

The complete set of infinite volume ground states for Kitaev's abelian quantum double models

Matthew Cha¹, Pieter Naaijken^{1,2}, and Bruno Nachtergaele¹

¹*Department of Mathematics, University of California, Davis, USA*

²*JARA Institute for Quantum Computing, RWTH Aachen University, Germany*

August 17, 2016

Abstract

We study the set of infinite volume ground states of Kitaev's quantum double model on \mathbb{Z}^2 for an arbitrary finite abelian group G . It is known that these models have a unique frustration-free ground state. Here we drop the requirement of frustration freeness, and classify the full set of ground states. We show that the ground state space decomposes into $|G|^2$ different charged sectors, corresponding to the different types of abelian anyons (also known as superselection sectors). In particular, all pure ground states are equivalent to one of the “single excitation” states studied by one of us in previous work. Our proof proceeds by showing that each ground state can be obtained as the weak* limit of finite volume ground states of the quantum double model with suitable boundary terms. The boundary terms allow for states which represent a pair of excitations, with one excitation in the bulk and one pinned to the boundary, to be included in the ground state space.

1 Introduction

The past two decades have witnessed a rising interest in topologically ordered states, mainly due to the realization that their properties could be useful for fault tolerant quantum computation [F, K]. The quantum double models introduced by Kitaev demonstrated the existence quantum spin models with short-range interactions that have ground states exhibiting topological order and anyonic excitation spectrum. A remarkable feature is that the ground space degeneracy depends on the genus of the surface on which the model is defined.

A characteristic of topologically ordered states in two dimensions is the appearance of elementary excitations with braid statistics, called anyons [Wi]. Perhaps the most well known model for anyons are the quasi-particle excitations of the fractional quantum hall effect [ASW, MR]. Braid statistics have also been studied in the context of local quantum physics [FRS, FG] and gauge theories [BVD]. The fusion rules and braiding of anyons is encoded algebraically as a unitary modular tensor category [BK]. In particular, the case of the representation theory of the quantum double for a finite group has been well studied [DPR, BVD, SV, K].

Kitaev's models are very special in the sense that they have frustration-free ground states and that the interactions terms in the Hamiltonian are all mutually commuting. Trivially, these models have a non-vanishing spectral gap above the ground state. The spectral gap is an important feature in the classification of topologically ordered ground states into a so-called topological phases [We, CGW], and has important implications such as exponential decay of correlations [NS, HK] and entanglement area laws [H, BH]. It was therefore important to show that this gap does not close for sufficiently small uniform perturbations of these models [BHM, MP].

Ground states of quantum lattice models is a well-studied subject. Knowing the set of ground states is essential for understanding the properties of quantum many-body system at sufficiently low temperatures. In the mathematical analysis of certain statistical mechanics phenomena, such as equilibrium states, phases transitions, superselection sectors and phase classification, it is often necessary or convenient to take the infinite volume limit (or thermodynamic limit) [BO, BR, N]. General existence and decomposition properties have been mastered for some time [BKR, BR]. The problem of proving that one has found the complete set of ground states for a given model, however, has been solved only in a few cases.

In this work, we study quantum double models for abelian groups, in their implementation as quantum spin Hamiltonians with short-range interactions as defined by Kitaev [K]. The simplest example is the toric code model, which corresponds to the choice $G = \mathbb{Z}_2$. The abelian quantum double model is particularly interesting because it has all of the characteristic features of topologically ordered systems, while at the same time being simple enough to be tackled directly. The main features of the model are: it is exactly solvable in the sense that the Hamiltonian can be explicitly diagonalized; the dimension of the space of ground states of the models defined on a compact orientable surface is a topological invariant and corresponds to the number of flat G -connections on the lattice (up to conjugation); there is a spectral gap above the ground state; the elementary excitations correspond to quasi-particles with braid statistics.

The focus of this paper is the set of infinite volume ground states of the model for finite abelian groups. Although the quantum double models are exactly solvable in finite volume, much less is known about the thermodynamic limit. The first results in this direction are due to Alicki, Fannes and Horodecki [AFH]. They showed that in the case $G = \mathbb{Z}_2$, also known as the toric code, there is a unique frustration-free ground state, which coincides with the translation invariant ground state. This uniqueness property is not general [GW], but is related to topological order in the ground state. The difficulty of solving the full ground state problem can be understood as follows. If δ is the derivation generating the dynamics, one has to find *all* states ω on the quasi-local algebra \mathcal{A} of observables that satisfy $\omega(A^*\delta(A)) \geq 0$ for all A in the domain of δ . It is possible to construct ground states as weak* limits of finite volume ground states, but even though the boundary goes to infinity in a sense, the resulting state strongly depends on the boundary conditions chosen for the finite volume ground states.

The main result of this paper is a complete classification of the set of infinite volume ground states for Kitaev's quantum double model for finite abelian groups. We find that the set of ground states can be decomposed into $|G|^2$ sectors. There is a one-to-one correspondence between the ground state sectors and the superselection sectors defined in [FN]. In particular, each sector corresponds to a different anyon type. The strategy of the proof is to reduce the infinite volume calculation to a finite volume calculation. In particular, we find a boundary term for every finite box such that the restriction of any infinite volume ground state to the box is a ground state of the finite volume Hamiltonian plus the boundary term. This strategy is motivated by the fact that infinite volume ground states minimize energy in a local region among all states that are equivalent in the complement of that region [BKR], and resembles the classical Dobrushin-Lanford-Ruelle theory of boundary conditions for the restriction of infinite volume equilibrium state [FW].

Our analysis is deeply connected to the notions of topological order, superselection sectors and the fact that the anyon quasi-particles of the quantum double models are time invariant. In analogy with the analysis of Doplicher, Haag and Roberts [DHR71, DHR72] in local quantum physics [Haag], it is possible to analyze the different types of anyons in the system, to obtain the modular tensor category $\text{Rep}(\mathcal{D}(G))$ describing all their properties. This was done in [N, FN] for the abelian quantum double models, and we will use some of these results here.

It is often surprisingly difficult to classify the full set of ground states in the thermodynamic limit. To our knowledge, the complete ground state problem has only been solved for the one-dimensional XY -model by Araki and Matsui [AM], for the one-dimensional XXZ -models by Matsui [M] and Koma and Nachtergaele [KN], and for the finite-range spin chains with a unique frustration free matrix product ground state by Ogata [O3]. To make progress, one typically has to pair the ground state problem with model specific notions; in the XY -model it was the Jordan-Wigner transformation to fermions and in the XXZ -model and the frustration-free spin chains it was a connection to zero-energy states [FNW, GW]. As far as we are aware, our result is the first solution to the ground state problem for a quantum model in two dimensions.

This paper is organized as follows: In Section 2 we introduce the model and state the main results. We describe the superselection sectors in Section 3 and present a new result that equates a boundary projector to a sum of products of local projectors. A detailed presentation of the main results and the proofs are in Section 4. We conclude with a discussion of the lack of stability for the infinite volume ground state condition, and some difficulties that appear in the analysis for the non-abelian case in Section 5.

Acknowledgments: PN has received funding from the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant agreement No 657004. BN was supported in part by the National Science Foundation under Grant DMS-1515850.

2 Main results

In this section we recall the definition of the quantum double models we study and present a precise statement of the main results. A more in-depth discussion can be found in the following sections.

Let G be a finite abelian group and consider the bonds (or edges) \mathcal{B} of the square lattice \mathbb{Z}^2 , i.e. the edges between nearest neighbors of points (or vertices) in \mathbb{Z}^2 . We give \mathcal{B} an orientation by having edges either point up or right. To each edge $e \in \mathcal{B}$ we associate a $|G|$ -dimensional Hilbert space with an orthonormal basis labeled by group elements and denoted by $|g\rangle$. Throughout the paper, we use the notation \bar{g} to denote the inverse element g^{-1} . In general, the model can be defined on any oriented metric graph. Reversing the orientation on a given edge corresponds to the unitary transformation that maps $|g\rangle$ to $|\bar{g}\rangle$ in the state space of that edge. We use the notation $\Lambda \subset_f \mathcal{B}$ to indicate that Λ is a *finite* subset of \mathcal{B} . The quantum spin system in Λ is defined on the Hilbert space $\mathcal{H}_\Lambda := \bigotimes_{x \in \Lambda} \mathcal{H}_x$ and its algebra of observables is $\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$. If $\Lambda_1 \subset \Lambda_2$ there is a natural inclusion $i_{\Lambda_2, \Lambda_1} : \mathcal{A}_{\Lambda_1} \hookrightarrow \mathcal{A}_{\Lambda_2}$ mapping $A \mapsto A \otimes I_{\Lambda_2 \setminus \Lambda_1}$. The maps i_{Λ_2, Λ_1} are isometric morphisms so we will often abuse notation by identifying $i_{\Lambda_2, \Lambda_1}(A)$ with simply A . For this net of algebras, we define the local algebra of observables and the quasi-local algebra of observables as, respectively,

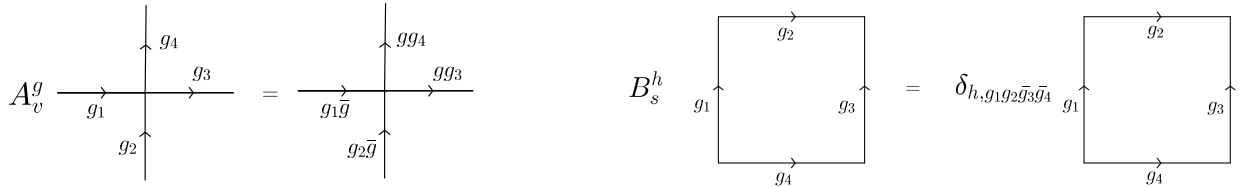
$$\mathcal{A}_{loc} = \bigcup_{\Lambda \subset_f \mathcal{B}} \mathcal{A}_\Lambda, \quad \mathcal{A} = \overline{\mathcal{A}_{loc}}^{\|\cdot\|}. \quad (2.1)$$

An observable $A \in \mathcal{A}$ is said to be supported on Λ if $A \in \mathcal{A}_\Lambda$. If Λ is the smallest set such that $A \in \mathcal{A}_\Lambda$ then we say Λ is the support of A .

To define the model we specify the local Hamiltonians and the Heisenberg dynamics on \mathcal{A} . The interaction terms of the local Hamiltonians are non-trivial only on certain subsets of \mathcal{B} , called stars and plaquettes. We define a *star* v to be a set of four edges sharing a vertex. Similarly, a *plaquette* f is the set of four edges forming a unit square in the lattice. Interaction terms are defined for each star and plaquette by

$$A_v = \frac{1}{|G|} \sum_{g \in G} A_v^g, \quad \text{and} \quad B_f = B_f^e, \quad (2.2)$$

where the terms A_v^g and B_f^h are defined by their action on basis of simple tensors as shown in the following diagram:



Here the group elements g_1, \dots, g_4 label the elements a tensor basis for the local Hilbert space corresponding to a star or plaquette.

It is easy to check that the operators A_v and B_f satisfy the following relations:

$$\begin{aligned} A_v^g A_v^{g'} &= A_v^{gg'}, & (A_v^{\bar{g}})^* &= A_v^g, \\ B_f^h B_f^{h'} &= \delta_{h, h'} B_f^h, & B_f^{h*} &= B_f^h, \\ A_v^g B_f^h &= B_f^{g\bar{h}} A_v^g & & \text{(if } v \text{ and } f \text{ share edges).} \end{aligned}$$

In all other cases the operators commute.

The interactions terms are mutually commuting projectors

$$A_v = A_v^* = A_v^2, \quad B_p = B_p^* = B_p^2, \quad [A_v, B_p] = 0 \text{ for all } v, p. \quad (2.3)$$

We caution the reader that in the case of the toric code model (which corresponds to $G = \mathbb{Z}_2$) one usually shifts the local interaction terms by a constant. This has no effect on the dynamics, but the algebraic relations are slightly different. Explicitly, the common convention is to define the toric code model in terms of star and plaquette operators A_v^{tc} and B_f^{tc} given by $2A_v - I = A_v^{tc}$ and $2B_p - I = B_p^{tc}$.

For $\Lambda \subset \mathcal{B}$ denote the subset of stars and plaquettes contained in Λ as

$$\mathcal{V}_\Lambda = \{v \subset \Lambda : v \text{ is a star}\}, \quad \mathcal{F}_\Lambda = \{f \subset \Lambda : f \text{ is a plaquette}\}. \quad (2.4)$$

If $\Lambda \subset_f \mathcal{B}$, the local Hamiltonian for the quantum double models defined by Kitaev [K] are given by

$$\sum_{v \in \mathcal{V}_\Lambda} (I - A_v) + \sum_{f \in \mathcal{F}_\Lambda} (I - B_f) = H_\Lambda \in \mathcal{A}_\Lambda. \quad (2.5)$$

Since the interaction terms are uniformly bounded and of finite range, the existence of global dynamics $t \mapsto \tau_t \in \text{Aut}(\mathcal{A})$ is readily established. For our analysis it will be enough to consider squares, $\Lambda_L \subset \mathcal{B}$, consisting of all edges in $[-L, L]^2$. We will denote $H_L = H_{\Lambda_L}$ and $\mathcal{H}_L = \mathcal{H}_{\Lambda_L}$. The generator of the dynamics is then defined by

$$\delta(A) = \lim_{L \rightarrow \infty} [H_L, A], \quad (2.6)$$

where \mathcal{A}_{loc} is a core for δ , and $\tau_t(A) = e^{it\delta}(A)$ for all $A \in \mathcal{A}_{loc}$.

We briefly describe the ground state space, \mathcal{G}_L , of the local Hamiltonians H_L . When defined on a torus, that is when we impose periodic boundary conditions, the ground states minimize the local energy of each interaction term, $\mathcal{G}_L^{per} = \ker H_L^{per} = \{\Omega \in \mathcal{H}_L : \forall s, p; A_s \Omega = \Omega, B_p \Omega = \Omega\}$ (see [K]). The dimension of \mathcal{G}_L^{per} is equal to the number of flat G -connections up to conjugation and independent of the size of the torus. For free boundary conditions, the frustration-free property still holds: $\mathcal{G}_L = \ker H_L$. It should be noted though that in this case the dimension grows exponentially with the perimeter of Λ_L . Other boundary conditions have also been considered [BK, FM, BSW].

Recall that a *state* on \mathcal{A} is a linear functional $\omega : \mathcal{A} \rightarrow \mathbb{C}$ such that $\omega(A) \geq 0$ if $A \geq 0$ and $\omega(I) = 1$. The set of all states is denoted $\mathcal{A}_{+,1}^*$ and is a convex set; its extremal points are called pure states. Since \mathcal{A} is unital, the set of states is compact in the weak* topology.

We are interested in the ground states of the infinite system defined as follows.

Definition 2.1. A state is a τ -ground state if

$$\omega(A^* \delta(A)) \geq 0 \quad \text{for all } A \in \mathcal{A}_{loc}. \quad (2.7)$$

We will refer to a τ -ground state as simply a ground state or an *infinite volume ground state*.

This definition can be interpreted as an infinite volume variational principle expressing that local perturbations do not decrease the energy of a ground state. At finite temperature T , equilibrium states are defined by the KMS-condition [HHW]. Definition 2.1 can be obtained as the zero-temperature limit $T \rightarrow 0$ of the KMS-condition. The set of all ground states is denoted by $K = \{\omega \in \mathcal{A}_{+,1}^* \mid \forall A \in \mathcal{A}_{loc}; \omega(A^* \delta(A)) \geq 0\}$. K is compact and closed in the weak* topology, and is a face in $\mathcal{A}_{+,1}^*$ (Theorem 5.3.37, [BR]).

Solutions to equation (2.7) satisfying in addition a frustration-free condition, i.e., $\omega(A_v) = \omega(B_f) = 1$ for all $v \in \mathcal{V}_\mathcal{B}$ and $f \in \mathcal{F}_\mathcal{B}$, were first studied in [AFH]. They showed that for $G = \mathbb{Z}_2$ there exists a unique frustration-free ground state, coinciding with the unique translation invariant ground state. This result was extended for all G in [FN]. The notion of topological order is crucial in the proof of these results.

