Math 156 Notes: Fourier Analysis on Finite Groups and Schur Orthogonality

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There are two related goals for these two lectures: first, to give a precise answer to the question, "in what sense is understanding the representation theory of a (finite) group like doing Fourier analysis on that group?", and second, to decompose the group algebra $\mathbf{C}G$ into a product of matrix algebras.

Remark. All of the material in these notes holds for arbitrary compact groups, given the right machinery.

1 Classical Fourier Analysis

In this section we recall some very basic facts about the classical Fourier series. Let $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ be the unit one-dimensional torus, so $\mathbf{T}$ is an abelian group, homeomorphic to the unit circle. Given a sufficiently nice (e.g., smooth) function $f : \mathbf{T} \to \mathbf{C}$, or equivalently, a smooth function $f : \mathbf{R} \to \mathbf{C}$ that is periodic with period one, we define its Fourier series $\widehat{f} : \mathbf{Z} \to \mathbf{C}$ by

$$\widehat{f}(n) = \int_{\mathbf{T}} f(t)\, e^{-2\pi i n t}\, dt.$$

Intuitively, if we think of f as a periodic signal, then $|\widehat{f}(n)|$ corresponds to "how much" of f has frequency $1/n$, and the argument of $\widehat{f}(n)$ is the phase. The fantastic result is that, for sufficiently nice f, we can reconstruct f from its Fourier series using the *Fourier inversion formula*:

$$f(t) = \sum_{n \in \mathbf{Z}} \widehat{f}(n)\, e^{2\pi i n t}.$$

This result is very important in analysis, and also in the real world, where it is fundamental to signal processing.

We can reinterpret the Fourier transform in terms of representation theory, using the following result:

Proposition 1.1. *Any continuous irreducible representation of $\mathbf{T}$ is one-dimensional, and is given by one of the characters*

$$\chi_n : \mathbf{T} \to \mathbf{C} \qquad \chi_n(t) = e^{2\pi i n t}.$$

Recall that for a finite group G, we have a Hermetian inner product $\langle \cdot, \cdot \rangle$ on the space $\mathrm{Maps}(G, \mathbf{C})$ of complex-valued functions on G, namely, $\langle f_1, f_2 \rangle = |G|^{-1} \sum_{g \in G} f_1(g)\overline{f_2(g)}$. With respect to this inner product, the irreducible characters of G form an orthonormal basis for the subspace of class functions. The analogous inner product on the space of nice functions on $\mathbf{T}$

is given by $\langle f_1, f_2\rangle = \int_{\mathbf{T}} f_1(t)\overline{f_2(t)}\,dt$, and as is easily checked, the irreducible characters χ_n are orthonormal. Note that $\widehat{f}(n) = \langle f, \chi_n\rangle$. The statement of Fourier inversion then becomes the analogous statement that the irreducible characters "span" the space of sufficiently nice complex-valued functions on $\mathbf{T}$, in the sense that for such f, we have $f = \sum_{n\in\mathbf{Z}}\langle f, \chi_n\rangle\,\chi_n$. (Since $\mathbf{T}$ is an abelian group, every element of $\mathbf{T}$ is its own conjugacy class.) This is the precise sense in which understanding the representation theory of $\mathbf{T}$ is equivalent to doing Fourier analysis on $\mathbf{T}$.

Example 1.2. The function $f(t) = \cos(2\pi t)$ is smooth and periodic on $\mathbf{T}$. Since $\cos(2\pi t) = \frac{1}{2}(e^{2\pi i t} + e^{-2\pi i t})$, its Fourier series is as follows:

$$\widehat{f}(n) = \begin{cases} \frac{1}{2} & \text{if } n = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

2 Discrete Fourier Analysis

As mentioned above, the Fourier series is fundamental to signal processing, including the signal processing done by computers. A computer cannot calculate using real numbers; it needs a discrete set of data points sampled from the input function. Given a complex-valued function f on $\mathbf{T}$, it is natural to sample f at the points $\frac{1}{n}\mathbf{Z}/\mathbf{Z} = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\}$. This is a motivation for the discrete Fourier transform.

