Representations and Spectra of Gauge Code Hamiltonians

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Abstract

What is an operator? From the perspective of a quantum code these are the things that we use to diagnose errors and perform error correction. We can also interpret these operators as the terms of a Hamiltonian, whose groundspace corresponds to the energetically protected codespace. In the case of mutually commuting operators we can easily diagonalize the Hamiltonian, but for gauge (subsystem) codes this does not hold. From the mathematical perspective we examine three different notions of representation theory, with a view to extracting spectral information about the Hamiltonian. Group theory representations give a block diagonalisation of the Hamiltonian as labeled by stabilizer eigenvalues. The coarser tool of Perron-Frobenius theory gives information about the spectral layout of these blocks in the case of CSS gauge codes. At the finest level, the operators in each of these blocks form a semisimple Lie algebra and ideals in this algebra correspond to tensor products of representations. Using all of these tools we perform exact diagonalisation on some large instances of the 3-dimensional gauge color code Hamiltonian. These numerics support the conjecture that these models are gapped, which in turn lends weight to the possibility that these may be self-correcting quantum memories.

Contents

1 Introduction

Physicists often like to solve Hamiltonians using a change of basis, or spin transform. But we can also work with transformations on the level of a group of operators, and later on figure out the spin transform (if needed). This is in line with the thinking of Gottesman and Heisenberg [13], where states are specified by the operators that act on them, instead of actually performing operations on states. This is often times harder than just manipulating the states themselves, but when it works it can yield new perspectives on the dynamics of the system. This is also the philosophy of category theory, where the goal is to lift information about elements of some mathematical object up to the level of the operators (morphisms) on the objects themselves. However, forgetting about the meaning of the symbols in this way leaves one with the question: "What is an operator?"

From the perspective of a quantum code these are the things that we use to diagnose errors and perform error correction. We can also interpret these operators as the terms of a Hamiltonian, whose groundspace corresponds to the energetically protected codespace. In the case of mutually commuting operators we can easily diagonalize the Hamiltonian, but for other systems of interest this does not hold.

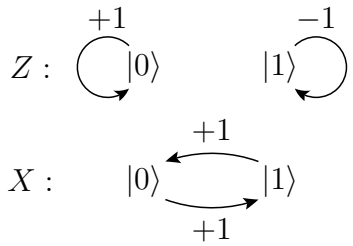
The mathematicians have a name for this question "what is an operator?" This is known as *representation theory.* We examine three different notions of such representations, with a view to extracting spectral information about the Hamiltonian. Group theory representations give a block diagonalization of the Hamiltonian as labeled by stabilizer eigenvalues. The coarser tool of Perron-Frobenius theory gives information about the spectral layout of these blocks in the case of CSS gauge codes. At the finest level, the operators in each of these blocks form a semisimple Lie algebra and ideals in this algebra correspond to tensor products of representations.

While there are some hints of this theory in the literature [2, 24] here we spell out in detail how this works and much more. Partly it's because these models are new and we don't have many examples.

Using all of these tools we perform exact diagonalization on some large instances of the 3-dimensional gauge color code Hamiltonian [3, 4, 19]. These numerics support the conjecture that these models are gapped, which in turn lends weight to the possibility that these may be self-correcting quantum memories. Having a constant gap (bound from below) is part of the story of topologically ordered phases [18, 7].

1.1 Some motivating examples

We start our journey considering a two-dimensional state space. This space is blessed with two basis vectors $|0\rangle$ and $|1\rangle$. The Z and X operators act on these states as:



From this picture we can see that Z acts by *stabilizing* the state $|0\rangle$ and anti-stabilizing the $|1\rangle$ state. The Z operator has been reduced to two operators each acting on a one dimensional subspace: $Z = +1 \oplus -1$. The X operator serves to "bitflip" the state between these two subspaces.

But what happens if we get confused and end up swapping the X and Z operators? We would like to see the X operator as stabilizing / anti-stabilizing two subspaces, together with the Z operator as bitflipping between these. The trick is to consider the *orbits* of the operator we hope to act as a stabilizer. In this case there is only one orbit, $|0\rangle + |1\rangle$ and indeed, the Z operator bitflips this to another state $|0\rangle - |1\rangle$ that is anti-stabilized by X.

We are going to be considering Hamiltonians built from summing operators of this form. In this paper we use a "neg-Hamiltonian" convention, to save complicating expressions with negative signs. The ground space corresponds to the *largest* eigenvalue.

So building a Hamiltonian from a single X or Z term, we find the ground space as the stabilized space by summing over the orbit of that term. The other operator, which we call the *adjacent operator*, acts to bitflip between the eigenspaces.

To further elucidate this idea we turn to another example, which is a Hamiltonian built from three commuting and independent operators:

$$H = XXI + IXX + ZZZ.$$

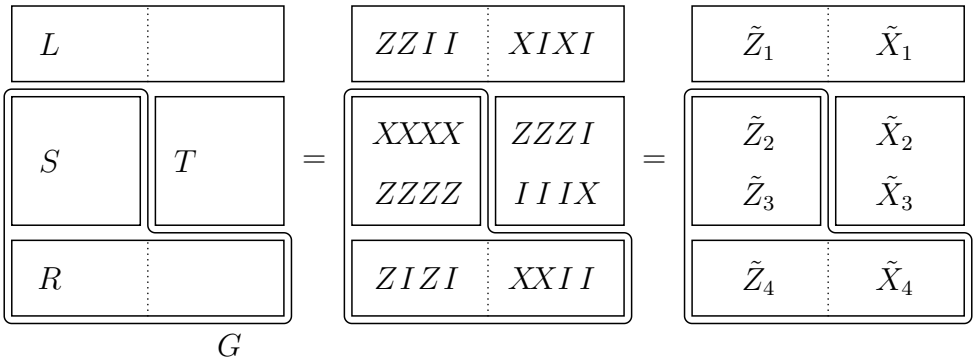
Starting with $|000\rangle$ we compute the orbit state as $|000\rangle + |011\rangle + |110\rangle + |101\rangle$. This time we have three adjacent operators ZII, IIZ, and IXI, one for each of the stabilizer operators. (We could also have chosen IZZ, ZZI, XXX.) For example, ZII sends the ground state to $|000\rangle + |011\rangle - |110\rangle - |101\rangle$ which is anti-stabilized by XXI. These adjacent operators form an abelian group of order $2^3 = 8$ and by applying each element of this group to the ground state we get a basis of our state space, which we call a *symmetry invariant basis.*

Now we consider a four qubit example:

$$H = XXII + IIXX + ZIZI + IZIZ.$$

This time the terms of the Hamiltonian do not generate an abelian group. We will call this group, as generated by the terms in the Hamiltonian, the *gauge group*, G. The *stabilizer* subgroup of G will be the elements of G that commute with every other element in G. By inspection we see this group is generated by $S_0 = \{XXXX, ZZZZ\}$. We can extend these generators to a complete independent generating set for G using the operators $R_0 = \{XXII, ZIZI\}$. These R_0 operators generate the *reduced gauge group* R. The operators adjacent to S_0 we call the *error operators* T_0. In this case we have $T_0 = \{ZZZI, IIIX\}$. The

logical operators are the n-qubit Pauli operators outside of the group G that commute with G. In this case they are generated by $L_0 = \{XIXI, ZZII\}$. All of this can be summarized in a table of adjacent pairs:



where the number of rows equals n, each operator commutes with operators on other rows, and anticommutes with the operator on the same row. If we take all the entries in the left column we get the operators $\{ZZII, XXXX, ZZZZ, ZIZI\}$. These generate an abelian group that stabilizes the state $|\psi\rangle = |0000\rangle + |1111\rangle$. Let r be the gauge operator $XXII$ adjacent to the stabilizer $ZIZI$, The state $|\psi\rangle$ then lies in the $G-$orbit

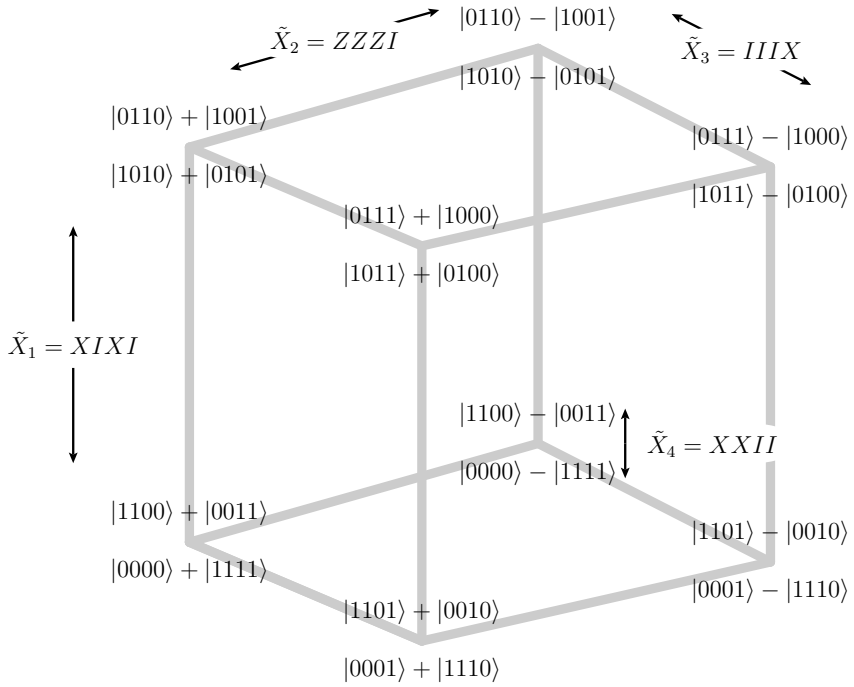
$$\{|\psi\rangle, r|\psi\rangle\} = \{|0000\rangle + |1111\rangle, |1100\rangle + |0011\rangle\}.$$

We use the T_0 operators $t_1 = ZZZI$ and $t_2 = IIIX$ to list three other $G-$orbits:

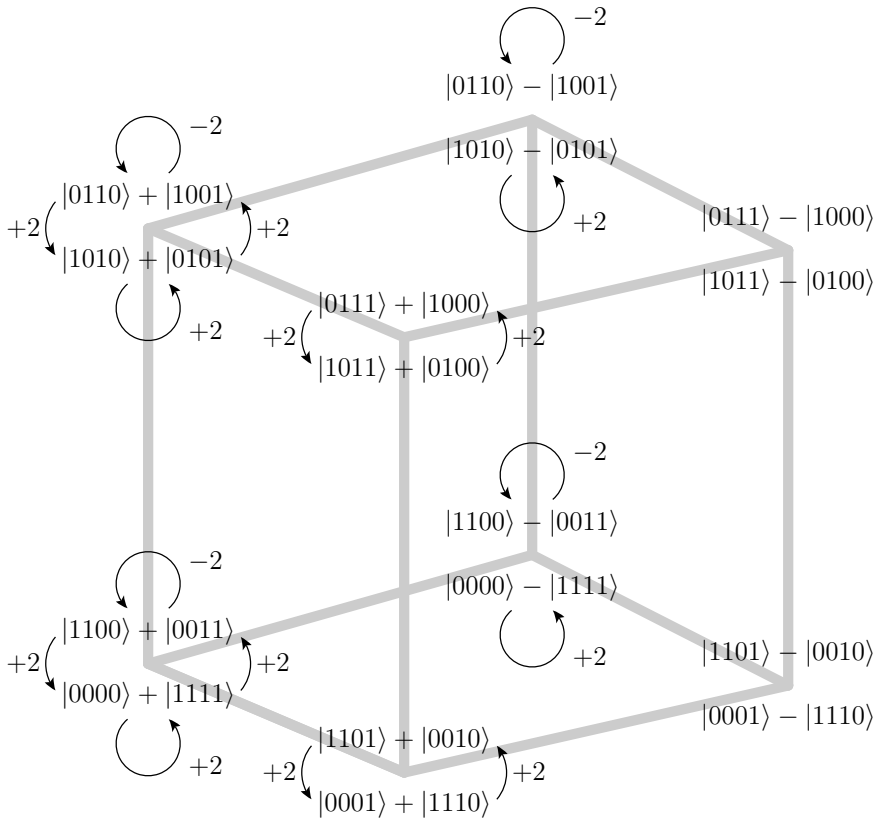
$$\{t_1|\psi\rangle, t_1 r|\psi\rangle\}, \{t_2|\psi\rangle, t_2 r|\psi\rangle\}, \{t_1 t_2|\psi\rangle, t_1 t_2 r|\psi\rangle\}.$$

So now we have sixteen vectors, forming an orthogonal basis for the state space. This is the symmetry invariant basis for this Hamiltonian.

We can now arrange these basis vectors on the vertices of a four dimensional hypercube, such that each dimension corresponds to one of the adjacent $\tilde{X}_i$ bitflip operators. Such an arrangement has a cartesian product structure which induces a tensor product decomposition of the original state space that corresponds to the $\tilde{X}_i$:



The Hamiltonian acts on states by left multiplication. Because this action is a sum of gauge group elements, it will decompose into blocks, one for each $G-$orbit. We depict this action as a weighted graph, where we omit edges with zero weight:



Equivalently we use this basis to write the matrix for the Hamiltonian in block diagonal form:

$$H = \begin{pmatrix} 2(X+Z) & 0 & 0 & 0 \\ 0 & 2X & 0 & 0 \\ 0 & 0 & 2Z & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes I$$

This block diagonal form will be worked out for general gauge group G below and summarized in Section 2.2.6.

