+, 4]$ have a common subgroup $[3, 4, 1^+, 4]^+ \cong [\infty, 3, 3, \infty]^+$, of index 2 in both and of index 4 in $[3, 4, 4]$, generated by the pararotations β and γ and the rotations σ and ϕ, where

$$\beta = \rho_{232}\rho_1 = \sigma_3\sigma_2^{-1}, \quad \gamma = \rho_3\rho_{212} = \sigma_3^{-1}\sigma_2, \quad \sigma = \rho_0\rho_1 = \sigma_1, \phi = \rho_0\rho_{212} = \sigma_1\sigma_2^2,$$

as in the embellished Coxeter diagram



In terms of these generators, defining relations for $[3, 4, 1^+, 4]^+$ are

$$\begin{aligned}\sigma^3 = \phi^3 = (\sigma\beta^{-1})^2 = (\phi\gamma^{-1})^2 = (\sigma^{-1}\phi)^2 \\ = (\beta\sigma^{-1}\phi)^2 = (\gamma\phi^{-1}\sigma)^2 = (\beta\sigma^{-1}\phi\gamma^{-1})^2 = 1.\end{aligned} \tag{32}$$

Since the corresponding matrices

$$B = S_3S_2^{-1}, \quad C = S_3^{-1}S_2, \quad S = S_1, \quad \text{and} \quad U = -iS_1S_2^2$$

all belong to the special linear group $\mathrm{SL}_2(\mathbb{G})$, $[3, 4, 1^+, 4]^+$ is a subgroup of the Gaussian modular group $\mathrm{PSL}_2(\mathbb{G})$. In fact, it is that very group, as we now show.

Theorem 6.1 *The Gaussian special linear group* $\mathrm{SL}_2(\mathbb{G})$ *is generated by the matrices*

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

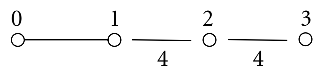
Proof Since B, C, and S are in $\mathrm{SL}_2(\mathbb{G}) \cong \langle A, B, C\rangle$ and since $A = SB^{-1}$, it follows that $\mathrm{SL}_2(\mathbb{G}) \cong \langle B, C, S\rangle$. ∎

The Gaussian modular group $\mathrm{PSL}_2(\mathbb{G}) \cong \langle \alpha, \beta, \gamma \rangle$ is thus generated by the corresponding isometries β, γ, and σ. It can be verified that $U = C^{-1}ACA^{-1} = C^{-1}SCS^{-1}$, from which it follows that $\phi = \gamma^{-1}\alpha\gamma\alpha = \gamma^{-1}\sigma\gamma\sigma^{-1}$. That is, $\mathrm{PSL}_2(\mathbb{G})$ is isomorphic to the group $[3, 4, 1^+, 4]^+ \cong \langle \beta, \gamma, \sigma, \phi \rangle$.

The identities $\phi = \gamma^{-1}\alpha\gamma\alpha = \gamma^{-1}\sigma\gamma\sigma^{-1}$ can be combined with the above relations to give a presentation for $\mathrm{PSL}_2(\mathbb{G})$ in terms of generators β, γ, and σ alone or, since $\sigma = \alpha\beta$, in terms of the generators α, β, and γ. A related presentation involves the half-turn $\tau = \alpha\phi\gamma^{-1} = \alpha\gamma^{-1}\alpha\gamma\alpha\gamma^{-1}$ corresponding to the matrix $T = AUC^{-1} = AC^{-1}ACA^{-1}C^{-1}$. Fine [8, p. 81] gives defining relations for $\mathrm{PSL}_2(\mathbb{G})$ satisfied by α, β, γ, and τ (his a, t, u, and l):

$$\alpha^2 = \tau^2 = (\alpha\tau)^2 = (\beta\tau)^2 = (\gamma\tau)^2 = (\alpha\beta)^3 = (\alpha\tau\gamma)^3 = 1, \quad \beta \rightleftharpoons \gamma. \tag{33}$$

The group $[\infty, (3, 3)^+, \infty]^+ \cong [3^+, 4, 1^+, 4, 1^+] \cong [3, 4, 4]^{+3} \cong \mathrm{P\bar{S}L}_2'(\mathbb{G})$, with Coxeter diagram

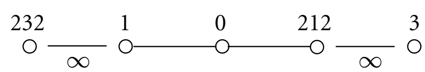


is the commutator subgroup of $[3, 4, 4]$ and $[3, 4, 4]^+ \cong \mathrm{P\bar{S}L}_2(\mathbb{G})$, of index 4 in $\mathrm{P\bar{S}L}_2(\mathbb{G})$ and of index 2 in $\mathrm{PSL}_2(\mathbb{G})$. It is generated by the rotations $\sigma = \rho_0\rho_1 = \sigma_1$, $\tau = \rho_{232}\rho_3 = \sigma_3^2$, and $\phi = \rho_0\rho_{212} = \sigma_1\sigma_2^2$, satisfying the relations

$$\sigma^3 = \tau^2 = \phi^3 = (\sigma^{-1}\phi)^2 = (\sigma^{-1}\tau\phi\tau)^2 = 1. \tag{34}$$

The corresponding matrices are $S = S_1$ $T = iS_3^2$, and $U = -iS_1S_2^2$.

The group $[3, 4, 1^+, 4] \cong [\infty, 3, 3, \infty]$ and its subgroups $[\infty, 3, 3, \infty]^+ \cong \mathrm{PSL}_2(\mathbb{G})$ and $[\infty, (3, 3)^+, \infty]^+ \cong \mathrm{P\bar{S}L}_2'(\mathbb{G})$ have a common commutator subgroup $[1^+, \infty, (3, 3)^+, \infty, 1^+] \cong [\infty, 3, 3, \infty]^{+3} \cong \mathrm{PSL}_2'(\mathbb{G}) \cong \mathrm{P\bar{S}L}_2''(\mathbb{G})$ with Coxeter diagram



This group, of index 4 in $\mathrm{PSL}_2(\mathbb{G})$ and of index 2 in $\mathrm{P\bar{S}L}_2'(\mathbb{G})$, is generated by the rotations $\sigma = \rho_0\rho_1 = \sigma_1$, $\phi = \rho_0\rho_{212} = \sigma_1\sigma_2^2$, $\psi = \rho_0\rho_3\rho_{212}\rho_3 = \sigma_3^{-1}\sigma_1\sigma_3$, and $\omega = \rho_o\rho_{232}\rho_1\rho_{232} = \sigma_3\sigma_1\sigma_2^2\sigma_3^{-1}$, satisfying the relations

$$\sigma^3 = \phi^3 = \psi^3 = \omega^3 = (\sigma^{-1}\phi)^2 = (\sigma^{-1}\psi)^2 = (\phi^{-1}\omega)^2 = (\psi^{-1}\omega)^2 = 1 \tag{35}$$

(*cf.* [9, pp. 770–771], [8, p. 139]). The corresponding matrices are $S = S_1$, $U = -iS_1S_2^2$, $V = S_3^{-1}S_1S_3$, and $W = -iS_3S_1S_2^2S_3^{-1}$. That is,

$$S = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

A 2×2 complex matrix can be converted into an equivalent 4×4 real matrix by replacing each entry $z = x + yi$ by a 2×2 real *duplex matrix*

$$Z = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

If the complex matrix has entries in $\mathbb{G}$, the corresponding real matrix has entries in $\mathbb{Z}$. Thus each group of linear or linear fractional transformations defined by certain Gaussian integer matrices has an equivalent representation as a subgroup of $\mathrm{SL}_4(\mathbb{Z})$ or $\mathrm{PSL}_4(\mathbb{Z})$.