The group $G = \frac{1}{n}\mathbf{Z}/\mathbf{Z}$ is cyclic of order n. We already understand the representation theory of G: there are n irreducible representations, given by the characters $\chi_0, \ldots, \chi_{n-1}$, where $\chi_j(t) = e^{2\pi i j t}$ for $t \in G$ (this is the correct formula since $\frac{1}{n}$ is a generator, and $\chi_j(\frac{1}{n}) = e^{2\pi i j/n}$). Note that this definition of χ_j agrees with the one above. Since $\chi_0, \chi_1, \ldots, \chi_{n-1}$ is an orthonormal basis for the finite-dimensional vector space $\mathrm{Maps}(G, \mathbf{C})$, we have that for any $f \in \mathrm{Maps}(G, \mathbf{C})$,

$$f = \sum_{j=0}^{n-1}\langle f, \chi_j\rangle\chi_j.$$

Hence it makes sense to define the *discrete Fourier transform* $\widehat{f} : \{0, 1, \ldots, n-1\} \to \mathbf{C}$ by $\widehat{f}(j) = \langle f, \chi_j\rangle$, so that the Fourier inversion formula holds.

Intuitively, the discrete Fourier transform picks out the components of f with frequencies less than about $\frac{2}{n}$; it makes sense that with only n sample points, one can only read information about that range of frequencies. Note however that as $n \to \infty$ one will recover the classical Fourier series, so that the discrete Fourier transform does approximate the Fourier series.

In the discussion above, we only used that $\frac{1}{n}\mathbf{Z}/\mathbf{Z}$ is abelian, and in fact, everything we said holds for arbitrary finite abelian groups. Indeed, let G be a finite abelian group, and for $f \in \mathrm{Maps}(G, \mathbf{C})$ and χ an irreducible character of G, define $\widehat{f}(\chi) = \langle f, \chi\rangle$ — note that the domain of the Fourier transform is now the set of irreducible characters. Then the Fourier inversion formula holds: $f = \sum_{\chi}\widehat{f}(\chi)\,\chi$.

With a view towards generalizing to nonabelian groups, we again reinterpret the above, this time in terms of the group algebra $\mathbf{C}G$. As a vector space, $\mathbf{C}G$ is canonically isomorphic to $\mathrm{Maps}(G, \mathbf{C})$; namely, the function $f \in \mathrm{Maps}(G, \mathbf{C})$ corresponds to the element

$$\frac{1}{|G|}\sum_{g\in G} f(g)\,g.$$

We have a decomposition $\mathbf{C}G = \bigoplus_\chi V_\chi$, where the sum is taken over all irreducible characters χ, and V_χ is the canonical one-dimensional χ-isotypic component of $\mathbf{C}G$. In fact, we can make the following stronger statement, proven in the text:

Proposition 2.1. *Let χ be an irreducible character of G, and let $e_\chi \in \mathbf{C}G$ be given by*

$$e_\chi = \frac{1}{|G|}\sum_{g\in G} \overline{\chi(g)} \cdot g.$$

Then e_χ is a central idempotent, and $V_\chi = e_\chi \mathbf{C}G = \mathbf{C} \cdot e_\chi$.

Using the above fact, we can obtain another formula for the Fourier transform:

Proposition 2.2. *Let $f \in \mathrm{Maps}(G, \mathbf{C})$, and let $x = |G|^{-1}\sum f(g)\, g$ be the corresponding element of $\mathbf{C}G$. Then*

$$x = \sum_\chi \widehat{f}(\overline{\chi})\, e_\chi,$$

where $\overline{\chi}(g) = \overline{\chi(g)} = \chi(g)^{-1}$.

Proof.

Clearly $f = \sum_\chi \widehat{f}(\overline{\chi})\overline{\chi}$; now take the image in $\mathbf{C}G$ under our isomorphism $\mathrm{Maps}(G, \mathbf{C}) \xrightarrow{\sim} \mathbf{C}G$.

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Example 2.3. Let $G = C_2 \times C_2 = \{1, (1, x), (x, 1), (x, x)\}$, where $x^2 = 1$. The character table is as follows:

$C_2 \times C_2$	1	$(1,x)$	$(x,1)$	(x,x)
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	1	-1	-1
χ_4	1	-1	-1	1

Let $f \in \mathrm{Maps}(G, \mathbf{C})$ correspond to the identity element in $\mathbf{C}G$, i.e., $f(g) = 4\delta_{1g}$. Then clearly $\langle f, e_\chi\rangle = 1$ for all χ, so we obtain

$$1 = e_{\chi_1} + e_{\chi_2} + e_{\chi_3} + e_{\chi_4}$$

as expected.

