On the Spectrum of Gauge Code Hamiltonians

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This is an attempt to apply some ideas from spectral graph theory to the study of Hamiltonians built from (possibly non-commuting) Pauli algebra terms.

Motivating examples of gapless and gapped models.

The XY model:

$$\mathcal{H} = \sum_i X_i X_{i+1} + Z_i Z_{i+1}.$$

Transverse field Ising model:

$$\mathcal{H} = \sum_i X_i X_{i+1} + Z_i.$$

In both these cases the existence of large weight stabilizer generators causes the gapless behaviour.

As another example, take

$$\mathcal{H} = \sum_i X_i + Z_i.$$

In this case there are no stabilizers and indeed the Hamiltonian is gapped.

We need to be a bit careful what we mean by the weight of a stabilizer generator. In this case, we define the weight of a stabilizer to mean the number of terms in the Hamiltonian required to build (by multiplication) a stabilizer. This rules out trivial cases such as $\mathcal{H} = \prod_{j=1}^{j=n} X_j$, where the Hamiltonian is one big stabilizer but is obviously gapped.

I. REPRESENTATIONS OF GAUGE CODES

A. Cayley graphs

The *Cayley graph* of a group G and generating set S is denoted Cayley(G, S). It has nodes G and edges $\{(g, sg) : g \in G, s \in S\}$. We will always assume S is closed under group inverse, and so we can consider the Cayley graph as undirected.

The adjacency matrix of a graph with N nodes is the N by N matrix A with non-zero entries $A_{ij} = 1$ corresponding to edges (i, j) of the graph. It follows that the adjacency matrix A of Cayley(G, S) is the linear operator $A : \mathbb{C}[G] \to \mathbb{C}[G]$ given by

$$A = \sum_{g \in S} \rho_{\text{reg}}(g)$$

where ρ_{reg} is the left regular representation of G on $\mathbb{C}[G]$.

Using representation theory we have that $\rho_{\text{reg}} : G \to \mathbb{C}[G]$ decomposes as the dirrect sum of irreducible representations $\rho_k : G \to \text{GL}(V_k)$ and so

$$A = \sum_{g \in S} \bigoplus_k \rho_k(g)$$

block diagonalizes A, with each ρ_k possibly appearing multiple times in the direct sum over k. See [7] and [3] for more details.

B. Gauge codes

The Pauli group $\mathcal{P}_1$ is normally defined as a set of matrices closed under matrix multiplication, but we can define it abstractly as the group generated by the (abstract) elements $\{m, X, Z\}$ with relations as follows:

$$m^2 = I,\ X^2 = I,\ Z^2 = I,\ mXmX = I,\ mZmZ = I, \text{ and } mZXZX = I,$$

where I is the group idenity. Actually, m is generated by X and Z, so it is not necessary to include m in the generating set, but here it simplifies the relations.

To define the n-qubit Pauli group $\mathcal{P}_n$, we use the $2n+1$ element generating set $\{m, X_1, .., X_n, Z_1, .., Z_n\}$ with relation $m^2 = I$ as before, and

$$X_i^2 = I,\ Z_i^2 = I,\ mX_imX_i = I,\ mZ_imZ_i = I,\ mZ_iX_iZ_iX_i = I, \text{ for } i = 1, ...n,$$
$$Z_iX_jZ_iX_j = I, \text{ for } i, j = 1, .., n,\ i \neq j.$$

Note that m commutes with all elements of $\mathcal{P}_n$ and squares to the idenity, so we will denote this element as -1. Similarly, ± 1 is thought of as the set $\{m, I\}$, and $-X$ is mX, etc.

The subgroup of $\mathcal{P}_n$ generated by the elements $\{X_1, ..., X_n\}$ is denoted $\mathcal{P}_n^X$. These are the X-type elements. Similarly, $\{Z_1, ..., Z_n\}$ generates the subgroup of Z-type elements $\mathcal{P}_n^Z$.