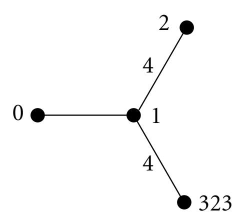
Groups involving antilinear or antilinear fractional transformations can likewise be represented by subgroups of $\bar{\mathrm{S}}\mathrm{L}_4(\mathbb{Z})$ or $\mathrm{P}\bar{\mathrm{S}}\mathrm{L}_4(\mathbb{Z})$. In particular, the group $[3,4,4]$ is isomorphic to $\mathrm{PO}_{3,1}(\mathbb{Z})$, the central quotient group of the group $O_{3,1}(\mathbb{Z})$ of 4×4 pseudo-orthogonal matrices with integral entries [7, pp. 428–429] (*cf.* [27, p. 301]).

7 Other Subgroups of $[3,4,4]$

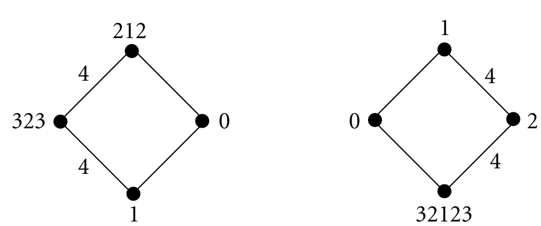
The group $[3,4,4]$ has a halving subgroup $[3,4,4,1^+]\cong[3,4^{1,1}]$, the symmetry group of the "half regular" honeycomb $\mathrm{h}\{4,4,3\}$, generated by the reflections ρ_0, ρ_1, ρ_2, and $\rho_{323}=\rho_3\rho_2\rho_3$, satisfying the relations

$$\rho_0^2=\rho_1^2=\rho_2^2=\rho_{323}^2=(\rho_0\rho_1)^3=(\rho_1\rho_2)^4=(\rho_1\rho_{323})^4=1,$$
$$\rho_0\rightleftharpoons\rho_2,\quad \rho_0\rightleftharpoons\rho_{323},\quad \rho_2\rightleftharpoons\rho_{323}, \tag{36}$$

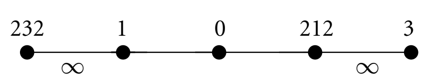
as indicated in the Coxeter diagram



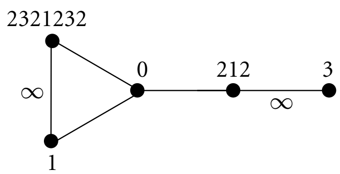
This group has two isomorphic halving subgroups of its own, $[(4^2,3^2)]$ and $[(3^2,4^2)]$, the former generated by the reflections ρ_0, ρ_1, ρ_{323}, and $\rho_{212}=\rho_2\rho_1\rho_2$ and the latter by the reflections ρ_0, ρ_1, ρ_2, and $\rho_{32123}=\rho_{323}\rho_1\rho_{323}$. These groups have the respective Coxeter diagrams

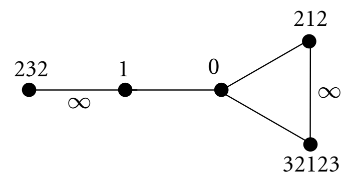


Likewise, the halving subgroup $[3,4,1^+,4]\cong[\infty,3,3,\infty]$, generated by the reflections $\rho_0,\rho_1,\rho_3,\rho_{212}=\rho_2\rho_1\rho_2$, and $\rho_{232}=\rho_2\rho_3\rho_2$, with Coxeter diagram



has two isomorphic halving subgroups $[1^+, \infty, 3, 3, \infty]$ and $[\infty, 3, 3, \infty, 1^+]$, one generated by the reflections ρ_0, ρ_1, ρ_{212}, ρ_3, and $\rho_{2321232} = \rho_{232}\rho_1\rho_{232}$, the other by the reflections ρ_0, ρ_1, ρ_{212}, ρ_{232}, and $\rho_{32123} = \rho_3\rho_{212}\rho_3$. The respective Coxeter diagrams are



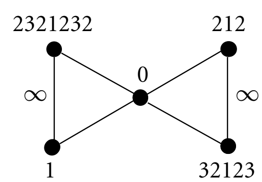


Each of the above subgroups of $[3, 4, 4]$ has an involutory automorphism, evident in the bilateral symmetry of its graph. For $[3, 4^{1,1}]$ this is conjugation by the reflection ρ_3, and for its halving subgroups it is conjugation by ρ_2 or ρ_{323}. For $[\infty, 3, 3, \infty]$ the automorphism is conjugation by ρ_2, and for its halving subgroups it is conjugation by ρ_{232} or ρ_3. Augmenting a group by its automorphism gives the parent group as a semidirect product.

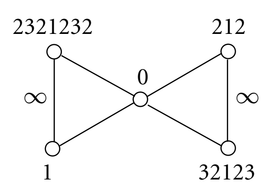
The two pairs of halving subgroups of $[3, 4^{1,1}]$ and $[\infty, 3, 3, \infty]$ have a common halving subgroup

$$[(3, 3, 4, 1^+, 4)] \cong [1^+, \infty, 3, 3, \infty, 1^+] \cong [(3, 3, \infty)^{1,1}],$$

of index 4 in $[3, 4^{1,1}]$ and $[\infty, 3, 3, \infty]$ and of index 8 in $[3, 4, 4]$, generated by the reflections $\rho_0, \rho_1, \rho_{212} = \rho_2\rho_1\rho_2$, $\rho_{32123} = \rho_3\rho_{212}\rho_3 = \rho_{323}\rho_1\rho_{323}$, and $\rho_{2321232} = \rho_2\rho_{32123}\rho_2 = \rho_{323}\rho_{212}\rho_{323} = \rho_{232}\rho_1\rho_{232}$, the Coxeter diagram being



The group $[(3, 3, \infty)^{1,1}]$, which has an automorphism group D_4 of order 8, is the radical subgroup $[3, 4, 4^*]$. Its direct subgroup $[(3, 3, \infty)^{1,1}]^+$, generated by the rotations $\sigma = \rho_0\rho_1$, $\phi = \rho_0\rho_{212}$, $\psi = \rho_0\rho_{32123}$, and $\omega = \rho_0\rho_{2321232}$, satisfying the relations (35), with Coxeter diagram

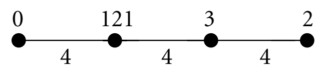


is the common commutator subgroup $[3, 4^{1,1}]^{+3} \cong [\infty, 3, 3, \infty]^{+3} \cong \mathrm{PSL}_2'(\mathbb{G})$ of all the above groups, of index 8 in both $[3, 4^{1,1}]$ and $[\infty, 3, 3, \infty]$ and of index 4 in their respective halving subgroups.

