Generic local Hamiltonians are gapless

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We prove that quantum local Hamiltonians, whose local terms are independent random matrices or certain random projectors, are gapless. The results apply to local Hamiltonians with translational invariance in a disorder (i.e., probabilistic) sense with some assumptions on the local distributions. Local terms may be drawn from standard random matrix ensembles. The Hamiltonian can be on a lattice in any spatial dimension or on a graph with a bounded maximum vertex degree. As a corollary there exist finite size partitions with respect to which the ground state is arbitrarily close to a product state. In addition to the lack of an energy gap, we prove that the ground state is degenerate when the local eigenvalue distribution is discrete. This work excludes the important class of truly translationally invariant Hamiltonians where the local terms are all equal.

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Random matrices have brought a number of fascinating phenomena into the world of physics from two-dimensional quantum gravity to spin glass phases to the localization problem [1, 2]. Random projectors as local interactions are of a particular interest in the quantum information community [3, 4], where they serve as random clauses for the satisfiability problem [5].

The gap of a quantum Hamiltonian is the positive difference of the two smallest eigenvalues. Ignoring degeneracies, finite systems are always gapped because they have a finite number of eigenvalues. Eigenvalue degeneracy is generically of low measure and one may be lead to intuit that closing of the gap in the limit of large system's size is too special and requires fine-tuning of the parameters (say to a critical point). The gap has important implications on the physics of quantum many-body systems; vanishing of the gap is necessary for quantum phase transitions [6]. The multiplicative growth of the dimension and non-commutativity make the analysis of the spectral properties of quantum systems difficult. In particular, the spectral gap problem is undecidable [7].

Since full rank orthogonal projections are trivial, one defines a generic projector to be rank deficient but with generic (e.g., Haar distributed) eigenvectors [8]. In spin chains, the ground state degeneracy of generic local projectors is well understood [8]. Their gap has been fully classified for frustration free spin-1/2 chains with strict translational invariance [9]. However, rigorous results are rare beyond these and especially difficult to obtain in higher dimensions.

In this paper we prove, under certain assumptions, that generic local Hamiltonians are almost surely gapless. The Hamiltonian can be on a lattice in any spatial dimension or a graph with bounded vertex degrees. Theorem 1 and Corollary 2 prove vanishingly small gap for continuous and discrete local eigenvalue distributions respectively. In addition, Corollary 1 proves that the ground state is degenerate with a high probability if the local eigenvalue distribution is discrete. Theorem 2, proves a vanishing gap for local terms that are random projectors of either certain fixed ranks or randomly varying ranks with Haar distributed eigenvectors.

Let the local Hamiltonian be the operator acting on the Hilbert space $(\mathbb{C}^d)^{\otimes n}$, where n is the total number of particles and d is the dimension of the local Hilbert space (e.g., number of spin states). Formally,

$$H(N) = \sum_{\langle i,j \rangle} \mathbb{I} \otimes H_{i,j} \tag{1}$$

where N is total number of summands, $\langle i,j \rangle$ denotes nearest neighbors, $H_{i,j}$'s are $d^2 \times d^2$ matrices and $\mathbb{I}$ is an identity matrix of size $(\mathbb{C}^d)^{\otimes(n-2)}$ on all particles other than i and j. Unless specified otherwise, below $\mathbb{I}$ indicates identity on all sites excluding the interaction. To meaningfully take a limit of arbitrary large N, we assume that $H_{i,j}$'s have bounded norms and that the number of nearest neighbors of any given $H_{i,j}$ is a constant independent of N. We work with finite dimensional Hilbert spaces and consider N to be arbitrary large. Below for simplicity we denote $H(N)$ simply by H and say matrices A and B are ϵ close if $\|A - B\| \le \epsilon$.

Definition. Let $\lambda_0(H)$ and $\lambda_1(H)$ be the smallest and second smallest eigenvalues of H respectively. We say H is gapless if for any $\epsilon > 0$ there exists $N > 0$ such that $\lambda_1(H) - \lambda_0(H) \le \epsilon$.