For the remainder of the paper, let G be a finite *abelian* group. In [N], single excitation states were constructed from the frustration-free ground state via localized endomorphisms. These states are labeled by a pair $(\chi, c) \in \hat{G} \times G$, where \hat{G} is the group of characters of G . Each label (χ, c) denotes a distinct charge, or superselection sector, of the model. We will show that the single excitation states are solutions to equation (2.7), and in fact that any pure ground state is equivalent to such a state.

In this paper we prove the set of ground states decomposes into $|G|^2$ sectors, $K^{\chi,c} \subset K$, corresponding to the superselection sectors defined in [FN]. The ground state sectors $K^{\chi,c}$ will be constructed explicitly in Section 4. Heuristically, states in $K^{\chi,c}$ are obtained by projecting ground states onto different charged sectors. Our main result is a complete characterization of the set of ground states of the abelian quantum double models, which is the content of the following theorem:

Theorem 2.2. Let $\omega \in K$ be a ground state of the quantum double model. Then there exists a convex decomposition of ω as

$$\omega = \sum_{\chi \in \hat{G}, c \in G} \lambda_{\chi,c}(\omega) \omega^{\chi,c} \quad \text{where} \quad \omega^{\chi,c} \in K^{\chi,c}. \quad (2.8)$$

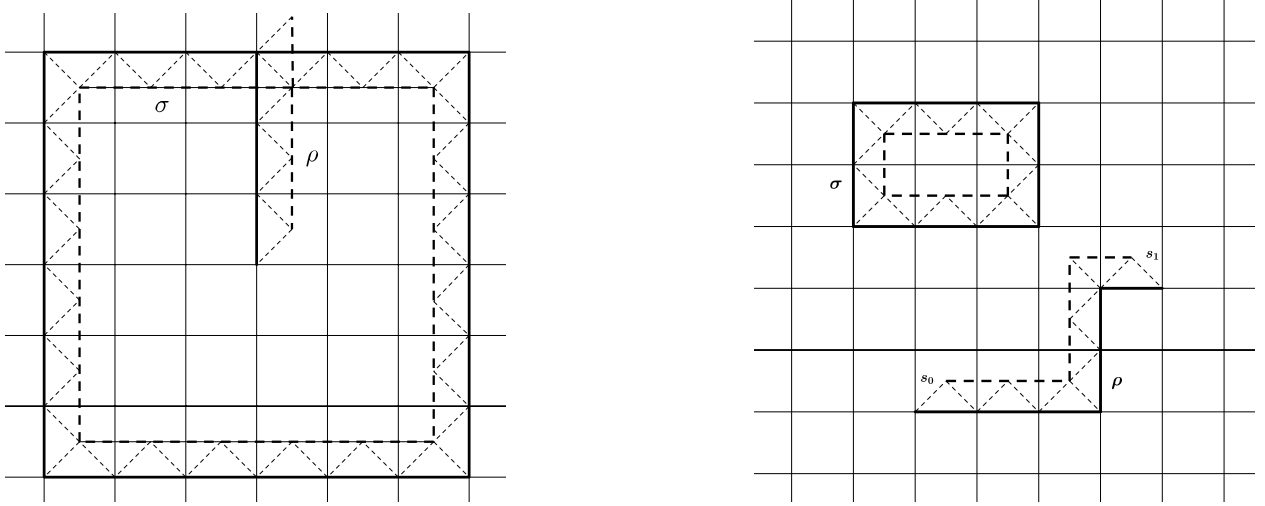


Figure 1: On the right, ρ is a typical open ribbon connecting sites (v_1, f_1) to (v_2, f_2) while σ is a closed ribbon. On the left, ρ now connects a site contained in the interior of the region to one that is not contained in the interior of the region, and intersects σ , a closed ribbon supported on the boundary.

For all $(\chi, c) \in \widehat{G} \times G$, $K^{\chi, c}$ is a face in the set of all states. In particular, if $\omega^{\chi, c} \in K^{\chi, c}$ is an extremal point of $K^{\chi, c}$ then $\omega^{\chi, c}$ is a pure state. If $\omega^{\chi, c} \in K^{\chi, c}$ is a pure state then $\omega^{\chi, c}$ is equivalent to a single excitation ground state defined in [FN].

The proof of this result is split up in different theorems, and can be found in Section 4. In particular, the statement and its proof is found as a combined result of Theorem 4.5, Corollary 4.6, and Theorem 4.7.

3 Excitations and superselection sectors

We study the charges, or superselection sectors, of the quantum double model. To this end, we study the excitations of the model, which can be obtained by using what are called *ribbon operators*. We first recall the definition of ribbon operators and review some properties. We will just state the properties that will be necessary for the proof of our results, and refer to the existing literature for proofs of these facts [BMD, FN, K]. A good understanding of these operators and how they can be used to build up the local Hilbert spaces is essential to our proof. In particular, we need to be able to detect such excitations with local projections, which we will call *charge projections*. We will prove the projections that measure the total charge in a box are supported on the boundary of the box.

After introducing the ribbon operators and charge projectors we turn to the main topic of interest: the infinite volume ground states. In this section we will introduce the unique frustration-free ground state. We will also construct ground states that are not frustration-free, namely, the single excitation states. It turns out that they can be obtained by judiciously choosing boundary terms of finite volume Hamiltonians, and taking weak* limits of finite volume ground states. As such, they can be identified with the superselection sectors of the theory, cf. [N, FN]. In the next section we will show that in fact all ground states are (quasi-)equivalent to such states.

3.1 Ribbon operators

We abuse notation and use v and f to also denote a vertex and face of the lattice \mathbb{Z}^2 . A *site* is a pair $s = (v, f)$ of a vertex $v \in \mathbb{Z}^2$ and neighboring face f . As we will see, excitations of the model are located at sites. Let the set of all sites contained in Λ_L be labeled by \mathcal{S}_L . A *ribbon* ρ is a sequence of adjacent sites connecting two sites s_0 and s_1 . We assume ribbons avoid self-crossing and label $\partial_0 \rho = s_0$ as the start of the ribbon and $\partial_1 \rho = s_1$ as its end. In particular, note that ribbons carry a direction (see [BMD] how this relates to the direction of the lattice). A ribbon is said to be *open* if $s_0 \neq s_1$ and *closed* if $s_0 = s_1$, see Figure 1.

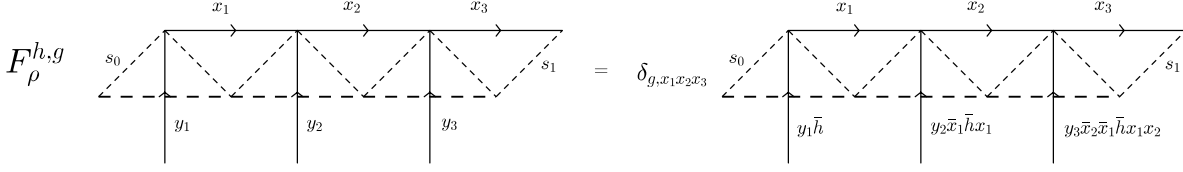


Figure 2: Definition of the ribbon operator $F_\rho^{h,g}$.

For any ribbon ρ and $g, h \in G$ the ribbon operator $F_\rho^{g,h}$ is defined as in Figure 2. The ribbon operators can also be defined recursively as concatenations of elementary triangle operators, see [BMD]. If ρ_0 and ρ_1 are two ribbons such that $\partial_1 \rho_0 = \partial_0 \rho_1$ then for the ribbon $\rho = \rho_0 \rho_1$, where the product is defined by concatenation of ribbons, the ribbon operator satisfies $F_\rho^{h,g} = \sum_{k \in G} F_{\rho_0}^{h,k} F_{\rho_1}^{g,h,k} g$. As we will see, the operators $F_\rho^{h,g}$ create excitations at the endpoints of the ribbons. However, in general $F_\rho^{h,g}$ will yield a superposition of different excitation types, and it is more convenient to choose a different basis. Essentially, what one does is to decompose the space of excitations as invariant subspaces with respect to a *local* action of the quantum double symmetry $\mathcal{D}(G)$ at each site. This symmetry is implemented by the star and plaquette operators A^g and B^h [BMD, K]. In this new basis the ribbon operators are labeled by pairs $(\chi, c) \in \hat{G} \times G$, where \hat{G} is the group of characters, and we define

$$F_\rho^{\chi,c} := \sum_{g \in G} \bar{\chi}(g) F_\rho^{\bar{c},g}. \quad (3.1)$$

If ρ consists of a single edge, then the family of ribbon operators generate the local algebra on that edge. Similarly, for any finite subset Λ the family of ribbon operators supported in Λ generate the local observable algebra \mathcal{A}_Λ .

We end this overview by listing some of the properties of the ribbon operators that we will use frequently. Most properties can be verified easily with a direct computation, but see [BMD, FN, K] for more information:

a.) For operators acting along the same ribbon:

$$F_\rho^{\chi,c} F_\rho^{\sigma,d} = F_\rho^{\chi\sigma,cd} \quad \text{and} \quad (F_\rho^{\chi,c})^* = F_\rho^{\bar{\chi},\bar{c}}. \quad (3.2)$$

b.) If $\rho \subset \Lambda_L$ is an open ribbon, then

$$A_{v_0}^k F_\rho^{\chi,c} = \bar{\chi}(k) F_\rho^{\chi,c} A_{v_0}^k, \quad A_{v_1}^k F_\rho^{\chi,c} = \chi(k) F_\rho^{\chi,c} A_{v_1}^k, \quad A_v^k F_\rho^{\chi,c} = F_\rho^{\chi,c} A_v^k \text{ if } v \neq v_0, v_1 \quad (3.3)$$

$$B_{f_0}^k F_\rho^{\chi,c} = F_\rho^{\chi,c} B_{f_0}^k, \quad B_{f_1}^k F_\rho^{\chi,c} = F_\rho^{\chi,c} B_{f_1}^k, \quad B_f^k F_\rho^{\chi,c} = F_\rho^{\chi,c} B_f^k \text{ if } f \neq f_0, f_1. \quad (3.4)$$

Here v_0 (resp. f_0) is the star (resp. plaquette) at the start of the ribbon ρ , and v_1 (f_1) those at the end, while $k \in G$.

c.) Let $\Omega \in \mathcal{G}_L$ be a ground state. We can compute the energy introduced by the ribbon operators using the relations above. If $\chi \neq \iota$ and $c \neq e$ then:

$$\text{if } \partial_i \rho \subset \Lambda, i = 0, 1 \text{ then} \quad H_L F_\rho^{\chi,e} \Omega = 2 F_\rho^{\chi,e} \Omega, \quad \text{and } H_L F_\rho^{\iota,c} \Omega = 2 F_\rho^{\iota,c} \Omega \quad (3.5)$$

$$\text{if } \partial_i \rho \subset \Lambda, \partial_{i+1} \rho \subsetneq \Lambda \text{ then} \quad H_L F_\rho^{\chi,e} \Omega = F_\rho^{\chi,e} \Omega, \quad \text{and } H_L F_\rho^{\iota,c} \Omega = F_\rho^{\iota,c} \Omega \quad (3.6)$$

$$\text{if } \partial_i \rho \subsetneq \Lambda, i = 0, 1 \text{ then} \quad H_L F_\rho^{\chi,e} \Omega = 0, \quad \text{and } H_L F_\rho^{\iota,c} \Omega = 0. \quad (3.7)$$

Thus, $\rho \subset \Lambda_L$ generates *excitations* at its endpoints.

d.) If ρ is a closed ribbon, then for all $\Omega \in \mathcal{G}_L$

$$F_\rho^{\chi,c} \Omega = \Omega \quad \forall (\chi, c) \in \hat{G} \times G \quad (3.8)$$

and

$$[F_\rho^{\chi,c}, A_v^g] = [F_\rho^{\chi,c}, B_f^c] = 0 \quad \forall v \in \mathcal{V}_B, f \in \mathcal{F}_B. \quad (3.9)$$

e.) If $\rho = \rho_0 \rho_1$, that is, $\partial_1 \rho_0 = \partial_0 \rho_1$. Then, the ribbon operators obey

$$F_\rho^{\chi,c} = F_{\rho_0}^{\chi,c} F_{\rho_1}^{\chi,c}. \quad (3.10)$$

f.) A complete set of eigenvectors of H_L for \mathcal{H}_L is,

$$\left\{ \prod_i F_{\rho_i}^{\chi_i, c_i} \Omega : \forall \Omega \in \mathcal{G}_L, (\chi_i, c_i) \in \widehat{G} \times G, \rho_i \text{ a ribbon} \right\}. \quad (3.11)$$

g.) When two ribbons intersect (as in left Figure 1),

$$F_\rho^{\chi,c} F_\sigma^{\xi,d} = \chi(d) \xi(c) F_\sigma^{\sigma,d} F_\rho^{\xi,c}. \quad (3.12)$$

h.) Ribbon operators satisfy *path independence* in the ground state, that is, if ρ and σ are ribbons such that $\partial_i \rho = \partial_i \sigma$ for $i = 0, 1$ then

$$F_\rho^{\chi,c} \Omega = F_\sigma^{\chi,c} \Omega \quad \text{for all} \quad \Omega \in \mathcal{G}_L, \text{ and } (\chi, c) \in \widehat{G} \times G. \quad (3.13)$$

3.2 Local and global charge projectors

We can detect the presence of an excitation and the charge type localized at a site $s = (v, f)$ with the orthogonal projectors

$$D_v^\chi := \frac{1}{|G|} \sum_{g \in G} \bar{\chi}(g) A_v^g \quad \text{for } \chi \in \widehat{G}, \quad (3.14)$$

$$D_f^c := B_f^c \quad \text{for } c \in G. \quad (3.15)$$

These can be obtained by considering the action of the quantum double at each site [BMD]. The first detects “electric” charges labeled by the characters of G , while the latter project on the “magnetic” charges labeled by group elements. The electric charges are located on the vertices, while the magnetic charges are located on faces. Since we only consider abelian models, these two types of excitations can be treated separately. One can check that, in the case of abelian groups G , the projectors commute. Thus there is no ambiguity in defining the operator $D_s^{\chi,c} := D_v^\chi D_f^c$ for the site $s = (v, f)$.

The local charge projectors have the following properties. They follow readily using the properties listed in the previous section.

$$D_v^\chi \Omega = \delta_{\chi, \iota} \Omega, \quad D_f^c \Omega = \delta_{c, e} \Omega \quad \text{for all } \Omega \in \mathcal{G}_L \quad (3.16)$$

$$D_v^\chi F_\rho^{\sigma,d} = F_\rho^{\sigma,d} D_v^{\chi\bar{\sigma}}, \quad D_f^c F_\rho^{\sigma,d} = F_\rho^{\sigma,d} D_f^{c\bar{d}} \quad \text{if } \partial_0 \rho = (v, f) \neq \partial_1 \rho \quad (3.17)$$

$$D_v^\chi F_\rho^{\sigma,d} = F_\rho^{\sigma,d} D_v^{\chi\bar{\sigma}}, \quad D_f^c F_\rho^{\sigma,d} = F_\rho^{\sigma,d} D_f^{c\bar{d}} \quad \text{if } \partial_1 \rho = (v, f) \neq \partial_0 \rho \quad (3.18)$$

$$D_v^\chi D_v^\sigma = \delta_{\chi, \sigma} D_s^\chi, \quad D_f^c D_f^d = \delta_{c, d} D_f^c \quad (3.19)$$

$$\sum_{\chi \in \widehat{G}} D_v^\chi = I, \quad \sum_{c \in G} D_f^c = I \quad \implies \quad \sum_{\chi \in \widehat{G}, c \in G} D_s^{\chi,c} = I. \quad (3.20)$$

Thus the projections onto the charges at a site s form a complete set of orthogonal projections, by equation (3.20). The ground state projector onto the ground state subspace \mathcal{G}_L is a product of all local charge projectors with trivial charge, $\prod_{s \in \mathcal{S}_L} D_s^{\iota, e}$.