The above result allows us to give a nice decomposition of $\mathbf{C}G$. First we note that the group-algebra multiplication on $\mathbf{C}G$ goes over to the convolution operation on $\mathrm{Maps}(G, \mathbf{C})$: that is, in $\mathbf{C}G$, we have

$$\begin{aligned}\big(|G|^{-1}\sum_g f_1(g)\, g\big) \cdot \big(|G|^{-1}\sum_h f_2(h)\, h\big) &= \frac{1}{|G|^2}\sum_{g,h} f_1(g) f_2(h)\, gh \\ &= \frac{1}{|G|^2}\sum_g \left(\sum_h f_1(gh^{-1}) f_2(h)\right) g,\end{aligned}$$

so that the product of $f_1, f_2 \in \mathrm{Maps}(G, \mathbf{C})$ corresponding to the product in $\mathbf{C}G$ is given by

$$(f_1 * f_2)(g) = \frac{1}{|G|}\sum_{h\in G} f_1(gh^{-1}) f_2(h).$$

This gives $\mathrm{Maps}(G, \mathbf{C})$ the structure of $\mathbf{C}$-algebra, with unit element given by $g \mapsto |G|\delta_{eg}$, where δ_{eg} is zero if $g \neq e$ and is 1 otherwise. We can now prove that the Fourier transform of the convolution is the product of the Fourier transforms, as in classical Fourier theory:

Corollary 2.4 (to Proposition 2.2). *Let $f_1, f_2 \in \mathrm{Maps}(G, \mathbf{C})$. Then*

$$(f_1 * f_2)^\wedge(\chi) = \widehat{f}_1(\chi)\widehat{f}_2(\chi).$$

Equivalently, the map

$$f \mapsto (\widehat{f}(\chi))_\chi : \mathrm{Maps}(G, \mathbf{C}) \to \prod_\chi \mathbf{C}$$

is an isomorphism of algebras, where the addition and multiplication on $\prod_\chi \mathbf{C}$ are done coordinate-wise.

Proof.

Let $x_i = |G|^{-1} \sum_g f_i(g)\, g \in \mathbf{C}G$ be the element corresponding to f_i, so that $x_1 x_2$ corresponds to $f_1 * f_2$. Since $e_\chi e_{\chi'}$ is zero when $\chi \neq \chi'$ and is e_χ otherwise, we have

$$\begin{aligned}\sum_\chi (f_1 * f_2)^\wedge(\overline{\chi}) e_\chi = x_1 x_2 &= \left(\sum_\chi \widehat{f}_1(\overline{\chi}) e_\chi\right)\left(\sum_{\chi'} \widehat{f}_1(\overline{\chi}') e_{\chi'}\right) \\ &= \sum_\chi \widehat{f}_1(\overline{\chi})\widehat{f}_2(\overline{\chi}) e_\chi.\end{aligned}$$

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Note that in the decomposition in the above Corollary, the χ-factor of $\mathbf{C}$ in $\prod_\chi \mathbf{C}$ really is just V_χ. We will generalize Corollary 2.4 in Theorem 4.1.

3 Non-abelian Groups and Schur Orthogonality

We would like somehow to generalize the previous section to non-abelian finite groups G, namely, to give an orthonormal basis for $\mathrm{Maps}(G, \mathbf{C})$ in terms of the representation theory of G. We immediately run into a problem: the characters of G only span the subspace of class functions. And whereas we can still decompose $\mathbf{C}G$ using central idempotents, an analogue of Proposition 2.2 would give us a Fourier transform with values in $\mathbf{C}G$ and not in $\mathbf{C}$, which would not span $\mathrm{Maps}(G, \mathbf{C})$ either.

The idea now is to use the matrix coefficients of the irreducible representations of G. Speaking informally, every matrix coefficient of an irreducible representation ρ of G — that is, a literal coefficient of the matrix $g\rho$ with respect to some fixed basis — is a function from G to $\mathbf{C}$, and there are exactly

$$\dim_{\mathbf{C}}(\mathrm{Maps}(G, \mathbf{C})) = |G| = \sum_\rho \dim(V_\rho)^2$$

such functions. The matrix coefficients will indeed provide us with an orthonormal basis.

