Bruhat decomposition via row reduction

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Let k be a field, $W \subset \mathrm{GL}_n(k)$ the group of permutation matrices, $B \subset \mathrm{GL}_n(k)$ the group of upper-triangular matrices.

Theorem. For every $g \in \mathrm{GL}_n(k)$ there is a unique $w \in W$ such that $g \in BwB$.

Proof. Let i_0 be the largest i such that $g_{i1} \neq 0$; define a matrix $b^{(1)}$ such that

$$b^{(1)}_{ii} = 1 \qquad b^{(1)}_{ii_0} = -g_{i1}g^{-1}_{i_0 1} \quad (i < i_0) \qquad b^{(1)}_{ij} = 0 \quad \text{(else)}.$$

Then $b^{(1)} \in B$ and $g' = b^{(1)}g$ has $g'_{i1} = \delta_{ii_0}$, and for any w, we have $g \in BwB$ if and only if $g' \in BwB$. Note that if we have $g' \in b^{(2)}wB$ for any w, then $b^{(2)}_{ii_0} = \delta_{ii_0} = w_{ii_0}$.

Say that a k-*expansion* of a matix M is any matrix obtained by adding a k'th row and column whose (i, k) and (k, j) entries are 0 for $i, j < k$. Note that any k-expansion of an upper-triangular matrix is upper-triangular.

Let h be the $(i_0, 1)$'th minor of g' and for $w \in W$, let v be the same minor of w, which is still a permutation matrix; then the first set following surjects onto the second one:

- Matrices $b^{(2)} \in B$ such that $g' \in b^{(2)}wB$;
- Matrices $b^{(3)} \in B \subset \mathrm{GL}_{n-1}(k)$ such that $h \in b^{(3)}vB$.

Indeed, we may take $b^{(2)}$ to be any i_0-expansion of $b^{(3)}$; conversely, every $b^{(2)}$ is such an expansion. By induction on n, such a $b^{(3)}$ exists for a unique v, hence $b^{(2)}$ exists for the unique w whose $(i_0, 1)$-minor is v. $\square$