2 Group representations

2.1 The Pauli group

The Pauli group $\mathcal{P}_1$ is normally defined as a set of matrices closed under matrix multiplication, but we can define it abstractly as the group generated by the (abstract) elements $\{\omega, X, Z\}$ with relations as follows:

$$\omega^2 = I,\ X^2 = I,\ Z^2 = I,\ \omega X \omega X = I,\ \omega Z \omega Z = I, \text{ and } \omega Z X Z X = I,$$

where I is the group identity. Actually, ω is generated by X and Z, so it is not necessary to include ω in the generating set, but here it simplifies the relations. This group has eight elements, and is isomorphic to the dihedral group D_4, the symmetry group of a square.

To define the *n-qubit Pauli group* $\mathcal{P}_n$, we use the $2n+1$ element generating set

$$\{\omega, X_1, .., X_n, Z_1, .., Z_n\}$$

with relation $\omega^2 = I$ as before, and

$$\begin{gathered} X_i^2 = I,\ Z_i^2 = I,\ \omega X_i \omega X_i = I,\ \omega Z_i \omega Z_i = I,\ \omega Z_i X_i Z_i X_i = I, \text{ for } i = 1, ...n, \\ Z_i X_j Z_i X_j = I, \text{ for } i, j = 1, .., n,\ i \neq j. \end{gathered} \tag{1}$$

This abstract approach to the definition of a group is known as a group *presentation*. In general, this is a set of generators together with a set of relations satisfied by these generators.

Note that each of the generators squares to the identity, and of these, only ω commutes with every element of $\mathcal{P}_n$. Therefore we will write ω as $-I$, similarly $\pm I$ will denote the set $\{\omega, I\}$, and $-X$ is ωX, etc.

We write the group commutator as $[[g, h]] := ghg^{-1}h^{-1}$ and note the important commutation relation:

$$[[Z_i, X_j]] = \begin{cases} -I & \text{if } i = j, \\ I & \text{if } i \neq j. \end{cases}$$

If we take an arbitrary $g \in \mathcal{P}_n$ written as a product of the generators, it follows that we can rewrite this product uniquely as $g = \pm g_1 ... g_n$ where each g_i is one of I, Z_i, X_i or $X_i Z_i$ for $i = 1, .., n$. Therefore, the size of the Pauli group is

$$|\mathcal{P}_n| = 2^{2n+1}.$$

The subgroup of $\mathcal{P}_n$ generated by the elements $\{X_1, ..., X_n\}$ is denoted $\mathcal{P}_n^X$. These are the X-type elements. Similarly, $\{Z_1, ..., Z_n\}$ generates the subgroup of Z-type elements $\mathcal{P}_n^Z$.

2.2 Subgroups of the Pauli group

We now define an $n-$*qubit gauge group* to be any non-abelian subgroup G of $\mathcal{P}_n$, defined by a set of generators $G_0 \subset \mathcal{P}_n$,

$$G := \langle G_0 \rangle.$$

Because G is not abelian, it follows that $-I \in G$. We also restrict G_0 to only contain Hermitian operators, which is equivalent to requiring that $g^2 = I$ for all $g \in G_0$.

Now let S be the largest subgroup of G not containing $-I$. S is then an abelian subgroup, also known as the *stabilizer* subgroup. G decomposes as a direct product:

$$G = S \times R,$$

where $R \cong P_r$ for some $1 \leq r \leq n$, and $S \cong \mathbb{Z}_2^m$ for $0 \leq m < n$. Therefore,

$$|G| = |S||R| = 2^{m+2r+1}.$$

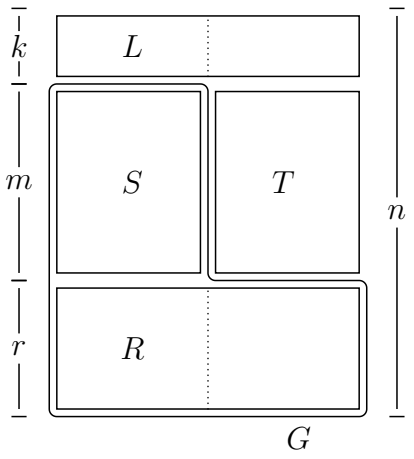
We call R the *reduced gauge group*. We consider both S and R to be subgroups of G. Let $\phi : R \to P_r$ be a group isomorphism, then $R_0 := \{\phi^{-1}(X_i), \phi^{-1}(Z_i)\}_{i=1,..,r}$ is a set of independent generators of R. We also let S_0 be a set of m independent generators of S.

To find the cosets of G in $\mathcal{P}_n$ we take the group closure of $G - \mathcal{P}_n$; when this is non-empty we only need to add I and $-I$. This is another gauge group, whose reduced gauge group is known as the *logical* operators L, and whose stabilizer subgroup is known as the *error* operators T. Now any coset of G can be written as ltG with $l \in L$ and $t \in T$. The size of T equals the size of S: $|T| = |S| = 2^m$. If we let L_0 be an independent generating set for L then we have the important formula:

$$n = \frac{1}{2}|L_0| + |S_0| + \frac{1}{2}|R_0| \tag{2}$$

$$= k + m + r \tag{3}$$

We summarize the information in this section in a table of Pauli group elements arranged in two columns and n rows:



Here we show the $2n$ generators of $\mathcal{P}_n$ arranged so that each row contains a pair of generators, where each such generator anti-commutes with the operator on the same row and commutes with all the other operators in the table. Note that this is exactly the definition of the Pauli group via a presentation given in the previous section. Furthermore, the table shows $2k$ generators of L, m generators each for S and T, and $2r$ generators of R. The gauge group G encloses R and S, and one can immediately see how L and T also form a gauge group.

2.3 Representations of the Pauli group

We now define the *Pauli representation* of the Pauli group as a group homomorphism:

$$\rho_{\text{pauli}} : \mathcal{P}_n \to \text{GL}(\mathbb{C}[2^n])$$

where $\mathbb{C}[2^n]$ is the 2^n-dimensional state space of n qubits. On the independent generators $\{X_1, .., X_n, Z_1, .., Z_n\}$, ρ_{pauli} is defined as the following tensor product of 2×2 matrices:

$$\rho_{\text{pauli}}(X_i) := \bigotimes_{j=1}^{n} \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{for } j = i \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } j \neq i \end{cases}$$

$$\rho_{\text{pauli}}(Z_i) := \bigotimes_{j=1}^{n} \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{for } j = i \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } j \neq i \end{cases}$$

Normally the image of ρ_{pauli} is thought of as the Pauli group itself, and we are indeed free to think that way because ρ_{pauli} is a group isomorphism.

Given a group representation $\rho : G \to GL(V)$ the *character* of ρ is a function $\chi_\rho : G \to \mathbb{C}$ given by

$$\chi_\rho(g) = \text{Tr } \rho(g).$$

Given two functions $u, v : G \to \mathbb{C}$ we define the following inner product:

$$\langle u, v \rangle := \frac{1}{|G|} \sum_{g \in G} u(g)\overline{v(g)}.$$

The character of the Pauli representation, $\chi_{pauli} : \mathcal{P}_n \to \mathbb{C}$ is given by:

$$\chi_{pauli}(g) = \sum_{v \in basis} \langle v|\rho_{pauli}(g)|v\rangle = \begin{cases} \pm 2^n & \text{if } g = \pm I \\ 0 & \text{otherwise} \end{cases}$$

Since $|P_n| = 2^{2n+1}$ it follows that $\langle \chi_{pauli}, \chi_{pauli} \rangle = 1$ and so ρ_{pauli} is an irreducible representation of $\mathcal{P}_n$.

The only other irreps of $\mathcal{P}_n$ are the 1-dimensional irreps $\rho : \mathcal{P}_n \to \mathbb{C}$ defined on the independent generators as:

$$\rho(X_i) = \pm 1, \quad \rho(Z_i) = \pm 1.$$

So we have 2^{2n} many 1-dimensional irreps, and a single 2^n-dimensional irrep. Summing the squares of the dimensions shows that we have a complete set of irreps of $\mathcal{P}_n$.

2.4 Representations of gauge groups

Although ρ_{pauli} restricted to a gauge group $G \subset \mathcal{P}_n$ serves as a representation of G it is no longer irreducible. Our aim will be to decompose ρ_{pauli} into irreps of G.

The 1-dimensional irreps $\rho : G \to \mathbb{C}$, are now defined by specifying the action of ρ on the independent generators:

$$\rho(h) = \pm 1 \text{ for } h \in S_0, \quad \rho(\phi^{-1}(X_i)) = \pm 1, \quad \rho(\phi^{-1}(Z_i)) = \pm 1.$$

This gives all 2^{m+2r} of the 1-dimensional irreps.

The 2^r-dimensional irreps are given by:

$$\rho(h) = \pm I^{\otimes r} \text{ for } h \in S_0, \quad \rho(\phi^{-1}(X_i)) = X_i, \quad \rho(\phi^{-1}(Z_i)) = Z_i.$$

We are free to choose the signs of the $\rho(h)$ for each $h \in S_0$. Hence there are 2^m many of these irreps. Each such choice corresponds to the choice of a *syndrome* vector $s(h) = \pm 1$, for $h \in S_0$, or alternatively, choice of an element $t \in T$:

$$\rho_t^1(h) = \begin{cases} I^{\otimes r} & \text{if } th = ht \\ -I^{\otimes r} & \text{if } th = -ht \end{cases}$$

Because G decomposes into a direct product $G = S \times \mathcal{P}_r$ we have the following representations:

$$\rho_t(g) = \rho_t^1(h)\rho_{pauli}^r(g'),$$

where $g = hg'$, $h \in S$, $g' \in \mathcal{P}_r$ and ρ_{pauli}^r is the r-qubit Pauli representation. The character for this representation is:

$$\chi_t(hg') = \rho_t^1(h) \sum_{v \in basis} \langle v|\rho_{pauli}^r(g')|v\rangle = \begin{cases} \pm 2^r \rho_t^1(h) & \text{if } g' = \pm I \\ 0 & \text{otherwise} \end{cases}$$

We have that $|G| = 2^{2r+m+1}$ and so $\langle \chi_t, \chi_t \rangle = 1$ and ρ_t is an irreducible representation of G. We now count the occurrences of this representation in ρ_{pauli}^r:

$$\begin{aligned} \langle \chi_{pauli}^r, \chi_t \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_{pauli}^r(g)\overline{\chi_t(g)} \\ &= \frac{1}{2^{2r+m+1}} \sum_{g=\pm I} 2^n 2^r = \frac{2^{n+1+r}}{2^{2r+m+1}} = 2^k \end{aligned}$$

where k is the number of logical qubits so that $n = r + m + k$.

In summary, the Pauli representation decomposes into 2^m many irreps ρ_t, each with dimension 2^r, and appearing with multiplicity 2^k :

$$\rho_{\text{pauli}} = \bigoplus_{t \in T} \rho_t \otimes I^{\otimes k}$$

2.5 Symmetry invariant basis

In general, given a representation $\rho : G \to \mathrm{GL}(V)$ and the character of some irreducible representation $\chi : G \to \mathbb{C}$ the following operator $P : V \to V$ projects onto the subspace on which this irreducible representation acts:

$$P := \frac{d}{|G|} \sum_{g \in G} \overline{\chi(g)} \rho(g).$$

where d is the dimension of the irreducible representation. We can use this to calculate projectors onto the irreps ρ_t in ρ_{pauli}:

$$\begin{aligned} P_t &= \frac{d}{|G|} \sum_{g \in G} \overline{\chi_t(g)} \rho_{pauli}(g) \\ &= \frac{d}{|G|} \sum_{h \in S} \sum_{g \in R} \overline{\chi_t(hg)} \rho_{pauli}(hg) \\ &= \frac{d}{|G|} 2^{2r} \sum_{h \in S} \rho_t^1(h) \rho_{pauli}(h) \\ &= \frac{1}{2^m} \sum_{h \in S} \rho_t^1(h) \rho_{pauli}(h). \end{aligned}$$

We can also write this as a product of projectors onto the ± 1 eigenspaces of stabilizers $\rho_{pauli}(h)$ for $h \in S$. Choose generators $h_1, ..., h_m$ of S and then the projectors onto the ± 1 eigenspace of $\rho_{pauli}(h_i)$ are

$$P_t^i = \frac{1}{2}\big(I^{\otimes n} \pm \rho_{pauli}(h_i)\big)$$

and we see that

$$P_t = \prod_{i=1,...,m} P_t^i = \frac{1}{2^m}\big(I^{\otimes n} \pm \rho_{pauli}(h_1)\big)...\big(I^{\otimes n} \pm \rho_{pauli}(h_m)\big).$$

This projector will have rank 2^{k+r} and

$$U := \sum_{t \in T} P_t$$

is a unitary transformation that sends physical qubits to encoded qubits.

