

# On the Spectrum of Gauge Code Hamiltonians

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This is an attempt to apply some ideas from spectral graph theory to the study of Hamiltonians built from (possibly non-commuting) Pauli algebra terms.

## **Motivating examples of gapless and gapped models.**

The  $XY$  model:

$$\mathcal{H} = \sum_i X_i X_{i+1} + Z_i Z_{i+1}.$$

Transverse field Ising model:

$$\mathcal{H} = \sum_i X_i X_{i+1} + Z_i.$$

In both these cases the existence of large weight stabilizer generators causes the gapless behaviour.

As another example, take

$$\mathcal{H} = \sum_i X_i + Z_i.$$

In this case there are no stabilizers and indeed the Hamiltonian is gapped.

We need to be a bit careful what we mean by the weight of a stabilizer generator. In this case, we define the weight of a stabilizer to mean the number of terms in the Hamiltonian required to build (by multiplication) a stabilizer. This rules out trivial cases such as  $\mathcal{H} = \prod_{j=1}^{j=n} X_j$ , where the Hamiltonian is one big stabilizer but is obviously gapped.

## **I. REPRESENTATIONS OF GAUGE CODES**

### **A. Cayley graphs**

The *Cayley graph* of a group  $G$  and generating set  $S$  is denoted  $\text{Cayley}(G, S)$ . It has nodes  $G$  and edges  $\{(g, sg) : g \in G, s \in S\}$ . We will always assume  $S$  is closed under group inverse, and so we can consider the Cayley graph as undirected.

The adjacency matrix of a graph with  $N$  nodes is the  $N$  by  $N$  matrix  $A$  with non-zero entries  $A_{ij} = 1$  corresponding to edges  $(i, j)$  of the graph. It follows that the adjacency matrix  $A$  of  $\text{Cayley}(G, S)$  is the linear operator  $A : \mathbb{C}[G] \rightarrow \mathbb{C}[G]$  given by

$$A = \sum_{g \in S} \rho_{\text{reg}}(g)$$

where  $\rho_{\text{reg}}$  is the left regular representation of  $G$  on  $\mathbb{C}[G]$ .

Using representation theory we have that  $\rho_{\text{reg}} : G \rightarrow \mathbb{C}[G]$  decomposes as the direct sum of irreducible representations  $\rho_k : G \rightarrow \text{GL}(V_k)$  and so

$$A = \sum_{g \in S} \bigoplus_k \rho_k(g)$$

block diagonalizes  $A$ , with each  $\rho_k$  possibly appearing multiple times in the direct sum over  $k$ . See [7] and [3] for more details.

## B. Gauge codes

The Pauli group  $\mathcal{P}_1$  is normally defined as a set of matrices closed under matrix multiplication, but we can define it abstractly as the group generated by the (abstract) elements  $\{m, X, Z\}$  with relations as follows:

$$m^2 = I, \quad X^2 = I, \quad Z^2 = I, \quad mXmX = I, \quad mZmZ = I, \quad \text{and} \quad mZXZX = I,$$

where  $I$  is the group identity. Actually,  $m$  is generated by  $X$  and  $Z$ , so it is not necessary to include  $m$  in the generating set, but here it simplifies the relations.

To define the  $n$ -qubit Pauli group  $\mathcal{P}_n$ , we use the  $2n + 1$  element generating set  $\{m, X_1, \dots, X_n, Z_1, \dots, Z_n\}$  with relation  $m^2 = I$  as before, and

$$X_i^2 = I, \quad Z_i^2 = I, \quad mX_imX_i = I, \quad mZ_imZ_i = I, \quad mZ_iX_iZ_iX_i = I, \quad \text{for } i = 1, \dots, n, \\ Z_iX_jZ_iX_j = I, \quad \text{for } i, j = 1, \dots, n, \quad i \neq j.$$

Note that  $m$  commutes with all elements of  $\mathcal{P}_n$  and squares to the identity, so we will denote this element as  $-1$ . Similarly,  $\pm 1$  is thought of as the set  $\{m, I\}$ , and  $-X$  is  $mX$ , etc.