Comment: The gap here refers to the positive difference of the smallest two distinct energies irrespective of possible finite size degeneracies of the ground state.

Below we assume that the joint distribution of the eigenvalues of a local term obeys a niceness property that most standard finite random matrix ensembles that we know of possess. Mathematically,

Assumption 1. *For all $\epsilon > 0$, there exists an open interval of size ϵ on which the joint eigenvalue distribution for all the eigenvalues being inside the interval is supported.*

Let us illustrate Assumption 1 by constructing a local term from the gaussian unitary ensemble (GUE). Let A be a $d^2 \times d^2$ matrix whose entries are standard complex normals. Then $(A + A^\dagger)/2$ is an instance of GUE whose eigenvalues we claim can get arbitrary close. In particular, A has a positive probability of being ϵ close to a multiple of the identity.

Theorem 1. *H is almost surely gapless if $H_{i,j}$'s are independent and each $H_{i,j}$ has a continuous joint distribution of eigenvalues that obeys Assumption 1.*

Proof. Let $H_{p,q}$ be a fixed local term and rewrite Eq. 1 as

$$H = \mathbb{I} \otimes H_{p,q} + \sum_{|\langle i,j \rangle|=1} \mathbb{I} \otimes H_{i,j} + \sum_{|\langle i,j \rangle| \geq 2} \mathbb{I} \otimes H_{i,j} \qquad (2)$$

where the first sum includes all the local terms that overlap with $H_{p,q}$ at a site (i.e., have distant 1) and the second sum all the terms with no overlap with $H_{p,q}$ (at least a distant 2). Let the number of overlapping terms be z. For example, on a square lattice $z = 4D - 2$ where D is the spatial dimension.

Because of Assumption 1 there is a positive probability that there exists $H_{p,q}$ whose two smallest eigenvalues are ϵ apart. In addition, by independence of the local terms there is a positive probability that every neighbor of $H_{p,q}$ is ϵ close to a multiple of the identity. At $\epsilon = 0$, we define H_0 to be

$$H_0 \equiv \mathbb{I} \otimes H^0_{p,q} + \sum_{|\langle i,j \rangle|=1} \beta_{i,j} \mathbb{I} \otimes \mathbb{I}_{i,j} + \sum_{|\langle i,j \rangle| \geq 2} \mathbb{I} \otimes H_{i,j}$$

where the superscript zero on the first term means that H^0_{pq} has a degenerate smallest eigenvalue denoted by λ_0.

Let λ_E be the smallest eigenvalue of $\sum_{|\langle i,j \rangle| \geq 2} \mathbb{I} \otimes H_{i,j}$. By assumption $H_{pq} = H^0_{pq} + \delta H_{pq}$, where $||\delta H_{pq}|| \leq \epsilon$ and the summands of distant 1 terms are $H_{i,j} = \beta_{i,j} \mathbb{I}_{i,j} + \delta H_{ij}$, where $||\delta H_{i,j}|| \leq \epsilon$. Since $||\mathbb{I} \otimes \delta H_{ij}|| = ||\delta H_{i,j}||$, by Weyl's inequalities [10, Chapter 3] the two smallest eigenvalues of H, denoted by $\lambda^{\epsilon,k}_{min}$ with $k \in \{1,2\}$, obey

$$\lambda_E + \beta + \lambda_0 - B \leq \lambda^{\epsilon,k}_{min} \leq \lambda_E + \beta + \lambda_0 + B, \qquad (3)$$

where $B = ||\delta H_{pq} + \sum_{|\langle i,j \rangle|=1} \delta H_{ij}|| \leq \epsilon(z+1)$, and $\beta \equiv \sum_{|\langle i,j \rangle|=1} \beta_{i,j}$. Since for any fixed ϵ the configuration just described has a positive probability, we can find a sufficiently large system's size N in which there is a site whose two smallest eigenvalues are ϵ apart and whose neighbors are ϵ close to a multiple of the identity. Since $z+1$ is finite and ϵ can be arbitrary small, we conclude that H is gapless. $\square$

Proof for $k-$local Hamiltonians with $k > 2$ is similar.