2.6 The Hamiltonian

The Hamiltonian of interest is an operator $H : \mathbb{C}[2^n] \to \mathbb{C}[2^n]$:

$$H := \sum_{g \in G_0} \rho_{\text{pauli}}(g).$$

Using the above decomposition we find:

$$\begin{aligned} H &= \sum_{g \in G_0} \bigoplus_{l \in L, t \in T} \rho_t(g) \\ &= \bigoplus_{l \in L, t \in T} \sum_{g \in G_0} \rho_t(g). \end{aligned}$$

We will notate each block as $H_t := \sum_{g \in G_0} \rho_t(g)$ for each irrep ρ_t appearing in H.

> **Fact 0:** The Hamiltonian is block diagonalized, with blocks indexed by operators t in the abelian group T and multiplicity 2^k :
>
> $$H = \bigoplus_{t \in T} H_t \otimes I^{\otimes k}.$$

More generally, we can assign real valued weights $J_g \in \mathbb{R}$ to each operator $g \in G_0$,

$$H = \sum_{g \in G_0} J_g \rho_{\text{pauli}}(g) = \bigoplus_{l \in L, t \in T} \sum_{g \in G_0} J_g \rho_t(g).$$

In other words, using weights does not change the block structure of H.

In the following sections we will forget the distinction between g and $\rho_{\text{pauli}}(g)$, so terms such as Z and X can be understood as the corresponding Pauli linear operators.

3 Applications

We now use the tools built so far to analyze two examples of gauge code Hamiltonians. The above procedure is not entirely automatic, it relies on extracting the isomorphism ϕ, but when this can be made to work it works surprisingly well.

3.1 The 2D compass model

Here we consider the two-dimensional compass model [1]. We coordinatize the qubits on a square lattice of $l \times l$ sites, (i, j) for $1 \leq i, j \leq l$. This gives $n = l^2$. For the single qubit Pauli

operators acting on site (i,j) we coordinatize with subscripts ij, with i and j understood modulo l. The generators of the gauge group are

$$G_0 = \big\{X_{ij}X_{i,j+1},\ Z_{ij}Z_{i+1,j} \text{ for } 1 \le i,j \le l\big\}.$$

We write generators of the reduced gauge group in anti-commuting pairs:

$$R_0 = \big\{X_{i1}X_{ij},\ Z_{1j}Z_{ij} \text{ for } 2 \le i,j \le l\big\}.$$

This makes it clear the isomorphism $\phi : R \to \mathcal{P}_r$ to use, and we again use pairs i,j to coordinatize $\mathcal{P}_r$:

$$\phi(X_{i1}X_{ij}) = X_{i-1,j-1}, \quad \phi(Z_{1j}Z_{ij}) = Z_{i-1,j-1}, \text{ for } 2 \le i,j \le l.$$

The generators for the stabilizers are

$$S_0 = \big\{\prod_{i=1}^{l} X_{ij}X_{i,j+1},\ \prod_{i=1}^{l} Z_{ji}Z_{j+1,i} \text{ for } 1 \le j \le l-1\big\}.$$

The logical operators are generated by $L_0 = \big\{\prod_i X_{i1}, \prod_j Z_{1j}\big\}$. These sets have cardinalities:

$$|G_0| = 2l^2,\ |R_0| = 2(l-1)^2,\ |S_0| = 2(l-1).$$

And we note that $k+m+r=n$ is satisfied. Now we write down the values of the irreps on the gauge operators. Here we define each irrep using a pair of syndrome vectors s_X and s_Z :

$$\rho(X_{i1}X_{i2}) = X_{i-1,1} \qquad \rho(Z_{1i}Z_{2i}) = Z_{1,i-1} \text{ for } 2 \le i \le l$$

$$\rho(X_{il}X_{i1}) = X_{i-1,l-1} \qquad \rho(Z_{li}Z_{1i}) = Z_{l-1,i-1} \text{ for } 2 \le i \le l$$

$$\rho(X_{ij}X_{i,j+1}) = X_{i-1,j-1}X_{i-1,j} \qquad \rho(Z_{ji}Z_{j,i+1}) = Z_{j-1,i-1}Z_{j,i-1} \text{ for } 2 \le i \le l, 2 \le j < l$$

$$\rho(X_{1j}X_{1,j+1}) = s_X(j-1)\prod_{i=1}^{l-1} X_{i,j-1}X_{ij} \qquad \rho(Z_{j1}Z_{j+1,1}) = s_Z(j-1)\prod_{i=1}^{l-1} Z_{j-1,i}Z_{ji} \text{ for } 2 \le j < l$$

$$\rho(X_{11}X_{12}) = \prod_{j=1}^{l-1} s_X(j)\prod_{i=1}^{l-1} X_{i1} \qquad \rho(Z_{11}Z_{21}) = \prod_{j=1}^{l-1} s_Z(j)\prod_{i=1}^{l-1} Z_{1i}.$$

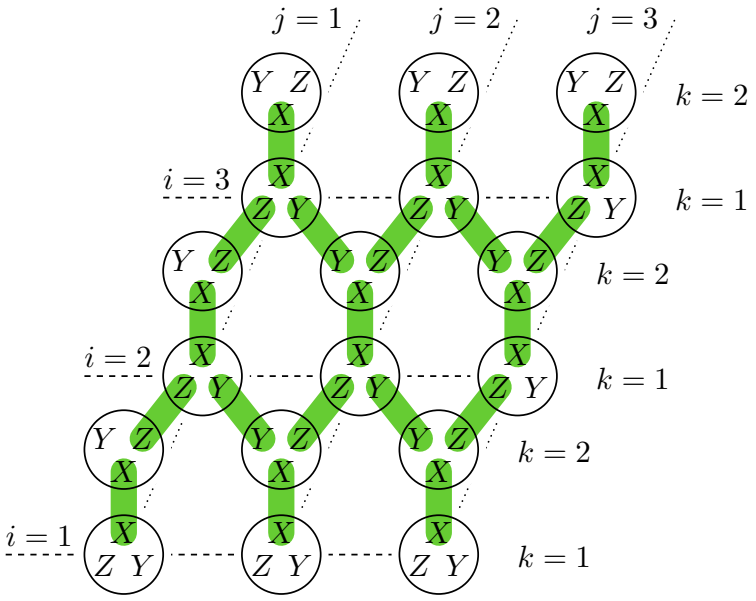
Note the transposition symmetry between the X and Z-type operators. We sum all these terms to find the form of the Hamiltonian in each block:

$$H_\rho = \sum_{g \in G_0} \rho(g) = \sum_{1 \le i,j < l} \rho(X_{ij}X_{i,j+1}) + \rho(Z_{ij}Z_{i+1,j}).$$

We note that in [8], they perform a spin transformation of the compass model which also results in an $(l-1)\times(l-1)$ lattice of spins and identical Hamiltonian up to some signs.

3.2 The Kitaev honeycomb model

The Kitaev honeycomb model [17] is built from spins on the sites of a hexagonal lattice. The lattice of linear size l has $n = 2l^2$ sites which we coordinatize using integer triples i, j, k with $1 \leq j, k \leq l$ and $k = 1, 2$. We use periodic boundary conditions so i, j are to be taken modulo l. Gauge generators have support on the edges of the honeycomb lattice, and we depict qubits here as circles:



The edges of the lattice are in one-to-one correspondence with the generators G_0:

$$G_0 := \{X_{ij1}X_{ij2},\ Z_{ij2}Z_{i+1,j1},\ Y_{ij1}Y_{i-1,j+1,2} \text{ for } 1 \leq i, j \leq l\}.$$

Note that we make the definition $Y := XZ$ for each site.

Stabilizers are generated from closed strings of gauge operators. For example, each hexagon gives a stabilizer

$$\begin{aligned} h_{ij} :&= X_{ij1}X_{ij2}Z_{ij2}Z_{i+1,j1}Y_{i+1,j1}Y_{i,j+1,2}X_{i,j+1,2}X_{i,j+1,1}Z_{i,j+1,1}Z_{i-1,j+1,2}Y_{i-1,j+1,2}Y_{ij1} \\ &= Z_{ij1}Y_{ij2}X_{i+1,j1}Z_{i,j+1,2}Y_{i,j+1,1}X_{i-1,j+1,2}. \end{aligned}$$

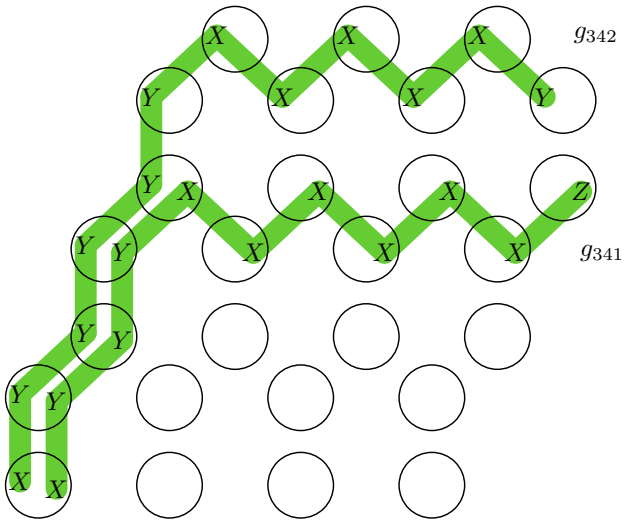
And the two homologically non-trivial loops give stabilizers:

$$h_v := \prod_{i=1}^{l} Y_{i11}Y_{i12}, \quad h_h := \prod_{j=1}^{l} X_{1j2}X_{2j1}.$$

This gives independent stabilizer generators S_0 from each hexagon, less one, as well as h_v and h_h. The number of hexagons is $\frac{1}{2}n$ and so we find $|S_0| = \frac{1}{2}n + 1$. There are no logical operators, so we must have $|R_0| = n - 2$.

Now we construct a set of string operators R_0, one for each site on the lattice, except for the two sites $(1,1,1)$ and $(1,1,2)$. Each string $g_{ijk} \in R_0$ is constructed as the product of gauge operators along a path starting at $(1,1,1)$ and terminating at (i,j,k).

Two elements of the set R_0 corresponding to $i=3$, $j=4$ and $k=1,2$.



Each such path is built from two "straight" path segments, first in the i direction and then in the j direction. The paths for operators g_{ij1} and g_{ij2} coincide along the i direction but become disjoint in the j direction: the g_{ij1} path goes around the bottom of the hexagons and the g_{ij2} path goes around the top. With periodic boundary conditions R_0 forms an independent generating set of R of size $n-2$.

We construct an isomorphism $\phi : R \to \mathcal{P}_r$ by sending elements of R_0 bijectively to the following independent generating set of $\mathcal{P}_r$:

$$\big\{c_{2j} := Z_1 ... Z_{j-1} X_j,\ c_{2j+1} := Z_1 ... Z_{j-1} Y_j \ \text{ for } 1 \le j \le r\big\}.$$

The bijection is constrained by setting $\phi(g_{ij1}) := c_{2j'+1}$ and $\phi(g_{ij2}) := c_{2j'}$ where j' is chosen uniquely for each i, j. The c_j are paired Majorana fermion operators [15, 17].

We check this is a group homomorphism by showing that relations satisfied by elements of R_0 are satisfied by their images under ϕ. All such relations are either of the form $g^2 = \pm I$, $gg' = \pm g'g$, or products thereof. So it is sufficient to check squares of elements and commutation relations. Every element of R_0 anticommutes with every other element of R_0, and this is true also of the c_j. Also, $g_{ij1}^2 = -I$ and $g_{ij2}^2 = I$ is preserved by ϕ because $c_{2j}^2 = I$ and $c_{2j+1}^2 = -I$. Finally, ϕ is an isomorphism because it is a bijection of two independent generating sets.

The next step is to write each element of G_0 as a product of reduced gauge operators and stabilizers. The key thing to note is that the product of two operators $g_{ijk}, g_{i'j'k'} \in R_0$ gives a string operator between the sites (i,j,k) and (i',j',k'). And *any* string operator between these two sites can then be generated by using stabilizers to "deform" the string $g_{ijk} g_{i'j'k'}$. For example, taking the product of two operators from R_0 that differ by one path segment

gives the following:

$$Z_{ij2}Z_{i+1,j,1} = g_{ij2}g_{i+1,j,1}$$
$$Y_{i+1,j,1}Y_{i,j+1,2} = g_{i+1,j,1}g_{i,j+1,2}$$

We need the homologically non-trivial stabilizers to get these:

$$Z_{lj2}Z_{1j1} = h_v g_{lj2}g_{1j1} \qquad \text{for } \; 2 \leq j \leq l$$

And the $X_{ij1}X_{ij2}$ gauge operators can be generated by the product of $g_{ij1}g_{ij2}$ and the enclosed hexagon stabilizers:

$$X_{ij1}X_{ij2} = g_{ij1}g_{ij2}\prod_{j'=1}^{j-1} h_{ij'}.$$

The only G_0 operators that are not quadratic in R_0 operators are the five operators that touch either of the sites $(1,1,1)$ or $(1,1,2)$.

So each block in the Hamiltonian is seen to be quadratic in the c_j plus five other Pauli operator terms which we denote as Λ_ρ:

$$H_\rho = \sum_{ij} \Gamma_{ij}(\rho)c_i c_j + \Lambda_\rho$$

The coefficients Γ_{ij} are dependant on the irrep ρ.

