# The Euler characteristic of a category

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#### Abstract

The Euler characteristic of a finite category is defined and shown to be compatible with Euler characteristics of other types of object, including orbifolds. A formula for the cardinality of the colimit of a diagram of sets is proved, generalizing the classical inclusion-exclusion formula. Both rest on a generalization of Möbius-Rota inversion from posets to categories.

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## Introduction

We first learn of Euler characteristic as 'vertices minus edges plus faces', and later as an alternating sum of ranks of homology groups. But Euler characteristic is much more fundamental than these definitions make apparent, as has been made increasingly explicit over the last fifty years; it is something akin to cardinality or measure. More precisely, it is the fundamental dimensionless quantity associated with an object.

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Finite sets provide the very simplest context for Euler characteristic, and of course the fundamental way to assign a quantity to a finite set is to count its elements. Indeed, Euler characteristic of topological spaces can usefully be thought of as a generalization of cardinality; for instance, it obeys the same laws with respect to unions and products.

A further example reinforces the point. A subset of  $\mathbb{R}^n$  is **polyconvex** if it is a finite union of compact convex subsets. Let  $V_n$  be the vector space of finitely additive measures, invariant under Euclidean transformations, defined on the polyconvex subsets of  $\mathbb{R}^n$ . Hadwiger's Theorem [KR] states that dim  $V_n = n+1$ . (See also [Sc2].) A natural basis consists of one d-dimensional measure for each  $d \in \{0, \ldots, n\}$ : for instance, {Euler characteristic, perimeter, area} when n = 2. Thus, up to scalar multiplication, Euler characteristic is the unique dimensionless measure on polyconvex sets.

Schanuel [Sc1] showed that in other contexts, Euler characteristic can be defined in a way that makes its fundamental nature transparent. He proved that for a certain category of polyhedra, Euler characteristic is determined by a straightforward universal property.

All of this makes clear the importance of defining and understanding Euler characteristic in new contexts. Here we do this for finite categories.

One might envisage simply transporting the definition from spaces to categories via the classifying space functor, as with other topological invariants: that is, given a category  $\mathbb{A}$ , define  $\chi(\mathbb{A})$  as the Euler characteristic of the classifying space  $B\mathbb{A}$ . The trouble with this is that the Euler characteristic of  $B\mathbb{A}$  is not always defined. Below we give a definition of the Euler characteristic of a category that agrees with the topological Euler characteristic when the latter exists, but is also valid in a range of situations when it does not. It is a rational number, not necessarily an integer.

A version of the definition can be given very succinctly. Let  $\mathbb{A}$  be a finite category; totally order its objects as  $a_1, \ldots, a_n$ . Let Z be the matrix whose (i,j)-entry is the number of arrows from  $a_i$  to  $a_j$ . Let  $M = Z^{-1}$ , assuming that this inverse exists. Then  $\chi(\mathbb{A})$  is the sum of the entries of M. Of course, it remains to convince the reader that this definition is the right one.

The foundation on which this work rests is a generalization of Möbius–Rota inversion (§1). Rota developed Möbius inversion for posets [R]; we develop it for categories. (A poset is viewed throughout as a category in which each hom-set has at most one element: the objects are the elements of the poset, and there is an arrow  $a \longrightarrow b$  if and only if  $a \le b$ .) This leads, among other things, to a 'representation formula': given any functor known to be a sum of representables, the formula tells us the representation explicitly. This in turn can be used to solve enumeration problems, in the spirit of Rota's paper.

However, the main application of this generalized Möbius inversion is to the theory of the Euler characteristic of a category ( $\S 2$ ). We actually use a different definition than the one just given; it agrees with the one above when Z is invertible, but is valid for a wider class of categories. It depends on the idea of the 'weight' of an object of a category. We justify the definition in two ways: by showing that it enjoys the properties that the name would lead one to expect

(behaviour with respect to products, fibrations, etc.), and by demonstrating its compatibility with Euler characteristics of other types of structure (groupoids, graphs, topological spaces, orbifolds).

The technology of Möbius inversion and weights also solves another problem: what is the cardinality of a colimit? For example, the union of a family of sets and the quotient of a set by a free action of a group are both examples of colimits of set-valued functors, and there are simple formulas for their cardinalities. (In the first case it is the inclusion-exclusion formula.) We generalize, giving a formula valid for any shape of colimit (§3).

Rota and his school proved a large number of results on Möbius inversion for posets. As we will repeatedly see, many are not truly order-theoretic: they are facts about categories in general. In particular, important theorems in Rota's original work [R] generalize from posets to categories (§4).

(The body of work on Möbius inversion in finite lattices is not, however, so ripe for generalization: a poset is a lattice just when the corresponding category has products, but a finite category cannot have products unless it is, in fact, a lattice.)

Other authors have considered different notions of Möbius inversion for categories; notably, there is that developed by Content, Lemay and Leroux [CLL] and independently by Haigh [H]. This generalizes both Rota's notion for posets and Cartier and Foata's for monoids [CF]. (Here a monoid is viewed as a one-object category.) The relation between their approach and ours is discussed in §4. A third approach, not discussed here, was taken by Dür [D].

In the case of groupoids, our Euler characteristic of categories agrees with Baez and Dolan's groupoid cardinality [BD]. That in turn interacts well with the species of Joyal [J, BLL]. Our definition of the Lefschetz number of an endofunctor (§2) may perhaps be related to Paré's definition of the cardinality of an endofunctor of the category of finite sets [Pa].

The view of Euler characteristic as generalized cardinality is promoted in [Sc1], [BD] and [Pr1]. The appearance of a non-integral Euler characteristic is nothing new: see for instance Wall [Wl], Bass [Ba] and Cohen [Co], and the discussion of orbifolds in §2.

Ultimately it would be desirable to have the Euler characteristic of categories described by a universal property, as Schanuel did for polyhedra [Sc1]. For this, it may be necessary to relax the constraints of the present work, where for simplicity our categories are required to be finite and the coefficients are required to lie in the rig (semiring) of rational numbers. Rather than asking, as below, 'does this category have Euler characteristic (in  $\mathbb{Q}$ )?', we should perhaps ask 'in what rig does the Euler characteristic of this category lie?' However, this is not pursued here.

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### 1 Möbius inversion

We consider a finite category  $\mathbb{A}$ , writing ob  $\mathbb{A}$  for its set of objects and, when a and b are objects,  $\mathbb{A}(a,b)$  for the set of maps from a to b.

**Definition 1.1** We denote by  $R(\mathbb{A})$  the  $\mathbb{Q}$ -algebra of functions ob  $\mathbb{A} \times$  ob  $\mathbb{A} \longrightarrow \mathbb{Q}$ , with pointwise addition and scalar multiplication, multiplication defined by

$$(\theta\phi)(a,c) = \sum_{b \in \mathbb{A}} \theta(a,b)\phi(b,c)$$

 $(\theta, \phi \in R(\mathbb{A}), a, c \in \mathbb{A})$ , and the Kronecker  $\delta$  as unit.

The **zeta function**  $\zeta_{\mathbb{A}} = \zeta \in R(\mathbb{A})$  is defined by  $\zeta(a,b) = |\mathbb{A}(a,b)|$ . If  $\zeta$  is invertible in  $R(\mathbb{A})$  then  $\mathbb{A}$  is said to **have Möbius inversion**; its inverse  $\mu_{\mathbb{A}} = \mu = \zeta^{-1}$  is the **Möbius function** of  $\mathbb{A}$ .

If a total ordering is chosen on the n objects of  $\mathbb{A}$  then  $R(\mathbb{A})$  can be regarded as the algebra of  $n \times n$  matrices over  $\mathbb{Q}$ . The defining equations of the Möbius function are

$$\sum_{b} \mu(a,b)\zeta(b,c) = \delta(a,c) = \sum_{b} \zeta(a,b)\mu(b,c)$$

for all  $a, c \in \mathbb{A}$ . By finite-dimensionality, either one implies the other.

The definitions above could be made for directed graphs rather than categories, since they do not refer to composition. However, this generality seems to be inappropriate. For example, the definition of Möbius inversion will lead to a definition of Euler characteristic, and if we use graphs rather than categories then we obtain something other than 'vertices minus edges'. Proposition 2.10 clarifies this point.

A different notion of Möbius inversion for categories has been considered; see  $\S 4$ .

**Examples 1.2** a. Any finite poset A has Möbius inversion; this special case was investigated by Rota [R] and others. We may compute  $\mu(a,c)$  by induction on the number of elements between a and c:

$$\mu(a,c) = \delta(a,c) - \sum_{b: a \le b < c} \mu(a,b).$$

In particular,  $\mu(a,c) = 0$  unless  $a \le c$ , and  $\mu(a,a) = 1$  for all a.

