Structure of 2D Topological Stabilizer Codes

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Abstract

We provide a detailed study of the general structure of translationally invariant two-dimensional topological stabilizer quantum error correcting codes, including subsystem codes. We show that they can be understood in terms of the homology of string operators that carry a certain topological charge. In subsystem codes two dual kinds of charges appear. We prove that two non-chiral codes are equivalent under local transformations iff if they have isomorphic topological charges. Our approach emphasizes local properties over global ones.

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1 Introduction

Quantum error correcting codes [1, 2, 3, 4] play a fundamental role in the quest to overcome the decoherence of quantum systems. Among them, stabilizer codes [5, 6] provide a large and flexible class of codes that are at the same time relatively easy to investigate. Interactions between physical qubits are sometimes subject to locality constraints in a geometrical sense. E.g., qubits might be placed in a D-dimensional array with interactions available only between nearest neighbors. In such cases the subclass of topological stabilizer codes (TSC) [7, 8, 9, 10, 11, 12, 13, 14] is a natural choice. TSCs not only have very nice locality properties, but are also flexible in terms of the manipulation strategies that they allow.

The purpose of this paper is to investigate the general structure of two-dimensional TSCs, covering both subspace codes and the more general subsystem codes [15, 16, 17]. The only constraint imposed is translational symmetry of the bulk of the code, which provides a way to list the codes.

A stabilizer quantum error correcting code is described in terms of certain 'check operators', a set of commuting observables that have to be measured in order to get information about what errors have affected an encoded state. In TSCs these measurements are local in a geometrical sense. In particular, a TSC is in practice usually given as a recipe to construct the check operators on a given lattice of qubits. The lattice can be arbitrarily large, but check operators are defined locally, with support on a set of qubits contained in a bounded region. What makes these codes topological, as opposed to simply local, is that no information about the encoded qubits can be recovered if only access to a local set of qubits is granted. Indeed, operators on encoded qubits have support on a number of physical qubits that grows with the lattice size.

An essential feature of known topological codes is that they have an error threshold [8, 18, 19, 20, 21]. I.e., in the limit of large lattices, for noise below a certain threshold error correction is asymptotically perfect. This is true either in a simple error correction scenario or in a fault-tolerant scenario, and also when qubit losses are taken into account [22]. It is also often true [23] —but for interesting exceptions, see[12, 14]— that the number of encoded qubits depends only on the homology of the qubit lattice. In particular a trivial homology gives no encoded qubits, but even in a planar setting it is possible to recover non-trivial codes by introducing suitable boundaries [24] and other defects such as twists [25]. Moreover, because in such 'homological' codes the lattices can be chosen very flexibly, it is possible to carry out certain computations by changing the code geometry over time, something called 'code deformation' [8, 26, 27]. Error correction turns out to be connected to classical statistical models [8, 19, 11] and this has given rise to fast algorithms to infer the errors [28]. Finally, there exist topological codes that are especially well suited to perform computations using transversal gates [9, 10], which minimizes error propagation.

General properties of TSCs in two or higher dimensions have already been explored to different degrees. General constraints on the code distance —directly related to the geometry of encoded operators— were first found in [29], and improved in [30]. The subsystem case was addresed in [31]. The geometry of logical operators in a subclass of TSCs, subject to constraints such as scale invariance of the number of encoded qubits, was studied in [23]. Constraints on the code distance for three-dimensional codes that do not satisfy this scale invariance condition have been recently developed [32].

One of the main results in this paper is that all two-dimensional TSCs can be understood in terms of 'string operators' that carry a 'topological charge'. In particular, homology plays an essential role and dictates the number of encoded qubits. In the case of subspace codes, the corresponding Hamiltonian model [7] —which has the code as its gapped ground state— exhibits anyonic excitations. Moreover, all subspace codes that give rise to the same anyon model turn out to be equivalent, in the sense that there exists a local transformation connecting them. The implications of this result, both from the condensed matter and quantum information perspectives, are explored in [33]. In the case of subsystem codes we no longer have a direct interpretation in terms of physical anyons. However, we find a nice duality structure between two kinds of charges. The first kind corresponds to an anyon model —possibly chiral—, and the second to fluxes with which the anyons interact topologically. We show that codes giving rise to the same non-chiral anyon model are locally equivalent. Computational implications of these results are discussed in the conclusions.

Our approach emphasizes local properties of the codes —such as the structure of check operators— over global ones —such as the number of encoded qubits. In practice we realize this by considering infinite versions of the codes that cover the plane. The idea is that all operators acting on a finite number of qubits become automatically local in this infinite picture. At the same time, the homology of the plane is trivial and this simplifies the analysis.

The paper is divided as follows. Section 2 informally summarizes the main results, providing the intuition behind the main definitions and proofs. Section 3 introduces in a formal manner topological stabilizer groups (TSGs), used to model TSCs. Subsequent sections develop diverse aspects of the structure of TSGs, culminating in a structure theorem in section 8 that we use to prove local equivalence. Finally, section 9 discusses natural extensions of this work.

To avoid repetition, we will often omit the qualifier "two-dimensional" when discussing lattices, TSCs, and so on, but it should be understood in all cases.

2 Approach and results

The purpose of this section is to summarize informally the approach taken to investigate TSCs and the results obtained. At the same time, it explains the motivation behind the main definitions and the intuition behind some proofs.

2.1 Stabilizer codes

Given a system with n qubits, its Pauli group is

$$\mathcal{P}_n := \langle i\mathbf{1}, X_1, Z_1, \dots, X_n, Z_n\rangle, \tag{1}$$

where X_i, Z_i denote the Pauli X and Z operators acting on the i-th qubit. A stabilizer code [5, 6] on n qubits is defined by a subgroup of Pauli operators $\mathcal{S} \subset \mathcal{P}$, called stabilizer group, such that $-1 \notin \mathcal{S}$. $\mathcal{S}$ is then abelian and its elements are self-adjoint. The code is composed by those states $|\psi\rangle \in \mathbf{C}^{2^n}$ such that $s|\psi\rangle = |\psi\rangle$ for every stabilizer $s \in \mathcal{S}$. It is enough to check these conditions for a set $\{s_i\}$ of independent generators of $\mathcal{S}$. Such stabilizer generators serve as check operators: they can be measured in order to recover information about any errors affecting the encoded states. If $\mathcal{S}$ has $n-k$ independent generators the code subspace has dimension 2^k, it encodes k logical qubits. The logical or encoded Pauli operators are recovered as the quotient

$$\frac{\mathcal{Z}_{\mathcal{P}_n}(\mathcal{S})}{\mathcal{S}} \simeq \mathcal{P}_k, \tag{2}$$

where $\mathcal{Z}_{\mathcal{P}_n}(\mathcal{S})$ denotes the centralizer of $\mathcal{S}$ in $\mathcal{P}_n$. This centralizer is the group of undetectable Pauli errors: those that do not change the error syndrome. The quotient is necessary because the elements of $\mathcal{S}$ are trivial undetectable errors, without any effect on encoded states.

Given a set $\{s_i\}$ of generators of $\mathcal{S}$ we can write down a Hamiltonian $H = -\sum_i s_i$. Its ground subspace is the code subspace, and there is a gap of two energy units to any excited state. This gap is important when we consider families of TSCs with local generators in systems of arbitrary size, because the size independence of the gap gives rise to a gapped phase. An error syndrome in the code amounts to an excitation configuration in the Hamiltonian system.

It is also possible to encode only in a subsystem of the code subspace. In the stabilizer formalism this logical subsystem can be described as that in which the action of a certain gauge group $\mathcal{G} \subset \mathcal{P}$ is trivial [16]. The gauge group satisfies

$$\mathcal{Z}_{\mathcal{G}}(\mathcal{G}) \propto \mathcal{S} \tag{3}$$

and we assume $\langle i\mathbf{1}\rangle \subset \mathcal{G}$. Logical operators are recovered from

$$\frac{\mathcal{Z}_{\mathcal{P}_n}(\mathcal{G})}{\mathcal{S}} \simeq \mathcal{P}_k, \tag{4}$$

where 2^k is no longer the dimension of the code subspace, which is instead 2^{k+r} for some r. Then $\mathcal{S}$ has $n-k-r$ independent generators and $\mathcal{G}$ has $n-k+r$

independent generators. Undetectable errors form the group $\mathcal{Z}_{\mathcal{P}_n}(\mathcal{S})$. They are, up to a product with an element of $\mathcal{G}$, logical operators. Indeed,

$$\frac{\mathcal{Z}_{\mathcal{P}_n}(\mathcal{G})}{\langle i\mathbf{1}\rangle\mathcal{S}} \simeq \frac{\mathcal{Z}_{\mathcal{P}_n}(\mathcal{S})}{\mathcal{G}}. \tag{5}$$

Stabilizer subsystem codes can be described just by giving $\mathcal{G}$, since then $\mathcal{S}$ is fixed up to signs that can be chosen arbitrarily.

For subsystem codes we no longer have such a straightforward interpretation in terms of a quantum Hamiltonian. A natural possibility is to take a set of generators $\{g_i\}$ of the gauge code as the Hamiltonian terms. This will guarantee that all the energy eigenvalues of the system have at least a degeneracy 2^k, since encoded operators commute with all Hamiltonian terms. However, there is no reason for an energy gap to persist in the limit of large system sizes.

2.2 Topological codes and lattice Pauli groups

For a TSC we mean a family of codes such that, loosely speaking, stabilizer generators are local, and non-trivial undetectable errors are non-local. In addition, for subsystem codes we impose that gauge generators are local. As we will see, certain topological subsystem codes naturally give rise to a few global generators either in the stabilizer or in the gauge group, but in such a way that it is irrelevant from the encoding perspective. There exist codes with local gauge generators and non-local stabilizer generators, such as Bacon-Shor codes [17], but we do not consider them topological. Indeed, they lack an essential feature of topological codes: a non-zero error threshold in the limit of large code size.

In order to characterize TSCs in a more concrete manner, let us first consider ordinary stabilizer codes, not subsystem ones. As discussed in the introduction, a TSC is usually given as a recipe to construct the check operators on a given lattice of qubits. We assume that this recipe, in the bulk of the code —e.g., ignoring possible boundaries or other defects— takes the form of a local and translationally invariant (LTI) set of check operators. Then we can consider an infinite version of the same lattice and construct using the recipe an infinite group with check operators as generators. We call this a lattice Pauli group (LPG). Translational invariance is supposed to hold only at a suitable scale.

In general, for a LPG we mean a group that has as generators a LTI set of Pauli operators on a certain lattice of qubits. By definition, the Pauli group $\mathcal{P}$ on the infinite lattice of qubits is itself a LPG, with generators all the single qubit Pauli operators. The goal is to understand which properties distinguish those LPGs that correspond to a TSC, which we call topological stabilizer groups (TSG). Then we can shift our study of TSCs to that of TSGs.

We define TSGs as LPGs $\mathcal{S}$ satisfying $-1 \notin \mathcal{S}$ and the topological condition

$$\mathcal{Z}_{\mathcal{P}}(\mathcal{S}) \propto \mathcal{S}. \tag{6}$$

Since in the infinite lattice all the Pauli operators are local, this condition reads:

Local undetectable errors do not affect encoded states.

To motivate (6) in detail, let $\mathcal{S}$ be the LPG corresponding to a certain TSC given by the family of stabilizer groups $\mathcal{S}_l$. Any operator $O \in \mathcal{Z}_{\mathcal{P}}(\mathcal{S})$ has an analog O' acting on the bulk of the l-th code in the family for l sufficiently large. Moreover, $O' \in \mathcal{Z}_{\mathcal{P}}(\mathcal{S}_l)$ due to the locality of the stabilizer generators, and no matter what detailed definition of locality we adopt, O' should be local for sufficiently large l. Since O' is undetectable and the codes $\mathcal{S}_l$ are topological, it follows that $\phi O' \in \mathcal{S}_l$ for some phase ϕ. Then if $\phi O \notin \mathcal{S}$ we can safely add ϕO, and all its translations, to an LTI generating set of $\mathcal{S}$, enlarging $\mathcal{S}$. The only question left then is whether one might have to keep adding larger and large generators, but this is not the case because $\mathcal{Z}_{\mathcal{P}}(\mathcal{S})$ has a LTI set of generators. This is guaranteed by corollary 17:

The centralizer of a LPG is a LPG.

Now consider the subsystem case. We define topological stabilizer subsystem groups (TSSGs) as LPGs $\mathcal{S}$ satisfying $-1 \notin \mathcal{S}$ and the topological condition

$$\mathcal{Z}_{\mathcal{P}}(\mathcal{Z}_{\mathcal{P}}(\mathcal{S})) \propto \mathcal{S}. \tag{7}$$

This condition is trivially satisfied when the number of qubits is finite, but is not generally true for LPGs, as an example in the next section shows. To motivate (7), let $\mathcal{S}$ be the LPG corresponding to a certain topological stabilizer subsystem code given by the family of stabilizer groups $\mathcal{S}_l$ and gauge groups $\mathcal{G}_l$. We first notice that it makes sense to define the gauge group as the LPG

$$\mathcal{G} := \mathcal{Z}_{\mathcal{P}}(\mathcal{S}). \tag{8}$$

Indeed, any operator $O \in \mathcal{Z}_{\mathcal{P}}(\mathcal{S})$ has a local analog $O' \in \mathcal{Z}_{\mathcal{P}}(\mathcal{S}_l)$ acting on the bulk of the l-th code in the family for l sufficiently large. Since O' is undetectable and the codes $(\mathcal{S}_l, \mathcal{G}_l)$ are topological, it follows that $\phi O' \in \mathcal{G}_l$ for some phase ϕ —but phases are unimportant in the gauge group. Conversely, any operator O' of $\mathcal{G}_l$ with support in the bulk corresponds to an operator $O \in \mathcal{Z}_{\mathcal{P}}(\mathcal{S})$ due to the locality of the stabilizer generators. Second, due to the definition (8) we have $\mathcal{S} \subset \mathcal{G}$ and from that $\mathcal{Z}_{\mathcal{P}}(\mathcal{G}) \subset \mathcal{G}$ and $\mathcal{Z}_{\mathcal{P}}(\mathcal{G}) = \mathcal{Z}_{\mathcal{G}}(\mathcal{G})$. Any operator $O \in \mathcal{Z}_{\mathcal{G}}(\mathcal{G})$ has an analog $O' \in \mathcal{Z}_{\mathcal{G}_l}(\mathcal{G}_l)$ for l sufficiently large, and thus we can if needed enlarge $\mathcal{S}$ as above to get $\mathcal{S} \propto \mathcal{Z}_{\mathcal{P}}(\mathcal{G})$, which does not affect (8).

2.3 Examples

The most important example of TSG is the *toric code*. In an infinite square lattice, we place one qubit per link and define two kinds of operators. For each vertex v there is a vertex operator $X_v = X_1X_2X_3X_4$ where $i = 1, 2, 3, 4$ identify the qubits lying on those links meeting at v. Similarly, for each face f there is a face operator $Z_f = Z_1Z_2Z_3Z_4$ with $i = 1, 2, 3, 4$ now identifying the qubits lying at those links composing the boundary of f. The stabilizer $\mathcal{S}$ is generated by all such vertex and face operators. Using homology theory it is not difficult to check that it satisfies (6).

To illustrate TSSGs, we can consider a variation of the toric code that, although useless as a quantum code, has nontrivial structure as we will see. For this *subsystem toric code* $\mathcal{S}$ is generated by face operators Z_f. It follows from the properties of the toric code that $\mathcal{G}$ has as generators vertex operators X_v and single qubit Z Pauli operators. Moreover, (7) is satisfied.

2.4 Independent generators and constraints

Given a LTI set G of generators of a LPG, a constraint is a subset of G such that the product of its elements is proportional to the identity —but none of the elements is. We distinguish between local constraints, with a finite number of elements, and global constraints, with a possibly infinite number of elements. The latter are well defined thanks to locality. Elements of G are independent if they are not part of a local constraint.

LPGs $\mathcal{A}$ that are the centralizer of some other LPG $\mathcal{B}$, so that $\mathcal{A} = \mathcal{Z}_{\mathcal{P}}(\mathcal{B})$, have very specific properties regarding independent generators and constraints:

The centralizer of a LPG admits a LTI set of independent generators. Such a set of generators has a finite number of global constraints.

These results are lemma 19 and theorem 22, but see section 2.10.

For general LPGs it is not true that there exists always a LTI set of independent generators. As a counterexample —that also shows that (7) is not satisfied for any abelian LPG—, one can consider a square lattice with one qubit per site, and the LPG generated by nearest neighbor Pauli operators of the form $Z_i Z_j$. There is one generator per link, and they are not independent because the product of all the link operators in a closed circuit is the identity operator. Moreover, it is not difficult to see that it is not possible to choose a translationally invariant set of independent generators. Similarly, the result does not remain true for higher dimensions. A counterexample in dimension three is given by the 3D toric code [8], where the product of the plaquette operators forming a closed surface is the identity operator.

In both of the examples offered above, the toric code and the subsystem toric code, the given generators for $\mathcal{S}$ and $\mathcal{G}$ are independent, as one can easily check. The set of all face operators gives rise to a global constraint, and the same is true for the set of all vertex operators. Thus, there are in total four global constraints, including the trivial empty set. In the subsystem case both the generators of $\mathcal{S}$ and $\mathcal{G}$ give rise to the same number of global constraints, two. This is always so.

2.5 Charge

A central notion in this paper is that of charge. To introduce it, first consider the commutator of two Pauli elements

$$(p; q) := pqp^{-1}q^{-1} = \pm 1. \tag{9}$$

Given $p \in \mathcal{P}$ we can construct a group morphism $(p;\,\cdot\,) : \mathcal{P} \longrightarrow \pm 1$. Obviously $\langle i\mathbf{1}\rangle$ is a subset of the kernel of $(p;\,\cdot\,)$. So let us consider more generally, given a LPG $\mathcal{A}$, the set of morphisms $\Phi(\mathcal{A})$ from $\mathcal{A}$ to ± 1 that satisfy this property. They form an abelian group with product $(\phi\phi')(a) := \phi(a)\phi'(a)$. We are especially interested in the subgroup $\Phi^0(\mathcal{A}) \subset \Phi(\mathcal{A})$ of those morphisms such that the preimage of -1 contains a finite number of elements of any LTI set of generators. E.g., if $\mathcal{A}$ is a TSG, then the elements of $\Phi^0(\mathcal{A})$ represent states of the corresponding Hamiltonian model with a finite number of excitations. Since $(p;\,\cdot\,)(q;\,\cdot\,) = (pq;\,\cdot\,)$, the elements of $\Phi^0(\mathcal{A})$ of the form $(p;\cdot)$ form a subgroup, that we denote $\mathrm{Com}_{\mathcal{A}}[\mathcal{P}]$. We are interested in those elements of $\Phi^0(\mathcal{A})$ that do not correspond to a Pauli operator. This leads to define the charge group

$$C_{\mathcal{A}} := \frac{\Phi^0(\mathcal{A})}{\mathrm{Com}_{\mathcal{A}}[\mathcal{P}]}. \tag{10}$$

The original motivation for this definition comes from the case in which $\mathcal{A}$ is a TSG. In that case the equivalence classes can be regarded as classes of excitation configurations up to local transformations. I.e., charge is conserved in any given region if the only transformations allowed are those that affect only that region. For a centralizer LPG the charge group is finite because it is dual to the group of global constraints. Interestingly, in the case of TSSGs it is useful to consider both the stabilizer charge group $C_{\mathcal{S}}$ and the gauge charge group $C_{\mathcal{G}}$, which turn out to be dual too.

Consider the toric code example. The Z_e single qubit Pauli operator corresponding to a given link e anticommutes with the two vertex operators corresponding to the vertices v, v' on the ends of the link, and commutes with all other vertex and face operators. Moreover, given any even set of vertices $v_1, \dots, v_{2n}$ we can find a set of links E such that $\prod_E Z_e$ anticommutes only the vertex operators X_{v_i}, $i = 1, \dots, 2n$ —indeed, consider any set of links forming n 'strings', with the j-th string linking the $2j-1$ and $2j$ vertices. The reasoning holds for face operators up to duality in the lattice. Therefore, the elements of $\mathrm{Com}_{\mathcal{S}}[\mathcal{P}]$ are exactly those elements of $\Phi^0(\mathcal{S})$ with value -1 on an even number of vertex operators and an even number of face operators. There are then four charges in $C_{\mathcal{S}}$, corresponding to the two possible values of vertex and face parities, in agreement with the number of global constraints.

In the subsystem toric code we only have face generators in $\mathcal{S}$, so that the number of charges in $C_{\mathcal{S}}$ is two as expected. As for $C_{\mathcal{G}}$, Z_e pauli operators can be individually 'flipped' with X_e without affecting other generators. Thus, $C_{\mathcal{G}}$ has two elements corresponding to the parity of vertex operators with value -1.

2.6 Strings and topological charge

Consider the charge group $C_{\mathcal{G}}$ of a TSSG. Given a LTI set of generators of $\mathcal{G}$, let us say that the support of $\phi \in \Phi(\mathcal{G})$ is the set of lattice sites for which there exist a generator g with support in the site and $\phi(g) = -1$. Since the number of charges is finite, we can coarse grain the lattice till each site can hold any of

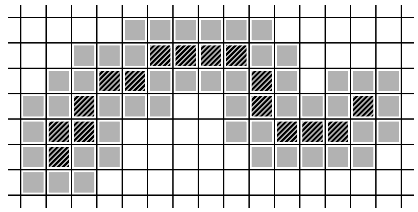


Figure 1: We represent sites as the faces of a square lattice. The striped sites form a path. Together with the grey sites they form the support of a corresponding string operator.

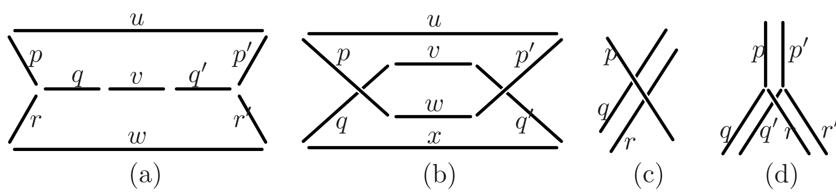


Figure 2: Geometries involved in the definition of string commutation rules.

the charges, in the sense that there exists for any site and for any charge $c \in C_{\mathcal{G}}$ a morphism $\phi \in \Phi^0(\mathcal{G})$ with charge c and support at that site. Consider two such morphisms ϕ, ϕ' with the same charge c but support at different sites σ, σ'. Their product has trivial charge so that $\phi\phi' = (p; \cdot)$ for some $p \in \mathcal{P}$. Moreover, if we coarse grain enough we can choose p to have support along a string-shaped region —a thickened path, see Fig. 1— connecting σ and σ'. This is a matter of choosing a suitable set of operators p_i for adjacent sites —one per valid $\phi\phi'$ pair—, coarse graining till all the p_i fit in sites adjacent to those involved, and finally forming string operators by composing these p_i operators. We say that p is a string operator with charge c and endpoints ϕ, ϕ'. In the case of a TSG $\mathcal{S}$ and from the perspective of the Hamiltonian model, a string operator p of charge c transports a charge c from one endpoint to another. E.g., when applied to a ground state p creates a pair of excitations with charge c on its endpoints.

Our next goal is to study the commutation properties of strings, which for many geometries only depend on their charges. We start considering three strings p, q, r common charge c and a common endpoint, see Fig. 2(a). The quantity

$$\theta(c) := (pq; pr) = (p; q)(p; r)(q; r) \tag{11}$$

only depends on the charge c, so that it is well defined. To check this, consider an alternative set of strings p', q', r' as in Fig. 2(a), together with the auxiliary strings u, v, w. Let $s = pqp'q'uv$ and $t = prp'r'uw$. Then $(s; t) = (pq; pr)(p'q'; p'r')$, but $s, t \in \mathcal{Z}_{\mathcal{P}}(\mathcal{G}) \propto \mathcal{S}$ and thus $(s; t) = 1$ because $\mathcal{S}$ is abelian. We can interpret (11) as the topological spin of the charge c [34]. We say that a charge c is bosonic if $\theta(c) = 1$ and fermionic if $\theta(c) = -1$.