The group $[3, 4, 4]$ also has a subgroup $[4, 4, 4]$, of index 3, generated by the reflections ρ_0, ρ_2, ρ_3, and $\rho_{121} = \rho_1\rho_2\rho_1$, satisfying the relations

$$\rho_0^2 = \rho_{121}^2 = \rho_2^2 = \rho_3^2 = (\rho_0\rho_{121})^4 = (\rho_{121}\rho_3)^4 = (\rho_2\rho_3)^4 = 1, \qquad (37)$$
$$\rho_0 \rightleftharpoons \rho_2, \quad \rho_0 \rightleftharpoons \rho_3, \quad \rho_{121} \rightleftharpoons \rho_2,$$

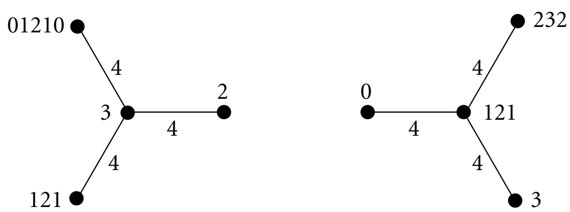
as indicated in the Coxeter diagram



This is the symmetry group of a self-dual regular honeycomb $\{4, 4, 4\}$. It has two halving subgroups,

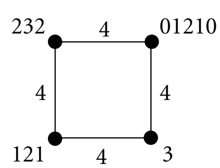
$$[1^+, 4, 4, 4] \cong [4^{1,1,1}] \cong [4, 4, 4, 1^+],$$

of index 6 in $[3, 4, 4]$, the first generated by the reflections ρ_2, ρ_3, ρ_{121}, and $\rho_{01210} = \rho_0\rho_{121}\rho_0$, the second by the reflections ρ_0, ρ_{121}, ρ_3, and $\rho_{232} = \rho_2\rho_3\rho_2$, satisfying the relations indicated in the diagrams



Each of these groups has an automorphism group D_3 of order 6, $[1^+, 4, 4, 4]$ being the radical subgroup $[3^*, 4, 4]$.

The two groups $[1^+, 4, 4, 4]$ and $[4, 4, 4, 1^+]$ have a common halving subgroup $[1^+, 4, 4, 4, 1^+] \cong [1^+, 4, 4^{1,1}] \cong [4^{[4]}]$, of index 4 in $[4, 4, 4]$ and of index 12 in $[3, 4, 4]$, generated by the reflections ρ_{121}, ρ_{232}, ρ_{01210}, and ρ_3, with Coxeter diagram



This group has an automorphism group D_4 of order 8.

The group $[4, 4, 4]$ has two other halving subgroups,

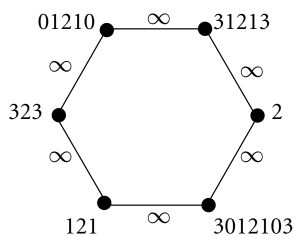
$$[4, 1^+, 4, 4] \cong [\infty, 4, 4, \infty] \cong [4, 4, 1^+, 4],$$

of index 6 in $[3, 4, 4]$, the first generated by the reflections $\rho_0, \rho_2, \rho_3, \rho_{12321} = \rho_{121}\rho_3\rho_{121}$, and $\rho_{1210121} = \rho_{121}\rho_0\rho_{121}$, the second by the reflections $\rho_0, \rho_2, \rho_{121}, \rho_{323} = \rho_3\rho_2\rho_3$, and $\rho_{31213} = \rho_3\rho_{121}\rho_3$, satisfying the relations indicated in the diagrams

Each of these groups has an involutory automorphism, conjugation by ρ_{121} or ρ_3.

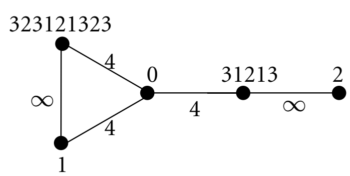
The groups $[1^+,4,4,4] \cong [4^{1,1,1}]$ and $[4,4,1^+,4] \cong [\infty,4,4,\infty]$ share the halving subgroup $[1^+,4,4,1^+,4] \cong [1^+,4^{1,1,1}] \cong [\infty,4,1^+,4,\infty] \cong [\infty^{[6]}]$, generated by the reflections $\rho_{121}, \rho_{323} = \rho_3\rho_2\rho_3$, $\rho_{01210} = \rho_0\rho_{121}\rho_0$, $\rho_{31213} = \rho_3\rho_{121}\rho_3$, ρ_2, and $\rho_{3012103} = \rho_3\rho_{01210}\rho_3 = \rho_0\rho_{31213}\rho_0$.

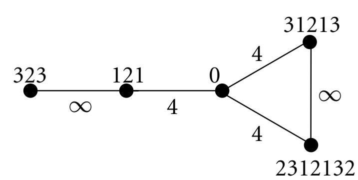
The Coxeter diagram is



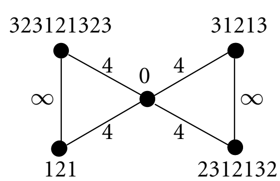
This group has an automorphism group D_6 of order 12.

The group $[4,4,1^+,4] \cong [\infty,4,4,\infty]$ has two isomorphic halving subgroups $[1^+,\infty,4,4,\infty]$ and $[\infty,4,4,\infty,1^+]$, the first generated by the reflections ρ_0, ρ_{121}, ρ_{31213}, ρ_2, and $\rho_{323121323} = \rho_{323}\rho_{121}\rho_{323}$ and the second by the reflections ρ_0, ρ_{121}, ρ_{31213}, ρ_{323}, and $\rho_{2312132} = \rho_2\rho_{31213}\rho_2$. The respective Coxeter diagrams are





The groups $[1^+,\infty,4,4,\infty]$ and $[\infty,4,4,\infty,1^+]$ have a common halving subgroup $[1^+,\infty,4,4,\infty,1^+] \cong [(4,4,\infty)^{1,1}]$, of index 4 in $[4,4,1^+,4]$, of index 8 in $[4,4,4]$, and of index 24 in $[3,4,4]$, generated by the reflections ρ_0, ρ_{121}, ρ_{31213}, $\rho_{2312132}$, and $\rho_{323121323}$. The Coxeter diagram is



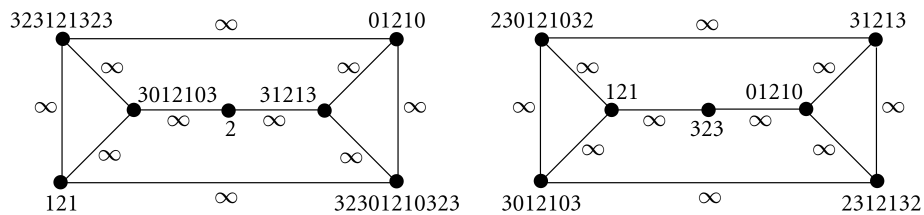
This group, with an automorphism group D_4 of order 8, is the radical subgroup $[4,4,4^*]$. The group $[1^+,4,4,4,1^+] \cong [4^{[4]}]$ has four halving subgroups $[(1^+,4^4)]$ of the same type, such as the one generated by $\rho_{232}, \rho_{121}, \rho_{01210}, \rho_{31213} = \rho_3\rho_{121}\rho_3$, and $\rho_{3012103} = \rho_3\rho_{01210}\rho_3$.

