Framework for classifying logical operators in stabilizer codes

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Entanglement, as studied in quantum information science, and non-local quantum correlations, as studied in condensed matter physics, are fundamentally akin to each other. However, their relationship is often hard to quantify due to the lack of a general approach to study both on the same footing. In particular, while entanglement and non-local correlations are properties of states, both arise from symmetries of global operators that commute with the system Hamiltonian. Here, we introduce a framework for completely classifying the local and non-local properties of all such global operators, given the Hamiltonian and a bi-partitioning of the system. This framework is limited to descriptions based on stabilizer quantum codes, but may be generalized. We illustrate the use of this framework to study entanglement and non-local correlations by analyzing global symmetries in topological order, distribution of entanglement and entanglement entropy.

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I. INTRODUCTION

In recent years, ideas from quantum information science have become increasingly useful in condensed matter physics [1–7]. In particular, it has been realized that many interesting physical systems in condensed matter physics may be described in the language of quantum coding schemes such as the stabilizer formalism [8–10]. What is emerging is a closeness between the two fields, which is a result of the intrinsic similarity between quantum correlations, as studied in condensed matter physics, and entanglement, as studied in quantum information science. The common building block for these studies is the locality and non-locality of correlations and entanglement in the system.

The notion of non-locality versus locality naturally arises in quantum information science through the study of quantum entanglement, as employed in many different constructions. For example, entangled states are essential as resources for quantum error-correcting codes, where encoded logical qubits are typically dispersedly entangled, and may be transformed among themselves by logical operators [11]. In the study of quantum codes, non-locality arises both in entangled states and in logical operators which act on entangled states. Particularly in quantum codes defined on geometric manifolds, how locally or non-locally logical operators may be defined is of particular interest since the geometric sizes of the logical operators determine properties such as the code distance and rate [8]. Also, such geometric sizes are closely related to the maximum tolerable rate of error in fault-tolerant constructions using the quantum code, the fault-tolerance threshold [12]. Entangled states can also provide serve as resources for perfect secret-sharing of classical or quantum information [13]. In such schemes, information is shared between one or more parties, using a logical operator which is defined jointly over all the parties in a non-local way. Again, the key construct is the logical operator. Clearly, for quantum codes, secret sharing, and for other quantum science applications, it is desirable to be able to determine whether a given local operator can be locally defined inside some subset of qubits or not. However, this is often difficult, both analytically or computationally. This is a challenge which underlies broad questions in quantum coding theory, including upper bounds on code distances of locally defined stabilizer codes [14, 15] and the feasibility of self-correcting memory [16, 17].

Non-locality and locality is also central to condensed matter physics, as seen in the study of spin systems, for example. The non-locality of correlations in such systems has been addressed through entanglement entropy [2–4, 18] and its generalizations [5–7] for systems commonly discussed in condensed matter physics. Renormalization of correlation length scales leads to novel, efficient numerical simulation schemes using, for example, the matrix product state [19] or tensor product state formalisms [20, 21]. Much current interest focuses on topologically ordered states whose correlation extends over the entire system [8, 22–24]. Such a global correlation is often a result of global symmetries of the system Hamiltonian which manifest themselves as the existence of global operators commuting with the Hamiltonian. These global operators resulting from the symmetry of the Hamiltonian are essentially akin to logical operators in quantum codes. Thus, one might hope that the analysis of logical operators and symmetry operators could provide a unified approach to understand the non-locality and locality of correlations and entanglement in condensed matter physics and quantum coding theory.

However, the connection between locality and non-locality in quantum coding theory and condensed matter physics is currently incomplete. In particular, the relation between the quantum entanglement discussed in quantum coding theory and the correlations studied in condensed matter physics has not been fully understood. A general framework to study the non-local and local properties of such systems will be a necessary first step to clarify the relation between two fields. Such a framework should be applicable to many classes of systems described through quantum coding schemes; and it should enable quantification of the degree of non-locality of correlations in any bi-partition or multi-partition of the system. Also, since many strongly entangled systems possess a degenerate ground state, the framework should study not only single states, but also the entire ground state space. Finally and most importantly, the framework would need to provide a systematic procedure to obtain all the logical operators in a computationally tractable way. The framework should classify all the logical operators according to their localities and non-localities so that the coding properties and their relation with the non-local correlation of the system can be studied on the same footing.

There are several pioneering works which have taken first steps toward such a framework. A first analysis leading in this direction was conducted for graph states, which are multi-partite entangled states corresponding to mathematical graphs [10]. It was found that multi-partite entanglement could be characterized and quantified completely in terms of the Schmidt measure. Later, the problem of finding a systematic framework was further investigated through the study of stabilizer states [25]. A complete characterization and quantification of the degree of non-locality of correlations in any bi-partite stabilizer state were provided. A generalization to multi-partite entanglement was also discussed in the same work. Recently, another important advance was made by extending this work to stabilizer codes in the context of two party quantum information encoding [26]. These works have initiated the investigation of non-local correlations of the systems described through quantum coding schemes, but largely only for individual states. The need is to generalize these approaches to an analysis of spaces of degenerate ground states, in order to connect quantum coding theory to condensed matter physics.

Here, we present a new framework which builds on the current techniques [10, 25] and reveals the non-local correlations in stabilizer codes through the analysis of logical operators in a bi-partition. In particular, the framework classifies all the logical operators according to their non-localities and localities, and computes all the logical operators along with the classification. The framework begins by identifying a group of operators, *the overlapping operator group*, which is at the heart of the correlations over two separated subsets of qubits. A new theoretical tool, a *canonical representation*, is included in the framework to analyze the overlapping operator group and to obtain logical operators along with their non-localities. This framework is specifically limited to stabilizer codes with a bi-partition into two complementary subsets.

The presentation of this framework is organized as follows. In section II, we give a brief review of the stabilizer formalism. Then, in section III, we introduce a classification of logical operators based on their localities and non-localities. We define the overlapping operator group and describe the theoretical tool to analyze it. In section IV, we provide a systematic procedure to compute all the logical operators based on the classification. And in section V, we illustrate the use of our framework to study the non-local property of the stabilizer formalism by showing three specific examples, considering the global symmetries in topological order, the distribution of multi-partite entanglement, and entanglement entropy.

II. REVIEW OF STABILIZER CODES

We begin with a review of stabilizer codes and their logical operators [27, 28]. Consider an N qubit system governed by the Hamiltonian

$$H_{stab} = -\sum_i S_i \tag{1}$$

where the interaction terms S_i are inside the *Pauli operator group*

$$\mathcal{P} = \langle iI, X_1, Z_1, \dots, X_N, Z_N \rangle \tag{2}$$

which is generated from local Pauli operators X_i and Z_i acting on each of the N single qubits. In *stabilizer codes*, interaction terms S_i are called *stabilizer generators* and they commute with each other, obeying $[S_i, S_j] = 0$ for $\forall i, j$. The *stabilizer group*

$$\mathcal{S} = \langle \{S_i,\ \forall i\} \rangle, \tag{3}$$

which is generated from all the stabilizer generators S_i, is a self-adjoint Abelian subgroup of the Pauli operator group which does not contain $-I$. Operators inside the stabilizer group $\mathcal{S}$ are called *stabilizers*. This *stabilizer Hamiltonian* is exactly solvable since stabilizer generators S_i commute each other and $S_i^2 = I$. The entire Hilbert space can be decomposed into a direct sum of subspaces with respect to $s_i = \pm 1$, where s_i are the eigenvalues of the stabilizer generators S_i, as

$$\mathcal{H} = \bigoplus_{\vec{s}} \mathcal{H}_{\vec{s}}. \tag{4}$$

Each of decomposed subspaces $\mathcal{H}_{\vec{s}}$ is characterized by eigenvalues $\vec{s} = (s_1, s_2, \cdots)$. The ground state space is $\mathcal{H}_{\vec{1}} \equiv \mathcal{H}_{1,1,\dots}$ where each ground state $|\psi\rangle$ is *stabilized* as $S_i|\psi\rangle = |\psi\rangle$ for $\forall i$ since the choice of $\vec{s} = \vec{1}$ minimizes the energy of the stabilizer Hamiltonian H_{stab}. Now let us denote the number of nontrivial generators for a group of operators $\mathcal{O}$ as $G(\mathcal{O})$. If the number of generators for $\mathcal{S}$ is $G(\mathcal{S}) = N - k$, we notice that there are 2^k ground states for this stabilizer code. Then, quantum information can be stored among degenerate ground states inside $\mathcal{H}_{\vec{1}}$ by assigning eigenstates of qubits $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$ to each pair of degenerate ground states. The encoded qubits in the ground state space are called *logical qubits*.

Logical qubits and their coding properties can be characterized by *logical operators*. Logical operators are defined as Pauli operators $\ell \in \mathcal{P}$ which commute with the entire Hamiltonian, $[\ell, H_{stab}] = 0$, but $\ell \notin \langle \mathcal{S}, iI \rangle$. Here,

iI is inserted since $\mathcal{P}$ includes iI while $\mathcal{S}$ does not. Since logical operators commute with the stabilizer Hamiltonian H_{stab}, when a logical operator ℓ is applied to one of the ground states $|\psi_0\rangle$ of the Hamiltonian, the resulting state $\ell|\psi_0\rangle$ is also a ground state of the Hamiltonian. Therefore, logical operators can transform the ground states among themselves and manipulate logical qubits encoded inside the ground state space $\mathcal{H}_{\vec{1}}$. One can also notice that the applications of stabilizers keep the ground state unchanged since they do not change the eigenvalues $\vec{s}$ of stabilizer generators S_i. Then, stabilizers inside $\mathcal{S}$ can be viewed as trivial logical operators. Since the application of stabilizers does not change the properties of logical operators which act on logical qubits, two logical operators ℓ and ℓ' are called *equivalent* when $\ell\ell' \in \langle \mathcal{S}, iI\rangle$. The equivalence between two logical operators ℓ and ℓ' are represented as $\ell \sim \ell'$.

In order to obtain all the logical operators which are not equivalent each other, it is convenient to consider the *centralizer group*

$$\mathcal{C} = \langle \{U \in \mathcal{P} \mid [U, S_i] = 0,\ \forall i\}\rangle \tag{5}$$

and its quotient group with respect to $\mathcal{S}$

$$\mathcal{C}/\mathcal{S} = \langle i\mathcal{S}, \ell_1\mathcal{S}, r_1\mathcal{S}, \ldots, \ell_k\mathcal{S}, r_k\mathcal{S}\rangle. \tag{6}$$

The centralizer group $\mathcal{C}$ includes all the Pauli operators which commute with any stabilizer generators S_i. All the stabilizer generators and logical operators are inside $\mathcal{C}$ from their definitions. The $2k$ nontrivial representatives ℓ_i and r_i $(i = 1, \cdots, k)$ for $\mathcal{C}/\mathcal{S}$ are *independent* logical operators which commute with each other, except for $\{\ell_i, r_i\} = 0$. Choices of $2k$ representatives are not unique (see discussion in section III C).

Each of the $2k$ logical operators can decompose the ground state space into a direct product of k subsystems

$$|\psi\rangle = \bigotimes_{i=1}^{k} (\alpha_i|\tilde{0}\rangle_i + \beta_i|\tilde{1}\rangle_i) \tag{7}$$

with

$$\begin{aligned} \ell_i|\tilde{0}\rangle_i &= |\tilde{0}\rangle_i \\ \ell_i|\tilde{1}\rangle_i &= -|\tilde{1}\rangle_i \\ r_i|\tilde{0}\rangle_i &= |\tilde{1}\rangle_i \\ r_i|\tilde{1}\rangle_i &= |\tilde{0}\rangle_i \end{aligned} \tag{8}$$

where $|\tilde{0}\rangle_i$ and $|\tilde{1}\rangle_i$ represent a subsystem which supports a logical qubit characterized by the i-th anti-commuting pair of logical operators ℓ_i and r_i. Therefore, ℓ_i and r_i can be viewed as Pauli operators Z and X acting on logical qubits spaned by $|\tilde{0}\rangle_i$ and $|\tilde{1}\rangle_i$.

The robustness of the quantum code can be measured by how far two encoded states are apart, which is quantified through the notion of *code distances*. Since logical operators transform encoded states among themselves, the existence of logical operators supported by a large number of qubits may imply that encoded states are far apart. The code distance of a stabilizer code is defined through the sizes of logical operators as follows:

$$d = \min_{U \in \mathcal{C}, \notin \mathcal{S}} w(U) \tag{9}$$

where $w(U)$ is the number of non-trivial Pauli operators in $U \in \mathcal{P}$. Essentially, $w(U)$ measures the *size* of Pauli operator U. The code distance d is the smallest possible size of all the logical operators in a stabilizer code. One of the ultimate goals in quantum coding theory is to achieve a larger code distance d for a fixed N in the presence of some physical constraints on stabilizer generators S_i, such as locality and translation symmetries of the stabilizer generators [17].