Now take two crossing strings p, q with charges c, c', as in Fig. 2(b). The mutual statistics of the charges c, d is given by the quantity [34]

$$\kappa(c, d) := (p; q). \tag{12}$$

It only depends on the charges c, d, so that it is well defined. The proof is analogous to the previous one, now with the geometry of Fig. 2(b). If $\kappa(c, d) = -1$ mutual statistics are said to be semionic, otherwise they are trivial. If in Fig. 2(c) p has charge c, q has charge d_1 and r has charge d_2, it follows that

$$\kappa(c, d_1 d_2) = \kappa(c, d_1)\kappa(c, d_2), \tag{13}$$

since $\kappa(c, d_1 d_2) = (p; qr) = (p; q)(p; r) = \kappa(c, d_1)\kappa(c, d_2)$.

The topological spin and mutual statistics are related. Consider the geometry of Fig. 2(d), where the strings p, q, r have charge c and the strings p', q', r' have charge d. The figure illustrates that

$$\theta(cd) = \theta(c)\theta(d)\kappa(c, d), \tag{14}$$

since $\theta(cd) = (pp'qq'; pp'rr') = (pq; pr)(p'q'; p'r')(r; q') = \theta(c)\theta(d)\kappa(c, d)$. In particular $\kappa(c, c) = 1$.

In summary, the charges in $C_{\mathcal{G}}$ can be regarded as the topological charges of an anyon model, with fusion given by the charge group product. But the topological structure does not stop there, because it is possible to define κ for a string with charge in $C_{\mathcal{S}}$ and another with charge in $C_{\mathcal{G}}$. This works because $\mathcal{Z}_{\mathcal{P}}(\mathcal{S}) \subset \mathcal{Z}_{\mathcal{P}}(\mathcal{Z}_{\mathcal{P}}(\mathcal{G}))$, which is all we need for the construction in Fig. 2(b) to make sense. We cannot define κ for two strings with charges in $C_{\mathcal{S}}$, nor can we define θ for charges in $C_{\mathcal{S}}$, unless of course we are dealing with a TSG rather than a general $TSSG$. Therefore, we should regard the charges in $C_{\mathcal{G}}$ as anyons, and the charges in $C_{\mathcal{S}}$ as fluxes that interact topogically with these anyons through an Aharonov-Bohm effect.

There is a natural morphism $C_\iota : C_{\mathcal{G}} \longrightarrow C_{\mathcal{S}}$ derived from the restriction of morphisms from $\mathcal{G}$ to morphisms from $\mathcal{S}$. It preserves commutation in the sense that $\kappa(c, d) = \kappa(C_\iota(c), d)$ for $c, d \in C_{\mathcal{G}}$. This is an important ingredient in the construction of canonical charge generators given in theorem 36:

$$\begin{aligned} C_{\mathcal{G}} &= \langle c_1, \dots, c_\alpha, d_1, \dots, d_\alpha, e_1, \dots, e_\beta \rangle, & \tilde{c}_i &= C_\iota(d_i), \\ C_{\mathcal{S}} &= \langle \tilde{c}_1, \dots, \tilde{c}_\alpha, \tilde{d}_1, \dots, \tilde{d}_\alpha, \tilde{e}_1, \dots, \tilde{e}_\beta \rangle, & \tilde{d}_i &= C_\iota(c_i), \end{aligned} \tag{15}$$

where $i = 1, \dots, \alpha$, the e_i generate the kernel of C_ι and all gauge charge generators are bosons except possibly e_1, c_1 and d_1, always with $\theta(c_1) = \theta(d_1)$. Among gauge charge generators the only nontrivial mutual statistcs are $\kappa(c_i, d_i) = -1$. The mutual statistics between gauge and stabilizer charges gives rise to the duality. Indeed, relabeling $C_{\mathcal{G}} = \langle c_1, \dots, c_{2\alpha+\beta} \rangle$, $C_{\mathcal{S}} = \langle \tilde{c}_1, \dots, \tilde{c}_{2\alpha+\beta} \rangle$,

$$\kappa(c_i, \tilde{c}_j) = 1 - 2\delta_{ij}, \qquad i, j = 1, \dots, 2\alpha + \beta. \tag{16}$$

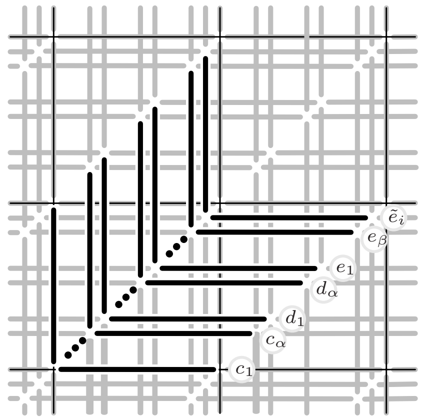


Figure 3: Framework of string segments. Each charge contributes a square lattice.

The anyon model attached to $C_\mathcal{G}$ only has a few parameters, α, β, $\chi = \theta(c_1)$ and $f = \theta(e_1)$. We call this the characteristic of the TSSG. The anyon model is chiral when $\chi = -1$, and when $f = -1$ there are fermions among the e_i. There exist TSSGs covering all possible combinations of these parameters, as shown in section 7.2. Since the subsystem toric code only has a nontrivial element in $C_\mathcal{G}$ it is an example of a TSSG with $\alpha = 0$ and $\beta = 1$. Moreover, string operators are products of single qubit Z Pauli operators, and thus $f = 1$.

In the case of TSGs $\beta = 0$, leaving the parameters α and χ. The toric code has $\alpha = 1$ and $\chi = 1$, and thus combining N toric codes together we get a code with $\alpha = N$ and $\chi = 1$. Chiral codes are not likely to exist [7].

2.7 Code structure and homology

With string operators and their geometrical commutation relations at hand, it is easy to uncover the general structure of TSSGs. Given a TSSG, we can put it in a standard form —its prototype is the toric code— through the following process. The first step is to find a set of canonical charges. Then we construct a translationally invariant framework of string operators, each in the form of a straight segment. For each charge there is a square lattice of segment operators, spatially disposed as indicated in Fig. 3. With suitable adjustments, the commutation relations between these segments operators are fixed by θ and κ. Strings with charge c_i (d_i) *also* have charge $\tilde{d}_i$ ($\tilde{c}_i$), and we adjust segment operators with charge e_i so that they are elements of $\mathcal{G}$ —in the technical part we impose more structure on those with charge $\tilde{e}_i$, but it does not affect the present discussion, only that of local equivalence.

Rather than the intricate framework of Fig. 3, it is enough to visualize, for each pair of dual charges c_i, d_i (or $e_i, \tilde{e}_i$) a single square lattice and its dual,

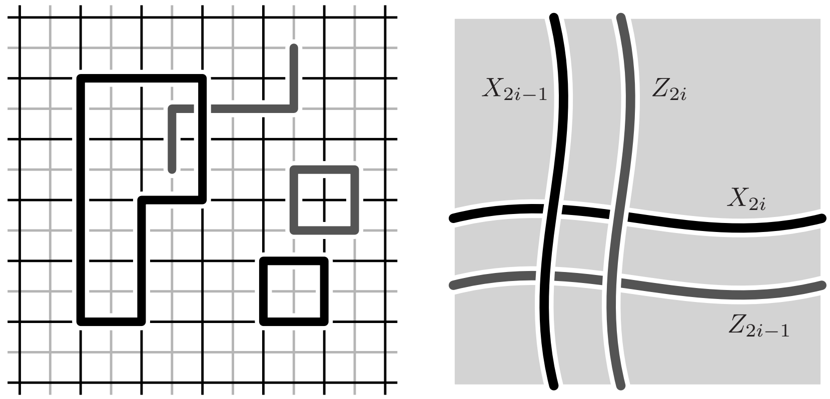


Figure 4: (Left) A square lattice and its dual. String operators can be visualized as collections of segments on these two lattices. We show for such strings: a direct plaquette, a dual plaquette, a direct boundary and a dual open string. The last two cross once. (Right) Logical operators corresponding to strings with charge c_i, in black, and strings with charge d_i, in grey, for any $i = 1, \ldots, \alpha$. The topology is that of a torus, with opposite boundaries identified.

see Fig. 4. Segment operators with charge c_i or e_i belong to the direct lattice, and those with charge d_i or $\tilde{e}_i$ to the dual lattice. We can form string operators by taking products of segment operators with the same charge. Operators from different lattices commute, and the commutation properties of string operators are simply dictated by κ and θ. E.g., a direct string with charge c_i anticommutes with a dual string of charge d_i iff the two strings cross an odd number of times.

Closed strings operators —or boundary strings since the homology is trivial in the plane— belong to $\mathcal{S}$ or at least to $\mathcal{G}$ —the former is the case of $\tilde{e}_i$ closed strings. Any boundary operator is a product of 'plaquette operators', each of them the product of the segment operators forming a plaquette. Crucially, it is possible to find LTI sets of independent generators of $\mathcal{G}$ and $\mathcal{S}$ that include these plaquette operators, which are the only ones subject to constraints —now obvious. In particular, those plaquette operators formed by $\tilde{c}_i, \tilde{d}_i$ or $\tilde{e}_i$ (c_i, d_i or e_i) strings are generators of $\mathcal{S}$ ($\mathcal{G}$).

The trivial stabilizer generators, i.e., those that are not plaquette operators, play no significant role, in the sense that it is always possible to find a local transformation that removes them as disentangled qubits. After this removal, we are left with a system where every Pauli operator can be decomposed as sg, where s is a product of string operators and g is a product of trivial gauge generators.

2.8 Back to codes

The homological perspective on string operators becomes most relevant when we go back to a code in a finite lattice, as we sketch here. The simplest possible way to do this is by considering a finite lattice with periodic boundary conditions, so that the topology is that of a torus. The decomposition in terms of string operators and trivial gauge generators still holds —the lattice must be large enough, though, in terms of the support of the segment operators and trivial gauge generators. But now closed string operators need not be boundaries, and this is an all-important feature.

Regarding gauge and stabilizer generators, each 'global constraint' becomes now a constraint of the form $\prod_i p_i \propto 1$, with p_i running over plaquette operators of a given lattice, direct or dual, so that we can remove one such plaquette from the corresponding generating sets. But this still does not give rise in general to a proper stabilizer and gauge group pair. Indeed, when $\beta \neq 0$ we have to either add nontrivial cycle strings with charge e_i to $\mathcal{S}$ or nontrivial cycle string with charge $\tilde{e}_i$ to $\mathcal{G}$. This does not affect the number of encoded qubits —but see below!. In fact, apart from this detail everything works as in a toric code. Each pair (c_i, d_i) contributes two logical qubits, so that $k = 2\alpha$, and logical operators correspond to nontrivial cycles, see Fig. 4.

From the above discussion it seems that the existence of the β charges e_j serves no purpose from a coding perspective. For quantum codes this is indeed true in the sense that they contribute no encoded qubits. However, interestingly they do provide encoded *bits*! The eigenvalues of nontrivial closed strings with charge $\tilde{e}_i$ label the classical states of the memory —and should be regarded as elements of $\mathcal{G}$—, and nontrivial closed string operators with charge e_i act as bit flip operators. Detecting errors thus amount to look for the endpoint of open string operators with charge e_i. I.e., the stabilizer —which must include only strings of trivial homology—, still provides the error syndrome through the eigenvalues of its generators. The subsystem toric code exemplifies all this.

2.9 Equivalence

We say that two TSSGs are locally equivalent if they can be related by a combination of LTI Clifford transformations, lattice coarse graining and addition/removal of disentangled qubits in a translationally invariant way. Such local transformations preserve the topological charge structure, and the converse also holds in some cases, see section 8.4:

Non-chiral TSSGs are locally equivalent iff they have the same characteristic.

This result, true also for chiral TSGs, is a consequence of the structure discussed above, together with the existence of certain 'elementary' TSSGs and the ability to 'complete' partially defined LTI Clifford transformations, see lemma 2.

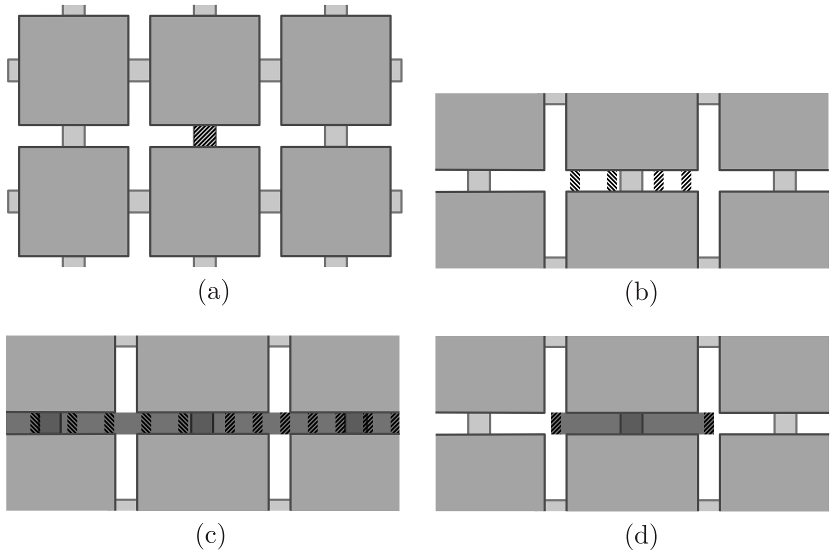


Figure 5: The regions involved in the construction of an LTI set of independent generators.

2.10 Centralizers and 2D topology

To finish this summary of results, let us explain the role played by the topology of the plane and the fact that the LPG is a centralizer of a LPG for the existence of an LTI set of independent generators. Let the support of a constraint be the union of the support of its elements. Consider a LPG $\mathcal{A}$ of the form $\mathcal{A} = \mathcal{Z}_{\mathcal{P}}(\mathcal{B})$. In order to recover a LTI set of independent generators from an arbitrary LTI set of generators of $\mathcal{A}$, all we need to do is remove in a translationally invariant way a sufficient amout of generators. If a generator is part of a constraint, it can be removed. The removal of constrained generators with support in the regions A, marked with dark grey in Fig. 5(a), can be done without much difficulty. It is enough to choose a large enough separation between these regions, so that generators in different regions are subject to independent constraints, which allows a translationally invariant removal. After the removal it is unclear whether the generators with support in different regions B, marked with light grey in Fig. 5(a), are also subject to independent constraints. But given a constraint R with support, say, in the region B_0 marked with stripes in Fig. 5(a), we can find —as shown below— another constraint R_0 containing the same generators as R in B_0 but with no support in the rest of regions B. This shows that generators with support in different regions B are indeed subject to independent constraints. Then we are left with the regions C, marked with white in Fig. 5(a), but these are disconnected from each other and we can deal with them as follows. The constraints that are left can only include generators with support in regions C now, and since they are distant enough and the generators are local, the restriction of any constraint to a particular region C is still a constraint. Thus, generators with support in different regions C are also subject to independent constraints.

Thus, the crucial point is the existence, for any given constraint R with support in regions B and C, of a constraint R_0 with the same support as R in B_0 but no support in other regions B. To guarantee it, we take the size of the regions A as large as needed, so that near B_0 any such R has support only in a long and narrow band. Then, up to translations, the elements in R must repeat along the band at certain stretches of a length larger than the size of the generators. At each side of the band we pick two such repeated stretches, marked with stripes in Fig. 5(b), and form a new constraint R_∞ that extendes to infinity by repeating the set of constraints in between the repeated stretches up to translations. This is depicted in Fig. 5(c), where the places where the repeated stretches overlap are marked with stripes and the support of R_∞ is darkened. We can then consider a subset of R_∞, containing only the elements on the darkened region in Fig. 5(d), which produces a set of generators S with product $p_1p_2 \in \mathcal{P}$, where p_i have each support in one of the two regions marked with stripes. The all-important property of these two regions is that they are sufficiently far away from region A. Since $p_1p_2 \in \mathcal{Z}_\mathcal{P}(\mathcal{B})$ and $\mathcal{B}$ has local generators, it follows that $p_1, p_2 \in \mathcal{Z}_\mathcal{P}(\mathcal{B})$ since the supports of p_1 and p_2 are sufficiently far appart. There exist sets of generators S_i of $\mathcal{A}$, $i = 1, 2$, with supports in a small neighborhood of the support of p_i and such that their product is proportional to p_i. Then to obtain R_0 it is enough to take the symmetric difference of S, S_1 and S_2.

3 Basic definitions

This section intends to put together most of the constructions needed later, with the aim of both presenting them and serving as a reference for later sections. It includes a list of basic examples of topological codes.

3.1 Notation

Sets — For integers we use the notation $\mathbf{N}_0 := \{0\} \cup \mathbf{N}^*$. $|S|$ denotes the cardinality of a set S, $\mathbf{P}(S)$ its power set and $\mathbf{P}_0(S) := \{s \in \mathbf{P}(S) : |s| < \aleph_0\}$. Given a mapping $f : A \longrightarrow B$ and any subsets $a \subset A$, $b \subset B$, we denote by $f[a] \subset B$ the image of a and by $f^{-1}[b] \subset A$ the preimage of b. Given any set S, we regard $\mathbf{P}(S)$ as the abelian group with addition given by the symmetric difference $S_1 + S_2 := S_1 \cup S_2 - S_1 \cap S_2$.

2D lattice — Since we are interested in topological aspects, we consider without loss of generality square lattices. Sites are labeled as usual with integer coordinates $(j, k) \in \mathbf{Z}^2$, with $\Sigma = \mathbf{Z}^2$ the set of all sites, and $\mathbf{0} := (0, 0)$. A block is a set of sites of the form

$$\square_{m,n}^{m',n'} := \{(j,k)\,:\, m \le j < m'; n \le k < n'\}. \tag{17}$$

Given a set of sites $\gamma \in \mathbf{P}_0(\Sigma)$, $\rho(\gamma)$ denotes the smallest $k \in \mathbf{N}_0$ such that $\gamma \subset \square_{m,n}^{m+k,n+k}$ for some $m,n \in \mathbf{Z}$. For $\gamma \subset \Sigma$, $l \in \mathbf{N}_0$, define a 'thickened' set

$$\mathrm{Thk}^l(\gamma) := \bigcup_{(i,j)\in\gamma} \square_{i-l,j-l}^{i+l+1,j+l+1}. \tag{18}$$

Connectedness — Two sites (i,j) and (m,n) are adjacent if $|m-i| \le 1$ and $|n-j| \le 1$. An ordered list of sites $(\sigma_k)_{k=1}^n = (\sigma_0, \cdots, \sigma_n)$ where σ_k is adjacent to σ_{k+1} for all $k = 1, \ldots, n$ is a path from σ_0 to σ_n. We define the inverse γ^{-1} of a path γ in the usual way, an also the composition of paths $\gamma' \circ \gamma$, where the last site of γ is the first site of γ'. A set of sites $\gamma \subset \Sigma$ is connected if for any $\sigma, \sigma' \in \gamma$ there exists a path $(\sigma_k)_{k=1}^n$ with $\sigma_0 = \sigma$, $\sigma_n = \sigma'$ and $\sigma_i \in \gamma$, $i = 1, \ldots, n$. Given an ordered set of sites $(\sigma_k)_{k=1}^n = ((a_k, b_k))_{k=1}^n$, let $\sigma'_k := (a_{k+1}, b_k)$, $k = 1, \ldots, n-1$ and define $\mathrm{Path}(\sigma_1, \ldots, \sigma_n)$ to be the path starting at σ_1, moving in a straight line to σ'_1, then in a straight line to σ_2, then σ'_2,..., and ending at σ_n. Sometimes we identify a path (σ_i) with the set $\{\sigma_i\}$.

Homology — Let Γ be an infinite square lattice. Γ_{fc}, Γ^*_{fc}, Γ_{edg} and Γ^*_{edg} denote, respectively, the set of faces, dual faces, edges and dual edges. We represent $\mathbf{Z}_2$ chains of these objects as elements of $\mathbf{P}(\Gamma_{\mathrm{fc}})$, $\mathbf{P}(\Gamma^*_{\mathrm{fc}})$, $\mathbf{P}(\Gamma_{\mathrm{edg}})$ and $\mathbf{P}(\Gamma^*_{\mathrm{edg}})$. ∂ denotes the usual boundary operation, which we express in particular as

$$\mathbf{P}(\Gamma_{\mathrm{fc}}) \xrightarrow{\partial} \mathbf{P}(\Gamma_{\mathrm{edg}}) \xrightarrow{\partial} \mathbf{P}(\Gamma^*_{\mathrm{fc}}), \qquad \mathbf{P}(\Gamma^*_{\mathrm{fc}}) \xrightarrow{\partial} \mathbf{P}(\Gamma^*_{\mathrm{edg}}) \xrightarrow{\partial} \mathbf{P}(\Gamma_{\mathrm{fc}}). \tag{19}$$

Given an edge e, it dual is denoted e^*. We name the elements $\mathbf{e}_1, \mathbf{e}_2 \in \Gamma_{\mathrm{edg}}$, $\mathbf{f} \in \Gamma_{\mathrm{fc}}, \mathbf{f}^* \in \Gamma^*_{\mathrm{fc}}$ according to Fig. 6. For faces, the symbol * is just a label.

Qubits and operators — Sites hold a common and finite number of qubits n. Let $X^\alpha_{j,k}$, $Z^\alpha_{j,k}$ denote single qubit X and Z operators acting on the qubit with label $\alpha = 1, \ldots, n$ on the site (j,k). They generate for each site (j,k) an algebra of operators $A_{(j,k)}$. Consider the infinite tensor product $A := \otimes_{(j,k)} A_{(j,k)}$, with elements $\otimes_{(j,k)} a_{(j,k)}$ with all but a finite number of the product terms identical $a_{(j,k)} = \mathbf{1}$. The Pauli group $\mathcal{P} \subset A$ is $\mathcal{P} := \langle i\mathbf{1}\rangle\langle X^\alpha_{ij}, Z^\alpha_{ij}\rangle_{i,j,\alpha}$. Given a Pauli operator $p \in \mathcal{P}$, $\mathrm{Supp}(p)$ denotes the set of sites containing qubits on the support of p, i.e., on which p acts nontrivially. Its size is measured by $\rho(p) := \rho(\mathrm{Supp}(p))$. For $\gamma \subset \Sigma$ we let $p|_\gamma$ be the restriction of p to qubits in γ, defined in the obvious way and only up to a global phase. Given $p, q \in \mathcal{P}$ their commutator is $(p;q) := pqp^{-1}q^{-1} = \pm 1$. For $\mathcal{A}, \mathcal{B} \subset \mathcal{P}$, the centralizer of $\mathcal{A}$ in $\mathcal{B}$ is

$$\mathcal{Z}_{\mathcal{B}}(\mathcal{A}) := \{b \in \mathcal{B}\,:\, \forall a \in \mathcal{A} \quad (a;b) = 1\}. \tag{20}$$

We write $\mathcal{A} \propto \mathcal{B}$ meaning $\langle i\mathbf{1}\rangle\mathcal{A} = \langle i\mathbf{1}\rangle\mathcal{B}$.

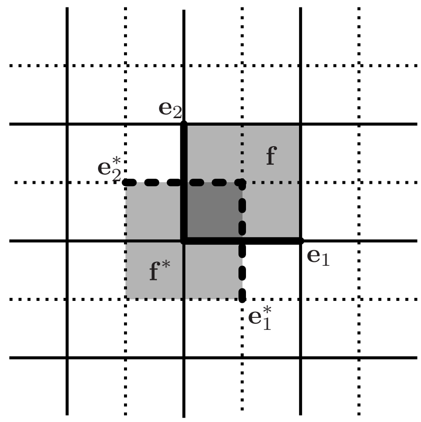


Figure 6: Naming some elements of Γ.