Every element $g \in \mathcal{P}_n$ can be written uniquely as a product $g = \pm g_X g_Z$, where g_X is an X-type operator and g_Z is a Z-type operator. This gives the size of the group as:

$$|\mathcal{P}_n| = 2^{2n+1}.$$

We now define the Pauli *representation* of the Pauli group as a group homomorphism:

$$\rho_{\text{pauli}} : \mathcal{P}_n \to \text{GL}(V)$$

where V is the 2^n dimensional state space of n qubits. On the independant generators $\{X_1, .., X_n, Z_1, .., Z_n\}$, ρ_{pauli} is defined as the following tensor product of 2×2 matrices:

$$\rho_{\text{pauli}}(X_i) := \bigotimes_{j=1}^{n} \left\{ \begin{array}{ll} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } j \neq i, \\ \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{for } j = i \end{array} \right\},\ \rho_{\text{pauli}}(Z_i) := \bigotimes_{j=1}^{n} \left\{ \begin{array}{ll} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } j \neq i, \\ \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{for } j = i \end{array} \right\}.$$

Normally the image of ρ_{pauli} is thought of as the Pauli group itself, and we are indeed free to think that way because ρ_{pauli} is a group isomorphism. It turns out that ρ_{pauli} is an irreducible representation (*irrep*) of $\mathcal{P}_n$ and the only other irreps of $\mathcal{P}_n$ are the 1-dimensional irreps $\rho : \mathcal{P}_n \to \mathbb{C}$ defined on the independant generators as:

$$\rho(X_i) = \pm 1, \quad \rho(Z_i) = \pm 1.$$

So we have 2^{2n} many 1-dimensional irreps, and a single 2^n-dimensional irrep. Summing the squares of the dimensions shows that we have a complete set of irreps of $\mathcal{P}_n$.

We now define a *gauge* subgroup G of $\mathcal{P}_n$ by choosing a set of generators $S \subset \mathcal{P}_n$,

$$G := \langle S \rangle.$$

We will assume G is not abelian, which is equivalent to the condition that $-I \in G$. We also restrict S to only contain Hermitian operators, which is equivalent to requiring that $g^2 = I$ for all $g \in S$. Now let H be the largest subgroup of G not containing $-I$. H is then an abelian subgroup, also known as the *stabilizer* subgroup. G decomposes as a direct product:

$$G = H \times R,$$

where $R \cong P_r$ for some $1 \leq r \leq n$, and $H \cong \mathbb{Z}_2^m$ for $0 \leq m < n$. Therefore, $|G| = |H||R| = 2^{m+2r+1}$. We call R the *reduced* gauge group. We consider both H and R to be subgroups of G. Let $\phi : P_r \to R$ be a group isomorphism, then $R_0 := \{\phi(X_i), \phi(Z_i)\}_{i=1,..,r}$ is a set of independant generators of R. We also let H_0 be a set of m independant generators of H.

The 1-dimensional irreps $\rho : G \to \mathbb{C}$, are now defined by specifying the action of ρ on the independant generators:

$$\rho(h) = \pm 1 \text{ for } h \in H_0, \quad \rho(\phi(X_i)) = \pm 1, \quad \rho(\phi(Z_i)) = \pm 1.$$

This gives all 2^{m+2r} of the 1-dimensional irreps. Finally, there are 2^m many 2^r-dimensional irreps given by:

$$\rho(h) = \pm I^{\otimes 2^r} \text{ for } h \in H_0, \quad \rho(\phi(X_i)) = X_i, \quad \rho(\phi(Z_i)) = Z_i.$$

The Hamiltonian of interest is normally defined as the negative sum of terms from S, but here we will reverse the sign:

$$\mathcal{H} := \sum_{g \in S} \rho_{\text{pauli}}(g).$$

The goal is to decompose $\mathcal{H}$ into blocks as

$$\mathcal{H} = \bigoplus_{\text{irrep}\rho} \sum_{g \in S} \rho(g),$$

and we will notate each block as $\mathcal{H}_\rho := \sum_{g \in S} \rho(g)$ for each irrep ρ appearing in $\mathcal{H}$.