The subgroup of  $\mathcal{P}_n$  generated by the elements  $\{X_1, \dots, X_n\}$  is denoted  $\mathcal{P}_n^X$ . These are the  $X$ -type elements. Similarly,  $\{Z_1, \dots, Z_n\}$  generates the subgroup of  $Z$ -type elements  $\mathcal{P}_n^Z$ .

Every element  $g \in \mathcal{P}_n$  can be written uniquely as a product  $g = \pm g_X g_Z$ , where  $g_X$  is an  $X$ -type operator and  $g_Z$  is a  $Z$ -type operator. This gives the size of the group as:

$$|\mathcal{P}_n| = 2^{2n+1}.$$

We now define the Pauli *representation* of the Pauli group as a group homomorphism:

$$\rho_{\text{pauli}} : \mathcal{P}_n \rightarrow \text{GL}(V)$$

where  $V$  is the  $2^n$  dimensional state space of  $n$  qubits. On the independant generators  $\{X_1, \dots, X_n, Z_1, \dots, Z_n\}$ ,  $\rho_{\text{pauli}}$  is defined as the following tensor product of  $2 \times 2$  matrices:

$$\rho_{\text{pauli}}(X_i) := \bigotimes_{j=1}^n \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } j \neq i, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{for } j = i \end{cases}, \quad \rho_{\text{pauli}}(Z_i) := \bigotimes_{j=1}^n \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } j \neq i, \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{for } j = i \end{cases}.$$

Normally the image of  $\rho_{\text{pauli}}$  is thought of as the Pauli group itself, and we are indeed free to think that way because  $\rho_{\text{pauli}}$  is a group isomorphism. It turns out that  $\rho_{\text{pauli}}$  is an irreducible representation (*irrep*) of  $\mathcal{P}_n$  and the only other irreps of  $\mathcal{P}_n$  are the 1-dimensional irreps  $\rho : \mathcal{P}_n \rightarrow \mathbb{C}$  defined on the independant generators as:

$$\rho(X_i) = \pm 1, \quad \rho(Z_i) = \pm 1.$$

So we have  $2^{2n}$  many 1-dimensional irreps, and a single  $2^n$ -dimensional irrep. Summing the squares of the dimensions shows that we have a complete set of irreps of  $\mathcal{P}_n$ .

We now define a *gauge* subgroup  $G$  of  $\mathcal{P}_n$  by choosing a set of generators  $S \subset \mathcal{P}_n$ ,

$$G := \langle S \rangle.$$

We will assume  $G$  is not abelian, which is equivalent to the condition that  $-I \in G$ . We also restrict  $S$  to only contain Hermitian operators, which is equivalent to requiring that  $g^2 = I$  for all  $g \in S$ . Now let  $H$  be the largest subgroup of  $G$  not containing  $-I$ .  $H$  is then an abelian subgroup, also known as the *stabilizer* subgroup.  $G$  decomposes as a direct product:

$$G = H \times R,$$

where  $R \cong P_r$  for some  $1 \leq r \leq n$ , and  $H \cong \mathbb{Z}_2^m$  for  $0 \leq m < n$ . Therefore,  $|G| = |H||R| = 2^{m+2r+1}$ . We call  $R$  the *reduced* gauge group. We consider both  $H$  and  $R$  to be subgroups of  $G$ . Let  $\phi : P_r \rightarrow R$  be a group isomorphism, then  $R_0 := \{\phi(X_i), \phi(Z_i)\}_{i=1,\dots,r}$  is a set of independant generators of  $R$ . We also let  $H_0$  be a set of  $m$  independant generators of  $H$ .