Definition 3.1. Let V be a finite-dimensional G-module, and let $V^* = \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$ be the linear dual of G (as a vector space). Pick $v \in V$ and $T \in V^*$. Then the map

$$f_{T,v} : G \to C \qquad \text{given by} \qquad f_{T,v}(g) = (v \circ g)T$$

is called a *matrix coefficient* of V.

Remark 3.2. With the above notation, let $v_1, \dots, v_n$ be a basis for V, and let $T_1, \dots, T_n$ be the dual basis of V^*, i.e., the T_i are defined by $v_j T_i = \delta_{ij}$. Let $\pi : G \to \mathrm{GL}(V)$ be the representation associated to V, and let $(\pi_{ij}(g))$ be the matrix associated to $g\pi$ with respect to the above basis. Then one can check that $\pi_{ij}(g) = (v_i \circ g)T_j = f_{T_j, v_i}(g)$, so that f_{T_j, v_i} is literally a matrix coefficient of G. The general $f_{T,v}$ will be a linear combination of matrix coefficients with respect to some basis.

Example 3.3. Let $G = S_3$, and let $V = \mathbf{C}^3$ be the natural S_3-module, with associated representation π. Then with respect to the canonical basis, the matrices for the elements of S_3 are as follows:

$$1\pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (12)\pi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (13)\pi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$(23)\pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (123)\pi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad (132)\pi = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

So for example, the top-left matrix coefficient is given by

$$f(1) = 1 \quad f((12)) = 0 \quad f((13)) = 0 \quad f((23)) = 1 \quad f((123)) = 0 \quad f((132)) = 0.$$

In order to generate orthonormal matrix coefficients, we need to add some more structure to our representations. Recall that a *Hermetian inner product* on a complex vector space V is a bilinear pairing $(\cdot,\cdot) : V \times V \to \mathbf{C}$ such that $(v,w) = \overline{(w,v)}$ for all $v, w \in \mathbf{C}$, and such that $(v,v) > 0$ for all nonzero v. If V is equipped with such an inner product, then any linear map $T : V \to \mathbf{C}$ is of the form $v \mapsto (v,w)$ for a unique $w \in V$. In particular, if V is a G-module, then any matrix coefficient of V is of the form $g \mapsto (v \circ g, w)$ for some $v, w \in V$.

It is important not to confuse a given inner product $(\cdot,\cdot)$ on a G-module with the canonical inner product $\langle\cdot,\cdot\rangle$ on $\mathrm{Maps}(G, \mathbf{C})$.

Definition 3.4. A *unitary G-module* is a G-module V equipped with an *invariant Hermetian inner product*, i.e., a Hermetian inner product satisfying

$$(v \circ g, w \circ g) = (v,w) \qquad \text{for all } v, w \in V \text{ and all } g \in G.$$

A *unitary representation* is defined similarly.

Note that V is a unitary G-module if and only if the matrix for each element of g is unitary (i.e., has orthonormal rows and columns) with respect to an orthonormal basis of V.

Proposition 3.5. *Any G-module V admits an invariant Hermetian inner product.*

Proof.

First note that V admits some Hermetian inner product: indeed, choose a basis $v_1, \dots, v_n$ for V, and define $(\cdot,\cdot)' : V \to \mathbf{C}$ by

$$\left(\sum_{i=1}^n a_i v_i,\ \sum_{j=1}^n b_j v_j \right)' = \sum_{i,j=1}^n a_i \overline{b_j}.$$

Now define $(\cdot,\cdot)$ by

$$(v,w) = \sum_{g \in G} (v \circ g,\ w \circ g)'.$$

It is clear that $(\cdot,\cdot)$ is a Hermetian inner product, and for $v, w \in V$ and $g \in G$, we have

$$(v \circ g,\ w \circ g) = \sum_{h \in G} (v \circ gh,\ w \circ gh)' = \sum_{h \in G} (v \circ h,\ w \circ h)' = (v, w).$$

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Example 3.6. Again consider the natural S_3-module $\mathbf{C}^3$, with the standard basis v_1, v_2, v_3, and Hermetian inner product $(v_i, v_j) = \delta_{ij}$. Since the permutation matrices are unitary, we see that the inner product $(\cdot,\cdot)$ is S_3-invariant.

As a nice consequence of Proposition 3.5, we can re-prove Maschke's Theorem:

Corollary 3.7 (Maschke's Theorem). *Let V be a G-module, and let $W \subset V$ be a proper G-submodule. Then there is a G-submodule W' such that $V = W \oplus W'$.*

Proof.