III. THE FRAMEWORK I

Now let us return to the central problem of this paper, concerning locality and non-locality in the stabilizer formalism: given a stabilizer code $\mathcal{S}$ which is split into two subsets of qubits A and $B = \bar{A}$, the problem is to unveil the local and non-local elements in the stabilizer group $\mathcal{S}$ and logical operators ℓ_i and r_i. By *local*, we mean that a given operator be represented only through qubits inside one of the two subsets, A or B. By *non-local*, we mean that a given operator cannot be defined either inside A or B alone.

The bi-partite entanglement of a stabilizer state, which is a subclass of stabilizer codes with $k = 0$, has been studied [25] by analyzing the local and non-local elements in the stabilizer group $\mathcal{S}$. However, the bi-partite properties of logical operators (versus states) are what interest us, as they are essential to the study of systems with degenerate ground states. The non-locality and locality of logical operators are crucial factors which determine the coding properties of quantum codes. Also, since logical operators may serve as indicators of the non-local correlations of the system, the study of the non-locality of logical operators addresses intrinsic relationships between condensed matter physics and quantum coding theory. Thus, we attempt to provide some unity to approaching the needs of both fields, by broadening prior studies to encompass the locality and non-locality of logical operators as well as logical states.

In this section and the next, we describe our approach by presenting a framework which provides a systematic and efficient method to classify logical operators according to their locality and non-locality in a bi-partition of stabilizer codes. This framework also obtains all the logical operators along with their classifications. The framework consists of three steps, depicted in Fig.1.

First, the framework classifies all the $2k$ logical operators by whether or not they have equivalent logical operators which can be defined separately inside two complementary subsets A or B. This classification is useful

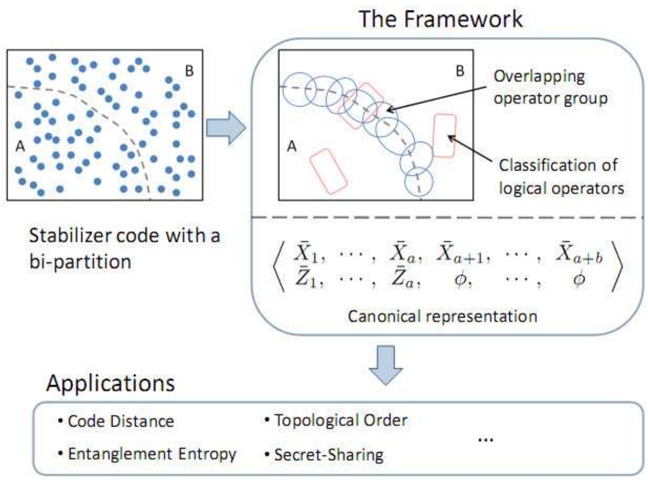

FIG. 1: The framework to study the non-local properties of stabilizer codes. The input to the framework is a stabilizer code with a bi-partition. The output of the framework is non-local properties of the code. On the left hand side of the figure, a stabilizer code with a bi-partition is described. The dots represents qubits in the system while the dotted line represents a bi-partition of qubits into two subsets A and B. The framework consists of the three elements. First, the logical operators are classified based on their localities and non-localities. Here, the rectangles (red online) represent logical operators. Then, the overlapping operator group is generated from the overlap of stabilizer generators S_i. The circles (blue online) represent stabilizer generators at the boundary between A and B. Finally, the framework analyzes the overlapping operator group through the canonical representation.

when coding properties of stabilizer codes are discussed. Also, with the help of the classification of logical operators, the analysis on logical qubits, which are described by a pair of anti-commuting logical operators, are simplified.

Second, the framework identifies a group of operators that arises naturally from the overlap of stabilizer generators at the boundary between A and B. We call this group of operators the *overlapping operator group*. The overlapping operator group includes all the necessary clues to study the correlations over A and B in a bi-partition and plays a central role in the study of logical operators in a bi-partition. However, this group contains elements that do not commute with each other. This fact makes the analysis of the overlapping operator group difficult.

Third, to deal with the challenge of analyzing the overlapping operator group, the framework introduces a new theoretical tool, a canonical representation which analyzes the overlapping operator group. The framework analyzes the overlapping operator group with a canonical representation and gives a method to compute all the logical operators in a computationally tractable way. Also, with the help of a canonical representation, all the logical operators obtained can be easily and systematically classified based on their localities and non-localities.

Below, we first introduce the classification of logical operators in section III A. Then, we define the overlapping operator group in section III B and describe the canonical representation in section III C. We also present a systematic method to obtain the canonical representation, not only for the overlapping operator group, but also for any subgroup of the Pauli operator group $\mathcal{P}$, in section III D. Later, we describe a method to compute logical operators along with the classification, in section IV.

A. Classification of Logical Operators

All the logical operators may be classified based on their localities and non-localities. The framework accomplishes this in the following way. First, the set of logical operators which can be supported only with qubits inside a subset A is defined as L_A. Let us denote the projection of an operator O onto A as $O|_A$. This keeps only the non-trivial Pauli operators which are inside A and truncates Pauli operators acting outside the subset A. Using this notation, L_A may be represented as

$$L_A = \{\ell \in \mathcal{C} | \exists U \in \mathcal{S},\ (U\ell)|_B = I\}. \tag{10}$$

L_A includes all the logical operators which can be *shrunk* into a subset A by applying an appropriate stabilizer generator. L_A also includes the stabilizer generators defined inside A, which may be viewed as trivial logical operators. Note that L_A forms a group. When the projections $(U\ell)|_B$ are considered, the arbitrariness resulting from trivial phase iI in the Pauli operator group $\mathcal{P}$ is neglected since it does not affect the properties of logical operators. We also define the set of logical operators which can be supported with qubits inside a complementary subset $B = \bar{A}$ as L_B in a way similar to L_A. Since logical operators inside L_A or L_B can be defined inside localized subsets in a bi-partition, we call them *localized logical operators*.

We may further classify localized logical operators in L_A by considering whether they can be also defined inside B or not. Let M_{AB} be the set of all the logical operators which can be defined both inside A and inside B, represented as

$$M_{AB} = \{\ell \in \mathcal{C} | \ell \in L_A, L_B\}. \tag{11}$$

Note that M_{AB} also forms a group. Next, let M_A be the set of all the logical operators which can be defined inside A, but cannot be defined inside B and define M_B similarly. M_A and M_B may be represented as

$$\begin{aligned} M_A &= \{\ell \in \mathcal{C} | \ell \in L_A, \notin L_B\} \\ M_B &= \{\ell \in \mathcal{C} | \ell \notin L_A, \in L_B\}. \end{aligned} \tag{12}$$

Finally, let M_ϕ be the set of all the logical operators which cannot be defined either inside A or B, represented as

$$M_\phi = \{\ell \in \mathcal{C} | \ell \notin L_A, L_B\}. \quad (13)$$

We explicitly show the classifications into four sets of logical operators in Fig.2. Logical operators in M_{AB} have two equivalent representations which can be defined only inside A or only inside B respectively. On the other hand, logical operators in M_ϕ always have support both inside A and B in a non-local way. We call them *non-local logical operators* since which are defined over A and B jointly. L_A and L_B are related to M_A, M_B, M_{AB} and M_ϕ as follows:

$$\begin{aligned} L_A &= M_A \cup M_{AB} \\ L_B &= M_B \cup M_{AB} \\ \mathcal{C} &= M_A \cup M_B \cup M_{AB} \cup M_\phi. \end{aligned} \quad (14)$$

When considering the coding properties of a stabilizer code, the localized logical operators inside L_A and L_B become important, since the code distance d can be upper bounded by the number of qubits inside each of complementary subsets A and B. On the other hand, non-local logical operators play an important role in the non-local correlations and entanglement over A and B in the ground state space. Therefore, *the classification of logical operators can serve as a guide to study the non-local properties of stabilizer codes.*

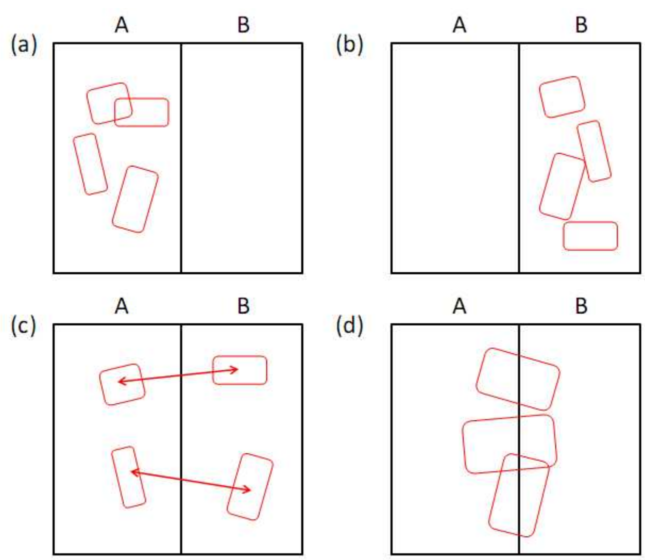

FIG. 2: The classification of logical operators. Each rectangle represents logical operators. (a) M_A. This subset includes logical operators defined only inside A. (b) M_B. (c) M_{AB}. Two logical operators connected through a red arrow are equivalent, each defined inside A or B. (d) M_ϕ. Non-local logical operators are defined jointly over A and B.

B. Overlapping Operator Group

Though the classification based on the localities and non-localities of logical operators are important concepts for describing the non-locality of the system, it is generally difficult to actually compute explicit logical operators and to computationally study their localities and non-localities. For example, if we want to see whether a given logical operator can be defined inside a localized subset A or not, using the most obvious "exhaustive elimination" approach, we would need to check whether or not there exists a stabilizer operator which can shrink the corresponding logical operator inside A, one by one. However, such an approach would not allow us to study the non-local properties of stabilizer codes systematically, since we would need to repeat the whole computation for each of logical operators. Clearly, such an direct approach is computationally intractable. Instead, what is needed is a more systematic procedure, which allows efficient computation of all the logical operators, in order to setup a general framework. In this subsection, we give a definition of the *overlapping operator group*, which will become essential in the computation of logical operators based on the classification of section IV.

The group we are interested in follows directly from considering a general property of localized logical operators inside A. Logical operators ℓ defined inside a subset A must commute with all the stabilizer generators S_i which have overlap with qubits inside A. So ℓ has the property that $\ell|_B = I$ with $[\ell, S_i|_A] = 0$ for $\forall i$. We define the group of operators generated from $S_i|_A$ as the *overlapping operator group*,

$$\mathcal{O}^A = \langle \{S_i|_A, \ \forall i\} \rangle. \quad (15)$$

Since localized logical operators in A can be computed so that they commute with all the operators in $\mathcal{O}^A$, the analysis of $\mathcal{O}^A$ is at the heart of computing localized logical operators.

Here, we emphasize the difference between the *restriction and overlap* of the stabilizer group (Fig.3). We define the *restriction* of a group of operators $\mathcal{G}$ into A as

$$\mathcal{G}|_A = \langle \{U \in \mathcal{G} \mid U|_{\bar{A}} = I\} \rangle. \quad (16)$$

Therefore, $\mathcal{G}|_A$ contains all the operators in $\mathcal{G}$ which are defined inside A. In this paper, we discuss $\mathcal{S}_A \equiv \mathcal{S}|_A$, $\mathcal{C}_A \equiv \mathcal{C}|_A$ and $\mathcal{P}_A \equiv \mathcal{P}|_A$ which are restrictions of the stabilizer group $\mathcal{S}$, the centralizer group $\mathcal{C}$ and the Pauli operator group $\mathcal{P}$.

In contrast to the restriction $\mathcal{S}_A$, the projected stabilizer generators $S_i|_A$ do not necessarily commute with each other. To see this explicitly, let us pick out two stabilizer generators S_1 and S_2 which are defined at the boundary between two subsets A and B (Fig.3(c)), where B is the complement of A. Represent the projections of

S_1 and S_2 as

$$\begin{aligned} S_1|_A &= U_1 \\ S_1|_B &= V_1 \\ S_2|_A &= U_2 \\ S_2|_B &= V_2. \end{aligned} \tag{17}$$

Note that $U_1, U_2 \in \mathcal{O}^A$ since U_1 and U_2 are the projections of stabilizer generators onto A. Since $[S_1, S_2] = 0$, we have $[U_1V_1, U_2V_2] = 0$. This leads to either $[U_1, U_2] = 0$ and $[V_1, V_2] = 0$ or $\{U_1, U_2\} = 0$ and $\{V_1, V_2\} = 0$. Thus, U_1 and U_2 may anti-commute with each other and the overlapping operator group $\mathcal{O}^A$ can be a group of anti-commuting Pauli operators.