- b. Let M be a finite monoid, regarded as a category with unique object  $\star$ . Then  $\zeta(\star,\star)=|M|$ , so  $\mu(\star,\star)=1/|M|$ .
- c. Let  $N \geq 0$ . Write  $\mathbb{D}_N^{\rm inj}$  for the category with objects  $0,\ldots,N$  whose maps  $a \longrightarrow b$  are the order-preserving injections  $\{1,\ldots,a\} \longrightarrow \{1,\ldots,b\}$ . Then  $\zeta(a,b)=\binom{b}{a}$ , and it is easily checked that  $\mu(a,b)=(-1)^{b-a}\binom{b}{a}$ . If we use surjections instead of injections then  $\zeta(a,b)=\binom{a-1}{b-1}$  and  $\mu(a,b)=(-1)^{a-b}\binom{a-1}{b-1}$ .

A category with Möbius inversion must be skeletal, for otherwise the matrix of  $\zeta$  would have two identical rows. The property of having Möbius inversion is not, therefore, invariant under equivalence of categories.

In general we cannot hope to just spot the Möbius function of a category. In 1.3-1.7 we build tools for computing Möbius functions. These cover large classes of categories, although not every finite skeletal category has Möbius inversion (1.11(d), (e)).

Let  $n \geq 0$ , let  $\mathbb{A}$  be a category or a directed graph, and let  $a, b \in \mathbb{A}$ . An n-path from a to b is a diagram

$$a = a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} a_n = b$$
 (1)

in  $\mathbb{A}$ . It is a **circuit** if a = b, and (when  $\mathbb{A}$  is a category) **nondegenerate** if no  $f_i$  is an identity.

**Lemma 1.3** The following conditions on a finite category  $\mathbb{A}$  are equivalent:

- a. every idempotent in  $\mathbb{A}$  is an identity
- b. every endomorphism in A is an automorphism
- c. every circuit in  $\mathbb{A}$  consists entirely of isomorphisms.

**Proof** (a)  $\Rightarrow$  (b) follows from the fact that if f is an element of a finite monoid then some positive power of f is idempotent. The other implications are straightforward.

**Theorem 1.4** Let  $\mathbb{A}$  be a finite skeletal category in which the only idempotents are identities. Then  $\mathbb{A}$  has Möbius inversion given by

$$\mu(a,b) = \sum (-1)^n / |\operatorname{Aut}(a_0)| \cdots |\operatorname{Aut}(a_n)|$$

where  $\operatorname{Aut}(a)$  is the automorphism group of  $a \in \mathbb{A}$  and the sum runs over all  $n \geq 0$  and paths (1) for which  $a_0, \ldots, a_n$  are all distinct.

**Proof** First observe that for a path (1) in  $\mathbb{A}$ , if  $a_0 \neq a_1 \neq \cdots \neq a_n$  then the  $a_i$ s are all distinct. Indeed, if  $0 \leq i < j \leq n$  and  $a_i = a_j$  then the sub-path running from  $a_i$  to  $a_j$  is a circuit, so by Lemma 1.3,  $f_{i+1}$  is an isomorphism, and by skeletality,  $a_i = a_{i+1}$ .

Now let  $a, c \in \mathbb{A}$  and define  $\mu$  by the formula above. We have

$$\begin{split} \sum_{b \in \mathbb{A}} \mu(a,b) \zeta(b,c) &= \mu(a,c) \zeta(c,c) + \sum_{b:b \neq c} \mu(a,b) \zeta(b,c) \\ &= |\mathrm{Aut}(c)| \left\{ \mu(a,c) + \sum_{b:b \neq c, \ g \in \mathbb{A}(b,c)} \mu(a,b) / |\mathrm{Aut}(c)| \right\}, \end{split}$$

and by definition of  $\mu$ , the term in braces collapses to  $\delta(a,c)/|\mathrm{Aut}(a)|$ .

**Corollary 1.5** Let  $\mathbb{A}$  be a finite skeletal category in which the only endomorphisms are identities. Then  $\mathbb{A}$  has Möbius inversion given by

$$\mu(a,b) = \sum_{n\geq 0} (-1)^n |\{nondegenerate\ n\text{-paths from } a\ to\ b\}| \in \mathbb{Z}.$$

When  $\mathbb{A}$  is a poset, this is Philip Hall's theorem (Proposition 3.8.5 of [St] and Proposition 6 of [R]).

Recall that an **epi-mono factorization system**  $(\mathcal{E}, \mathcal{M})$  on a category  $\mathbb{A}$  consists of a class  $\mathcal{E}$  of epimorphisms in  $\mathbb{A}$  and a class  $\mathcal{M}$  of monomorphisms in  $\mathbb{A}$ , satisfying axioms [FK]. The axioms imply that every map f in  $\mathbb{A}$  can be expressed as me for some  $e \in \mathcal{E}$  and  $m \in \mathcal{M}$ , and that this factorization is essentially unique: the other pairs  $(e', m') \in \mathcal{E} \times \mathcal{M}$  satisfying m'e' = f are those of the form  $(ie, mi^{-1})$  where i is an isomorphism.

**Theorem 1.6** Let  $\mathbb{A}$  be a finite skeletal category with an epi-mono factorization system  $(\mathcal{E}, \mathcal{M})$ . Then  $\mathbb{A}$  has Möbius inversion given by

$$\mu(a,b) = \sum (-1)^n / |\operatorname{Aut}(a_0)| \cdots |\operatorname{Aut}(a_n)|$$

where the sum is over all  $n \geq r \geq 0$  and paths (1) such that  $a_0, \ldots, a_r$  are distinct,  $a_r, \ldots, a_n$  are distinct,  $f_1, \ldots, f_r \in \mathcal{M}$ , and  $f_{r+1}, \ldots, f_n \in \mathcal{E}$ .

**Proof** The objects of  $\mathbb{A}$  and the arrows in  $\mathcal{E}$  determine a subcategory of  $\mathbb{A}$ , also denoted  $\mathcal{E}$ ; it satisfies the hypotheses of Theorem 1.4 and therefore has Möbius inversion. The same is true of  $\mathcal{M}$ .

Any element  $\alpha \in \mathbb{Q}^{\text{ob}\,\mathbb{A}}$  gives rise to an element of  $R(\mathbb{A})$ , also denoted  $\alpha$  and defined by  $\alpha(a,b) = \delta(a,b)\alpha(b)$ . The resulting map from  $\mathbb{Q}^{\text{ob}\,\mathbb{A}}$  to  $R(\mathbb{A})$  preserves multiplication (where the multiplication on  $\mathbb{Q}^{\text{ob}\,\mathbb{A}}$  is pointwise). We have elements |Aut| and 1/|Aut| of  $\mathbb{Q}^{\text{ob}\,\mathbb{A}}$ , where, for instance, |Aut|(a) = |Aut(a)|.

By the essentially unique factorization property,  $\zeta_{\mathbb{A}} = \zeta_{\mathcal{E}} \cdot \frac{1}{|\operatorname{Aut}|} \cdot \zeta_{\mathcal{M}}$ . Hence  $\mathbb{A}$  has Möbius function  $\mu_{\mathbb{A}} = \mu_{\mathcal{M}} \cdot |\operatorname{Aut}| \cdot \mu_{\mathcal{E}}$ . Theorem 1.4 then gives the formula claimed.

**Example 1.7** Let  $N \geq 0$  and write  $\mathbb{F}_N$  for the full subcategory of **Set** with objects  $1, \ldots, N$ , where n denotes a (chosen) n-element set. Let  $\mathcal{E}$  be the set of surjections in  $\mathbb{F}_N$  and  $\mathcal{M}$  the set of injections; then  $(\mathcal{E}, \mathcal{M})$  is a factorization system. Theorem 1.6 gives a formula for the inverse of the matrix  $(i^j)_{i,j}$ . For instance, take N = 3; then  $\mu(1, 2)$  may be computed as follows:

Here '1  $\stackrel{2}{\longmapsto}$  2' means that there are 2 monomorphisms from 1 to 2, '3  $\stackrel{6}{\Longrightarrow}$  2' that there are 6 epimorphisms from 3 to 2, etc. Hence  $\mu(1,2) = -1 + 3/2 - 3 = -5/2$ .

One of the uses of the Möbius function is to calculate Euler characteristic (§2). Another is to calculate representations. Specifically, suppose that we have a **Set**-valued functor known to be **familially representable**, that is, a coproduct of representables. The Yoneda Lemma tells us that the family of representing objects is unique (up to isomorphism); better, if we have Möbius inversion, there is actually a formula for it:

**Proposition 1.8** Let  $\mathbb{A}$  be a finite category with Möbius inversion and let X:  $\mathbb{A} \longrightarrow \mathbf{Set}$  be a functor satisfying

$$X \cong \sum_{a} r(a) \mathbb{A}(a, -)$$

for some natural numbers r(a)  $(a \in A)$ . Then

$$r(a) = \sum_{b} |Xb| \mu(b, a)$$

for all  $a \in \mathbb{A}$ .