Much more restrictive are local terms with *discrete* eigenvalue distributions, in which any two local eigenvalues are either equal or a constant apart; i.e., cannot be ϵ close. Also the neighbors are either exactly a multiple of the identity or a constant distant apart. Corollary 2 proves that such Hamiltonians are also gapless. Now we prove that their ground states are degenerate.

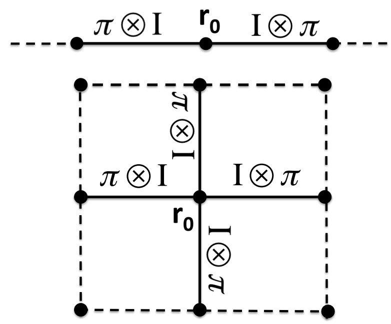

Figure 1: Examples of H_0 (Eq. 6) on a line and a square lattice, where for simplicity we dropped the subscript $\mathbf{r_0}$ on $\mathbb{I}$

Corollary 1. *If the local eigenvalues have a discrete distribution satisfying Assumption 1, then the ground state is almost surely degenerate and can be represented as a product state.*

Proof. If the eigenvalue distribution is discrete, in the proof of the theorem above, we would have a finite probability of existence of a $H_{p,q}$ whose smallest eigenvalue is *exactly* $k-$fold degenerate and its nearest neighbors are exactly proportional to the identity matrix. Let the ground state of all the terms in the Hamiltonian excluding $H_{p,q}$ be $|\theta_E\rangle$, and let us denote the eigenvectors of the smallest eigenvalue of $H_{p,q}$ by $|\psi^{(\ell)}_{pq}\rangle$ where $\ell \in \{1, \ldots, k\}$. Then the ground states of H are product states $|\theta_E\rangle \otimes |\psi^{(\ell)}_{pq}\rangle$ and are $k-$fold degenerate. $\square$

If the discrete distribution has a finite number of atoms in its distribution, then the above corollary suggests a *large* degeneracy of the ground state. We now turn to the gap of random projectors which does not obey Assumption 1.

Theorem 2. *H is almost surely gapless if the local terms have independent and Haar distributed eigenvectors, and $H_{i,j}$ in Eq. 1 are random projectors whose individual ranks are either:*
1. Fixed and at most $d(d-1)$.
2. Vary randomly among the terms in the Hamiltonian.

Proof. Let an arbitrary vertex on the lattice or the graph be at the fixed position $\mathbf{r_0}$ and rewrite Eq. 1 as

$$H = \sum_{|\langle i,\mathbf{r_0} \rangle|=1} \mathbb{I} \otimes H_{i,\mathbf{r_0}} + \sum_{|\langle i,j \rangle| \geq 2} \mathbb{I} \otimes H_{i,j} \qquad (4)$$

where the first sum includes all the terms that act on $\mathbf{r_0}$ and its neighboring vertices (distant 1), and the second sum includes the rest of the interactions.

To prove the first statement take $H_{i,j}$ in Eq. 1 to be random projectors with the rank $d(d-1)$. Let $\pi = \mathrm{diag}(0, 1, \ldots, 1)$ be the diagonal projector of rank $d-1$ and $\mathbb{I}_{\mathbf{r_0}}$ an $d \times d$ identity matrix acting on $\mathbf{r_0}$. Since eigenvectors are Haar, there is a positive probability density that all the terms in the first sum in Eq. 4 take the form $\pi \otimes \mathbb{I}_{\mathbf{r_0}}$. Examples of this are illustrated in Fig. 1.

By independence of local terms and Haar-distribution local eigenvectors, for sufficiently large N almost surely there will be a site $\mathbf{r_0}$, whose neighbors are ϵ close to projectors of the form just described. By assumption $\epsilon = 0$ corresponds to neighbors having the exact form.