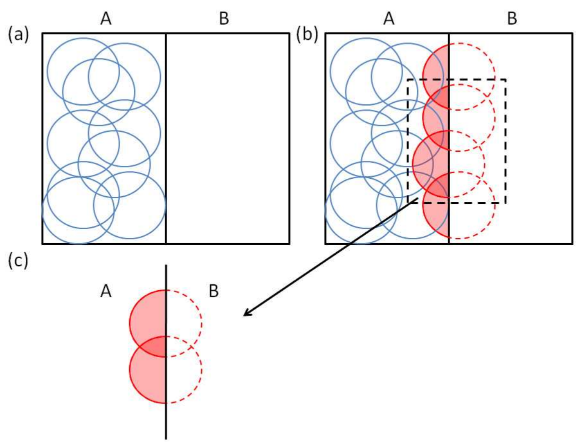

FIG. 3: An illustration of the differences between the projection (restriction) $\mathcal{S}_A$ and the overlap $\mathcal{O}^A$. Each circle represents the stabilizer generators supported by qubits inside the circle. (a) The restriction $\mathcal{S}_A$ generated from all the stabilizer generators defined inside A. (b) The overlapping operator group $\mathcal{O}^A$ generated from all the overlaps of stabilizer generators with A. $\mathcal{O}^A$ includes left parts (shaded regions) of the stabilizer generators at the boundary between A and B in addition to stabilizer generators inside A. (c) Two stabilizer generators at the boundary between A and B. Though stabilizer generators commute with each other, their projections onto A do not necessarily commute with each other.

C. Canonical Representation

Since the overlapping operator group $\mathcal{O}^A$ may include elements which anti-commute with each other, the analysis of $\mathcal{O}^A$ becomes more complicated than the analysis of the original stabilizer group $\mathcal{S}$. Here, we introduce a new approach which we call the *canonical representation* of an operator group. The canonical representation captures a concept which naturally arises in order to extract physical properties of logical operators from a group of anti-commuting Pauli operators. This representation leads to the computation of all the logical operators in each of four sets M_A, M_B, M_{AB} and M_ϕ as we shall see in the next section.

Let us first give the definition of the canonical representation for the overlapping operator group $\mathcal{O}^A$. In the canonical representation, $\mathcal{O}^A$ can be represented as

$$\mathcal{O}^A = \langle iI, \bar{X}_1, \ldots, \bar{X}_{a+b}, \bar{Z}_1, \ldots, \bar{Z}_a \rangle \tag{18}$$

with $G(\mathcal{O}^A) = 2a+b$ independent generators which commute with each other except $\{\bar{X}_i, \bar{Z}_i\} = 0$ $(i = 1, \cdots, a)$. Therefore, independent generators are separated into the commuting generators $\bar{X}_{a+1} \cdots \bar{X}_{a+b}$ and the anti-commuting generators $\bar{X}_1, \cdots, \bar{X}_a$ and $\bar{Z}_1, \cdots, \bar{Z}_a$. We call $G(\mathcal{O}^A)$ independent generators which satisfy these commutation relations *canonical generators* for $\mathcal{O}^A$. Later, we shall show that any subgroup of Pauli operator group $\mathcal{P}$ can be represented through canonical generators. For simplicity, we introduce a symbolic notation for canonical generators of $\mathcal{O}^A$ represented as

$$\mathcal{O}^A = \left\langle \begin{array}{ccccccc} \bar{X}_1, & \cdots, & \bar{X}_a, & \bar{X}_{a+1}, & \cdots, & \bar{X}_{a+b} \\ \bar{Z}_1, & \cdots, & \bar{Z}_a, & \phi, & \cdots, & \phi \end{array} \right\rangle. \tag{19}$$

Here, generators in the same column anti-commute while any other pair of generators in the different columns commute each other. The *null operator* ϕ below $\bar{X}_i$ means that $\bar{X}_i$ has *no anti-commuting operator pair* in $\mathcal{O}^A$.

The commutation relations between canonical generators can be concisely captured by considering them as if they act like Pauli operators X_i and Z_i defined for each of single qubits. To grasp commutation relations between canonical generators more clearly, it is convenient to consider a Clifford transformation U which transforms $\mathcal{O}^A$ to a group of operators whose generators are represented only with local Pauli operators X_i and Z_i. Let us consider the Clifford transformation U obeying the following conditions

$$\bar{X}_i = UX_iU^\dagger \qquad (1 \le i \le a+b) \tag{20}$$

$$\bar{Z}_i = UZ_iU^\dagger \qquad (1 \le i \le a) \tag{21}$$

and define a group $\mathcal{Q}$ as

$$\mathcal{Q} = \left\langle \begin{array}{cccccc} X_1, & \cdots, & X_a, & X_{a+1}, & \cdots, & X_{a+b} \\ Z_1, & \cdots, & Z_a, & \phi, & \cdots, & \phi \end{array} \right\rangle. \tag{22}$$

Then, $\mathcal{O}^A$ can be represented through the Clifford transformation U as

$$\mathcal{O}^A = U\mathcal{Q}U^\dagger \tag{23}$$

where the Clifford transformation U acts on each of elements inside the groups of operators.

D. Derivation of Canonical Generators

Now, let us provide an explicit procedure to obtain canonical generators, not only for the overlapping operator group, but also for an arbitrary subgroups of the Pauli operator group $\mathcal{O} \in \mathcal{P}$. Before describing each step to obtain canonical generators in detail, let us begin by looking at some simple examples.

As the most familiar example, the centralizer group $\mathcal{C}$ is represented in a canonical form as

$$\mathcal{C} = \left\langle \begin{array}{cccccc} \ell_1, & \cdots, & \ell_k, & S_1, & \cdots, & S_{G(\mathcal{S})} \\ r_1, & \cdots, & r_k, & \phi, & \cdots, & \phi \end{array} \right\rangle \tag{24}$$

where $S_1, \cdots, S_{G(\mathcal{S})}$ represent $G(\mathcal{S})$ independent stabilizer generators. One can easily see that logical operators commute with all the stabilizer generators, but not inside the stabilizer group $\mathcal{S}$ in this representation.

Next, for $\mathcal{O} = \langle O_1, O_2, iI \rangle$ generated from two commuting Pauli operators O_1 and O_2 with $[O_1, O_2] = 0$, we can set $\bar{X}_1 = O_1$ and $\bar{X}_2 = O_2$. Then, $\mathcal{O}$ is represented as

$$\mathcal{O} = \left\langle \begin{array}{cc} O_1, & O_2 \\ \phi, & \phi \end{array} \right\rangle. \tag{25}$$

On the other hand, if O_1 and O_2 are anti-commuting, we can set $\bar{X}_1 = O_1$ and $\bar{Z}_1 = O_2$ and $\mathcal{O}$ is represented as

$$\mathcal{O} = \left\langle \begin{array}{c} O_1 \\ O_2 \end{array} \right\rangle. \tag{26}$$

Now let us consider a more complicated example, where $\mathcal{O} = \langle O_1, O_2, O_3, iI \rangle$ is generated from three independent generators with commutations relations characterized as

$$\begin{aligned} [O_1, O_2] &= 0 \\ \{O_1, O_3\} &= 0 \\ \{O_2, O_3\} &= 0. \end{aligned} \tag{27}$$

We start by representing a group of operators $\mathcal{O}' = \langle O_1, O_2, iI \rangle$ as

$$\mathcal{O}' = \left\langle \begin{array}{cc} O_1, & O_2 \\ \phi, & \phi \end{array} \right\rangle. \tag{28}$$

Though we might be tempted to put O_3 below O_1 in place of the null operator ϕ due to anti-commutation between O_1 and O_3, we soon notice that O_3 also anti-commutes with O_2. Therefore, by replacing O_2 with $O_1 O_2$, we obtain the canonical representation of $\mathcal{O}$ as

$$\mathcal{O} = \left\langle \begin{array}{cc} O_1, & O_1 O_2 \\ O_3, & \phi \end{array} \right\rangle. \tag{29}$$

Finally, here is a general procedure to obtain canonical generators for an arbitrary group of operators $\mathcal{O} = \langle iI, \{O_i, \ \forall i\} \rangle$. Let us suppose that an operator group $\mathcal{M}$ is already represented in a canonical form as

$$\mathcal{M} = \left\langle \begin{array}{cccccc} \bar{X}_1, & \cdots, & \bar{X}_a, & \bar{X}_{a+1}, & \cdots, & \bar{X}_{a+b} \\ \bar{Z}_1, & \cdots, & \bar{Z}_a, & \phi, & \cdots, & \phi \end{array} \right\rangle. \tag{30}$$

We derive a canonical representation for an operator group $\mathcal{M}' = \langle \mathcal{M}, U, iI \rangle \supset \mathcal{M}$ with $U \notin \mathcal{M}$. Therefore, by iterating the procedure we shall describe in the below, one can represent any subgroup of the Pauli operator group in canonical representations. Let us represent the commutation relations between U and the original canonical generators as

$$\bar{X}_i U = (-1)^{p_i} U \bar{X}_i \qquad (1 \leq i \leq a) \tag{31}$$
$$\bar{Z}_i U = (-1)^{q_i} U \bar{Z}_i \qquad (1 \leq i \leq a) \tag{32}$$
$$\bar{X}_{a+i} U = (-1)^{t_i} U \bar{X}_{a+i} \qquad (1 \leq i \leq b) \tag{33}$$

with $p_i, q_i, t_i = 0, 1$ where 0 represents the commutations between U and corresponding canonical generators while 1 represents the anti-commutations. Here, we define a new operator

$$U' = \prod_i \bar{X}_i^{q_i} \bar{Z}_i^{p_i} U \tag{34}$$

such that U' commutes with $\bar{X}_i$ and $\bar{Z}_i$ for $\forall i$. This can be easily verified by directly checking the commutation relations

$$\begin{aligned} [U', \bar{X}_i] &= [\bar{Z}_i^{p_i} U, \bar{X}_i] = 0 \\ [U', \bar{Z}_i] &= [\bar{X}_i^{q_i} U, \bar{Z}_i] = 0. \end{aligned} \tag{35}$$

First, if $t_i = 0$ for $\forall i$, we have the canonical representation of $\mathcal{M}'$ as

$$\mathcal{M}' = \left\langle \begin{array}{ccccccc} \bar{X}_1, & \cdots, & \bar{X}_a, & \bar{X}_{a+1}, & \cdots, & \bar{X}_{a+b}, & U' \\ \bar{Z}_1, & \cdots, & \bar{Z}_a, & \phi, & \cdots, & \phi, & \phi \end{array} \right\rangle \tag{36}$$

since U' commutes with all the operators in $\mathcal{M}$. Next, we consider the case where there exist integers j with $t_j = 1$ $(j \leq b)$. Without loss of generality, we can assume that $t_1 = 1$. Then, the canonical representation of $\mathcal{M}$ is

$$\mathcal{M}' = \left\langle \begin{array}{cccccc} \bar{X}_1, & \cdots, & \bar{X}_a, & \bar{X}'_{a+1}, & \bar{X}'_{a+2}, & \cdots \\ \bar{Z}_1, & \cdots, & \bar{Z}_a, & U', & \phi, & \cdots \end{array} \right\rangle \tag{37}$$

where

$$\bar{X}'_{a+i} = \bar{X}_{a+i} \bar{X}_{a+1} \qquad (r_i = 1, \ i \neq 1) \tag{38}$$
$$\bar{X}'_{a+i} = \bar{X}_{a+i} \qquad (r_i = 0). \tag{39}$$

We can easily see that $[\bar{X}'_i, U] = 0$ for $\forall i$ by directly checking the commutation relations. Therefore, given the canonical representation of $\mathcal{M}^{'}$, we can find the canonical representation of $\mathcal{M}$ by following the above procedure. By treating each O_i as U, we can obtain canonical generators for $\mathcal{O} = \langle iI, \{O_i, \ \forall i\} \rangle$.

Note that this procedure can be viewed as a generalization of Gaussian elimination, applied to commuting subgroups of the Pauli operator group [28], but now made effective also for a group of operators consisting of Pauli operators which may not commute each other. The above procedure clearly defines a general procedure to obtain canonical generators for any subgroup of the Pauli operator group $\mathcal{P}$. Moreover, in contrast to the computational cost of the "exhaustive elimination" approach, which scales with $|S|$, the computational cost of this procedure grows as $\sim \log(|S|)$. Specifically, each of the iteration steps can be performed in a number of steps which is of the order of $G(\mathcal{M})$. Therefore, in total, this procedure only requires a number of steps which scales linearly in $G(\mathcal{O})^2$.