The groups $[1^+,\infty,4,4,\infty]$ and $[\infty,4,1^+,4,\infty] \cong [\infty^{[6]}]$ have a common halving subgroup generated by the reflections $\rho_2, \rho_{121}, \rho_{31213}, \rho_{01210} = \rho_0\rho_{121}\rho_0$, $\rho_{3012103} =$

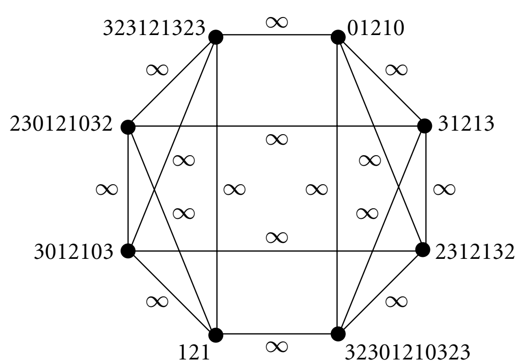
$\rho_0\rho_{31213}\rho_0$, $\rho_{323121323} = \rho_{323}\rho_{121}\rho_{323}$ and $\rho_{32301210323} = \rho_0\rho_{323121323}\rho_0 = \rho_{323}\rho_{01210}\rho_{323}$. Likewise $[\infty^{[6]}] \cong [\infty, 4, 1^+, 4, \infty]$ and $[\infty, 4, 4, \infty, 1^+]$ have a common halving subgroup generated by the reflections $\rho_{323}, \rho_{121}, \rho_{31213}, \rho_{01210} = \rho_0\rho_{121}\rho_0$, $\rho_{3012103} = \rho_0\rho_{312132}\rho_0$, $\rho_{2312132} = \rho_2\rho_{31213}\rho_2$, and $\rho_{230121032} = \rho_2\rho_{3012103}\rho_2 = \rho_0\rho_{2312132}\rho_0$. The two halving subgroups,

$$[1^+, \infty, 4, 1^+, 4, \infty] \cong [(1^+, \infty^6)] \cong [\infty, 4, 1^+, 4, \infty, 1^+],$$

have the respective Coxeter diagrams



These two groups, together with $[1^+, \infty, 4, 4, \infty, 1^+] \cong [(4, 4, \infty)^{1,1}]$, have their own halving subgroup $[1^+, \infty, 4, 1^+, 4, \infty, 1^+]$, of index 4 in the groups $[1^+, \infty, 4, 4, \infty]$, $[\infty, 4, 1^+, 4, \infty] \cong [\infty^{[6]}]$, $[\infty, 4, 4, \infty, 1^+]$, and $[1^+, 4, 4, 4, 1^+] \cong [4^{[4]}]$, of index 8 in $[1^+, 4, 4, 4]$, $[4, 1^+, 4, 4]$, $[4, 4, 1^+, 4]$, and, $[4, 4, 4, 1^+]$, of index 16 in $[4, 4, 4]$, and of index 48 in $[3, 4, 4]$, being conjugate to the radical subgroup $[(3, 4)^*, 4]$. Generators and relations for this group are evident in the Coxeter diagram



The group $[(3, 3, \infty)^{1,1}] \cong [1^+, \infty, 3, 3, \infty, 1^+]$ has a trionic subgroup $[(1^+, \infty^6)]^+ \cong [1^+, \infty, (3, 3)^{\triangle}, \infty, 1^+]$, of index 6 in $[(3, 3, \infty)^{1,1}]$ and of index 3 in $[(3, 3, \infty)^{1,1}]^+ \cong [1^+, \infty, (3, 3)^+, \infty, 1^+]$. This is the direct subgroup of the group $[(1^+, \infty^6)] \cong [\infty, 4, 1^+, 4, \infty, 1^+]$ defined above, as well as the commutator subgroup $\mathrm{PSL}_2''(\mathbb{G})$ of $[(3, 3, \infty)^{1,1}]^+ \cong \mathrm{PSL}_2'(\mathbb{G})$. It is generated by the half-turns x_1, x_2, x_3, and x_4 and the pararotations x_5 and x_6, where

$$x_1 = \rho_{31213}\rho_{323}, \quad x_2 = \rho_{323}\rho_{3012103}, \quad x_3 = \rho_{2312132}\rho_{323}, \quad x_4 = \rho_{323}\rho_{230121032},$$
$$x_5 = \rho_{01210}\rho_{323}, \quad x_6 = \rho_{323}\rho_{121}.$$

Defining relations for the group $[(1^+,\infty^6)]^+$ are

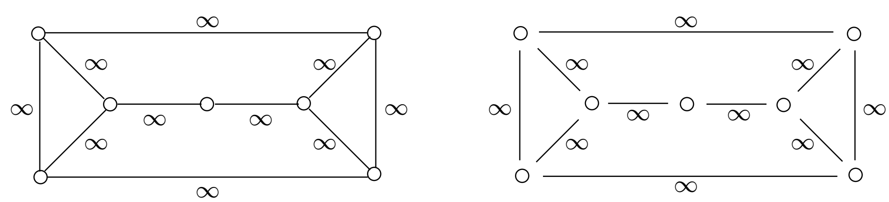
$$
\begin{aligned}
x_1^2 = x_2^2 = x_3^2 = x_4^2 &= (x_1x_2)^2 = (x_3x_4)^2 = (x_5x_6)^2 \\
&= (x_1x_6)^2 = (x_2x_5)^2 = (x_3x_6)^2 = (x_4x_5)^2 = 1
\end{aligned}
\tag{38}
$$

(*cf.* [9, p. 771], [8, p. 140]). Products of generators of $[(1^+,\infty^6)]^+$ are commutators of the generators of $[(3,3,\infty)^{1,1}]^+$:

$$
x_1x_2 = \sigma^{-1}\psi\sigma\psi^{-1},\quad x_3x_4 = \phi^{-1}\omega\phi\omega^{-1},\quad x_5x_6 = \sigma^{-1}\phi\sigma\phi^{-1},\quad x_6x_5 = \psi^{-1}\omega\psi\omega^{-1},
$$
$$
x_1x_6x_4x_5 = \sigma^{-1}\omega\sigma\omega^{-1},\quad x_3x_6x_2x_5 = \phi^{-1}\psi\phi\psi^{-1}.
$$

The commutator subgroup of both $[(1^+,\infty^6)]$ and $[(1^+,\infty^6)]^+$ is an ionic subgroup $[(1^+,\infty^6)]^{+7} \cong \mathrm{PSL}_2'''(\mathbb{G})$, of index 128 in $[(1^+,\infty^6)]$ and hence of index 64 in $[(1^+,\infty^6)]^+ \cong \mathrm{PSL}_2''(\mathbb{G})$. It is of index 192 in $[(3,3,\infty)^{1,1}]^+ \cong [\infty,3,3,\infty]^{+3} \cong \mathrm{PSL}_2'(\mathbb{G})$, of index 768 in $[\infty,3,3,\infty]^+ \cong [3,4,1^+,4]^+ \cong \mathrm{PSL}_2(\mathbb{G})$, and of index 3072 in $[3,4,4]$. All further members of the derived series for $\mathrm{PSL}_2(\mathbb{G})$ have infinite index [9, p. 772], [8, p. 141].