**Proof** Follows from the definition of Möbius function.

In the spirit of Rota's programme, this can be applied to solve counting problems, as illustrated by the following standard example.

**Example 1.9** A **derangement** is a permutation without fixed points. We calculate  $d_n$ , the number of derangements of n letters.

Fix  $N \geq 0$ . Take the category  $\mathbb{D}_N^{\rm inj}$  of Example 1.2(c) and the functor  $S: \mathbb{D}_N^{\rm inj} \longrightarrow \mathbf{Set}$  defined as follows: S(n) is  $S_n$ , the underlying set of the nth symmetric group, and if  $f \in \mathbb{D}_N^{\rm inj}(m,n)$  and  $\tau \in S_m$ , the induced permutation  $S_f(\tau) \in S_n$  acts as  $\tau$  on im f and fixes all other points. Any permutation consists of a derangement together with some fixed points, so

$$S_n \cong \sum_m d_m \mathbb{D}_N^{\text{inj}}(m, n).$$

Then by Proposition 1.8,

$$d_n = \sum_m |S_m| \mu(m,n) = \sum_m m! (-1)^{n-m} \binom{n}{m} = n! \left(\frac{1}{0!} - \frac{1}{1!} + \dots + \frac{(-1)^n}{n!}\right).$$

To set up the theory of Euler characteristic we will not need the full strength of Möbius invertibility; the following suffices.

**Definition 1.10** Let  $\mathbb{A}$  be a finite category. A **weighting** on  $\mathbb{A}$  is a function  $k^{\bullet}: \operatorname{ob} \mathbb{A} \longrightarrow \mathbb{Q}$  such that for all  $a \in \mathbb{A}$ ,

$$\sum_{b} \zeta(a,b)k^b = 1.$$

A **coweighting**  $k_{\bullet}$  on  $\mathbb{A}$  is a weighting on  $\mathbb{A}^{op}$ .

Note that  $\mathbb{A}$  has Möbius inversion if and only if it has a unique weighting, if and only if it has a unique coweighting; they are given by

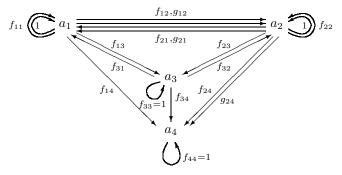
$$k^{a} = \sum_{b} \mu(a, b), \qquad k_{b} = \sum_{a} \mu(a, b).$$

**Examples 1.11** a. Let  $\mathbb{L}$  be the category



Then the unique weighting  $k^{\bullet}$  on  $\mathbb{L}$  is  $(k^a, k^{b_1}, k^{b_2}) = (-1, 1, 1)$ .

- b. Let M be a finite monoid, regarded as a category with unique object  $\star$ . Again there is a unique weighting  $k^{\bullet}$ , with  $k^{\star} = 1/|M|$ .
- c. If  $\mathbb{A}$  has a terminal object 1 then  $\delta(-,1)$  is a weighting on  $\mathbb{A}$ .
- d. A finite category may admit no weighting at all (even if Cauchy-complete). An example is the category  $\mathbb A$  with objects and arrows



where if  $a_i \xrightarrow{p} a_j \xrightarrow{q} a_k$  and neither p nor q is an identity then  $q \circ p = f_{ik}$ .

e. A category may certainly have more than one weighting: for instance, if  $\mathbb{A}$  is the category consisting of two objects and a single isomorphism between them, a weighting on  $\mathbb{A}$  is any pair of rational numbers whose sum is 1. But even a skeletal category may admit more than one weighting. Indeed, the full subcategories  $\mathbb{B} = \{a_1, a_2\}$  and  $\mathbb{C} = \{a_1, a_2, a_3\}$  of the category  $\mathbb{A}$  of the previous example both have infinitely many weightings.

In contrast to Möbius invertibility, the property of admitting at least one weighting is invariant under equivalence:

**Lemma 1.12** Let  $\mathbb{A}$  and  $\mathbb{B}$  be equivalent finite categories. Then  $\mathbb{A}$  admits a weighting if and only if  $\mathbb{B}$  does.

**Proof** Let  $F: \mathbb{A} \longrightarrow \mathbb{B}$  be an equivalence. Given  $a \in \mathbb{A}$ , write  $C_a$  for the number of objects in the isomorphism class of a, and similarly  $C_b$  when  $b \in \mathbb{B}$ . Take a weighting  $l^{\bullet}$  on  $\mathbb{B}$  and put  $k^a = (C_{Fa}/C_a)l^{Fa}$ ; then  $k^{\bullet}$  is a weighting on  $\mathbb{A}$ .

Weightings and Möbius functions are compatible with sums and products of categories; the following lemma is easily verified.

**Lemma 1.13** Let  $n \geq 0$  and let  $\mathbb{A}_1, \ldots, \mathbb{A}_n$  be finite categories.

a. If each  $\mathbb{A}_i$  has a weighting  $k_i^{\bullet}$  then  $\sum_i \mathbb{A}_i$  has a weighting  $l^{\bullet}$  given by  $l^a = k_i^a$  whenever  $a \in \mathbb{A}_i$ . If each  $\mathbb{A}_i$  has Möbius inversion then so does  $\sum_i \mathbb{A}_i$ , where for  $a \in \mathbb{A}_i$  and  $b \in \mathbb{A}_j$ ,

$$\mu(a,b) = \left\{ \begin{array}{ll} \mu_{\mathbb{A}_i}(a,b) & \textit{if } i = j \\ 0 & \textit{otherwise}. \end{array} \right.$$

b. If each  $\mathbb{A}_i$  has a weighting  $k_i^{\bullet}$  then  $\prod_i \mathbb{A}_i$  has a weighting  $l^{\bullet}$  given by  $l^{(a_1,\ldots,a_n)}=k_1^{a_1}\cdots k_n^{a_n}$ . If each  $\mathbb{A}_i$  has Möbius inversion then so does  $\prod_i \mathbb{A}_i$ , with

$$\mu((a_1,\ldots,a_n),(b_1,\ldots,b_n)) = \mu(a_1,b_1)\cdots\mu(a_n,b_n).$$

Recall that any functor X taking values in **Set** or **Cat** has a category of elements  $\mathbb{E}(X)$ , and that in the **Cat**-valued case, this applies equally when X is a weak (or 'pseudo') functor; see the Appendix. We call X **finite** if  $\mathbb{E}(X)$  is

finite. When the domain category  $\mathbb{A}$  is finite, this just means that each set or category Xa is finite.

**Lemma 1.14** Let  $\mathbb{A}$  be a finite category and  $X : \mathbb{A} \longrightarrow \mathbf{Cat}$  a finite weak functor. Suppose that we have weightings on  $\mathbb{A}$  and on each Xa, all written  $k^{\bullet}$ . Then there is a weighting on  $\mathbb{E}(X)$  defined by  $k^{(a,x)} = k^a k^x$   $(a \in \mathbb{A}, x \in Xa)$ .

**Proof** Let  $a \in \mathbb{A}$  and  $x \in Xa$ . Then

$$\sum_{(b,y)\in\mathbb{E}(X)} \zeta((a,x),(b,y))k^bk^y = \sum_b \sum_{f\in\mathbb{A}(a,b)} \left(\sum_{y\in Xb} \zeta((Xf)x,y)k^y\right)k^b$$
$$= \sum_b \zeta(a,b)k^b = 1.$$

This result will be used to show how Euler characteristic behaves with respect to fibrations.

9

### 2 Euler characteristic

In this section, the Euler characteristic of a category is defined and its basic properties are established. The definition is justified by a series of propositions showing its compatibility with the Euler characteristics of other types of object: graphs, topological spaces, and orbifolds. There follows a brief discussion of the Lefschetz number of an endofunctor.

**Lemma 2.1** Let  $\mathbb{A}$  be a finite category,  $k^{\bullet}$  a weighting on  $\mathbb{A}$ , and  $k_{\bullet}$  a coweighting on  $\mathbb{A}$ . Then  $\sum_a k^a = \sum_a k_a$ .

Proof

$$\sum_{b} k^{b} = \sum_{b} \left( \sum_{a} k_{a} \zeta(a, b) \right) k^{b} = \sum_{a} k_{a} \left( \sum_{b} \zeta(a, b) k^{b} \right) = \sum_{a} k_{a}.$$

**Definition 2.2** A finite category A has Euler characteristic if it admits both a weighting and a coweighting. Its Euler characteristic is then

$$\chi(\mathbb{A}) = \sum_{a} k^{a} = \sum_{a} k_{a}$$

for any weighting  $k^{\bullet}$  and coweighting  $k_{\bullet}$ .

Any category  $\mathbb A$  with Möbius inversion has Euler characteristic,  $\chi(\mathbb A)=\sum_{a,b}\mu(a,b),$  as in the Introduction.