Pauli group subsets — For $S \subset \mathcal{P}$, $\gamma \subset \Sigma$ and $q \in \mathcal{P}$ we define

$$\begin{aligned} \mathrm{Supp}(S) &:= \bigcup \mathrm{Supp}[S], \qquad & S|_\gamma &:= \{s \in S \,:\, \mathrm{Supp}(s) \subset \gamma\}, \\ S||_\gamma &:= \{s \in S \,:\, \mathrm{Supp}(s) \cap \gamma \neq \emptyset\}, \qquad & (S;q) &:= \prod_{p\in S_q} (p;q), \end{aligned} \tag{21}$$

where $S_q = \{p \in S \,:\, \mathrm{Supp}(p) \cap \mathrm{Supp}(q) \neq \emptyset\}$ and only for S_q finite the definition holds. We set $\rho(S) := \rho(\mathrm{Supp}(S))$. We say that a set of generators $\mathcal{A}_g$ of a group $\mathcal{A} \subset \mathcal{P}$ is independent if the product of any finite subset of nontrivial generators is nontrivial —trivial elements of $\mathcal{P}$ are those in $\langle i\mathbf{1}\rangle$. We define a 'product' group morphism

$$\begin{aligned} \mathrm{Pro} \,:\, \mathbf{P}_0(\mathcal{P}) &\longrightarrow \mathcal{P}/\langle i\mathbf{1}\rangle \\ S &\longmapsto \langle i\mathbf{1}\rangle \prod_{p\in S} p. \end{aligned} \tag{22}$$

Morphisms — Let $\mathcal{A} \subset \mathcal{P}$ be a group. The group morphisms $\phi \,:\, \mathcal{A} \longrightarrow \{1,-1\}$ such that $\phi(i) = 1$ form an abelian group $\Phi(\mathcal{A})$ with $(\phi\phi')(a) := \phi(a)\phi'(a)$. Given a subset $\mathcal{A}_g$ of $\mathcal{A}$, we define a group morphism

$$\begin{aligned} \mathrm{Neg}_{\mathcal{A}_g} \,:\, \Phi(\mathcal{A}) &\longrightarrow \mathbf{P}(\mathcal{A}_g - \langle i\mathbf{1}\rangle) \\ \phi &\longmapsto \mathcal{A}_g \cap \phi^{-1}[-1], \end{aligned} \tag{23}$$

which has an inverse $\mathrm{Neg}_{\mathcal{A}_g}^{-1}$ if $\mathcal{A}_g$ is an independent set of generators of $\mathcal{A}$. The support of $\phi \in \Phi(\mathcal{A})$ with respect to $\mathcal{A}_g$ is

$$\mathrm{Supp}_{\mathcal{A}_g}(\phi) := \mathrm{Supp}(\mathrm{Neg}_{\mathcal{A}_g}(\phi)). \tag{24}$$

We define the subgroup of morphisms with finite support

$$\Phi^0(\mathcal{A}_g) := \{\phi \in \Phi(\langle \mathcal{A}_g \rangle) \,:\, |\mathrm{Supp}_{\mathcal{A}_g}(\phi)| < \aleph_0\}. \tag{25}$$

Given $\phi \in \Phi^0(\mathcal{A}_g)$, we will denote with the same name the group morphism

$$\begin{aligned} \phi \,:\, \mathbf{P}(\mathcal{A}_g) &\longrightarrow \{1, -1\} \\ S &\longmapsto (-1)^{|S \cap \mathrm{Neg}_{\mathcal{A}_g}(\phi)|} \end{aligned} \tag{26}$$

The commutation structure gives rise to morphisms that need a name,

$$\begin{aligned} \mathrm{Com}_{\mathcal{A}} \,:\, \mathcal{P} &\longrightarrow \Phi(\mathcal{A}) & \qquad \mathrm{Com}_{\mathcal{A}} \,:\, \mathbf{P}(\mathcal{P}) &\longrightarrow \Phi(\mathcal{A}) \\ p &\longmapsto (p;\cdot) & S &\longmapsto (S;\cdot) \end{aligned} \tag{27}$$

For $p \in \mathcal{P}$ or $p \in \mathbf{P}(\mathcal{P})$ we set $\mathrm{Supp}_{\mathcal{A}_g}(p) := \mathrm{Supp}_{\mathcal{A}_g}((p;\cdot))$.

Translation operators — We define translation operators for sites setting

$$T_{m,n}\,(j,k) := (j+m, k+n). \tag{28}$$

The action on edges or faces of Γ is analogous. For Pauli operators we define $T_{m,n} \,:\, \mathcal{P} \longrightarrow \mathcal{P}$ as group morphisms with $T_{m,n}\,(i\mathbf{1}) := i\mathbf{1}$ and

$$T_{m,n}\,(\sigma^\alpha_{j,k}) := \sigma^\alpha_{j+m,k+n}, \qquad \sigma = X, Z. \tag{29}$$

For $\phi \in \Phi(\mathcal{A})$ we set

$$(T_{m,n}\,(\phi))(\cdot) := \phi(T_{-m,-n}\,(\cdot)). \tag{30}$$

To generalize translation to sets of translatable objects, we recursively define whenever it makes sense

$$T_{m,n}\,(A) := \{T_{m,n}\,(a) \,:\, a \in A\}. \tag{31}$$

Finally, we set for $d \in \mathbf{N}^*$

$$T^{(d)}\,(\cdot) := \bigcup_{m,n \in \mathbf{Z}} T_{md,nd}\,(\cdot). \tag{32}$$

Coarse graining and composition — A qubit lattice can be coarse grained by identifying each block $\square^{(m+1)l,(n+1)l}_{ml,nl}$ as the site (m,n) of the new lattice, for some $l \in \mathbf{N}^*$. We denote the effect of coarse graining on a given object x by $\mathrm{Crs}_l(x)$, be it an operator, a set of sites, a group or any other structure.

Two disjoint qubit lattices can be put together to form a single lattice. If $\mathcal{A}$ is a Pauli subgroup on the first lattice and $\mathcal{B}$ is a Pauli subgroup on the second lattice we define a new group $\mathcal{A} \otimes \mathcal{B} := \{a \otimes b \,:\, a \in \mathcal{A}, b \in \mathcal{B}\}$.

3.2 Lattice Pauli morphism

Consider Pauli groups $\mathcal{P}_i$ over square lattices.

Definition 1 *A lattice Pauli (iso)morphism is a group (iso)morphism $F : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ such that $F(i) = i$ and, for any $m, n \in \mathbf{Z}$ and some $d \in \mathbf{N}^*$,*

$$T_{md,nd} \circ F = F \circ T_{md,nd}\,. \tag{33}$$

We say that F has period d and define

$$\rho(F) := \max_{\sigma \in \square_{0,0}^{d,d}} \rho(\{\sigma\} \cup \mathrm{Supp}(F[\mathcal{P}_1|_{\{\sigma\}}])). \tag{34}$$

Such a morphism is always injective. $\rho(F)$ is finite and

$$\mathrm{Supp}(F(p)) \subset \mathrm{Thk}^{\rho(F)-1}(\mathrm{Supp}(p)) \tag{35}$$

for every $p \in \mathcal{P}_1$. By coarse graining we can always have $d = 1$ and $\rho(F) = 2$. If n_i is the number of qubits per site in $\mathcal{P}_i$, $n_1 \leq n_2$ and the equality holds iff F is an isomorphism. To see way, take $\gamma = \square_{0,0}^{L,L}$ and let $g_1 = 2n_1L^2$ be the number of generators of $\mathcal{P}'_1 := \mathcal{P}_1|_\gamma$ and $g_2 = 2n_2(L + 2\rho(F) - 2)^2$ be the number of generators of $\mathcal{P}'_2 := \mathcal{P}_2|_{\mathrm{Thk}^{\rho(F)-1}(\gamma)}$. Then $F[\mathcal{P}'_1] \subset \mathcal{P}'_2$ gives $g_1 \leq g_2$ and thus in the limit of large L we get the desired result. The following 'extension lemma' will be very useful later and is true for any spatial dimensionality.

Lemma 2 *For every lattice Pauli morphism $F : \mathcal{P}_1 \longrightarrow \mathcal{P}_3$ there exist $\mathcal{P}_2$ and a lattice Pauli isomorphism $F_{\mathrm{ext}} : \mathcal{P}_1 \otimes \mathcal{P}_2 \longrightarrow \mathcal{P}_3$ with the same period and*

$$F_{\mathrm{ext}}(p \otimes \mathbf{1}) = F(p). \tag{36}$$

Since F_{ext} preserve centralizers, there exists $G : \mathcal{P}_2 \longrightarrow \mathcal{P}_3$ with

$$F_{\mathrm{ext}}(p \otimes q) = F(p)G(q), \quad G[\mathcal{P}_2] = \mathcal{Z}_{\mathcal{P}_3}(F[\mathcal{P}_1]). \tag{37}$$

Proof — In this proof we need to deal with 'regions' that are collections of qubits, rather than of complete sites. So let $\Sigma(\alpha)$ denote all qubits with label α and $\mathcal{P}_i^\alpha := \mathcal{P}_i|_{\Sigma(\alpha)}$. We can assume w.l.o.g. $d = 1$. To avoid trivial cases we assume $1 \leq n_1 < n_3$, and set $n_2 = n_3 - n_1$. If we can construct a lattice Pauli isomorphism $F' : \mathcal{P}_3 \longrightarrow \mathcal{P}_1 \otimes \mathcal{P}_2$ with period 1 and such that $F'(F(\cdot)) = F''(\cdot) \otimes \mathbf{1}$ for some isomorphism $F'' : \mathcal{P}_1 \longrightarrow \mathcal{P}_1$, the result follows taking $F_{\mathrm{ext}} = F'^{-1} \circ (F'' \otimes \mathbf{I}_{\mathcal{P}_2})$. We construct F' as a composite $F' = F_b \circ F_a$. In turn, $F_a : \mathcal{P}_3 \longrightarrow \mathcal{P}_3$ is composed of two kinds of maps. The first kind is that of single qubit Clifford gates, preserving translational invariance. The second corresponds to CNot gates applied to pairs of qubits with different label α —but possibly in different sites— again preserving translational invariance. Composing such 'moves', it is a routine computation —we are essentially back to a finite Pauli group case, with the restriction of not being able to perform CNot gates between qubits with the same label— to construct an isomorphism

F_1 such that $F_1'[\mathcal{P}_1^1|_{\mathbf{0}}] \subset \mathcal{P}_3^1$ for $F_1' := F_1 \circ F$. Then we have $F_1'[\mathcal{P}_1^1] = \mathcal{P}_3^1$ —translational invariance only implies inclusion, but the restriction of F_1' to $\mathcal{P}_1^1$ and $\mathcal{P}_3^1$ is actually an isomorphism, since $\mathcal{P}_1^1$, $\mathcal{P}_3^1$ can be regarded as lattices with 1 qubit per site. Now let $u \in F_1'[\mathcal{P}_1^1]$, $v \in F_1'[\mathcal{P}_1^\alpha]$, $1 < \alpha \leq n_1$. Then $1 = (u; v) = (u; v|_{\Sigma(1)})$ and thus $v|_{\Sigma(1)} \subset \mathcal{Z}_{\mathcal{P}_3^1}(\mathcal{P}_3^1) \propto \langle i\mathbf{1}\rangle$. The idea is then to repeat the procedure and successively construct for $i = 2, \ldots, n_1$, isomorphisms F_i from F_{i-1} without using any elementary moves involving qubits with labels $\alpha < i$, so that $(F_i \circ F)[\mathcal{P}_1^\beta] = \mathcal{P}_3^\beta$ for $\beta \leq i$. It suffices then to set $F_a = F_{n_1}$ and let $F_b : \mathcal{P}_3 \longrightarrow \mathcal{P}_1 \otimes \mathcal{P}_2$ be the isomorphism such that $F_b(\sigma_{i,j}^\alpha \sigma_{i,j}^{n_1+\beta}) = \sigma_{i,j}^\alpha \otimes \sigma_{i,j}^\beta$, $\alpha = 1, \ldots, n_1$, $\beta = 1, \ldots, n_2$. ■

3.3 Lattice Pauli groups

Definition 3 *Let S be a set such that $\mathrm{Supp}[S]$ and $T_{m,n}[S]$ are defined. S is local and translationally invariant (LTI) if there exist $k, l \in \mathbf{N}^*$ such that for any $s \in S$, $m, n \in \mathbf{Z}$,*

$$\rho(s) \leq k, \qquad T_{ml,nl}(s) \in S. \tag{38}$$

In that case we say that S is k-bounded and has period l.

Definition 4 *A subgroup $\mathcal{A} \subset \mathcal{P}$ is a lattice Pauli group (LPG) if it admits a LTI set of generators $\mathcal{A}_g \subset \mathcal{A}$.*

When we say that $\mathcal{A}_g$ generates a LPG $\mathcal{A}$ it is understood that $\mathcal{A}_g$ is a LTI set. The following result is an immediate consequence of the definition of LPGs.

Proposition 5 *Consider a set of LPGs $\bar{\mathcal{A}}^1, \ldots, \bar{\mathcal{A}}^n$ on a given lattice. There exists $L \in \mathbf{N}^*$ such that the coarse grained LPGs*

$$\mathcal{A}^k = \mathrm{Crs}_L(\bar{\mathcal{A}}^k) \subset \mathcal{P}, \qquad k = 1, \ldots, n, \tag{39}$$

admit LTI sets of independent generators $\mathcal{A}_g^k$ such that

1. *$\mathcal{A}_g^i$ has period 1, and*
2. *$\rho(a) \leq 2$ for any $a \in \mathcal{A}_g^i$.*

We will assume that all the LPGs that we discuss satisfy these properties, unless otherwise stated.

Definition 6 *Given two LPGs $\mathcal{A}, \mathcal{B}$ defined on disjoint qubit lattices, their composition is the LPG $\mathcal{A} \otimes \mathcal{B}$ defined on the union of the two lattices.*

$\mathcal{A} \otimes \mathcal{B}$ is indeed an LPG because $\mathcal{A}_g \cup \mathcal{B}_g$ is a valid LTI generating set.

Definition 7 *A LPG (iso)morphism from $\mathcal{A}_1 \subset \mathcal{P}_1$ to $\mathcal{A}_2 \subset \mathcal{P}_2$, denoted $\mathcal{A}_1 \xrightarrow{F} \mathcal{A}_2$, is a lattice pauli (iso)morphism $F : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ with $F[\mathcal{A}_1] = \mathcal{A}_2$.*

Two LPGs $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic if there exists such an isomorphism F. Given a LPG morphism $\mathcal{A}_1 \xrightarrow{F} \mathcal{A}_2$ we can coarse grain both lattices, that of $\mathcal{A}$ and that of $\mathcal{B}$, and obtain a morphism

$$\mathrm{Crs}_l(\mathcal{A}) \xrightarrow{\mathrm{Crs}_l(F)} \mathrm{Crs}_l(\mathcal{B}), \tag{40}$$

where $l \in \mathbf{N}^*$. Given two LPG morphisms $\mathcal{A}_1 \xrightarrow{F} \mathcal{A}_2, \mathcal{B}_1 \xrightarrow{G} \mathcal{B}_2$, if the corresponding lattices are disjoint we form in the obvious way the morphism

$$\mathcal{A}_1 \otimes \mathcal{B}_1 \xrightarrow{F \otimes G} \mathcal{A}_2 \otimes \mathcal{B}_2. \tag{41}$$

3.4 Topological stabilizer groups

Definition 8 *A lattice stabilizer group is a LPG $\mathcal{S} \subset \mathcal{P}$ such that $-1 \notin \mathcal{S}$.*

Definition 9 *A topological stabilizer group (TSG) is a lattice stabilizer group $\mathcal{S} \subset \mathcal{P}$ such that $\mathcal{Z}_{\mathcal{P}}(\mathcal{S}) \propto \mathcal{S}$.*

Definition 10 *A topological stabilizer subsystem group (TSSG) is a lattice stabilizer group $\mathcal{S} \subset \mathcal{P}$ such that $\mathcal{Z}_{\mathcal{P}}(\mathcal{Z}_{\mathcal{P}}(\mathcal{S})) \propto \mathcal{S}$. Its gauge group is $\mathcal{G} := \mathcal{Z}_{\mathcal{P}}(\mathcal{S})$.*

TSGs are TSSGs. As we will show in corollary 17, $\mathcal{G}$ is a LPG. The product $\mathcal{S}_1 \otimes \mathcal{S}_2$ of two TSSGs is also a TSSG. A LPG isomorphism F from $\mathcal{S}_1$ to $\mathcal{S}_2$ is also a LPG isomorphism from $\mathcal{G}_1$ to $\mathcal{G}_2$.

3.5 Examples

Some examples of TSSGs follow. Among them, two subsystem variants of the toric code that we will find extremely useful. All the generator sets below are independent. Lattices are displayed in Fig. 7.

Empty code — The simplest TSG, with no qubits.

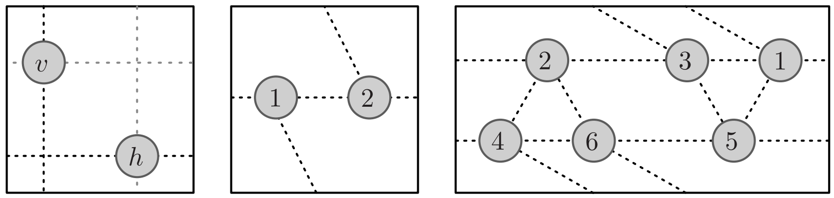
Trivial code — This TSG is constructed on a lattice with one qubit per site, with $\mathcal{P}_{\mathrm{T}}$ denoting the pauli group. The stabilizer $\mathcal{S}_{\mathrm{T}}$ has generators

$$\mathcal{S}_g^{\mathrm{T}} := T^{(1)}\left(\{Z_{0,0}\}\right). \tag{42}$$

Subsystem trivial code — This TSSG is constructed on the same lattice as the trivial code. The stabilizer is trivial, $\mathcal{S}_{\mathrm{S}} = \{\mathbf{1}\}$ and $\mathcal{G}_{\mathrm{S}} = \mathcal{P}_{\mathrm{T}}$.

Toric code — This TSG [7] is constructed on a lattice with two qubits per site, labeled as h and v. We denote by $\mathcal{P}_{\mathrm{TC}}$ the corresponding pauli group. The stabilizer $\mathcal{S}_{\mathrm{TC}}$ has generators

$$\begin{aligned} \mathcal{S}_g^{\mathrm{TC}} &:= T^{(1)}\left(\{S^X, S^Z\}\right), \qquad S^Z := Z^h_{0,0} Z^h_{0,1} Z^v_{0,0} Z^v_{1,0}, \\ S^X &:= X^h_{0,0} X^h_{-1,0} X^v_{0,0} X^v_{0,-1}. \end{aligned} \tag{43}$$



(a) (b) (c)

Figure 7: Unit cells for the TPGs in the examples. (a) In the toric code, qubits can be identified with horizontal and vertical links in a square lattice, or with vertical and horizontal links in its dual. Stabilizers are related to faces and dual faces. (b) In the honeycomb subsystem code qubits can be identified with the vertices of a honeycomb lattice. Gauge generators are related to links. (c) In the topological subsystem color code qubits can be identified with the vertices of a lattice derived from the honeycomb. The numbering is obtained by moving counterclockwise along any hexagonal plaquette. Gauge generators are related to links.

Subsystem toric code — This TSSG is constructed on the same lattice as the toric code. The stabilizer $\mathcal{S}_{\mathrm{STC}}$ and the gauge group $\mathcal{G}_{\mathrm{STC}}$ have generators

$$\mathcal{S}_g^{\mathrm{STC}} := T^{(1)}\left(\{S^Z\}\right), \qquad \mathcal{G}_g^{\mathrm{STC}} := \langle i\mathbf{1}\rangle T^{(1)}\left(\{S^X, Z^h_{0,0}, Z^v_{0,0}\}\right). \tag{44}$$

Fermionic subsystem toric code — This TSSG is constructed on the same lattice as the toric code. $\mathcal{S}_{\mathrm{FTC}}$ and $\mathcal{G}_{\mathrm{FTC}}$ have generators

$$\begin{aligned} \mathcal{S}_g^{\mathrm{FTC}} &:= T^{(1)}\left(\{S^Z_f\}\right), & \mathcal{G}_g^{\mathrm{FTC}} &:= \langle i\mathbf{1}\rangle T^{(1)}\left(\{S^X, Z^h_f, Z^v_f\}\right), \\ Z^h_f &:= Z^h_{0,0}X^h_{-1,0}X^v_{0,-1}X^v_{1,-1}, & S^Z_f &:= Z^h_{0,0}Z^h_{0,1}Y^v_{0,0}Y^v_{1,0}X^h_{1,0}X^h_{-1,1}, \\ Z^v_f &:= Z^v_{0,0}X^h_{-1,0}X^v_{0,-1}X^h_{0,0}. \end{aligned} \tag{45}$$

Honeycomb subsystem code — This TSSG [13] is constructed on a lattice with two qubits per site. $\mathcal{S}_{\hexagon}$ and $\mathcal{G}_{\hexagon}$ have generators

$$\begin{aligned} \mathcal{S}_g^{\hexagon} &:= T^{(1)}\left(\{S^{\hexagon}\}\right), & \mathcal{G}_g^{\hexagon} &:= \langle i\mathbf{1}\rangle T^{(1)}\left(\{G^X_{0,0}, G^Y_{0,0}, G^Z_{0,0}\}\right), \\ G^X_{i,j} &:= X^1_{i,j}X^2_{i,j}, & S^{\hexagon} &:= X^2_{0,0}Z^1_{1,0}Y^2_{1,0}X^1_{1,1}Y^1_{0,1}Z^2_{0,1}, \\ G^Y_{i,j} &:= Y^1_{i+1,j}Y^2_{i,j}, & G^Z_{i,j} &:= Z^1_{i,j+1}Z^2_{i,j}. \end{aligned} \tag{46}$$

Topological subsystem color code — This TSSG [11] is constructed on a lattice with six qubits per site, labeled 1-6. $\mathcal{S}_{\mathrm{col}}$ and $\mathcal{G}_{\mathrm{col}}$ have generators

$$\begin{aligned}
&\mathcal{S}_g^{\mathrm{col}} := T^{(1)}\left(\{S_{0,0}^1, S_{0,0}^2\}\right), \qquad \mathcal{G}_g^{\mathrm{col}} := \langle i\mathbf{1}\rangle T^{(1)}\left(\{G_{0,0}^1, G_{0,0}^2, \ldots, G_{0,0}^{10}\}\right),\\
&S_{i,j}^2 := X_{i,j}^1 Y_{i+1,j}^2 X_{i+1,j}^3 Y_{i+1,j+1}^4 X_{i,j+1}^5 Y_{i,j+1}^6 Y_{i,j+1}^1 Y_{i,j+1}^2 Y_{i,j}^3 \cdot\\
&\quad \cdot Y_{i+1,j}^4 Y_{i+1,j}^5 Y_{i+1,j+1}^6 X_{i+1,j}^1 X_{i+1,j+1}^2 X_{i,j+1}^3 X_{i,j+1}^4 X_{i,j}^5 X_{i+1,j}^6,\\
&S_{i,j}^1 := Z_{i,j}^1 Z_{i+1,j}^2 Z_{i+1,j}^3 Z_{i+1,j+1}^4 Z_{i,j+1}^5 Z_{i,j+1}^6, \qquad G_{i,j}^{10} := Z_{i,j}^2 Z_{i,j}^6,
\end{aligned}$$

$$\begin{aligned}
G_{i,j}^1 &:= X_{i,j}^1 Y_{i+1,j}^2, & G_{i,j}^2 &:= X_{i+1,j}^2 Y_{i+1,j}^3, & G_{i,j}^3 &:= X_{i+1,j}^3 Y_{i+1,j+1}^4,\\
G_{i,j}^4 &:= X_{i+1,j+1}^4 Y_{i,j+1}^5, & G_{i,j}^5 &:= X_{i,j+1}^5 Y_{i,j+1}^6, & G_{i,j}^6 &:= X_{i,j+1}^6 Y_{i,j}^1,\\
G_{i,j}^7 &:= Z_{i,j}^1 Z_{i,j}^5, & G_{i,j}^8 &:= Z_{i,j}^3 Z_{i,j}^5, & G_{i,j}^9 &:= Z_{i,j}^2 Z_{i,j}^4. \qquad (47)
\end{aligned}$$

3.6 Local equivalence

Two TSSGs are locally equivalent when they are the same up to local transformations, i.e., coarse graining, LPG isomorphisms and composition with a trivial TSG $\mathcal{S}_{\mathrm{T}}$ or a trivial TSSG $\mathcal{S}_{\mathrm{S}}$.