The form of $\mathcal{H}$ is seen to be very similar to the adjacency matrix of Cayley(G, S) but instead of the regular representation we are using the Pauli representation. We use the following map to relate these two representations:

Theorem. Define a linear map $f : \mathbb{C}[G] \to \mathbb{C}[G_X]$ as follows. Any element $g \in \mathcal{P}_n$ can be written as $g = \pm g_x g_z$ and we set $f(g) = f(\pm g_x g_z) := \pm g_x$. Then,

$$fA = \mathcal{H} f.$$

Proof. *(Proof??)...* □

It is easy to see that any eigenvector v of A with eigenvalue λ is either in the kernel of f, or otherwise fv is an eigenvector of $\mathcal{H}$ with eigenvalue λ. Furthermore, f is full-rank *(Proof??)*, and so *all* the eigenvectors of $\mathcal{H}$ are of the form fv for some eigenvector v of A.

We write the (distinct) eigenvalues of $\mathcal{H}$ in decreasing order:

$$\lambda_1 > \lambda_2 > ...$$

Of particular interest is the gap between the first and second eigenvalues, $\epsilon := \lambda_1 - \lambda_2$.

There is a well understood theory of expansion in Cayley graphs that shows how the structure of the group G leads to gapped behaviour of A. Unfortunately, the top eigenvector of A is in the kernel of f and so these results do not help us show gapped behaviour of $\mathcal{H}$.

The above commutation relation for f is the definition of an *intertwining* map, and it is a general result that such maps either preserve an irreducible representation or send them to zero.

Theorem. The (images of) all the one-dimensional irreps are contained in the kernel of f. All the other irreps are preserved.

Proof. *(Proof??)...* □

In the sequal we will make the identification between g and $\rho_{pauli}(g)$. So terms such as Z and X are understood to be the corresponding Pauli linear operators.

C. Example: 2D compass model

Here we consider the two dimensional compass model. We coordinatize the qubits on a square lattice of $l \times l$ sites, (i, j) for $1 \leq i, j \leq l$. This gives $n = l^2$. For the single qubit Pauli operators acting on site (i, j) we coordinatize with subscripts ij, with i and j understood modulo l. The generators of the gauge group are

$$S = \big\{X_{ij}X_{i,j+1},\ Z_{ij}Z_{i+1,j} \text{ for } 1 \leq i, j \leq l\big\}.$$

We write generators of the reduced gauge group in anti-commuting pairs:

$$R_0 = \big\{X_{i1}X_{ij},\ Z_{1j}Z_{ij} \text{ for } 2 \leq i, j \leq l\big\}.$$

This makes it clear the isomorphism $\phi : R_0 \to \mathcal{P}_r$ to use, and we again use pairs i, j to coordinatize $\mathcal{P}_r$:

$$\phi(X_{i1}X_{ij}) = X_{i-1,j-1}, \quad \phi(Z_{1j}Z_{ij}) = Z_{i-1,j-1}, \text{ for } 2 \leq i, j \leq l.$$

The generators for the stabilizers are

$$H_0 = \big\{\prod_{i=1}^{l} X_{ij}X_{i,j+1},\ \prod_{i=1}^{l} Z_{ji}Z_{j+1,i} \text{ for } 1 \leq j \leq l-1\big\}.$$

The logical operators are generatred by $L_0 = \big\{\prod_i X_{i1}, \prod_j Z_{1j}\big\}$. These sets have cardinalities:

$$|S| = 2l^2,\ |R_0| = 2(l-1)^2,\ |H_0| = 2(l-1).$$

And we note that $\frac{1}{2}|L_0|+|H_0|+\frac{1}{2}|R_0| = n$. Now we write down the values of the irreps on the gauge operators. Note the transposition symmetry between the X and Z-type operators:

$$\begin{aligned}
\rho(X_{i1}X_{i2}) &= X_{i-1,1} & \rho(Z_{1i}Z_{2i}) &= Z_{1,i-1} & &\text{for } 2 \le i \le l \\
\rho(X_{ij}X_{i,j+1}) &= X_{i-1,j-1}X_{i-1,j} & \rho(Z_{ji}Z_{j,i+1}) &= Z_{j-1,i-1}Z_{j,i-1} & &\text{for } 2 \le i,j \le l \\
\rho(X_{1j}X_{1,j+1}) &= \pm\prod_{i=1}^{l} X_{i,j-1}X_{ij} & \rho(Z_{j1}Z_{j+1,1}) &= \pm\prod_{i=1}^{l} Z_{j-1,i}Z_{ji} & &\text{for } 2 \le j < l \\
\rho(X_{11}X_{12}) &= \pm\prod_{i=1}^{l} X_{i1} & \rho(Z_{11}Z_{21}) &= \pm\prod_{i=1}^{l} Z_{1i}. & &
\end{aligned}$$

We sum all these terms to find the form of the hamiltonian in each block:

$$\mathcal{H}_\rho = \sum_{g\in S}\rho(g) = \sum_{1\le i,j<l}\rho(X_{ij}X_{i,j+1}) + \rho(Z_{ij}Z_{i+1,j}).$$

We note that in [2], they perform a (ad-hoc?) transformation of the compass model which results in a similar $(l-1)\times(l-1)$ lattice of spins, but they have more terms in their Hamiltonian. It is not clear if these modifications are essential to their analysis.

D. Example: Kitaev honeycomb model

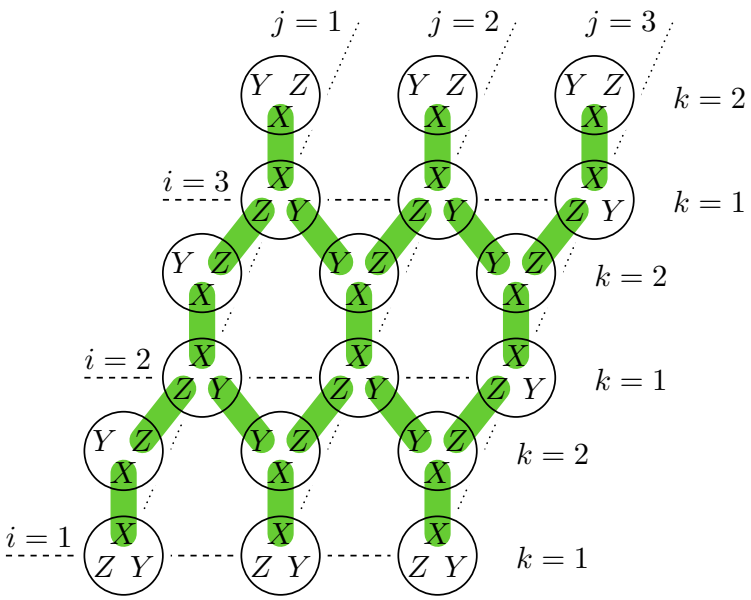

FIG. 1: Gauge generators have support on the edges of the honeycomb lattice. Qubits here are depicted as circles.

The Kitaev honeycomb model [8] is built from spins on the sites of a hexagonal lattice. The lattice of linear size l has $n = 2l^2$ sites which we coordinatize using integer triples i, j, k with $1 \le j, k \le l$ and $k = 1, 2$. We use periodic boundary conditions so i, j are to be taken

modulo l. See figure 1. The edges of the lattice are in one-to-one correspondence with the generators S:

$$S := \big\{X_{ij1}X_{ij2},\ Z_{ij2}Z_{i+1,j1},\ Y_{ij1}Y_{i-1,j+1,2} \text{ for } 1 \le i,j \le l\big\}.$$

Stabilizers are generated from closed strings of gauge operators. For example, each hexagon gives a stabilizer

$$\begin{aligned} h_{ij} :&= X_{ij1}X_{ij2}Z_{ij2}Z_{i+1,j1}Y_{i+1,j1}Y_{i,j+1,2}X_{i,j+1,2}X_{i,j+1,1}Z_{i,j+1,1}Z_{i-1,j+1,2}Y_{i-1,j+1,2}Y_{ij1} \\ &= Z_{ij1}Y_{ij2}X_{i+1,j1}Z_{i,j+1,2}Y_{i,j+1,1}X_{i-1,j+1,2}. \end{aligned}$$

This gives independant stabilizer generators from each hexagon, less one, as well as two homologically non-trivial loops. The number of hexagons is $\frac{1}{2}n$ and so we find $|H_0| = \frac{1}{2}n+1$. There are no logical operators, so we must have $|R_0| = n-2$.