**Examples 2.3** a. If  $\mathbb{A}$  is a finite discrete category then  $\chi(\mathbb{A}) = |\operatorname{ob} \mathbb{A}|$ .

- b. If M is a finite monoid then  $\chi(M) = 1/|M|$ . When M is a group, this can be understood as follows: M acts freely on the contractible space EM, which has Euler characteristic 1; one would therefore expect the quotient space BM to have Euler characteristic 1/|M|. (Compare [Wl] and [Co].)
- c. By Corollary 1.5, a finite poset A has Euler characteristic  $\sum_{n\geq 0} (-1)^n c_n$ , where  $c_n$  is the number of chains in A of length n. (See [Pu], [Fo], [R] and [Fa] for connections with poset homology, and §4 for further comparisons with the Rota theory.) More generally, the results of §1 give formulas for the Euler characteristic of any finite category that either has no non-trivial idempotents or admits an epi-mono factorization system.
- d. If  $\mathbb{A}$  has Euler characteristic and either an initial or a terminal object then  $\chi(\mathbb{A}) = 1$ , by 1.11(c) and its dual. In particular, any finite category with both an initial and a terminal object has Euler characteristic 1. This applies, for instance, to the category  $\mathbb{C}$  of 1.11(e). Hence having Möbius inversion is a strictly stronger property than having Euler characteristic.

e. Euler characteristic is not invariant under Morita equivalence. For example, the two-element monoid consisting of the identity and an idempotent is Morita equivalent to the category freely generated by a split epimorphism, but their respective Euler characteristics are 1/2 and 1.

Clearly  $\chi(\mathbb{A}^{op}) = \chi(\mathbb{A})$ , one side being defined when the other is. The next few propositions set out further basic properties of Euler characteristic.

**Proposition 2.4** Let  $\mathbb{A}$  and  $\mathbb{B}$  be finite categories.

- a. If there is an adjunction  $\mathbb{A} \longrightarrow \mathbb{B}$  and both  $\mathbb{A}$  and  $\mathbb{B}$  have Euler characteristic then  $\chi(\mathbb{A}) = \chi(\mathbb{B})$ .
- b. If  $\mathbb{A} \simeq \mathbb{B}$  then  $\mathbb{A}$  has Euler characteristic if and only if  $\mathbb{B}$  does, and in that case  $\chi(\mathbb{A}) = \chi(\mathbb{B})$ .

In (a), it may be that one category has Euler characteristic but the other does not: consider, for instance, the unique functor from the category  $\mathbb{A}$  of 1.11(d) to the terminal category.

#### Proof

- a. Suppose that  $\mathbb{A} \xrightarrow{F} \mathbb{B}$  with  $F \dashv G$ . Then  $\zeta(Fa, b) = \zeta(a, Gb)$  for all  $a \in \mathbb{A}, b \in \mathbb{B}$ ; write  $\zeta(a, b)$  for their common value. Take a coweighting  $k_{\bullet}$  on  $\mathbb{A}$  and a weighting  $k^{\bullet}$  on  $\mathbb{B}$ . Then  $\sum_{a} k_{a} = \sum_{b} k^{b}$  by the same proof as that of Lemma 2.1.
- b. The first statement follows from Lemma 1.12 and its dual, and the second from (a).  $\Box$

**Example 2.5** If  $\mathbb{B}$  is a category with an initial or a terminal object then  $\chi(\mathbb{A}^{\mathbb{B}}) = \chi(\mathbb{A})$  for all  $\mathbb{A}$ , provided that both Euler characteristics exist. Indeed, if 0 is initial in  $\mathbb{B}$  then evaluation at 0 is right adjoint to the diagonal functor  $\mathbb{A} \longrightarrow \mathbb{A}^{\mathbb{B}}$ .

**Proposition 2.6** Let  $n \geq 0$  and let  $\mathbb{A}_1, \ldots, \mathbb{A}_n$  be finite categories that all have Euler characteristic. Then  $\sum_i \mathbb{A}_i$  and  $\prod_i \mathbb{A}_i$  have Euler characteristic, with

$$\chi\left(\sum_{i} \mathbb{A}_{i}\right) = \sum_{i} \chi(\mathbb{A}_{i}), \qquad \chi\left(\prod_{i} \mathbb{A}_{i}\right) = \prod_{i} \chi(\mathbb{A}_{i}).$$

**Proof** Follows from Lemma 1.13.

**Example 2.7** Let  $\mathbb{A}$  be a finite groupoid. Choose one object  $a_i$  from each connected-component of  $\mathbb{A}$ , and write  $G_i$  for the automorphism group of  $a_i$ . Then  $\mathbb{A} \simeq \sum_i G_i$ , so by 2.3(b), 2.4(b) and 2.6, we have  $\chi(\mathbb{A}) = \sum_i 1/|G_i|$ . This is what Baez and Dolan call the cardinality of the groupoid  $\mathbb{A}$  [BD].

One might also ask whether  $\chi(\mathbb{A}^{\mathbb{B}}) = \chi(\mathbb{A})^{\chi(\mathbb{B})}$ . By 2.3(d), 2.5 and 2.6, the answer is yes if every connected-component of  $\mathbb{B}$  has an initial or a terminal object (and all the Euler characteristics exist). But in general the answer is no: for instance, take  $\mathbb{A}$  to be the 2-object discrete category and  $\mathbb{B}$  to be the category of 3.4(b). See also Propp [Pr2], Speed [Sp], and §5, 6 of Rota [R].

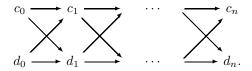
An important property of topological Euler characteristic is its behaviour with respect to fibrations. Let  $E \longrightarrow A$  be a topological fibration. If A has connected-components  $A_1, \ldots, A_n$  and  $X_i$  is the fibre in the ith component then, under suitable hypotheses,  $\chi(E) = \sum_i \chi(A_i) \chi(X_i)$ . Lemma 1.14 implies the following categorical analogue.

**Proposition 2.8** Let  $\mathbb{A}$  be a finite category and  $X : \mathbb{A} \longrightarrow \mathbf{Cat}$  a finite weak functor. Let  $k^{\bullet}$  be a weighting on  $\mathbb{A}$  and suppose that  $\mathbb{E}(X)$  and each Xa have Euler characteristic. Then

$$\chi(\mathbb{E}(X)) = \sum_{a} k^{a} \chi(Xa).$$

**Examples 2.9** a. When X is a finite **Set**-valued functor,  $\chi(\mathbb{E}(X)) = \sum_a k^a |Xa|$ . For example, let M be a finite monoid. A finite functor  $X: M \longrightarrow \mathbf{Set}$  is a finite set S with a left M-action. Following [BD], we write  $\mathbb{E}(X)$  as  $S/\!/M$ , the **weak quotient** of S by M. (Its objects are the elements of S, and the arrows  $s \longrightarrow s'$  are the elements  $m \in M$  satisfying ms = s'.) Then  $\chi(S/\!/M) = |S|/|M|$ .

b. Define a sequence  $(\mathbb{S}^n)_{n\geq -1}$  of categories inductively as follows.  $\mathbb{S}^{-1}$  is empty. Let  $\mathbb{L}$  be the category of 1.11(a); define  $X: \mathbb{L} \longrightarrow \mathbf{Cat}$  by  $Xa = \mathbb{S}^{n-1}$  and  $Xb_1 = Xb_2 = \mathbf{1}$  (the terminal category); put  $\mathbb{S}^n = \mathbb{E}(X)$ . Explicitly,  $\mathbb{S}^n$  is the poset



(If we take the usual expression of the topological n-sphere  $S^n$  as a CW-complex with two cells in each dimension  $\leq n$  then  $\mathbb{S}^n$  is the set of cells ordered by inclusion;  $S^n$  is the classifying space of  $\mathbb{S}^n$ .)

Each  $\mathbb{S}^n$  is a poset, so has Euler characteristic. By Proposition 2.8,

$$\chi(\mathbb{S}^n) = -\chi(\mathbb{S}^{n-1}) + 2\chi(\mathbf{1}) = 2 - \chi(\mathbb{S}^{n-1})$$

for all  $n \geq 0$ ; also  $\chi(\mathbb{S}^{-1}) = 0$ . Hence  $\chi(\mathbb{S}^n) = 1 + (-1)^n$ .

The next three propositions show how the Euler characteristics of various types of structure are compatible with that of categories.

First, Euler characteristic of categories extends Euler characteristic of graphs. More precisely, let  $G = (G_1 \Longrightarrow G_0)$  be a directed graph, where  $G_1$  is the set of edges and  $G_0$  the set of vertices. We will show that if F(G) is the free category on G then  $\chi(F(G)) = |G_0| - |G_1|$ . This only makes sense if F(G) is finite, which is the case if and only if G is finite and circuit-free; then F(G) is also circuit-free. (A directed graph is **circuit-free** if it contains no circuits of non-zero length, and a category is **circuit-free** if every circuit consists entirely of identities.)