Definition 11 *The TSSGs $\mathcal{S}_1$, $\mathcal{S}_2$ are locally equivalent if there exists a LPG isomorphism $F : \mathcal{P}_1 \otimes \mathcal{P}_{\mathrm{T}}^{k_1+m_1} \longrightarrow \mathcal{P}_2 \otimes \mathcal{P}_{\mathrm{T}}^{k_2+m_2}$*

$$\mathrm{Crs}_{l_1}(\mathcal{S}_1) \otimes \mathcal{S}_{\mathrm{T}}^{\otimes k_1} \otimes 1 \xrightarrow{F} \mathrm{Crs}_{l_2}(\mathcal{S}_2) \otimes \mathcal{S}_{\mathrm{T}}^{\otimes k_2} \otimes 1, \qquad (48)$$

where $k_1, k_2, m_1, m_2 \in \mathbf{N}_0$ and $l_1, l_2 \in \mathbf{N}^*$.

For convenience, call this a $(k_1, k_2, m_1, m_2, l_1, l_2)$-equivalence.

Definition 12 *The TSGs $\mathcal{S}_1$ and $\mathcal{S}_2$ are locally equivalent if they are $(k_1, k_2, 0, 0, l_1, l_2)$-equivalent as TSSGs.*

Two TSGs that are equivalent as TSSGs are also equivalent as TSGs. Indeed, if F gives the TSSG equivalence it maps gauge elements not in the stabilizer to gauge elements not in the stabilizer. I.e., denoting by $\mathcal{P}_{\mathrm{T}}^{\otimes m_i}$ the copies of $\mathcal{P}_{\mathrm{T}}$ holding $\mathcal{S}_{\mathrm{S}}^{\otimes m_i}$, we have $F[\mathcal{P}_{\mathrm{T}}^{\otimes m_1}] = \mathcal{P}_{\mathrm{T}}^{\otimes m_2}$. But centralizers are mapped to centralizers, and the centralizer of $\mathcal{P}_{\mathrm{T}}^{\otimes m_i}$ is $\mathcal{P}_i \otimes \mathcal{P}_{\mathrm{T}}^{\otimes k_i}$, so that a restriction of F will give the equivalence.

We still have to check transitivity. But given a $(k_1, k_2, m_1, m_2, l_1, l_2)$-equivalence F of $\mathcal{S}_1$ and $\mathcal{S}_2$, and a $(k_2, k_3, m_2, m_3, l_2, l_3)$-equivalence G of $\mathcal{S}_2$ and $\mathcal{S}_3$ we can define an isomorphism $G \circ F$ from $\mathcal{S}_1$ to $\mathcal{S}_3$

$$G \circ F := (\mathrm{Crs}_{l_2}(G) \otimes 1_{\mathrm{T}}^{\otimes t_2}) \circ S \circ (\mathrm{Crs}_{l_2'}(F) \otimes 1_{\mathrm{T}}^{\otimes t_1}) \qquad (49)$$

where $t_1 = k_2' l_2 + m_2' l_2$, $t_2 = k_2 l_2' + m_2 l_2'$, 1_{T} is the identity morphism $\mathcal{P}_{\mathrm{T}} \xrightarrow{1_{\mathrm{T}}} \mathcal{P}_{\mathrm{T}}$ and S swaps the last t_2 qubit labels with the previous t_1 qubit labels involved in the trivial codes. This is a $(k_1', k_3', m_1', m_3', l_1', l_3')$-equivalence with

$$\begin{aligned}
k_1' &:= k_1 l_2' + k_2' l_2, & l_1' &:= l_1 l_2', & m_1' &:= m_1 l_2' + m_2' l_2,\\
k_3' &:= k_3 l_2 + k_2 l_2', & l_3' &:= l_3 l_2. & m_3' &:= m_3 l_2 + m_2 l_2', \qquad (50)
\end{aligned}$$

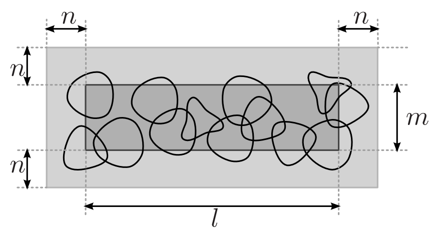


Figure 8: Illustration of proposition 14. γ_l appears in darker grey. The closed shapes represents the support of elements of $\mathrm{Neg}_{\mathcal{B}_g}(\phi)$. The support of S is contained in the larger rectangle.

4 Constraints and independent generators

This section shows how a LTI set of independent generators can be constructed for any LPG that is the centralizer of a LPG. $\mathcal{A} = \langle \mathcal{A}_g \rangle$, $\mathcal{B} = \langle \mathcal{B}_g \rangle$ are LPGs on a given 2D lattice. $\mathcal{P}_g$ denotes the set of single qubit X, Z Pauli operators.

4.1 Constraints

Definition 13 *The groups of $\mathcal{B}$-constraints of $\mathcal{A}_g$ are*

$$\begin{aligned} \mathrm{Cnstr}_{\mathcal{B}}(\mathcal{A}_g) &:= \{S \subset \mathcal{A}_g - \langle i\mathbf{1} \rangle \,:\, \mathrm{Com}_{\mathcal{B}}(S) = 1\}, \\ \mathrm{Cnstr}^0_{\mathcal{B}}(\mathcal{A}_g) &:= \mathbf{P}_0(\mathcal{A}_g) \cap \mathrm{Cnstr}_{\mathcal{B}}(\mathcal{A}_g) \end{aligned} \tag{51}$$

When $\mathcal{B} = \mathcal{P}$ we drop the label. These constraints give us a measure of how non-independent the elements of $\mathcal{A}_g$ are with respect to their action in $\mathcal{B}$. $\mathcal{A}_g$ is independent iff $\mathrm{Cnstr}^0(\mathcal{A}_g) = \{\emptyset\}$.

The following proposition, illustrated in Fig. 8, states that given $a \in \mathcal{A}$ such that it commutes with all generators of $\mathcal{B}$ outside of a block of a fixed width but arbitrary length, then there exist another $a' \in \mathcal{A}$ that anticommutes with the same elements of $\mathcal{B}$ as a does but such that it is the product of a set of generators of $\mathcal{A}$ inside a slightly larger block.

Proposition 14 *For any $m \in \mathbf{N}^*$ there exists $n \in \mathbf{N}_0$ such that for every $a \in \mathcal{A}$ and $l \in \mathbf{N}_0$ with*

$$\mathrm{Neg}_{\mathcal{B}_g}(a) \subset \mathcal{B}_g||_{\gamma_l}, \qquad \gamma_l := \square^{l,m}_{0,0}, \tag{52}$$

there exists $S \in \mathbf{P}_0(\mathcal{A}_g)$ with

$$\mathrm{Com}_{\mathcal{B}}(S) = \mathrm{Com}_{\mathcal{B}}(a), \qquad \mathrm{Supp}(S) \subset \mathrm{Thk}^n(\gamma_l). \tag{53}$$

We denote by $N(\mathcal{A}_g, \mathcal{B}_g, m) \in \mathbf{N}_0$ the smallest of such n.

Proof — Let $\mathcal{B}_1 := \langle \mathcal{B}_g||_{\gamma_1} \rangle$. For $\phi \in \Phi(\mathcal{B}_1)$ let

$$A_\phi := \{a \in \mathcal{A} \,:\, \mathrm{Neg}_{\mathcal{B}_g}(a) \subset \mathcal{B}_g||_{\gamma_\infty}, \mathrm{Com}_{\mathcal{B}_1}(a) = \phi\} \tag{54}$$

and $\Phi_1 := \{\phi \in \Phi_{\mathcal{B}_1} \,:\, A_\phi \neq \emptyset\}$. Φ_1 is finite because $\mathcal{B}_1$ is finite. For each $\phi \in \Phi_1$ we choose $a_\phi \in A_\phi$ that satisfies $\mathrm{Neg}_{\mathcal{B}_g}(a_\phi) \subset \mathcal{B}_g||_{\gamma_l}$ for a minimal l and let l_ϕ be this minimum value. We also choose $S_\phi \in \mathbf{P}_0(\mathcal{A}_g)$ with $\mathrm{Com}_{\mathcal{B}}(a_\phi) = \mathrm{Com}_{\mathcal{B}}(S_\phi)$. Finally, we choose $n \in \mathbf{N}_0$ so that $\mathrm{Supp}(S_\phi) \subset \mathrm{Thk}^n(\gamma_{l_\phi})$ for any $\phi \in \Phi_1$.

We now prove the proposition by induction on l. Let $\phi = \mathrm{Com}_{\mathcal{B}_1}(a)$, so that $a \in A_\phi$. If $l = 1$, $S = S_\phi$ satisfies the required properties because $\mathrm{Neg}_{\mathcal{B}_g}(a_\phi) = \mathrm{Neg}_{\mathcal{B}_1}(a_\phi) = \mathrm{Neg}_{\mathcal{B}_1}(a) = \mathrm{Neg}_{\mathcal{B}_g}(a)$. If $l > 1$, let $a' = T_{-1,0}(aa_\phi)$. Since $l_\phi \leq l$, $\mathrm{Neg}_{\mathcal{B}_g}(a') \subset \mathcal{B}_g||_{\gamma_{l-1}}$ and by induction there exists $S' \in \mathbf{P}_0(\mathcal{A}_g)$ with $\mathrm{Com}_{\mathcal{B}}(S') = \mathrm{Com}_{\mathcal{B}}(a')$ and $\mathrm{Supp}(S') \subset \mathrm{Thk}^n(\gamma_{l-1})$. Then $S = S_\phi T_{1,0}(S')$ satisfies the required properties. ■

It is worth rewriting this in the special case $\mathcal{B} = \mathcal{P}$.

Corolary 15 *For any $m \in \mathbf{N}^*$ there exists $n \in \mathbf{N}_0$ such that for every $a \in \mathcal{A}$ and $l \in \mathbf{N}_0$ with*

$$\mathrm{Supp}(a) \subset \gamma_l := \square_{0,0}^{m,l}, \tag{55}$$

there exists $S \in \mathbf{P}_0(\mathcal{A}_g)$ with $a \in \mathrm{Pro}(S)$ and

$$\mathrm{Supp}(S) \subset \mathrm{Thk}^n(\gamma_l). \tag{56}$$

We denote by $N(\mathcal{A}_g, m) \in \mathbf{N}_0$ the smallest of such n.

Lemma 16 $\mathrm{Cnstr}^0_{\mathcal{B}}(\mathcal{A}_g)$ *admits a LTI set of generators.*

Proof — Let $n_1 = N(\mathcal{A}_g, \mathcal{B}_g, 1)$ according to proposition 14 and $n_2 = N(\mathcal{A}_g, \mathcal{B}_g, 1)$ according to the axis-exchanged version of proposition 14. Set $N = \max(n_1, n_2)$. We claim that

$$G := \{S \in \mathrm{Cnstr}^0_{\mathcal{B}}(\mathcal{A}_g) \,:\, \rho(S) \leq L\}, \tag{57}$$

where $L := 8N + 2$, generates $\mathrm{Cnstr}^0_{\mathcal{B}}(\mathcal{A}_g)$.

Take any $S \in \mathrm{Cnstr}^0_{\mathcal{B}}(\mathcal{A}_g)$. If $\rho(S) \leq L$ then $S \in \langle G \rangle$. In other case, without lost of generality we assume $\mathrm{Supp}(R) \subset \square_{0,0}^{a,b}$ with $a > b$ and $a > L$. Let $l = 4N + 1$ and set $S = S_1 + S_2$ with $S_1 = S|_{\square_{0,0}^{l,b}}$. Then $\mathrm{Supp}(S_2) \subset \square_{l-1,0}^{a,b}$, $\mathrm{Com}_{\mathcal{B}}(S_1) = \mathrm{Com}_{\mathcal{B}}(S_2) =: \phi \in \mathrm{Com}_{\mathcal{B}}[\mathcal{A}]$, and

$$\mathrm{Supp}_{\mathcal{B}_g}(\phi) \subset \mathrm{Supp}_{\mathcal{B}_g}(S_1) \cap \mathrm{Supp}_{\mathcal{B}_g}(S_2) \subset \mathcal{B}_g||_{\square_{l-1,0}^{l,b}}. \tag{58}$$

Due to proposition **??** (up to a translation), there exists $S_3 \in \mathbf{P}_0(\mathcal{A}_g)$ with $\phi = \mathrm{Com}_{\mathcal{B}}(S_3)$ and $\mathrm{Supp}(S_3) \subset \mathrm{Thk}^N(\square_{l-1,0}^{l,b})$. Then $S = S_1' + S_2'$, where $S_1' = S_1 + S_3$ and $S_2' = S_2 + S_3$ are elements of $\mathrm{Cnstr}^0_{\mathcal{B}}(\mathcal{A}_g)$. But $S_i' \in \square_{u_i+a_i, v_i+b_i}^{a_i,b_i}$, $i = 1, 2$, with $a_1 = l + N$, $b_1 = b + 2N$, $a_2 = a + 1 + N$ and $b_2 = b + 2N$. Thus $a_i + b_i < a + b$ and by induction on $a + b$ it follows that $R \in \langle G \rangle$. ■

Since $\mathcal{Z}_{\mathcal{A}}(\mathcal{B}) = \mathcal{A} \cap \bigcup \mathrm{Pro}[\mathrm{Cnstr}^0_{\mathcal{B}}(\mathcal{A}_g)]$, we have:

Corolary 17 *$\mathcal{Z}_{\mathcal{A}}(\mathcal{B})$ is a LPG.*

When $a \in \mathcal{Z}_{\mathcal{P}}(\mathcal{B})$ does not have support on the boundary of a region, the restriction of a to this region still belongs to the centralizer:

Proposition 18 *Let $\mathcal{A} = \mathcal{Z}_{\mathcal{P}}(\mathcal{B})$. For any $a \in \mathcal{A}$ and $\gamma \in \mathbf{P}_0(\Sigma)$*

$$\mathrm{Supp}(a) \cap (\mathrm{Thk}^1(\gamma) - \gamma) = \emptyset \quad \Longrightarrow \quad a|_\gamma \in \langle i\mathbf{1}\rangle\mathcal{A}. \tag{59}$$

Proof — Since $\mathcal{B}_g$ is 2-bounded, for any $b \in \mathcal{B}_g$ with $\mathrm{Supp}(b) \cap \gamma \neq \emptyset$ we have $\mathrm{Supp}(b) \cap (\mathrm{Supp}(a) - \gamma) = \emptyset$ and thus $(b; a|_\gamma) = (b; a) = 1$. ■

Lemma 19 *Let $\mathcal{A} = \mathcal{Z}_{\mathcal{P}}(\mathcal{B})$ and $\mathcal{A}_g$ be independent.* $\mathrm{Cnstr}(\mathcal{A}_g)$ *is finite.*

Proof — Let $n_1 := N(\mathcal{A}_g, 2)$ according to corollary 15 and $n_2 := N(\mathcal{A}_g, 2)$ according to the axis-exchanged version of corollary 15. Set $N = \max(n_1, n_2)$ and $D = 2N + 2$. First, we want to show that for any $R \in \mathrm{Cnstr}(\mathcal{A}_g)$ we have

$$R||_{\square_{0,0}^{2,D}} = \emptyset \quad \Longrightarrow \quad R||_{\square_{2,N}^{\infty,D-N}} = \emptyset. \tag{60}$$

We start observing that for any $l \in \mathbf{N}^* - \{1\}$, there exist $k_1, k_2 \in \mathbf{N}^*$ with $k_2 - 2 > k_1 > l$ and such that

$$(T_{k_2-k_1,0}\,(R) + R)\,||_{\square_{k_2,0}^{k_2+2,D}} = \emptyset, \tag{61}$$

because $\{S \subset \mathcal{A}_g \;:\; S = S||_{\square_{0,0}^{2,D}}\}$ is finite. Let $R_0 := R||_{\square_{2,0}^{\infty,D}}$, $R_1 := T_{k_2-k_1,0}\,(R_0) + R_0$. Clearly we have $R_1||_{\square_{0,0}^{2,D} \cup \square_{k_2,0}^{k_2+2,D}} = \emptyset$. Then $R_2 := R_1||_{\square_{2,0}^{k_2,D}}$ satisfies $\mathrm{Supp}(\mathrm{Pro}(R_2)) \subset \gamma_1 \cap \gamma_2$ for $\gamma_1 := \square_{0,-2}^{k_2+2,0}$, $\gamma_2 := \square_{0,D}^{k_2+2,D+2}$. According to proposition 18 there exist $a_i \in \mathcal{A}$, $i = 1, 2$, such that $a_1 a_2 \in \mathrm{Pro}(R_2)$. Applying corollary 15, there exist $S_i \in \mathbf{P}_0(\mathcal{A}_g)$, $i = 1, 2$ with $a_i \in \mathrm{Pro}(S_i)$ and $\mathrm{Supp}(S_i) \subset \mathrm{Thk}^N(\gamma_i)$. Then $R_3 = R_2 + S_1 + S_2 \in \mathrm{Cnstr}^0(\mathcal{A}_g) = \{\emptyset\}$ and $R_1||_{\square_{2,N}^{l,D-N}} = R_3||_{\square_{2,N}^{l,D-N}} = \emptyset$. This can only be true if $R||_{\square_{2,N}^{l,D-N}} = \emptyset$.

We next show that for any $R_1, R_2 \in \mathrm{Cnstr}(\mathcal{A}_g)$ with $R_1||_{\square_{0,0}^{2,D}} = R_2||_{\square_{0,0}^{2,D}}$, we have $R_1 = R_2$, which implies the finiteness of $\mathrm{Cnstr}(\mathcal{A}_g)$. Let $R = R_1 + R_2$. Then according to the above result $R||_{\square_{2,N}^{\infty,D-N}} = \emptyset$. Moreover, we can apply the result with the first coordinate inverted and a suitable translation, getting $R||_{\square_{-\infty,N}^{\infty,D-N}} = \emptyset$. But now for any $j \in \mathbf{Z}$ we have $R||_{\square_{j,N}^{j+D,2+N}} = \emptyset$. Applying the same reasoning with the coordinates exchanged gives $R||_{\square_{j+N,-\infty}^{j+D-N,\infty}} = \emptyset$ for any $j \in \mathbf{Z}$. Thus $R_1 + R_2 = \emptyset$. ■

4.2 Independent generators

We need a couple of lemmas as a preparation for the main result of this section. The first states that given a constraint R without support on the 'sides' of a 'corridor', as long as this sides are wide enough and the corridor long enough, there will be another constraint S that coincides with R in the centre of the corridor but that has support only on an infinite version of the corridor, as illustrated in Fig. 9.

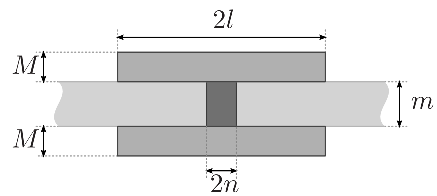


Figure 9: Illustration of proposition 20. R has no support on the region γ, in medium grey. The support of S is contained in the light grey area, which extends to infinity on both sides. S and R coincide in the region γ_{-n}^{n}, in dark grey.

Proposition 20 *Let $\mathcal{A}_g$ be M-bounded. For any $m, n \in \mathbf{N}_0$ there exists $l \in \mathbf{N}_0$ such that the following holds. For any $R \in \mathrm{Cnstr}(\mathcal{A}_g)$ satisfying*

$$R||_\gamma = \emptyset, \qquad \gamma := \square_{-l,-M}^{l,0} \cup \square_{-l,m}^{l,m+M}, \tag{62}$$

there exists $S \in \mathrm{Cnstr}(\mathcal{A}_g)$ such that

$$S \subset \mathcal{A}_g|_{\gamma_{-\infty}^{\infty}}, \quad (S+R)||_{\gamma_{-n}^{n}} = \emptyset, \quad \gamma_a^b := \square_{a,0}^{b,m}. \tag{63}$$

We denote by $L(\mathcal{A}_g, m, n)$ the minimal such l.

Proof — For $a < b$, we set $\gamma_b^a := \gamma_a^b$. Let $l \in \mathbf{N}_0$ be such that for any $R \in \mathrm{Cnstr}(\mathcal{A}_g)$ satisfying (62) there exist $m_x, n_x \in \mathbf{N}_0$, $x = \pm$, with $n \leq xm_x < xm_x + M \leq xn_x \leq r - M$ and such that

$$(T_{d_x,0}(R) + R)||_{\gamma_{n_x}^{n_x+xM}} = \emptyset, \qquad d_x := n_x - m_x. \tag{64}$$

There exists such l because the set $\mathcal{A}_g||_{\gamma_0^M}$ is finite and $\mathcal{A}_g||_{\gamma_t^M} = T_{t,0}(\mathcal{A}_g||_{\gamma_0^M})$. We claim that this choice for l satisfies the statement in the lemma. To check it, given R, m_x, n_x as above let

$$\begin{aligned} R_x &:= R||_{\gamma_{m_x}^{n_x}} - R||_{\gamma_{m_x-xM}^{m_x}}, \quad R_x^\infty := \bigcup_{a\in\mathbf{N}^*} T_{ad_x,0}(R_x), \\ S &:= R||_{\gamma_{n_-}^{n_+}} \cup R_-^\infty \cup R_+^\infty. \end{aligned} \tag{65}$$

Due to (62), $R||_{\gamma_{n_-}^{n_+}} \subset R|_{\mathrm{Thk}^M(\gamma_{n_-}^{n_+})} \subset R|_{\gamma\cup\gamma_{-r}^{r}} \subset \mathcal{A}_g|_{\gamma_{-\infty}^{\infty}}$ and thus $S \subset \mathcal{A}_g|_{\gamma_{-\infty}^{\infty}}$. The definition of S ensures that

$$(S+R)||_{\gamma_{n_--M}^{n_++M}} = \emptyset, \tag{66}$$

and thus in particular $(S+R)||_{\gamma_{-n}^{n}} = \emptyset$. The definition is also such that for any $a \in \mathbf{N}^*$, $x = \pm$,

$$(T_{ad_x,0}(S) + S)||_{\gamma_{m_x+ad_x}^{\infty}} = \emptyset \tag{67}$$

To complete the proof, it suffices to show that for any $p \in \mathcal{P}_g$ we have $(S;p) = 1$. Let $\mathrm{Supp}(p) \subset \square_{t,-\infty}^{t+1,\infty}$ for some $t \in \mathbf{N}_0$. W.l.o.g. we set $t \geq 0$. If $t \leq n_+$, $(S;p) = (R;p) = 1$ according to (66). In other case $t = m_+ + ad_+ + t'$ for some $a \in \mathbf{N}^*$, $0 \leq t' < d_+$. Using (67) we get $(S;p) = (T_{ad_+,0}(S);p) = (S;T_{-ad_+,0}(p)) = 1$. ■

Next we give a procedure to remove dependent generators of $\mathcal{A}_g$ from selected regions preserving translational invariance. The constraints must have generators that do not have support on more than one of these regions at a time.

Proposition 21 *Let $\mathcal{A}_g$ have period d and $\mathrm{Cnstr}^0(\mathcal{A}_g) = \langle R_0 \rangle$ for some LTI set $R_0 \subset \mathbf{P}_0(\mathcal{A}_g)$. Let $\gamma \in \mathbf{P}_0(\Sigma)$ be such that for any $S \in R_0$ there is at most a pair $(m,n) \in \mathbf{Z}^2$ such that*

$$S||_{T_{md,nd}(\gamma)} \neq \emptyset. \tag{68}$$

Then there exists $A \subset \mathcal{A}_g||_\gamma$ such that

$$\mathcal{A} = \langle \mathcal{A}'_g \rangle, \qquad \mathcal{A}'_g := \mathcal{A}_g - T^{(d)}(A), \tag{69}$$

and $S||_{T^{(d)}(\gamma)} = \emptyset$ for any constraint $S \in \mathrm{Cnstr}^0(\mathcal{A}'_g)$.