Respective Coxeter diagrams for the groups $[(1^+,\infty^6)]^+$ and $[(1^+,\infty^6)]^{+7}$ are



8 The Eisenstein Modular Group

The integral domain $\mathbb{E} = \mathbb{Z}[\omega] = \mathbb{Z}^2(-3)$ of *Eisenstein integers* comprises the complex numbers $e = e_0 + e_1\omega$, where $(e_0, e_1) \in \mathbb{Z}^2$ and $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-3}$ is a primitive cube root of unity, so that $\omega^2 + \omega + 1 = 0$. Quadratic integers of this type were investigated by Gotthold Eisenstein (1823–1852). Each Eisenstein integer e has a norm $N(e) = |e|^2 = e_0^2 - e_0e_1 + e_1^2$. The units of $\mathbb{E}$ are the six numbers with norm 1, namely ± 1, $\pm\omega$, $\pm\omega^2$, which form the Eisenstein *unit scalar* group $\bar{\mathrm{S}}\mathrm{Z}(\mathbb{E}) \cong C_6 \cong \langle -\omega\rangle$, with proper subgroups $S_3Z(\mathbb{E}) \cong C_3 \cong \langle\omega\rangle$ and $S_2Z(\mathbb{E}) \cong C_2 \cong \langle -1\rangle$.

The *special linear* group of $\mathrm{SL}_2(\mathbb{E})$ of 2×2 Eisenstein integer matrices of determinant 1 is generated by the matrices

$$
A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\quad C = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}
$$

[2, p. 316]. The *semispecial linear* group $S_2L_2(\mathbb{E})$ of 2×2 matrices S over $\mathbb{E}$ with $(\det S)^2 = 1$ is generated by A, B, C, and $L = \backslash 1, -1\backslash$ [29, p. 231]. The *ternispecial linear* group $S_3L_2(\mathbb{E})$

of matrices T with $(\det T)^3 = 1$ is generated by A, B, and $M = \backslash 1, \omega^2 \backslash$. The *unit linear* group $\bar{\mathrm{S}}\mathrm{L}_2(\mathbb{E})$ of matrices U with $|\det U| = 1$ is generated by A, B, and $N = \backslash 1, -\omega \backslash$. Note that $C = M^{-1}BM$, $L = N^3$, and $M = N^2$. The centre of both $\mathrm{SL}_2(\mathbb{E})$ and $S_2L_2(\mathbb{E})$ is the *special scalar* group $\mathrm{SZ}_2(\mathbb{E}) \cong \langle -I \rangle$, and the centre of both $\bar{\mathrm{S}}\mathrm{L}_2(\mathbb{E})$ and $S_3L_2(\mathbb{E})$ is the *unit scalar* group $\bar{\mathrm{S}}\mathrm{Z}(\mathbb{E}) \cong \langle -\omega I \rangle$.

The *Eisenstein modular* group

$$\mathrm{PSL}_2(\mathbb{E}) \cong \mathrm{SL}_2(\mathbb{E}) / \mathrm{SZ}_2(\mathbb{E}) \cong S_3L_2(\mathbb{E}) / \bar{\mathrm{S}}\mathrm{Z}(\mathbb{E}),$$

defined as the group of cosets of $\mathrm{SZ}_2(\mathbb{E})$ in $\mathrm{SL}_2(\mathbb{E})$, is generated in H^3 by the half-turn $\alpha = \cdot\langle A \rangle$ and the pararotations $\beta = \cdot\langle B \rangle$ and $\gamma = \cdot\langle C \rangle$; being alternatively the groups of cosets of $\bar{\mathrm{S}}\mathrm{Z}(\mathbb{E})$ in $S_3L_2(\mathbb{E})$, it is also generated by the half-turn α, the pararotation β, and the rotation $\mu = \cdot\langle M \rangle$ (of period 3). The *Eisenstein extended modular* group

$$\mathrm{P}\bar{\mathrm{S}}\mathrm{L}_2(\mathbb{E}) \cong \bar{\mathrm{S}}\mathrm{L}_2(\mathbb{E}) / \bar{\mathrm{S}}\mathrm{Z}(\mathbb{E}) \cong S_2L_2(\mathbb{E}) / \mathrm{SZ}_2(\mathbb{E}),$$

is similarly generated either by the half-turn α, the pararotation β, and the rotation $\nu = \cdot\langle N \rangle$ (of period 6) or by the half-turn α, the pararotations β and γ, and the half-turn $\lambda = \cdot\langle L \rangle$.

When the complex field $\mathbb{C}$ is regarded as a two-dimensional vector space over $\mathbb{R}$, the Eisenstein integers constitute a two-dimensional lattice A_2. The points of A_2 are the vertices of a regular tessellation $\{3, 6\}$ of the Euclidean plane E^2, whose symmetry group $[3, 6]$ is generated by three reflections ρ_1, ρ_2, ρ_3, satisfying the relations

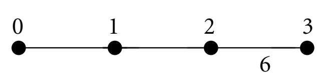
$$\rho_1^2 = \rho_2^2 = \rho_3^2 = (\rho_1\rho_2)^3 = (\rho_1\rho_3)^2 = (\rho_2\rho_3)^6 = 1. \tag{39}$$

The tessellation $\{3, 6\}$ is the vertex figure of a regular honeycomb $\{3, 3, 6\}$ of hyperbolic 3-space H^3, the cell polyhedra of which are regular tetrahedra $\{3, 3\}$ whose vertices all lie on the abolute sphere.

The symmetry group $[3, 3, 6]$ of the honeycomb $\{3, 3, 6\}$ is generated by four reflections ρ_0, ρ_1, ρ_2, ρ_3, satisfying (39) as well as

$$\rho_0^2 = (\rho_0\rho_1)^3 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = 1. \tag{40}$$

The combined relations (39) and (40) are indicated in the Coxeter diagram



The generators ρ_0, ρ_1, ρ_2, ρ_3 can be represented by antilinear fractional transformations $\bar{\cdot}\langle R_0 \rangle$, $\bar{\cdot}\langle R_1 \rangle$, $\bar{\cdot}\langle R_2 \rangle$, $\bar{\cdot}\langle R_3 \rangle$, determined by the matrices

$$R_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -\omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The direct subgroup $[3, 3, 6]^+$ is generated by three rotations $\sigma_1 = \rho_0\rho_1$, $\sigma_2 = \rho_1\rho_2$, $\sigma_3 = \rho_2\rho_3$, with the defining relations

$$\sigma_1^3 = \sigma_2^3 = \sigma_3^6 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1. \tag{41}$$

The generators $\sigma_1, \sigma_2, \sigma_3$ can be represented by linear fractional transformation $\cdot\langle S_1\rangle$, $\cdot\langle S_2\rangle$, $\cdot\langle S_3\rangle$, corresponding to the unit matrices

$$S_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} \omega & 0 \\ -\omega & \omega^2 \end{pmatrix}, \quad S_3 = \begin{pmatrix} \omega^2 & 0 \\ 0 & -\omega \end{pmatrix},$$

with entries in $\mathbb{E} = \mathbb{Z}[\omega]$ and determinants in $S_2Z(\mathbb{E}) \cong \langle -1 \rangle$ [29, pp. 234, 246], [23, pp. 102–103].

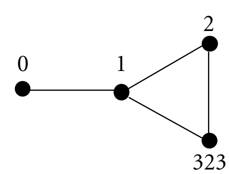
The matrices S_1, S_2, S_3 belong to and generate the semispecial linear group $S_2L_2(\mathbb{E}) \cong \langle A, B, C, L\rangle$, since

$$S_1S_2S_3^{-2} = A, \quad S_3^2S_2^{-1} = B, \quad S_2S_3^2S_2 = C, \quad \text{and} \quad S_3^3 = L.$$

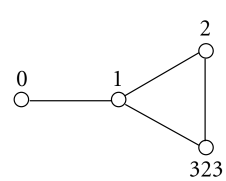
Thus the group $[3,3,6]^+$, generated by σ_1, σ_2, and σ_3 is the Eisenstein extended modular group $\mathrm{P\bar{S}L}_2(\mathbb{E}) \cong \langle \alpha, \beta, \gamma, \lambda\rangle$.