**Proposition 2.10** Let G be a finite circuit-free directed graph. Then  $\chi(F(G))$  is defined and equal to  $|G_0| - |G_1|$ .

**Proof** Given  $a, b \in G_0$ , write  $\zeta_G(a, b)$  for the number of edges from a to b in G. Then  $\zeta_{F(G)} = \sum_{n \geq 0} \zeta_G^n$  in R(F(G)), the sum being finite since G is circuit-free. Hence  $\mu_{F(G)} = \delta - \zeta_G$ , and the result follows.

This suggests that in the present context, it is more fruitful to view a graph as a special category (via F) than a category as a graph with structure. Compare the comments after Definition 1.1.

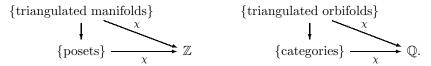
The second result compares the Euler characteristics of categories and topological spaces. We show that under suitable hypotheses,  $\chi(B\mathbb{A}) = \chi(\mathbb{A})$ , where  $B\mathbb{A}$  is the classifying space of a category  $\mathbb{A}$  (that is, the geometric realization of its nerve  $N\mathbb{A}$ ). To ensure that  $B\mathbb{A}$  has Euler characteristic, we assume that  $N\mathbb{A}$  contains only finitely many nondegenerate simplices; then

$$\chi(B\mathbb{A}) = \sum_{n>0} (-1)^n |\{\text{nondegenerate } n\text{-simplices in } N\mathbb{A}\}|.$$

An n-simplex in  $N\mathbb{A}$  is just an n-path in  $\mathbb{A}$ , and is nondegenerate in the sense of simplicial sets if and only if it is nondegenerate as a path, so  $\mathbb{A}$  must contain only finitely many nondegenerate paths. This is the case if and only if  $\mathbb{A}$  is circuit-free, if and only if  $\mathbb{A}$  is skeletal and contains no endomorphisms except identities. So by Corollary 1.5, we have:

**Proposition 2.11** Let  $\mathbb{A}$  be a finite skeletal category containing no endomorphisms except identities. Then  $\chi(B\mathbb{A})$  is defined and equal to  $\chi(\mathbb{A})$ .

For the final compatibility result, consider the following schematic diagrams:



On the left, we start with a compact manifold M equipped with a finite triangulation. As shown in §3.8 of [St], the topological Euler characteristic of M is equal to the Euler characteristic of the poset of simplices in the triangulation, ordered by inclusion. We generalize this result from manifolds to orbifolds, which entails replacing posets by categories and  $\mathbb{Z}$  by  $\mathbb{Q}$ .

Let M be a compact orbifold equipped with a finite triangulation. (See [MP] for definitions.) The simplices in the triangulation form a poset P, and if  $p \in P$  is a d-dimensional simplex then  $\downarrow p = \{q \in P \mid q \leq p\}$  is isomorphic to the poset  $\mathbb{P}_{d+1}$  of nonempty subsets of  $\{1, \ldots, d+1\}$ , with  $p \in \downarrow p$  corresponding to  $\{1, \ldots, d+1\} \in \mathbb{P}_{d+1}$ . Every  $p \in P$  has a stabilizer group G(p), and

$$\chi(M) = \sum_{p \in P} (-1)^{\dim p} / |G(p)|.$$

On the other hand, the groups G(p) fit together to form a **complex of finite** groups on  $P^{\text{op}}$ , that is, a weak functor  $G: P^{\text{op}} \longrightarrow \mathbf{Cat}$  taking values in finite groups (regarded as one-object categories) and injective homomorphisms; see §3 of [M]. This gives a finite category  $\mathbb{E}(G)$ . For example, when M is a manifold, each group G(p) is trivial and  $\mathbb{E}(G) \cong P$ .

The following result is joint with Ieke Moerdijk.

**Proposition 2.12** Let M be a compact orbifold equipped with a finite triangulation. Let G be the resulting complex of groups. Then  $\chi(\mathbb{E}(G))$  is defined and equal to  $\chi(M)$ .

**Proof** Every arrow in  $\mathbb{E}(G)$  is monic, so by Theorem 1.4,  $\mathbb{E}(G)$  has Euler characteristic. Moreover, P is a finite poset, so has a unique coweighting  $k_{\bullet}$ , and  $\chi(\mathbb{E}(G)) = \sum_{p} k_{p}/|G(p)|$  by the dual of Proposition 2.8.

The coweight of p in P is equal to the coweight of p in  $\downarrow p \cong \mathbb{P}_{d+1}$ , where  $d = \dim p$ . The unique coweighting  $k_{\bullet}$  on  $\mathbb{P}_{d+1}$  is given by  $k_J = (-1)^{|J|-1}$ , so  $k_p = (-1)^{(d+1)-1} = (-1)^{\dim p}$ . The result follows.

There is an accompanying theory of Lefschetz number. Let  $\mathbb{A}$  be a finite category and  $F: \mathbb{A} \longrightarrow \mathbb{A}$  an endofunctor of  $\mathbb{A}$ . The category  $\mathbf{Fix} F$  has as objects the (strict) fixed points of F, that is, the objects  $a \in \mathbb{A}$  such that Fa = a; a map  $a \longrightarrow b$  in  $\mathbf{Fix} F$  is a map  $f: a \longrightarrow b$  in  $\mathbb{A}$  such that Ff = f.

**Definition 2.13** Let F be an endofunctor of a finite category. Its **Lefschetz** number L(F) is  $\chi(\mathbf{Fix}\,F)$ , when this exists.

The Lefschetz number is, then, the sum of the (co)weights of the fixed points. This is analogous to the standard Lefschetz fixed point formula, (co)weight playing the role of index. The following results further justify the definition.

**Proposition 2.14** Let  $\mathbb{A}$  be a finite category.

- a.  $L(1_{\mathbb{A}}) = \chi(\mathbb{A})$ , one side being defined if and only if the other is.
- b. If  $\mathbb{B}$  is another finite category and  $\mathbb{A} \xrightarrow{F} \mathbb{B}$  are functors then L(GF) = L(FG), one side being defined if and only if the other is.
- c. Let  $F: \mathbb{A} \longrightarrow \mathbb{A}$  and write  $BF: B\mathbb{A} \longrightarrow B\mathbb{A}$  for the induced map on the classifying space of  $\mathbb{A}$ . If  $\mathbb{A}$  is skeletal and contains no endomorphisms except identities then L(F) = L(BF), with both sides defined.

In the special case that A is a poset, part (c) is Theorem 1.1 of [BB].

**Proof** For (a) and (b), just note that  $\mathbf{Fix} 1_{\mathbb{A}} \cong \mathbb{A}$  and  $\mathbf{Fix} GF \cong \mathbf{Fix} FG$ . For (c), recall from the proof of Proposition 2.11 that  $N\mathbb{A}$  has only finitely many nondegenerate simplices; then

$$L(BF) = \sum_{n\geq 0} (-1)^n |\{\text{nondegenerate } n\text{-simplices in } N\mathbb{A} \text{ fixed by } NF\}|$$

$$= \sum_{n\geq 0} (-1)^n |\{\text{nondegenerate } n\text{-paths in } \mathbf{Fix} F\}|$$

$$= L(F),$$

using Corollary 1.5 in the last step.

An **algebra** for an endofunctor F of  $\mathbb{A}$  is an object  $a \in \mathbb{A}$  equipped with a map  $h : Fa \longrightarrow a$ . With the evident structure-preserving morphisms, algebras for F form a category  $\mathbf{Alg} \ F$ . There is a dual notion of  $\mathbf{coalgebra}$  (where now  $h : a \longrightarrow Fa$ ), giving a category  $\mathbf{Coalg} \ F$ .

**Proposition 2.15** Let  $\mathbb{A}$  be a finite skeletal category containing no endomorphisms except identities. Then  $\chi(\mathbf{Alg}\,F) = L(F) = \chi(\mathbf{Coalg}\,F)$ , with all three terms defined.

**Proof** First observe that  $\mathbb{A}$  is circuit-free. Now, the inclusion  $\mathbf{Fix} F \longrightarrow \mathbf{Alg} F$  has a right adjoint R: given an algebra (a,h), circuit-freeness implies that  $F^n a$  is a fixed point for all sufficiently large n, and  $R(a,h) = F^n a$ . The Euler characteristics of  $\mathbf{Alg} F$  and  $\mathbf{Fix} F$  exist, by Corollary 1.5, and are equal, by Proposition 2.4(a). The statement on coalgebras follows by duality.  $\square$ 

For example, if f is an endomorphism of a finite poset A then the sub-posets

$$\{a \in A \mid f(a) \le a\}, \qquad \{a \in A \mid f(a) = a\}, \qquad \{a \in A \mid f(a) \ge a\}$$

all have the same Euler characteristic.