Proof — If $S||_\gamma = \emptyset$ for any constraint $S \in \mathrm{Cnstr}^0(\mathcal{A}_g)$, clearly $A = \emptyset$ suffices. In other case, choose $a \in \mathcal{A}_g||_\gamma$ and $S \in R_0$ such that $a \in S$. We claim that $\mathcal{A} = \langle \mathcal{A}'_g \rangle$ with $\mathcal{A}'_g := \mathcal{A}_g - T^{(d)}(a)$. Due to translational invariance, it suffices to show $a \in \langle \mathcal{A}'_g \rangle$. But $a \in \mathrm{Pro}(S - \{a\})$, and since $T^{(d)}(a) \cap S = \{a\}$ we have $(S - \{a\}) \subset \mathcal{A}'_g$. Let $S_{m,n} := T_{md,nd}(S)$. Given $S' \in R_0$, let $f(S') = S' + S_{m,n}$ if $T_{md,nd}(a) \in S'$, $f(S') = S'$ otherwise. Then the set $R'_0 = f[R_0]$ generates $\mathrm{Cnstr}^0(\mathcal{A}'_0)$. Moreover, $\mathcal{A}'_g$ and R'_0 satisfy the conditions of the lemma, because $\mathrm{Supp}(S + S_{m,n}) \subset \mathrm{Supp}(S) \cup \mathrm{Supp}(S_{m,n})$. Since $|(\mathcal{A}'_g||_\gamma)| = |(\mathcal{A}_g||_\gamma)| - 1$, the result follows by induction on $|(\mathcal{A}_g||_\gamma)|$. ■

Theorem 22 *Let $\mathcal{A} = \mathcal{Z}_\mathcal{P}(\mathcal{B})$. Then $\mathcal{A}$ admits LTI independent generators.*

Proof — We will construct from $\mathcal{A}_g$ a LTI set of independent generators $\mathcal{A}_3$ through a series of intermediate sets $\mathcal{A}_i$ so that $\mathcal{A}_3 \subset \mathcal{A}_2 \subset \mathcal{A}_1 \subset \mathcal{A}_0 := \mathcal{A}_g$. At each step we will apply proposition 21 to a given region γ_i and obtain

$$\mathcal{A}_{i+1} = \mathcal{A}_i - T^{(D)}(A_i) \tag{70}$$

for a suitable $D \in \mathbf{N}^*$, $A_i \subset \mathcal{A}_i||_{\gamma_i}$. Let us define $R_i := \mathrm{Cnstr}^0(\mathcal{A}_i)$. Since we apply proposition 21, we have $S||_{T^{(D)}(\gamma_i)} = \emptyset$ for any constraint $S \in R_j$ if $j > i$. The goal is thus to have $\{\emptyset\} = R_3 \subset R_2 \subset R_1 \subset R_0$. R_g^i will denote a LTI set of generators of R_i of period D.

0) W.l.o.g. we assume that $\mathcal{A}_g^0$ and R_g^0 have period 1 and are 2-bounded. Let $n_1 := N(\mathcal{A}_g, 2)$ according to corollary 15 and $n_2 := N(\mathcal{A}_g, 2)$ according to the axis-exchanged version of corollary 15. Set $N = \max(1, n_1, n_2)$ and $K := 2N+2$.

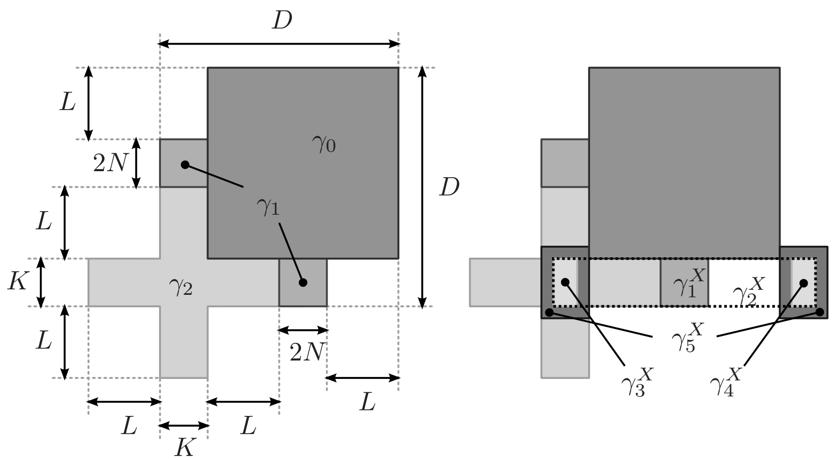


Figure 10: The regions involved in the proof of theorem 22. γ_2^X is outlined with dashed lines.

Let $l_1 = L(\mathcal{A}_g, K, 2)$ according to proposition 20 and $l_2 = L(\mathcal{A}_g, K, 2)$ according to the axis-exchanged version of proposition 20. Set $L := \max(l_1, l_2)$ and $D := K + 2L + 2N$. The geometry of the proof, see Fig. 10, involves the regions

$$
\begin{aligned}
\gamma_0 &:= \square_{K,K}^{D,D}, & \gamma_1 &:= \gamma_1^X \cup \gamma_1^Y, & \gamma_2 &:= \square_{2-L,0}^{K+L-2,K} \cup \square_{0,2-L}^{K,K+L-2}, \\
\gamma_1^X &:= \square_{K+L-2,0}^{K+L+2,K}, & \gamma_1^Y &:= \square_{0,K+L-2}^{K,K+L+2}, & \gamma_2^X &:= \square_{N,0}^{D+K-N,K}, \\
\gamma_2^Y &:= \square_{0,N}^{K,D+K-N}, & \gamma_3^X &:= \square_{N,0}^{K-N,K}, & \gamma_3^Y &:= \square_{0,N}^{K,K-N}, \\
\gamma_4^X &:= T_{D,0}\,(\gamma_3^X), & \gamma_4^Y &:= T_{D,0}\,(\gamma_3^X), & \gamma_5^X &:= \mathrm{Thk}^N(\gamma_3^X \cup \gamma_4^X), \\
\gamma_5^Y &:= \mathrm{Thk}^N(\gamma_3^Y \cup \gamma_4^Y), & \gamma_6^X &:= \gamma_2^X \cup \gamma_5^X, & \gamma_6^Y &:= \gamma_2^Y \cup \gamma_5^Y.
\end{aligned} \tag{71}
$$

1) $\mathcal{A}_0$, R_g^0 and γ_0 satisfy the conditions of proposition 21 and we get $\mathcal{A}_1$ setting $i = 0$ in (70). Let for $\alpha = X, Y$

$$
Q^\alpha := \{S \in R_1 \,:\, S||_{\gamma_1^\alpha} \neq \emptyset, S|_{\gamma_6^\alpha} \neq S\}. \tag{72}
$$

We claim that $R_1 = \langle R_g^1 \rangle$ with $R_g^1 := R^1 - T^{(D)}\,(Q^X \cup Q^Y)$. Take any $R \in R_1$. The set

$$
\{\gamma \in T^{(D)}\,(\{\gamma_1^X, \gamma_1^Y\}) \,:\, R||_\gamma \neq \emptyset\} \tag{73}
$$

is finite, and thus we proceed by induction on its cardinality. If it is empty $R \in R_1$. In other case, due to translation and axis-exchange symmetries we can assume that $S||_{\gamma_1^X} \neq \emptyset$. Then it suffices to show that there exists $R' \in R_1$ such that $(R + R')||_{\gamma_1^X} = \emptyset$. To construct R' we first notice that $R \in R_0$ and apply proposition 20 —up to a translation $T_{K+L,0}$— to obtain $S \in R_0$ such that

$$
S \subset \mathcal{A}_g^0|_{\gamma_{-\infty}^{\infty}}, \quad (S + R)||_{\gamma_{-M}^{M}} = \emptyset, \quad \gamma_a^b := \square_{K+L+a,0}^{K+L+b,K}. \tag{74}
$$

Setting $S' := S|_{\gamma_2^X}$ we get $(S'+R)||_{\gamma_{-M}^M} = \emptyset$ and $\mathrm{Supp}(S') \subset \gamma_3^X \cup \gamma_4^X$. From proposition 18, corollary 15 and the involved geometry it follows the existence of $S_3, S_4 \subset \mathbf{P}_0(\mathcal{A}_0)$ such that $\mathrm{Supp}(S_i) \subset \mathrm{Thk}^N \gamma_i^X$, $i = 3, 4$, and $S'+S_3+S_4 \in R_0$. We can take $R' := S' + S_1 + S_2$ because $R'|_{\gamma_6^X} = R'$ and $(R+R')||_{\gamma_{-M}^M} = \emptyset$.

2) $\mathcal{A}_1$, R_g^1 and γ_1 satisfy the conditions of proposition 21 and we get $\mathcal{A}_1$ setting $i = 1$ in (70). We claim that $R_g^2 := T^{(D)}\{S \in R_2 : \mathrm{Supp}(S) \subset \gamma_2\}$ generates R_2. Indeed,

$$\Sigma = T^{(D)}(\gamma_0 \cup \gamma_1 \cup \gamma_2), \tag{75}$$

gives $\mathrm{Supp}(S) \subset T^{(D)}(\gamma_2)$ for any $S \in R_2$. Then $R_2 \ni S_{m,n} := S|_{T_{m,n}(\gamma_2)}$, because for any $b \in \mathcal{B}_g$ there is at most one pair $(m,n) \in \mathbf{Z}^2$ with $\mathrm{Supp}(b) \cap T_{m,n}(\gamma_2) \neq \emptyset$.

3) $\mathcal{A}_3$, R_g^2 and γ_2 satisfy the conditions of proposition 21 and we get $\mathcal{A}_3$ setting $i = 2$ in (70). Due to (75) we have $R_4 = \{\emptyset\}$ as desired. ■

5 Charge and strings

This section defines charges, establishes their duality with constraints and introduces strings.

5.1 Charge groups

Let $\mathcal{A}$ be a LPG with independent generators $\mathcal{A}_g$ of a LPG. We can define

$$\Phi^0(\mathcal{A}) := \Phi^0(\mathcal{A}_g) \tag{76}$$

because the choice of the generating set $\mathcal{A}_g$ is immaterial.

Definition 23 *The charge group of $\mathcal{A}$ is*

$$C_{\mathcal{A}} := \frac{\Phi^0(\mathcal{A})}{\mathrm{Com}_{\mathcal{A}}[\mathcal{P}]}. \tag{77}$$

Its elements are the charges of $\mathcal{A}$.

When $\phi_1, \phi_2 \in c \in C_{\mathcal{A}}$ we write $\phi_1 \sim \phi_2$ and $\mathrm{Chg}(\phi) = c$. A LPG isomorphism F from $\mathcal{A} \subset \mathcal{P}_A$ to $\mathcal{B} \subset \mathcal{P}_B$ induces an isomorphism

$$\begin{aligned} F^* \,:\, \Phi^0(\mathcal{A}) &\longrightarrow \Phi^0(\mathcal{B}) \\ \phi &\longmapsto \phi \circ F^{-1} \end{aligned} \tag{78}$$

that maps charges to charges because $F^*[\mathrm{Com}_{\mathcal{A}}[\mathcal{P}_A]] = \mathrm{Com}_{\mathcal{B}}[\mathcal{P}_B]$, giving rise to an isomorphism

$$C_F : C_{\mathcal{A}} \longrightarrow C_{\mathcal{B}}. \tag{79}$$

Coarse graining does not affect the group of charges, in the sense that there exists a natural isomorphism

$$C_{\mathcal{A}} \simeq C_{\mathrm{Crs}_l(\mathcal{A})} \tag{80}$$

for any $l \in \mathbf{N}^*$. LPG composition gives yet another natural isomorphism

$$C_{\mathcal{A}\otimes\mathcal{B}} \simeq C_{\mathcal{A}} \times C_{\mathcal{B}}. \tag{81}$$

5.2 Charge of generators

Definition 24 *We say that $a \in \mathcal{A}_g - \langle i\mathbf{1}\rangle$ has charge $c \in C_{\mathcal{A}}$ if $\mathrm{Neg}^{-1}_{\mathcal{A}_g}(\{a\}) \in c$.*

The charge of a, denoted $\mathrm{Chg}_{\mathcal{A}_g}(a)$, depends on the whole $\mathcal{A}_g$ and

$$\mathrm{Chg}(\phi) = \prod_{a\in \mathrm{Neg}_{\mathcal{A}_g}(\phi)} \mathrm{Chg}_{\mathcal{A}_g}(a). \tag{82}$$

We need a prescription that tell us how the charges of generators change as the generating set changes preserving translational invariance.

Proposition 25 *Let $\mathcal{A}$ be a LPG with a LTI set of independent generators $\mathcal{A}_g$ of period L. Let $a, b \in \mathcal{A}_g - \langle i\mathbf{1}\rangle$ satisfy $b \notin T^{(L)}(\{a\})$. Then*

$$\mathcal{A}'_g := \mathcal{A}_g \cup T^{(L)}(\{ab\}) - T^{(L)}(\{a\}) \tag{83}$$

is a LTI set of independent generators of $\mathcal{A}$ and

$$\mathrm{Chg}_{\mathcal{A}'_g}(ab) = \mathrm{Chg}_{\mathcal{A}_g}(a), \qquad \mathrm{Chg}_{\mathcal{A}'_g}(b) = \mathrm{Chg}_{\mathcal{A}_g}(a)\mathrm{Chg}_{\mathcal{A}_g}(b). \tag{84}$$

Proof — Since $ab \notin \mathcal{A}_g$ because $\mathcal{A}_g$ is independent, we have $b, ab \in \mathcal{A}'_g$. Then $a \in \langle \mathcal{A}'_g\rangle$ and by translational invariance $\mathcal{A} = \langle \mathcal{A}'_g\rangle$. Given $S' \in \mathrm{Cnstr}^0(\mathcal{A}'_g)$, let

$$z := \{(i,j) \in \mathbf{Z}^2 \,:\, T_{iL,jL}(ab) \in S'\}. \tag{85}$$

Then we can construct $S \in \mathrm{Cnstr}^0(\mathcal{A}_g)$ setting

$$S := S' + \sum_{(i,j)\in z} T_{iL,jL}(\{a,b\}) - \sum_{(i,j)\in z} T_{iL,jL}(\{ab\}). \tag{86}$$

But $|S| \geq |S'|$ because $T^{(L)}(\{a\}) \cap S' = \emptyset$, and thus $\mathcal{A}'_g$ is independent. Let

$$\phi_a := \mathrm{Neg}^{-1}_{\mathcal{A}_g}(\{a\}), \qquad \phi_b := \mathrm{Neg}^{-1}_{\mathcal{A}_g}(\{b\}). \tag{87}$$

To recover (84), apply (82) to both $\mathcal{A}_g$ and $\mathcal{A}'_g$ to get

$$\begin{aligned} \mathrm{Chg}(\phi_a) &= \mathrm{Chg}_{\mathcal{A}_g}(a) = \mathrm{Chg}_{\mathcal{A}'_g}(ab), \\ \mathrm{Chg}(\phi_b) &= \mathrm{Chg}_{\mathcal{A}_g}(b) = \mathrm{Chg}_{\mathcal{A}'_g}(ab)\mathrm{Chg}_{\mathcal{A}'_g}(b). \end{aligned} \tag{88}$$

∎

5.3 Charge-constraint duality

Given $R \in \mathrm{Cnstr}(\mathcal{A}_g)$ and $c \in C_{\mathcal{A}}$ we can define

$$c \cdot R := \phi(R), \qquad \phi \in c, \tag{89}$$

because the choice of ϕ is inmaterial.

Proposition 26 *If* $\mathrm{Cnstr}(\mathcal{A}_g)$ *is finite,* $\mathrm{Cnstr}(\mathcal{A}_g)$ *and* $C_{\mathcal{A}}$ *are dual.*

Proof — We construct dual sets of generators for $\mathrm{Cnstr}(\mathcal{A}_g)$ and $C_{\mathcal{A}}$. Let $C \subset C_{\mathcal{A}}$ be the set of charges c such that $\mathrm{Chg}_{\mathcal{A}_g}(a_c) = c$ for some $a_c \in \mathcal{A}_g$. Clearly $C_{\mathcal{A}} = \langle C \rangle$ and we can choose some countable $C_g \subset C$ as an independent set of generators of C. For each $c \in C_g$ define the set $R_c \subset \mathcal{A}_g$ as follows

$$R_c := \bigcup_{c' \in \bar{c}} A_{c'}, \qquad \bar{c} := C_{\mathcal{A}} - \langle C_g - \{c\} \rangle,$$
$$A_c := \{a \in \mathcal{A}_g - \langle i\mathbf{1} \rangle \,:\, \mathrm{Chg}_{\mathcal{A}_g}(a) = c\}. \tag{90}$$

Checking that $R_c \in \mathrm{Cnstr}(\mathcal{A}_g)$ amounts to show that $\phi(R_c) = 1$ for every $\phi \in \mathrm{Com}_{\mathcal{A}_g}[\mathcal{P}]$. But given $c_1, c_2 \in \bar{c}$, $c_3, c_4 \notin \bar{c}$ we have $c_1c_2, c_3c_4 \notin \bar{c}$ and $c_1c_3 \in \bar{c}$. Then $\mathrm{Chg}(\phi) = 1 \notin \bar{c}$ gives $|R_c \cap \mathrm{Neg}_{\mathcal{A}_g}(\phi)|$ and thus $\phi(R_c) = 1$ as desired. Since $a_c \in R_{c'}$ iff $c = c'$, for $c, c' \in C_g$ we have $c \cdot R_{c'} = \mathrm{Neg}^{-1}_{\mathcal{A}_g}(\{a_c\})(R_{c'}) = 1 - 2\delta_{c,c'}$. Thus C_g is finite. To show that the R_c generate $\mathrm{Cnstr}(\mathcal{A}_g)$, consider that $b \in R \in \mathrm{Cnstr}(\mathcal{A}_g)$ with $a_c \notin R$ for every $c \in C_g$. There exists $A \subset \{a_c \,:\, c \in C_g\}$ such that $\mathrm{Chg}_{\mathcal{A}_g}(b) = \prod_{a \in A} \mathrm{Chg}_{\mathcal{A}_g}(a)$. Thus $\phi(R) = -1$ for $\phi = \mathrm{Neg}^{-1}_{\mathcal{A}_g}(\{b\} \cup A) \in \mathrm{Com}_{\mathcal{A}}[\mathcal{P}]$, a contradiction, showing that R is empty. ■

5.4 Coarse graining

We next show that, given a collection of LPGs with independent generators and finite charge groups, by coarse graining the lattice it is possible to gain charge translational symmetry and other properties. First, there exist generators with any given charge —among a generating set of charges— and support in a single site. Second, given a Pauli operator p that commutes with all generators in $\mathcal{A}_g$ with support outside a connected region γ, there exist another operator p' that anticommutes with the same generators as p but has support in a region only slightly larger that γ —preparing the ground for string operators. Third, given an element of one of this LPGs with support in a block, its generators have support in a slightly larger block.

Proposition 27 *Consider a set of LPGs* $\bar{\mathcal{A}}_1, \dots, \bar{\mathcal{A}}^n$*, on a given qubit lattice, all admiting LTI sets of independent generators and with finite charge groups. There exists* $L \in \mathbf{N}^*$ *such that the coarse grained LPGs*

$$\mathcal{A}^k = \mathrm{Crs}_L(\bar{\mathcal{A}}^k) \subset \mathcal{P}, \qquad k = 1, \dots, n, \tag{91}$$

for any independent set of generators $\{c^k_l\}_{l=1}^{m_k}$ *of* $C_{\mathcal{A}^k}$*,* $m_k \in \mathbf{N}_0$*, admit LTI sets of independent generators* $\mathcal{A}^k_g$ *such that the properties 1 and 2 in proposition 5 and the following ones are satisfied.*

3. *For any $c \in C_{\mathcal{A}^k}$ there exists $\phi \in c$ such that* $\mathrm{Supp}_{\mathcal{A}_g^k}(\phi) \subset \{\mathbf{0}\}$.

4. $\phi \sim T_{i,j}(\phi)$ *for any* $\phi \in \Phi^0(\mathcal{A}^k)$ *and* $i, j \in \mathbf{Z}$.

5. *For fixed k, for any $p \in \mathcal{P}$ and connected set of sites $\gamma \in \mathbf{P}_0(\Sigma)$ with* $\mathrm{Supp}_{\mathcal{A}_g^k}(p) \subset \gamma$ *there exists $p' \in \mathcal{P}$ with* $\mathrm{Com}_{\mathcal{A}^k}(pp') = \emptyset$, $\mathrm{Supp}(p') \subset \mathrm{Thk}^1(\gamma)$.

6. *For any $A \in \mathbf{P}_0(\mathcal{A}_g^k)$ and $\gamma = \square_{i,j}^{i+L,j+L}$, where $i, j \in \mathbf{Z}$ and $L \in \mathbf{N}_0$,*

$$\mathrm{Supp}(\mathrm{Pro}(A)) \subset \gamma \quad \Longrightarrow \quad \mathrm{Supp}(A) \subset \mathrm{Thk}^1(\gamma). \tag{92}$$

7. *For every $l = 1, \dots, m_k$ there exists $a_l^k \in \mathcal{A}_g^k$ with charge c_l^k,* $\mathrm{Supp}(a_l^k) = \{\mathbf{0}\}$.