In [16] they introduce a similar set of mutually anti-commuting string operators R_0.

4 Symplectic representations

This is a way of "brute-forcing" the representations when we cannot find a way of writing them down in a closed form expression. For finite systems this yields an algorithm that is efficiently implementable.

In this section, and the remainder of this paper, we restrict our attention to gauge groups formed from terms in $\mathcal{P}_n^X \cup \mathcal{P}_n^Z$. We call these *CSS gauge codes.* We next turn to a discussion of the symplectic structure of these operators.

Let $\mathcal{F}$ denote the finite field with two elements 0 and 1. Both $\mathcal{P}_n^X$ and $\mathcal{P}_n^Z$ are abelian groups, and can be identified with the additive group structure of the n dimensional vector space over $\mathcal{F}$:

$$\mathcal{P}_n^X \cong \mathcal{F}^n, \quad \mathcal{P}_n^Z \cong \mathcal{F}^n.$$

We do this in the obvious way by sending X_i to the basis vector with 1 in the $i-$th entry, and similarly for each Z_i. We also identify the computational basis of our statespace $\mathbb{C}[2^n]$ with $\mathcal{F}^n$ in the obvious way:

$$\mathbb{C}[2^n] \cong \mathbb{C}[\mathcal{F}^n].$$

This has the potential to be very confusing, and so where appropriate we use X and Z subscripts.

X-type operators act on the $\mathbb{C}[\mathcal{F}^n]$ basis vectors using $\mathcal{F}$ addition:

$$g_X \in \mathcal{P}_n^X \cong \mathcal{F}^n, \quad g_X : v \longmapsto g_X + v$$

Z-type operators act on the $\mathbb{C}[\mathcal{F}^n]$ basis vectors using $\mathcal{F}$ inner product:

$$g_Z \in \mathcal{P}_n^Z \cong \mathcal{F}^n, \quad g_Z : v^\top \longmapsto g_Z v^\top$$

This is an $\mathcal{F}$ scalar, just zero or one. We think of this as a "syndrome". This suggests that actually these Z-type operators live in the dual vector space $\mathcal{F}_n$. Because of the underlying symmetry (and notational confusion) between the X and Z-type operators, we make the convention that by default all our $\mathcal{F}$-vectors come as row vectors (ie. dual vectors). This means we use the transpose operator $^\top$ to indicate a primal (column) vector.

It doesn't make sense to add an X-type operator and a Z-type operator:

$$g_Z + g_X \quad \textit{no no!!!}$$

but it does make sense to take the inner product:

$$g_Z g_X^\top = g_X g_Z^\top.$$

This is an $\mathcal{F}$ scalar which gives the commutator of the two operators.

An $\mathcal{F}$-linear operator such as $A : \mathcal{F}^n \to \mathcal{F}^m$ acts on the left as $u^\top \mapsto Au^\top$. It also acts on dual vectors as $A : \mathcal{F}_m \to \mathcal{F}_n$ which corresponds to acting on the right: $v \mapsto vA$. We call the rowspace of A the *span* and denote it as

$$\langle A \rangle = \{vA | v \in \mathcal{F}_m\}$$

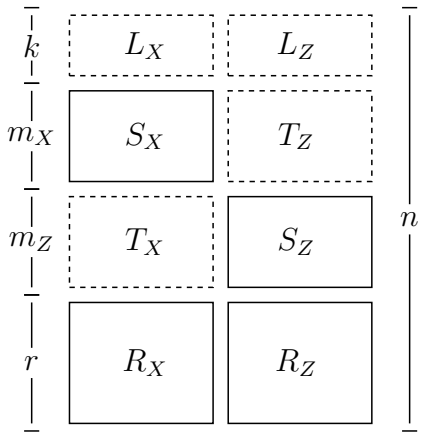
The kernel of A is defined as

$$\ker(A) = \{u^\top | u^\top \in \mathcal{F}^n, \quad Au^\top = 0\}.$$

We wish to use this language to decompose a CSS gauge group G. First we write the gauge group generators in terms of X-type and Z-type operators:

$$G_0 = G_X \cup G_Z.$$

Following the theory from the previous section, we are going to rewrite the gauge group generators as a union of stabilizer generators $S_0 = S_X \cup S_Z$ and reduced gauge generators $R_0 = R_X \cup R_Z$. Similarly, the error operators will be split into X and Z type operators T_X and T_Z and finally the logical operators L_X and L_Z. We summarize all of these sets in the following table:



The solid rectangles indicate operators that span the X and Z parts of the gauge group, and the dashed rectangles indicate operators that do not live inside the gauge group.

We consider each of these blocks $L_X, L_Z, S_X, T_Z, T_X, S_Z, R_X, R_Z$, as well as G_X, G_Z, as either a set of $\mathcal{F}_n$ vectors (the rows) or as an $\mathcal{F}$-linear operator. For example, we write $u \in R_X$ to mean u is a row of the matrix R_X.

We first find the stabilizers S_Z. These are built out of $\mathcal{F}_n$ vectors from the span of G_Z that commute with the rows of G_X :

$$\begin{aligned}\langle S_Z \rangle &= \{\ vG_Z \mid vG_Z G_X^\top = 0, \quad v \in F_{|G_Z|}\ \} \\ &= \{\ vG_Z \mid v^\top \in \ker(G_X G_Z^\top)\ \}.\end{aligned}$$

The generators (rows of S_Z) are then extracted from this span by row reduction. We swap the role of X and Z to find S_X.

Once we have the stabilizers, in order to complete the above table as a presentation of the Pauli group we solve the following $\mathcal{F}$-linear block matrix equation,

$$\begin{pmatrix} L_X \\ S_X \\ T_X \\ R_X \end{pmatrix} \begin{pmatrix} L_Z \\ T_Z \\ S_Z \\ R_Z \end{pmatrix}^\top = I,$$

subject to the restriction that the rows of R_X lie in the span of G_X and the rows of L_X do not. Similarly for R_Z and L_Z. This set of 16 equations is quadratic in the unknown variables and so it is not obvious how to proceed, but it turns out a systematic way can be found.

We begin by finding L_Z. These operators satisfy the following *homology* condition:

$$l_Z \in L_Z \ \text{ is given by } \ l_Z^\top \in \ker(G_X) \ \mod \ \langle S_Z \rangle.$$

In other words, L_Z is formed from a basis for the kernel of G_X row-reduced using S_Z. To be more specific we take any direct sum decomposition

$$\mathcal{F}_n = \langle S_Z \rangle \oplus V$$

then the operation of mod $\langle S_Z \rangle$ is the projection onto V. We can explicitly write such a projector as the $n \times n$ matrix given by

$$P_Z = I + A^\top S_Z$$

where the matrix A is the $m_Z \times n$ matrix consisting of the leading 1's in any row-reduction of S_Z. We define P_X similarly for the operation of mod $\langle S_X \rangle$.

To find L_X we solve the following $\mathcal{F}$-linear system:

$$\begin{pmatrix} L_Z \\ G_Z \end{pmatrix} L_X^\top = \begin{pmatrix} I \\ 0 \end{pmatrix}$$

The reduced gauge group matrix R_X is found as a row-reduction of $G_X P_X$. We cannot merely set R_Z to be $G_Z P_Z$ because we also require $R_X R_Z^\top = I$. Instead we define the auxiliary matrix $\widetilde{R}_Z$ to be a row-reduction of $G_Z P_Z$.

The error operators T_X are then found as the solution to the $\mathcal{F}$-linear system:

$$\begin{pmatrix} L_Z \\ S_Z \\ \widetilde{R}_Z \end{pmatrix} T_X^\top = \begin{pmatrix} 0 \\ I \\ 0 \end{pmatrix}$$

And then the operators T_Z solve the $\mathcal{F}$-linear system:

$$\begin{pmatrix} L_X \\ S_X \\ T_X \\ R_X \end{pmatrix} T_Z^\top = \begin{pmatrix} 0 \\ I \\ 0 \\ 0 \end{pmatrix}$$

Finally at this point R_Z is given as the solution to

$$\begin{pmatrix} L_X \\ S_X \\ T_X \\ R_X \end{pmatrix} R_Z^\top = \begin{pmatrix} 0 \\ 0 \\ 0 \\ I \end{pmatrix}$$

Note that R_Z and $\widetilde{R}_Z$ have identical span and so we have $R_Z T_X^\top = 0$.

We call this array of eight $\mathcal{F}$-linear matrices an (L, S, T, R)-decomposition of the gauge group. In general this will not be unique for any given gauge group.

4.1 The Hamiltonian

The complex Hilbert state space of our Hamiltonian has 2^n dimensions and we write this space as $\mathbb{C}[2^n]$. This notation is meant to suggest that we are forming a $\mathbb{C}$ vector space using 2^n "points" as basis vectors. Working in the computational basis, we do indeed have 2^n such points; these are the elements of $\mathcal{F}_n$. And so we make the identification

$$\mathbb{C}[2^n] \cong \mathbb{C}[\mathcal{F}_n].$$

In other words, we are labeling our basis vectors with elements of $\mathcal{F}_n$ and therefore such notation as

$$\langle u|H|v\rangle$$

with $u, v \in \mathcal{F}_n$ makes sense. We will make further use of this below, by writing $\mathcal{F}$-vector space computations inside the Dirac brackets.

Returning to the code (L, S, T, R)-decomposition above, given the Pauli operator $t \in T$ such that $t = t_X t_Z$ (in $\mathcal{P}_n$) we get a basis for the irrep ρ_t:

$$\{|vR_X + t_X\rangle \text{ such that } v \in \mathcal{F}_r\}.$$

In other words, the basis of the irrep ρ_t is an affine subspace of $\mathcal{F}_n$. Each such affine subspace is indexed by an element of $\mathcal{F}_r$ and all of these are translates of each other, so we make the following identification:

$$\mathbb{C}[\{vR_X + t_X\}_{v\in\mathcal{F}_r}] \cong \mathbb{C}[\mathcal{F}_r].$$

This will allow us to write the components of each block H_t of the Hamiltonian as $\langle u|H_t|v\rangle$ for $u, v \in \mathcal{F}_r$. We make this identification of affine subspaces not out of laziness but because it will help us to compare each of the Hamiltonian blocks H_t below.

Important: The computational basis identifies basis vectors of $\mathbb{C}[2^n]$ with elements of a finite vector space $\mathcal{F}_n$:

$$\mathbb{C}[2^n] \cong \mathbb{C}[\mathcal{F}_n].$$

The (L, S, T, R)-decomposition naturally decomposes $\mathcal{F}_n$ into 2^{m_Z+k} affine subspaces:

$$\{vR_X + t_X + l_X\}_{v\in\mathcal{F}_r}$$

for each $t_X \in \langle T_X\rangle, l_X \in \langle L_X\rangle$. Each such affine subspace forms a basis for the irreducible blocks H_{t_X,t_Z} of H, and can be naturally identified with $\mathcal{F}_r$:

$$H_{t_X,t_Z} : \mathbb{C}[\mathcal{F}_r] \to \mathbb{C}[\mathcal{F}_r].$$

We now wish to understand the action of the gauge group on each of its irreps. Starting with the $t_X, t_Z = 0, 0$ irrep, this is where each of the stabilizers has a trivial action. In $\mathcal{F}^n$ this corresponds to the additive action of the zero vector.

States $u \in \langle R_X\rangle$ can be built from a vector matrix product

$$u = vR_X$$

with $v \in \mathcal{F}_r$. Since $R_X R_Z^\top = I$ we can write $v = uR_Z^\top$. Each $g_X \in G_X$ acts on u to give

$$\begin{aligned} u_1 &= (u + g_X) \bmod \langle S_X\rangle \\ &= (u + g_X)P_X \\ &= (vR_X + g_X)P_X. \end{aligned}$$

writing $u_1 = v_1 R_X$ we then have

$$\begin{aligned} v_1 &= (vR_X + g_X)P_X R_Z^\top \\ &= v + g_X R_Z^\top. \end{aligned}$$

So we have that working in the computational basis, the action of the X part of the gauge group in the $t_X, t_Z = 0, 0$ irrep is to send $v \in \mathcal{F}_r$ to $v + g_X R_Z^\top$. In summary, we have the following contributions from the G_X terms of the Hamiltonian:

$$\langle v|H_{0,0}|v + g_X R_Z^\top\rangle \mathrel{+}= 1, \quad \text{for } g_X \in G_X, v \in \mathcal{F}_r$$

where we use the $+=$ notation because there may be other contributions to the same component. These terms will always be off the diagonal unless g_X is a stabilizer.

The action of the G_Z gauge group contributes to the diagonal of H. These diagonal terms apply a kind of "potential energy" penalty to the basis states that depends on the *syndrome* vector:

$$\text{syndrome}(u) = G_Z u^\top$$

for $u^\top \in \mathcal{F}^n$. This is an $\mathcal{F}$-vector that has one component for each row of G_Z. Writing $|G_Z|$ for the number of these rows, and using a *weight* function w that just counts the number of non-zero entries in any $\mathcal{F}$-vector we have the following contributions to the Hamiltonian:

$$\langle v|H_{0,0}|v\rangle \mathrel{+}= |G_Z| - 2w(G_Z R_X^\top v^\top),$$

for $v \in \mathcal{F}_r$.