A *global* (or *total*) *charge projector* selects the total charge (χ, c) in the region Λ_L . Heuristically, if for each face $f \in \mathcal{F}_L$ there is a local charge c_f , then the total charge of magnetic type is $\prod_{f \in \mathcal{F}_L} c_f = c$ (again, we use that our group is abelian). For example, if magnetic charges c and \bar{c} are located on two faces and all other faces carry trivial charge, then the total charge in the region is $c\bar{c} = e$. Thus, for a charge c the *conjugate charge* is given by the inverse group element \bar{c} . Here “conjugate charge” is standard terminology: it means that you can combine the two charges to obtain a trivial charge. It has nothing to do with conjugation in the group.

Note that the example above in particular shows that trivial global charge does not mean that there are no excitations in the region Λ_L . Rather, it means that all charges add up to the trivial charge (similarly,

the total charge of a state with an electron and a positron is trivial). An open ribbon operator with both endpoints in Λ generates a charge c (see equation (3.17)), and a conjugate charge \bar{c} (from equation (3.18)) pair at each of its endpoints. Thus, charge is created locally at its endpoints but the total charge of the initial state is preserved. However, an open ribbon operator with only one of its endpoints in the region Λ_L does *not* conserve the global charge in the region. The same is true for charges of electric type, where the multiplication is in the dual group \widehat{G} .

We define the global charge projectors by

$$D_L^\chi := \sum_{\prod_v \chi_v = \chi} \prod_{v \in \mathcal{V}_L} D_v^{\chi_v}, \quad D_L^c := \sum_{\prod_f c_f = c} \prod_{f \in \mathcal{F}_L} D_f^{c_f}, \quad (3.21)$$

where the sums are over all configurations $\{\chi_v\}_{v \in \mathcal{V}_L}$ such that $\prod_v \chi_v = \chi$ and configurations $\{c_f\}_{f \in \mathcal{F}_L}$ such that $\prod_f c_f = c$, respectively.

To project onto (non-trivial) electric ϵ or magnetic μ charge types we have the projectors, respectively,

$$D_L^\epsilon := \sum_{\chi \neq \iota} D_L^\chi = I - D_L^\iota, \quad (3.22)$$

$$D_L^\mu := \sum_{c \neq e} D_L^c = I - D_L^e. \quad (3.23)$$

From the definitions above it appears that the global charge projectors are supported on the entire region Λ_L . We will show that they are actually boundary operators.

To this end, we consider closed ribbon operators encircling the boundary of Λ_L given by the operators

$$V_L^\epsilon := \frac{1}{|G|} \sum_{c \in G} (I - F_{\partial L}^{c, \bar{c}}), \quad V_L^\mu := \frac{1}{|G|} \sum_{\chi \in \widehat{G}} (I - F_{\partial L}^{\chi, \bar{\chi}}), \quad (3.24)$$

where we use ∂L to denote the closed ribbon running along the boundary of Λ_L . Physically this can be thought of as an interferometry experiment [BSS]: we create a conjugate pair of charges from the ground state, and take one around the region. Due to the anyonic nature of the charges, if there is a charge present in the region Λ_L , this is a non-trivial operation for at least one of the charges. This in principle can be detected and used to detect the total charge in Λ_L .

We first show that V_L^ϵ and V_L^μ are in fact projections (cf. [BMD, Sect. B.9]).

Proposition 3.1. *The operators V_L^ϵ and V_L^μ are orthogonal projections.*

Proof. From equation (3.2), we have that

$$\sum_{\chi, \chi'} F_{\partial L}^{\chi, e} F_{\partial L}^{\chi', e} = \sum_{\chi, \chi'} F_{\partial L}^{\chi \chi', e} = |G| \sum_{\chi} F_{\partial L}^{\chi, e}.$$

Thus,

$$\begin{aligned} (V_L^\epsilon)^2 &= \frac{1}{|G|^2} \sum_{\chi, \chi'} \left(I - F_{\partial L}^{\chi, e} - F_{\partial L}^{\chi', e} + F_{\partial L}^{\chi, e} F_{\partial L}^{\chi', e} \right) \\ &= \frac{1}{|G|} \sum_{\chi} (I - F_{\partial L}^{\chi, e}) = V_L^\epsilon. \end{aligned}$$

Also with equation (3.2) we find

$$\begin{aligned} (V_L^\epsilon)^* &= \frac{1}{|G|} \sum_{\chi} I - F_{\partial L}^{\chi, e*} \\ &= \frac{1}{|G|} \sum_{\chi} I - F_{\partial L}^{\bar{\chi}, e} = V_L^\epsilon. \end{aligned}$$

The calculation to show that $V_L^\mu = (V_L^\mu)^2 = (V_L^\mu)^*$ is similar. \square

We can now prove our claim that the operators D_L^ϵ and D_L^μ are supported on the boundary of Λ_L , by showing that they are equal to the projections in the proposition above. We do this by showing that equality holds on a spanning set of vectors.

Lemma 3.2. *The global charge projectors D_L^ϵ and D_L^μ , defined above, are supported on the boundary of the region Λ_L . In particular, they are equal to the boundary ribbon operators on \mathcal{H}_L , defined in equations (3.24):*

$$D_L^\epsilon = V_L^\epsilon \in \mathcal{A}_{\Lambda_L \setminus \Lambda_{L-1}}, \quad D_L^\mu = V_L^\mu \in \mathcal{A}_{\Lambda_L \setminus \Lambda_{L-1}}. \quad (3.25)$$

Proof. Let $\Omega \in \mathcal{G}_L$. From equation (3.8), $F_{\partial L}^{\chi,c} \Omega = \Omega$ for all χ, c . Thus, $V_L^\epsilon \Omega = V_L^\mu \Omega = 0$. On the other hand, from equation (3.16), $D_L^\epsilon \Omega = \sum_{\prod_s \chi_s \neq \iota} \prod_v D_v^{\chi_v} \Omega = \sum_{\prod_v \chi_v \neq \iota} \prod_v \delta_{\chi_v, \iota} \Omega = 0$, where each term in the sum vanishes because the requirement $\prod_v \chi_v \neq \iota$ forces a non-trivial charge to exist at at least one star. Similarly, $D_L^\mu \Omega = 0$. Hence the operators agree on the subspace \mathcal{G}_L .

Let ρ be a ribbon and suppose ρ has both endpoints contained in Λ_L . Label its endpoints as $\partial_0 \rho = (v_0, f_0)$ and $\partial_1 \rho = (v_1, f_1)$. By path independence in the ground state, equation (3.13), we can assume without loss of generality that ρ does not intersect the boundary ribbon, implying $V_L^\epsilon F_\rho^{\chi,c} \Omega = F_\rho^{\chi,c} V_L^\epsilon \Omega$. Also,

$$D_L^\epsilon F_\rho^{\chi,c} = \sum_{\prod_v \chi_v \neq \iota} \prod_{v \in \mathcal{V}_L} D_v^{\chi_v} F_\rho^{\chi,c} \quad (3.26)$$

$$= F_\rho^{\chi,c} \sum_{\prod_v \chi_v \neq \iota} D_{v_0}^{\chi_{v_0} \bar{\chi}} D_{v_1}^{\chi_{v_1} \chi} \prod_{\substack{v \in \mathcal{V}_L \\ v \neq v_0, v_1}} D_v^{\chi_v} \quad (3.27)$$

$$= F_\rho^{\chi,c} D_L^\epsilon. \quad (3.28)$$

The third equality comes from the fact that introducing a charge conjugate-charge pair does not add any total charge to the system. In other words, the effect is just a relabeling in the summation. Thus, if ρ has both endpoints contained in Λ_L , $F_\rho^{\chi,c}$ commutes with both D_L^ϵ and V_L^ϵ .

Now suppose ρ has both endpoints contained on the boundary. Since the endpoints of ρ are not contained in Λ_L , that is, $\partial_i \rho \not\subseteq \Lambda_L$ the corresponding vertex and face projectors will not be involved in the products defining D_L^ϵ . It follows from (3.3) that $D_L^\epsilon F_\rho^{\chi,c} = F_\rho^{\chi,c} D_L^\epsilon$. On the other hand, from the ribbon crossing relations (3.12) we can compute

$$V_L^\epsilon F_\rho^{\chi,c} = \left(\frac{1}{|G|} \sum_{g \in G} I - F_{\partial L}^{\iota, g} \right) F_\rho^{\chi,c} \quad (3.29)$$

$$= F_\rho^{\chi,c} \left(\frac{1}{|G|} \sum_{\chi \neq \iota} I - \chi(g) \chi(\bar{g}) F_{\partial L}^{\iota, g} \right) \quad (3.30)$$

$$= F_\rho^{\chi,c} V_L^\epsilon. \quad (3.31)$$

Finally, we consider the action of V_L^ϵ and D_L^ϵ on the spanning set of vectors of the form $\prod_i F_{\rho_i}^{\chi_i, c_i} \Omega$. From the above arguments, without loss of generality, we can consider the product of ribbon operators for ribbons which have one endpoint on the boundary and one in the interior of Λ , since the other path operators commute with both V_L^ϵ and D_L^ϵ . We also assume $\partial_0 \rho_i \subset \Lambda_L$. This can always be achieved by reversing the direction of the ribbon (called “inversion” in [BMD]). By concatenation of ribbons (3.10), without loss of generality we can assume that all endpoints are distinct, that is, $\partial_i \rho_k \neq \partial_j \rho_l$ for all $i, j = 0, 1$ and $k, l = 0, 1, \dots, n$. Let $\{(v_i, f_i)\}_{i=1}^n$ enumerate the endpoints of ρ_i contained in the interior of Λ_L .

Applying the ribbon operator relations (3.12),

$$V_L^\epsilon \left(\prod_{i=1}^n F_{\rho_i}^{\chi_i, c_i} \right) \Omega = \frac{1}{|G|} \left(\sum_{g \in G} I - F_{\partial L}^{\iota, g} \right) \left(\prod_{i=1}^n F_{\rho_i}^{\chi_i, c_i} \right) \Omega \quad (3.32)$$

$$= \frac{1}{|G|} \left(\prod_{i=1}^n F_{\rho_i}^{\chi_i, c_i} \right) \left(\sum_{g \in G} I - \prod_{i=1}^n \chi_i(g) F_{\partial L}^{\iota, g} \right) \Omega \quad (3.33)$$

$$= \left(\prod_{i=1}^n F_{\rho_i}^{\chi_i, c_i} \right) (I - \delta_{\prod_{i=1}^n \chi_i, \iota}) \Omega, \quad (3.34)$$

where we use orthogonality of characters, $\frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^n \chi_i(g) = \delta_{\prod_{i=1}^n \chi_i, \iota}$. On the other hand it follows that

$$D_L^\epsilon \left(\prod_{i=1}^n F_{\rho_i}^{\chi_i, c_i} \right) \Omega = \left(\sum_{\prod_v \chi_v \neq \iota} \prod_v D_v^{\chi_v} \right) \left(\prod_{i=1}^n F_{\rho_i}^{\chi_i, c_i} \right) \Omega \quad (3.35)$$

$$= \left(\prod_{i=1}^n F_{\rho_i}^{\chi_i, c_i} \right) \left(\sum_{\prod_v \chi_v \neq \iota} \prod_{i=1}^n D_{v_i}^{\chi_i \chi_{v_i}} \prod_{v \neq v_i} D_v^{\chi_v} \right) \Omega \quad (3.36)$$

$$= \left(\prod_{i=1}^n F_{\rho_i}^{\chi_i, c_i} \right) (I - \delta_{\prod_{i=1}^n \chi_i, \iota}) \Omega, \quad (3.37)$$

where for the last equality we apply an extension of (3.17).

We have shown $D_L^\epsilon \left(\prod_{i=1}^n F_{\rho_i}^{\chi_i, c_i} \Omega \right) = V_L^\epsilon \left(\prod_{i=1}^n F_{\rho_i}^{\chi_i, c_i} \Omega \right)$ for any arbitrary spanning vector of \mathcal{H}_L . Therefore, $D_L^\epsilon = V_L^\epsilon$ as operators on \mathcal{H}_L . A similar argument gives $D_L^\mu = V_L^\mu$ as operators on \mathcal{H}_L . \square

The following lemma states that localized ribbon operators do not change the total charge. In other words, local operation cannot change the charged sector.

Lemma 3.3. *Let $L' > L$. Then, $[D_{L'}^\chi, A] = [D_{L'}^c, A] = 0$ for all $\chi \in \widehat{G}, c \in G$ and $A \in \mathcal{A}_L$.*

Proof. Suppose $A = F_\rho^{\sigma, d}$ is a ribbon operator. If ρ is a closed ribbon, then by equation (3.9), $[D_{L'}^\chi, F_\rho^{\sigma, d}] = [D_{L'}^c, F_\rho^{\sigma, d}] = 0$.

If ρ is an open ribbon let $\partial_i \rho = s_i = (v_i, f_i)$ for $i = 0, 1$. Since, $L' > L$ it follows that $v_i \in \mathcal{V}_{L'}$ and $s_i \in \mathcal{S}_{L'}$ for $i = 0, 1$. Thus,

$$D_{L'}^\chi F_\rho^{\sigma, d} = \sum_{\prod_v \chi_v = \chi} \prod_{v \in \mathcal{V}_{L'}} D_v^{\chi_v} F_\rho^{\chi, c} \quad (3.38)$$

$$= F_\rho^{\chi, c} \sum_{\prod_v \chi_v = \chi} \prod_{\substack{v \in \mathcal{V}_{L'} \\ v \neq v_0, v_1}} D_v^{\chi_v} D_{s_0}^{\chi_{v_0} \bar{\sigma}} D_{s_1}^{\chi_{v_1} \sigma} \quad (3.39)$$

$$= F_\rho^{\sigma, d} D_{L'}^\chi. \quad (3.40)$$

The lemma follows from the fact that ribbon operators generate \mathcal{A}_L .

A similar calculation holds to show $[D_{L'}^c, A] = 0$. \square

Before we continue let us briefly summarize the physical interpretation of the properties of ribbon operators and charge projectors. The ribbon operators create a pair of excitations, one at each end of the ribbon. In addition, the different types of excitations are labeled by pairs $(\chi, c) \in \widehat{G} \times G$ and the charges at each end are *conjugate* to each other. In other words, the *global* charge does not change after applying a ribbon operator. Moreover, when acting with a ribbon operator on the ground state, the resulting state only depends on the endpoints of the ribbon, i.e. the location of the excitations. A similar thing is true for configurations of multiple charges, up to a phase. This phase can be explained by the anyonic nature of the charges: exchanging two of them gives an overall phase, much like interchanging two fermions yield a minus sign. The local Hilbert spaces can be obtained completely by such operations, so that we can define a basis by specifying the charge at each site (with an additional constraint on the total charge). This observation will play an important role in our proof.

3.3 Infinite volume ground states

Let $\Omega_L \in \mathcal{G}_L$ be a sequence of finite volume ground states for the regions Λ_L . A frustration-free ground state of the quantum double model ω^0 , can be constructed by taking a converging subsequence of the family of states $\{\omega_L\}_{L=2}^\infty$ as $L \rightarrow \infty$, where ω_L is an arbitrary extension of the state $\langle \Omega_L, \cdot \rangle$ to the quasi-local algebra \mathcal{A} (in particular, we could choose a product state). For any v and f choose L large enough such that $v, f \subset \Lambda_L$. Since ω_L is a ground state for the finite model it follows that $\omega^0(I - A_v) = \omega_L(I - A_v) = 0$ and $\omega^0(I - B_f) = \omega_L(I - B_f) = 0$. This is the frustration-freeness property.

Ground states of the quantum double model are locally indistinguishable, so weak* limits of sequences of such states will always converge to the same state. It turns out that this is the only frustration-free ground state (that is, satisfying equation (2.7)) of the model. We list some properties of this state in the following proposition:

Proposition 3.4. *([AFH, N, FN]) Let ω^0 be the frustration-free ground state of the quantum double model obtained as a weak* limit of finite volume ground states. Then,*

- (i) *if ω is a frustration-free ground state then $\omega = \omega^0$,*
- (ii) *ω^0 is a pure state,*
- (iii) *Let $(\pi_0, \Omega_0, \mathcal{H}_0)$ be a GNS-representation for ω^0 and H_0 be the Hamiltonian in this GNS representation. Then, $\text{spec}(H_0) = \mathbb{Z}^{\geq 0}$ with a simple ground state eigenvector Ω_0 .*

The first property is in fact a general property for frustration-free models satisfying a local indistinguishability condition, (for the definition see [BHM]). The frustration-free ground state of the quantum double model satisfies the local indistinguishability condition [FN]. The last property follows from an application of strong resolvent convergence (see for example Theorem VIII.24 of [RS]) and shows that the quantum double model has a spectral gap in its frustration-free ground state. As a consequence of local indistinguishability, the gap is stable under local perturbations [BHM, MP].