Let $(\cdot,\cdot)$ be an invariant Hermetian inner product on V, and define

$$W' = \{v \in V\ :\ (v, w) = 0 \text{ for all } w \in W\}.$$

Then since $(v, w) = (v \circ g, w \circ g)$ for $g \in G$, we see that W' is a G-submodule. If $v \in W$ and $(v, w) = 0$ for all $w \in W$, then in particular, $(v, v) = 0$, so $v = 0$; hence $W \cap W' = \{0\}$. We leave as an exercise to show that $V = W + W'$ (hint: choose an orthonormal basis for W and extend).

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We can now give the statement of Schur Orthogonality.

Theorem 3.8 (Schur Orthogonality). *Let G be a finite group, and let V_1 and V_2 be non-isomorphic irreducible G-modules.*

(i) *Every matrix coefficient of V_1 is orthogonal to every matrix coefficient of V_2 (with respect to $\langle\cdot,\cdot\rangle$).*

(ii) *Let $(\cdot,\cdot)$ be an invariant inner product on V_1, let $v_1, v_2, w_1, w_2 \in V_1$, and let $f_1(g) = (w_1 \circ g, v_1)$ and $f_2(g) = (w_2 \circ g, v_2)$ be matrix coefficients of V_1. Then*

$$\langle f_1, f_2 \rangle = \frac{1}{\dim V_1}(w_1, w_2)(v_2, v_1).$$

More explicitly,

$$\frac{1}{|G|}\sum_{g \in G} (w_1 \circ g, v_1)\overline{(w_2 \circ g, v_2)} = \frac{1}{\dim V_1}(w_1, w_2)(v_2, v_1).$$

Before we begin the proof, we show how Schur Orthogonality can provide us with an orthonormal basis of $\mathrm{Maps}(G, \mathbf{C})$. I think this is the closest thing there is to doing Fourier analysis over non-abelian groups.

Corollary 3.9. *Let $V_1, \ldots, V_r$ be the distinct irreducible G-modules. For each V_i, choose an invariant inner product $(\cdot,\cdot)_i$, and choose an orthonormal basis $v_{i,1}, \ldots, v_{i,d_i}$ for V_i with respect to $(\cdot,\cdot)_i$, where $d_i = \dim V_i$. Let*

$$f_{ijk}(g) = \sqrt{\dim V_i} \cdot (v_{i,k} \circ g, v_{i,j})_i$$

for $i = 1, \ldots, r$ and $j, k = 1, \ldots, d_i$. Then the f_{ijk} are an orthonormal basis for $\mathrm{Maps}(G, \mathbf{C})$.

Note that $f_{ijk}(g)$ is $\sqrt{\dim V_i}$ times the (k, j)th entry of the matrix corresponding to g on the representation V_i with respect to the basis $v_{i,1}, \ldots, v_{i,d_i}$.

Proof.

We have defined $|G| = \sum_{i=1}^{r} (\dim V_i)^2$ total matrix coefficients, so it suffices to show that the set $\{f_{ijk}\}$ is orthonormal. By Theorem 3.8 (i), we know that matrix coefficients coming from distinct V_i are orthogonal, and by Theorem 3.8 (ii),

$$\langle f_{ijk}, f_{ij'k'} \rangle = \frac{1}{\dim V_i} \cdot (\dim V_i)\,(v_{i,k}, v_{i,k'})\,(v_{i,j'}, v_{i,j}),$$

which is one exactly when $j = j'$ and $k = k'$, and zero otherwise.

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Example 3.10. Returning again to our S_3 example, we have three distinct irreducible S_3-modules, namely, the trivial module V_1, the sign module V_2, and the two-dimensional module V_3, which we realize as the sub-representation of the natural S_3-module $\mathbf{C}^3$ spanned by $v_1 - v_2$ and $v_2 - v_3$. Choosing bases for V_1 and V_2 to get isomorphisms $V_1 \cong \mathbf{C}$ and $V_2 \cong \mathbf{C}$, we see that the inner products $(u, v)_1 = u \cdot v$ and $(u, v)_2 = u \cdot v$ (multiplication of complex numbers) are invariant. For the module V_3, we can restrict the invariant inner product from Example 3.6 on the natural G-module $\mathbf{C}^3$ to V_3 to obtain $(\cdot,\cdot)_3$. With respect to this inner product, an orthonormal basis for V_3 is given by $v = (v_1 - v_2)/\sqrt{2}$ and $w = (v_1 + v_2 - 2v_3)/\sqrt{6}$. With respect to this basis, the matrices for the elements of S_3 are as follows:

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (12) \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (13) \mapsto \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$(23) \mapsto \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \quad (123) \mapsto \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \quad (132) \mapsto \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

With respect to the above choices of basis, our orthonormal basis for $\mathrm{Maps}(S_3, \mathbf{C})$ is as follows:

S_3	1	(12)	(13)	(23)	(123)	(132)
f_{111}	1	1	1	1	1	1
f_{211}	1	-1	-1	-1	1	1
f_{311}	$\sqrt{2}$	$-\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	$-1/\sqrt{2}$	$-1/\sqrt{2}$
f_{312}	0	0	$-\sqrt{3/2}$	$\sqrt{3/2}$	$-\sqrt{3/2}$	$\sqrt{3/2}$
f_{321}	0	0	$-\sqrt{3/2}$	$\sqrt{3/2}$	$\sqrt{3/2}$	$-\sqrt{3/2}$
f_{322}	$\sqrt{2}$	$\sqrt{2}$	$-1/\sqrt{2}$	$-1/\sqrt{2}$	$-1/\sqrt{2}$	$-1/\sqrt{2}$

One can check that the above functions are indeed orthonormal with respect to $\langle \cdot, \cdot \rangle$.

Now we proceed to prove Theorem 3.8. The following treatment can be found in Chapter 2 of Daniel Bump's *Lie Groups*. We require a lemma:

Lemma 3.11. *Let V_1 and V_2 be two G-modules, and let $(\cdot,\cdot)$ be any Hermetian inner product on V_1. For $v_1 \in V_1$ and $v_2 \in V_2$, the map $T : V_1 \to V_2$ defined by*

$$wT = \sum_{g \in G} (w \circ g, v_1)\, v_2 \circ g^{-1}$$

is a G-module homomorphism.

Proof.

It is clear that T is a linear transformation. For $h \in G$ we have

$$(w \circ h)T = \sum_{g \in G} (w \circ hg, v_1)\, v_2 \circ g^{-1}$$

Making the change of variables $g \mapsto h^{-1}g$, this is equal to

$$\sum_{g \in G} (w \circ g, v_1)\, v_2 \circ g^{-1}h = (wT) \circ h.$$

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Proof of Schur Orthogonality.

First we prove part (i) of the Theorem. Let $(\cdot, \cdot)_i$ be an invariant bilinear form on V_i, and let $v_i, w_i \in V_i$ for $i = 1, 2$, so $f_i(g) = (w_i \circ g, v_i)_i$ is an arbitrary matrix coefficient for V_i. By Lemma 3.11, the map $T : V_1 \to V_2$ given by

$$wT = \sum_{g \in G} (w \circ g, v_1)_1\, v_2 \circ g^{-1}$$

is a G-module homomorphism. Since V_1 and V_2 are nonisomorphic irreducible G-modules, there exist no nonzero G-module homomorphisms between them, so $wT = 0$ for all w. In particular,

$$\sum_{g \in G} (w_1 \circ g, v_1)_1\, v_2 \circ g^{-1} = 0,$$

so taking the inner product of the left-hand side with w_2 and using linearity, we have

$$0 = \left(\sum_{g \in G} (w_1 \circ g, v_1)_1\, v_2 \circ g^{-1}, w_2 \right)_2 = \sum_{g \in G} (w_1 \circ g, v_1)_1 \cdot (v_2 \circ g^{-1}, w_2)_2.$$

But by invariance of $(\cdot, \cdot)_2$, we have

$$(v_2 \circ g^{-1}, w_2)_2 = (v_2 \circ g^{-1} \circ g, w_2 \circ g)_2 = \overline{(w_2 \circ g, v_2)}_2,$$

so we obtain

$$0 = \sum_{g \in G} (w_1 \circ g, v_1)_1 \cdot \overline{(w_2 \circ g, v_2)}_2 = |G| \langle f_1, f_2 \rangle.$$