Proof — All properties are preserved under coarse graining. We show, for properties $x = 3, \dots, 7$, that if properties 1 to $x-1$ are satisfied then property x is also satisfied by further coarse graining or changing the sets of generators.
3) Since $C_{\mathcal{A}^k}$ is finite, there exist $m \in \mathbf{N}^*$ such that for any k and $c \in C_{\mathcal{A}^k}$ there exists $\phi \in c$ with $\mathrm{Supp}_{\mathcal{A}_g^k}(\phi) \subset \square_{0,0}^{m,m}$. Then the LPGs $\mathrm{Crs}_m(\mathcal{A}^k)$ satisfy point 3.
4) The key observation here is that for any $\phi, \phi' \in \Phi^0(\mathcal{A}^k)$, $i, j \in \mathbf{Z}$, we have $\phi \sim \phi'$ iff $T_{i,j}(\phi) \sim T_{i,j}(\phi')$. For each k and $c \in C_{\mathcal{A}^k}$, choose $\phi_c^k \in \Phi^0(\mathcal{A}^k)$ such that $\mathrm{Supp}_{\mathcal{A}_g^k}(\phi_c^k) \subset \{\mathbf{0}\}$. Since $C_{\mathcal{A}^k}$ is finite, there exist $m_1, m_2 \in \mathbf{N}^*$, $m_1 < m_2$, such that $T_{m_1,0}(\phi_c^k) \sim T_{m_2,0}(\phi_c^k)$ for any k, $c \in C_{\mathcal{A}^k}$ and $i, j \in \mathbf{Z}$. Or equivalently, $\phi_c^k \sim T_{m,0}(\phi_c^k)$ with $m = m_2 - m_1$. Then for any $\phi \in c \in C_{\mathcal{A}^k}$ we have $\phi \sim \phi_c^k \sim T_{m,0}(\phi_c) \sim T_{m,0}(\phi)$. The same reasoning in the other axis gives $m' \in \mathbf{N}^*$, and the LPGs $\mathrm{Crs}_{mm'}(\mathcal{A}^k) = \langle \mathrm{Crs}_{mm'}(\mathcal{A}_g^k) \rangle$ satisfy properties 1-4.
5) For each k and for each $\phi \in \mathrm{Com}_{\mathcal{A}^k}[\mathcal{P}]$ with $\mathrm{Supp}_{\mathcal{A}_g^k}(\phi) \subset \square_{0,0}^{3,3}$, choose $m' \in \mathbf{N}_0$ such that there exists $p \in \mathcal{P}$ with $\mathrm{Com}_{\mathcal{A}^k}(p) = \phi$ and $\mathrm{Supp}(p) \subset \mathrm{Thk}^{m'}(\square_{0,0}^{3,3})$. Let $m-1$ equal the largest of such m'. We will show that for any k, $\phi \in \mathrm{Com}_{\mathcal{A}^k}[\mathcal{P}]$ and $\gamma \in \mathbf{P}_0(\Sigma)$ with $\mathrm{Supp}_{\mathcal{A}_g^k}(\phi) \subset \gamma$ and γ connected, there exists $p \in \mathcal{P}$ with $\mathrm{Com}_{\mathcal{A}^k}(p) = \phi$ and $\mathrm{Supp}(p) \subset \mathrm{Thk}^m(\gamma)$. Then the LPGs $\mathrm{Crs}_m(\mathcal{A}^k) = \langle \mathrm{Crs}_m(\mathcal{A}_g^k) \rangle$ satisfy properties 1-5. Let $\sigma \in \Sigma$ be such that the set $\gamma' := \gamma - \{\sigma\}$ is connected. There is always such a site if $\gamma \neq \emptyset$. If $|\gamma| = 1$, just notice that $\rho(\mathrm{Thk}^1(\gamma)) = 3$ and $\mathrm{Thk}^m(\gamma) = \mathrm{Thk}^{m-1}(\mathrm{Thk}^1(\gamma))$. In other case, choose $\sigma' \in \Sigma$ adjacent to σ. Set

$$\phi_0 := \mathrm{Neg}_{\mathcal{A}_g^k}^{-1}(\mathrm{Neg}_{\mathcal{A}_g^k}(\phi)||_{\{\sigma\}}). \tag{93}$$

Due to the second property of proposition 5 $\mathrm{Supp}_{\mathcal{A}_g^k}(\phi_0) \subset \gamma_0 := \mathrm{Thk}^1(\{\sigma\})$. Choose $\phi_1 \in \Phi^0(\mathrm{Crs}_{l_2}(\mathcal{A}^k))$ and $p \in \mathcal{P}$ with $\phi_0 \sim \phi_1$ and

$$\mathrm{Supp}_{\mathcal{A}_g^k}(\phi_1) \subset \{\sigma'\}, \qquad \mathrm{Com}_{\mathcal{A}^k}(p) = \phi_0\phi_1, \qquad \mathrm{Supp}(p) \subset \mathrm{Thk}^{m-1}(\gamma_0). \tag{94}$$

This is always possible because $\sigma' \in \mathrm{Thk}^1(\{\sigma\})$. The result follows by induction on $|\gamma|$ observing that

$$\mathrm{Supp}_{\mathcal{A}_g^k}(\phi\phi_0\phi_1) \subset \gamma', \qquad \mathrm{Thk}^{m-1}(\gamma_0) \cup \mathrm{Thk}^m(\gamma') = \mathrm{Thk}^m(\gamma). \tag{95}$$

6) We will show that for any k, $A \in \mathbf{P}_0(\mathcal{A}_g^k)$ and $\gamma = \square_{i,j}^{i+L,j+L}$, where $i, j \in \mathbf{Z}$ and $L \in \mathbf{N}_0$,

$$\mathrm{Supp}(\mathrm{Pro}(A)) \subset \gamma \quad \Longrightarrow \quad \mathrm{Supp}(A) \subset \mathrm{Thk}^2(\gamma). \tag{96}$$

Then the LPGs $\mathrm{Crs}_2(\mathcal{A}^k) = \langle \mathrm{Crs}_2(\mathcal{A}_g^k)\rangle$ satisfy properties 1-6. Let $a \in A$ satisfy $\mathrm{Supp}(a) \not\subset \mathrm{Thk}^2(\square_{i,j}^{i+L,j+L})$. W.l.o.g, we assume that $\mathrm{Supp}(a) \subset \square_{u,v}^{u+2,v+2}$ with $u \geq i+1+L$. Choose $w \in \mathbf{N}^*$ such that $T_{w,0}(a) \notin A$ and $p \in \mathcal{P}$ such that $\mathrm{Supp}(p) \subset \mathrm{Thk}^1(\square_{u,v}^{u+w+2,v+2})$, $\mathrm{Com}_{\mathcal{A}^k}(p) = \phi\, T_{w,0}(\phi)$, $\phi := \mathrm{Neg}^{-1}_{\mathcal{A}_g^k}(\{a\})$. Then $\mathrm{Supp}(p) \cap \mathrm{Supp}(\mathrm{Pro}(A)) = \emptyset$ implies $(p; A) = 1$, but at the same time $(p; A) = \phi(A) = -1$, a contradiction.
7) For each k, let us show that there exist a list of LTI sets of independent generators $\mathcal{A}_g^k = \mathcal{A}_g^{k,0}, \mathcal{A}_g^{k,1}, \ldots, \mathcal{A}_g^{k,m_k} = \tilde{\mathcal{A}}_g^k$ such that for any $t = 1, \ldots, m_k$ and any $l = 1, \ldots, t$ there exists $a_l \in \mathcal{A}_g^{k,t}$ with charge c_l^k and satisfying (**??**). Moreover, we do it in such a way that all these generating sets preserve properties 1-6, so that the final generating sets $\tilde{\mathcal{A}}_g^k$ satisfy all the desired properties. We construct $\mathcal{A}_g^{k,t}$ from $\mathcal{A}_g^{k,t-1}$ as follows. There exists $\phi \in c_t^k$ with $\mathrm{Supp}_{\mathcal{A}_g^{k,t-1}}(\phi) = \{\mathbf{0}\}$. Also, there exists $\hat{a}_0 \in \mathrm{Neg}_{\mathcal{A}_g^{k,t-1}}(\phi)$ such that $\hat{a}_0 \neq a_k$ for any $1 \leq k < t$, because the charge c_t^k is independent of the charges $c_1^k, \ldots, c_{t-1}^k$. Label the elements of $\mathrm{Neg}_{\mathcal{S}_g^{k,t-1}}(\phi) - \{\hat{a}_0\}$ as $\{\hat{a}_i\}_{i=1}^r$ and apply proposition 25 repeatedly to perform the substitutions $\hat{a}_i \to \hat{a}_i' = \hat{a}_0\hat{a}_i$, $i = 1, \ldots, r$. Let $\mathcal{A}_g^{k,t}$ be the resulting set of generators. Then $\mathrm{Chg}_{\mathcal{A}_g^{k,t}}(\hat{a}_0) = c_t^k$ and the rest of generators with support in $\mathbf{0}$, updated or not, preserve their charge. ■

5.5 Strings

Let $\mathcal{A}$ satisfy the properties in proposition 27. We define the following sets of 'string operators'.

Definition 28 *Given $\phi \in \Phi^0(\mathcal{A})$, $\gamma \subset \Sigma$, we set*

$$\mathrm{Str}(\phi; \gamma) := \{p \in \mathcal{P} \,:\, \mathrm{Com}_{\mathcal{A}_g}(p) = \phi, \mathrm{Supp}(p) \subset \mathrm{Thk}^1(\gamma)\}. \tag{97}$$

Given $\phi_1, \phi_2 \in \Phi^0(\mathcal{A})$ with $\mathrm{Supp}_{\mathcal{A}_g}(\phi_i) = \{\sigma_i\} \subset \Sigma$, we set

$$\begin{aligned} \mathrm{Str}(\phi_1, \phi_2) &:= \mathrm{Str}(\phi_1\phi_2; \mathrm{Path}(\sigma_1, \sigma_2)), \qquad \mathrm{Str}(1; 1) := \{\mathbf{1}\}, \\ \mathrm{Str}(1, \phi_1) &:= \mathrm{Str}(\phi_1; 1) := \mathrm{Str}(\phi_1; \{\sigma_1\}). \end{aligned} \tag{98}$$

Finally, we set for $c \in C_{\mathcal{A}}$ and a path $\gamma := (\sigma_i)_{i=1}^n$,

$$\mathrm{Str}(c; \gamma) := \bigcup \{\mathrm{Str}(\phi_1\phi_2; \gamma) \,:\, \phi_1, \phi_2 \in c, \mathrm{Supp}(\phi_1) \subset \sigma_1, \mathrm{Supp}(\phi_2) \subset \sigma_n\}. \tag{99}$$

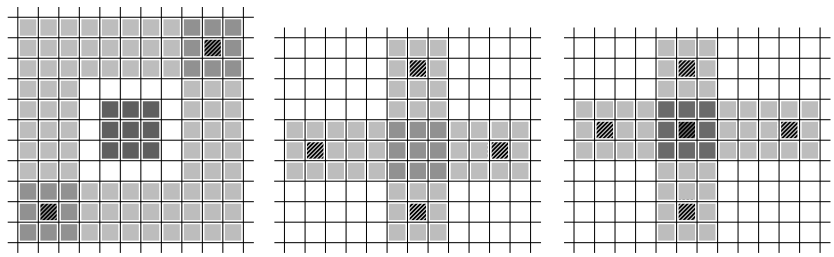


Figure 11: (Left) The geometry of proposition 29 for $l=8$. The striped sites are the common endpoints of two string operators p_1 and p_2, the first running trough the bottom and right side, the second through the top and right side. The central shaded set of sites is $\square_{3,3}^{l-2,l-2}$. (Center) The geometry of proposition 30 for $a_1=-5$, $a_2=4$, $b_1=-3$ and $b_2=4$. The striped sites are the endpoints of two string operators p_A and p_B. (Right) The geometry of proposition 33 for $a_1=-4$, $a_2=5$, $b_1=-5$ and $b_2=3$. The striped sites are the endpoints of four string operators p_i, each with an endpoint in the central striped site.

In general $\mathrm{Str}(\phi;\gamma)$ can be empty. However, suppose that for some $n\in\mathbf{N}_0$

$$\phi=\prod_{k=1}^{n}\phi_k,\quad \phi_k\in\mathrm{Com}_{\mathcal{A}}[\mathcal{P}],\qquad \gamma=\bigcup_{k=1}^{n}\gamma_k,\quad \gamma_k\in\mathbf{P}_0(\Sigma), \tag{100}$$

with the sets γ_k connected and $\mathrm{Supp}_{\mathcal{A}_g}(\phi_k)\subset\gamma_k$ for $k=1,\ldots,n$. Then due to property 5 $\mathrm{Str}(\phi;\gamma)$ is nonempty. In particular, $\mathrm{Str}(\phi_1,\phi_2)$ is nonempty iff $\phi_1\sim\phi_2$, and $\mathrm{Str}(c;\gamma)$ is always nonempty.

The product of two strings with common endpoints gives a 'closed' string operator. Such a Pauli operator belongs to the centralizer of $\mathcal{A}$. We next show that when the two strings have nontrivial charge and enclose a block, the support of the generators of the resulting centralizer element must cover entirely the interior of the block. The geometry is displayed in Fig. 11.

Proposition 29 *Let $\mathcal{B}=\mathcal{Z}_{\mathcal{P}}(\mathcal{A})$, $l\in\mathbf{N}^*$. Let $\phi\in c\in C_{\mathcal{A}}$ with* $\mathrm{Supp}_{\mathcal{A}_g}(\phi)=$ $\{\mathbf{0}\}$ *and c nontrivial, and $p_1,p_2\in\mathcal{P}$, $B\in\mathbf{P}_0(\mathcal{B}_g)$ with*

$$p_1\in\mathrm{Str}(\phi,T_{l,l}(\phi)),\qquad p_2\in\mathrm{Str}(T_{l,l}(\phi),\phi),\qquad p_1p_2\in\mathrm{Pro}(B). \tag{101}$$

Then $\square_{3,3}^{l-2,l-2}\subset\mathrm{Supp}(B)$.

Proof — It is enough to consider $l\geq 6$. Let $\gamma:=\square_{3,3}^{l-2,l-2}$ and suppose that there is a site $(i,j)\in\gamma-\mathrm{Supp}(B)$. Define the sets of sites

$$\gamma_1:=\square_{i,j}^{i+1,\infty},\quad \gamma_2:=\square_{-\infty,-\infty}^{i,\infty},\quad \gamma_3:=\square_{i-1,-1}^{i+2,3}. \tag{102}$$

Choose $b\in\mathrm{Pro}(B||_{\gamma_1})$ and set $p_3:=bp_1p_2$, so that $\mathrm{Supp}(b)\subset\square_{i-1,j}^{i+2,\infty}$ and $\gamma_1\cap\mathrm{Supp}(p_3)=\emptyset$. Choose $p,q,r\in\mathcal{P}$ with $p\propto p_1|_{\gamma_2}$, $q\propto p_3|_{\gamma_2}$. Let $\Sigma_2:=$

$\{\square_{a,b}^{a+2,b+2} : (a,b) \in \Sigma\}$. Since $p_3 \in \mathcal{Z}_{\mathcal{P}}(\mathcal{A})$, $\mathrm{Supp}_{\mathcal{A}_g}(q)$ is contained in the set

$$\bigcup\{\gamma \in \Sigma_2 : \mathrm{Supp}(p_3) \cap \gamma_2 \cap \gamma \neq \emptyset, \mathrm{Supp}(p_3) \cap (\Sigma - \gamma_2) \cap \gamma \neq \emptyset\} \subset \gamma_3. \quad (103)$$

Let $\phi_1 \in c$ be such that $\mathrm{Com}_{\mathcal{A}}(p) = \phi_1\, \phi$. Then $\mathrm{Supp}_{\mathcal{A}_g}(\phi_1)$ is contained in

$$\begin{aligned}&\bigcup\{\gamma \in \Sigma_2 : \mathrm{Supp}(p) \cap \gamma_2 \cap \gamma \neq \emptyset, \mathrm{Supp}(p) \cap (\Sigma - \gamma_2) \cap \gamma \neq \emptyset\} \cup \\ &\cup (\mathrm{Supp}(T_{l,l}(\phi)) \cap \gamma_2) \subset \gamma_3. \end{aligned} \quad (104)$$

But from $\mathrm{Thk}(\gamma_3) \cap (\mathrm{Supp}(b) \cup \mathrm{Supp}(p_2)) = \emptyset$ we get $\mathrm{Supp}_{\mathcal{A}_g}(q) = \mathrm{Supp}_{\mathcal{A}_g}(q)|_{\gamma_3} = \mathrm{Supp}_{\mathcal{A}_g}(p)|_{\gamma_3}$. Thus $\mathrm{Com}_{\mathcal{A}_g}(q) = \phi \in c$, a contradiction. ■

6 Topological charge

This section explores how the commutation properties of string operators sometimes only depend on the charges and topology of the involved strings.

6.1 String commutation rules

We first consider the case of two crossing string operators, with charges possibly in two different *LPGs*. When the two *LPG*-s are suitably related, whether the string operators commute depends only on the respective charges. From a physical perspective, this corresponds to the 'mutual statistics' of topological charges. The geometry involved is displayed in Fig. 11.

Proposition 30 *Let $\mathcal{A}, \mathcal{B}$ be LPGs on a given lattice satisfying the properties in proposition 27 and*

$$\mathcal{Z}_{\mathcal{P}}(\mathcal{B}) \subset \mathcal{Z}_{\mathcal{P}}(\mathcal{Z}_{\mathcal{P}}(\mathcal{A})). \quad (105)$$

Let $p_A, p_B \in \mathcal{P}$ be such that, see Fig. 11,

$$\begin{aligned} p_A &\in \mathrm{Str}(c_A; \gamma_h), \quad \gamma_h = \mathrm{Path}((a_1, 0), (a_2, 0)), \\ p_B &\in \mathrm{Str}(c_B; \gamma_v), \quad \gamma_v = \mathrm{Path}((0, b_1), (0, b_2)), \end{aligned} \quad (106)$$

for some integers $a_1, b_1 \leq -3$, $a_2, b_2 \geq 3$, and $c_A \in C_{\mathcal{A}}$, $c_B \in C_{\mathcal{B}}$. The quantity

$$\kappa(c_A, c_B) := (p_A; p_B), \quad (107)$$

only depends on c_A and c_B. Moreover, for any $c'_A \in C_{\mathcal{A}}$,

$$\kappa(c_A, c_B) = \kappa(c_B, c_A), \qquad \kappa(c_A c'_A, c_B) = \kappa(c_A, c_B)\kappa(c'_A, c_B). \quad (108)$$

Proof — Let $\bar{p}_A, \bar{p}_B, \bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2$ be an alternative choice for the definition of $\kappa(c_A, c_B)$, with the same properties as $p_A, p_B, a_1, a_2, b_1, b_2$. Set $a := a_2 - \bar{a}_1$, $b := b_2 + \bar{b}_2 + 3$, and choose $q_A, q_B \in \mathcal{P}$ such that

$$\begin{aligned} &q_A \in \mathrm{Str}(\mathrm{Com}_{\mathcal{A}}(p_A \bar{p}_A); \gamma_A), \quad q_B \in \mathrm{Str}(\mathrm{Com}_{\mathcal{B}}(p_B \bar{p}_B); \gamma_B), \\ &\gamma_A := \mathrm{Path}((a_1, 0), (a_1, b), (\bar{a}_1 + a, 0)) \cup \square_{a_2,0}^{a_2+1,1}, \\ &\gamma_B := \mathrm{Path}((0, b_1), (a, \bar{b}_1)) \cup \mathrm{Path}((0, b_2), (a, \bar{b}_2)). \end{aligned} \quad (109)$$

We set $\tilde{p}_A := T_{a,0}\,(\bar{p}_A)$, $\tilde{p}_B := T_{a,0}\,(\bar{p}_B)$, so that $p_A\tilde{p}_A q_A \in \mathcal{Z}_{\mathcal{P}}(\mathcal{A})$, $p_B\tilde{p}_B q_B \in \mathcal{Z}_{\mathcal{P}}(\mathcal{B})$. Simple support considerations give

$$(p_A;p_B)(\bar{p}_A;\bar{p}_B) = (p_A;p_B)(\tilde{p}_A;\tilde{p}_B) = (p_A\tilde{p}_A q_A^1 q_A^2; p_B\tilde{p}_B q_B^1 q_B^2) = 1. \qquad (110)$$

Regarding the first property in (108), first notice that condition (105) is symmetric, so that $\kappa(c_B, c_A)$ is defined. Choose $\phi_A \in C_{\mathcal{A}}$, $\phi_B \in C_{\mathcal{B}}$ with support in $\{\mathbf{0}\}$. Let $p_A, q_A, r_A, p_B, q_B, r_B \in \mathcal{P}$ be such that

$$\begin{aligned}
&p_A \in \mathrm{Str}(T_{-6,0}\,(\phi_A), T_{0,0}\,(\phi_A)), && q_A \in \mathrm{Str}(T_{0,-6}\,(\phi_A), T_{0,0}\,(\phi_A)),\\
&r_A \in \mathrm{Str}(T_{0,-6}\,(\phi_A), T_{-6,0}\,(\phi_A)), && p_B \in \mathrm{Str}(T_{3,-3}\,(\phi_B), T_{-3,-3}\,(\phi_B)),\\
&q_B \in \mathrm{Str}(T_{-3,3}\,(\phi_B), T_{-3,-3}\,(\phi_B)), && r_B \in \mathrm{Str}(T_{-3,3}\,(\phi_B), T_{3,-3}\,(\phi_B)). \qquad (111)
\end{aligned}$$

Again from support considerations

$$\kappa(c_A, c_B)\kappa(c_B, c_A) = (p_A;p_B)(q_A;q_B) = \kappa(p_A q_A r_A, p_B q_B r_B) = 1. \qquad (112)$$

As for the second property in (108), just notice that for $c_1, c_2 \in C_{\mathcal{A}}$, $\sigma, \sigma' \in \Sigma$,

$$\begin{aligned}
p_1 \in &\mathrm{Str}(c_1;\mathrm{Path}(\sigma,\sigma')) \wedge p_2 \in \mathrm{Str}(c_2;\mathrm{Path}(\sigma,\sigma')) \Longrightarrow\\
&\Longrightarrow p_1 p_2 \in \mathrm{Str}(c_1 c_2;\mathrm{Path}(\sigma,\sigma')) \qquad (113)
\end{aligned}$$

■

Proposition 31 *Let $\mathcal{A}, \mathcal{B}$ be as in proposition 30. Let $c_A \in C_{\mathcal{A}} - \{1\}$. There exists $c_B \in C_{\mathcal{B}}$ with $\kappa(c_A, c_B) = -1$.*

Proof — Choose $\mathcal{B}_g$ independent, and $\phi \in c$, $h, v \in \mathcal{P}$, $B \in \mathbf{P}_0(\mathcal{B}_g - \langle i\mathbf{1}\rangle)$ with

$$\begin{aligned}
&h \in \mathrm{Str}(\phi, T_{6,0}\,(\phi)), && \mathrm{Pro}(B) \ni h\, T_{0,6}\,(h)\, v\, T_{6,0}\,(v),\\
&v \in \mathrm{Str}(\phi, T_{0,6}\,(\phi)), && \mathrm{Supp}_{\mathcal{A}_g}(\phi) = \mathbf{0}. \qquad (114)
\end{aligned}$$

According to proposition 29 we can choose $b \in B$ with $\mathrm{Supp}(b) \subset \square^{4,4}_{3,3}$. Let c_B be the charge of b in $\mathcal{B}_g$ and choose $L \in \mathbf{N}^*$ such that $T_{-L,0}\,(b) \notin B$ and $p \in \mathrm{Str}(\phi; T_{0,-L}\,(\phi))$, $\phi := \mathrm{Neg}^{-1}_{\mathcal{B}_g}(b)$. Then $\kappa(c_A, c_B) = (p;v) = (p;r) = -1$ for any $r \in \mathrm{Pro}(B)$ because $\mathrm{Supp}(p) \cap \mathrm{Supp}(hT_{0,6}\,(h)T_{6,0}\,(v)) = \emptyset$. ■

Corolary 32 *$C_{\mathcal{A}}$ is dual to $C_{\mathcal{B}}$.*

Next we consider the case of string operators with common charge and a common endpoint. The geometry involved is displayed in Fig. 11. From a physical perspective, the resulting invariant $\theta(c)$ corresponds to the 'topological spin' of the topological charge c. We say that c is a boson if $\theta(c) = 1$ and a fermion if $\theta(c) = -1$.