Adding up all of the above we have in summary,

$$H_{0,0} = \sum_{\substack{v\in\mathcal{F}_r \\ g_X\in G_X}} |v + g_X R_Z^\top\rangle\langle v| + \sum_{v\in\mathcal{F}_r} \left(|G_Z| - 2w(G_Z R_X^\top v^\top)\right) |v\rangle\langle v|.$$

For any $t_X \in \langle T_X\rangle$ the Hamiltonian block $H_{t_X,0}$ has components indexed by basis vectors:

$$u = vR_X + t_X$$

this means that the G_X gauge terms have the same effect on $H_{t_X,0}$ as $H_{0,0}$ and only the diagonal has changed:

$$H_{t_X,0} = \sum_{\substack{v\in\mathcal{F}_r \\ g_X\in G_X}} |v + g_X R_Z^\top\rangle\langle v| + \sum_{v\in\mathcal{F}_r} \left(|G_Z| - 2w(G_Z R_X^\top v^\top + G_Z t_X^\top)\right) |v\rangle\langle v|.$$

The general form of each the Hamiltonian block is:

$$\begin{aligned} H_{t_X,t_Z} = &\sum_{\substack{v\in\mathcal{F}_r \\ g_X\in G_X}} \eta(t_Z g_X^\top)\ |v + g_X R_Z^\top\rangle\langle v| \\ &+ \sum_{v\in\mathcal{F}_r} \left(|G_Z| - 2w(G_Z R_X^\top v^\top + G_Z t_X^\top)\right) |v\rangle\langle v|. \end{aligned}$$

Here we use η to send $t_Z g_X^\top$ which is an $\mathcal{F}$ value to the multiplicative subgroup $\{\pm 1\}$ of $\mathbb{C}$:

$$\eta(0) = 1,\ \eta(1) = -1.$$

The $\eta(t_Z g_X^\top)$ term is a kind of parity check that picks up one phase flip for (some of) the X type stabilizers found in g_X. This works because T_Z is a left inverse of $S_X^\top$. The $t_Z \in \langle T_Z \rangle$ selects which X type stabilizers act as -1 in this irrep.

In summary, we have the complete representation theory for CSS gauge code Hamiltonians.

5 Gapless 1D models

In this section we briefly introduce two important one dimensional models that fit into the CSS gauge code framework.

The XY-model lives on a one dimensional chain of n qubits. We write the gauge group generators as

$$G_0 = \{X_i X_{i+1}, Z_i Z_{i+1} \ \text{ for } \ i = 1, ..., n\}$$

with periodic boundary conditions. For n even this model has no logical operators, one X-type stabilizer and one Z-type stabilizer. With n odd, there are no stabilizers and $k = 1$. Normally the gauge operators are written as $\{X_i X_{i+1}, Y_i Y_{i+1}$ for $i = 1, ..., n\}$ but note that there is an automorphism of the Pauli group that sends G_0 to these operators. For n even, this model is exactly solvable, and the gap goes to zero as the system size grows [20].

For the one dimensional transverse field Ising model, we have

$$G_0 = \{X_i, Z_i Z_{i+1} \ \text{ for } \ i = 1, ..., n\},$$

with periodic boundary conditions. This model has one X-type stabilizer and no logical operators. This model is also exactly solvable, with gap going to zero as the system size grows [23].

6 Perron-Frobenius theory

We begin this section with the following two definitions. A CSS gauge code is *self-dual* when the X and Z type gauge generators are equal:

$$G_X = G_Z.$$

The XY-model is an example of a self-dual CSS gauge code. A CSS gauge code is *weakly self-dual* when there is a permutation P on the set of n qubits that induces equality of the gauge generators:

$$G_X P = G_Z,$$

where we write P as an $n \times n$ permutation matrix. The compass model is then weakly self-dual when we transpose the square lattice of $l \times l$ qubits.

We now turn to another notion of reducibility, coarser than the group theoretic reducibility. One way to understand this is via graph theory. Given a CSS gauge code Hamiltonian H, we see that the diagonal terms (working in the computational basis) come from the Z operators and the off-diagonal terms come from X type operators. We think of the Z operators as potential energy, and the X operators as kinetic terms. This suggests the following definition. We define a graph Γ with vertices the 2^n computational basis elements, and edges:

$$|v\rangle \mapsto g_X|v\rangle, \quad \text{for all} \quad v \in \mathcal{F}_n, g_X \in G_X.$$

These are undirected edges because $g_X^2 = I$. We also add weighted loops corresponding to the Z-type gauge operators:

$$|v\rangle \mapsto \sum_{g_Z \in G_Z} g_Z|v\rangle, \quad \text{for all} \quad v \in \mathcal{F}_n.$$

In this way we can consider H and Γ interchangeably, as either a matrix or a graph. An irreducible matrix is one whose corresponding graph is connected. Using the $\mathcal{F}$-linear (L, S, T, R)-decomposition of the gauge group we then have the following:

In the computational basis, any CSS gauge code Hamiltonian H is the direct sum of 2^{m_Z} irreducible matrices Γ_{t_X} indexed by $t_X \in \langle T_X \rangle$ with multiplicity 2^k:

$$H = \bigoplus_{\substack{t_X \in \langle T_X \rangle \\ l_X \in \langle L_X \rangle}} \Gamma_{t_X}.$$

A basis for each Γ_{t_X} is given by a coset of G_X in $\mathcal{F}_n$:

$$\{vS_X + uR_X + t_X\}_{v \in \mathcal{F}_{m_X}, u \in \mathcal{F}_r}.$$

The off-diagonal entries of H are all positive. If we use a spectral shift operator, a constant multiple of the identity $+cI$, we find that $H + cI$ is a non-negative matrix. We call such a matrix *Perron-Frobenius.* These are also known as *Stoquastic* Hamiltonians in the literature [6]. Each of the blocks Γ_{t_X} is also Perron-Frobenius and in combination with their irreducibility, the Perron-Frobenius theorem gives:

For each $t_X \in \langle T_X \rangle$ the largest eigenvalue of Γ_{t_X},

$$\lambda_1(\Gamma_{t_X})$$

is non-degenerate, and is associated with an eigenvector

$$v_1(\Gamma_{t_X})$$

with positive components.

Now we make the restriction that H comes from a weakly self-dual gauge code. Our goal will be to show that the groundspace of H is spanned by vectors which are stabilized. This will imply that $\lambda_1(H) = \lambda_1(H_{0,0})$. To begin, let v_1 be the top eigenvector of Γ_{t_X} with $t_X \in \langle T_X \rangle$. Because v_1 has all positive components it will be fixed by any operator $s \in \langle S_Z \rangle$. To show that the X type stabilizers also fix v_1 we use weak self-duality and see that by change of basis we can swap the roles of the X and Z type operators.

Fact 1:
For any weakly self-dual gauge code Hamiltonian H every ground-state is stabilized and so

$$\lambda_1(H) = \lambda_1(H_{0,0})$$

and for $t_X \in \langle T_X \rangle, t_Z \in \langle T_Z \rangle$ with $t_X \neq 0$ or $t_Z \neq 0$

$$\lambda_1(H) > \lambda_1(H_{t_X,t_Z}).$$

Notice that $H_{0,0}$ is also Perron-Frobenius (in the computational basis) and so has non-degenerate groundspace, but it appears with multiplicity 2^k within the Hamiltonian H and this accounts for the degeneracy of the groundspace of H.

The next goal is to search for the second eigenvalue of H, $\lambda_2(H)$. Using a basis change and the above equation for $H_{t_X,0}$ we have

Fact 2: Given a weakly self-dual gauge code Hamiltonian H and $t_X \in \langle T_X \rangle$, $t_X \in \langle T_Z \rangle$ the blocks $H_{t_X,0}$ and H_{0,t_Z} are Perron-Frobenius.

We extend the above argument:

For a weakly self-dual gauge code Hamiltonian,

$$\lambda_1(H_{t_X,0}) < \lambda_1(H_{t_X,t_Z}) \quad \text{and}$$
$$\lambda_1(H_{0,t_Z}) < \lambda_1(H_{t_X,t_Z}),$$

where $t_X \neq 0$ and $t_Z \neq 0$.

In summary, to find the spectral gap of a weakly self-dual gauge code Hamiltonian, which is the difference of the top two eigenvalues of the Hamiltonian, we need only examine the top two eigenvalues of $H_{0,0}$ and the top eigenvalue of $H_{t_X,0}$ for each $t_X \in T_X$.

7 The gauge color code model

We now turn to the central animal that motivated the theory developed in this paper.

The three dimensional gauge color code [3, 4, 19] is a self-dual CSS gauge code. It is based on the following geometric construction known as a *colex* [5]. We begin with a tetrahedron and subdivide it into finitely many convex 3-dimensional polytopes, or *bodies*. Each body has a boundary consisting of 0-dimensional cells which we call *vertices*, 1-dimensional cells called *edges* and 2-dimensional cells called *faces*. By a *cell* we mean any of these 0,1,2 or 3-dimensional convex polytopes. Any two bodies in this tetrahedral subdivision will have either empty intersection or otherwise intersect on a common vertex, edge or face. When the intersection is on a face these two bodies are called *adjacent*. Two vertices in the same edge will also be called adjacent. Each body is colored by one of four *colors*, either taken to be red, green, blue, yellow or otherwise an element of the set $\{1, 2, 3, 4\}$. The four exterior triangular faces of the bounding tetrahedron are called *regions,* the intersection of two regions is called a *border* and the intersection of three regions is called a *corner*. A cell not contained within any region is called an interior cell.

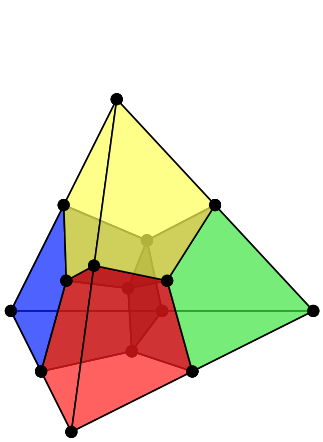
This colored cellulation is required to have the following further properties:

1. Adjacent bodies have different colors.

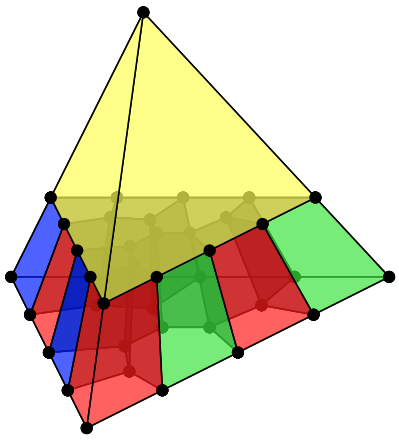
2. Each region has a unique color such that no bodies intersecting that region has that color.

3. All vertices are adjacent to four other vertices, except for the corner vertices which are adjacent to three other vertices.

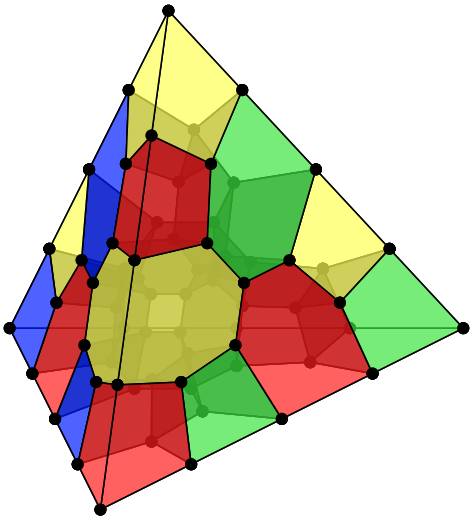
Here we show some instances of this construction, along with the colors of the unobscured bodies. Each instance is labeled by the number of vertices n.



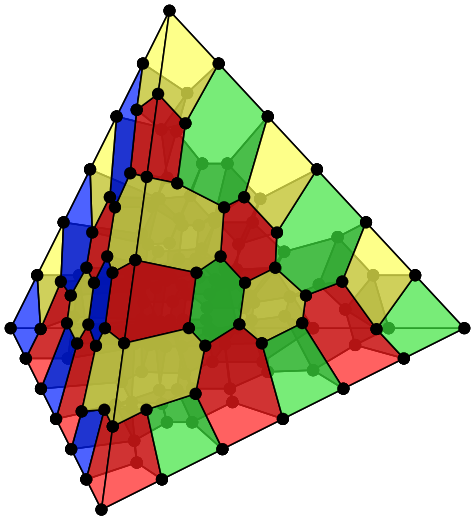
$n = 15$



$n = 39$



$n = 65$



$n = 175$

We note the following consequences of the above conditions. Every face supports an even number of vertices. We now think of each region as corresponding to a "missing" body. Each edge is then contained in the boundary of three bodies. This means we can associate a unique color to each edge, which is also the color of two bodies intersecting a vertex of the edge. That is, each edge joins two bodies of the same color. Each face bounds two bodies, and so we color each face with the two colors of these bodies.