3.3.1 Single excitation ground states

In the finite volume, elementary excitations are constructed by violating one of the frustration-freeness ground state conditions. These excitations must come in pairs since they are generated by open ribbon operators. By introducing a boundary condition to the Hamiltonian, we generate ground states which have one excitation in the bulk and one on the boundary. In the infinite volume, this is equivalent to moving one of the excitations off to infinity, thereby isolating a single excitation in the bulk. By construction, these states will be ground states in the infinite volume. One way to understand this intuitively is that even though there is an excitation, we cannot lower the energy of the system with local operations. It is possible to move the excitation around (thereby locally decreasing the energy), but we cannot get rid of it completely with local operations, and the moved excitation will increase the local energy density at its new location.

The idea is to use the projections that were introduced in the previous section to define boundary conditions which can compensate for the existence of an excitation in the bulk. Recall that these projections are supported on the boundary of Λ_L .

Definition 3.5. Define the following Hamiltonians with boundary condition

$$H_L^\epsilon := H_L - V_L^\epsilon, \quad (3.41)$$

$$H_L^\mu := H_L - V_L^\mu, \quad (3.42)$$

$$H_L^{\epsilon, \mu} := H_L - V_L^\epsilon - V_L^\mu. \quad (3.43)$$

We will sometimes use the index k to denote either ϵ, μ , or (ϵ, μ) , and set $V_L^{\epsilon, \mu} := V_L^\epsilon + V_L^\mu$.

Recall that the boundary terms V_L^k are linear combination of closed ribbon operators and thus commute with each interaction term, and hence the Hamiltonian:

$$[V_L^k, B_f] = [V_L^k, A_v] = [V_L^k, H_L] = 0 \quad \text{for all } k, f, v. \quad (3.44)$$

The ground states of the Hamiltonian with boundary conditions include the ground states of the model with open boundary conditions. To see this, let $\Omega_L \in \mathcal{G}_L$ be a ground state of H_L . From equation (3.8), $H_L^k \Omega_L = 0$ for all k . Thus, $\mathcal{G}_L \subset \mathcal{G}_L^k$ for all k .

Now consider an open ribbon ρ connecting a site $\partial_0 \rho = (v, f)$ in the interior of Λ_L to a site on the boundary, for instance ρ as the ribbon in Figure 1, and its corresponding ribbon operator, $F_\rho^{\chi, c} \subset \mathcal{A}_L$. Then, for (χ, c) we have that

$$\begin{aligned} H_L F_\rho^{\chi, c} \Omega_L &= F_\rho^{\chi, c} \left(I - \frac{1}{|G|} \sum_{g \in G} \bar{\chi}(g) A_v^g + I - B_f^{\bar{c}} \right) \Omega_L \\ &= F_\rho^{\chi, c} (2 - \delta_{\bar{\chi}, \iota} - \delta_{\bar{c}, e}) \Omega_L. \end{aligned}$$

In the last line we used orthogonality of characters: $\langle \chi_1, \chi_2 \rangle := \frac{1}{|G|} \sum_{g \in G} \bar{\chi}_1(g) \chi_2(g) = \delta_{\chi_1, \chi_2}$. Similar calculations as in the proof of Lemma 3.2 yield:

$$\begin{aligned} V_L^\epsilon F_\rho^{\chi, c} \Omega_L &= F_\rho^{\chi, c} \left[\frac{1}{|G|} \sum_{c' \in G} \left(I - \chi(c') F_{\partial L}^{\epsilon, c'} \right) \right] \Omega_L \\ &= F_\rho^{\chi, c} (I - \delta_{\chi, l}) \Omega_L, \end{aligned}$$

and for the magnetic charges,

$$\begin{aligned} V_L^\mu F_\rho^{\chi, c} \Omega_L &= F_\rho^{\chi, c} \left[\frac{1}{|G|} \sum_{\chi' \in \widehat{G}} \left(I - \chi'(c) F_{\partial L}^{\chi', e} \right) \right] \Omega_L \\ &= F_\rho^{\chi, c} (I - \delta_{c, e}) \Omega_L. \end{aligned}$$

Therefore, together with equation (3.6) we find

$$H_L^{\epsilon, \mu} F_\rho^{\chi, c} \Omega_L = 0. \quad (3.45)$$

If ρ connects sites in the interior of Λ_L then $[V_L^k, F_\rho^{\chi, c}] = 0$. Thus, combined with a similar calculation from above for multiple charges on the boundary, we can conclude that

$$H_L^{\epsilon, \mu} \left(\prod_i F_{\rho_i}^{\chi_i, c_i} \right) \Omega_L \geq 0.$$

Therefore we can conclude:

Lemma 3.6. *Let H_L^k and \mathcal{G}_L^k be given as above for $k = \epsilon, \mu$ and (ϵ, μ) . Then,*

- (i) $H_L^{\epsilon, \mu} \geq 0$ and $H_L^k \mathcal{G}_L^k = 0$,
- (ii) \mathcal{G}_L^ϵ is spanned by $\{F_\rho^{\chi, e} \Omega_L : \rho \text{ connects an interior site to the boundary, } \Omega_L \in \mathcal{G}_L, \chi \in \widehat{G}\}$.
 \mathcal{G}_L^μ is spanned by $\{F_\sigma^{\epsilon, c} \Omega_L : \sigma \text{ connects an interior site to the boundary, } \Omega_L \in \mathcal{G}_L, c \in G\}$.
 $\mathcal{G}_L^{\epsilon, \mu}$ is spanned by $\{F_\rho^{\chi, e} F_\sigma^{\epsilon, c} \Omega_L : \rho, \sigma \text{ connects interior sites to the boundary, } \Omega_L \in \mathcal{G}_L, (\chi, c) \in \widehat{G} \times G\}$.

From the decompositions given in Lemma 3.6, it is clear that

$$\text{if } L' > L \text{ then } H_{L'}^{\epsilon, \mu} (\mathcal{G}_{L'}^{\epsilon, \mu}) = 0. \quad (3.46)$$

Note that \mathcal{G}^μ and \mathcal{G}_L^ϵ are subspaces of $\mathcal{G}_L^{\epsilon, \mu}$. This result allows us to decompose the ground state space into different sectors corresponding to the different charges:

Corollary 3.7. *The ground state space has a natural decomposition*

$$\mathcal{G}_L^{\epsilon, \mu} = \bigoplus_{\chi \in \widehat{G}, c \in G} D_L^{\chi, c} \mathcal{G}_L^{\epsilon, \mu} \quad (3.47)$$

Proof. This follows from Lemma 3.6 and the relation (3.20). \square

We now come to states in the thermodynamic limit that describe a single excitation. Such states may be constructed on the quasi-local algebra by moving one of the excitations in a pair off to infinity. Let ρ be a ribbon extending to infinity such that $\partial_0 \rho = s$ and $\partial_1 \rho = \infty$, where $\partial_1 \rho = \infty$ means that the ribbon goes to infinity in any direction. We assume that it does not “loop back”, in the sense that if ρ_n is the ribbon consists of the first n parts of ρ , then for any fixed point in the lattice, the distance to the endpoint of ρ_n that is not fixed goes to infinity as n goes to infinity.

We denote $\rho_L = \rho \cap \Lambda_L$. Define the state $\omega_s^{\chi, c}$ on \mathcal{A}_{loc} , and its unique continuous extension to \mathcal{A} , by

$$\omega_s^{\chi, c}(A) := \lim_{L \rightarrow \infty} \langle F_{\rho_L}^{\chi, c} \Omega_L, A F_{\rho_L}^{\chi, c} \Omega_L \rangle. \quad (3.48)$$

The limit converges because the sequence is eventually constant for fixed local A , by locality and local indistinguishability. By path independence in the ground state, the state $\omega_s^{X,c}$ is independent of the path ρ takes to infinity and depends only on the basepoint s . Indeed, $\omega_s^{X,c}$ is an infinite volume ground state since for all $A \in \mathcal{A}_{loc}$ we have that $\delta(A) = \lim_{L \rightarrow \infty} [H_L, A] = \lim_{L \rightarrow \infty} [H_L^{\epsilon, \mu}, A]$, and from Lemma 3.6, $\omega_s^{X,c}(H_L^{\epsilon, \mu}) = 0$. Since $H_L^{\epsilon, \mu}$ is a positive operator, the result follows.

The states $\omega_s^{X,c}$ were first introduced in [N, FN]. They showed the states can be constructed from the frustration-free ground state via an automorphism, $\omega_s^{X,c} = \omega^0 \circ \alpha_{\rho}^{X,c}$, where

$$\alpha_{\rho}^{X,c}(A) = \lim_{L \rightarrow \infty} F_{\rho L}^{X,c*} A F_{\rho L}^{X,c}. \quad (3.49)$$

The limit converges in norm for each $A \in \mathcal{A}$ and defines an outer automorphism.

3.4 Superselection criterion

Superselection sectors arise because the quasi-local algebra has many inequivalent representations. Most representations do not have any physical relevance (for example, because the energy is unbounded), so it is important to restrict the class of representations of interest. For example, a theory may have different, inequivalent particle types, like the excitations in the quantum double. Another example would be electric charge. Here we will use the term “charge” in a generalized sense, as a label of the different particle types. Once we can identify different classes of representations with charges, it is reasonable to impose additional constraints. In particular, we can impose certain locality conditions, and demand that we are able to move the localization regions around. A superselection criterion is a rule that tells us precisely which representations we select in the end.

The Doplicher-Haag-Roberts (DHR) analysis in algebraic quantum field theory showed that starting from a vacuum state and a physically motivated superselection criterion, one could recover a family of superselection sectors corresponding to the global gauge group [DHR71, DHR72]. This allows one to recover all physically relevant properties of the charges, such as their particle statistics. A similar analysis can be done for the quantum double models, producing the single excitations ground states as the irreducible objects in each superselection sector [N, FN]. The role of the vacuum is played by the translation invariant frustration-free ground state.

For the quantum double models the relevant criterion is as follows. Let $\Lambda \subset \mathcal{B}$ be an infinite cone region (the precise shape is not that important). We consider representations π which satisfy the following criterion for *any* such Λ :

$$\pi_0 \upharpoonright \mathcal{A}_{\Lambda^c} \cong \pi \upharpoonright \mathcal{A}_{\Lambda^c}. \quad (3.50)$$

Here π_0 is the GNS representation of the frustration-free ground state and $\pi_0 \upharpoonright \mathcal{A}_{\Lambda^c}$ means that we restrict the representation to \mathcal{A}_{Λ^c} , the C^* -algebra generated by all local observables supported outside Λ . Physically, to detect the charge of a state in the representation π one needs to measure the value of a “Wilson loop”. If such loops around the charge are not allowed (as in the selection criterion, due to the absence of the cone), the charge cannot be detected.

The superselection structure of the quantum double model can be analyzed in the same spirit as the DHR program. The sector structure is summarized in the following proposition

Proposition 3.8. *[N, FN] Let $(\pi_s^{X,c}, \Omega_s^{X,c}, \mathcal{H}_s^{X,c})$ be the GNS triple for $\omega_s^{X,c}$. Then,*

- (i) $\pi_s^{X,c}$ are irreducible representations satisfying the criterion (3.50),
- (ii) $\pi_s^{X,c} \cong \pi_{s'}^{X,c}$,
- (iii) if $(\chi, c) \neq (\chi', c')$ then $\pi_s^{X,c}$ and $\pi_s^{X',c'}$ belong to different superselection sectors (and hence are inequivalent),
- (iv) if π is irreducible and satisfies (3.50) then there exists ρ and (χ, c) such that $\pi \cong \pi_s^{X,c}$.

Pushing this analysis further, all properties of the charges such as their fusion and braiding rules can be recovered [FN]. It follows that the structure is completely described by the representation theory of the quantum double, $\text{Rep}(\mathcal{D}(G))$. It is interesting to see that the charge superselection structure is closely related to the classification of ground states of the quantum double, as will become even clearer in the next section.

4 The complete set of ground states

In this section we prove our main result: a complete classification of the ground states of the quantum double model for abelian groups. Our strategy is to find a boundary condition such that any infinite volume ground state has energy zero on the Hamiltonian with this boundary condition. It turns out that this is possible with the boundary conditions introduced in Section 3.2. The classification of infinite volume ground states then simplifies to a classification of infinite volume limits of finite volume ground states. These finite volume ground states are well understood by the results in the previous section, and this allows us to obtain our classification. This strategy is similar to the solution of the complete ground state for the XXZ chain given in [KN].

We begin by introducing some notation.

Definition 4.1. Let $K := \{\omega \in \mathcal{A}_{+,1}^* : \omega(A^*\delta(A)) \geq 0\}$ denote the set of infinite volume ground states, where δ is the generator of the dynamics for the quantum double model. Similarly, for the set of all finite volume ground state functionals of $H_L^{\epsilon,\mu}$ we write $K_L := \{\omega_L : \mathcal{A}_L \rightarrow \mathbb{C} \mid \omega_L(H_L^{\epsilon,\mu}) = 0\}$.

The first step is to show that any infinite volume ground state minimizes the energy of the finite volume Hamiltonians $H_L^{\epsilon,\mu}$ of Definition 3.5. Here we will use the formulation of the boundary term in terms of a sum of products of local charge projections. This gives us precise control on the location of possible excitations.

Lemma 4.2. *Let $\omega \in K$ be a ground state of the quantum double model for abelian group G . Then,*

$$\omega(H_L - D_L^\epsilon - D_L^\mu) = 0, \quad (4.1)$$

for all $L \geq 2$.

Proof. We will show that $\omega(\sum_{v \in \mathcal{V}_L} (I - A_v)) = \omega(V_L^\epsilon)$ and $\omega(\sum_{f \in \mathcal{F}_L} (I - B_f)) = \omega(V_L^\mu)$. The result then follows from Lemma 3.2.

Let $L \geq 2$ be given. Consider an arbitrary enumeration of the set of plaquettes, $\mathcal{F}_L = \{f_i\}_{i=1}^{n_L}$, and a configuration of magnetic charges, $\{c_i \in G\}_{i=1}^{n_L}$ such that $\prod_i c_i = e$. In the following, sums and products indexed by i, j and k will run from 1 to n_L unless otherwise stated. Pair f_i with a neighboring vertex v_i and let $\rho_i \subset \Lambda_L$ be a ribbon such that $\partial_0 \rho_i = (v_i, f_i) \subset \Lambda_L$ and $\partial_1 \rho_i = (v_{i+1}, f_{i+1})$. With this choice, consider the operator $A = (\prod_i F_{\rho_i}^{\iota, c_i})(\prod_i (I - B_{f_i}))$, where the family $\{\tilde{c}_i\}$ is chosen such that

$$B_{f_i} \left(\prod_k F_{\rho_f}^{\iota, \tilde{c}_k} \right) = \left(\prod_k F_{\rho_f}^{\iota, \tilde{c}_k} \right) B_{f_i}^{c_i}, \quad \forall i. \quad (4.2)$$

The condition that $\prod_i c_i = e$ guarantees that such a family $\{\tilde{c}_i\}_i$ always exists.