Proposition 33 *Let $\mathcal{A}$ be a LPG satisfying the properties in proposition 27 and*

$$\mathcal{Z}_{\mathcal{P}}(\mathcal{A}) \subset \mathcal{Z}_{\mathcal{P}}(\mathcal{Z}_{\mathcal{P}}(\mathcal{A})). \qquad (115)$$

Let $c \in C_{\mathcal{A}}$, $P = \{p_1, p_2, p_3, p_4\} \subset \mathcal{P}$ be such that for $k, l = 1, 2, 3, 4$

$$\begin{aligned} p_k \in \mathrm{Str}(c; \gamma_k), \quad \gamma_1 = \mathrm{Path}(\mathbf{0}, (a_1, 0)), \qquad \gamma_2 = \mathrm{Path}(\mathbf{0}, (a_2, 0)), \\ p_k p_l \in \mathrm{Str}(c; \gamma_l \circ \gamma_k^{-1}), \quad \gamma_3 = \mathrm{Path}(\mathbf{0}, (0, b_1)), \qquad \gamma_4 = \mathrm{Path}(\mathbf{0}, (0, b_2)), \end{aligned} \tag{116}$$

for some integers $a_1, b_1 \leq -3$, $a_2, b_2 \geq 3$. Let $p, q, r \in P$ be all different. The quantity

$$\theta(c) := (p; q)(p; r)(q; r) = (pq; pr), \tag{117}$$

only depends on c. Moreover, for any $c_1, c_2 \in C_{\mathcal{A}}$,

$$\theta(c_1 c_2) = \theta(c_1)\theta(c_2)\kappa(c_1, c_2). \tag{118}$$

Proof — There exist $\phi_k \in \Phi^0(\mathcal{A})$ such that

$$\begin{aligned} &\mathrm{Supp}(\phi_0) \subset \square^{1,1}_{0,0}, \qquad \mathrm{Supp}(\phi_1) \subset \square^{a_1+1,1}_{a_1,0}, \qquad \mathrm{Supp}(\phi_2) \subset \square^{a_2+1,1}_{a_2,0}, \\ &\mathrm{Supp}(\phi_3) \subset \square^{1,b_1+1}_{0,b_1}, \qquad \mathrm{Supp}(\phi_4) \subset \square^{1,b_2+1}_{0,b_2}, \qquad p_k \in \mathrm{Str}(\phi_0, \phi_k), \end{aligned} \tag{119}$$

where $k = 1, 2, 3, 4$. Let $\bar{\phi}_k, \bar{p}_{k'}, \bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2$ be an alternative choice for the definition of $\kappa(c_A, c_B)$, with the same properties as $\bar{\phi}_k, p_{k'}, a_1, a_2, b_1, b_2$, $k = 0, \ldots, 4$, $k' = 1, \ldots, 4$. We show that $(p_1 p_4; p_2 p_4) = (\bar{p}_1 \bar{p}_3; \bar{p}_3 \bar{p}_4)$, the other combinations are similar. Set $b := b_2 - \bar{b}_1$ and choose $q_1, q_2, q_3 \in \mathcal{P}$ such that

$$\begin{aligned} q_1 &\in \mathrm{Str}(\phi_1 T_{0,b}(\bar{\phi}_1); \mathrm{Path}((a_1, 0), (\bar{a}_1, b))), \\ q_2 &\in \mathrm{Str}(\phi_2 T_{0,b}(\bar{\phi}_4); \mathrm{Path}((0, b + \bar{a}_2), (a_2, 0))), \\ q_3 &\in \mathrm{Str}(\phi_4 T_{0,b}(\bar{\phi}_3); \mathrm{Path}((0, b_2), (0, b + \bar{b}_1))). \end{aligned} \tag{120}$$

Then, from support considerations

$$(p_1 p_4; p_2 p_4)(T_{0,b}(\bar{p}_1 \bar{p}_3); T_{0,b}(\bar{p}_3 \bar{p}_4)) = \tag{121}$$

$$= (p_1 p_4 T_{0,b}(\bar{p}_1 \bar{p}_3) q_1 q_3; p_2 p_4 T_{0,b}(\bar{p}_3 \bar{p}_4) q_2 q_3) = 1. \tag{122}$$

We next prove (118). Choose $\phi_1 \in c_1, \phi_2 \in c_2$ with support in $\{\mathbf{0}\}$. Let $p, q, r, p', q', r' \in \mathcal{P}$ be such that

$$\begin{aligned} p &\in \mathrm{Str}(T_{0,0}(\phi_1), T_{0,-11}(\phi_1)), \qquad & q &\in \mathrm{Str}(T_{0,0}(\phi_1), T_{12,0}(\phi_1)), \\ r &\in \mathrm{Str}(T_{0,0}(\phi_1), T_{0,12}(\phi_1)), \qquad & p' &\in \mathrm{Str}(T_{3,3}(\phi_2), T_{3,-8}(\phi_2)), \\ q' &\in \mathrm{Str}(T_{3,3}(\phi_2), T_{15,3}(\phi_2)), \qquad & r' &\in \mathrm{Str}(T_{3,3}(\phi_2), T_{3,15}(\phi_2)). \end{aligned} \tag{123}$$

Then $(pp'qq'; pp'rr') = (pq; qr)(p'q'; q'r')(p'; q) = \theta(c)\theta(c')\kappa(c, c')$. But, setting $\phi := \mathrm{Crs}_4(\phi_1 T_{3,3}(\phi_2))$,

$$\begin{aligned} &\mathrm{Crs}_4(pp') \in \mathrm{Str}(T_{0,0}(\phi), T_{0,-3}(\phi)), \qquad \mathrm{Crs}_4(qq') \in \mathrm{Str}(T_{0,0}(\phi), T_{3,0}(\phi)), \\ &\mathrm{Crs}_4(qq') \in \mathrm{Str}(T_{0,0}(\phi), T_{0,3}(\phi)), \end{aligned} \tag{124}$$

and, using the observation (126) below, we get

$$\begin{aligned} \kappa(c, c') &= \kappa(\mathrm{Crs}_4(c), \mathrm{Crs}_4(c)) = (\mathrm{Crs}_4(pp')\mathrm{Crs}_4(qq'); \mathrm{Crs}_4(pp')\mathrm{Crs}_4(rr')) = \\ &= (pp'qq'; pp'rr') = (pq; qr)(p'q'; q'r')(p'; q) = \theta(c)\theta(c')\kappa(c, c'). \end{aligned} \tag{125}$$

■

Clearly $\kappa(c, 1) = \theta(1) = 1$. Then (118) gives $\kappa(c, c) = 1$.

6.2 Local transformations

κ and θ are invariant under coarse graining, up to the natural isomorphism (80),

$$\kappa = \kappa \circ (\mathrm{Crs}_l \times \mathrm{Crs}_l), \qquad \theta = \theta \circ \mathrm{Crs}_l, \tag{126}$$

because $\mathrm{Crs}_l(\mathrm{Str}(c;\gamma)) \subset \mathrm{Str}(\mathrm{Crs}_l(c);\mathrm{Crs}_l(\gamma))$ for any $c \in C_{\mathcal{A}}$, path γ and $l \in \mathbf{N}^*$. As for the behavior under LPG composition, consider the projections $p_i : C_{\mathcal{A}_1\otimes\mathcal{A}_2} \longrightarrow C_{\mathcal{A}_i}$, constructed according to the isomorphism (81). When both sides are defined,

$$\kappa = (\kappa \circ (p_1 \times p_1))(\kappa \circ (p_2 \times p_2)), \qquad \theta = (\kappa \circ p_1)(\kappa \circ p_2). \tag{127}$$

Finally, given an LPG isomorphism F from $\mathcal{A}$ to $\mathcal{B}$ that can be regarded also as an LPG isomorphism F' from $\mathcal{A}'$ to $\mathcal{B}'$, whenever both sides are defined

$$\kappa = \kappa \circ (C_F \times C_{F'}), \qquad \theta = \theta \circ C_F, \tag{128}$$

because, e.g., for any $c \in C_{\mathcal{A}}$, $L \in \mathbf{Z}$,

$$\mathrm{Crs}_{2r+1}(F[\mathrm{Str}(c;\gamma_1)]) \subset \mathrm{Str}(\mathrm{Crs}_{2r+1}(C_F(c));\gamma_2) \tag{129}$$

where $r = \rho(F) - 1$, $\gamma_1 := \mathrm{Path}((r,r),((2r+1)L+r,r)), \gamma_2 := \mathrm{Path}(\mathbf{0},(L,0))$.

7 Duality and canonical generators

This section uncovers the relationship between $C_{\mathcal{S}}$ and $C_{\mathcal{G}}$ and provides a classification of the topological charges that a TSSG may exhibit.

7.1 Injection morphism

Let $\mathcal{S}$ be a TSSG with gauge group $\mathcal{G}$, which is a LPG according to corollary 17. Using theorem 22, lemma 19, proposition 26 and corollary 32 we learn that $C_{\mathcal{S}}$ and $C_{\mathcal{G}}$ are finite and dual through κ. The functions

$$\kappa : C_{\mathcal{G}} \times C_{\mathcal{G}} \longrightarrow \pm 1, \qquad \theta : C_{\mathcal{G}} \longrightarrow \pm 1, \qquad \kappa : C_{\mathcal{G}} \times C_{\mathcal{S}} \longrightarrow \pm 1, \tag{130}$$

are all defined. The charges in $C_{\mathcal{G}}$ and $C_{\mathcal{S}}$ are naturally related. Since $\mathcal{S} \subset \mathcal{G}$, there is an injection morphism $\iota : \mathcal{S} \longrightarrow \mathcal{G}$ that gives rise to the morphism

$$\begin{aligned} \iota^* \ :\ \Phi^0(\mathcal{G}) &\longrightarrow \Phi^0(\mathcal{S}) \\ \phi &\longmapsto \phi \circ \iota. \end{aligned} \tag{131}$$

Since $\iota^*[\mathrm{Com}_{\mathcal{G}}[\mathcal{P}]] \subset \mathrm{Com}_{\mathcal{S}}[\mathcal{P}]$, we get a natural morphism

$$C_\iota : C_{\mathcal{G}} \longrightarrow C_{\mathcal{S}}. \tag{132}$$

Proposition 34 *$\kappa(c,d) = \kappa(c, C_\iota(d))$ for any $c, d \in C_{\mathcal{G}}$.*

Proof — By coarse graining, we can always get $\mathcal{S}_g$ and $\mathcal{G}_g$ such that proposition 5 holds for them and for every $s \in \mathcal{S}_g$ there exists $G_s \subset \mathcal{G}_g$ with $s \in \mathrm{Pro}(G_s)$ and $\mathrm{Supp}(G_s) \subset \mathrm{Thk}^1(\mathrm{Supp}(s))$. But for any $c \in C_\mathcal{G}$, $L \in \mathbf{Z}$,

$$\mathrm{Crs}_5(\mathrm{Str}(c; \mathrm{Path}((2,2),(5L+2,2)))) \subset \mathrm{Str}(\mathrm{Crs}_5(C_\iota(c)); \mathrm{Path}(\mathbf{0},(L,0))). \tag{133}$$

■

Proposition 35 *For any $c \in C_\mathcal{G}$, $C_\iota(c) = 1$ iff $\kappa(c,d) = 1$ for every $d \in C_\mathcal{G}$.*

Proof — If $C_\iota(c) = d \neq 1$ for some $d \in C_\mathcal{S}$, by duality there exists $e \in C_\mathcal{G}$ such that $\kappa(e,d) = -1 = \kappa(e,c)$. ■

7.2 Canonical charge generators

We want dual canonical sets of generators that reflect the properties of the injection morphism (132).

Theorem 36 *Let $\mathcal{S}$ be a TSSG with gauge group $\mathcal{G}$. For some $\alpha, \beta \in \mathbf{N}_0$ and $\chi, f = \pm 1$ the groups $C_\mathcal{S}$ and $C_\mathcal{G}$ admit independent set of generators*

$$\begin{aligned} C_\mathcal{G} &= \langle c_1, \dots, c_\alpha, d_1, \dots, d_\alpha, e_1, \dots, e_\beta \rangle = \langle c_i \rangle_{i=1}^{2\alpha+\beta}, \\ C_\mathcal{S} &= \langle \tilde{c}_1, \dots, \tilde{c}_\alpha, \tilde{d}_1, \dots, \tilde{d}_\alpha, \tilde{e}_1, \dots, \tilde{e}_\beta \rangle = \langle \tilde{c}_i \rangle_{i=1}^{2\alpha+\beta}, \end{aligned} \tag{134}$$

such that for $i = 1, \dots, \alpha$, $k = 1, \dots, \beta$ and $u, v = 1, \dots, 2\alpha+\beta$,

$$\begin{aligned} &C_\iota(c_i) = \tilde{d}_i, && C_\iota(e_k) = 1, && \theta(c_i) = \theta(d_i) = \chi^{\delta_{i1}}, \\ &C_\iota(d_i) = \tilde{c}_i, && \theta(e_k) = f^{\delta_{k1}}, && \kappa(c_u, \tilde{c}_v) = 1 - 2\delta_{uv}. \end{aligned} \tag{135}$$

Moreover, α, β, χ and f only depend on the TSSG, not the choice of generators.

Before proceeding with the proof, notice that propositions 34 and 35 imply

$$\kappa(c_i, c_j) = \kappa(d_i, d_j) = \kappa(c_u, e_k) = 1, \quad \kappa(c_i, d_j) = 1 - 2\delta_{ij}. \tag{136}$$

Proof — We find the generators in three steps.
1) Let K be the kernel of C_ι in $C_\mathcal{G}$. $K \simeq \mathbf{Z}_2^\beta$ for some $\beta \in \mathbf{N}_0$ and it admits a set of independent generators $K = \langle e_1, \dots, e_\beta \rangle$. If $\theta(e_k) = 1$ for every $k = 1, \dots, n$, we do nothing and set $f = 1$. In other case, suppose w.l.o.g. that $\theta(e_1) = -1$ and take new generators $\bar{e}_k$ with $\bar{e}_k = e_k$ if $\theta(e_k) = 1$ or $k = 1$, $\bar{e}_k = e_k e_\beta$ otherwise, remove the bars and set $f = -1$. Since $f = 1$ iff all the elements of K are bosons, f does not depend on the choice of generators.
2) $C_\mathcal{G} \simeq \mathbf{Z}_2^{\beta+n}$ for some $n \in \mathbf{N}_0$ and we choose a set of independent generators $\mathcal{G} = \langle e_1, \dots, e_\beta, g_1, \dots, g_n \rangle$. According to proposition 35, for each g_i there exists g_j such that $\kappa(g_i, g_j) = -1$, $i, j = 1, \dots, n$. In a similar way that a canonical basis of the Pauli group is obtained, we can obtain from the g_i a set of generators of $C_\mathcal{G}$ as in (134) and with the mutual statistics of (136). If $\theta(c_i)\theta(d_i) = 1$ does

not hold, we can always find suitable generators. E.g., if $\theta(c_i) = -\theta(d_i) = 1$ we set $\bar{c}_i := c_i, \bar{d}_i := c_i d_i$ and remove the bars. Similarly, suppose that there exist $1 \leq i < j \leq n$ with $\theta(c_i) = \theta(c_j) = -1$. Then we can set $\bar{c}_i := c_i c_j$, $\bar{d}_i := d_i c_j$, $\bar{c}_j := c_i d_i d_j$ and $\bar{d}_j := c_i d_i c_j d_j$ so that $\theta(\bar{c}_i) = \theta(\bar{d}_i) = \theta(\bar{c}_j) = \theta(\bar{d}_j) = 1$, and remove the bars. χ only depends on the TSG because the total number of bosos in $C_\mathcal{G}$ is $2^{\alpha+\beta-1}(2^{\alpha+1} + \chi + \chi f)$.
3) We define $\tilde{c}_i$ and $\tilde{d}_i$ according to (135). The existence of the e_k is a consequence of duality, which also guarantees the independence of all these generators.

■

Definition 37 *The characteristic of a TSSG is given by the numbers $\alpha, \beta \in \mathbf{N}_0$ and $\chi, f = \pm 1$ in theorem 36, with $\chi = 1$ if $\alpha = 0$ and $f = 1$ if $\beta = 0$.*

The composition of two TSSGs with characteristics $\alpha^k, \beta^k, \chi^k, f^k$, $k = 1, 2$, has characteristic

$$\alpha = \alpha^1 + \alpha^2, \quad \beta = \beta^1 + \beta^2, \quad \chi = \chi^1 \chi^2, \quad 2(1 + f) = (1 + f^1)(1 + f^2). \tag{137}$$

The examples in table 1 show that TSSGs of arbitrary characteristic exist.

Code	α	β	χ	f
Empty code, trivial code, trivial subsystem code	0	0	1	1
Toric code	1	0	1	1
Subsystem toric code	0	1	1	1
Fermionic subsystem toric code, honeycomb code	0	1	1	-1
Topological subsystem color code	1	0	-1	1

Table 1: The characteristic of several codes.

7.3 Local equivalence

Two TSSGs $\mathcal{S}_1, \mathcal{S}_2$ have the same characteristic iff there exists a group isomorphism $\lambda : C_{\mathcal{G}_1} \longrightarrow C_{\mathcal{G}_2}$ such that

$$\kappa \circ (\lambda \times \lambda) = \kappa, \qquad \theta \circ \lambda = \theta. \tag{138}$$

In view of (128) a LPG isomorphism F from $\mathcal{S}_1$ to $\mathcal{S}_2$ induces such an isomorphism $\lambda = C_F$, with F regarded as an isomorphism from $\mathcal{G}_1$ to $\mathcal{G}_2$. The same is true for coarse graining, see (126). Finally, the composition of a TSSG with a trivial code also gives a charge isomorphism in view of (127). In summary:

Proposition 38 *Locally equivalent TSSGs have the same characteristic.*

7.4 Chirality

A TSG $\mathcal{S}$ can be regarded as a TSSG with $\beta = 0$, so that its characteristic is given by α and χ. But all known TSGs have $\chi = 1$ (e.g., for toric codes $\alpha = 1$ and for color codes $\alpha = 2$). Indeed, from the condensed matter perspective the Hamiltonian model related to a TSG is chiral if $\chi = -1$ [33], and this is thought to be incompatible with the fact that the stabilizer generators commute with each other. Therefore, unlike in TSSGs not all characteristics may admit a realization. On the other hand, if we consider only non-chiral TSGs, those with $\chi = 1$, there exists TSGs with arbitrary values of α. We will generally refer to TSSGs with $\chi = -1$ as chiral.

8 Structure

This section shows how any TSSG can be put in a standard form by means of a framework of string operators.

8.1 More coarse graining

We will find useful the following notation for the indices of canonical charges as given in theorem 36:

$$\begin{aligned} K_c &:= \{1, \ldots, \alpha\}, & K_e &:= \{2\alpha+1, \ldots, 2\alpha+\beta\}, \\ K_d &:= \{\alpha+1, \ldots, 2\alpha\}, & K &:= K_c \cup K_d \cup K_e, \end{aligned}$$

$$k^* := \begin{cases} k+\alpha & \text{if } k \in K_c, \\ k-\alpha & \text{if } k \in K_d, \\ k & \text{if } k \in K_e, \end{cases} \tag{139}$$

Before proceeding with the main result we need to gain an additional property by coarse graining. Namely, for $k \in K_c \cup K_d$ and given canonical charge generators, not only we should be able to find, in any given site, separately a stabilizer generator s_k with charge $\tilde{c}_k$ and a gauge generator g_{k^*} with charge c_{k^*}. Rather, both should correspond to the same ϕ up to the morphism ι^*.

Proposition 39 *Any TSSG can be coarse grained to a TSSG $\mathcal{S}$, with gauge group $\mathcal{G}$ and LTI set of independent generators $\mathcal{S}_g$, $\mathcal{G}_g$, satisfying the properties in proposition 27 and also the following, for a choice of canonical charge generators. There exist $s_k \in \mathcal{S}_g$ and $g_k \in \mathcal{G}_g - \langle i\mathbf{1}\rangle$, $k \in K$, with*

$$\mathrm{Chg}_{\mathcal{S}_g}(s_k) = \tilde{c}_k, \qquad \mathrm{Chg}_{\mathcal{G}_g}(g_k) = c_k, \qquad \mathrm{Supp}(s_k) = \mathrm{Supp}(g_k) = \{\mathbf{0}\}, \tag{140}$$

and, for $k \in K_c \cup K_d$,

$$\{s_k\} = \mathrm{Neg}_{\mathcal{S}_g}(\iota^*(\mathrm{Neg}^{-1}_{\mathcal{G}_g}(\{g_{k^*}\}))). \tag{141}$$

Proof — Coarse graining and choosing $\mathcal{G}'_g, \mathcal{S}'_g$ according to proposition 27 for certain $c'_k, \tilde{c}'_k$, there exist s'_k, g'_k that satisfy all conditions except (141). Let $l \in \mathbf{N}^*$ be the minimal integer such that for any $k \in K_c \cup K_d$

$$\mathrm{Supp}_{\mathcal{S}'_g}(\iota^*(\mathrm{Neg}^{-1}_{\mathcal{G}'_g}(\{g'_k\}))) \subset \mathrm{Thk}^l(\{\mathbf{0}\}). \tag{142}$$

In order that all properties except (141) are met, let

$$\begin{aligned} \mathcal{G}_g &:= \mathrm{Crs}_{2l+1}(\mathcal{G}'_g), & c_k &:= \mathrm{Crs}_{2l+1}(c'_k), & g_k &:= \mathrm{Crs}_{2l+1}(T_{l,l}\,(g'_k)), \\ \mathcal{S}''_g &:= \mathrm{Crs}_{2l+1}(\mathcal{S}'_g), & \tilde{c}_k &:= \mathrm{Crs}_{2l+1}(\tilde{c}'_k), & s''_k &:= \mathrm{Crs}_{2l+1}(T_{l,l}\,(s'_k)). \end{aligned} \tag{143}$$

For any $k \in K_c \cup K_d$ we have $\mathrm{Supp}_{\mathcal{S}''_g}(\iota^*(\mathrm{Neg}^{-1}_{\mathcal{G}'_g}(g_k)) = \{\mathbf{0}\}$. For $k \in K_e$ we set $s_k := s''_k$. Let us show that there exist a list of LTI sets of independent generators $\mathcal{S}''_g = \mathcal{S}^0_g, \mathcal{S}^1_g, \dots, \mathcal{S}^{2\alpha}_g = \mathcal{S}_g$ such that for any $k \in K$ and $\mathcal{S}^t_g$ with $k > 2\alpha - t$ there exist a suitable s_k. We construct $\mathcal{S}^{t+1}_g$ from $\mathcal{S}^t_g$ as follows. Set $u := 2\alpha - t$, $\phi := \iota^*(\mathrm{Neg}^{-1}_{\mathcal{G}_g}(g_{u^*}))$ and choose $s_u \in \mathrm{Neg}_{\mathcal{S}^t_g}(\phi)$ such that $s_u \neq s_k$ for any $k \in K$ with $k > u$. This is always possible because the charge $\tilde{c}_u$ is independent of the charges $\tilde{c}_{u+1}, \dots, \tilde{c}_{2\alpha+\beta}$. Label the elements of $\mathrm{Neg}_{\mathcal{S}^t_g}(\phi) - \{s_u\}$ as $\{\hat{s}_i\}_{i=1}^r$ and apply proposition 25 repeatedly to perform the substitutions $\hat{s}_i \to \hat{s}_u \hat{s}_i$, $i = 1, \dots, r$. Letting $\mathcal{S}^{t+1}_g$ be the resulting set of generators, we get as needed

$$\mathrm{Neg}^{-1}_{\mathcal{S}^{t+1}_g}(\{s_u\}) = \iota^*(\mathrm{Neg}^{-1}_{\mathcal{G}_g}(\{g_{u^*}\})) \in \tilde{c}_u. \tag{144}$$

Notice that the transformations $\hat{s}_i \to \hat{s}_0 \hat{s}_i$ may affect the s_k, $k > u$, that had the required properties in $\mathcal{S}^t_g$, but does not affect these properties. ■

8.2 Homological structure

It turns out to be useful to consider an extension of κ and θ to all charges $C_{\mathcal{S}}$. We fix part of this extension setting $\kappa(\tilde{c}_k, \cdot) = \kappa(c_{k^*}, \cdot)$ for $k \in K$ and $\theta(\tilde{c}_k) = \theta(c_{k^*})$ for $k \in K_c \cup K_d$. This still leaves room for choosing $\theta(\tilde{c}_k) = \pm 1$ for $k \in K_e$ in any way that we please.

With such an extension in hand, we now give a 'homological' construction from which the generators of $\mathcal{G}$ and $\mathcal{S}$ with nontrivial charge can be recovered in a convenient way. The idea is as follows. For each $k \in K_c \cup K_e$ we visualize an infinite square lattice Γ and its dual Γ^*, and attach dual charges to them. For each edge and dual edge there is an operator in $\mathcal{P}$, and the commutation relations of these operators are those that we would expect were they string operators with the corresponding charge. In particular, for $k \in K_c$ direct edges carry both charge c_k and $\tilde{d}_k$, and dual edges both charge d_k and $\tilde{c}_k$. As for $k \in K_e$, direct edges carry charge e_l and dual edges charge $\tilde{e}_l$, where $k = l + 2\alpha$. Edge operators from different pairs of dual lattices commute. Closed strings in a given lattice give rise to elements of $\mathcal{G}$ or $\mathcal{S}$, and it is possible to find $\mathcal{S}_g$ and $\mathcal{G}_g$ such that all charged elements are 'face operators'.

Theorem 40 *Let $\mathcal{S}$ be a TSSG with gauge group $\mathcal{G}$, characteristic α, β, χ, f and canonical generators (134). There exist $L \in \mathbf{N}^*$, independent generator sets $\mathcal{S}_g$, $\mathcal{G}_g$, and mappings*

$$\epsilon_k : \Gamma_{\text{edg}} \cup \Gamma^*_{\text{edg}} \longrightarrow \mathcal{P}, \qquad k \in K_c \cup K_e \tag{145}$$

that satisfy the following properties.