The 1-dimensional irreps  $\rho : G \rightarrow \mathbb{C}$ , are now defined by specifying the action of  $\rho$  on the independant generators:

$$\rho(h) = \pm 1 \text{ for } h \in H_0, \quad \rho(\phi(X_i)) = \pm 1, \quad \rho(\phi(Z_i)) = \pm 1.$$

This gives all  $2^{m+2r}$  of the 1-dimensional irreps. Finally, there are  $2^m$  many  $2^r$ -dimensional irreps given by:

$$\rho(h) = \pm I^{\otimes 2^r} \text{ for } h \in H_0, \quad \rho(\phi(X_i)) = X_i, \quad \rho(\phi(Z_i)) = Z_i.$$

The Hamiltonian of interest is normally defined as the negative sum of terms from  $S$ , but here we will reverse the sign:

$$\mathcal{H} := \sum_{g \in S} \rho_{\text{pauli}}(g).$$

The goal is to decompose  $\mathcal{H}$  into blocks as

$$\mathcal{H} = \bigoplus_{\text{irrep } \rho} \sum_{g \in S} \rho(g),$$

and we will notate each block as  $\mathcal{H}_\rho := \sum_{g \in S} \rho(g)$  for each irrep  $\rho$  appearing in  $\mathcal{H}$ .

The form of  $\mathcal{H}$  is seen to be very similar to the adjacency matrix of Cayley( $G, S$ ) but instead of the regular representation we are using the Pauli representation. We use the following map to relate these two representations:

**Theorem.** Define a linear map  $f : \mathbb{C}[G] \rightarrow \mathbb{C}[G_X]$  as follows. Any element  $g \in \mathcal{P}_n$  can be written as  $g = \pm g_x g_z$  and we set  $f(g) = f(\pm g_x g_z) := \pm g_x$ . Then,

$$fA = \mathcal{H}f.$$

**Proof.** (*Proof??*)... □

It is easy to see that any eigenvector  $v$  of  $A$  with eigenvalue  $\lambda$  is either in the kernel of  $f$ , or otherwise  $fv$  is an eigenvector of  $\mathcal{H}$  with eigenvalue  $\lambda$ . Furthermore,  $f$  is full-rank (*Proof??*), and so *all* the eigenvectors of  $\mathcal{H}$  are of the form  $fv$  for some eigenvector  $v$  of  $A$ .

We write the (distinct) eigenvalues of  $\mathcal{H}$  in decreasing order:

$$\lambda_1 > \lambda_2 > \dots$$

Of particular interest is the gap between the first and second eigenvalues,  $\epsilon := \lambda_1 - \lambda_2$ .

There is a well understood theory of expansion in Cayley graphs that shows how the structure of the group  $G$  leads to gapped behaviour of  $A$ . Unfortunately, the top eigenvector of  $A$  is in the kernel of  $f$  and so these results do not help us show gapped behaviour of  $\mathcal{H}$ .

The above commutation relation for  $f$  is the definition of an *intertwining* map, and it is a general result that such maps either preserve an irreducible representation or send them to zero.

**Theorem.** The (images of) all the one-dimensional irreps are contained in the kernel of  $f$ . All the other irreps are preserved.

**Proof.** (*Proof??*)... □

In the sequel we will make the identification between  $g$  and  $\rho_{\text{pauli}}(g)$ . So terms such as  $Z$  and  $X$  are understood to be the corresponding Pauli linear operators.

### C. Example: 2D compass model

Here we consider the two dimensional compass model. We coordinatize the qubits on a square lattice of  $l \times l$  sites,  $(i, j)$  for  $1 \leq i, j \leq l$ . This gives  $n = l^2$ . For the single qubit Pauli operators acting on site  $(i, j)$  we coordinatize with subscripts  $ij$ , with  $i$  and  $j$  understood modulo  $l$ . The generators of the gauge group are

$$S = \{X_{ij}X_{i,j+1}, Z_{ij}Z_{i+1,j} \text{ for } 1 \leq i, j \leq l\}.$$

We write generators of the reduced gauge group in anti-commuting pairs:

$$R_0 = \{X_{i1}X_{ij}, Z_{1j}Z_{ij} \text{ for } 2 \leq i, j \leq l\}.$$

This makes it clear the isomorphism  $\phi : R_0 \rightarrow \mathcal{P}_r$  to use, and we again use pairs  $i, j$  to coordinatize  $\mathcal{P}_r$ :

$$\phi(X_{i1}X_{ij}) = X_{i-1,j-1}, \quad \phi(Z_{1j}Z_{ij}) = Z_{i-1,j-1}, \quad \text{for } 2 \leq i, j \leq l.$$