We want to apply the ground state condition to the operator A , hence we compute

$$\begin{aligned} A^*\delta(A) &= A^*[H_L, A] \\ &= \prod_i (I - B_{f_i}) \left(\prod_i F_{\rho_i}^{\iota, \tilde{c}_i} \right)^* \left(\sum_j \left[-B_{f_j}, \prod_i F_{\rho_i}^{\iota, \tilde{c}_i} \right] \right) \prod_i (I - B_{f_i}) \\ &= \prod_i (I - B_{f_i}) \left(\prod_i F_{\rho_i}^{\iota, \tilde{c}_i} \right)^* \prod_i F_{\rho_i}^{\iota, \tilde{c}_i} \left(\sum_j B_{f_j} - B_{f_j}^{c_j} \right) \prod_i (I - B_{f_i}) \\ &= \prod_i (I - B_{f_i}) \left(\sum_j B_{f_j} - B_{f_j}^{c_j} \right) \\ &= - \prod_i (I - B_{f_i}) \left(\sum_j B_{f_j}^{c_j} \right). \end{aligned}$$

The operator $\prod_i (I - B_{f_i}) \left(\sum_j B_{f_j}^{c_j} \right)$ is a product of commuting positive operators and, hence, it is positive. But this implies that $A^*[H_L, A] \leq 0$.

Because of the ground state condition, equation (2.7), and the calculation above, $\omega(A^*[H_L, A]) = 0$. We can then sum over each configuration c_i with trivial product. Note that if we fix c_j for $j = 1, \dots, n_L - 1$, this

fixes c_{n_L} by the condition that their product should be trivial. Hence the summation over all configurations gives

$$0 = \sum_{(c_1, \dots, c_{n_L-1}) \in G^{n_L-1}} \omega \left(\prod_i (I - B_{f_i}) \left(B_{f_{n_L}}^{\prod c_i} + \sum_{k=1}^{n_L-1} B_{f_k}^{c_k} \right) \right). \quad (4.3)$$

Here we separated the n_L face from the others in the summation, since its magnetic charge is fixed by the others. We now do the summation over c_1 . Note that as c_1 runs over the group G , so does $\prod_{i=1}^{n_L-1} c_i$. Also note that for any j , $\sum_{c_j \in G} B_{f_j}^{c_j} = I$. This yields, by repeating this procedure,

$$0 = \sum_{(c_2, \dots, c_{n_L-1}) \in G^{n_L-2}} \omega \left(\prod_i (I - B_{f_i}) \left(2I + \sum_{k=2}^{n_L-1} B_{f_k}^{c_k} \right) \right) \quad (4.4)$$

$$= c(G, n_L) \omega \left(\prod_i I - B_{f_i} \right), \quad (4.5)$$

where $c(G, n_L)$ is some non-zero constant depending only on $|G|$ and the number of plaquettes. Therefore,

$$\omega \left(\prod_{i=1}^{n_L} (I - B_{f_i}) \right) = 0. \quad (4.6)$$

Equation (4.6) generally holds for a finite subset $\Lambda \subset \mathcal{B}$, where we assume that the subset is contained in some box Λ_L . We will need this fact for the following argument.

We proceed by induction to show that

$$\omega \left(\sum_{f \in \mathcal{F}_L} I - B_f \right) = \omega \left(1 - \prod_{f \in \mathcal{F}_L} B_f \right). \quad (4.7)$$

For the case of two faces, f_1 and f_2 , we have from equation (4.6)

$$\begin{aligned} 0 &= \omega((I - B_{f_1})(I - B_{f_2})) \\ &= \omega(I - B_{f_1} - B_{f_2} + B_{f_1} B_{f_2}), \end{aligned}$$

so that $\omega(I - B_{f_1}) + \omega(I - B_{f_2}) = \omega(I - B_{f_1} B_{f_2})$.

Suppose that equation (4.7) holds if X is a finite collection of faces with $|X| \leq n$. Now let X be a finite collection of n faces making up a region in Λ_L and enumerate the elements, $X = \{f_i\}_{i=1}^n$ and let $f_{n+1} \notin X$ be a face in X but otherwise arbitrary. From equation (4.6) it follows that

$$\omega \left(\left(\prod_{i \in X} I - B_{f_i} \right) (I - B_{f_{n+1}}) \right) = 0.$$

Expanding the product and using the hypothesis we have,

$$\begin{aligned} 0 &= \omega \left(\left(\prod_{f_i \in X} I - B_{f_i} \right) (I - B_{f_{n+1}}) \right) \\ &= \omega \left(I - \sum_{i=1}^{n+1} B_{f_i} + \sum_{i < j \leq n+1} B_{f_i} B_{f_j} - \sum_{i < j < k \leq n+1} B_{f_i} B_{f_j} B_{f_k} + \dots + (-1)^{n+1} \left(\prod_{f_i \in X} B_{f_i} \right) B_{f_{n+1}} \right) \\ &= 1 + \left[\sum_{i=1}^{n+1} \omega(I - B_{f_i}) - \binom{n+1}{1} \right] - \left[\sum_{i < j \leq n+1} \omega(I - B_{f_i} B_{f_j}) - \binom{n+1}{2} \right] + \dots \\ &\quad + (-1)^{n+1} \omega \left(\left(\prod_{f_i \in X} B_{f_i} \right) B_{f_{n+1}} \right) \\ &= -(-1)^{n+1} + \sum_{i=1}^{n+1} \omega(I - B_{f_i}) + \sum_{i < j \leq n+1} \omega(I - B_{f_i} B_{f_j}) + \dots + (-1)^{n+1} \omega \left(\left(\prod_{f_i \in X} B_{f_i} \right) B_{f_{n+1}} \right), \end{aligned}$$

where in the last step we use the elementary equation $\sum_{k=1}^{n-1} (-1)^k \binom{n}{k} = -(1 + (-1)^n)$. We can then apply the induction hypothesis to all but the last terms. Note that for the term with k products of B_{f_i} , after applying the summation in the induction hypothesis, each term $\omega(I - B_{f_i})$ appears exactly $\binom{n}{k-1}$ times. Hence we obtain

$$\begin{aligned} 0 &= -(-1)^{n+1} + \sum_{k=1}^n (-1)^{k+1} \binom{n}{k-1} \left(\sum_{i=1}^{n+1} \omega(I - B_{f_i}) \right) + (-1)^{n+1} \omega \left(\left(\prod_{f_i \in X} B_{f_i} \right) B_{f_{n+1}} \right) \\ &= -(-1)^{n+1} \omega \left(I - \prod_{i=1}^{n+1} B_{f_i} \right) + (-1)^{n+1} \omega \left(\sum_{i=1}^{n+1} (I - B_{f_i}) \right), \end{aligned}$$

where we used that $\sum_{k=1}^n (-1)^{k+1} \binom{n}{k-1} = (-1)^{n+1}$. Therefore equation (4.7) holds.

Now consider a configuration of magnetic charges, $\{c_i \in G\}_{i=1}^{n_L}$ such that $\prod_i c_i = e$ and ρ_i and \tilde{c}_i are as defined earlier. Let $A' = \prod_{i=1}^{n_L} F_{\rho_i}^{\iota, \tilde{c}_i} \prod_{i=1}^{n_L} B_{f_i}^{c_i}$ and let $l = \#\{i : c_i \neq e\}$. We compute

$$\begin{aligned} A'^* \delta(A') &= A'^* [H_L, A'] \\ &= \prod_i B_{f_i}^{c_i} \left(\prod_i F_{\rho_i}^{\iota, \tilde{c}_i} \right)^* \sum_j \left[-B_{f_j}, \prod_i F_{\rho_i}^{\iota, \tilde{c}_i} \right] \prod_i B_{f_i}^{c_i} \\ &= \prod_i B_{f_i}^{c_i} \sum_j (B_{f_j} - B_{f_j}^{c_j}) \\ &= -l \prod_i B_{f_i}^{c_i} \leq 0. \end{aligned}$$

Therefore, applying the ground state condition gives

$$\text{if } l > 0 \quad \text{then} \quad \omega \left(\prod_{i=1}^{n_L} B_{f_i}^{c_i} \right) = 0. \quad (4.8)$$

Finally, applying the equivalence in Lemma 3.2 with equations (4.7) and (4.8) gives the result,

$$\begin{aligned} \omega \left(\sum_{i=1}^{n_L} I - B_{f_i} \right) &= \omega \left(I - \prod_{i=1}^{n_L} B_{f_i} \right) \\ &= \omega \left(I - \sum_{\prod_i c_i = e} \prod_{i=1}^{n_L} B_{f_i}^{c_i} \right) \\ &= \omega(D_L^\mu) = \omega(V_L^\mu). \end{aligned}$$

A similar argument gives

$$\omega \left(\sum_{v \in \mathcal{V}_L} I - A_v \right) = \omega(V_L^\epsilon). \quad (4.9)$$

This concludes the proof. \square

We now state and prove the main result of the paper, starting with the definitions of the infinite volume ground state subsets:

Definition 4.3. Define the following convex subset of states for each $(\chi, c) \in \hat{G} \times G$:

$$\begin{aligned} K^{\chi, c} &:= \left\{ \omega^{\chi, c} \in K : \exists \omega \in K \text{ such that } \lim_{L \rightarrow \infty} \omega(D_L^{\chi, c}) > 0 \text{ exists, and} \right. \\ &\quad \left. \omega^{\chi, c} = \text{w}^* \text{-} \lim_{L \rightarrow \infty} \frac{\omega(\cdot D_L^{\chi, c})}{\omega(D_L^{\chi, c})} \right\}. \end{aligned} \quad (4.10)$$

By Lemma 4.2, if $\omega \in K$ then $\omega|_{\mathcal{A}_L}$ is a finite volume ground state for $H_L^{\epsilon,\mu}$ for all $L \geq 2$. Thus, if $L' > L$ and $\omega(D_{L'}^{\chi,c}) > 0$ then

$$\frac{\omega(\cdot D_{L'}^{\chi,c})}{\omega(D_{L'}^{\chi,c})} \Big|_{\mathcal{A}_L} = \frac{\omega(D_{L'}^{\chi,c} \cdot D_{L'}^{\chi,c})}{\omega(D_{L'}^{\chi,c})} \Big|_{\mathcal{A}_L} \quad (4.11)$$

is a finite volume ground state for $H_L^{\epsilon,\mu}$ for all L' , where the equality follows from Lemma 3.3. Note that in particular we see that $\omega(\cdot D_{L'}^{\chi,c})$ is a positive linear functional. Thus, $K^{\chi,c} \subset K$ is a subset of the set of infinite volume ground states. The interpretation of a state in $K^{\chi,c}$ is that it has a *global* excitation of type (χ, c) , hence the projection onto the charge (χ, c) in the region Λ_L has a positive expectation value as L goes to infinity. The assumption that $\lim_{L \rightarrow \infty} \omega(D_L^{\chi,c})$ exists is always satisfied, as follows from the next lemma.

Lemma 4.4. *The limit $\lambda_{\chi,c}(\omega) := \lim_{L \rightarrow \infty} \omega(D_L^{\chi,c})$ exists for all ground states ω and we have $\lambda_{\chi,c}(\omega) \geq 0$. Furthermore, if $\omega^{\chi,c} \in K^{\chi,c}$ then $\lambda_{\sigma,d}(\omega^{\chi,c}) = \delta_{(\sigma,d),(\chi,c)}$.*

Proof. Let $L'' > L' > L$. First, we claim that

$$D_L^{\chi}(\mathcal{G}_{L''}^{\epsilon,\mu}) \subset D_{L''}^{\chi}(\mathcal{G}_{L''}^{\epsilon,\mu}) \quad \text{if } \chi \neq \iota \quad (4.12)$$

$$D_L^c(\mathcal{G}_{L''}^{\epsilon,\mu}) \subset D_{L''}^c(\mathcal{G}_{L''}^{\epsilon,\mu}) \quad \text{if } c \neq e \quad (4.13)$$

$$D_{L'}^{\iota}(\mathcal{G}_{L''}^{\epsilon,\mu}) \subset D_L^{\iota}(\mathcal{G}_{L''}^{\epsilon,\mu}) \quad \text{if } \chi = \iota \quad (4.14)$$

$$D_{L'}^e(\mathcal{G}_{L''}^{\epsilon,\mu}) \subset D_L^e(\mathcal{G}_{L''}^{\epsilon,\mu}) \quad \text{if } c = e \quad (4.15)$$

(see Lemma 3.6 for a description of $\mathcal{G}_{L''}^{\epsilon,\mu}$). Note the reversal of L and L' in the last two equations. The reason is that while in the first two equations, the *presence* of a charge in the region is measured, in the last two equations it is the *absence* of any charge that is important.

To see why these equations are true, consider first $\chi \neq \iota$ and note that the subspace $D_L^{\chi}(\mathcal{G}_{L''}^{\epsilon,\mu})$ is spanned by $\{F_{\rho}^{\chi,e} F_{\sigma}^{\iota,c} \Omega : \forall \Omega \in \mathcal{G}_{L''}; c \in G; \rho, \sigma \text{ path connecting sites from the interior of } \Lambda_{L''} \text{ to that boundary such that } \partial_0 \rho \subset \Lambda_L\}$. The same statement is true if we replace L by L' . Thus, D_L^{χ} selects for a χ -excitation in the region Λ_L whereas $D_{L'}^{\chi}$ selects for a χ -excitation in the region $\Lambda_{L'}$. The later condition is weaker. Therefore, $D_L^{\chi}|_{\mathcal{G}_{L''}^{\epsilon,\mu}} \leq D_{L'}^{\chi}|_{\mathcal{G}_{L''}^{\epsilon,\mu}}$ as projections. A similar argument gives $D_L^c|_{\mathcal{G}_{L''}^{\epsilon,\mu}} \leq D_{L'}^c|_{\mathcal{G}_{L''}^{\epsilon,\mu}}$ as projections. If $\chi = \iota$, $D_{L'}^{\iota}$ selects for a trivial ϵ -type charge (i.e., the absence of an electric charge) in the region $\Lambda_{L'}$ while D_L^{ι} selects for a trivial ϵ -type charge in the region Λ_L . The later condition is weaker. Therefore, $D_{L'}^{\iota}|_{\mathcal{G}_{L''}^{\epsilon,\mu}} \leq D_L^{\iota}|_{\mathcal{G}_{L''}^{\epsilon,\mu}}$ as projections. A similar argument gives $D_{L'}^e|_{\mathcal{G}_{L''}^{\epsilon,\mu}} \leq D_L^e|_{\mathcal{G}_{L''}^{\epsilon,\mu}}$ as projections. This shows that (4.12)–(4.15) hold.

Let $\omega \in K$ be an infinite volume ground state. As remarked below Definition 4.3, $\omega|_{\mathcal{A}_{L''}} \in K_{L''}$ is a ground state functional for $H_{L''}^{\epsilon,\mu}$. Consider the sequence $\{\omega(D_L^{\chi})\}_{L=2}^{\infty}$. If $\chi \neq \iota$, the inclusion (4.12) gives that $\omega(D_{L'}^{\chi} - D_L^{\chi}) \geq 0$, thus the sequence is increasing. The sequence is also bounded, $\omega(D_L^{\chi}) \leq \|D_L^{\chi}\| = 1$. Hence we have a uniformly bounded and increasing sequence, and therefore the limit $\lim_{L \rightarrow \infty} \omega(D_L^{\chi})$ exists. A similar argument gives that the limit $\lim_{L \rightarrow \infty} \omega(D_L^{\chi,c})$ exists if $\chi \neq \iota$ and $c \neq e$.

If $\chi \neq \iota$ and $c = e$, where there is a non-trivial electric charge and the magnetic charge is trivial, we can rewrite the projector $D_L^{\chi,e}$ as

$$D_L^{\chi,e} = D_L^{\chi} D_L^e = D_L^{\chi} \left(I - \sum_{c \neq e} D_L^c \right) = D_L^{\chi} - \sum_{c \neq e} D_L^{\chi,c}.$$

This is enough to show the limit $\lim_{L \rightarrow \infty} \omega(D_L^{\chi,e})$ exists. By similar arguments, the limits exist for the cases $\chi = \iota$ but $c \neq e$, and when $\chi = \iota$ with $c = e$. The limits are always positive, since $\omega(D_L^{\chi,c}) \geq 0$ for all L .