Fine [8, pp. 75–76, 81–82, 89–95] gives finite presentations for the Eisenstein modular group $\mathrm{PSL}_2(\mathbb{E})$, and Schulte & Weiss [29, p. 246] have shown that $\mathrm{PSL}_2(\mathbb{E})$ is a subgroup of index 2 in $[3,3,6]^+$. We now show that $\mathrm{PSL}_2(\mathbb{E})$ is the commutator subgroup of $[3,3,6]$.

The group $[3,3,6]$ has a halving subgroup $[3,3,6,1^+] \cong [3,3^{[3]}]$, generated by the reflections ρ_0, ρ_1, ρ_2, and $\rho_{323} = \rho_3\rho_2\rho_3$, satisfying the relations indicated in the diagram



This group has an involutory automorphism, conjugation by ρ_3, interchanging generators ρ_2 and ρ_{323}. There is also a semidirect subgroup $[(3,3)^+,6]$, generated by the rotations $\sigma_1 = \rho_0\rho_1$ and $\sigma_2 = \rho_1\rho_2$ and the reflection ρ_3. The groups $[3,3,6]$, $[3,3,6]^+$, $[3,3,6,1^+]$, and $[(3,3)^+,6]$ have a common commutator subgroup $[3,3,6,1^+]^+ \cong [(3,3)^+,6,1^+] \cong [3,3,6]^{+2}$, of index 4 in $[3,3,6]$ and of index 2 in the others, generated by the three rotations σ_1, σ_2, and $\sigma_{33} = \sigma_3^2 = (\rho_2\rho_3)^2 = \rho_2\rho_{323} = \sigma_2^{-1}\rho_3\sigma_2\rho_3$, with Coxeter diagram



Defining relations for the group $[(3,3)^+,6,1^+]$ are

(42) $$\sigma_1^3 = \sigma_2^3 = \sigma_{33}^3 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_{33})^3 = (\sigma_1\sigma_2\sigma_{33})^2 = 1.$$

Since the corresponding matrices S_1, S_2, and $S_{33} = S_3^2$ belong to the special linear group $\mathrm{SL}_2(\mathbb{E}) \cong \langle A, B, C\rangle$, $[(3,3)^+, 6, 1^+]$ is a subgroup of the Eisenstein modular group $\mathrm{PSL}_2(\mathbb{E})$. Indeed, we find it to be the whole group.

Theorem 8.1 *The Eisenstein special linear group* $\mathrm{SL}_2(\mathbb{E})$ *is generated by the matrices*

$$S_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} \omega & 0 \\ -\omega & \omega^2 \end{pmatrix}, \quad S_3 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}.$$

Proof Since S_1, S_2, and S_{33} each belong to $\mathrm{SL}_2(\mathbb{E}) \cong \langle A, B, C\rangle$, and since

$$S_1S_2S_{33}^{-1} = A, \quad S_{33}S_2^{-1} = B, \quad S_2S_{33}S_2 = C,$$

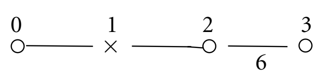
it follows that $\mathrm{SL}_2(\mathbb{E}) \cong \langle S_1, S_2, S_{33}\rangle$. ∎

The Eisenstein modular group $\mathrm{PSL}_2(\mathbb{E}) \cong \langle \alpha, \beta, \gamma\rangle$ is thus generated by the corresponding isometries σ_1, σ_2, and σ_{33}. That is, $\mathrm{PSL}_2(\mathbb{E})$ is isomorphic to the group $[(3,3)^+, 6, 1^+] \cong \langle \sigma_1, \sigma_2, \sigma_{33}\rangle$.

If we define matrices S and T by $S = S_1$ and $T = S_1S_2$, then it can be verified that $S = AB$ and that $T = C^{-1}ABCA^{-1}C^{-1}$. Likewise, taking $\sigma = \sigma_1$ and $\tau = \sigma_1\sigma_2$, we have $\sigma = \alpha\beta$ and $\tau = \gamma^{-1}\alpha\beta\gamma\alpha\gamma^{-1}$. These identites can be combined with the above relations to give a presentation for $\mathrm{PSL}_2(\mathbb{E})$ in terms of the generators α, σ, and τ or, on replacing σ and τ with $\alpha\beta$ and $\gamma^{-1}\alpha\beta\gamma\alpha\gamma^{-1}$, in terms of the generators α, β, and γ. Alperin [1, p. 2935] gives defining relations for $\mathrm{PSL}_2(\mathbb{E})$ satisfied by $a = \sigma_1$, $b = \sigma_1\sigma_2$, and $c = \sigma_1\sigma_2\sigma_{33}$:

$$a^3 = b^2 = c^2 = (ab)^3 = (a^{-1}c)^3 = (bc)^3 = 1. \tag{43}$$

The trionic subgroup $[(3,3)^{\triangle}, 6, 1^+] \cong \mathrm{PSL}_2'(\mathbb{E})$, with Coxeter diagram



is the commutator subgroup of $[(3,3)^+, 6, 1^+] \cong \mathrm{PSL}_2(\mathbb{E})$, of index 3. It is generated by the four half-turns $\sigma_{12} = \sigma_1\sigma_2$, $\sigma_{21} = \sigma_2\sigma_1$, $\bar{\sigma}_{12} = \sigma_1\sigma_2\sigma_{33}$, and $\bar{\sigma}_{21} = \sigma_2\sigma_{33}\sigma_1$, satisying the relations

$$\begin{aligned} \sigma_{12}^2 = \sigma_{21}^2 = \bar{\sigma}_{12}^2 = \bar{\sigma}_{21}^2 &= (\sigma_{12}\sigma_{21})^2 = (\bar{\sigma}_{12}\bar{\sigma}_{21})^2 \\ &= (\sigma_{12}\bar{\sigma}_{12})^3 = (\sigma_{21}\bar{\sigma}_{21})^3 = (\sigma_{12}\sigma_{21}\bar{\sigma}_{12}\bar{\sigma}_{21})^3 = 1. \end{aligned} \tag{44}$$

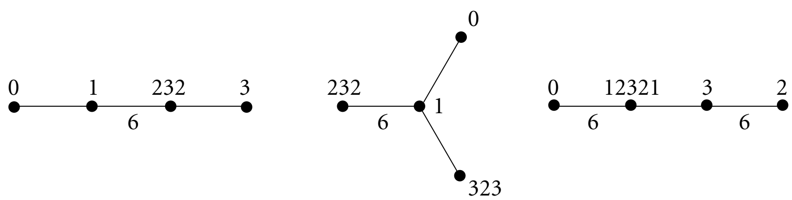
The corresponding matrices are $S_{12} = S_1S_2$, $S_{21} = S_2S_1$, $\bar{S}_{12} = S_1S_2S_{33}$, and $\bar{S}_{21} = S_2S_{33}S_1$, which evaluate as

$$S_{12} = \begin{pmatrix} 0 & \omega^2 \\ -\omega & 0 \end{pmatrix}, \quad S_{21} = \begin{pmatrix} \omega & \omega \\ 1 & -\omega \end{pmatrix}, \quad \bar{S}_{12} = \begin{pmatrix} 0 & \omega \\ -\omega^2 & 0 \end{pmatrix}, \quad \bar{S}_{21} = \begin{pmatrix} \omega^2 & \omega^2 \\ 1 & -\omega^2 \end{pmatrix}.$$

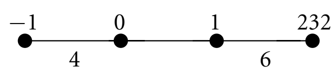
By combining the identities $\sigma_{12} = b$, $\bar{\sigma}_{12} = c$, $\sigma_{21} = a^{-1}ba$, and $\bar{\sigma}_{21} = a^{-1}ca$ with the above relations, we obtain a presentation for $\mathrm{PSL}_2'(\mathbb{E})$ in terms of the alternative generators a, b, and c (*cf.* [1, p. 2937].