The generators for the stabilizers are

$$H_0 = \left\{ \prod_{i=1}^l X_{ij}X_{i,j+1}, \prod_{i=1}^l Z_{ji}Z_{j+1,i} \text{ for } 1 \leq j \leq l-1 \right\}.$$

The logical operators are generated by  $L_0 = \{ \prod_i X_{i1}, \prod_j Z_{1j} \}$ . These sets have cardinalities:

$$|S| = 2l^2, \quad |R_0| = 2(l-1)^2, \quad |H_0| = 2(l-1).$$

And we note that  $\frac{1}{2}|L_0| + |H_0| + \frac{1}{2}|R_0| = n$ . Now we write down the values of the irreps on the gauge operators. Note the transposition symmetry between the  $X$  and  $Z$ -type operators:

$$\begin{aligned}
\rho(X_{i1}X_{i2}) &= X_{i-1,1} & \rho(Z_{1i}Z_{2i}) &= Z_{1,i-1} & \text{for } 2 \leq i \leq l \\
\rho(X_{ij}X_{i,j+1}) &= X_{i-1,j-1}X_{i-1,j} & \rho(Z_{ji}Z_{j,i+1}) &= Z_{j-1,i-1}Z_{j,i-1} & \text{for } 2 \leq i, j \leq l \\
\rho(X_{1j}X_{1,j+1}) &= \pm \prod_{i=1}^l X_{i,j-1}X_{ij} & \rho(Z_{j1}Z_{j+1,1}) &= \pm \prod_{i=1}^l Z_{j-1,i}Z_{ji} & \text{for } 2 \leq j < l \\
\rho(X_{11}X_{12}) &= \pm \prod_{i=1}^l X_{i1} & \rho(Z_{11}Z_{21}) &= \pm \prod_{i=1}^l Z_{1i}.
\end{aligned}$$

We sum all these terms to find the form of the hamiltonian in each block:

$$\mathcal{H}_\rho = \sum_{g \in S} \rho(g) = \sum_{1 \leq i, j < l} \rho(X_{ij}X_{i,j+1}) + \rho(Z_{ij}Z_{i+1,j}).$$

We note that in [2], they perform a (ad-hoc?) transformation of the compass model which results in a similar  $(l-1) \times (l-1)$  lattice of spins, but they have more terms in their Hamiltonian. It is not clear if these modifications are essential to their analysis.

#### D. Example: Kitaev honeycomb model

The Kitaev honeycomb model is build from spins on the sites of a hexagonal lattice. The lattice of linear size  $l$  has  $n = 2l^2$  sites which we coordinatize using integer triples  $i, j, k$  with  $1 \leq j, k \leq l$  and  $k = 1, 2$ . We use periodic boundary conditions so  $i, j$  are to be taken modulo  $l$ . The edges of the lattice are in one-to-one correspondence with the generators  $S$ :

$$S = \{X_{ij1}X_{ij2}, Z_{ij2}Z_{i+1,j1}, Y_{ij1}Y_{i-1,j+1,2} \text{ for } 1 \leq i, j \leq l\}.$$

Stabilizers are generated from closed strings of gauge operators. This gives independant stabilizer generators from each hexagon, less one, as well as two homologically non-trivial loops. The number of hexagons is  $\frac{1}{2}n$  and so we find  $|H_0| = \frac{1}{2}n + 1$ . There are no logical operators, so we must have  $|R_0| = n - 2$ .

In [8] they introduce a set of mutually anti-commuting string operators  $K_0$ . With periodic boundary conditions  $K_0$  forms an independant generating set of  $R$  of size  $n - 2$ . To construct an isomorphism  $\phi : R \rightarrow \mathcal{P}_r$  we map elements of  $K_0$  to the following independant generating set of  $\mathcal{P}_r$ :

$$\{c_{2j} := Z_1 \dots Z_{j-1} X_j, c_{2j+1} := Z_1 \dots Z_{j-1} Y_j \text{ for } 1 \leq j \leq r\}.$$

*XXX Prove this is an isomorphism*

The  $c_j$  are now paired Majorana fermion operators, and each block in the Hamiltonian is seen to be quadratic in these operators:

$$\mathcal{H}_\rho = \sum_{ij} \Gamma_{ij}(\rho) c_i c_j$$

where the coefficients  $\Gamma_{ij}$  are dependant on the irrep  $\rho$ .