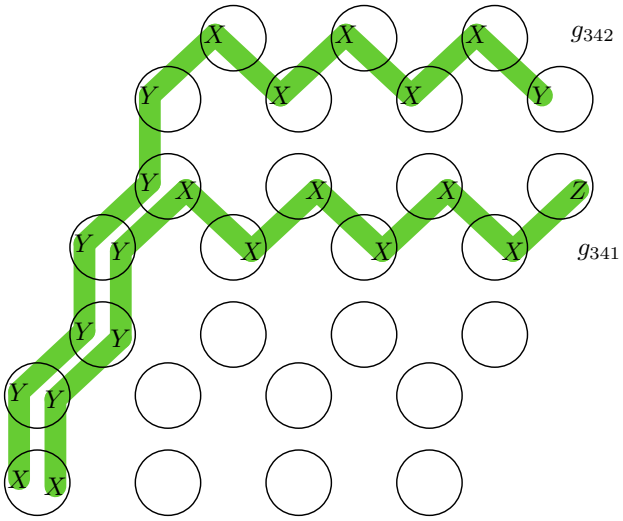

FIG. 2: Two elements of the set K_0 corresponding to $i = 3$, $j = 4$ and $k = 1, 2$.

Now we construct a set of string operators K_0, one for each site on the lattice, except for the two sites $(1, 1, 1)$ and $(1, 1, 2)$. Each string $g_{ijk} \in K_0$ is constructed as the product of gauge operators along a path starting at $(1, 1, 1)$ and terminating at (i, j, k). Each such path is built from two "straight" path segments, first in the i direction and then in the j direction. The paths for operators g_{ij1} and g_{ij2} coincide along the i direction but become disjoint in the j direction: the g_{ij1} path goes around the bottom of the hexagons and the g_{ij2} path goes around the top. See figure 2. With periodic boundary conditions K_0 forms an independant generating set of R of size $n-2$.

We construct an isomorphism $\phi : R \to \mathcal{P}_r$ by setting $\phi(g_{ij1}) := c_{2j'}$ and $\phi(g_{ij2}) := c_{2j'+1}$ where the c_j form the following independant generating set of $\mathcal{P}_r$:

$$\big\{c_{2j} := Z_1 ... Z_{j-1}X_j,\ c_{2j+1} := Z_1 ... Z_{j-1}Y_j \text{ for } 1 \le j \le r\big\}.$$

We check this is a group homomorphism by showing that relations satisfied by elements of K_0 are satisfied by their images under ϕ. All such relations are either of the form $g^2 = \pm I$, $gg' = \pm g'g$, or products thereof. So it is sufficient to check squares of elements and

commutation relations. Every element of K_0 anticommutes with every other element of K_0, and this is true also of the c_j. Also, $g_{ijk}^2 = I$ and $g_{ijk}^2 = -I$ is preserved by ϕ because $c_{2j}^2 = I$ and $c_{2j}^2 = -I$. Finally, ϕ is an isomorphism because it is a bijection of two independant generating sets.

The c_j are now paired Majorana fermion operators [8], and each block in the Hamiltonian is seen to be quadratic in these operators:

$$\mathcal{H}_\rho = \sum_{ij} \Gamma_{ij}(\rho) c_i c_j$$

where the coefficients Γ_{ij} are dependant on the irrep ρ. To find these coefficients we see that the $X_{ij1}X_{ij2}$ gauge operators can be generated by the product of $g_{ij1}g_{ij2}$ and the enclosed stabilizers (hexagons):

$$X_{ij1}X_{ij2} = g_{ij1}g_{ij2} \prod_{j'=1}^{j-1} h_{ij'}.$$

And similarly, we take the product of two operators from K_0 that differ by one path segment (gauge operator) to find the other gauge operators:

$$Z_{ij2}Z_{i+1,j1} = g_{ij2}g_{i+1,j1}$$
$$Y_{i+1,j1}Y_{i,j+1,2} = g_{i+1,j1}g_{i,j+1,2}.$$