Now we prove part (ii). Let the notation be as in the statement of part (ii) of the theorem. Define $T : V_1 \to V_1$ as above. By Schur's Lemma, there is a constant $c = c(v_1, v_2)$ depending only on v_1 and v_2 such that $wT = cw$ for all $w \in V_1$. In particular, $w_1 T = c w_1$, so as in the proof of part (i), we have

$$\begin{aligned} c(v_1, v_2) \cdot (w_1, w_2) = (w_1 T, w_2) &= \sum_{g \in G} (w_1 \circ g, v_1) \cdot (v_2 \circ g^{-1}, w_2) \\ &= \sum_{g \in G} (w_1 \circ g, v_1) \cdot \overline{(w_2 \circ g, v_2)} = |G| \langle f_1, f_2 \rangle. \end{aligned}$$

On the other hand, substituting g^{-1} for g and using the invariance of $(\cdot, \cdot)$, we have

$$\begin{aligned} c(v_1, v_2) \cdot (w_1, w_2) = |G| \langle f_1, f_2 \rangle &= \sum_{g \in G} (w_1 \circ g^{-1}, v_1) \cdot \overline{(w_2 \circ g^{-1}, v_2)} \\ &= \sum_{g \in G} (v_2 \circ g, w_2) \cdot \overline{(v_1 \circ g, w_1)} = c(w_1, w_2) \cdot (v_1, v_2) \end{aligned}$$

by the above argument, with v_i and w_i exchanged. Choosing v_1' and v_2' such that $(v_1', v_2') = 1$, we have $c(w_1, w_2) = C \cdot (w_1, w_2)$, where $C = c(v_1', v_2')$ is a constant depending only on V_1. Therefore,

$$|G|\langle f_1, f_2\rangle = c(w_1, w_2) \cdot (v_1, v_2) = C \cdot (v_1, v_2) \cdot (w_1, w_2).$$

To calculate C, we choose an orthonormal basis $u_1, \ldots, u_d$ for V_1, and we let $m_i(g) = (g \circ u_i, u_i)$. If π is the representation associated to the G-module V_1, then $m_i(g)$ is the (i, i)th coefficient of the matrix for $g\pi$ with respect to the basis $u_1, \ldots, u_d$. Hence if χ is the character for π, we have $\chi(g) = \mathrm{Tr}(g\pi) = \sum_{i=1}^{d} m_i(g)$, and therefore since V_1 is irreducible,

$$\begin{aligned}
1 = \langle \chi, \chi \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} = \frac{1}{|G|} \sum_{g \in G} \sum_{i,j=1}^{d} m_i(g)\overline{m_j(g)} \\
&= \frac{1}{|G|} \sum_{i,j=1}^{d} \sum_{g \in G} m_i(g)\overline{m_j(g)} = \sum_{i,j=1}^{d} \langle m_i, m_j \rangle = \frac{C}{|G|} \sum_{i,j=1}^{d} (u_i, u_j)(u_i, u_j) \\
&= \frac{C}{|G|} \sum_{i,j=1}^{d} \delta_{ij} = \frac{C}{|G|} \cdot \dim V_1.
\end{aligned}$$

In the last step we used the orthonormality of the u_i. Hence $C/|G| = 1/\dim V_1$. ❐

4 Decomposition of $\mathrm{Maps}(G, \mathbf{C})$

We can use Schur orthogonality to prove the following decomposition theorem, which generalizes Corollary 2.4 to non-abelian groups:

Theorem 4.1. *Let G be a finite group, let $W_1, \ldots, W_r$ be the distinct irreducible G-modules, and decompose $\mathbf{C}G = \bigoplus_{i=1}^{r} V_i$, where $V_i \cong W_i^{\oplus \dim W_i}$ is the sum of all submodules of $\mathbf{C}G$ isomorphic to W_i. Then V_i is a $\mathbf{C}$-algebra which is isomorphic to the algebra $M_{d_i}(\mathbf{C})$ of $d_i \times d_i$ matrices, where $d_i = \dim W_i$.*

Proof.

Fix $V = V_i$ and $W = W_i$, and set $d = d_i$. The matrix algebra $M_d(\mathbf{C})$ is spanned by the d^2 matrices $\{m_{ij}\}_{i,j=1,\ldots,d}$, where m_{ij} is the matrix whose (i, j)th entry is 1 and whose other entries are zero. The rule for multiplying the $m_{i,j}$ is as follows: $m_{i,j}m_{i',j'} = \delta_{i'j}m_{ij'}$. By linearity, if we can find linearly independent elements $x_{ij} \in V$ such that $x_{ij}x_{i'j'} = \delta_{i'j}x_{ij'}$, then the map $M_d(\mathbf{C}) \to V$ taking $m_{ij} \mapsto x_{ij}$ is an algebra isomorphism.