We emphasize that the choices of canonical generators are not unique for a given group of operator $\mathcal{O}$. First, note that the largest Abelian subgroup of $\mathcal{O}$ is uniquely defined as

$$\mathcal{O}_c = \langle \bar{X}_{a+1}, \cdots, \bar{X}_{a+b} \rangle. \tag{40}$$

Therefore, the commuting canonical generators $\bar{X}_{a+1}, \cdots, \bar{X}_{a+b}$ can be chosen freely from $\mathcal{O}_c$ as long as each of them is independent. Also, there is a freedom to apply an element from $\mathcal{O}_c$ to the anti-commuting canonical generators $\bar{X}_1, \cdots, \bar{X}_a$ and $\bar{Z}_1, \cdots, \bar{Z}_a$ since this does not change the commutation relations. Finally, for two given pairs of anti-commuting canonical generators $\bar{X}_1$, $\bar{Z}_1$, $\bar{X}_2$ and $\bar{Z}_2$, the following choices of anti-commuting canonical generators

$$\begin{aligned} \bar{X}_1 &\leftrightarrow \bar{X}_1 \bar{X}_2 \\ \bar{Z}_1 &\leftrightarrow \bar{Z}_1 \\ \bar{X}_2 &\leftrightarrow \bar{X}_2 \\ \bar{Z}_2 &\leftrightarrow \bar{Z}_1 \bar{Z}_2 \end{aligned} \tag{41}$$

also satisfy the commutation relations of canonical generators. This arbitrariness in the choices of anti-commuting canonical generators leads to the arbitrariness of the definitions of logical qubits. We will treat this problem carefully when non-local properties of logical qubits are discussed in section V.

IV. FRAMEWORK II

In the previous section, we described the classification of logical operators based on their localities and non-localities. Then, we defined the overlapping operator group and presented a theoretical tool, the canonical representation, to analyze the overlapping operator group. In this section, we connect the classification of logical operators with the overlapping operator group by using the canonical representation to compute logical operators in each of four sets M_A, M_B, M_{AB} and M_ϕ.

Computation of logical operators in M_A, M_B and M_{AB} can be performed by directly analyzing the overlapping operator groups $\mathcal{O}^A$ and $\mathcal{O}^B$ through a canonical representation, as we see in section IV A and IV B. For the computation of logical operators in M_ϕ, the relation between $\mathcal{O}^A$ and $\mathcal{O}^B$ needs to be analyzed. We show that there is a one-to-one correspondence between anti-commuting canonical generators for $\mathcal{O}^A$ and $\mathcal{O}^B$, in section IV C. This correspondence leads to an efficient means to compute the logical operators in M_ϕ, shown in section IV D.

A. Localized Logical Operators; M_{AB}

Let us begin by obtaining all the localized logical operators in M_{AB}, through the canonical representation of the overlapping operator group $\mathcal{O}^A$. We can represent the overlapping operator group $\mathcal{O}^A$ as

$$\mathcal{O}^A = \left\langle \begin{array}{ccccccc} \bar{X}_1, & \cdots, & \bar{X}_a, & \bar{X}_{a+1}, & \cdots, & \bar{X}_{a+b} \\ \bar{Z}_1, & \cdots, & \bar{Z}_a, & \phi, & \cdots, & \phi \end{array} \right\rangle. \tag{42}$$

Since $\mathcal{S}_A \subseteq \mathcal{O}^A$ and can be represented as

$$\mathcal{S}_A = \langle S_1, \cdots, S_{G(\mathcal{S}_A)} \rangle \tag{43}$$

with $G(\mathcal{S}_A)$ independent stabilizer generators, we can represent $\mathcal{O}^A$ as

$$\mathcal{O}^A = \left\langle \begin{array}{cccccccc} \bar{X}_1, & \cdots, & \bar{X}_a, S_1, & \cdots, & S_{G(\mathcal{S}_A)}, & \ell_1, & \cdots, & \ell_d \\ \bar{Z}_1, & \cdots, & \bar{Z}_a, \phi, & \cdots, & \phi, & \phi, & \cdots, & \phi \end{array} \right\rangle \tag{44}$$

by extracting stabilizer generators for $\mathcal{S}_A$. We may then note that operators $\ell_1, \cdots, \ell_d$ commute with all the operators in $\mathcal{O}^A$, but not inside $\mathcal{S}_A$. Therefore, operators $\ell_1, \cdots, \ell_d$ are actually *localized logical operators in* L_A.

In fact, we can easily see that logical operators $\ell_1, \cdots, \ell_d$ are in M_{AB} by representing $\ell_1, \cdots, \ell_d$ as

$$\ell_i = \prod_{j \in R_i} S_j|_A \in \mathcal{O}^A \tag{45}$$

for some sets of integers R_i for $i = 1, \cdots, d$. Then, there exist localized logical operators

$$\ell_i' = \prod_{j \in R_i} S_j|_A S_j = \prod_{j \in R_i} S_j|_B \in \mathcal{O}^B \tag{46}$$

which are now defined inside B, but equivalent to ℓ_i (Fig.4). Therefore, $\ell_1, \cdots, \ell_d$ are in M_{AB}. Soon, we shall see that logical operators $\ell_1, \cdots, \ell_d$ are all the independent logical operators in M_{AB}.

B. Localized Logical Operators; M_A and M_B

Though we obtained localized logical operators in M_{AB} through the analysis of the overlapping operator groups, this approach does not exhaust all the localized logical operators in L_A. In order to find logical operators in M_A, let us represent the restriction of the centralizer group

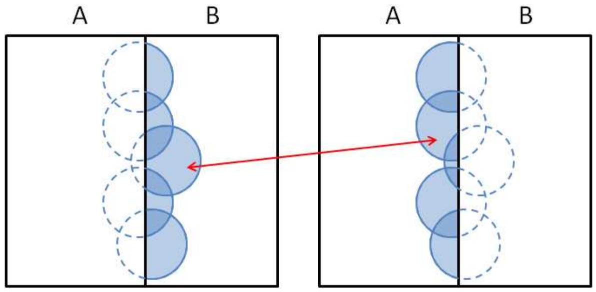

FIG. 4: Two equivalent logical operators in M_{AB} found both inside $\mathcal{O}^A$ and $\mathcal{O}^B$. Each circle represents stabilizer generators which overlap at the boundary between A and B.

into A in a canonical form as

$$\mathcal{C}_A = \langle \{U \in \mathcal{P}_A \mid [U, O] = 0,\ O \in \mathcal{O}^A\} \rangle \tag{47}$$
$$= \left\langle \begin{array}{cccccccccc} S_1, & \cdots, & S_{G(\mathcal{S}_A)}, & \ell_1, & \cdots, & \ell_d, & \alpha_1, & \cdots, & \alpha_c \\ \phi, & \cdots, & \phi, & \phi, & \cdots, & \phi, & \alpha'_1, & \cdots, & \alpha'_c \end{array} \right\rangle .$$

Thus, we immediately notice that $\alpha_1 \cdots \alpha_c$ and $\alpha'_1 \cdots \alpha'_c$ are localized logical operators in M_A. Since $\mathcal{C}_A$ includes all the operators defined inside A which commute with all the stabilizer generators, $\ell_1, \cdots, \ell_d$, $\alpha_1 \cdots \alpha_c$ and $\alpha'_1 \cdots \alpha'_c$ are all the localized logical operators in L_A.

Now we show that $\ell_1, \cdots, \ell_d$ are all the independent logical operators in M_{AB} while $\alpha_1 \cdots \alpha_c$ and $\alpha'_1 \cdots \alpha'_c$ are logical operators in M_A. Since logical operators in M_{AB} can be defined both inside A and B, they must commute with all the localized logical operators in L_A. Therefore, logical operators from $\alpha_1 \cdots \alpha_c$ and $\alpha'_1 \cdots \alpha'_c$ cannot be inside M_{AB} since they have anti-commuting pairs inside L_A. Thus, we notice that $\ell_1, \cdots, \ell_d$ are in M_{AB}, while $\alpha_1 \cdots \alpha_c$ and $\alpha'_1 \cdots \alpha'_c$ are in M_A.

C. One-to-one Correspondence between $\mathcal{O}^A$ and $\mathcal{O}^B$

Unlike for the localized logical operators obtained above, the computation of non-local logical operators is somewhat complicated, since we need to discuss the commutation relations of operators inside both A and B at the same time. Therefore, we need a more sophisticated approach, which allows us to discuss two overlapping operator groups $\mathcal{O}_A$ and $\mathcal{O}_B$ simultaneously.

There is an intrinsic one-to-one correspondence between anti-commuting canonical generators in $\mathcal{O}_A$ and $\mathcal{O}_B$. Surprisingly, this correspondence between $\mathcal{O}_A$ and $\mathcal{O}_B$ enables us to compute all the non-local logical operators *only through the computations performed locally inside each of complementary subsets A and B.*

First, let us recall the canonical representation for the overlapping operator group $\mathcal{O}^A$ with

$$\mathcal{O}^A = \left\langle \begin{array}{cccccccc} \bar{X}_1, & \cdots, & \bar{X}_a, & \ell_1, & \cdots, & \ell_d, & \mathcal{S}_A \\ \bar{Z}_1, & \cdots, & \bar{Z}_a, & \phi, & \cdots, & \phi, & \phi \end{array} \right\rangle \tag{48}$$

where $\mathcal{S}_A$ represents the generators for $\mathcal{S}_A$ symbolically. We can represent $\bar{X}_i$, $\bar{Z}_i$ and ℓ_i as

$$\bar{X}_i = \prod_{j \in R_i^{(x)}} S_j|_A \in \mathcal{O}^A \tag{49}$$

$$\bar{Z}_i = \prod_{j \in R_i^{(z)}} S_j|_A \in \mathcal{O}^A \tag{50}$$

$$\ell_i = \prod_{j \in R_i} S_j|_A \in \mathcal{O}^A. \tag{51}$$

through the overlaps $S_i|_A$ and some sets of integers $R_i^{(x)}$, $R_i^{(x)}$ and R_i. Then, we notice that the operators

$$\bar{X}'_i = \prod_{j \in R_i^{(x)}} S_j|_B \in \mathcal{O}^B \tag{52}$$

$$\bar{Z}'_i = \prod_{j \in R_i^{(z)}} S_j|_B \in \mathcal{O}^B \tag{53}$$

$$\ell'_i = \prod_{j \in R_i} S_j|_B \in \mathcal{O}^B \tag{54}$$

actually form canonical generators for the overlapping operator group $\mathcal{O}^B$. In fact, simple calculations lead to that

$$\mathcal{O}^B = \left\langle \begin{array}{cccccccc} \bar{X}'_1, & \cdots, & \bar{X}'_a, & \ell'_1, & \cdots, & \ell'_d, & \mathcal{S}_B \\ \bar{Z}'_1, & \cdots, & \bar{Z}'_a, & \phi, & \cdots, & \phi, & \phi \end{array} \right\rangle . \tag{55}$$

We can easily see that $\bar{X}'_i$, $\bar{Z}'_i$ and ℓ'_i are independent generators since we can also construct the canonical generators for $\mathcal{O}^A$ by starting from the canonical generators for $\mathcal{O}^B$. This construction also ensures that there are the same numbers of anti-commuting canonical generators for $\mathcal{O}^A$ and $\mathcal{O}^B$. We can easily check the commutation relations between $\bar{X}'_i$, $\bar{Z}'_i$ and ℓ'_i which are exactly the same as the commutation relations between $\bar{X}_i$, $\bar{Z}_i$ and ℓ_i. Therefore, there is a *one-to-one correspondence* between canonical generators for $\mathcal{O}^A$ and $\mathcal{O}^B$.

We can see the origin of this correspondence by extracting $\mathcal{S}_A$ and $\mathcal{S}_B$ from the stabilizer group $\mathcal{S}$ as

$$\mathcal{S} = \langle \mathcal{S}_A, \mathcal{S}_B, \mathcal{S}_{AB} \rangle \tag{56}$$

where the *non-local stabilizer group* $\mathcal{S}_{AB}$ satisfies

$$G(\mathcal{S}) = G(\mathcal{S}_A) + G(\mathcal{S}_B) + G(\mathcal{S}_{AB}). \tag{57}$$

All the generators for $\mathcal{S}_{AB}$ are non-locally defined jointly over A and B. Here, we note that the definition of the non-local stabilizer group $\mathcal{S}_{AB}$ is not unique since we have a freedom to apply stabilizers inside $\mathcal{S}_A$ and $\mathcal{S}_B$ to generators for $\mathcal{S}_{AB}$. Now, we notice that the following stabilizer generators are all the independent generators for $\mathcal{S}_{AB}$;

$$S_i^{(x)} \equiv \prod_{j\in R_i^{(x)}} S_j = \bar{X}_i\bar{X}_i' \quad (58)$$

$$S_i^{(z)} \equiv \prod_{j\in R_i^{(z)}} S_j = \bar{Z}_i\bar{Z}_i' \quad (59)$$

$$S_i^{(L)} \equiv \prod_{j\in R_i} S_j = \ell_i\ell_i'. \quad (60)$$

Note that the analysis of the overlapping operator group $\mathcal{O}^A$ automatically generates anti-commuting canonical generators for the overlapping operator group $\mathcal{O}^B$. In particular, a one-to-one correspondence we revealed here provides independent generators for $\mathcal{S}_{AB}$ explicitly.