Using this cellulation we now construct the gauge code. Qubits are associated with the n vertices. We associate operators to other cells, or union of cells, by using the contained vertices as support. Because this is a self-dual code, the same goes for both the X and Z type operators. The $X/Z-$type gauge group is generated by $X/Z-$type operators supported on each face. The $X/Z-$type stabilizer group is generated by $X/Z-$type operators supported on each body. There is one $X/Z-$type logical operator and these are generated by $X/Z-$type operators supported on any region.

8 The orbigraph

Performing numerics on small gauge code models makes it evident that there is a great deal more symmetry than we have found using the above techniques. In particular, the components of eigenvectors have many identical values. This motivates the following exploration of the symmetry of the Hamiltonian.

A *graph automorphism* is a permutation matrix P such that

$$P^T H P = H.$$

The set of all such graph automorphisms form a group $\mathcal{A}$. The goal here is to find eigenvectors of H that are invariant under the action of $\mathcal{A}$. Such an eigenvector will have components that are constant on the orbits of $\mathcal{A}$, so therefore it will live naturally on the vector space whose

basis is these orbits. We can do better than this, and actually project the graph itself down onto this space. We do this by constructing a new graph which we call the *orbigraph*. The matrix for the orbigraph is written $H/\mathcal{A}$ and acts on the vector space with basis consisting of the orbits of $\mathcal{A}$. It follows that the components of $H/\mathcal{A}$, indexed by a pair of $\mathcal{A}-$orbits i and j, is defined by

$$(H/\mathcal{A})_{ij} = |\{g \in G \text{ s.t. } gv \in j\}| \text{ where } v \in i.$$

In other words, $(H/\mathcal{A})_{ij}$ counts the number of gauge group elements that sends any particular element v of the $\mathcal{A}-$orbit i to the $\mathcal{A}-$orbit j.

Here is a simple example. We take as gauge group

$$G = \{XII, IXI, IIX, ZII, IZI, IIZ\}.$$

The Hamiltonian $H = \sum_{g\in G} g$ has matrix:

$$H = \begin{pmatrix} 3 & 1 & 1 & 1 & . & . & . & . \\ 1 & 1 & . & . & 1 & 1 & . & . \\ 1 & . & 1 & . & 1 & . & 1 & . \\ 1 & . & . & 1 & . & 1 & 1 & . \\ . & 1 & 1 & . & -1 & . & . & 1 \\ . & 1 & . & 1 & . & -1 & . & 1 \\ . & . & 1 & 1 & . & . & -1 & 1 \\ . & . & . & . & 1 & 1 & 1 & -3 \end{pmatrix}$$

where we indicate zero entries with a dot. In this case, the automorphism group is $\mathcal{A} = S_3$, there are four $\mathcal{A}-$orbits, and

$$H/\mathcal{A} = \begin{pmatrix} 3 & 3 & . & . \\ 1 & 1 & 2 & . \\ . & 2 & -1 & 1 \\ . & . & 3 & -3 \end{pmatrix}.$$

In this case the automorphism group $\mathcal{A}$ of the graph is the same as the automorphism group $Aut(G)$ of the gauge group, but in general it is possible that $Aut(G) < \mathcal{A}$.

Note that the orbigraph is no longer Hermitian, and in general will not even be normal.

We can also write $H/\mathcal{A} = QHP$ using the following two matrices for P and Q :

$$Q = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & . & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & 1 & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & 1 & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix}.$$

The idea is that each column of P sums over an orbit, and each row of Q chooses one member of each orbit.

In general, we will apply this idea to each of the blocks $H_{t_X,0}$. From **Fact 2** above, and using the fact that we are summing over a trivial representation of $\mathcal{A}$ we have the following:

> **Fact 3:** The spectrum of the orbigraph of $H_{t_X,0}$ contains the ground eigenvalue of $H_{t_X,0}$.

8.1 The compass model

For the next example we take the $l = 3$ compass model. $H_{0,0}$ acts on a 16 dimensional space. The order of $\mathcal{A}$ is 72, and we find three $\mathcal{A}-$orbits. The orbigraph method can be applied in the case where the Hamiltonian weights for X and Z are uniform as w_X and w_Z. We separate the X and Z terms of the orbigraph to show how this works:

$$H_{0,0}/\mathcal{A} = \begin{pmatrix} . & 9 & . \\ 1 & 4 & 4 \\ . & 6 & 3 \end{pmatrix} + \begin{pmatrix} 9 & . & . \\ . & 1 & . \\ . & . & -3 \end{pmatrix} = \begin{pmatrix} 9 & 9 & . \\ 1 & 5 & 4 \\ . & 6 & . \end{pmatrix}$$

This can be solved analytically and we find $\lambda_1 = 4 + 2\sqrt{13} \cong 11.21110255$. Keep in mind that the original state space has dimension $2^9 = 512$ so we have come a long way down to 3.

By exactnumerical diagonalization we get the spectrum of $H_{0,0}$ and note that the orbigraph lifts all degeneracy as well as missing some excited eigenspaces:

λ	$H_{0,0}$ degeneracy	$H_{0,0}/\mathcal{A}$ degeneracy
11.2111025509	1	1
6.0	1	1
2.0	4	
0.0	4	
-3.21110255093	1	1
-4.0	4	
-6.0	1	

The reason we miss some excited spaces is that they do not contain any trivial irrep of $\mathcal{A}$. Note that we miss the eigenspace with $\lambda = -6$ even though this is one dimensional. It must contain some other non-trivial one dimensional irrep. If we want to make an orbigraph for these other spaces we would construct the orbigraph by summing over orbits using different characters of $\mathcal{A}$ (these are *momenta* in the abelian terminology). See [10] chapter 5 for more details.

8.2 The gauge color code model

The smallest gauge color code has $n = 15$ qubits, G_0 has 18 each of X/Z-type gauge operators, and 4 each of X/Z-type stabilizer generators. $H_{0,0}$ acts on a 64 dimensional space, and $\mathcal{A}$

has order 720. We find 7 orbits:

$$H_{0,0}/\mathcal{A} = \begin{pmatrix} 18 & 18 & . & . & . & . & . \\ 3 & 12 & 15 & . & . & . & . \\ . & 6 & 6 & 12 & . & . & . \\ . & . & 9 & . & 9 & . & . \\ . & . & . & 12 & -6 & 6 & . \\ . & . & . & . & 15 & -12 & 3 \\ . & . & . & . & . & 18 & -18 \end{pmatrix}$$

The eigenvalue equation results in the recurrence relation:

$$\lambda a_k = 3k a_{k-1} + (18 - 6k)a_k + (18 - 3k)a_{k+1},$$

which has largest solution $\lambda_1 = 18\sqrt{2} \cong 25.4558441$.

Numerics give the full spectrum of $H_{0,0}$ and we note that the orbigraph lifts all degeneracy as well as preserving every eigenvalue:

λ	$H_{0,0}$ degeneracy	$H_{0,0}/\mathcal{A}$ degeneracy
25.4558441227	1	1
16.9705627485	6	1
8.48528137424	15	1
0.0	20	1
-8.48528137424	15	1
-16.9705627485	6	1
-25.4558441227	1	1

The second eigenvalue of H comes from a recurrence relation in two variables which has solution $\lambda_2 = 9\sqrt{2} + 3\sqrt{10} \cong 22.21475504$. So the gap for this Hamiltonian is

$$\lambda_1 - \lambda_2 = 9\sqrt{2} - 3\sqrt{10} \cong 3.24108908.$$

8.3 A table of orbigraphs

Here we tabulate the order of the graph automorphism group $\mathcal{A}$ of $H_{t_X,0}$ and the resulting orbigraph sizes, which is the number of $\mathcal{A}-$orbits. We use the software library `nauty`[21] for computing graph automorphisms.

model	n	r	t_X	$\lvert\mathcal{A}\rvert$	$\lvert\mathcal{A}-\text{orbits}\rvert$	$\lvert\text{Aut(code)}\rvert$
1D XY	9	8	0	18	23	18
	10	8	0	200	10	20
	11	10	0	22	63	22
	12	10	0	288	36	24
2D compass	9	4	0	72	3	36
	9	4		12	4	
	16	9	0	128	24	64
	16	9		16	48	
	25	16	0	200	430	100
	25	16		20	3418	
3D compass	27	22	0	216	20609	
	27	22		72	60283	
3D gauge color	15	6	0	720	7	$\lvert S_4\rvert = 24$
	15	6		36	16	
	39	18	0	36	14400	$\lvert\mathbb{Z}_3\rvert = 3$
	65	32	0			$\lvert\mathbb{Z}_4\rvert = 4$

We also show the order of Aut(code) which is the automorphism group of the gauge code, defined as follows. Elements of this group act by permuting the n qubits. Such an action is given by the $\mathcal{F}$-linear permutation matrices P_n, P_Z and P_X, such that the following $\mathcal{F}$-linear equations hold:

$$P_X G_X P_n = G_X,$$
$$P_Z G_Z P_n = G_Z.$$

This implies that the action of Aut(code) commutes with the Hamiltonian and preserves eigenvalues of the stabilizers. Therefore this group action restricts to an action on $H_{0,0}$.

Mystery: the graph automorphism group is often bigger, sometimes much bigger, than the automorphism group of the underlying code. So where is the extra symmetry coming from?

A crucial hint is provided by the fact that these graph symmetries respect the symplectic structure in the following sense. Recall that we defined graph symmetries via permutation matrices P such that $P^\top H P = H$. In other words, P is a permutation on the set of basis vectors $\mathcal{F}^n$. It turns out that not only are these maps $\mathcal{F}$-linear, but they also preserve syndromes. By this we mean we can find an $\mathcal{F}$-linear map Q such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}^n & \xrightarrow{G_z} & \mathcal{F}^{|G_Z|} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathcal{F}^n & \xrightarrow{G_z} & \mathcal{F}^{|G_Z|} \end{array}$$

That P has this extra $\mathcal{F}$-linear behaviour is not true in general, but it holds for the orbigraphs in the above table.

All this suggests a further examination of the commutation structure of the code. Indeed, this is captured by the notion of a Lie algebra, which we turn to next.

9 Lie algebra representations

We now turn to a finer notion of representation theory, being the representation theory of semi-simple Lie algebras. We outline the theory of representations of semi-simple Lie algebras [12] and show how this applies to CSS gauge code Hamiltonians.

An abstract *Lie algebra* $\mathfrak{g}$ is a vector space together with a bilinear form:

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that $[A, B] = -[B, A]$ and $[A, [B, C] + [B, [C, A]] + [C, [A, B]] = 0$.

A *representation* of a Lie algebra $\mathfrak{g}$ on a vector space V is a linear map

$$\rho : \mathfrak{g} \to \mathrm{GL}(V)$$

that sends the abstract bracket to the concrete one:

$$\rho([A, B]) = \rho(A)\rho(B) - \rho(B)\rho(A).$$

A Lie algebra requires us to "forget" about multiplication of operators, and only allow the taking of brackets (and linear combinations.) This is not as crazy as it may at first seem. The fundamental calculation in the theory of quantum stabilizer codes is actually a Lie algebra calculation. Consider a state $|\psi\rangle$ that is stabilized by some operator $s \in S$:

$$s|\psi\rangle = |\psi\rangle,$$

We wish to understand the effect of an error operator $t \in T$ on our state $|\psi\rangle$, where we have $st = -ts$:

$$st|\psi\rangle = ts|\psi\rangle + [s, t]|\psi\rangle = -ts|\psi\rangle = -t|\psi\rangle,$$

which shows that $t|\psi\rangle$ is a -1 eigenvalue of s. The key point here is that nowhere did we need to multiply (compose) two operators, it was all done using the bracket.

We continue the analysis of CSS gauge code Hamiltonians. The terms of the Hamiltonian block H_{t_X,t_Z} form a Lie algebra which we denote $\mathfrak{g}_{t_X,t_Z}$. The basis for this Lie algebra is formed from all iterated brackets of the terms in H_{t_X,t_Z}. This is a concrete Lie algebra, or in other words, it comes with a representation on the vector space $\mathbb{C}[\mathcal{F}_r]$.

The simplest such example of this is the one qubit Lie algebra which is generated by X and Z. This will have basis $\{X, Z, 2XZ = [X, Z]\}$ and so is a three dimensional Lie algebra. In fact, it is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ the Lie algebra of traceless two by two matrices. Notice that we do not include I in these algebras as this is associated to the multiplicative (group) structure of the operators. Moreover we never need consider Hamiltonians with such terms

as these just shift the spectrum by a constant. Notice also that if we try to build a larger Lie algebra from taking iterated brackets of the n-qubit operators $\{X_i, Z_i\}$ we still only get a direct sum of n copies of $\mathfrak{sl}_2(\mathbb{C})$. So while the group generated by products of $\{X_i, Z_i\}$ is the whole Pauli group $\mathcal{P}_n$, as a Lie algebra we get something finer. This reflects the fact that these terms break up into n commuting "pieces", which motivates the following definition.

An *ideal* of a Lie algebra $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ that "eats" all the other elements of $\mathfrak{g}$:

$$[A, B] \in \mathfrak{h}, \quad \text{for all} \quad A \in \mathfrak{h}, B \in \mathfrak{g}.$$

In general the structure of the Lie algebra $\mathfrak{g}_{t_X,t_Z}$ will be more complicated than the corresponding group structure. But we do have the following:

The Lie algebra $\mathfrak{g}_{t_X,t_Z}$ is a semi-simple Lie algebra.