The theory of Euler characteristic presented here can be extended in at least two directions.

First, we can relax the finiteness assumption. For instance, the category of finite sets and bijections should have Euler characteristic  $\sum_{n=0}^{\infty} 1/|S_n| = e$ , as observed in [BD]. See the remarks after Corollary 4.3.

Second, note that the Euler characteristic of categories is defined in terms of the cardinality of finite sets; it is then clear that the theory can be developed for  $\mathcal{V}$ -enriched categories when there is a suitable notion of cardinality or Euler characteristic of objects of  $\mathcal{V}$ . For example,  $\mathcal{V}$  might be the category of finite-dimensional vector spaces, with dimension playing the role of cardinality, and this leads to an Euler characteristic for finite linear categories. For another, recall that a **0-category** is a set and an **n-category** is a category enriched in (n-1)-categories; iterating, we obtain an Euler characteristic for finite n-categories. In particular, if  $\mathbf{S}^n$  is the n-category consisting of two parallel n-cells then  $\chi(\mathbf{S}^n) = 1 + (-1)^n$ .

### 3 The cardinality of a colimit

The main theorem of this section generalizes the formulas

$$|X \cup Y| = |X| + |Y| - |X \cap Y|,$$
  $|S/G| = |S|/|G|$ 

where X and Y are subsets of some larger set and S is a set acted on freely by a group G.

Take a finite functor  $X : \mathbb{A} \longrightarrow \mathbf{Set}$ . The colimit  $\varinjlim X$  can be viewed as the gluing-together of the sets Xa. Its cardinality depends on the way in which these sets are glued together, which in turn is determined by the action of X on morphisms, so in general there is no formula for  $|\varinjlim X|$  purely in terms of the cardinalities |Xa|  $(a \in \mathbb{A})$ .

Suppose, however, that we are in the extreme case that there are no 'unforced' equations of the type (Xf)(x) = (Xf')(x'). For pushouts, this means that the two functions along which we are pushing out are injective; when  $\mathbb{A}$  is a group G, so that X is a G-action, it means that the action is free. In this extreme case,  $|\lim_{\longrightarrow} X|$  can be calculated as a weighted sum of the cardinalities |Xa|.

We now make this precise. Recall that a **Set**-valued functor is said to be familially representable if it is a sum of representables.

**Proposition 3.1** Let  $\mathbb{A}$  be a finite category and  $k^{\bullet}$  a weighting on  $\mathbb{A}$ . If  $X : \mathbb{A} \longrightarrow \mathbf{Set}$  is finite and familially representable then  $|\lim X| = \sum_a k^a |Xa|$ .

**Proof** The result holds if X is representable, since then  $|\lim_{\longrightarrow} X| = 1$ . On the other hand, the class of functors X for which the conclusion holds is clearly closed under finite sums.

To make use of this, we need a way of recognizing familially representable functors. Carboni and Johnstone [CJ1, CJ2] show that when  $\mathcal{A}$  satisfies certain hypotheses, including having all limits, a functor  $\mathcal{A} \longrightarrow \mathbf{Set}$  is familially representable if and only if it preserves connected limits. This is of little help, because our categories  $\mathbb{A}$  are finite, and a finite category does not have even all finite limits unless it is a lattice.

However, a standard philosophy applies: when  $\mathbb{A}$  fails to have all limits of a certain type, it is rarely useful to consider the functors  $\mathbb{A} \longrightarrow \mathbf{Set}$  preserving limits of that type; the correct substitute is the class of functors that are suitably 'flat'. The notion of flatness appropriate here will be called nondegeneracy. (This is unrelated to the usage of 'nondegenerate' in §1.)

**Definition 3.2** Let  $\mathbb{A}$  be a small category. A functor  $X : \mathbb{A} \longrightarrow \mathbf{Set}$  is **nondegenerate** if  $\mathbb{E}(X)$  has the following diagram-completion properties:



Explicitly, this means that (i) given arrows  $a \xrightarrow{f} b \xrightarrow{f'} a'$  in  $\mathbb{A}$  and  $x \in Xa$ ,  $x' \in Xa'$  satisfying (Xf)(x) = (Xf')(x'), there exist arrows  $a \xrightarrow{g} c \xrightarrow{g'} a'$  and  $z \in Xc$  satisfying fg = f'g', (Xg)(z) = x, and (Xg')(z) = x', and (ii) given arrows  $a \xrightarrow{f} b$  in  $\mathbb{A}$  and  $x \in Xa$  satisfying (Xf)(x) = (Xf')(x), there exist  $c \xrightarrow{g} a$  and  $z \in Xc$  satisfying fg = f'g and (Xg)(z) = x.

This is the most concrete form of the definition. For further explanation and a proof that nondegeneracy is equivalent to familial representability, see the Appendix; for references, see [Ln].

From Lemma 5.2 we deduce a more applicable form of Proposition 3.1:

**Theorem 3.3** Let  $\mathbb{A}$  be a finite Cauchy-complete category and  $k^{\bullet}$  a weighting on  $\mathbb{A}$ . If  $X : \mathbb{A} \longrightarrow \mathbf{Set}$  is finite and nondegenerate then  $|\lim_{\longrightarrow} X| = \sum_a k^a |Xa|$ .

Recalling that  $\varinjlim X$  is the set of connected-components of  $\mathbb{E}(X)$ , this may be rephrased as  $\pi_0(\mathbb{E}(X)) = \sum k^a |Xa|$ . On the other hand, Proposition 2.8 implies that  $\chi(\mathbb{E}(X)) = \sum k^a |Xa|$ . Indeed, under the hypotheses of the Theorem, X is familially representable, so each connected-component of  $\mathbb{E}(X)$  has an initial object, so  $\chi(\mathbb{E}(X)) = \pi_0(\mathbb{E}(X))$ .

**Examples 3.4** a. Let  $\mathbb{L}$  be the category of 1.11(a). A functor  $X : \mathbb{L} \longrightarrow \mathbf{Set}$  is nondegenerate if and only if both functions  $Xa \longrightarrow Xb_i$  are injective. In that case, Theorem 3.3 says that

$$|Xb_1 + Xa Xb_2| = |Xb_1| + |Xb_2| - |Xa|$$

where the set on the left-hand side is a pushout.

b. Let  $\mathbb{B}$  be the category  $\left(a \xrightarrow{f} b\right)$ . A functor  $X : \mathbb{B} \longrightarrow \mathbf{Set}$  is nondegenerate if and only if the two functions Xf, Xg are injective and have disjoint images. The unique weighting  $k^{\bullet}$  on  $\mathbb{B}$  is  $(k^a, k^b) = (-1, 1)$ , and

$$|(Xb)/\sim| = |Xb| - |Xa|$$

where  $\sim$  is the equivalence relation generated by  $(Xf)(x) \sim (Xg)(x)$  for all  $x \in Xa$ .

- c. Let G be a group. A functor  $X: G \longrightarrow \mathbf{Set}$  is a set S equipped with a left G-action; the functor is nondegenerate if and only if the action is free. Theorem 3.3 then says that the number of orbits is |S|/|G|.
- d. The Theorem can be viewed as a generalized inclusion-exclusion principle. (Compare [R].) Let  $n \geq 0$  and let  $\mathbb{P}_n$  be the poset of nonempty subsets of  $\{1,\ldots,n\}$ , ordered by inclusion. (So  $\mathbb{P}_2^{\text{op}}$  is the category  $\mathbb{L}$  of (a).)

Its unique coweighting  $k_{\bullet}$  is defined by  $k_J = (-1)^{|J|-1}$ . Given subsets  $S_1, \ldots, S_n$  of some set, there is a nondegenerate functor  $X : \mathbb{P}_n^{\text{op}} \longrightarrow$  **Set** defined on objects by  $X(J) = \bigcap_{j \in J} S_j$  and on maps by inclusion. Theorem 3.3 gives the inclusion-exclusion formula,

$$|S_1 \cup \dots \cup S_n| = \sum_{\emptyset \neq J \subseteq \{1,\dots,n\}} (-1)^{|J|-1} \left| \bigcap_{j \in J} S_j \right|.$$

**Corollary 3.5** Let  $\mathbb{A}$  be a finite Cauchy-complete category admitting a weighting. Let  $X, Y : \mathbb{A} \longrightarrow \mathbf{Set}$  be finite nondegenerate functors satisfying |Xa| = |Ya| for all  $a \in \mathbb{A}$ . Then  $|\lim X| = |\lim Y|$ .

The condition that  $\mathbb{A}$  admits a weighting cannot be dropped: consider the category  $\mathbb{A}$  of Example 1.11(d) and the functors  $X = \mathbb{A}(a_1, -) + \mathbb{A}(a_4, -)$ ,  $Y = \mathbb{A}(a_2, -)$ .