D. Non-Local Logical Operators; M_ϕ

Having uncovered the relations between $\mathcal{O}^A$ and $\mathcal{O}^B$, let us finally obtain non-local logical operators. First, we define a group of operators which commute with all the stabilizer generators inside A as

$$\mathcal{C}(\mathcal{S}_A) = \langle\{U \in \mathcal{P}_A \mid [U,O] = 0,\ \forall O \in \mathcal{S}_A\}\rangle \quad (61)$$

and a similar group of operators inside B as

$$\mathcal{C}(\mathcal{S}_B) = \langle\{U \in \mathcal{P}_B \mid [U,O] = 0,\ \forall O \in \mathcal{S}_B\}\rangle. \quad (62)$$

Their canonical representations can be obtained as

$$\mathcal{C}(\mathcal{S}_A) = \left\langle \begin{array}{cccccc} \{\bar{X}_i\} & ,\mathcal{S}_A, & \ell_1, & \cdots, & \ell_d, & \{\alpha_i\} \\ \{\bar{Z}_i\} & ,\phi, & r_1, & \cdots, & r_d, & \{\alpha_i'\} \end{array} \right\rangle \quad (63)$$

and

$$\mathcal{C}(\mathcal{S}_B) = \left\langle \begin{array}{cccccc} \{\bar{X}_i'\} & ,\mathcal{S}_B, & \ell_1', & \cdots, & \ell_d', & \{\beta_i\} \\ \{\bar{Z}_i'\} & ,\phi, & r_1', & \cdots, & r_d', & \{\beta_i'\} \end{array} \right\rangle \quad (64)$$

where r_i and r_i' are anti-commuting pairs of ℓ_i and ℓ_i' defined inside A and B respectively. Here, we used $\{\bar{X}_i\}$ symbolically to represent canonical generators $\bar{X}_1, \cdots, \bar{X}_a$.

Now, let us consider the following operators δ_i

$$\delta_i = r_i r_i' \quad (65)$$

which are defined jointly over A and B for $i = 1, \cdots, d$. Let us define the set of d different δ_i as

$$\Delta = \{\delta_1, \cdots, \delta_d\}. \quad (66)$$

We may now show that the set of jointly defined operators Δ consists only of non-local logical operators in M_ϕ. First, we can easily see that δ_i commute with all the stabilizers in $\mathcal{S}_A$ and $\mathcal{S}_B$. We can also check the commutations with stabilizer generators in $\mathcal{S}_{AB}$ by seeing

$$[\delta_i, S_i^{(x)}] = [r_i r_i', \bar{X}_i\bar{X}_i'] = 0 \quad (67)$$

$$[\delta_i', S_i^{(z)}] = [r_i r_i', \bar{Z}_i\bar{Z}_i'] = 0 \quad (68)$$

$$[\delta_i', S_i^{(L)}] = [r_i r_i', \ell_i\ell_i'] = 0. \quad (69)$$

Since δ_i commute with all the stabilizer generators, but not inside the stabilizer group $\mathcal{S}$, δ_i are logical operators. We can show that δ_i are in M_ϕ by considering the anti-commutations with ℓ_i

$$\{\delta_i, \ell_i\} = \{\delta_i, \ell_i'\} = 0. \quad (70)$$

Suppose that δ_i can be also defined inside A. Then, δ_i commute with all the logical operators in M_{AB} which leads to a contradiction. Therefore, δ_i in Δ are non-local logical operators in M_ϕ. Later, we shall see that these δ_i are all the independent non-local logical operators. Since the proof requires further analysis on localized logical operators, we postpone it until section V A.

Here, let us mention the reduction of computational cost of obtaining non-local logical operators as a result of the one-to-one correspondence between canonical generators between the overlapping operator groups $\mathcal{O}^A$ and $\mathcal{O}^B$. For the computations of localized logical operators, one only needs to see whether a given operator commutes with the overlaps of the stabilizer generators, which can be efficiently checked through our analysis based on the overlapping operator group. However, for the computations of non-local logical operators, one needs to discuss the commutation relationships with stabilizer generators defied globally over a bi-partition since non-local logical operators cannot be defined locally. Though a naive approach is to check the commutation relations for all the stabilizer generators, due to the one-to-one correspondence between the overlapping operator groups $\mathcal{O}^A$ and $\mathcal{O}^B$, we only need to consider the commutation relations inside each of subsets A and B.

We summarize our results graphically in Fig.5 which shows all the logical operators along with the canonical generators for the overlapping operator groups. The procedures to compute logical operators in M_A, M_B, M_{AB} and M_ϕ described in this section complete our framework.

V. EXAMPLES: NON-LOCAL PROPERTIES OF STABILIZER CODES

The two sections above present a framework providing computational tractable means for computing and classifying logical operators based on their non-localities and localities. In particular, this framework includes procedures to derive all the logical operators in each of four sets M_{AB}, M_A, M_B and M_ϕ through the analysis of the overlapping operator groups, using canonical representations. We now turn to a demonstration of the usefulness

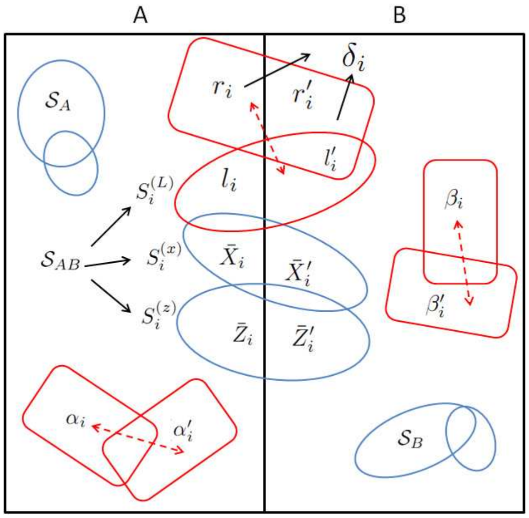

FIG. 5: An illustration of stabilizer generators and logical operators in a bi-partition. Circles represent stabilizer generators while squares represent logical operators except that ℓ_i and ℓ_i' are found inside stabilizer generators. The dotted arrows show the possibility of anti-commutations. Logical operators α_i and α_i' (β_i and β_i') in M_A (M_B) form pairs just inside A (B). Stabilizer generators for non-local stabilizer group $\mathcal{S}_{AB}$ are represented as products of canonical generators for $\mathcal{O}^A$ and $\mathcal{O}^B$. While logical operators ℓ_i and ℓ_i' in M_{AB} appear as the projection of some stabilizer generators $S_i^{(L)}$ for $\mathcal{S}_{AB}$, non-local logical operators δ_i are products of anti-commuting pairs of ℓ_i and ℓ_i'.

of this framework, by studying some interesting non-local properties of the stabilizer formalism. Below, we consider global symmetries in topological order, the distribution of multi-partite entanglement, and entanglement entropy.

A. Geometric duality and topological order

A valuable application of our framework is the study of geometric properties of logical operators. The framework has required no geometry so far, since our discussion has concentrated on properties of logical operators and stabilizers in the Pauli operator space, without considering the geometric locations of qubits in real physical space. Though the framework discussed locality and non-locality of logical operators, the bi-partition was placed inside the Pauli operator group $\mathcal{P}$. However, the notion of geometry becomes particularly important, for example, when the practical implementation of quantum codes is the issue. When quantum codes are defined on some geometric manifold, a bi-partition in the Pauli operator space becomes a bi-partition in real physical space consisting of physical qubits.

An interesting problem where geometries of logical operators become essential is the study of topological order emerging in quantum codes. The mathematical notion of *topology* is widely appreciated as playing a crucial role in many interesting physical phenomena [23, 29–32], including some quantum codes [8]. Topological order also possesses potential practical importance, since a system with topological order may support dissipationless currents [33] and serve as a resource for universal quantum computation [34]. Topological properties of various systems have been analyzed using a range of quantities, including Berry phases [35], anyonic excitations, and topological entropy [6, 7].

Topological order is commonly believed to be a result of global symmetries in the system [32]. Such a global symmetry emerges as the existence of global operators which commute with the system Hamiltonian. If this observation is interpreted in the language of quantum codes, it asserts that the existence of large logical operators implies the existence of topological order in the system. However, why large logical operators may give rise to topological order is not well understood. To answer this fundamental question about topological order and symmetries, topological aspects of logical operators need to be analyzed. Thus, the analysis of the geometric invariance (and variance) of shapes of logical operators is a necessary step to understand an intrinsic connection between global symmetries and topological order.

Here, we add geometry to the discussion of quantum codes by proving a theorem which indicates the existence of an intrinsic duality on the geometric shapes of logical operators in stabilizer codes. This theorem can be directly proven using our framework. We apply this theorem to the Toric code [8] which supports topological order in the ground state space. We give a general discussion on the Toric code from a viewpoint of symmetries of the system by analyzing geometric properties of logical operators.

The theorem we prove in this subsection also complements our framework. Using this theorem, we show that non-local logical operators δ_i in Δ in section IV D are all the possible independent non-local logical operators.

1. Duality of logical operators in a bi-partition

Our main goal in this subsection is to discuss the geometry of logical operators. Consider a stabilizer code defined with some physical qubits on some geometric manifold. The geometric shape of a logical operator may be simply captured by considering the shape of qubits where the logical operator has non-trivial support. However, logical operators have many equivalent representations, since the application of stabilizers does not change the properties of logical operators. This makes it difficult to uniquely define the geometry of a logical operator, or to determine the geometrically invariant properties of a given logical operator.

To avoid this difficulty, we take the following approach. If a logical operator is defined inside some localized region A, then, the shape of the logical operator may be bounded by the shape of A. By analyzing the geometric shapes of possible regions where each logical operator can be supported, geometric properties of logical operators may be studied. For this purpose, localized logical operators obtained in our framework become essential. We start our discussion of geometries of logical operators by proving the following theorem governing localized logical operators in a bi-partition. This theorem can be also be easily derived from the discussions independently done elsewhere [26].

Theorem 1. *For a stabilizer code with k logical qubits, let g_A and g_B be the numbers of independent logical operators which can be supported only by qubits inside A and B respectively with $B = \bar{A}$. Then the following formula holds:*

$$g_A + g_B = 2k. \tag{71}$$

Theorem 1 states that by naively counting the number of independent logical operators which can be defined inside each region and taking the sum of them, the sum is equal to $2k$. Therefore, the sum of the number of localized logical operators inside each subset A and B is always conserved. The proof is immediate with our framework.

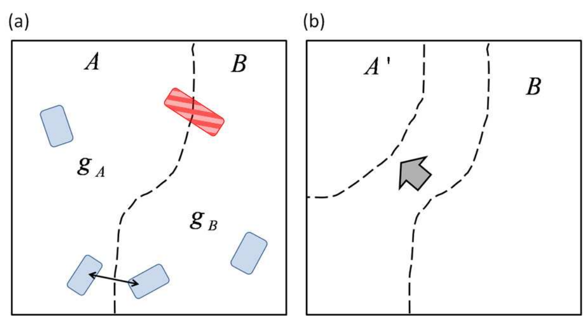

FIG. 6: (a) Geometric duality on stabilizer codes ($g_A + g_B = 2k$). Rectangles represent logical operators. g_A and g_B represent the number of localized logical operators inside A and B. A hatched rectangle (red online) at the boundary between A and B is a non-local logical operator which is not counted either in g_A or in g_B. Rectangles connected by a two-sided arrow are equivalent and counted both in g_A and in g_B. Other rectangles are localized logical operators defined only inside A or B and counted once in g_A or in g_B. (b) Shrinking logical operators. When $g_A = g_{A'}$ while A' is smaller than A, all the logical operators in A have equivalent logical operators in A'. Application of appropriate stabilizers shrinks the shapes of logical operators from A to A'.

Proof. Since all the localized logical operators are defined inside $\mathcal{C}_A$ and $\mathcal{C}_B$, we have

$$g_A = G(\mathcal{C}_A) - G(\mathcal{S}_A) \tag{72}$$
$$g_B = G(\mathcal{C}_B) - G(\mathcal{S}_B). \tag{73}$$

By looking at the relation between $\mathcal{O}^A$ and $\mathcal{C}_A$, we have

$$G(\mathcal{C}_A) = 2V_A - G(\mathcal{O}_A) \tag{74}$$

where V_A is the number of qubits inside A. Therefore, we have

$$\begin{aligned} g_A + g_B =& 2(V_A + V_B) - G(\mathcal{O}_A) - G(\mathcal{O}_B) \\ &- G(\mathcal{S}_A) - G(\mathcal{S}_B). \end{aligned} \tag{75}$$

Now, by using the generators for $\mathcal{S}_{AB}$ represented through canonical generators in $\mathcal{O}_A$ and $\mathcal{O}_B$, we have

$$G(\mathcal{O}^A) = G(\mathcal{S}_A) + G(\mathcal{S}_{AB}) \tag{76}$$
$$G(\mathcal{O}^B) = G(\mathcal{S}_B) + G(\mathcal{S}_{AB}). \tag{77}$$

Finally, from $V_A + V_B = N$ and $G(\mathcal{S}_A) + G(\mathcal{S}_B) + G(\mathcal{S}_{AB}) = N - k$, we have

$$g_A + g_B = 2k. \tag{78}$$

□

Note that Theorem 1 is generally true for all the stabilizer codes with any bi-partitions. Before introducing geometries, we make a few remarks on Theorem 1. First, let us discuss the relationship of this theorem with our classification of logical operators. g_A counts the number of localized logical operators in A while g_B counts the number of localized logical operators in B. Localized logical operators in M_{AB} are counted twice both in g_A and g_B while non-local logical operators are not counted either in g_A or g_B. Localized logical operators in M_A and M_B are counted once in g_A and g_B respectively.