9 Other Subgroups of $[3, 3, 6]$

Besides the halving subgroup $[3, 3^{[3]}]$ just discussed, the group $[3, 3, 6]$ has several other subgroups of interest. The subgroup $[3, 6, 3]$, of index 4, is generated by the reflections ρ_0, ρ_1, ρ_3, and $\rho_{232} = \rho_2\rho_3\rho_2$. The subgroup $[6, 3^{1,1}]$, of index 5, is generated by the reflections ρ_0, ρ_1, ρ_{232}, and $\rho_{323} = \rho_3\rho_2\rho_3$. The subgroup $[6, 3, 6]$, of index 6, is generated by the reflections ρ_0, ρ_2, ρ_3, and $\rho_{12321} = \rho_1\rho_{232}\rho_1$. The three groups $[3, 6, 3]$, $[6, 3^{1,1}]$, and $[6, 3, 6]$ have the respective Coxeter diagrams



As evidenced by the bilateral symmetry of their graphs, each of these groups has an involutory automorphism. For the group $[6, 3^{1,1}]$ this is conjugation by a reflection ρ_{-1} in a plane bisecting the fundamental region. Augmenting $[6, 3^{1,1}]$ by this automorphism, we get another Coxeter group $[4, 3, 6]$, generated by the reflections ρ_{-1}, ρ_0, ρ_1, and ρ_{232}, as in the diagram



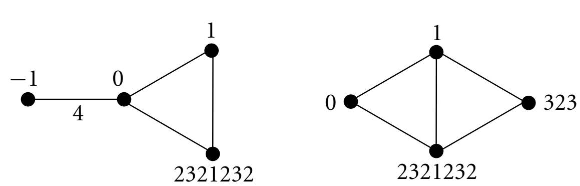
The generators $\rho_{-1}, \rho_0, \rho_1, \rho_{232}$ can be represented by antilinear fractional transformations $\bar{\cdot}\langle R_{-1}\rangle$, $\bar{\cdot}\langle R_0\rangle$, $\bar{\cdot}\langle R_1\rangle$, $\bar{\cdot}\langle R_{232}\rangle$, determined by the matrices

$$R_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\omega^2 & 1 \\ 1 & \omega \end{pmatrix}, \quad R_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad R_{232} = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega \end{pmatrix}.$$

(Note that the entries of R_{-1} are not Eisenstein integers.) Another representation of this group is given by Nostrand, Schulte & Weiss [25, p. 167].

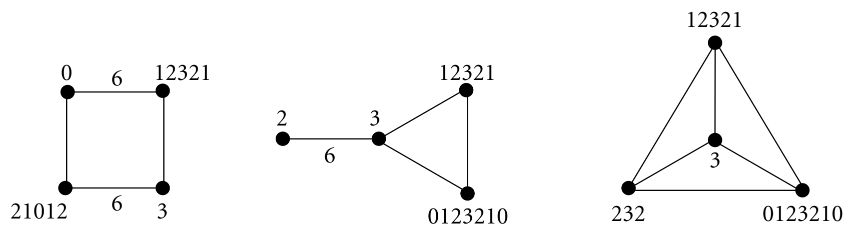
The groups $[3, 3, 6]$, $[3, 6, 3]$, $[6, 3, 6]$, and $[4, 3, 6]$ are the symmetry groups of regular honeycombs. The group $[4, 3, 6]$ has two halving subgroups, $[1^+, 4, 3, 6] \cong [6, 3^{1,1}]$ and $[4, 3, 6, 1^+] \cong [4, 3^{[3]}]$, the respective symmetry groups of the "half regular" honeycombs $\mathrm{h}\{4, 3, 6\}$ and $\mathrm{h}\{6, 3, 4\}$. These two groups have a common halving subgroup $[1^+, 4, 3, 6, 1^+] \cong [3^{[\,]\times[\,]}]$, of index 4 in $[4, 3, 6]$, the symmetry group of the "quarter regular" honeycomb $\mathrm{q}\{4, 3, 6\} = \mathrm{q}\{6, 3, 4\}$.

Generators for $[4, 3^{[3]}]$ are ρ_{-1}, ρ_0, ρ_1, and $\rho_{2321232} = \rho_{232}\rho_1\rho_{232}$, and generators for $[3^{[\,]\times[\,]}]$ are ρ_0, ρ_1, $\rho_{2321232}$, and $\rho_{323} = \rho_{(-1)0(-1)} = \rho_{-1}\rho_0\rho_{-1}$, as indicated in the Coxeter diagrams



The groups $[6, 3^{1,1}]$ and $[3, 6, 3]$ have a common subgroup $[(3, 6)^{[2]}]$, of index 4 in $[6, 3^{1,1}]$, of index 5 in $[3, 6, 3]$, of index 8 in $[4, 3, 6]$, and of index 20 in $[3, 3, 6]$. The halving subgroup of $[6, 3, 6]$ is $[6, 3, 6, 1^+] \cong [6, 3^{[3]}]$, a subgroup of index 3 in $[3, 6, 3]$ and of index 12 in $[3, 3, 6]$. This group has its own halving subgroup $[1^+, 6, 3, 6, 1^+] \cong [1^+, 6, 3^{[3]}] \cong [3^{[3,3]}]$, of index 4 in $[6, 3, 6]$, of index 6 in $[3, 6, 3]$, and of index 24 in $[3, 3, 6]$.

Generators for $[(3, 6)^{[2]}]$ are ρ_0, ρ_{12321}, ρ_3, and $\rho_{21012} = \rho_2\rho_1\rho_0\rho_1\rho_2$, generators for $[6, 3^{[3]}]$ are ρ_2, ρ_3, ρ_{12321}, and $\rho_{0123210} = \rho_0\rho_{12321}\rho_0$, and generators for $[3^{[3,3]}]$ are $\rho_{0123210}$, ρ_{12321}, ρ_{232}, and ρ_3, as indicated in the diagrams



The group $[3^{[3,3]}]$ is the radical subgroup $[(3, 3)^*, 6] \cong [3^*, 6, 3]$.

With the exception of $[3, 3, 6]$ and $[4, 3, 6]$, all of the above groups have nontrivial automorphism groups, of order 2 in most cases. The generators of $[3^{[\ ]\times[\ ]}]$ and $[(3, 6)^{[2]}]$ are each permuted by an automorphism group D_2 of order 4, and $[3^{[3,3]}]$ has an automorphism group S_4 of order 24. Adjoining such automorphisms to a given group yields other Coxeter groups, subgroups, or supergroups as semidirect products.