We write

$$\sum_{|\langle i,\mathbf{r_0}\rangle|=1} \mathbb{I}\otimes H_{i,\mathbf{r_0}} = \sum_{|\langle i,\mathbf{r_0}\rangle|=1} \mathbb{I}\otimes (H^0_{i,\mathbf{r_0}} + \delta H_{i,\mathbf{r_0}}), \quad (5)$$

where the superscript zero denotes $\epsilon = 0$, and by assumption $H^0_{i,\mathbf{r_0}} = \pi_i \otimes \mathbb{I}_{\mathbf{r_0}}$ and $||\mathbb{I}\otimes \delta H_{i,\mathbf{r_0}}|| \le \epsilon$.

We define the $\epsilon = 0$ Hamiltonian by $H_0 \equiv \sum_{|\langle i,\mathbf{r_0}\rangle|=1} \mathbb{I}\otimes H^0_{i,\mathbf{r_0}} + \sum_{|\langle i,j\rangle|\ge 2} \mathbb{I}\otimes H_{i,j}$, which is equal to

$$\begin{aligned} H_0 \;&=\; \mathbb{I}_{\mathbf{r_0}} \otimes \{ \sum_{|\langle i,\mathbf{r_0}\rangle|=1} \pi_i \otimes \mathbb{I} + \sum_{|\langle i,j\rangle|\ge 2} \mathbb{I}\otimes H_{i,j}\} \\ &\equiv\; \mathbb{I}_{\mathbf{r_0}} \otimes H_E, \end{aligned} \quad (6)$$

where H_E is the Hamiltonian acting on all sites other than $\mathbf{r_0}$ (inside braces) and $\mathbb{I}$ in the distant-2 terms excludes i, j and $\mathbf{r_0}$. By eigenvalue decomposition we have $H_E = \Theta_E^{-1}\Lambda_E\Theta_E$, where Λ_E is the diagonal matrix of eigenvalues and Θ_E is the unitary matrix of eigenvectors. We arrive at the desired result where $H_0 = \mathbb{I}_{r_0}\otimes\Theta_E^{-1}\Lambda_E\Theta_E$; the eigenvalues of this Hamiltonian all have a $d-$fold algebraic multiplicity.

Let $|\theta_E\rangle$ denote the ground state of H_E with the energy $\lambda_E \ge 0$, the ground states of H_0 can be taken to be

$$|\Psi_k\rangle = |e_k\rangle \otimes |\theta_E\rangle, \quad (7)$$

where $|e_k\rangle$ are the standard vectors supported at $\mathbf{r_0}$ and the corresponding eigenvalues of H_0 are $\lambda^{0,(k)}_{min} = \lambda_E$, where $k \in \{1, 2, ..., d\}$. For $\epsilon > 0$ what used to be degenerate ground state energies of H_0, generically split, i.e., $\lambda^{\epsilon,(k)}_{min} \ne \lambda^{\epsilon,(\ell)}_{min}$, where $k, \ell \in \{1, \dots, d\}$ and $k \ne \ell$. Since $||\sum_{|\langle i,\mathbf{r_0}\rangle|=1} \mathbb{I}\otimes\delta H_{i,\mathbf{r_0}}|| \le z||\delta H_{max}||$, Weyl inequalities [10, Chapter 3] imply that for all k

$$-z\epsilon \le \lambda^{\epsilon,(k)}_{min} - \lambda^{(0)}_E \;\le\; z\epsilon \quad (8)$$

where, as before, $||\mathbb{I}\otimes\delta H_{i,\mathbf{r_0}}|| = ||\delta H_{i,\mathbf{r_0}}||$, z is the number of nearest neighbor terms to $\mathbf{r_0}$ (e.g., $z = 2D$ on a square lattice in D spatial dimensions) and δH_{max} is the overlapping term with the maximum error norm.