Then, a simple number counting argument leads to the following corollary.

Corollary 1. *The number of independent localized logical operators in M_{AB} and the number of independent non-local logical operators are the same.*

The proof is immediate with Theorem 1. Let us denote the number of independent logical operators in M_A, M_B, M_{AB} and M_ϕ as m_A, m_B, m_{AB} and m_ϕ where

$$m_{AB} = G(M_{AB}) - G(\mathcal{S}) \tag{79}$$
$$m_A = G(L_A) - G(M_{AB}) \tag{80}$$
$$m_B = G(L_B) - G(M_{AB}) \tag{81}$$
$$m_\phi = 2k - (m_A + m_B + m_{AB}). \tag{82}$$

Since $g_A = m_A + m_{AB}$ and $g_B = m_B + m_{AB}$, theorem 1 asserts that

$$m_A + m_B + 2m_{AB} = 2k. \tag{83}$$

Therefore, we have

$$m_{AB} = m_\phi. \tag{84}$$

This corollary complements our framework by showing that non-local logical operators δ_i in Δ are all the possible independent non-local logical operators due to $m_{AB} = m_{\phi}$.

Second, let us discuss some consequences of Theorem 1 for coding properties. Theorem 1 allows us to derive the quantum singleton bound easily [36]. Let us define three subsets such that $V_A = d-1$, $B = \bar{A}$ and $B' \subseteq B$ with $V_{B'} = d-1$. Then, we have

$$g_A = g_{B'} = 0 \tag{85}$$
$$g_B = 2k. \tag{86}$$

Since we have

$$g_B \leq 2(V_B - V_{B'}), \tag{87}$$

we obtain

$$k \leq N - 2(d-1). \tag{88}$$

This is the quantum singleton bound.

2. Geometric interpretation of the theorem

Now, let us apply Theorem 1 to a stabilizer code which is implemented with some physical qubits where qubits are located at some specific positions in geometric manifolds. To begin with, let us consider two regions A and $A' \subset A$ which can support the same number of independent logical operators with $g_A = g_{A'}$ (Fig.6(b)). Note that all the logical operators defined inside A can be transformed into equivalent logical operators defined inside A' by applying appropriate stabilizers. Therefore, *all the logical operators in A can be deformed into other equivalent logical operators defined inside a smaller subset A'.*

Theorem 1 clearly shows limitations on the possible geometric shapes of equivalent logical operators. In order to create a quantum code with large logical operators, we need to have a small g_A for large region A. However, the effort of decreasing g_A for large A results in increasing g_B for small B for $B = \bar{A}$ since we have $g_A + g_B = 2k$. Thus, our theorem shows a clear restriction on the *sizes of logical operators*, and indicates the *intrinsic duality on geometric shapes of localized logical operators* of stabilizer codes in a bi-partition.

Now that we have the restriction $g_A + g_B = 2k$ in hand, let us discuss the problem of giving an upper bound on the code distance for a local stabilizer code. This problem has been addressed in the literature [17], by a beautiful construction of logical operators in which logical operators can be shorten to equivalent logical operators defined in smaller subsets. The *cleaning lemma*, at the heart of this method, can now be understood as resulting from an application of our formula. Let us suppose that there is no logical operators defined inside A at all. Then, one has $g_A = 0$ and $g_B = 2k$ from Theorem 1. Since B can contain $2k$ logical operators, we notice that *all the logical operators in the system can be defined in B* by applying appropriate stabilizers. Therefore, by finding a region A such that $g_A = 0$, we can deform logical operators to its complement A. This eventually gives an upper bound on the sizes of logical operators and the code distances.

3. Application to topological order

Global symmetries of the system, which emerge as the existence of large logical operators, are at the heart of topologically ordered systems. To understand this underlying connection between global symmetries and topological order, logical operators need to be analyzed. For this purpose, our framework, and Theorem 1, give useful insights about the geometric properties of logical operators. We now apply these to analyze the geometric properties of logical operators in the Toric code.

The Toric code is the simplest known, exactly solvable, model which is described in the stabilizer formalism, supporting a degenerate ground state, with topological order. Consider an $L \times L$ square lattice on the torus. The Toric code is defined qubits which live on edges of bonds (Fig.7 (a)). There are $N = 2(L \times L)$ qubits in total. For simplicity of discussion, we set periodic boundary conditions. The Hamiltonian is:

$$H = -\sum_{s,p}(\mathcal{A}_s + \mathcal{B}_p) \tag{89}$$
$$\mathcal{A}_s = \prod_{i \in s} X_i \tag{90}$$
$$\mathcal{B}_p = \prod_{i \in p} Z_i \tag{91}$$

where s represent "stars" and p represent "plaquettes" (Fig.7(a)). The Toric code has $2k = 4$ independent logical operators since $G(\mathcal{S}) = N - 2$. Each of independent logical operators ℓ_1, r_1, ℓ_2 and r_2 are shown in Fig.7(b). These four logical operators obey the following commutation relations:

$$\left\{ \begin{array}{ll} \ell_1 & , \ell_2 \\ r_1 & , r_2 \end{array} \right\} \tag{92}$$

where commutation relations are defined in a way similar to the canonical representation.

Consider several bi-partitions of the system and discuss how the geometric properties of the logical operators may be understood. Let us define the following regions. Q_x is a region which circles the lattice in $\hat{x}$-direction (Fig.8(a)). Q_y is a region which circles the lattice in the $\hat{y}$-direction (Fig.8(b)). Finally, $R_1 = Q_x \cup Q_y$ is a region which is the union of Q_x and Q_y (Fig.8(c)).

Consider first a bi-partition into two subsets Q_x and $\bar{Q}_x$ described in Fig.8(a). Logical operators r_2 and ℓ_1 are defined inside Q_x, and we have $g_{Q_x} \geq 2$. r_2 and ℓ_1 also have equivalent logical operators in $\bar{Q}_x$ since translations of r_2 and ℓ_1 in $\hat{y}$-direction are equivalent to original

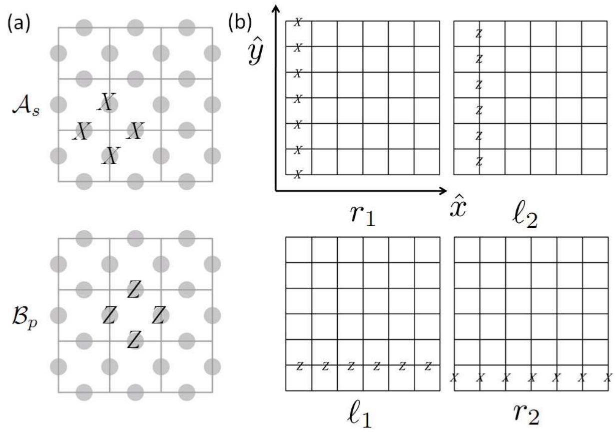

FIG. 7: The Toric Code. Qutbis live on edges on the bonds. Periodic boundary conditions are set in x and y directions. (a) Stabilizers (interaction terms) in the Toric code. Stabilizers act on four qubits in either stars or plaquettes. (b) Logical operators in the Toric code.

logical operators r_2 and ℓ_1 respectively. Then, we have $g_{\bar{Q}_x} \geq 2$.

Now apply theorem 1 to this bi-partition. Since $g_{Q_x} + g_{\bar{Q}_x} = 4$ due to the theorem, we must have $g_{Q_x} = g_{\bar{Q}_x} = 2$. Let us interpret these equations and discuss the geometries of logical operators. Since $g_{\bar{Q}_x} = 2$, r_1 and ℓ_2 are all the independent logical operators which can be defined inside Q_x. In other words, *logical operators r_2 and ℓ_1 cannot both be defined inside Q_x*. Without discussing the properties of r_2 and ℓ_1, one can analyze geometric properties of r_2 and ℓ_1 through Theorem 1. This observation can be explained through the classifications of logical operators in our framework. Logical operators r_1 and ℓ_2 are localized logical operators in a set $M_{Q_x\bar{Q}_x}$ while logical operators r_2 and ℓ_1 are non-local logical operators in a set M_ϕ in a bi-partition into Q_x and $\bar{Q}_x$.

It is more illuminating when we consider the equation $g_{\bar{Q}_x} = 2$. Even when we expand the region Q_x to $\bar{Q}_x$, logical operators r_2 and ℓ_1 still cannot be defined inside $\bar{Q}_x$ since r_1 and ℓ_2 are all the independent logical operators which can be defined inside $\bar{Q}_x$. Therefore, one can conclude that logical operators r_2 and ℓ_1 can be defined *only inside regions which circle around the lattice in the $\hat{y}$ direction*. The similar discussion holds for logical operators r_1 and ℓ_2 by considering a bi-partition into Q_y and $\bar{Q}_y$ (Fig.8(b)). Logical operators r_1 and ℓ_2 can be defined *only inside regions which circle around the lattice in the $\hat{x}$ direction.*

Next, let us consider a bi-partition into two subsets $R_1 = Q_x \cup Q_y$ and $\bar{R}_1$ described in Fig.8(c). Since all the four independent logical operators can be defined inside R_1, we have $g_{R_1} = 4$. Then, there is no logical operator which can be defined inside $\bar{R}_1$ since $g_{\bar{R}_1} = 0$. One notices that $\bar{R}_1$ has no winding either in x or y direction. Therefore, *there is no logical operator defined inside a region which does not circle around the lattice in any direction.*

These discussions clarify that the geometric shapes of logical operators have universal, *topological* properties, which are invariant under the application of stabilizers. Specifically, whether a region circles around the lattice in x and y directions can be quantified by the *winding numbers* of regions. Define the winding numbers w_i such that $w_i = 1$ if a region circles around the lattice in i direction where $i = x, y$ and w_i otherwise. The winding numbers w_i of geometric shapes of logical operators are quantities which are invariant among all the equivalent logical operators. Thus, *the winding numbers of logical operators may be viewed as topological invariants.*

This analysis of the logical operators of the Toric code shows that the signature of quantum order in the system can be found in the geometric properties of symmetry operators which commute with the system Hamiltonian. Though logical operators are originally used as indicators for coding properties of quantum codes, logical operators can actually be the indicators for quantum order, including topological order. Our framework for logical operators, coupled with appropriate geometric considerations, can be used to study such order, and indeed may be useful for classifying quantum phases of such systems.

B. Non-Locality of Logical Qubits and Secret-Sharing

Perhaps as interesting as the geometric properties of logical operators is the nature of entanglement of degenerate ground states. The entanglement of single stabilizer states may be characterized by entanglement entropy [25]. However, in stabilizer codes, the nature of the entanglement in each of several degenerate ground states can be different. This makes a stabilizer code capable of providing a kind of entanglement which is essentially different from the entanglement of a single state. Specifically, the *distributed nature of entanglement in ground state space* is of interest.

The study of entanglement distributed across two or more parties is enabled by the analysis of logical operators provided in our framework. Below, using the framework, we discuss how entanglement is distributed in a bi-partition by extending the classification of logical operators to the classification of logical qubits. In particular, we quantify how much entanglement is distributed across a bi-partition and discuss secret-sharing of information [13] between two parties.