## II. BOUNDING THE GAP

In this section we view the Hamiltonian as the adjacency matrix of a *weighted* graph. We restrict our attention to Hamiltonians whose off-diagonal entries are non-negative. This can be achieved by considering Hamiltonians where each term involves either  $X$ -type operators or  $Z$ -type operators but not both. That is,  $S$  consists only of  $X$ -type operators and  $Z$ -type operators. We also shift the Hamiltonian by a constant energy, so that the diagonal entries are non-negative:

$$\mathcal{H} := \sum_{g \in S} \rho_{\text{pauli}}(g) + kI.$$

This does not change the spectral gap or eigenvalues.

A simple variational argument shows that the top eigenvector (the *groundstate*) can be chosen to have all positive entries (this is the Perron-Frobenius theorem) and therefore is stabilized:

**Theorem.** Every groundstate is stabilized.

**Proof.** □

In [4], they derive the following cheeger inequality by considering bi-partitions of the graph. We will do the same, but using matrix block notation.

Let  $v_2$  be a second eigenvector,  $\mathcal{H}v_2 = \lambda_2 v_2$  and  $\|v_2\| = 1$ . We bi-partition the space so that  $v_2$  has (vector) blocks:

$$v_2 = \begin{pmatrix} x \\ y \end{pmatrix}$$

with  $x \geq 0$  and  $y \leq 0$ , component-wise. Let the blocks of  $\mathcal{H}$  under the same partition be:

$$\mathcal{H} = \begin{pmatrix} A & C \\ C^\top & B \end{pmatrix}.$$

If we denote  $\lambda_1(A)$  as the top eigenvalue of  $A$  and  $\lambda_1(B)$  as the top eigenvalue of  $B$ , then

$$\begin{aligned} \lambda_2 &= v_2^\top \mathcal{H} v_2 = x^\top A x + 2x^\top C y + y^\top B y \\ &\leq x^\top A x + y^\top B y \leq \|x\|^2 \lambda_1(A) + \|y\|^2 \lambda_1(B) \\ &\leq \min(\lambda_1(A), \lambda_1(B)) \leq \lambda_1. \end{aligned}$$

Defining the following constant as a maximisation over all bi-partitions of  $\mathcal{H}$ :

$$\nu(\mathcal{H}) := \max_{A, B} \min(\lambda_1(A), \lambda_1(B))$$

the above calculation shows that

$$\lambda_2 \leq \nu(\mathcal{H}) \leq \lambda_1.$$

To bound  $\lambda_2$  from below, we use a variational argument. For any unit vector  $v$  orthogonal to the top eigenspace of  $\mathcal{H}$  we have  $v^\top \mathcal{H} v \leq \lambda_2$ .

See also, [1]. And [6].

## APPENDIX A: GAP SOURCE CODE

This is source code for the GAP system [5] to display the irreducible representations of  $\mathcal{P}_2$ .

```
F := FreeGroup("m", "xi", "zi", "ix", "iz");;
m := F.1;; xi := F.2;; zi := F.3;; ix := F.4;; iz := F.5;;
G := F / [ix*xi*ix*xi, m*iz*ix*iz*ix, m*zi*m*zi, xi*xi, xi*m*xi*m,
          zi*iz*zi*iz, m*xi*zi*xi*zi, ix*m*ix*m, m*xi*m*xi, xi*ix*xi*ix,
          iz*iz, ix*zi*ix*zi, m*m, zi*m*zi*m, ix*ix,
          zi*ix*zi*ix, xi*iz*xi*iz, m*iz*m*iz, m*ix*m*ix, m*zi*xi*zi*xi,
          iz*zi*iz*zi, m*ix*iz*ix*iz, iz*m*iz*m, iz*xi*iz*xi, zi*zi];;
Print(Order(G));
LoadPackage( "repsn" );;
chi := Irr(G);
Print(chi);
for c in chi do
  rep := IrreducibleAffordingRepresentation(c);
  Print(rep);
  Print("\n");
od;
```

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