To prove the second claim, let $\omega^{\chi,c} \in K^{\chi,c}$ and choose $\omega \in K$ such that

$$\omega^{\chi,c} = \text{w}^* \text{-} \lim_{L' \rightarrow \infty} \frac{\omega(\cdot D_{L'}^{\chi,c})}{\omega(D_{L'}^{\chi,c})}.$$

We use freely that the charge projectors commute. Equations (4.12)–(4.13) imply that if $\chi \neq \iota$ and $c \neq e$ then $\omega(D_L^{\chi,c} D_{L'}^{\chi,c}) = \omega(D_L^{\chi,c})$. Equations (4.14)–(4.15) imply that $\omega(D_L^{\iota,e} D_{L'}^{\iota,e}) = \omega(D_{L'}^{\iota,e})$. If $\chi \neq \iota$, (4.12)

and (4.14) imply that $\omega(D_L^{\chi,e} D_{L'}^{\chi,e}) = \omega(D_L^{\chi} D_{L'}^e)$. From the Cauchy-Schwarz inequality it follows that

$$\begin{aligned} |\omega(D_L^{\chi,e} D_{L'}^{\chi,e}) - \omega(D_L^{\chi,e})| &= |\omega(D_L^{\chi}(D_{L'}^e - D_L^e))| \\ &\leq \sqrt{\omega(D_L^{\chi})} \sqrt{|\omega((D_{L'}^e - D_L^e)^2)|} \\ &= \sqrt{\omega(D_L^{\chi})} \sqrt{|\omega(D_{L'}^e + D_L^e - 2D_L^e D_{L'}^e)|} \\ &= \sqrt{\omega(D_L^{\chi})} \sqrt{|\omega(D_{L'}^e - D_L^e)|} \end{aligned}$$

and similarly for $c \neq e$, $|\omega(D_L^{\iota,c} D_{L'}^{\iota,c}) - \omega(D_L^{\iota,c})| \leq \sqrt{\omega(D_L^c)} \sqrt{|\omega(D_{L'}^{\iota} - D_L^{\iota})|}$.

Let $\epsilon > 0$ be given. The previous arguments show that for all $(\chi, c) \in \widehat{G} \times G$ there exists l such that if $L' > L > l$ then

$$|\omega(D_L^{\chi,c} D_{L'}^{\chi,c}) - \omega(D_L^{\chi,c})| < \epsilon. \quad (4.16)$$

Thus,

$$\begin{aligned} \lambda^{\chi,c}(\omega^{\chi,c}) &= \lim_{L \rightarrow \infty} \lim_{L' \rightarrow \infty} \frac{\omega(D_L^{\chi,c} D_{L'}^{\chi,c})}{\omega(D_{L'}^{\chi,c})} \\ &= \lim_{L \rightarrow \infty} \lim_{L' \rightarrow \infty} \frac{\omega(D_L^{\chi,c})}{\omega(D_{L'}^{\chi,c})} \\ &= 1. \end{aligned}$$

Therefore, combining $\sum_{\sigma,d} \lambda_{\sigma,d}(\omega^{\chi,c}) = 1$ and $\lambda_{\chi,c}(\omega^{\chi,c}) = 1$ gives $\lambda_{\sigma,d}(\omega^{\chi,c}) = \delta_{(\sigma,d),(\chi,c)}$. \square

From the arguments given in the previous lemma, we can achieve a slightly stronger bound which will be used later. Let $\epsilon > 0$ be given. Then, for all $(\chi, c) \in \widehat{G} \times G$ we show that there exists l such that if $L' > L > l$ then

$$|\omega((D_{L'}^{\chi,c} - D_L^{\chi,c})^2)| < \epsilon. \quad (4.17)$$

If $\chi \neq \iota$ and $c \neq e$, (4.12) and (4.13) give that $\omega((D_{L'}^{\chi,c} - D_L^{\chi,c})^2) = \omega(D_{L'}^{\chi,c} - D_L^{\chi,c})$. Similarly, $\omega((D_{L'}^{\iota,e} - D_L^{\iota,e})^2) = |\omega(D_{L'}^{\iota,e} - D_L^{\iota,e})|$. If $\chi \neq \iota$ then

$$\begin{aligned} |\omega((D_{L'}^{\chi,e} - D_L^{\chi,e})^2)| &= |\omega(D_{L'}^{\chi,e} + D_L^{\chi,e} - 2D_L^{\chi,e} D_{L'}^{\chi,e})| \\ &\leq |\omega(D_{L'}^{\chi,e} - D_L^{\chi} D_{L'}^e)| + |\omega(D_L^{\chi,e} - D_L^{\chi} D_{L'}^e)| \\ &= |\omega(D_{L'}^e (D_L^{\chi} - D_L^{\chi}))| + |\omega(D_L^{\chi} (D_{L'}^e - D_L^e))|, \end{aligned}$$

with a similar bound holding if $\chi = \iota$ and $c \neq e$. Thus, (4.17) holds.

Lemma 4.4 allows us to distinguish ground states with different charges and makes it possible to decompose any ground state into charged ground states. This is the first part of our main result.

Theorem 4.5. *Let $\omega \in K$ be a ground state. Then there exists a convex decomposition of ω as*

$$\omega = \sum_{\chi \in \widehat{G}, c \in G} \lambda_{\chi,c}(\omega) \omega^{\chi,c} \quad \text{where} \quad \omega^{\chi,c} \in K^{\chi,c}. \quad (4.18)$$

Furthermore, we can calculate the coefficients explicitly as

$$\lambda_{\chi,c}(\omega) = \lim_{L \rightarrow \infty} \omega(D_L^{\chi,c}). \quad (4.19)$$

If $\lambda_{\chi,c}(\omega) > 0$ then

$$\omega^{\chi,c} = \text{w}^* - \lim_{L \rightarrow \infty} \frac{\omega(\cdot D_L^{\chi,c})}{\omega(D_L^{\chi,c})}. \quad (4.20)$$

Proof. For convenience, in this proof we denote $\lambda_{\chi,c}(\omega) = \lambda_{\chi,c}$.

By Lemma 4.4 the values $\lambda_{\chi,c} \geq 0$ are well-defined, so we will have to show that the limit in equation (4.20) exists and that the decomposition in equation (4.18) agrees with the state ω .

Let $L' > L \geq 2$. Since for each L' the charge projections add up to the identity, by equation (3.20), we have $\omega = \sum_{\chi,c} \omega(\cdot D_{L'}^{\chi,c})$ for all L' . By Lemma 4.2, $\omega|_{\mathcal{A}_L}$ is a finite volume ground state for $H_L^{\epsilon,\mu}$ for all $L \geq 2$ (see also the remark after Definition 4.3).

Now suppose $\lambda_{\chi,c} > 0$. Let $\epsilon > 0$ be given and suppose ϵ is small enough such that $\lambda_{\chi,c} > \epsilon > 0$. By Lemma 4.4, and inequalities (4.16) and (4.17) there exists $L > 0$ such that if $L'' > L' > L$ then

$$|\omega(D_{L'}^{\chi,c}) - \lambda_{\chi,c}| < \epsilon, \quad |\omega(D_{L''}^{\chi,c} - D_{L'}^{\chi,c})| < \epsilon, \quad \text{and} \quad |\omega((D_{L''}^{\chi,c} - D_{L'}^{\chi,c})^2)| < \epsilon.$$

We also demand that $\omega(D_{L'}^{\chi,c}) > 0$ for all $L' > L$, which can always be achieved by choosing L big enough. Note that $\lambda_{\chi,c} \leq 1$, so we can restrict to $\epsilon < 1$. Let $A \in \mathcal{A}_L$, then

$$\left| \frac{\omega(AD_{L'}^{\chi,c})}{\omega(D_{L'}^{\chi,c})} - \frac{\omega(AD_{L''}^{\chi,c})}{\omega(D_{L''}^{\chi,c})} \right| = \frac{1}{\omega(D_{L'}^{\chi,c})\omega(D_{L''}^{\chi,c})} |\omega(AD_{L'}^{\chi,c})\omega(D_{L''}^{\chi,c}) - \omega(AD_{L''}^{\chi,c})\omega(D_{L'}^{\chi,c})| \quad (4.21)$$

$$\leq \frac{1}{\omega(D_{L'}^{\chi,c})\omega(D_{L''}^{\chi,c})} \left(|\omega(AD_{L'}^{\chi,c})| |\omega(D_{L''}^{\chi,c}) - \omega(D_{L'}^{\chi,c})| \right. \quad (4.22)$$

$$\left. + \omega(D_{L'}^{\chi,c}) |\omega(A(D_{L''}^{\chi,c} - D_{L'}^{\chi,c}))| \right). \quad (4.23)$$

Recall that by Lemma 3.3 we have that $\omega(AD_{L'}^{\chi,c}) = \omega(D_{L'}^{\chi,c}AD_{L'}^{\chi,c})$. It follows that $|\omega(AD_{L'}^{\chi,c})| \leq \|A\|\omega(D_{L'}^{\chi,c})$. We also note the estimate $\frac{1}{\omega(D_{L''}^{\chi,c})} \leq \frac{1}{\lambda_{\chi,c} - \epsilon}$. The last term can be estimated using the Cauchy-Schwarz inequality,

$$|\omega(A(D_{L''}^{\chi,c} - D_{L'}^{\chi,c}))| \leq \sqrt{\omega(A^*A)} \sqrt{|\omega((D_{L''}^{\chi,c} - D_{L'}^{\chi,c})^2)|} \leq \|A\|\sqrt{\epsilon}.$$

Combining these estimates we obtain the bound

$$\left| \frac{\omega(AD_{L'}^{\chi,c})}{\omega(D_{L'}^{\chi,c})} - \frac{\omega(AD_{L''}^{\chi,c})}{\omega(D_{L''}^{\chi,c})} \right| \leq \frac{\|A\|(\epsilon + \sqrt{\epsilon})}{\lambda_{\chi,c} - \epsilon}. \quad (4.24)$$

Because $\lambda_{\chi,c} > 0$, this goes to zero as ϵ goes to zero.

Thus for each pair (χ, c) there is a sequence of states

$$\omega_L^{\chi,c} = \frac{\omega(\cdot D_L^{\chi,c})}{\omega(D_L^{\chi,c})}, \quad (4.25)$$

converging in the weak* limit (or $\omega^{\chi,c}$ is the zero functional if $\lambda_{\chi,c} = 0$). We have the following properties (cf. equations (4.48–4.50) in [KN]):

$$\text{w}^*\text{-}\lim_{L \rightarrow \infty} \omega_L^{\chi,c} := \omega^{\chi,c} \quad \text{exists,} \quad (4.26)$$

$$\text{w}^*\text{-}\lim_{L \rightarrow \infty} \left| \omega - \sum_{\chi,c} \lambda_{\chi,c} \omega_L^{\chi,c} \right| = 0 \quad (4.27)$$

$$\omega_L^{\chi,c}(H_L^{\epsilon,\mu}) = 0 \quad \text{for all } L' > L \geq 2. \quad (4.28)$$

The last property follows directly from the fact that $\omega_L^{\chi,c} \in K_L$ for all $L' > L$ and $\omega^{\chi,c}$ is an infinite volume ground state for all (χ, c) . Thus we have proven the ground state decomposition as in equation (4.18). \square

Corollary 4.6. *For all $(\chi, c) \in \widehat{G} \times G$, $K^{\chi,c}$ is a face in the set of all states. In particular, if $\omega^{\chi,c} \in K^{\chi,c}$ is an extremal point of $K^{\chi,c}$ then $\omega^{\chi,c}$ is a pure state.*

Proof. Let $\omega^{\chi,c} \in K^{\chi,c}$ and suppose $\phi \leq \lambda \omega^{\chi,c}$. Since the set of ground states is a face this implies ϕ is a ground state. By Theorem 4.5, decompose ϕ as $\phi = \sum \lambda_{\sigma,d}(\phi) \phi^{\sigma,d}$. By Lemma 4.4,

$$\begin{aligned} \lambda_{\sigma,d}(\phi) &= \lim_{L \rightarrow \infty} \phi(D_L^{\sigma,d}) \\ &\leq \lambda \lim_{L \rightarrow \infty} \omega^{\chi,c}(D_L^{\sigma,d}) \\ &= \delta_{(\sigma,d),(\chi,c)}. \end{aligned}$$

Therefore, $\phi = \phi^{\chi,c} \in K^{\chi,c}$, which shows $K^{\chi,c}$ is a face in the set of all states.

Suppose $\omega^{\chi,c}$ is an extremal state of $K^{\chi,c}$ and that $\omega^{\chi,c}$ can be decomposed as a convex combination of states

$$\omega^{\chi,c} = c_1\omega_1 + c_2\omega_2.$$

It follows that $c_i\omega_i \leq \omega^{\chi,c}$, so by the face property of $K^{\chi,c}$, $\omega_i \in K^{\chi,c}$. The supposition that the state $\omega^{\chi,c}$ is extremal in $K^{\chi,c}$ gives that $\omega_1 = \omega_2 = \omega^{\chi,c}$. \square

We note that the definition we take for a face does not require it to be a closed set in the weak* topology (but see also Theorem 4.8). The decomposition above suggests that each ground state can be decomposed into ground states that are related to the superselection sectors. Indeed, the *pure* states in $K^{\chi,c}$ are equivalent to one of the charged states that we constructed before:

Theorem 4.7. *If $\omega^{\chi,c} \in K^{\chi,c}$ is a pure state then $\omega^{\chi,c}$ is equivalent to a single excitation ground state $\omega_s^{\chi,c}$, as defined in equation (3.48).*

Proof. First, we notice that $\omega_s^{\chi,c} \in K^{\chi,c}$ for all sites s since $\frac{\omega_s^{\chi,c}(AD_{L'}^{\chi,c})}{\omega_s^{\chi,c}(D_{L'}^{\chi,c})} = \omega_s^{\chi,c}(A)$ for all $A \in \mathcal{A}_L$ and $L' > L$, where L is chosen big enough so that $s \subset \Lambda_L$.

Let $\omega^{\chi,c} \in K^{\chi,c}$ and let $\omega \in K$ be a ground state such that $\omega^{\chi,c} = \text{w}^*\text{-lim } \frac{\omega(\cdot D_{L'}^{\chi,c})}{\omega(D_{L'}^{\chi,c})}$ and $\lambda_{\chi,c} > 0$. Let $\epsilon > 0$ be given and suppose ϵ is small enough such that $\lambda_{\chi,c} > \epsilon > 0$. By Lemma 4.4 and equation (4.16), there is an $L > 0$ such that if $L'' \geq L' \geq L$ then $|\omega(D_{L''}^{\chi,c}) - \omega(D_{L'}^{\chi,c} D_{L''}^{\chi,c})| < \epsilon$ and $|\omega(D_{L''}^{\chi,c}) - \omega(D_{L'}^{\chi,c})| < \epsilon$. Fix an operator $A \in \mathcal{A}_{loc} \cap \mathcal{A}_{(L)^c}$. Then, there is an $L' > L + 1$ such that A is supported on the annulus $A \in \mathcal{A}_{L'-2} \cap \mathcal{A}_{(L)^c}$, and an $L'' > L' + 1$ such that

$$\left| \omega^{\chi,c}(A) - \frac{\omega(AD_{L''}^{\chi,c})}{\omega(D_{L''}^{\chi,c})} \right| < \|A\|\epsilon. \quad (4.29)$$

The estimate (4.24) also holds. Applying (4.16) and the estimates

$$\frac{1}{\omega(D_{L'}^{\chi,c})} \leq \frac{1}{\lambda_{\chi,c} - \epsilon} \quad \text{and} \quad \frac{1}{\omega(D_L^{\chi,c} D_{L'}^{\chi,c})} \leq \frac{1}{\lambda_{\chi,c} - \epsilon}$$

gives

$$\begin{aligned} \left| \frac{\omega(AD_{L'}^{\chi,c})}{\omega(D_{L'}^{\chi,c})} - \frac{\omega(AD_L^{\chi,c} D_{L'}^{\chi,c})}{\omega(D_L^{\chi,c} D_{L'}^{\chi,c})} \right| &\leq \left| \frac{\omega(AD_{L'}^{\chi,c})}{\omega(D_{L'}^{\chi,c})} - \frac{\omega(AD_L^{\chi,c} D_{L'}^{\chi,c})}{\omega(D_{L'}^{\chi,c})} \right| + \left| \frac{\omega(AD_L^{\chi,c} D_{L'}^{\chi,c})}{\omega(D_{L'}^{\chi,c})} - \frac{\omega(AD_L^{\chi,c} D_{L'}^{\chi,c})}{\omega(D_L^{\chi,c} D_{L'}^{\chi,c})} \right| \\ &\leq \frac{\|A\|(\epsilon + \sqrt{\epsilon})}{(\lambda_{\chi,c} - \epsilon)^2} \end{aligned}$$