II. BOUNDING THE GAP

In this section we view the Hamiltonian as the adjacency matrix of a *weighted* graph. We restrict our attention to Hamiltonians whose off-diagonal entries are non-negative. This can be achieved by considering Hamiltonians where each term involves either X-type operators or Z-type operators but not both. That is, S consists only of X-type operators and Z-type operators. We also shift the Hamiltonian by a constant energy, so that the diagonal entries are non-negative:

$$\mathcal{H} := \sum_{g \in S} \rho_{\text{pauli}}(g) + kI.$$

This does not change the spectral gap or eigenvalues.

A simple variational argument shows that the top eigenvector (the *groundstate*) can be chosen to have all positive entries (this is the Perron-Frobenius theorem) and therefore is stabilized:

Theorem. Every groundstate is stabilized.

Proof. □

In [4], they derive the following cheeger inequality by considering bi-partitions of the graph. We will do the same, but using matrix block notation.

Let v_2 be a second eigenvector, $\mathcal{H}v_2 = \lambda_2 v_2$ and $||v_2|| = 1$. We bi-partition the space so that v_2 has (vector) blocks:

$$v_2 = \begin{pmatrix} x \\ y \end{pmatrix}$$

with $x \geq 0$ and $y \leq 0$, component-wise. Let the blocks of $\mathcal{H}$ under the same partition be:

$$\mathcal{H} = \begin{pmatrix} A & C \\ C^\top & B \end{pmatrix}.$$

If we denote $\lambda_1(A)$ as the top eigenvalue of A and $\lambda_1(B)$ as the top eigenvalue of B, then

$$\begin{aligned}\lambda_2 = v_2^\top \mathcal{H} v_2 &= x^\top A x + 2x^\top C y + y^\top B y \\ &\leq x^\top A x + y^\top B y \ \leq \ ||x||^2 \lambda_1(A) + ||y||^2 \lambda_1(B) \\ &\leq \min(\lambda_1(A), \lambda_1(B)) \ \leq \ \lambda_1.\end{aligned}$$

Defining the following constant as a maximisation over all bi-partitions of $\mathcal{H}$:

$$\nu(\mathcal{H}) := \max_{A,B} \ \min((\lambda_1(A), \lambda_1(B))$$

the above calculation shows that

$$\lambda_2 \leq \nu(\mathcal{H}) \leq \lambda_1.$$

To bound λ_2 from below, we use a variational argument. For any unit vector v orthogonal to the top eigenspace of $\mathcal{H}$ we have $v^\top \mathcal{H} v \leq \lambda_2$.

See also, [1]. And [6].

APPENDIX A: GAP SOURCE CODE

This is source code for the GAP system [5] to display the irreducible representations of $\mathcal{P}_2$.

```
F := FreeGroup("m", "xi", "zi", "ix", "iz");;
m := F.1;; xi := F.2;; zi := F.3;; ix := F.4;; iz := F.5;;
G := F / [ix*xi*ix*xi, m*iz*ix*iz*ix, m*zi*m*zi, xi*xi, xi*m*xi*m,
    zi*iz*zi*iz, m*xi*zi*xi*zi, ix*m*ix*m, m*xi*m*xi, xi*ix*xi*ix,
    iz*iz, ix*zi*ix*zi, m*m, zi*m*zi*m, ix*ix,
    zi*ix*zi*ix, xi*iz*xi*iz, m*iz*m*iz, m*ix*m*ix, m*zi*xi*zi*xi,
    iz*zi*iz*zi, m*ix*iz*ix*iz, iz*m*iz*m, iz*xi*iz*xi, zi*zi];;
Print(Order(G));
LoadPackage( "repsn" );;
chi := Irr(G);
Print(chi);
for c in chi do
    rep := IrreducibleAffordingRepresentation(c);
    Print(rep);
    Print("\n");
od;
```

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