This follows from the characterization of semi-simple Lie algebras as those having no nonzero abelian ideals. Any such ideal would correspond to the existence of stabilizers, and we already got rid of these.

We also have a ready made *Cartan subalgebra* of $\mathfrak{g}_{t_X,t_Z}$. This is the algebra $\mathfrak{h}_{t_X,t_Z}$ generated by the $Z-$type terms of H_{t_X,t_Z}. The *weight spaces* are the simultaneous eigenspaces of the operators in the Cartan subalgebra. These eigenspaces are labeled by what we called syndromes previously, and these are all one dimensional because the span of R_X does not intersect the kernel of R_Z. Therefore the representation of $\mathfrak{g}_{t_X,t_Z}$ on $\mathbb{C}[\mathcal{F}_r]$ is irreducible.

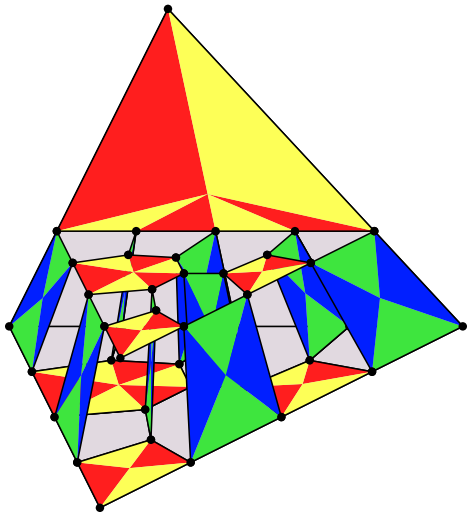
Note that a decomposition of a Lie algebra into disjoint (apart from zero) ideals will give a direct sum decomposition of the Lie algebra. Also, any irreducible representation of a direct sum of Lie algebras can be considered as the tensor product of irreducible representations of the individual summands. This is key to the numerical algorithms below: we examine the ideals generated by the terms in the Hamiltonian block H_{t_X,t_Z}. Each such ideal corresponds to a direct summand of $\mathfrak{g}_{t_X,t_Z}$ and so the spectrum of H_{t_X,t_Z} can be written as a sum over spectra of smaller gauge code Hamiltonians corresponding to each ideal.

9.1 Ideal structure of gauge codes

Ideals are generated by anti-commuting operators, and so to find these ideals we search for a partition of the gauge group operators such that operators from different partitions commute.

The XY-model has gauge group terms X_iX_{i+1}, Z_iZ_{i+1} for $i = 1, ..., n$. When $n = 2k$ is even these terms generate two Lie algebra ideals. For $i = 1, ..., k$ the terms $X_{2i}X_{2i+1}$ and $Z_{2i+1}Z_{2i+2}$ generate one ideal, and the other ideal comes from switching X and Z.

We next examine the Lie algebra ideal structure of the gauge color code. Two faces operators, of X and Z type, will anti-commute only when they intersect on a single vertex. This only happens when such faces have disjoint coloring. Here we show an example of this in the $n = 39$ model:



There are three of these arrangements, each corresponding to the three ways of partitioning the set of colors into two sets of two. It follows that $\mathfrak{g}_{t_X,t_Z}$ is the direct sum of 6 disjoint ideals, and specifically, that each Hamiltonian term in H_{t_X,t_Z} lies in a single one of these ideals. This result is crucial for obtaining the exact diagonalization numerical results below.

9.2 Lie algebra classification

Here we explicitly compute decompositions of $\mathfrak{g}_{0,0}$ into the direct sum of simple Lie algebras. First we review the classificiation of simple Lie algebras.

Let n be the rank of a simple lie algebra $\mathfrak{g}$. This is the dimension of the Cartan subalgebra $\mathfrak{h}$.

The simple Lie algebras are classified into four infinite series A_n, B_n, C_n, D_n as well as five other exceptional Lie algebras that we will not need.

The A_n series can be constructed as $\mathfrak{sl}_{n+1}$ which are the traceless $(n+1)\times(n+1)$ matrices. Therefore the algebra dimension is n^2+2n.

For $n \geq 2$ the B_n series comes from the Lie algebras $\mathfrak{so}_{2n+1}$. These can be constructed as $(2n+1)\times(2n+1)$ matrices in block form

$$\begin{pmatrix} P & Q & T \\ R & S & U \\ V & W & 0 \end{pmatrix}$$

with Q, R anti-symmetric and $P^\top = -S, T = -W^\top, U = -V^\top$. This algebra therefore has dimension $2n^2+n$.

For $n \geq 3$ the C_n series comes from the Lie algebras $\mathfrak{sp}_{2n}$. These can be constructed as $2n \times 2n$ matrices in block form

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

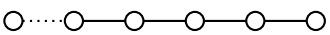
with Q, R symmetric and $P^\top = -S$. It follows that this algebra has dimension $2n^2+n$.

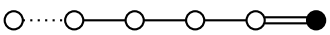
For $n \geq 4$ the D_n series is $\mathfrak{so}_{2n}$. These can be constructed as $2n \times 2n$ matrices in block form

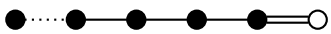
$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

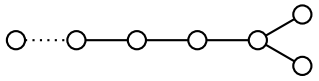
with Q, R anti-symmetric and $P^\top = -S$. Therefore this algebra has dimension $2n^2 - n$.

The Dynkin diagrams of the classical Lie algebras are as follows. The dark nodes correspond to the short roots.

For $n \geq 1$: A_n 

For $n \geq 2$: B_n 

For $n \geq 3$: C_n 

For $n \geq 4$: D_n 

9.3 A table of gauge code Lie algebras

Using brute-force computation we now find the dimension of the ideals in $\mathfrak{g}_{0,0}$ and therefore which simple Lie algebra these correspond to. Note that here we switch back to using n to denote the number of qubits, which in general is not the rank of the Lie algebra.

model	n	r	t_X	$\mathfrak{g}_{t_X,0}$
1D XY	9	8	0	D_9
	10	8	0	$D_5 \oplus D_5$
	11	10	0	D_{11}
	12	10	0	$D_6 \oplus D_6$
1D Ising	$4, ..., 16$	$n-1$	0	D_n
2D compass	9	4	0	A_{15}
	16	9	0	D_{256}
3D gauge color	15	6	0	$6A_1$
	39	18	0	$6A_7$
	65	32	0	$4A_{31} \oplus 2A_{63}$

This verifies the ideal decompositions we found in the previous section, and also corroborates the large amount of symmetry found with the orbigraph method. For example, the S_6 symmetry of the $n = 15$ gauge color code corresponds to permutations of the six A_1 ideals.

There is a remaining mystery of where the extra $\mathbb{Z}_2$ symmetry of the 2D compass model is coming from. This symmetry was found with the orbigraph method in section 8.3. It would be interesting if this symmetry turns out to be an element of the Weyl group of the Lie algebra $\mathfrak{g}_{t_X,0}$.

10 Numerical results

Here we show tables for the first and second eigenvalues of the compass and gauge color code models. These results are obtained using exact diagonalization methods. For each instance we indicate the groundspace eigenvalue λ_1 which is obtained from $H_{0,0}$. Then we list the second eigenvalue of $H_{0,0}$ as well as the first eigenvalue of $H_{t_X,0}$ for $t_X \neq 0$. The weight of the corresponding frustrated stabilizer is $w(s_Z)$. The eigenvalue closest to $\lambda_1(H_{0,0})$ is marked with a tick, along with the value of the gap, $\lambda_1 - \lambda_2$. We only show the results for a single frustrated stabilizer generator, as it was confirmed numerically that adding further frustrated stabilizers never produces a better candidate for λ_2. Also, we only show non-isomorphic stabilizer generators, under the lattice symmetry of the model. We use the iterative solvers in software library `SLEPc` [14] to find these eigenvalues.

2D compass code model

n	t_X	$w(s_Z)$	λ_1	λ_2 ?	gap
16	0		19.012903	16.335705	
		8		18.369300 ✓	0.643603
25	0		29.076200	27.597280	
		10		28.624004 ✓	0.452196
36	0		41.410454	40.585673	
		12		41.094532 ✓	0.315922

Such numerics for the 2D compass model have been previously found using similar methods [8].

3D compass code model

n	t_X	$w(s_Z)$	λ_1	λ_2 ?	gap
27	0		60.295471	58.382445	
		18		59.757677 ✓	0.53779

3D gauge code model

n	t_X	$w(s_Z)$	λ_1	λ_2 ?	gap
15	0		25.455844	16.970563	
		8		22.214755 ✓	3.241089
65	0		104.076026	99.014097	
		8		100.429340	
		12		100.585413	
		12		101.602340	
		18		102.382483 ✓	1.693543
175	0		267.197576	264.250644	
		8		263.171190	
		8		263.324858	
		8		263.340832	
		12		264.269635	
		12		264.617135	
		12		264.745548	
		18		264.843629	
		18		265.413935	
		18		265.754772	
		24		266.148188 ✓	1.04939

The gap of the 3D gauge color code is clearly far more robust than the other models, see Figure 1. It does decrease with n, but note also that the stabilizers in the code are also growing. For larger codes in this family the stabilizers do not get bigger than weight at most 24. To emphasize this point we show in Figure 2 the ground eigenvalues of all of these blocks $H_{t_X,0}$.

There are two main points to make about these numerics. The first is that the gap of the compass model is decreasing much faster than the gap in the gauge color model. In fact, there is strong evidence [9] that the gap of the compass model tends to zero as the lattice size grows. The second point to make is that the gap always corresponds to frustrating a stabilizer ($t_X \neq 0$.) Moreover, the stabilizer that gives rise to the gap is the one with largest weight. This is a crucial connection to make because the stabilizers of the compass model grow with the linear size of the model while those of the gauge color model do not need to grow beyond a constant bound. This would suggest that if this is the mechanism for gapless behaviour that the gauge color model may be gapped.

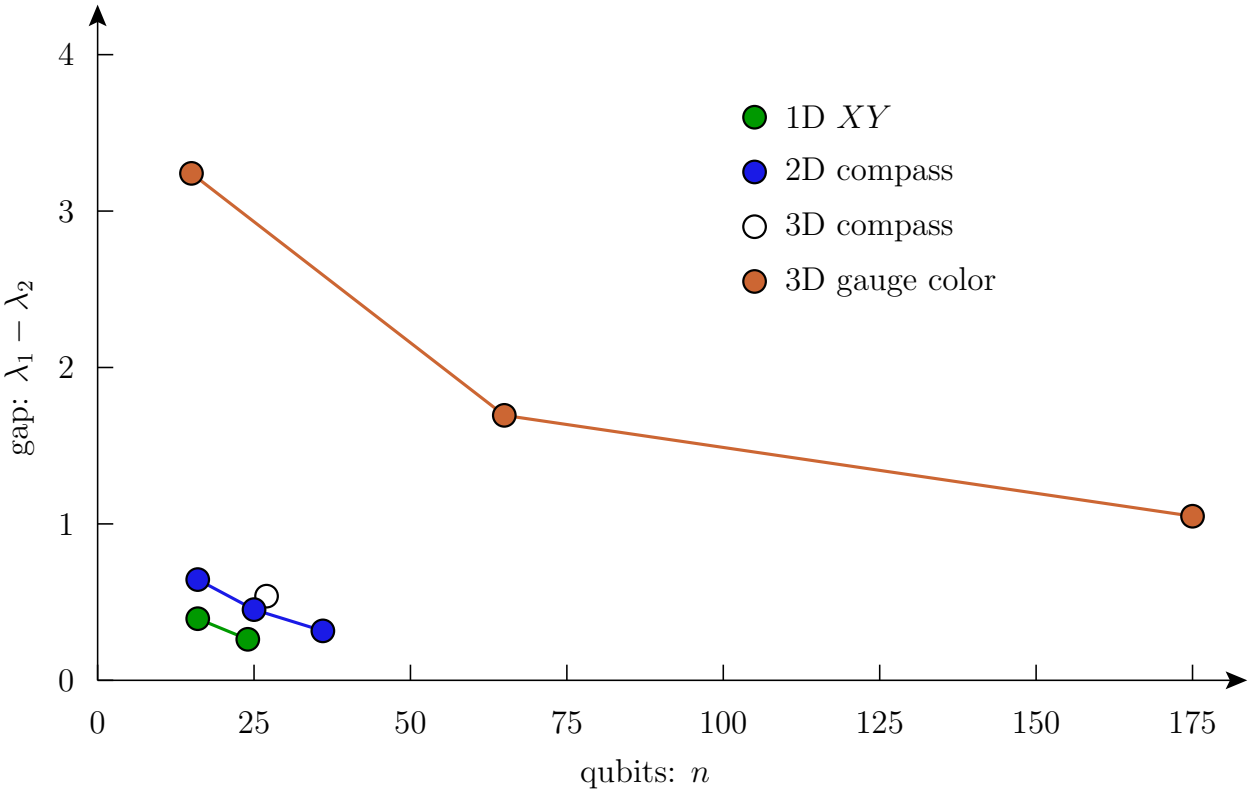


Figure 1: The spectral gap of four different gauge code Hamiltonians, versus the number of qubits n. The gap is defined as the difference between the ground eigenvalue and the first excited eigenvalue. These results are obtained by exact diagonalization.