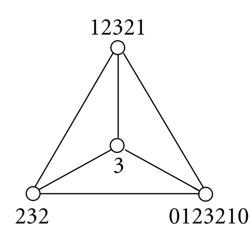
The group $[(3, 3)^{\triangle}, 6, 1^+]$, which is generated by four half-turns σ_{12}, σ_{21}, $\bar\sigma_{12}$, and $\bar\sigma_{21}$, satisfying the relations (44), has a subgroup $[3^{[3,3]}]^+ \cong [(3, 3)^*, 6, 1^+]$ of index 4. The direct subgroup of the group $[3^{[3,3]}]$ and the commutator subgroup $\mathrm{PSL}_2''(\mathbb{E})$ of $[(3, 3)^{\triangle}, 6, 1^+] \cong \mathrm{PSL}_2'(\mathbb{E})$, it is generated by the rotations u, v, and w, where

$$u = \rho_{232}\rho_3 = \sigma_{12}\bar\sigma_{12}, \quad v = \rho_{12321}\rho_3 = \sigma_{21}\bar\sigma_{21}, \quad w = \rho_{0123210}\rho_3 = \sigma_{12}\sigma_{21}\bar\sigma_{12}\bar\sigma_{21},$$

or $u = bc$, $v = a^{-1}bca$, and $w = abca^{-1}$. Defining relations for $\mathrm{PSL}_2''(\mathrm{E})$ are

$$u^3 = v^3 = w^3 = (uv^{-1})^3 = (vw^{-1})^3 = (wu^{-1})^3 = 1 \tag{45}$$

(*cf.* [1, p. 2937]). As the group $[3^{[3,3]}]^+$, its Coxeter diagram is



The group $[3^{[3,3]}]$ has hypercompact subgroup $[3^{[[3,3]]}]^+$ of index 5, generated by six reflections, the fundamental region being a regular hexahedron $\{4, 3\}$ whose vertices all lie on the absolute sphere. As the radical subgroup $[6, (3^{1,1})^*] \cong [(4, 3)^*, 6]$, it is of index 24 in $[6, 3^{1,1}]$, of index 48 in $[4, 3, 6]$, and of index 120 in $[3, 3, 6]$. The direct subgroup $[3^{[[3,3]]}]^+$ is a subgroup of index 60 in $[(3, 3)^+, 6, 1^+] \cong \mathrm{PSL}_2(\mathbb{E})$. If we let $x = abcb = \sigma_1\sigma_{33}\sigma_{12}$, $[3^{[[3,3]]}]^+$ is the normal subgroup of $\mathrm{PSL}_2(\mathbb{E})$ generated by x^2 [1, p. 2939].

The commutator subgroup of $[3^{[3,3]}]^+$ is a group $[3^{[3,3]}]^{\triangle} \cong \mathrm{PSL}_2'''(\mathbb{E})$, the normal subgroup of $\mathrm{PSL}_2(\mathbb{E})$ generated by x^3. It is of index 27 in $[3^{[3,3]}]^+ \cong [(3, 3)^*, 6, 1^+] \cong \mathrm{PSL}_2''(\mathbb{E})$, of index 108 in $[(3, 3)^{\triangle}, 6, 1^+] \cong \mathrm{PSL}_2'(\mathbb{E})$, of index 324 in $[(3, 3)^+, 6, 1^+] \cong \mathrm{PSL}_2(\mathbb{E})$, and of index 1296 in $[3, 3, 6]$. All subsequent members of the derived series for $\mathrm{PSL}_2(\mathbb{E})$ have infinite index [1, p. 2938].

10 Summary

Through the systematic application of the theory of discrete groups operating in hyperbolic space, we have provided a unified description of linear fractional transformations over rings of rational or quadratic integers. The following theorems summarize the isomorphisms established here between real or complex linear fractional groups (and their derived subgroups) and subgroups of hyperbolic Coxeter groups.

Theorem 10.1 *The rational modular group* $\mathrm{PSL}_2(\mathbb{Z})$ *and its commutator subgroup are isomorphic to subgroups of the symmetry group of the regular hyperbolic tessellation* $\{3, \infty\}$*:*

$$P\bar{S}L_2(\mathbb{Z}) \cong [3, \infty],$$

$$\mathrm{PSL}_2(\mathbb{Z}) \cong [3, \infty]^+,$$

$$\mathrm{PSL}_2'(\mathbb{Z}) \cong [3^+, \infty, 1^+].$$

Theorem 10.2 *The semiquadratic modular groups* $\mathrm{PSL}_{1+1}(\mathbb{Z}[\sqrt{d}])$*,* $d = 2$ *or* 3*, and their commutator subgroups are isomorphic to subgroups of the symmetry groups of the regular hyperbolic tessellations* $\{4, \infty\}$ *and* $\{6, \infty\}$*:*

$$P\bar{S}L_{1+1}(\mathbb{Z}[\sqrt{2}]) \cong [4, \infty], \quad P\bar{S}L_{1+1}(\mathbb{Z}[\sqrt{3}]) \cong [6, \infty],$$

$$\mathrm{PSL}_{1+1}(\mathbb{Z}[\sqrt{2}]) \cong [4, \infty]^+, \quad \mathrm{PSL}_{1+1}(\mathbb{Z}[\sqrt{3}]) \cong [6, \infty]^+,$$

$$\mathrm{PSL}_{1+1}'(\mathbb{Z}[\sqrt{2}]) \cong [4, \infty]^{+3}, \quad \mathrm{PSL}_{1+1}'(\mathbb{Z}[\sqrt{3}]) \cong [6, \infty]^{+3}.$$

Theorem 10.3 *The Gaussian modular (Picard) group* $\mathrm{PSL}_2(\mathbb{G})$ *and the Eisenstein modular group* $\mathrm{PSL}_2(\mathbb{E})$ *and their derived subgroups are isomorphic to subgroups of the symmetry groups of the regular honeycombs* $\{3, 4, 4\}$ *and* $\{3, 3, 6\}$ *of hyperbolic 3-space:*

$$P\bar{S}L_2(\mathbb{G}) \cong [3, 4, 4]^+, \quad P\bar{S}L_2(\mathbb{E}) \cong [3, 3, 6]^+,$$

$$\mathrm{PSL}_2(\mathbb{G}) \cong [3, 4, 1^+, 4]^+, \quad \mathrm{PSL}_2(\mathbb{E}) \cong [(3, 3)^+, 6, 1^+],$$

$$\mathrm{PSL}_2'(\mathbb{G}) \cong [\infty, 3, 3, \infty]^{+3}, \quad \mathrm{PSL}_2'(\mathbb{E}) \cong [(3, 3)^{\triangle}, 6, 1^+],$$

$$\mathrm{PSL}_2''(\mathbb{G}) \cong [(1^+, \infty^6)]^+, \quad \mathrm{PSL}_2''(\mathbb{E}) \cong [3^{[3,3]}]^+,$$

$$\mathrm{PSL}_2'''(\mathbb{G}) \cong [(1^+, \infty^6)]^{+7}. \quad \mathrm{PSL}_2'''(\mathbb{E}) \cong [3^{[3,3]}]^{\triangle}.$$

In passing we have found explicit or implicit matrix representations for every crystallographic Coxeter group whose fundamental region is the closure of a Koszul (asymptotic) triangle or tetrahedron. Except for the mixed groups $[4,3,6]$ and $[4,3^{[3]}]$, the isometries of each such paracompact group can be represented by 2×2 matrices over the rational integers $\mathbb{Z}$ or some ring $\mathbb{Z}^2(d)$ of quadratic integers.

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