If  $\mathbb{A}$  not only has a weighting but admits Möbius inversion then a stronger statement can be made: Proposition 1.8.

### 4 Relations with Rota's theory

In 1964, Gian-Carlo Rota published his seminal paper [R] on Möbius inversion in posets. The name is motivated as follows: in the poset of positive integers ordered by divisibility,  $\mu(a,b) = \mu(b/a)$  whenever a divides b, where the  $\mu$  on the right-hand side is the classical Möbius function. He was not the first to define Möbius inversion in posets—Weisner, Hall, and Ward preceded him—but Rota's contribution was the decisive one; in particular, he realized the power of the method in enumerative combinatorics. The history of Möbius inversion is well described in [R], [G] and [St].

In this section we see that some of the principal results in Rota's theory are the order-theoretic shadows of more general categorical facts. We also consider a different generalization of Möbius–Rota inversion.

Given a poset A, Rota considered its **incidence algebra** I(A), which is the subring of R(A) consisting of the integer-valued  $\theta \in R(A)$  such that  $\theta(a,b) = 0$  whenever  $a \not\leq b$ . By Example 1.2(a) or Corollary 1.5,  $\mu \in I(A)$ .

In posets, then,  $\zeta(a,b)=0 \Rightarrow \mu(a,b)=0$ . More generally:

**Theorem 4.1** If  $\mathbb{A}$  is a finite category with Möbius inversion then, for  $a, b \in \mathbb{A}$ ,

$$\zeta(a,b) = 0 \Rightarrow \mu(a,b) = 0.$$

The proof uses a combinatorial lemma.

**Lemma 4.2** Let  $n \geq 2$  and  $\sigma \in S_{n-1}$ . Then there exist  $k \geq 1$  and  $p_0, \ldots, p_k$  such that

$$p_0 = 1, \quad p_1, \dots, p_{k-1} \in \{1, \dots, n-1\}, \quad p_k = n,$$

and  $p_r = \sigma(p_{r-1}) + 1$  for each  $r \in \{1, ..., k\}$ .

**Proof** Suppose not; then there is an infinite sequence  $(p_r)_{r\geq 0}$  of elements of  $\{1,\ldots,n-1\}$  satisfying  $p_0=1$  and  $p_r=\sigma(p_{r-1})+1$  for all  $r\geq 1$ . Let  $\varepsilon$  be the endomorphism of the finite set  $\{p_r\mid r\geq 0\}$  defined by  $\varepsilon(p)=\sigma(p)+1$ . Then  $\varepsilon$  is injective but not surjective (since  $1\not\in \operatorname{im}\varepsilon$ ), contradicting finiteness.

**Proof of Theorem 4.1** Write the objects of  $\mathbb{A}$  as  $a_1, \ldots, a_n$ . There is an  $n \times n$  matrix Z defined by  $Z_{ij} = \zeta(a_i, a_j)$ , and Z is invertible over  $\mathbb{Q}$  with  $(Z^{-1})_{ij} = \mu(a_i, a_j)$ . Suppose that  $i, j \in \{1, \ldots, n\}$  and  $Z_{ij} = 0$ . Certainly  $i \neq j$ , so  $n \geq 2$  and we may assume that (i, j) = (1, n). By Cramer's formula for the inverse of a matrix, our task is to prove that the (n, 1)-minor Z' of Z has determinant zero.

The (p,q)-entry of Z' is  $Z_{p,q+1}$ , so

$$\det Z' = \sum_{\sigma \in S_{n-1}} \pm Z_{1,\sigma(1)+1} \cdots Z_{n-1,\sigma(n-1)+1}.$$

It suffices to prove that each summand is 0. Indeed, let  $\sigma \in S_{n-1}$ . Take  $p_0, \ldots, p_k$  as in the Lemma. By hypothesis, there is no map  $a_1 \longrightarrow a_n$  in  $\mathbb{A}$ . Categories have composition, so there is no diagram

$$a_1 = a_{p_0} \longrightarrow a_{p_1} \longrightarrow \cdots \longrightarrow a_{p_k} = a_n$$

in A. Hence  $\zeta(a_{p_{r-1}},a_{p_r})=0$  for some  $r\in\{1,\ldots,k\}$ , giving  $Z_{p_{r-1},\sigma(p_{r-1})+1}=0$ , as required.  $\square$ 

Given objects a, c of a category  $\mathbb{A}$ , let  $\mathbb{A}_{a,c}$  be the full subcategory consisting of those  $b \in \mathbb{A}$  for which there exist arrows  $a \longrightarrow b \longrightarrow c$ . Theorem 4.1 easily implies:

**Corollary 4.3** Let  $\mathbb{A}$  be a finite category. Then  $\mathbb{A}$  has Möbius inversion if and only if  $\mathbb{A}_{a,c}$  has Möbius inversion for all  $a, c \in \mathbb{A}$ , and in that case the Möbius function of  $\mathbb{A}_{a,c}$  is the restriction of that of  $\mathbb{A}$ .

These results suggest a way of relaxing the finiteness assumption on our categories. It extends to categories the local finiteness condition on posets used in the Rota theory. Let  $\mathbb A$  be a category for which each subcategory  $\mathbb A_{a,c}$  is finite. Then each hom-set  $\mathbb A(a,b)$  has finite cardinality,  $\zeta(a,b)$ , and there is a  $\mathbb Q$ -algebra

$$\hat{R}(\mathbb{A}) = \{\theta : \text{ob } \mathbb{A} \times \text{ob } \mathbb{A} \longrightarrow \mathbb{Q} \mid \text{for } a, b \in \mathbb{A}, \ \zeta(a, b) = 0 \Rightarrow \theta(a, b) = 0\}$$

with operations defined as for  $R(\mathbb{A})$ . Evidently  $\zeta \in \hat{R}(\mathbb{A})$ , and  $\mathbb{A}$  may be said to have Möbius inversion if  $\zeta$  has an inverse  $\mu$  in  $\hat{R}(\mathbb{A})$ . By Theorem 4.1, this extends the definition for finite categories. For example, the skeletal category  $\mathbb{D}^{\text{inj}}$  of finite totally ordered sets and order-preserving injections has Möbius inversion; compare Example 1.2(c).

The main theorem in Rota's paper [R] relates the Möbius functions of two posets linked by a Galois connection. Viewing a poset as a special category, a Galois connection is nothing but a (contravariant) adjunction, which suggests the following generalization of Rota's theorem.

**Proposition 4.4** Let  $\mathbb{A}$  and  $\mathbb{B}$  be finite categories with Möbius inversion. Let  $\mathbb{A} \xrightarrow{F} \mathbb{B}$  be an adjunction,  $F \dashv G$ . Then for all  $a \in \mathbb{A}$ ,  $b \in \mathbb{B}$ ,

$$\sum_{a':Fa'=b}\mu(a,a')=\sum_{b':Gb'=a}\mu(b',b).$$

**Proof** Write  $\zeta(a,b) = \zeta(Fa,b) = \zeta(a,Gb)$ . Then for all  $a \in \mathbb{A}, b \in \mathbb{B}$ ,

$$\sum_{a':Fa'=b}\mu(a,a')=\sum_{a'\in\mathbb{A}}\mu(a,a')\delta(Fa',b)=\sum_{a'\in\mathbb{A},\,b'\in\mathbb{B}}\mu(a,a')\zeta(a',b')\mu(b',b).$$

The result follows by symmetry.

For example, when l is an element of a finite lattice L, the inclusion of the sub-poset  $\{x \in L \mid x \leq l\}$  into L has right adjoint  $(- \land l)$ , giving Weisner's Theorem (p.351 of [R]).

The Euler characteristic of posets has been studied extensively; see [St] for references. Given a finite poset A, the classifying space BA always has Euler characteristic, which by Proposition 2.11 is equal to the Euler characteristic of the category A. On the other hand, we may form a new poset  $\widetilde{A}$  by adjoining to A a least element 0 and a greatest element 1, and then  $\chi(A) = \mu_{\widetilde{A}}(0,1) + 1$ ; see [R] or §3.8 of [St]. This result can be extended from posets to categories:

**Proposition 4.5** Let  $\mathbb{A}$  be a finite category. Write  $\widetilde{\mathbb{A}}$  for the category obtained from  $\mathbb{A}$  by freely adjoining an initial object 0 and a terminal object 1. If  $\mathbb{A}$  has Möbius inversion then  $\widetilde{\mathbb{A}}$  does too, and  $\mu_{\widetilde{\mathbb{A}}}(0,1) = \chi(\mathbb{A}) - 1$ .