In stabilizer codes, degenerate ground states of the Hamiltonian are the same as logical qubits of the corresponding stabilizer code. In order to understand entanglement of degenerate ground states in a stabilizer code, logical qubits need to be analyzed. Since logical qubits are described by a pair of anti-commuting logical operators, commutation relationships between all the classified logical operators, including non-local logical operators,

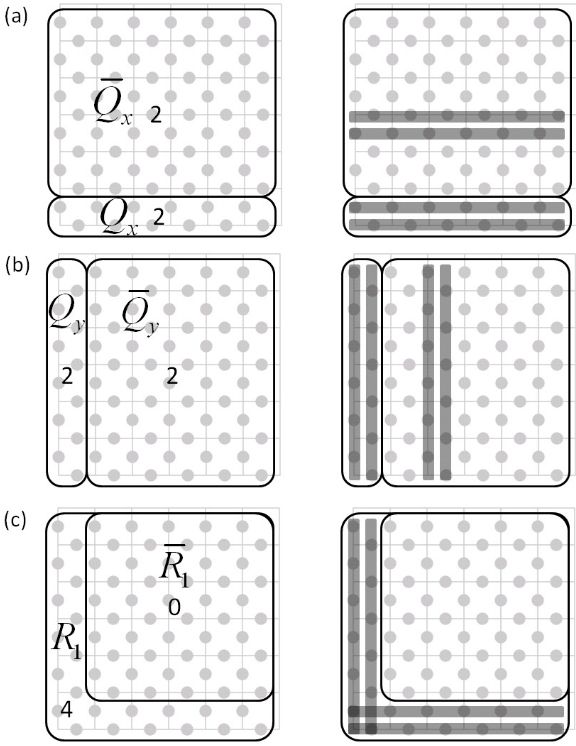

FIG. 8: Various bi-partitions and geometric properties of logical operators in the Toric code. Rectangles represent logical operators and circles represent qubits. Periodic boundary conditions are set in the $\hat{x}$ and $\hat{y}$ directions. Figures on the left hand side show bi-partitions and the number of logical operators defined inside each of complementary regions. Figures on the right hand side show the shapes of logical operators. (a) A bi-partition into Q_x and $\bar{Q}_x$. Each dot represents each qubit. Q_x is a region of qubits which extends in the $\hat{x}$ direction. Logical operators connected by two-sided arrows are equivalent. Both Q_x and $\bar{Q}_x$ support only r_1 and ℓ_2. r_2 and ℓ_1 cannot be defined wither inside Q_x or $\bar{Q}_x$. (b) A bi-partition into Q_x and $\bar{Q}_x$. Both Q_y and $\bar{Q}_y$ support only r_2 and ℓ_1. r_1 and ℓ_2 cannot be defined wither inside Q_y or $\bar{Q}_y$. (c) A bi-partition into $R_1 = Q_x \cup Q_y$. R_1 supports all the independent logical operators r_1, ℓ_1, r_2 and ℓ_2. There is no logical operator defined inside $\bar{R}_1$.

must be understood. For this purpose, all the logical operators need to be listed along with their classifications.

In section V B 1, we start our discussion by analyzing the commutation relations between each of $2k$ logical operators which are classified and computed through our framework. We introduce a classification of logical qubits by extending the classification of logical operators. In section V B 2, we discuss secret-sharing of classical and quantum information and show that our classification of logical qubits clearly capture distribution of entanglement in a stabilizer code.

1. Non-local logical qubit

In order to discuss properties of logical qubits, properties of pairs of anti-commuting logical operators need to be analyzed. Hence, let us start by listing all the classified logical operators obtained through our framework. Logical operators in M_A, M_B, M_{AB} and M_ϕ are found in the following group of operators:

$$\mathcal{C}(\mathcal{S}_A) = \left\langle \begin{array}{c} \{\bar{X}_i\} \\ \{\bar{Z}_i\} \end{array} \begin{array}{c} ,\mathcal{S}_A, \\ ,\phi, \end{array} \begin{array}{c} \ell_1, \\ r_1, \end{array} \begin{array}{c} \cdots, \\ \cdots, \end{array} \begin{array}{c} \ell_d, \\ r_d, \end{array} \begin{array}{c} \{\alpha_i\} \\ \{\alpha_i'\} \end{array} \right\rangle \tag{93}$$

and

$$\mathcal{C}(\mathcal{S}_B) = \left\langle \begin{array}{c} \{\bar{X}_i'\} \\ \{\bar{Z}_i'\} \end{array} \begin{array}{c} ,\mathcal{S}_B, \\ ,\phi, \end{array} \begin{array}{c} \ell_1', \\ r_1', \end{array} \begin{array}{c} \cdots, \\ \cdots, \end{array} \begin{array}{c} \ell_d', \\ r_d', \end{array} \begin{array}{c} \{\beta_i\} \\ \{\beta_i'\} \end{array} \right\rangle \tag{94}$$

where ℓ_i are in M_{AB}, α_i and α_i' are in M_A, β_i and β_i' are in M_B, and $\delta_i = r_i r_i'$ are in M_ϕ.

Then, in order to specify commutation relationships between all the classified logical operators, we list all the logical operators symbolically as follows:

$$\left\{ \begin{array}{cccccccccc} \alpha_1 & ,\cdots, & \alpha_{m_A'}, & \beta_1 & ,\cdots, & \beta_{m_B'}, & \ell_1, & \cdots, & \ell_{m_\phi} \\ \alpha_1' & ,\cdots, & \alpha_{m_A'}', & \beta_1' & ,\cdots, & \beta_{m_B'}', & \delta_1, & \cdots, & \delta_{m_\phi}' \end{array} \right\} \tag{95}$$

where two logical operators in the same column anti-commute each other as in the canonical representation and $m_A' = \frac{1}{2} m_A$ and $m_B' = \frac{1}{2} m_B$. We write down the commutation relations between logical operators which describe logical qubits symbolically as follows

$$\begin{aligned} M_A &\leftrightarrow M_A \\ M_B &\leftrightarrow M_B \\ M_\phi &\leftrightarrow M_{AB} \end{aligned} \tag{96}$$

where the two-directional arrow "$\leftrightarrow$" represents the possibility of anti-commutations between corresponding sets of logical operators. We summarize the above commutation relationships graphically in Fig.9.

This list of logical operators in Eq.(95) defines k logical qubits. Let us analyze the locality and non-locality of these logical qubits in detail. First, consider a logical qubit described by a pair of anti-commuting logical operators α_i and α_i' (Fig.9(a)). This logical qubit is described by a pair of two localized logical operators. There also exist logical qubits described by a pair of localized logical operators in M_B (Fig.9(b)). We call these logical qubits *local logical qubits*. Such local logical qubits can be completely manipulated through local operations on physical qubits inside either A or B.

Next, let us consider a logical qubit described by a pair of anti-commuting logical operators ℓ_i and δ_i (Fig.9(c)). This logical qubit is described by one localized logical operator and one non-local logical operator. We call such a logical qubit *local logical qubit*. Non-local logical qubits cannot be completely manipulated through local operations on physical qubits either inside A or inside B. Therefore, k logical qubits defined in Eq.(95) can be classified into two types, local and non-local logical qubits.

Recall the problem of the arbitrariness in the definition of logical qubits which we mentioned at the end of section III. Though the above choice of anti-commuting logical operators defines k different logical qubits, one can choose a different set of logical operators to define logical qubits. However, we shall see that the above definition of logical qubits and classification into local and non-local logical qubits are quite useful when distributions of entanglement are to be analyzed.

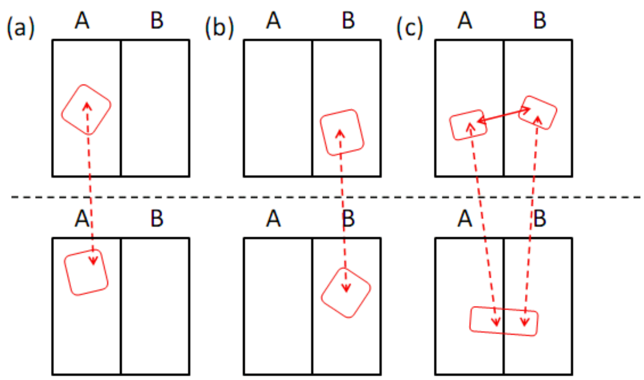

FIG. 9: The commutation relations between the four subsets of logical operators. The dotted arrows represent the anti-commutations between corresponding logical operators. We may have anti-commutations between two logical operators in the same column while other pairs always commute each other as in the symbolic form of canonical representations. (a) Anti-commutations inside M_A. (b) Anti-commutations inside M_B. (c) The pairing between M_{AB} and M_ϕ. (a)(b) define local logical qubits while (c) defines non-local logical qubits.

2. Secret-sharing

Let us now analyze how secret-charing schemes work with a stabilizer code. We also explain our results in terms of local and non-local logical qubits.

Secret-sharing is a scheme which allows sharing of information between two parties or multiple parties so that each party cannot access the encoded information individually. Shared information can be accessed only when all the parties agree and execute a protocol together. Such encoded information is shared by multiple parties as a secret among all the parties. Some entangled quantum system can be used as a resource for secret-charing of classical or quantum information [13, 37]. Below, we consider secret-sharing between two parties with a stabilizer code.

First, we discuss secret-sharing of classical information between two parties using a stabilizer code. Consider a situation where one party A possesses a subset of qubits A and the other party B possesses a complementary subset of qubits B. When ℓ is a non-local logical operator with $\ell \in M_\phi$, a bit of information can be shared between A and B by assigning 0 and 1 to each of eigenstates of ℓ. Such encoded information cannot be read out by individual access either from A or from B since ℓ does not have an equivalent logical operator defined either inside A or inside B. Therefore, a bit of information can be shared between A and B only if there exists a non-local logical operators with $m_\phi \neq 0$.

The method for sharing one bit of information can be easily extended to the method for sharing multiple bits of information. In order to share m bits of information between A and B, there must be m independent non-local logical operators. Suppose that m bits of information are encoded with respect to $u_1, \cdots, u_m \in M_\phi$. If $m > m_\phi$, there exists a set of integers R such that

$$\prod_{i \in R} u_i \in L_A \cup L_B. \tag{97}$$

Then, a local measurement can read out a bit of information out of shared at most m bits of information. Therefore, only m_ϕ bits of information can be shared between A and B.

Let us interpret the above discussion in terms of logical qubits defined in Eq.(95). A local logical qubit described by a pair of logical operators α_i and α_i' or β_i and β_i' cannot be used for sharing classical information. A non-local logical qubit described by a pair of δ_i and ℓ_i can be used for sharing a bit of information. The classification of logical qubits introduced in section V B 1 directly corresponds to their abilities as resources for sharing classical information.

Next, let us discuss secret-sharing of quantum information. In order to encode quantum information, a pair of anti-commuting logical operators ℓ and r is required. Suppose that a qubit is encoded with respect to ℓ and r. In order for this encoded qubit to be shared by A and B, the following condition is necessary:

$$\ell, r \in M_\phi. \tag{98}$$

However, this condition is not sufficient to provide a logical qubit for sharing a qubit. If there exists a localized logical operator $\ell' \in L_A \cup L_B$ which satisfies $\{\ell', \ell\} = 0$ and $\{\ell', r\} = 0$, a measurement of ℓ' can have the same effect as a measurement of ℓr on encoded qubit information. Similarly, a measurement of $\ell' \in L_A \cup L_B$ with $\{\ell', \ell\} = 0$ and $[\ell', r] = 0$ have the same effect as a measurement of r on encoded qubit information and a measurement of $\ell' \in L_A \cup L_B$ with $[\ell', \ell] = 0$ and $\{\ell', r\} = 0$ have the same effect as a measurement of ℓ on encoded qubit information. Therefore, necessary conditions for a logical qubit described by ℓ and r to be shared between

A and B are:

$$\begin{aligned} [\ell, \ell'] &= 0 \\ [r, \ell'] &= 0 \\ \{\ell, r\} &= 0 \end{aligned} \tag{99}$$

for all the logical operators $\ell' \in L_A \cup L_B$, along with the condition in Eq.(98).

Now, let us show that there cannot exist a pair of logical operators ℓ and r which satisfies the above condition in Eq.(99). All the logical operators can be represented as a product of $2k$ logical operators α_i, α_i', β_i, β_i', ℓ_i and δ_i except the trivial contribution from $\mathcal{S}$. Then ℓ and r must be a product of ℓ_i in order for ℓ and r to commute with all the localized logical operators. This contradicts the fact that ℓ and r are logical operators in M_ϕ. Therefore, secret-sharing of quantum information is impossible the bi-partitioning of a stabilizer code. Our result can be summarized in the following theorem.

Theorem 2. *One cannot share quantum information secretly between two parties inside the ground state space of stabilizer codes.*

Let us interpret this observation with respect to the classification of logical qubits defined for the choices of logical operators in Eq.(95). In order for quantum information to be shared between A and B, there should exist a logical qubit described by a pair of non-local logical operators. However, there are only two types of logical qubits, local logical qubits and non-local logical qubits. There cannot exist a logical qubit which can be used for sharing quantum information. We note that this is a direct consequence of $m_\phi = m_{AB}$.

We have seen that non-local logical operators in non-local logical qubits are essential in secret-sharing of classical information. It might be natural to think that non-local logical operators are responsible for sharing information in a non-local way between two separated parties. However, a surprising consequence of the equation $m_\phi = m_{AB}$ in secret-sharing of quantum information is the fact that the properties of non-local logical operators are completely governed by localized logical operators in M_{AB}. In fact, a trivial, but very insightful interpretation of $m_\phi = m_{AB}$ in terms of secret-sharing of classical information is obtained:

Corollary 2. *The necessary and sufficient condition for a stabilizer code to be useful for classical information sharing is $m_{AB} \neq 0$.*

This corollary states the following: The existence of localized logical operators in M_{AB} automatically guarantees the existence of non-local logical operators and non-local logical qubits. Therefore, the existence of localized logical operators in M_{AB} can be used as a criteria to check whether a stabilizer code and a given bi-partition can give a resource for secret-charing of classical information.