Combining the above estimates, it follows that for a site s contained in Λ_L

$$\begin{aligned} |\omega^{\chi,c}(A) - \omega_s^{\chi,c}(A)| &\leq \left| \omega^{\chi,c}(A) - \frac{\omega(AD_{L''}^{\chi,c})}{\omega(D_{L''}^{\chi,c})} \right| + \left| \frac{\omega(AD_{L''}^{\chi,c})}{\omega(D_{L''}^{\chi,c})} - \frac{\omega(AD_{L'}^{\chi,c})}{\omega(D_{L'}^{\chi,c})} \right| \\ &\quad + \left| \frac{\omega(AD_{L'}^{\chi,c})}{\omega(D_{L'}^{\chi,c})} - \frac{\omega(AD_L^{\chi,c} D_{L'}^{\chi,c})}{\omega(D_L^{\chi,c} D_{L'}^{\chi,c})} \right| + \left| \frac{\omega(AD_L^{\chi,c} D_{L'}^{\chi,c})}{\omega(D_L^{\chi,c} D_{L'}^{\chi,c})} - \frac{\omega(D_{L'}^{\chi,c} D_L^{\chi,c} AD_L^{\chi,c} D_{L'}^{\chi,c})}{\omega(D_L^{\chi,c} D_{L'}^{\chi,c})} \right| \\ &\quad + \left| \frac{\omega(D_{L'}^{\chi,c} D_L^{\chi,c} AD_L^{\chi,c} D_{L'}^{\chi,c})}{\omega(D_L^{\chi,c} D_{L'}^{\chi,c})} - \omega_s^{\chi,c}(A) \right| \\ &\leq \|A\|\epsilon + 2 \frac{\|A\|(\epsilon + \sqrt{\epsilon})}{(\lambda_{\chi,c} - \epsilon)^2} + \left| \frac{\omega(D_{L'}^{\chi,c} D_L^{\chi,c} AD_L^{\chi,c} D_{L'}^{\chi,c})}{\omega(D_L^{\chi,c} D_{L'}^{\chi,c})} - \omega_s^{\chi,c}(A) \right|, \end{aligned} \quad (4.30)$$

where the fourth term vanishes by Lemma 3.3. The last term will be shown to be identically zero.

Recall that $\omega_s^{\chi,c}|_{\mathcal{A}_{L'}} = \omega^0(F_{\rho_{L'}}^{\chi,c*} \cdot F_{\rho_{L'}}^{\chi,c})|_{\mathcal{A}_{L'}}$, where $\rho_{L'}$ is a ribbon that connects the site s to the boundary of $\Lambda_{L'}$. Denote the subspace $\mathcal{G}_{L',L}^{\chi,c} := D_L^{\chi,c} D_{L'}^{\chi,c}(\mathcal{G}_{L'}^{\epsilon,\mu})$; it is spanned by simple vectors of the form $F_{\sigma}^{\chi,c} F_{\xi}^{\iota,c} \Omega$ where $\Omega \in \mathcal{G}_{L'}^0$, and σ and ι are ribbons connecting site s and t , respectively, in the interior of Λ_L to the boundary of $\Lambda_{L'}$, (see Lemma 3.6).

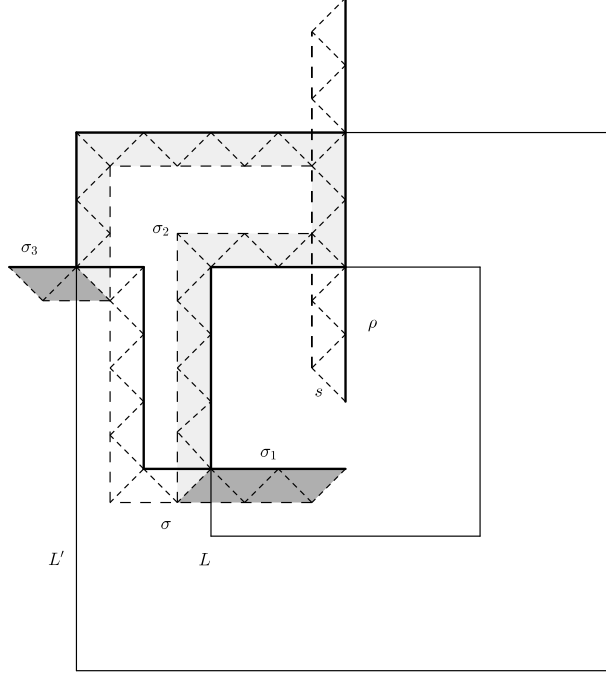


Figure 3: A depiction of the ribbons ρ , σ and $\sigma' = \sigma_1\sigma_2\sigma_3$. The ribbon σ' is shaded, σ_2 is distinguished by a lighter shade.

First, we consider the case $\psi \in \mathcal{G}_{L',L}^{X,c}$ and $\psi = F_{\sigma}^{X,c}\Omega$. Indeed, for each σ as above, we can find a new ribbon, $\sigma' = \sigma_1\sigma_2\sigma_3$, see Figure 3, connecting y to the boundary of $\Lambda_{L'}$ with the following properties

$$F_{\sigma_1}^{X,c} \in \mathcal{A}_L \text{ and } \sigma_1 \cap \Lambda_L = \sigma \cap \Lambda_L \quad (4.31)$$

$$F_{\sigma_2}^{X,c} \in \mathcal{A}_{L'-1} \cap \mathcal{A}_{(L+1)^c} \text{ and } \sigma_2 \cap \Lambda_{L'-2} \cap \Lambda_{(L+2)^c} = \rho_{L'} \cap \Lambda_{L'-2} \cap \Lambda_{(L+2)^c} \quad (4.32)$$

$$F_{\sigma_3}^{X,c} \in \mathcal{A}_{L'} \cap \mathcal{A}_{(L'-1)^c} \text{ and } \sigma_3 \cap \Lambda_{L''} \cap \Lambda_{(L'-1)^c} = \sigma \cap \Lambda_{L''} \cap \Lambda_{(L'-1)^c} \quad (4.33)$$

$$F_{\sigma'}^{X,c}\Omega = F_{\sigma_1}^{X,c}F_{\sigma_2}^{X,c}F_{\sigma_3}^{X,c}\Omega. \quad (4.34)$$

Here we used that the state only depends on the endpoints of the ribbon, not on the path. Decompose $\psi = F_{\sigma}^{X,c}\Omega = F_{\sigma_1}^{X,c}F_{\sigma_2}^{X,c}F_{\sigma_3}^{X,c}\Omega$.

Suppose A is a product of ribbon operators. If A is not a product of closed paths, then its action on $\mathcal{G}_{L',L}^{X,c}$ yields a subspace with strictly higher energy with respect to $H_{L'}^{X,c}$. Thus, if $\psi \in \mathcal{G}_{L',L}^{X,c}$ then $\langle \psi, A\psi \rangle = 0 = \omega_s^{X,c}(A)$. If A is a product of closed paths, then A leaves the frustration free ground state invariant, $A\Omega = \Omega$. Let $k \in \mathbb{C}$ be such that $AF_{\rho_{L'}}^{X,c} = kF_{\rho_{L'}}^{X,c}A$; k can be computed from ribbon intertwining relations (3.12). It follows that

$$\begin{aligned} \omega_s^{X,c}(A) &= \langle F_{\rho_{L'}}^{X,c}\Omega, AF_{\rho_{L'}}^{X,c}\Omega \rangle \\ &= k\langle \Omega, A\Omega \rangle \\ &= k. \end{aligned}$$

Now going back to the properties of σ_l in equations (4.31)-(4.34) we have that

$$[A, F_{\sigma_1}^{X,c}] = 0, \quad [A, F_{\sigma_3}^{X,c}] = 0, \quad \text{and} \quad AF_{\sigma_2}^{X,c} = kF_{\sigma_2}^{X,c}A. \quad (4.35)$$

Thus, if A is a product of closed ribbon operators then

$$\begin{aligned} A\psi &= AF_{\sigma_1}^{X,c}F_{\sigma_2}^{X,c}F_{\sigma_3}^{X,c}\Omega \\ &= kF_{\sigma_1}^{X,c}F_{\sigma_2}^{X,c}F_{\sigma_3}^{X,c}A\Omega \\ &= k\psi \end{aligned}$$

and ψ has eigenvalue k . For the case $\psi = F_{\sigma}^{\chi,e} F_{\xi}^{\iota,c} \Omega$, the decomposition $\sigma = \sigma_1 \sigma_2 \sigma_3$ and $\xi = \xi_1 \xi_2 \xi_3$ as above then we can choose $\sigma_2 = \xi_2$. Therefore, the same argument as above shows $A\psi = k\psi$.

For a general $\psi \in \mathcal{G}_{L',L}^{\chi,c}$, ψ is a linear combination of the simple vectors $F_{\sigma}^{\chi,e} F_{\xi}^{\iota,c} \Omega$. Thus, by linearity $A\psi = k\psi$ for all $\psi \in \mathcal{G}_{L',L}^{\chi,c}$. Therefore, if ψ is normalized

$$\langle \psi, A\psi \rangle = k = \omega_s^{\chi,c}(A). \quad (4.36)$$

Note that we already established this equation for A an open ribbon operator.

Since ribbon operators span the algebra $\mathcal{A}_{L'-2} \cap \mathcal{A}_{(L+2)^c}$, we extend the above argument by linearity so that

$$\langle \psi, A\psi \rangle = \omega_s^{\chi,c}(A) \quad \text{for all } A \in \mathcal{A}_{L'-2} \cap \mathcal{A}_{(L+2)^c}. \quad (4.37)$$

A general mixed state supported on $\mathcal{G}_{L',L}^{\chi,c}$ is of the form

$$\phi = \frac{\omega_{L'}(D_{L'}^{\chi,c} D_L^{\chi,c} \cdot D_L^{\chi,c} D_{L'}^{\chi,c})}{\omega_{L'}(D_L^{\chi,c} D_{L'}^{\chi,c})} = \sum c_{\psi} \langle \psi, A\psi \rangle,$$

where $\omega_{L'} \in K_{L'}$ and each ψ is a linear combination of vectors of the form $F_{\sigma}^{\chi,e} F_{\xi}^{\iota,c} \Omega$. Since the c_{ψ} add up to one, it follows that $\phi(A) = \omega_s^{\chi,c}(A)$ for all $A \in \mathcal{A}_{L'-2} \cap \mathcal{A}_{(L+2)^c}$.

For the ground state ω , Lemma 4.2 gives $\omega|_{\mathcal{A}_{L'}} \in K_{L'}$. Therefore,

$$\frac{\omega(D_{L'}^{\chi,c} D_L^{\chi,c} A D_L^{\chi,c} D_{L'}^{\chi,c})}{\omega(D_L^{\chi,c} D_{L'}^{\chi,c})} = \omega_s^{\chi,c}(A) \quad \text{for all } A \in \mathcal{A}_{L'-2} \cap \mathcal{A}_{(L+2)^c}. \quad (4.38)$$

Since L' was chosen such that $L' > L$, and otherwise arbitrary, the estimate in (4.30) becomes

$$|\omega^{\chi,c}(A) - \omega_s^{\chi,c}(A)| \leq \|A\| \left(\epsilon + 2 \frac{(\epsilon + \sqrt{\epsilon})}{(\lambda_{\chi,c} - \epsilon)^2} \right) \quad \text{for all } A \in \mathcal{A}_{loc} \cap \mathcal{A}_{L^c}. \quad (4.39)$$

Now suppose further that $\omega^{\chi,c}$ is a pure state. Proposition 3.8 also gives that the states $\omega_s^{\chi,c}$ are pure states. Therefore, applying the criterion for equivalence of pure states (Corollary 2.6.11, [BR]) we have

$$\omega^{\chi,c} \approx \omega_s^{\chi,c} \text{ for all } (\chi, c) \in \widehat{G} \times G. \quad (4.40)$$

This completes the proof. \square

The above two theorems give a complete characterization of the ground states of the quantum double model. The sets of states $K^{\chi,c}$ played an important role in the analysis. We end our discussion by finding the weak* closure of these sets in the set of all states. Recall that each state in $K^{\chi,c}$ has a charge (χ, c) . Now consider a sequence of states where the χ charge is gradually moved off to infinity. The resulting weak* limit will be a state with only a charge c , so the weak closures of the sets $K^{\chi,c}$ are strictly larger than $K^{\chi,c}$ (unless $\chi = \iota$ and $c = e$). The final result is that this procedure of moving away charges gives the weak* closures of the sets of charged ground states.

Theorem 4.8. *The closures in the weak* topology are given by*

$$\overline{K^{\chi,c}}^w = \text{Conv} (K^0 \cup K^{\chi,e} \cup K^{\iota,c} \cup K^{\chi,c}), \quad (4.41)$$

where Conv denotes the convex hull.

Proof. First, we show that $K^{\chi,e} \subset \overline{K^{\chi,c}}^w$.

Let $\omega^{\chi,e} \in K^{\chi,e}$ and ρ be a path extending to infinity based at site s . Consider the automorphism $\alpha_{\rho}^{\iota,c}$ (see (3.49)) that generates a charge of type (ι, c) located at the site s . We claim that the state $\omega^{\chi,e} \circ \alpha_{\rho}^{\iota,c} \in K^{\chi,c}$. To see this, from Theorem 4.5 and Lemma 4.4, write $\omega^{\chi,e}(A) = \lim_{L \rightarrow \infty} \omega^{\chi,e}(A D_L^{\chi,e}) / \omega^{\chi,e}(D_L^{\chi,e})$ for $A \in \mathcal{A}$. Notice that for $L' > L$ large enough such that $s \subset \Lambda_L$, equation (3.17) gives $F_{\rho_{L'}}^{\iota,c} D_L^{\chi,e} = D_L^{\chi,e} F_{\rho_{L'}}^{\iota,c}$ and $(\alpha_{\rho}^{\iota,c})^{-1}(D_L^{\chi,e}) = D_L^{\chi,e}$. Thus,

$$\omega^{\chi,e} \circ \alpha_{\rho}^{\iota,c}(A) = \lim_{L \rightarrow \infty} \frac{\omega^{\chi,e}(\alpha_{\rho}^{\iota,c}(A) D_L^{\chi,e})}{\omega^{\chi,e}(D_L^{\chi,e})} = \lim_{L \rightarrow \infty} \frac{\omega^{\chi,e} \circ \alpha_{\rho}^{\iota,c}(A D_L^{\chi,e})}{\omega^{\chi,e} \circ \alpha_{\rho}^{\iota,c}(D_L^{\chi,e})}. \quad (4.42)$$

To finish the claim, we need to show $\omega^{\chi,e} \circ \alpha_{\rho}^{\iota,c}$ is a ground state. Recall that $D_{L'}^{\chi,e} \mathcal{G}_{L'}^{\epsilon,\mu}$ is spanned by simple vectors of the form $F_{\sigma}^{\chi,e} \Omega$ where $\Omega \in \mathcal{G}_{L'}^0$, and σ is a ribbon connecting a site $s \in \Lambda_{L'}$ to the boundary of $\Lambda_{L'}$. Let $\psi \in D_{L'}^{\chi,e} \mathcal{G}_{L'}^{\epsilon,\mu}$ and write $\psi = \sum_j b_j F_{\sigma_j}^{\chi,e} \Omega_j$. Since $F_{\rho_{L'}}^{\iota,c} \psi = \sum_j b_j F_{\rho_{L'}}^{\iota,c} F_{\sigma_j}^{\chi,e} \Omega_j \in D_{L'}^{\chi,e} \mathcal{G}_{L'}^{\epsilon,\mu}$, it follows that $H_L^{\epsilon,\mu} F_{\rho_{L'}}^{\iota,c} \psi = 0$. Indeed, we compute $\omega^{\chi,e} \circ \alpha_{\rho}^{\iota,c}$ is an infinite volume ground state:

$$\begin{aligned} \omega^{\chi,e} \circ \alpha_{\rho}^{\iota,c}(H_L^{\epsilon,\mu}) &= \lim_{L' \rightarrow \infty} \frac{\omega^{\chi,e} \circ \alpha_{\rho}^{\epsilon,\mu}(H_L^{\epsilon,\mu} D_{L'}^{\chi,c})}{\omega^{\chi,e} \circ \alpha_{\rho}^{\iota,c}(D_{L'}^{\chi,c})} = \lim_{L' \rightarrow \infty} \frac{\omega^{\chi,e} \circ \alpha_{\rho}^{\epsilon,\mu}(D_{L'}^{\chi,c} H_L^{\epsilon,\mu} D_{L'}^{\chi,c})}{\omega^{\chi,e} \circ \alpha_{\rho}^{\iota,c}(D_{L'}^{\chi,c})} \\ &= \lim_{L' \rightarrow \infty} \frac{\omega^{\chi,e}(D_{L'}^{\chi,e} F_{\rho_{L'}}^{\iota,c*} H_L^{\epsilon,\mu} F_{\rho_{L'}}^{\iota,c} D_{L'}^{\chi,e})}{\omega^{\chi,e}(D_{L'}^{\chi,e})} \\ &= 0, \end{aligned}$$

where the last equality is true since the state

$$\frac{\omega^{\chi,e}(D_{L'}^{\chi,e} F_{\rho_{L'}}^{\iota,c*} \cdot F_{\rho_{L'}}^{\iota,c} D_{L'}^{\chi,e})}{\omega^{\chi,e}(D_{L'}^{\chi,e})}$$

is a mixed state supported on $D_{L'}^{\chi,c} \mathcal{G}_L^{\epsilon,\mu}$.