Using Schur orthogonality, it is easy to find $f_{ij} \in \mathrm{Maps}(G, \mathbf{C})$ satisfying the above multiplication relations. (They are literal matrix entries for W). Indeed, let $(\cdot, \cdot)$ be a G-invariant Hermetian inner product on W, let $w_1, \ldots, w_d$ be an orthonormal basis for W, and let

$$f_{ij}(g) = d \cdot (w_i \circ g, w_j).$$

By Corollary 3.9, the f_{ij} are linearly independent, and

$$\begin{aligned}
(f_{ij} * f_{i'j'})(g) &= \frac{1}{|G|} \sum_{h \in G} f_{ij}(gh^{-1}) f_{i'j'}(h) \\
&= \frac{d^2}{|G|} \sum_{h \in G} (w_i \circ gh^{-1}, w_j) \cdot (w_{i'} \circ h, w_{j'}) \\
&= \frac{d^2}{|G|} \sum_{h \in G} (w_{i'} \circ h, w_{j'}) \cdot \overline{(w_j \circ h, w_i \circ g)} \\
&= d \cdot (w_{i'}, w_j)(w_i \circ g, w_{j'}) = \delta_{i'j} f_{ij'}(g),
\end{aligned}$$

where the next-to-last equality is by Schur Orthogonality.

Let

$$x_{ij} = \frac{1}{|G|} \sum_{g \in G} \overline{f_{ij}(g)}\, g = \frac{d}{|G|} \sum_{g \in G} (w_j, w_i \circ g)\, g$$

be the image of $\overline{f_{ij}}$ in $\mathbf{C}G$, and note that $\overline{f_{ij}} * \overline{f_{i'j'}} = \overline{f_{ij} * f_{i'j'}}$, so that the x_{ij} satisfy the same multiplication relations as the f_{ij}. It remains to show that $x_{ij} \in V$; it suffices to prove that $W_i = \mathrm{Span}(x_{i1}, \ldots, x_{id})$ is isomorphic to W. Indeed, define $\theta : W \to W_i$ by $w_j\theta = x_{ij}$ and extending linearly. Then θ is an isomorphism of vector spaces. Let $g \in G$, and let $w_j g = \sum_{k=1}^{d} a_{jk} w_k$. Then

$$(w_j \circ g)\theta = \sum_{k=1}^{d} a_{jk}\, x_{ik}.$$

On the other hand,

$$\begin{aligned}
x_{ij} g &= \frac{d}{|G|} \sum_{h \in G} (w_j, w_i \circ h)\, hg = \frac{d}{|G|} \sum_{h \in G} (w_j, w_i \circ hg^{-1})\, h \\
&= \frac{d}{|G|} \sum_{h \in G} (w_j \circ g, w_i \circ h)\, h = \frac{d}{|G|} \sum_{h \in G} \sum_{k=1}^{d} a_{jk} \cdot (w_k, w_i \circ h)\, h \\
&= \sum_{k=1}^{d} a_{jk} \frac{d}{|G|} \sum_{h \in G} (w_k, w_i \circ h)\, h = \sum_{k=1}^{d} a_{jk}\, x_{ik}.
\end{aligned}$$

❒

The proof of Theorem 4.1 shows that, in fact, after choosing a basis for W_i, the component V_i becomes a $d_i \times d_i$ matrix algebra, with G acting on the right by multiplication by the matrix for g acting on W_i with respect to that basis. One then decomposes V_i into $W_i^{\oplus \dim W_i}$ by noting that the subspace of $V_i \cong M_{d_i}(\mathbf{C})$ spanned by the entries of a single row is invariant and is obviously isomorphic to W. A simple argument with central idempotents shows that furthermore, a choice of basis for each W_i gives an isomorphism $\mathbf{C}G \cong \prod_{i=1}^{r} M_{d_i}(\mathbf{C})$ of $\mathbf{C}$-algebras (where addition and multiplication on the right-hand side is done coordinate-wise), with $g \in G$ acting on the ith component by right-multiplication by the matrix for g acting on W_i, with respect to the chosen basis.