**Proof** Suppose that  $\mathbb{A}$  has Möbius inversion. Let  $\mathbb{A}_0$  be the category obtained from  $\mathbb{A}$  by freely adjoining an initial object 0. Extend  $\mu \in R(\mathbb{A})$  to a function  $\mu \in R(\mathbb{A}_0)$  by defining

$$\mu(0,b) = -\sum_{a \in \mathbb{A}} \mu(a,b), \qquad \mu(a,0) = 0, \qquad \mu(0,0) = 1$$

 $(b, a \in \mathbb{A})$ . It is easily checked that this is the Möbius function of  $\mathbb{A}_0$ .

Dually, if  $\mathbb{B}$  is a finite category with Möbius inversion then the category  $\mathbb{B}_1$  obtained from  $\mathbb{B}$  by freely adjoining a terminal object 1 also has Möbius inversion, with  $\mu(c,1) = -\sum_{b \in \mathbb{B}} \mu(c,b)$  for all  $c \in \mathbb{B}$ . Take  $\mathbb{B} = \mathbb{A}_0$ : then  $\mathbb{A}_{01} = \widetilde{\mathbb{A}}$  has Möbius inversion, and

$$\mu(0,1) = -\sum_{b \in \mathbb{A}_0} \mu(0,b) = -\sum_{b \in \mathbb{A}} \mu(0,b) - \mu(0,0) = \sum_{a,b \in \mathbb{A}} \mu(a,b) - 1 = \chi(\mathbb{A}) - 1.$$

**Remark** Recall [CKW] that given categories  $\mathbb{B}$ ,  $\mathbb{A}$  and a functor  $M : \mathbb{B}^{op} \times \mathbb{A} \longrightarrow \mathbf{Set}$ , the **collage** of M is the category  $\mathbb{C}$  formed by taking the disjoint union of  $\mathbb{B}$  and  $\mathbb{A}$  and adjoining one arrow  $b \longrightarrow a$  for each  $b \in \mathbb{B}$ ,  $a \in \mathbb{A}$  and  $m \in M(b, a)$ , with composition defined using M. Assuming finiteness, if  $\mathbb{B}$  and  $\mathbb{A}$  have Möbius inversion then so does  $\mathbb{C}$ :

$$\mu_{\mathbb{C}}(b,b') = \mu_{\mathbb{B}}(b,b'), \qquad \mu_{\mathbb{C}}(a,a') = \mu_{\mathbb{A}}(a,a'), \qquad \mu_{\mathbb{C}}(a,b) = \emptyset,$$

$$\mu_{\mathbb{C}}(b,a) = -\sum_{b',a'} \mu_{\mathbb{B}}(b,b') |M(b',a')| \, \mu_{\mathbb{A}}(a',a)$$

 $(b, b' \in \mathbb{B}, a, a' \in \mathbb{A})$ . In the proof above, the calculation of the Möbius function of  $\mathbb{A}_0$  is the special case where  $\mathbb{B}$  is the terminal category and M has constant value 1. The ordinal sum of posets is another special case. Moreover, one easily deduces a formula for the Euler characteristic of a collage, which in the special case of posets is essentially Theorem 3.1 of Walker [Wk].

Let us now look at the different generalization of Rota's Möbius inversion proposed, independently, by Content, Lemay and Leroux [CLL] and by Haigh [H]. (See also [Lr] and §4 of [La]. Haigh briefly considered the same generalization as here, too; see 3.5 of [H].) Given a sufficiently finite category  $\mathbb{A}$ , they take the algebra  $I(\mathbb{A})$  of functions from {arrows of  $\mathbb{A}$ } to  $\mathbb{Q}$  (or more generally, to some base commutative ring), with a convolution product:

$$(\theta\phi)(f) = \sum_{hg=f} \theta(g)\phi(h).$$

Taking  $\zeta \in I(\mathbb{A})$  to have constant value 1, they call the **Möbius function** of  $\mathbb{A}$  the inverse  $\mu = \zeta^{-1}$  in  $I(\mathbb{A})$ , if it exists. When  $\mathbb{A}$  is a poset, this agrees with Rota; when  $\mathbb{A}$  is a monoid, it agrees with Cartier and Foata [CF].

They seek to solve a harder problem than we do: if a finite category  $\mathbb{A}$  has Möbius inversion in their sense then it does in ours (with  $\mu(a,b) = \sum_{f \in \mathbb{A}(a,b)} \mu(f)$ ), but not conversely. For instance, a non-trivial finite group never has Möbius inversion in their sense, but always does in ours.

# 5 Appendix: category theory

Here follows a skeletal account of some standard notions: category of elements, flat functors, and Cauchy-completeness. Details can be found in texts such as [Bo]. Throughout, A denotes a small category.

Let  $X : \mathbb{A} \longrightarrow \mathbf{Set}$ . The category of elements  $\mathbb{E}(X)$  of X has as objects all pairs (a, x) where  $a \in \mathbb{A}$  and  $x \in Xa$ , and as maps  $(a, x) \longrightarrow (a', x')$  all maps  $f : a \longrightarrow a'$  in  $\mathbb{A}$  such that (Xf)(x) = x'.

Similarly, if  $X : \mathbb{A} \longrightarrow \mathbf{Cat}$  (where  $\mathbf{Cat}$  is the category of small categories and functors) then X has a **category of elements**  $\mathbb{E}(X)$ ; its objects are pairs (a, x) where  $a \in \mathbb{A}$  and  $x \in Xa$ , and its maps  $(a, x) \longrightarrow (a', x')$  are pairs  $(f, \xi)$ 

where  $f: a \longrightarrow a'$  in  $\mathbb{A}$  and  $\xi: (Xf)(x) \longrightarrow x'$  in Xa'. This definition can be made even when X is a **weak functor** or **pseudofunctor**, that is, only preserves composition and identities up to coherent isomorphism. The weak functors  $\mathbb{A} \longrightarrow \mathbf{Cat}$  correspond to the fibrations over  $\mathbb{A}^{\mathrm{op}}$ ; see [Bo].

The definition for **Cat**-valued functors extends that for **Set**-valued functors if a set is viewed as a discrete category (one with no maps other than identities).

Any two functors  $Y: \mathbb{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$  and  $X: \mathbb{A} \longrightarrow \mathbf{Set}$  have a tensor product  $Y \otimes X$ , a set, defined by

$$Y \otimes X = \left( \coprod_{a \in \mathbb{A}} Ya \times Xa \right) / \sim$$

where  $\sim$  is the equivalence relation generated by  $(y,(Xf)(x)) \sim ((Yf)(y),x)$  whenever  $f: a \longrightarrow b, x \in Xa$  and  $y \in Yb$ . (It may be helpful to think of X and Y as left and right A-modules.) A functor  $X: A \longrightarrow \mathbf{Set}$  is **flat** if

$$-\otimes X:[\mathbb{A}^{\mathrm{op}},\mathbf{Set}] \longrightarrow \mathbf{Set}$$

preserves finite limits. An equivalent condition is that  $\mathbb{E}(X)$  is **cofiltered**, that is, every finite diagram in  $\mathbb{E}(X)$  admits at least one cone.

**Proposition 5.1** The following conditions on a functor  $X : \mathbb{A} \longrightarrow \mathbf{Set}$  are equivalent:

- a. X is nondegenerate (in the sense of 3.2)
- b. every connected-component of  $\mathbb{E}(X)$  is cofiltered
- c. X is a sum of flat functors.
- $d. \otimes X : [\mathbb{A}^{\mathrm{op}}, \mathbf{Set}] \longrightarrow \mathbf{Set}$  preserves finite connected limits

**Proof** See [Ln] or [ABLR].

An idempotent  $e: a \longrightarrow a$  in  $\mathbb{A}$  splits if there exist  $a \xrightarrow{s} b$  such that si = 1 and is = e. The category  $\mathbb{A}$  is **Cauchy-complete** if every idempotent in  $\mathbb{A}$  splits. All of the examples of categories in this paper are Cauchy-complete, except that a finite monoid is Cauchy-complete if and only if it is a group.

**Lemma 5.2** Let  $\mathbb{A}$  be a Cauchy-complete category and  $X : \mathbb{A} \longrightarrow \mathbf{Set}$  a finite functor. Then X is familially representable if and only if X is nondegenerate.

**Proof** By Proposition 5.1, it is enough to prove that a finite functor X is representable if and only if it is flat. 'Only if' is immediate.

For 'if', suppose that X is flat. Then  $\mathbb{E}(X)$  is cofiltered, and finite by hypothesis, so the identity functor  $1_{\mathbb{E}(X)}$  admits a cone. Also,  $\mathbb{E}(X)$  is Cauchy-complete since  $\mathbb{A}$  is. Now, if  $\mathbb{C}$  is a Cauchy-complete category and  $(j \xrightarrow{p_c} c)_{c \in \mathbb{C}}$  is a cone on  $1_{\mathbb{C}}$  then  $p_j$  is idempotent, and the object through which it splits is initial. Hence  $\mathbb{E}(X)$  has an initial object; equivalently, X is representable.  $\square$ 

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