Now consider a sequence s_n of sites such that $s_1 = s$ and $s_n \rightarrow \infty$ as $n \rightarrow \infty$. Let ρ_n be a ribbon extending to infinity based at the site s_n and then define the sequence of states

$$\omega_n = \omega^{\chi,e} \circ \alpha_{\rho_n}^{\iota,c} \in K^{\chi,c}.$$

For $A \in \mathcal{A}_{loc}$, choose n large enough so that $\alpha_{\rho_n}^{\iota,c}(A) = A$. It follows that,

$$\omega_n(A) = \omega^{\chi,e} \circ \alpha_{\rho_n}^{\iota,c}(A) = \omega^{\chi,e}(A).$$

Therefore, $w^*\text{-}\lim \omega_n = \omega^{\chi,e} \in K^{\chi,e}$. By similar arguments one can show the inclusion,

$$\text{Conv}(K^0 \cup K^{\chi,e} \cup K^{\iota,c} \cup K^{\chi,c}) \subset \overline{K^{\chi,c}}^w.$$

Now, to show the reverse inclusion, suppose $\widehat{\omega}^{\chi,c} \in \overline{K^{\chi,c}}^w$ and let $\omega_{\lambda}^{\chi,c} \in K^{\chi,c}$ be a net in $K^{\chi,c}$ such that $w^*\text{-}\lim_{\lambda} \omega_{\lambda}^{\chi,c} = \widehat{\omega}^{\chi,c}$. For each λ we can write

$$\omega_{\lambda}^{\chi,c} = \lim_{L \rightarrow \infty} \frac{\omega_{\lambda}(\cdot D_L^{\chi,c})}{\omega_{\lambda}(D_L^{\chi,c})}.$$

The proof of Lemma 4.4 gives that $D_L^{\sigma,d} D_{L'}^{\chi,c} |_{\mathcal{G}_{L'}^{\epsilon,\mu}} = 0$ if $(\sigma, d) \notin \{(\chi, c), (\chi, e), (\iota, c), (\iota, e)\}$. Thus, in that case,

$$\widehat{\omega}^{\chi,c}(D_L^{\sigma,d}) = \lim_{\lambda} \omega_{\lambda}^{\chi,c}(D_L^{\sigma,d}) \quad (4.43)$$

$$= \lim_{\lambda} \lim_{L' \rightarrow \infty} \frac{\omega_{\lambda}^{\chi,c}(D_L^{\sigma,d} D_{L'}^{\chi,c})}{\omega_{\lambda}^{\chi,c}(D_{L'}^{\chi,c})} \quad (4.44)$$

$$= 0. \quad (4.45)$$

This holds for all L , hence $\lambda_{\sigma,d} = 0$. Since the set of infinite volume ground states is closed in the weak* topology, we apply equation (4.43) to the ground state decomposition (4.18) of $\widehat{\omega}^{\chi,c}$,

$$\widehat{\omega}^{\chi,c} = \lambda_0 \omega^0 + w^*\text{-}\lim_{L \rightarrow \infty} \left(\lambda_{\chi,e} \frac{\widehat{\omega}^{\chi,c}(\cdot D_L^{\chi,e})}{\widehat{\omega}^{\chi,c}(D_L^{\chi,e})} + c_{\iota,c} \frac{\widehat{\omega}^{\chi,c}(\cdot D_L^{\iota,c})}{\widehat{\omega}^{\chi,c}(D_L^{\iota,c})} + c_{\chi,c} \frac{\widehat{\omega}^{\chi,c}(\cdot D_L^{\chi,c})}{\widehat{\omega}^{\chi,c}(D_L^{\chi,c})} \right). \quad (4.46)$$

Therefore, $\widehat{\omega}^{\chi,c} \in \text{Conv}(K^0 \cup K^{\chi,e} \cup K^{\iota,c} \cup K^{\chi,c})$. \square

5 Concluding remarks

We have proved that the set of known ground states for Kitaev's quantum double model is complete in the case of finite abelian groups. A natural question is if the results can be extended to non-abelian groups. The

main technical challenge is that the quantum double $\mathcal{D}(G)$ has higher dimensional irreducible representations. In physical terms, this manifests itself in the presence of *non-abelian* anyons. As a result, the structure of the ribbon operators and quasi-particles is much richer than that of the abelian case. In particular, the quasi-particle excitations no longer decompose into simply electric and magnetic type, since they have to account for the more complicated structure of $\text{Rep}(\mathcal{D}(G))$. For instance, the boundary operators we use in Lemmas 3.2 and 4.2 would need to be generalized to account for this structure. Since the fusion rules are non-abelian, the fusion of two excitations does not always have a definite outcome. This makes it more difficult to sum over all admissible configurations that lead to a given charge in the region. In addition, the charged sectors are generated by endomorphism that are not automorphisms in general, and that are less straightforward to construct [N2].

A current challenge in mathematical physics is the classification of gapped ground state phases [BMNS, BO, O1, O2, O3]. One approach to classifying a phase is to construct a complete set of invariants. By definition, an invariant is a quantity that is constant within a phase. Consequently, if an invariant is computed for two systems and is found to take different values, the systems must be in different phases. In the literature a topological phase is often defined as an open region in a space of Hamiltonians where there is a non-vanishing gap above the ground state [CGW]. Therefore, the construction of invariants can be expected to rely on the existence of a spectral gap. In the quantum double models, while it is known that the gap above the vacuum state is stable under small uniform perturbations, we do not expect that the charged ground states will survive a generic perturbation of this model since the anyon quasi-particles will in general not appear as time-invariant states [K2]. Our classification of the complete ground state space of the quantum double model for abelian groups gives an example that shows that the set of infinite volume ground states is generally not an invariant of a phase. From the physical point of view, however, the invariance of the structure of anyon quasi-particles is usually taken as fact. There are few mathematically rigorous results in this direction [Haah]. We hope that our results are a first step in rigorously studying the stability properties of the superselection structure of quantum double models.

References

- [AFH] Alicki, R., Fannes, M., Horodecki, M.: *A statistical mechanics view on Kitaev's proposal of quantum memories*. J. Phys. A, **40**(24), 6451–6467 (2007)
- [AM] Araki, H., Matsui, T.: *Ground states of the XY-model*. Commun. Math. Phys. **101**, 213–245 (1985)
- [ASW] Arovas, D., Schrieffer, J.R., Wilczek, F.: *Fractional statistics and the quantum Hall effect*. Phys. Rev. Lett. **53**, 722–723 (1984)
- [BMNS] Bachmann, S., Michalakis, S., Nachtergaele, B., Sims, R.: *Automorphic equivalence within gapped phases of quantum lattice systems*. Commun. Math. Phys. **309**, 835–871 (2012)
- [BO] Bachmann, S., Ogata, Y.: *C^1 -classification of gapped parent Hamiltonians of quantum spin chains*. Commun. Math. Phys. **338**, 1011–1042 (2015)
- [BVD] Bais, F.A., van Driel, P., and de Wild Propitius, M.: *Anyons in discrete gauge theories with Chern-Simons terms*. Nucl. Phys. B **393**, 547–570 (1993)
- [BK] Bakalov, B., Kirillov, Jr., A.: *Lectures on Tensor Categories and Modular Functors* (University Lecture Series 21). American Mathematical Society, Providence, RI (2001)
- [BSW] Beigi, S., Shor, P.W., Whalen, D.: *The quantum double model with boundary: condensations and symmetries*. Commun. Math. Phys. **306**, 663–694 (2011)
- [BMD] Bombin, H., Martin-Delgado, M.A.: *A family of non-abelian Kitaev models on the lattice: topological condensation and confinement*. Phys. Rev. B **78**, 115421 (2008)
- [BSS] Bonderson, P., Shtengel, K., Slingerland, J.K.: *Interferometry of non-abelian anyons*. Ann. Physics **323**, 2709–2755 (2008)

- [BH] Brandao, F.G.S.L, Horodecki, M.: *Exponential Decay of Correlations Implies Area Law*. Commun. Math. Phys. **333**, 761 (2015)
- [BKR] Bratteli, O., Kishimoto, A., Robinson, D.: *Ground states of infinite quantum spin systems*. Commun. Math. Phys. **64**, 41–48 (1978)
- [BR] Bratteli, O., Robinson, D.W.: *Operator algebras and quantum statistical mechanics 1 and 2*, Second Edition. Springer Verlag (1987)
- [BHM] Bravyi, S., Hastings, M., Michalakis, S.: *Topological quantum order: stability under local perturbations*. J. Math. Phys. **51**, 093512 (2011)
- [BK] Bravyi, S., Kitaev, A.: *Quantum codes on a lattice with boundary*. arXiv:quant-ph/9811052v1 (1998)
- [CGW] Chen, X., Gu, Z.-C., Wen, X.-G.: *Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order*, Phys. Rev. B **84**, 155138 (2011)
- [DPR] Dijkgraaf, R., Pasquier, V., Roche, P.: *Quasi Hopf algebras, group cohomology and orbifold models*. Nucl. Phys. B. Proc. Suppl. **18B**, 60–72 (1991)
- [DHR71] Doplicher, S., Haag, R., Roberts, J.E.: *Local observables and particle statistics. I*. Commun. Math. Phys. **23**, 199–230 (1971)
- [DHR72] Doplicher, S., Haag, R., Roberts, J.E.: *Local observables and particle statistics. II*. Commun. Math. Phys. **35**, 49–85 (1974)
- [FNW] Fannes, M., Nachtergaele, B., Werner, R. F.: *Finitely correlated states of quantum spin chains*. Commun. Math. Phys. **144**, 443–490 (1992)
- [FW] Fannes, M., Werner, R.F.: *Boundary Conditions for Quantum Lattice Systems*. Helv. Phys. Acta **68**, 635–657 (1995)
- [FN] Fiedler, L., Naaijken, P.: *Haag duality for Kitaev’s quantum double model for abelian groups*. Rev. Math. Phys. **27**, 1550021 (2015)
- [FRS] Fredenhagen, K., Rehren, K.H., Schroer, B.: *Superselection sectors with braid group statistics and exchange algebras*. Commun. Math. Phys. **125**, 201–226 (1989)
- [F] Freedman, M.: *P/NP, and the quantum field computer*. Proc. Natl. Acad. Sci. USA **95**, 98–101 (1998)
- [FM] Freedman, M., Meyer, D.A.: *Projective plane and planar quantum codes*. Found. Comput. Math. **1**, 325–332 (2001)
- [FG] Fröhlich, J., Gabbiani, F.: *Braid statistics in local quantum theory*. Rev. Math. Phys. **2**, 251–353 (1990)
- [GW] Gottstein, C.T., Werner, R.F.: *Ground states of the q-deformed Heisenberg ferromagnet*. arXiv:cond-mat/9501123 (1995)
- [Haag] Haag, R.: *Local quantum physics: Fields, particles, algebras*, Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition (1996)
- [HHW] Haag, R., Hugenholtz, N.M., Winnink, M.: *On the equilibrium states in quantum statistical mechanics*. Commun. Math. Phys. **5**, 215–236 (1967)
- [Haah] Haah, J.: *An invariant of topologically ordered states under local unitary transformations*. Commun. Math. Phys. **342(3)**, 771–801 (2016)
- [H] Hastings, M.B.: *An area law for one-dimensional quantum systems*. J. Stat. Mech. **2007**, P08024 (2007)

- [HK] Hastings, M.B., Koma, T.: *Spectral gap and exponential decay of correlations*. Commun. Math. Phys. **265**, 781–804 (2006)
- [K] Kitaev, A.Y.: *Fault-tolerant quantum computation by anyons*. Ann. Physics, **303**, 2–30 (2003)
- [K2] Kitaev, A.: *Anyons in an exactly solved model and beyond*. Ann. Physics **321**, 2–111 (2006)
- [KN] Koma, T., Nachtergaele, B.: *The complete set of ground states of the ferromagnetic XXZ chains*. Adv. Theor. Math. Phys. **2**, 533–558 (1998)
- [M] Matsui, T.: *On ground states of the one-dimensional ferromagnetic XXZ chain*. Let. Math. Phys. **37**, 397–403 (1996)
- [MP] Michalakis, S., Zwolak, J.P.: *Stability of frustration-free Hamiltonians*. Commun. Math. Phys. **322**, 277–302 (2013)
- [MR] Moore, G., Read, N.: *Nonabelions in the fractional quantum hall effect*. Nuc. Phys. B **360**, 362–396 (1990)
- [N] Naaijken, P.: *Localized endomorphisms in Kitaev’s toric code on the plane*. Rev. Math. Phys. **23**, 347–373 (2011)
- [N2] Naaijken, P.: *Kitaev’s quantum double model from a local quantum physics point of view*. In: R. Brunetti, C. Dappiaggi, K. Fredenhagen, J. Yngvason (eds.), *Advances in Algebraic Quantum Field Theory*, pp. 365–395. Springer (2015)
- [NOS] Nachtergaele, B., Ogata, Y., Sims, R.: *Propagation of correlations in quantum lattice systems*. J. Stat. Phys. **124**, 1–13 (2006)
- [NS] Nachtergaele, B., Sims, R.: *Lieb-Robinson bounds and the exponential clustering theorem*. Commun. Math. Phys. **265**, 119–130 (2006)
- [O1] Ogata, Y.: *A class of asymmetric gapped Hamiltonians on quantum spin chains and its Characterization I*. Commun. Math. Phys. (2016) doi:10.1007/s00220-016-2696-6
- [O2] Ogata, Y.: *A class of asymmetric gapped Hamiltonians on quantum spin chains and its Characterization II*. Commun. Math. Phys. (2016) doi:10.1007/s00220-016-2697-5
- [O3] Ogata, Y.: *A class of asymmetric gapped Hamiltonians on quantum spin chains and its characterization III*, arXiv:1606.05508 [math-ph] (2016)
- [RS] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics Vol. I: Functional Analysis*, Revised and Enlarged edition. Academic Press, 400 p. (1980)
- [SV] Szlachányi, K., Vecsernyés, P.: *Quantum symmetry and braid group statistics in G-spin models*. Commun. Math. Phys. **156**, 127–168 (1993)
- [We] Wen, X.-G.: *Vacuum degeneracy of chiral spin states in compactified space*. Phys. Rev. Lett. **B40**, 7387–7390 (1989)
- [Wi] Wilczek, F.: *Fractional statistics and anyon superconductivity*. World Scientific Publishing Co., Inc., Teaneck, NJ (1990)