C. Entanglement Entropy

As our third, and final example of application of our framework, we turn to *entanglement entropy.* Entanglement entropy plays important roles in condensed matter physics and quantum information science [2–7]. We may analyze the entanglement entropies of states inside the ground state space, and obtain some nice bounds using our framework.

For simplicity, we compute entanglement entropies for two ground states $|\psi_0\rangle$ and $|\psi_1\rangle$ which satisfy

$$\ell_i|\psi_0\rangle = |\psi_0\rangle \tag{100}$$
$$r_i|\psi_1\rangle = |\psi_1\rangle \tag{101}$$

for $\forall i$. Therefore, $|\psi_0\rangle$ is characterized by logical operators ℓ_i in M_{AB} while $|\psi_1\rangle$ is characterized by logical operators r_i in M_ϕ. Since logical operators in M_A and M_B do not affect the entanglement over A and B, without loss of generality, we can assume that

$$f_i|\psi_0\rangle = |\psi_0\rangle \tag{102}$$
$$f_i|\psi_1\rangle = |\psi_1\rangle \tag{103}$$

where f_i are $m_A' + m_B'$ independent logical operators in M_A and M_B. Then, we can represent $|\psi_0\rangle$ and $|\psi_1\rangle$ as

$$|\psi_0\rangle\langle\psi_0| = \frac{1}{2^N}\prod_{i=1}^{m_\phi}(I+\ell_i)\prod_{i=1}^{m_A'+m_B'}(I+f_i)\prod_{i=1}^{N-k}(I+S_i) \tag{104}$$

$$|\psi_1\rangle\langle\psi_1| = \frac{1}{2^N}\prod_{i=1}^{m_\phi}(I+r_i)\prod_{i=1}^{m_A'+m_B'}(I+f_i)\prod_{i=1}^{N-k}(I+S_i) \tag{105}$$

with a set of N independent commuting Pauli operators which consist of $N-k$ independent stabilizer generators S_i and logical operators. In order to compute the entanglement entropy of $|\psi_0\rangle$ and $|\psi_1\rangle$, we consider the following groups of operators

$$\mathcal{S}(0) = \langle \{\ell_i\}, \{f_i\}, \{S_i\}, \forall i\rangle \tag{106}$$
$$\mathcal{S}(1) = \langle \{r_i\}, \{f_i\}, \{S_i\}, \forall i\rangle \tag{107}$$

with the decompositions

$$\mathcal{S}(0) = \langle \mathcal{S}(0)_A, \mathcal{S}(0)_B, \mathcal{S}(0)_{AB}\rangle \tag{108}$$
$$\mathcal{S}(1) = \langle \mathcal{S}(1)_A, \mathcal{S}(1)_B, \mathcal{S}(1)_{AB}\rangle \tag{109}$$

in a way similar to the decomposition of $\mathcal{S}$ into $\mathcal{S} = \langle \mathcal{S}_A, \mathcal{S}_B, \mathcal{S}_{AB}\rangle$. Then, the entanglement entropies of $|\psi_0\rangle$ and $|\psi_1\rangle$ are represented as follows [25]:

$$E_A(|\psi_0\rangle) = \frac{1}{2}G(\mathcal{S}(0)_{AB}) \tag{110}$$

$$E_B(|\psi_1\rangle) = \frac{1}{2}G(\mathcal{S}(1)_{AB}). \tag{111}$$

Since $\mathcal{S}(0)$ includes ℓ_i, generators $\ell_i\ell'_i$ for $\mathcal{S}_{AB}$ can be decomposed into ℓ_i and ℓ'_i. Therefore, we have

$$E_A(|\psi_0) = \frac{1}{2}G(\mathcal{S}(0)_{AB}) \tag{112}$$

$$= \frac{1}{2}(G(\mathcal{S}_{AB}) - m_\phi). \tag{113}$$

On the other hand, since $\mathcal{S}(1)_{AB}$ includes r_i in addition to the generators for $\mathcal{S}_{AB}$, we have

$$E_A(|\psi_1\rangle) = \frac{1}{2}(G(\mathcal{S}_{AB}) + m_\phi). \tag{114}$$

Thus, one can easily see that the entanglement entropy of an arbitrary ground state $|\psi\rangle$ in the degenerate ground state space satisfies

$$\frac{1}{2}(G(\mathcal{S}_{AB}) - m_\phi) \leq E_A(|\psi\rangle) \leq \frac{1}{2}(G(\mathcal{S}_{AB}) + m_\phi). \tag{115}$$

Therefore, the entanglement entropy of degenerate ground states has an arbitrariness m_ϕ, which is the same as the number of non-local logical operators.

Now, let us apply this bound on the entanglement entropy to two problems we have addressed in this section. We begin with the relation between this bound and secret-sharing. Since the maximal entropy of degenerate ground states is $\frac{1}{2}(G(\mathcal{S}_{AB}) + m_\phi)$ while the minimal entropy is $\frac{1}{2}(G(\mathcal{S}_{AB}) - m_\phi)$, the entropy may vary up to m_ϕ inside the degenerate ground state space. Therefore, the ground state space of the Hamiltonian is capable of encoding m_ϕ bits of information. This is another interpretation of secret-sharing of classical information with logical qubits described by a pair of logical operators in M_ϕ and M_{AB}. Thus, the distribution of entanglement can be accessed in terms of the entanglement entropy and its relation with non-local logical operators in M_ϕ.

Next, let us consider the entanglement entropy of degenerate ground states of the Toric code. The entanglement entropy for a region without windings is particularly important in the discussion of the entanglement area law [18]. From the discussion on the Toric code in section V A, there is no logical operator inside a region without any winding. Then, one can tightly bound the entanglement entropy since $m_\phi = 0$:

$$E_A(|\psi\rangle) = \frac{1}{2}G(\mathcal{S}_{AB}). \tag{116}$$

Through a direct computation, we have

$$E_A = 2(n_x + n_y) - 2 \tag{117}$$

where A is an $n_x \times n_y$ square region of the lattice, with $2n_x \times n_y$ qubits. Since there are no logical operators inside A for $n_x, n_y < L$ where L is the length of the entire lattice, the entanglement entropy takes a single value. Noticing that $2(n_x + n_y)$ is the length of the perimeter of A, the term independent of length, the factor of two can be identified as being the topological entropy of the Toric code [6, 7].

Unlike secret-sharing, the entanglement entropy of the Toric code takes only a single value. This is a consequence of the fact that degenerate ground states of the Toric code are robust against local perturbations. In the language of logical operators, the robustness of ground states results from the fact that there are no logical operators inside local regions. Since all the logical operators are defined globally, any local perturbation cannot couple different ground states, so ground states are stable. In our discussion of the Toric code, we found that there is no logical operator inside regions without windings. This is the underlying reason behind the robustness of the Toric code and the resulting uniqueness of the entanglement entropy. Thus, the uniqueness of the entanglement entropy reflects the robustness of the Toric code, demonstrating the usefulness of analyzing the locality and non-locality of logical operators.

VI. SUMMARY AND OUTLOOK

In this paper, we have provided a systematic framework to study the non-local properties of logical operators and ground states in the stabilizer formalism, given a bi-partition. It is our hope that this will open the door to further unite the studies of correlations in condensed matter physics with the studies of entanglement in quantum information science. The framework can likely be broadened in many ways, five of which are discussed below.

Though we have studied only bi-partite systems in this paper, many-qubit systems can provide rich varieties of entanglement that may not be quantified by a bi-partition. Even for three qubit states, there are two inequivalent states, the W state and the GHZ state, which can be classified only by multi-partite entanglement [38]. Several multi-partite entanglement measures have been proposed to characterize many-qubit entangled states, and each of them successfully reveals different aspects of quantum entanglement [39, 40]. Our framework can be also generalized to the study of stabilizer codes with a multi-partition by extending the classification of logical operators based on their localities with respect to each of subsets constituting a multi-partition.

Multi-partite correlations are also important in the study of condensed matter physics, as seen in the investigation of spin systems. One interesting example in which multi-partition plays a crucial role is the study of topological order. In fact, the study of topological order lies at the interface between condensed matter physics and quantum information theory. However, currently, topological order is interpreted in somewhat different ways in condensed matter physics and quantum coding theory.

When topological order is discussed in quantum codes, properties of a single state are frequently studied, rather than properties of the whole system. When a quantum code possesses topological order, it is known that there usually exist large logical operators that are defined non-locally in a multi-partition. The existence of such large logical operator often results in a degenerate ground state with topological order [8, 41]. Topological properties of a state that corresponds to a codeword in a quantum code can be studied through its topological entropy [6, 7]. It has also been suggested that the preparation time of a corresponding state from a product state through local operations can be used to distinguish globally entangled states [1].

On the other hand, when topological order is discussed in condensed matter physics, the entire spectrum of the Hamiltonian is discussed, rather than a single state. Topological order often appears along with small ground state degeneracy [32]. This degeneracy can usually explained by the existence of symmetry operators that commute with the entire Hamiltonian. Topological orders can be analyzed by scattering of anyonic excitations [34].

Recently, the connection between these two different views of topological order in condensed matter physics and quantum coding theory has begun to be understood. The relation between anyonic excitations and topological entropy is discussed [6, 7]. The existence of degenerate ground states can be understood through logical operators and symmetry operators which commute with the entire Hamiltonian. Also, it is suggested that the entanglement spectrum of a ground state of the Hamiltonian may reveal the properties of the whole energy spectrum [5]. A study of stabilizer codes in a multi-partition will provide further insights on this connection since entanglement measures and logical operators can be computed efficiently with our framework.

Hand in hand with the global symmetries of topological order are local, physical symmetries of the system, such as translation symmetries and geometric symmetries. These symmetries can limit the properties of quantum codes, but also may simplify their analysis. Quantum codes with constrained global symmetries are particularly important and need to be studied since these symmetries may reflect the nature of real physical systems. These additional constraints translate into constraints on the structure of the overlapping operator groups in our framework. For example, translation symmetries of the system result in translation symmetries of the overlapping operator groups. Therefore, these additional constraints make it easier to implement our framework.

An example of the situation when realistic physical systems become desirable is the study of self-correcting quantum memories. A self-correcting memory is an ideal memory device which corrects errors by itself through thermal dissipations. There exists a theoretical proposal of a self-correcting quantum memory in four-dimensional space [42]. However, it was proven that there does not exist a self-correcting quantum memory within the stabilizer formalism [17]. The feasibility of three-dimensional self-correcting quantum memory is an important open question which remains unsolved at this moment. The memory's feasibility is related to whether random thermal noises can accidentally create one of logical operators which describe a logical qubit or not. The probability for thermal noise to destroy encoded information can be estimated by analyzing the subsets where each of logical operators is defined. Our framework can study whether a logical operator can be defined inside a subset or not. Therefore, it provides a useful tool to investigate this interesting open question [43].

Though we have limited our considerations to stabilizer codes, there are many quantum codes and spin systems which cannot be described through the stabilizer formalism. One novel class of quantum codes, now called *subsystem codes* [44, 45], replaces the commuting interaction terms of stabilizer codes (Eq.(1)) with interaction terms which may anti-commute with each other. These subsystem codes are interesting, particularly because they may potentially provide a means to realize a self-correcting quantum memory in three-dimensions, for example, using the Bacon-Shor subsystem code [16]. The two-dimensional Bacon-Shor subsystem code also possess several promising features, such as a lower fault-tolerance threshold [46] compared with certain codes utilizing similar space and time resources. And in the condensed matter physics community, the Bacon-Shor subsystem code is known as the quantum compass model [47]. Physical properties arising from the quantum compass model have been studied numerically [48–50], and are interesting, for example, in the notable role they play in explaining the effects of the orbital degree of freedom of atomic electrons on the properties of transition-metal oxides [51].

Despite this promising progress with subsystem codes, the underlying mechanism for the general capabilities of and physical properties arising from stabilizer codes is not fully understood. In subsystem codes, the logical operators can be computed by analyzing all the elements in the Hamiltonian, but this is generally computationally difficult. The canonical representations of our framework could allow efficient computation of logical operators in subsystem codes by allowing the extraction of commuting operators from the elements in the Hamiltonian. Also, the analysis of logical operators through our framework may be extended to subsystem codes and give insights on the underlying mechanism of subsystem codes.

These subsystem codes also have interesting interpretations in terms of condensed matter physics. The Hamiltonian corresponding to a subsystem code may contain interaction terms that anti-commute with each other. Unlike a Hamiltonian in the stabilizer formalism, a ground state of such a Hamiltonian cannot be obtained by separately minimizing each interaction term. This is an analogue of frustration in condensed matter physics. A frustrated Hamiltonian with anti-commuting interaction terms is not solvable in general. However, by extracting the operator elements that commute with all the interaction terms in the Hamiltonian, one can obtain some insight about the physical properties appearing in the Hamiltonian even without solving it. These symmetry operators can be computed in a way similar to the method employed in our framework for the computation of logical operators through canonical representations.

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