Since the eigenvectors are Haar and the dimension of the local kernels is d, for any fixed ϵ we can find a sufficiently large N in which there is a site whose all neighbors are ϵ close to $\mathbb{I}_{\mathbf{r_0}}\otimes\pi$. Since z is finite and ϵ can be arbitrary small, by Eq. 8 the d smallest eigenvalues of H_0 will be $\mathcal{O}(\epsilon)$ apart giving an arbitrary small gap.

Lastly if we allow variable ranks of H_{ij}, then the same construction as above guarantees having a site on which the Hamiltonian acts trivially and the above argument guarantees ϵ splitting of the eigenvalues and lack of an energy gap in the limit. □

The proof generalizes for local projectors of fixed ranks $d(d-h)$, with $d > h \ge 1$ on lattices or graphs whose degrees are constants independent of N. All that needs to be changed in the proof above is to let $2 \le \dim(\mathrm{Ker}(\pi)) < d$. We believe this proof technique can be generalized to projectors with higher ranks, where configurations other than the ones constructed here (e.g., Fig. 1) would be needed.

In Corollary 1 and under Assumption 1, we proved that the ground state is degenerate. Now we show that the gap closes without using Assumption 1; i.e., irrespective of possible ground state degeneracy.

Corollary 2. *The Hamiltonian is almost surely gapless if the local eigenvalues have a discrete eigenvalue distribution and eigenvectors are Haar distributed.*

Proof. This follows a construction similar to that given in Theorem 2. Suppose $\lambda^1_{pq}, \lambda^2_{pq}, \dots, \lambda^d_{pq}$ are d discrete eigenvalues of the distribution that are not all equal. Then with an arbitrarily high probability we can find a site, $\mathbf{r_o}$, whose neighbors are all ϵ close to having the form $\mathbb{I}_{\mathbf{r_o}}\otimes\mathrm{diag}(\lambda^1_{pq}, \lambda^2_{pq}, \dots, \lambda^d_{pq})$. In Theorem 2 replacing π with $\mathrm{diag}(\lambda^1_{pq}, \lambda^2_{pq}, \dots, \lambda^d_{pq})$ and following the same argument that lead to Eq. 8, we arrive at the bound on the gap being $|\lambda^{\epsilon,(k)}_{min} - \lambda^{(0)}_E| \le 2z\epsilon$, where ϵ can be arbitrary small. □

Remark. In the proofs above we used Weyl's theorem. Had we used first order perturbation theory, for example in Eq. 8, instead of $||\sum_{|\langle i,\mathbf{r_0}\rangle|=1}\mathbb{I}\otimes\delta H_{i\mathbf{r_0}}||$ we would have $\lambda^{\epsilon,k}_{min} = \lambda^{(0)}_E + \sum_{|\langle ij\rangle|=1}\langle\theta_E\otimes e_k|\delta H_{i\mathbf{r_0}}|\theta_E\otimes e_k\rangle + O(\epsilon^2)$. *However, the general issue with using standard perturbation theory to prove lack of an energy gap is that one would have to prove that a constant gap does not open up in all higher order terms*. This is not obviously so, since the k^{th} correction multiplying ϵ^k involves a sum of combinatorially many terms and may actually become comparable to ϵ^{-k} in the magnitude. This in turn causes the series to diverge beyond some order. This issue is often met with in (particle) physics.

Although the general spectral gap problem is undecidable [7], we have proved that under certain genericity assumptions local Hamiltonians are gapless in any dimension. In addition, we showed that there are finite partitions with respect to which the ground state is arbitrary close to a product state (Corollary 1 and Eq. 7) and proved exact degeneracy when the local interactions have a discrete distribution. This work applies to Hamiltonians whose local terms are drawn from random matrix ensembles or are random projectors. The actual scaling of the gap with the system's size, depends on the actual distribution of the local eigenpairs and the lattice geometry. It would be interesting to classify the gap when the local terms are equal. This has only been achieved for spin-1/2 frustration free spin chains [9].

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