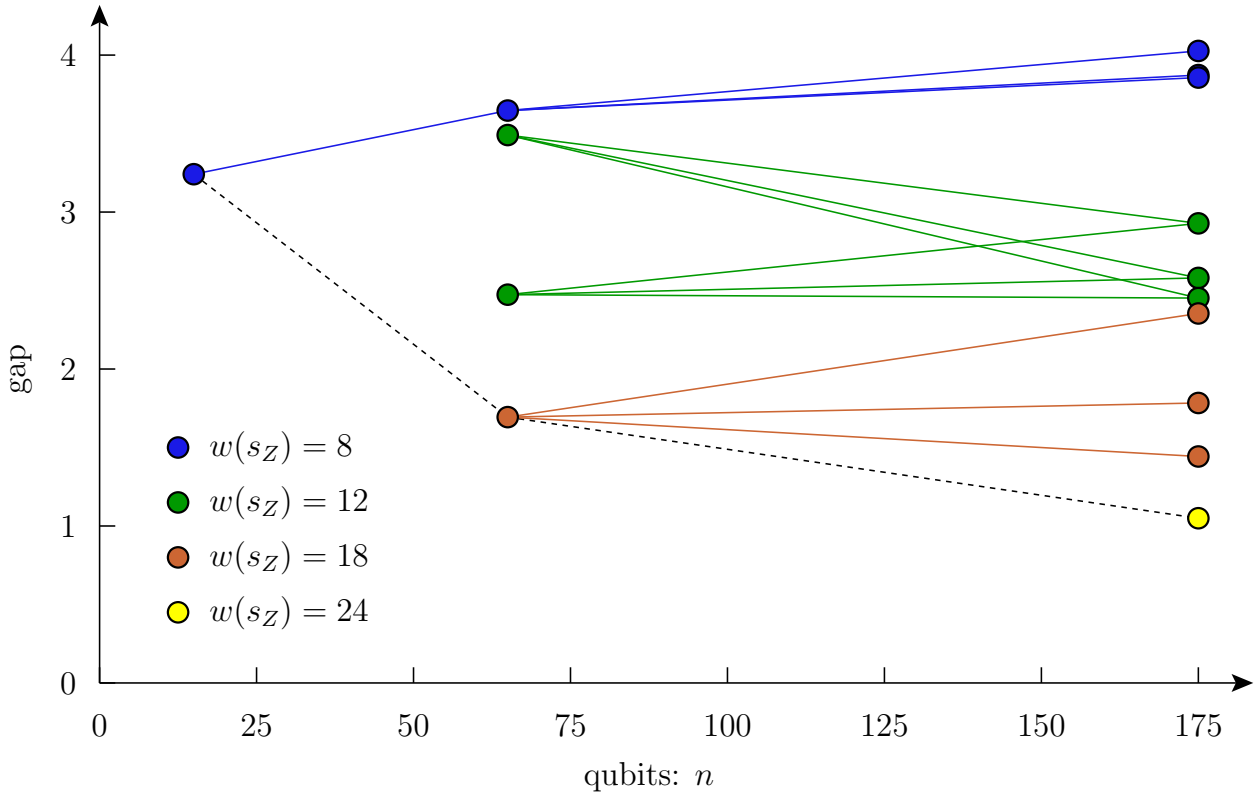


Figure 2: Here we show the spectral gap for each Hamiltonian block $H_{t_X,0}$ of the 3D gauge color code models of sizes $n = 15$, $n = 65$ and $n = 175$. This gap is defined as $\lambda_1(H) - \lambda_1(H_{t_X,0})$. Each point is colored according to the weight of the frustrated stabilizer.

11 Cheeger cuts

In this final section of the paper we give some heuristic arguments for why the size of the stabilizers is related to the gap of the code.

The Perron-Frobenius structure theory places strong constraints on the first and second eigenvectors of Γ_{t_X} : the first eigenvector has all positive entries, and therefore all vectors orthogonal to the first eigenvector will have both positive and negative entries. In general, the set of edges of Γ_{t_X} where such a vector changes sign we call a Cheeger cut. (We ignore the possibility that this vector may have zero entries.) The Cheeger cut associated to the second eigenvector is particularly important, and we next show an example of how this cut relates to the gap.

11.1 The double well model is gapless

We consider a linear graph Hamiltonian with a "double-well" potential. This does not correspond to any gauge code Hamiltonian. The state space will be d dimensional with basis vectors numbered $|1\rangle, ..., |d\rangle$. We take $H = A + U$ with

$$A_{ij} = \begin{cases} 1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad U_{ij} = \begin{cases} 2 & \text{if } i = j = 1 \text{ or } i = j = n, \\ 0 & \text{otherwise.} \end{cases}$$

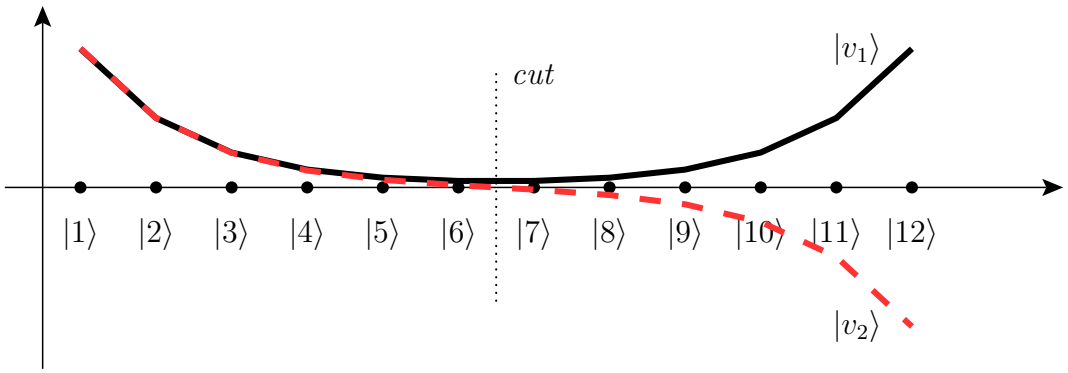
A here is a kind of transition matrix, and U is a diagonal potential energy term.

For $d \gg 1$, the largest eigenvalue is $\lambda_1 \cong \frac{5}{2}$. The corresponding eigenvector $|v_1\rangle$ has all positive components that decay exponentially away from the well sites at $|1\rangle$ and $|d\rangle$:

$$\langle i|v_1\rangle \cong 2^{i-1}\langle 1|v_1\rangle \quad \text{for} \quad i \ll \frac{d}{2}.$$

For the second eigenvalue, λ_2 we also have $\lambda_2 \cong \frac{5}{2}$ and indeed, as d grows the gap $\lambda_1 - \lambda_2 \to 0$ and so this model is gapless.

Here we depict the wavefunctions for the first two eigenvectors for a system with $d = 12$:



The simplest way to show this model is gapless is using a variational argument. Any another vector $|u\rangle$ that is orthogonal to the groundspace vector will have $\langle u|H|u\rangle \leq \lambda_2$. To construct

a candidate for $|u\rangle$ partition the basis vectors into two parts:

$$\Gamma = \Gamma_A \cup \Gamma_B$$

and write $|v_1\rangle = |v_A\rangle \oplus |v_B\rangle$ as well as Hamiltonian with this decomposition as

$$H = \begin{pmatrix} H_{AA} & H_{AB} \\ H_{BA} & H_{BB} \end{pmatrix}.$$

Now let

$$|u\rangle = |v_A\rangle \oplus -|v_B\rangle$$

And then

$$\begin{aligned} \lambda_2 \geq \langle u|H|u\rangle &= \langle v_A|H_{AA}|v_A\rangle + \langle v_B|H_{BB}|v_B\rangle - \langle v_B|H_{BA}|v_A\rangle - \langle v_A|H_{AB}|v_B\rangle \\ &= \lambda_1 - 4\langle v_B|H_{BA}|v_A\rangle. \end{aligned}$$

So if we can show that $\langle v_B|H_{BA}|v_A\rangle$ tends to zero we are done. This term involves the dynamical coupling between the groundstate wavefunction along the cut between A and B. To succeed we must find such a cut where the wavefunction is small. In general this appears to be quite difficult, even though in the models we are considering numerics show that not only is the wavefunction small away from potential wells but it is exponentially small.

11.2 The cut and symmetry

We now study the cut associated to the second eigenvector of a weakly self-dual gauge Hamiltonian H, and relate this to the stabilizers of the code. The key realization is that Γ_{t_X} is like the double well potential above, but now we have 2^{m_X} such wells, that is, one for every $s_X \in \langle S_X\rangle$. This is clear from examining the basis vectors for Γ_{t_X}. These are

$$vS_X + uR_X + t_X, \quad \text{where} \quad v \in \mathcal{F}_{m_X}, u \in \mathcal{F}_r$$

and those that satisfy the most G_Z terms are precisely those with $u = 0$.

We already know this is either the second eigenvector of $H_{0,0}$ or otherwise the first eigenvector of $H_{t_X,0}$ for some $t_X \neq 0$. To relate this to the Perron-Frobenius theory we note the decomposition:

$$\Gamma_{t_X} = \bigoplus_{t_Z \in \langle T_Z\rangle} H_{t_X,t_Z}.$$

This gives the spectral decomposition of each graph Γ_{t_X} in terms of "momenta" t_Z.

We focus on Γ_0. This must contain the second eigenvector of H by weak self-duality of the code. X type stabilizers $s_X \in S_X$ act on the $0, t_Z$ irreps in Γ_0 by ± 1 according to the commutator $[[s_X, t_Z]]$. Suppose the second eigenvector of H lives in H_{0,t_Z} for $t_Z \neq 0$. Let $s_X \in S_X$ with $[[s_X, t_Z]] = -1$. Then we must have an odd number of Cheeger cuts on every Γ_0 path between $|v\rangle$ and $s_X|v\rangle$ for all basis vectors $|v\rangle$, that is, $v \in \langle S_X\rangle \oplus \langle R_X\rangle$.

In a similar vein, if the second eigenvector of H lives in $H_{0,0}$ then we must have an even number of Cheeger cuts on every Γ_0 path between $|v\rangle$ and $s_X|v\rangle$ for all stabilizers $s_X \in S_X$ and basis vectors $|v\rangle$.

In summary, the idea is that large stabilizers lead to widely separated well potentials and hence gapless behaviour, while stabilizers of bounded weight force the cuts to appear close to the wells and hence maintain a gap. Even though numerics show the wavefunction becoming exponentially small away from well potentials, it is also exponentially wide. So making these arguments rigorous appears to be difficult.

The following fact would appear to be true under certain conditions, but is not at all true for example when T is trivial:

Proto-fact: For a sufficiently "well-behaved" weakly self-dual gauge code Hamiltonian H

$$\begin{aligned}\lambda_2(H) &= \min_{t_X \neq 0} \lambda_1(H_{t_X,0}) \\ &= \min_{t_Z \neq 0} \lambda_1(H_{0,t_Z}).\end{aligned}$$

Indeed, contrary to this proto-fact we suspect that $H_{0,0}$ will not be gapped in the generic case. Numerics suggest that there is no lower bound on the gap of randomly constructed stabilizer-less gauge code Hamiltonians. Perhaps double well behaviour can still be imitated even without stabilizers: merely having a large region of almost-stabilizer behaviour (large shallow well) could be enough to send the gap to zero.

There are also results that state that generic local Hamiltonians are gapless [22].

11.3 Cheeger inequalities

We saw above how the Cheeger cut gives a variational ansatz for building a second eigenvector to the Hamiltonian and hence an upper bound on the gap.

In this section we show how the Cheeger cut also yields a lower bound on the gap.

In [11], they derive the following Cheeger inequality by considering bi-partitions of the graph. We will do the same, but using matrix block notation.

Let v_2 be a second eigenvector, $Hv_2 = \lambda_2 v_2$ and $||v_2|| = 1$. We bi-partition the space so that v_2 has (vector) blocks:

$$v_2 = \begin{pmatrix} x \\ y \end{pmatrix}$$

with $x \geq 0$ and $y \leq 0$, component-wise. Let the blocks of H under the same partition be:

$$H = \begin{pmatrix} A & C \\ C^\top & B \end{pmatrix}.$$

If we denote $\lambda_1(A)$ as the top eigenvalue of A and $\lambda_1(B)$ as the top eigenvalue of B, then

$$\begin{aligned}\lambda_2 = v_2^\top H v_2 &= x^\top A x + 2x^\top C y + y^\top B y \\ &\leq x^\top A x + y^\top B y \ \leq \ ||x||^2 \lambda_1(A) + ||y||^2 \lambda_1(B) \\ &\leq \min(\lambda_1(A), \lambda_1(B)) \ \leq \ \lambda_1.\end{aligned}$$

Defining the following constant as a maximization over all bi-partitions of H :

$$\nu(H) := \max_{A,B} \ \min(\lambda_1(A), \lambda_1(B))$$

the above calculation shows that

$$\lambda_2 \leq \nu(H) \leq \lambda_1.$$

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