1. *The mappings are translationally invariant,*
$$T_{iL,jL} \circ \epsilon_k = \epsilon_k \circ T_{i,j}\,. \tag{146}$$

2. *For any $d, e \in \Gamma_{\text{edg}}$ and $k, l \in K_c \cup K_e$,*
$$(\epsilon_k(d); \epsilon_l(e)) = \begin{cases} \theta(c_k) & \text{if } l = k \text{ and } |\partial d \cap \partial e| = 1, \\ 1 & \text{otherwise,} \end{cases}$$
$$(\epsilon_k(d^*); \epsilon_l(e^*)) = \begin{cases} \theta(\tilde{c}_k) & \text{if } l = k \text{ and } |\partial d^* \cap \partial e^*| = 1, \\ 1 & \text{otherwise,} \end{cases}$$
$$(\epsilon_k(d); \epsilon_l(e^*)) = \begin{cases} -1 & \text{if } l = k \text{ and } d = e, \\ 1 & \text{otherwise.} \end{cases} \tag{147}$$

3. *Let $\mathcal{S}_g(c) \subset \mathcal{S}_g$ contain all elements of charge c, and similarly for $\mathcal{G}_g(c)$. For $k \in K_c$, $l \in K_e$,*
$$\mathcal{G}_g(c_k) = \mathcal{S}_g(\tilde{d}_k) \propto \bigcup_{f^* \in \Gamma^*_{\text{fc}}} \text{Pro}(\epsilon_k[\partial f^*]), \quad \mathcal{S}_g(\tilde{c}_l) \propto \bigcup_{f \in \Gamma_{\text{fc}}} \text{Pro}(\epsilon_l[\partial f]),$$
$$\mathcal{G}_g(d_k) = \mathcal{S}_g(\tilde{c}_k) \propto \bigcup_{f \in \Gamma_{\text{fc}}} \text{Pro}(\epsilon_k[\partial f]), \qquad \mathcal{G}_g(c_l) \propto \bigcup_{f^* \in \Gamma^*_{\text{fc}}} \text{Pro}(\epsilon_l[\partial f^*]). \tag{148}$$

4. *Let $\mathcal{P}_k := \langle \epsilon_k(e) \rangle_{e \in \Gamma_{\text{edg}}}$ and $\mathcal{P}^*_k := \langle \epsilon_k(e) \rangle_{e \in \Gamma^*_{\text{edg}}}$. Then*
$$\mathcal{G}_g(1) \subset \mathcal{Z}_{\mathcal{P}}(\mathcal{P}'_\Gamma) \qquad \mathcal{P}'_\Gamma := \prod_{k \in K_c} \mathcal{P}_k \mathcal{P}^*_k \prod_{l \in K_d} \mathcal{P}_l,$$
$$\mathcal{S}_g(1) \subset \mathcal{Z}_{\mathcal{P}}(\mathcal{P}_\Gamma), \qquad \mathcal{P}_\Gamma := \prod_{k \in K_c \cup K_d} \mathcal{P}_k \mathcal{P}^*_k. \tag{149}$$

Proof — We proceed in three steps. **0)** We assume that $\mathcal{G} = \langle \mathcal{G}'_g \rangle$ and $\mathcal{S} = \langle \mathcal{S}'_g \rangle$, with $\mathcal{G}'_g$ and $\mathcal{S}'_g$ LTI sets of independent generators that have been coarse grained and chosen according to proposition 39. We can assume such a coarse graining w.l.o.g because it can be absorbed in the value of L.
1) We first give the construction of ϵ_k, and then explain it. Let $L = 3(2\alpha + 5\beta)$ and set $\phi_k := \text{Neg}^{-1}_{\mathcal{G}'_g}(\{g_k\})$, $\tilde{\phi}_k := \text{Neg}^{-1}_{\mathcal{S}'_g}(\{s_k\})$ for $k \in K$. Choose for $k \in K_c \cup K_d$, $l \in K_e$,

$$\begin{aligned} &H_k \in \text{Str}(\phi_k, T_{L,0}\,(\phi_k)), && V_k \in \text{Str}(\phi_k, T_{0,L}\,(\phi_k)), && h_l \in \text{Str}(\phi_l, T_{6,0}\,(\phi_l)), \\ &H_l \in \text{Str}(\phi_l, T_{L,0}\,(\phi_l)), && V_l \in \text{Str}(\phi_l, T_{0,L}\,(\phi_l)), && v_l \in \text{Str}(\phi_l, T_{0,6}\,(\phi_l)), \\ &\tilde{H}_l \in \text{Str}(\tilde{\phi}^l, T_{L,0}\,(\tilde{\phi}_l)), && \tilde{V}_l \in \text{Str}(\tilde{\phi}_k, T_{0,L}\,(\tilde{\phi}_l)), && p_l \in \text{Str}(\iota^*(\phi_l), 1). \end{aligned} \tag{150}$$

Define for $k \in K_c \cup K_d$, $l, l' \in K_e$,

$$\begin{aligned} 2\,\theta_0^k &:= 1 - \theta(c_k)\,(H_k; T_{L,0}\,(H_k)), & 2\,\theta_{i,j}^l &:= 1 - \theta(\tilde{c}_l)\,(D_i^l; D_j^l), \\ 2\,\theta_1^k &:= 1 - (1-2\theta_0^k)\,(H_k; V_k), & 2\,\tau_{l,l'} &:= 1 - (H_l; T_{3,3}\,(V_{l'})), \\ 2\,\theta_2^k &:= 1 - (1-2\theta_1^k)\,(V_k; T_{0,L}\,(V_k)), \end{aligned} \tag{151}$$

where $i, j \in \{\mathrm{E(ast)}, \mathrm{N(orth)}, \mathrm{W(est)}, \mathrm{S(outh)}\}$, and

$$D_{\mathrm{E}}^l := \tilde{H}_l, \quad D_{\mathrm{N}}^l := \tilde{V}_l, \quad D_{\mathrm{W}}^l := T_{-L,0}\,(\tilde{H}_l), \quad D_{\mathrm{S}}^l := T_{0,-L}\,(\tilde{V}_l). \tag{152}$$

Let $\Delta := -18\alpha + 3\beta$. Define ϵ_k according to (146) setting for $k \in K_c$, $l \in K_e$

$$\begin{aligned} \epsilon_k(\mathbf{e}_1) &:= \Upsilon_{3k,3k}^{L,0}(s_k, \theta_0^k, 0, H_k), & \epsilon_k(\mathbf{e}_1^*) &:= \Upsilon_{3k^*,3k^*-L}^{0,L}(s_{k^*}, \theta_1^{k^*}, \theta_2^{k^*}, V_{k^*}), \\ \epsilon_k(\mathbf{e}_2) &:= \Upsilon_{3k,3k}^{0,L}(s_k, \theta_1^k, \theta_2^k, V_k), & \epsilon_k(\mathbf{e}_2^*) &:= \Upsilon_{3k^*-L,3k^*}^{L,0}(s_{k^*}, \theta_0(k^*), 0, H_{k^*}), \\ \epsilon_l(\mathbf{e}_1) &:= \Upsilon_{3l,3l}^{L,0}(p_l, 1, 1, H_l), & \epsilon_l(\mathbf{e}_1^*) &:= T_{12l+\Delta,12l+\Delta-L}\,(\tilde{v}_l \tilde{V}_l), \\ \epsilon_l(\mathbf{e}_2) &:= \Upsilon_{3l,3l}^{0,L}(p_l, 1, 1, V_l), & \epsilon_l(\mathbf{e}_2^*) &:= T_{12l+\Delta-L,12l+\Delta}\,(\tilde{h}_l \tilde{H}_l), \end{aligned} \tag{153}$$

$$\begin{aligned} \tilde{h}_l &:= \hat{h}_l^{\theta_{\mathrm{E,S}}^l}\, T_{L,0}\,(\hat{h}_l^{\theta_{\mathrm{W,S}}^l} \hat{v}_l^{\theta_{\mathrm{W,E}}^l}) \prod_{d=1}^{2\alpha+\beta-l} T_{3d-L,3d,(}\,\hat{v}_l), & \hat{h}_l &:= \Upsilon_{-3,6}^{6,0}(p_l, 1, 1, h_l), \\ \tilde{v}_l &:= \hat{h}_l^{\theta_{\mathrm{N,S}}^l} \hat{v}_l^{\theta_{\mathrm{N,E}}^l}\, T_{-L,0}\,(\hat{v}_l^{\theta_{\mathrm{N,W}}^l}) \prod_{d=1}^{2\alpha+\beta-l} T_{3d,3d,(}\,\hat{h}_l), & \hat{v}_l &:= \Upsilon_{6,-3}^{0,6}(p_l, 1, 1, v_l), \end{aligned}$$

$$\Upsilon_{x,y}^{u,v}(c, t, t', W) := T_{x,y}\,\left(c^t T_{u,v}\,(c^{t'}) W\right). \tag{154}$$

The main idea behind this constructions is to build the edge operators from suitable string operators, as those in (150), placing them in lattices that are shifted from each other to guarantee suitable commutation relations. As for the details, firstly we seek $\epsilon_l(e) \in \mathcal{G}$ for $e \in \Gamma_{\mathrm{edg}}$, which is the reason for the introduction of p_l. Secondly, we want the equations in (147) to hold, so we introduce the binary values in (151) that tell us what adjustments are needed. Regarding corrections involving the topological spin, marked θ, different values of k require different adjustments. For $k \in K_c$ only three numbers describe the adjustments because of the constraints imposed by proposition 33, and the adjustment can be made using s_k. For $k \in K_e$ and dual edges there are six adjustments to make, and they can be made with the help of suitably placed 'tiny' string operators $\hat{h}_l, \hat{v}_l \in \mathcal{G}$ that only share support with one of the 'edge' string operators in the construction. The reason for the factor 12 in (153) — rather than 3, the width of a string—, is indeed the need to make room for these tiny strings. For $k \in K_e$ and direct edges no adjustments are needed. E.g., if we take $h = \epsilon_l(\mathbf{e}_1)$, $\quad v = \epsilon_l(\mathbf{e}_2)$, $\quad e = T_{-L,0}\,(h)$ and $s := \mathrm{Pro}(\epsilon_l[\mathbf{f}]) \in \langle i\mathbf{1}\rangle\mathcal{S}$, then $(e; h)(e; v) = (e; hv) = (e; s) = 1$ and thus $(e; h) = (e; v)$. Similarly one can get $(h; v) = (e; v)$ and thus $(e; h) = \theta(c_l)$ using proposition 33. Finally, corrections

regarding mutual statistics, marked τ, are only needed for $k \in K_e$ and dual edges, and we perform them using $\hat{h}_l, \hat{v}_l \in \mathcal{G}$ again.
2) We next construct $\mathcal{S}_g$, $\mathcal{G}_g$. Let for $i, j \in \mathbf{Z}$, $k \in K$,

$$s_{i,j}^k := \begin{cases} T_{iL+3k^*, jL+3k^*}\,(s_k) & \text{if } k \in K_c \cup K_d, \\ T_{iL+12k+\Delta, jL+12K+\Delta}\,(s_k) & \text{otherwise,} \end{cases}$$
$$g_{i,j}^k := T_{iL+3k, jL+3k}\,(g_k), \qquad p_{i,j}^k := T_{iL+3k, jL+3k}\,(p_k), \tag{155}$$

Choose $S_k \in \mathcal{S}$ and $G_k \in \mathcal{G}$ for $k \in K$ such that

$$S_k \propto \begin{cases} \mathrm{Pro}(\epsilon_k[\mathbf{f}]) & \text{if } k \in K_c, \\ \mathrm{Pro}(\epsilon_{k^*}[\mathbf{f}^*]) & \text{if } k \in K_d, \\ \mathrm{Pro}(\epsilon_k[\mathbf{f}]) & \text{if } k \in K_e. \end{cases} \qquad G_k \quad \propto \begin{cases} \mathrm{Pro}(\epsilon_k[\mathbf{f}^*]) & \text{if } k \in K_c, \\ \mathrm{Pro}(\epsilon_{k^*}[\mathbf{f}]) & \text{if } k \in K_d, \\ \mathrm{Pro}(\epsilon_k[\mathbf{f}^*]) & \text{if } k \in K_e. \end{cases} \tag{156}$$

As we show below, for $k, k' \in K$, $i, j \in \mathbf{Z}$,

$$s_{i,j}^k \in \hat{S}^{k'} \iff g_{i,j}^k \in \hat{G}^{k'} \iff i = j = k - k' = 0, \tag{157}$$

where $\hat{S}^k \in \mathbf{P}_0(\mathcal{S}_g')$, $\hat{G}^k \in \mathbf{P}_0(\mathcal{G}_g')$ are such that $S^k \in \mathrm{Pro}(\hat{S}^k)$, $G^k \in \mathrm{Pro}(\hat{G}^k)$. Applying repeatedly proposition 25 we can define new independent generators

$$\mathcal{S}_g := \mathcal{S}_g' \cup \bigcup_{k \in K} T^{(L)}\,(\{S_k\}) - \bigcup_{k \in K} T^{(L)}\,(\{s_{0,0}^k\}),$$
$$\mathcal{G}_g'' := \mathcal{G}_g' \cup \bigcup_{k \in K} T^{(L)}\,(\{G_k\}) - \bigcup_{k \in K} T^{(L)}\,(\{g_{0,0}^k\}). \tag{158}$$

such that $\mathrm{Chg}_{\mathcal{S}_g}(S_k) = \tilde{c}_k$ and $\mathrm{Chg}_{\mathcal{G}_g}(G_k'') = c_k$. The rest of generators have trivial charge because for any $k \in K$

$$T^{(L)}\,(\{S_k\}) \in \mathrm{Cnstr}(\mathcal{S}_g), \quad T^{(L)}\,(\{G_k\}) \in \mathrm{Cnstr}(\mathcal{G}_g''). \tag{159}$$

For $l \in K_e$, let $P_l = \mathrm{Neg}_{\mathcal{G}_g''}(p_{0,0}^l)$. Clearly —consider the support— $G_k \notin T^{(L)}\,P_l$ for any $k \in K$. Then applying repeatedly proposition 25 we can define

$$\mathcal{G}_g := \mathcal{G}_g^{0''} \cup \bigcup_{l \in K_e} \bigcup_{p \in P_l} T^{(L)}\,(pG_l) - \bigcup_{l \in K_e} \bigcup_{p \in P_l} T^{(L)}\,(p). \tag{160}$$

This last change does not involve charged generators and thus completes the construction, which clearly satisfies the desired properties.

To prove (157) we define the semi-infinite strings

$$E_{i,j} := \bigcup_{n \in \mathbf{N}_0} T_{i+n,j}\,(\mathbf{e}_1), \quad E_{i,j}^* := \bigcup_{n \in \mathbf{N}^*} T_{i+n,j}\,(\mathbf{e}_2^*). \tag{161}$$

Then for any $k, k' \in K_c$, $l, l' \in K_e$, by construction

$$\mathrm{Neg}_{\mathcal{G}_g'}(\epsilon_k[E_{i,j}]) = \mathrm{Neg}_{\mathcal{S}_g'}(\epsilon_k[E_{i,j}]) = \{g_{i,j}^k\} = \{s_{i,j}^{k^*}\}, \quad \mathrm{Neg}_{\mathcal{G}_g'}(\epsilon_l[E_{i,j}]\,p_{i,j}^l) = \{g_{i,j}^l\},$$
$$\mathrm{Neg}_{\mathcal{G}_g'}(\epsilon_k[E_{i,j}^*]) = \mathrm{Neg}_{\mathcal{S}_g'}(\epsilon_k[E_{i,j}^*]) = \{g_{i,j}^{k^*}\} = \{s_{i,j}^k\}, \quad \mathrm{Neg}_{\mathcal{S}_g'}(\epsilon_l[E_{i,j}^*]) = \{s_{i,j}^l\}, \tag{162}$$

and due to (147)

$$(\epsilon_{k'}[E_{i,j}];G_k)=(\epsilon_{k'}[E_{i,j}];S_{k^*})=(\epsilon_{k'}[E^*_{i,j}];G^{k^*})=(\epsilon_{k'}[E^*_{i,j}];S^k)=1-2\delta_{i,0}\delta_{j,0}\delta_{k,k'},$$
$$(\epsilon_{l'}[E_{i,j}]p^l_{i,j};G^l)=(\epsilon_{l'}[E^*_{i,j}];S^l)=1-2\delta_{i,0}\delta_{j,0}\delta_{l,l'}, \tag{163}$$

which together give the desired result by inspection. ■

According to (148) we can label non-trivially charged elements of $\mathcal{S}_g$ and $\mathcal{G}_g$ with a face or dual face, so that $\mathcal{S}_g(\tilde{c}_k)=\{s^k_f\}_f$ and $\mathcal{G}_g(c_k)=\{g^k_f\}_f$, where for each k the index f takes values either in Γ_{fc} or Γ^*_{fc} and $g^k_f=s^{k^*}_f$ for $k\in K_c$. Then for $E\subset\Gamma_{\mathrm{edg}}$, $k\in K_c\cup K_e$,

$$\mathrm{Neg}_{\mathcal{F}_g}(\epsilon_k[E])=\{s^k_{f^*}\,:\,f^*\in\partial E\},\quad \mathrm{Neg}_{\mathcal{F}_g}(\epsilon_k[E^*])\quad=\{g^k_f\,:\,f\in\partial E^*\}, \tag{164}$$

where $\mathcal{F}_g:=\mathcal{S}_g\cup(\mathcal{G}_g-\mathcal{G}_g(1))$. Define

$$\hat{\epsilon}_k(e):=\epsilon_k(e^*)(g^k_fg^k_h)^{b(c_k)},\qquad \partial e^*=\{f,h\},$$
$$\hat{\epsilon}_k(e^*):=\epsilon_k(e)(s^k_{f^*}s^k_{h^*})^{b(\tilde{c}_k)},\qquad \partial e=\{f^*,h^*\}, \tag{165}$$

where $b(\cdot)=(1-\theta(\cdot))/2$, $k\in K_c\cup K_e$ and $e\in\Gamma_{\mathrm{edg}}$. Then for $d,e\in\Gamma_{\mathrm{edg}}\cup\Gamma^*_{\mathrm{edg}}$ and $k,l\in K_c\cup K_e$,

$$(\hat{\epsilon}_k(d);\epsilon_l(e))=1-2\delta_{k,l}\delta_{d,e}, \tag{166}$$

which implies that the edge operators $\epsilon_k(e)$ form a LTI set $\mathcal{E}$ of independent generators of $\mathcal{P}_\Gamma$ with $\mathrm{Cnstr}(\mathcal{E})=\emptyset$. Dual edge operators $\hat{\epsilon}_k(e)\in\mathcal{P}_\Gamma$ satisfy the same properties as the edge operators $\epsilon_k(e^*)$. More importantly, (166) implies that any element of $\mathcal{P}$ can be uniquely decomposed —up to a phase— as pq with $p\in\mathcal{P}_\Gamma$ and $q\in\mathcal{Z}_{\mathcal{P}}(\mathcal{P}_\Gamma)$. This paves the way for the following complement to theorem 40, which isolates the trivial part of the stabilizer.

Proposition 41 *There exists a LPG morphism* $F:\mathcal{P}_{\mathrm{T}}^{\otimes n}\longrightarrow\mathrm{Crs}_L(\mathcal{P})$

$$\mathcal{S}_{\mathrm{T}}^{\otimes n}\xrightarrow{F}\mathrm{Crs}_L(\langle\mathcal{S}_g(1)\rangle), \tag{167}$$

with period 1 and $F[\mathcal{P}_{\mathrm{T}}^{\otimes n}]\subset\mathrm{Crs}_L(\mathcal{Z}_{\mathcal{P}}(\mathcal{P}_\Gamma))$*, for some* $n\in\mathbf{N}_0$.

Proof — Due to the coarse graining, we can assume w.l.o.g. that $L=1$. Due to the above considerations, for each $s\in\mathcal{S}_g(1)$ we can choose $\overline{s}\in\mathcal{Z}_{\mathcal{P}}(\mathcal{P}_\Gamma)$ such that $\mathrm{Neg}_{\mathcal{S}_g}(\mathrm{Com}_{\mathcal{S}}(\overline{s}))=\{s\}$. Moreover, the choice can be done in a translationally invariant way so that $\overline{T_{i,j}\,(s)}=T_{i,j}\,(\overline{s})$. Consider a translationally invariant ordering of $\mathcal{S}_g(1)$, so that for $r,s\in\mathcal{S}_g(1)$ we have $r<s$ iff $T_{i,j}\,(r)<T_{i,j}\,(s)$. For $s\in\mathcal{S}^0_g$, introduce the *finite* sets $\mu(s):=\{r\in\mathcal{S}_g(1)\,:\,r<s,(\overline{r};\overline{s})=-1\}$, and let $\hat{s}:=\overline{s}\prod_{r\in\mu(s)}r$, which clearly is a translationally invariant definition. Then for any $r,s\in\mathcal{S}_g(1)$ we have

$$(r;s)=1,\quad(\hat{r};\hat{s})=1,\quad(r;\hat{s})=1-2\delta_{r,s}. \tag{168}$$

Given $s\in\mathcal{S}_g(1)$, let $\sigma_s\in\Sigma$ be the minimal element of $\mathrm{Supp}(s)$ in lexicographical ordering, which is translationally invariant. Let us label the elements of $\mathcal{S}_1$ with

$\sigma_s = \mathbf{0}$ as s^m, with $m = 1, \ldots, n$ for some $n \in \mathbf{N}_0$. We can then define F setting $F(Z^m_{i,j}) = s^m$ and $F(X^m_{i,j}) = \psi^m \hat{s}^m$, where $\psi^m = 1, i$ as needed and $X^m_{i,j}, Z^m_{i,j}$ denote $X_{i,j}, Z_{i,j}$ on the m-th copy of $\mathcal{P}_\mathrm{T}$. ■

Corolary 42 *Every element of $\mathcal{P}$ admits a unique decomposition —up to phases— as pqg with $p \in \mathcal{P}_\Gamma$, $q \in \mathcal{P}_1 \subset \mathcal{P}$, $g \in \mathcal{G}_1 \subset \mathcal{G} - \mathcal{S}$, where*

$$\mathrm{Crs}_L(\mathcal{P}_1) := F[\mathcal{P}_\mathrm{T}^{\otimes n}], \qquad \mathcal{G}_1 := \mathcal{Z}_\mathcal{P}(\mathcal{P}_\Gamma \mathcal{P}_1). \tag{169}$$

8.3 Examples

Some examples of the construction in theorem 40 follow. Only their nontrivial content is given. For the topological spin extensions we choose $\theta(\tilde{c}_1) = 1$.

- Toric code and subsystem toric code ($L = 1$)

$$\epsilon_1(\mathbf{e}_1) = Z^h_{0,0}, \quad \epsilon_1(\mathbf{e}_1^*) = X^h_{0,0}, \quad \epsilon_1(\mathbf{e}_2) = Z^v_{0,0}, \quad \epsilon_1(\mathbf{e}_2^*) = X^v_{0,0}. \tag{170}$$

- Fermionic subsystem toric code ($L = 1$)

$$\epsilon_1(\mathbf{e}_1) = Z^h_f, \quad \epsilon_1(\mathbf{e}_1^*) = X^h_{0,0}, \quad \epsilon_1(\mathbf{e}_2) = Z^v_f, \quad \epsilon_1(\mathbf{e}_2^*) = X^v_{0,0}, \tag{171}$$

- Honeycomb subsystem code ($L = 2$)

$$\begin{aligned}
\epsilon_1(\mathbf{e}_1) &= G^Y_{0,0} G^X_{1,0} G^Y_{1,0} G^X_{2,0}, & \epsilon_1(\mathbf{e}_1^*) &= Y^1_{1,0} Y^1_{1,-1}, \\
\epsilon_1(\mathbf{e}_2) &= G^Z_{0,0} G^X_{0,1} G^Z_{0,1} G^X_{0,2}, & \epsilon_1(\mathbf{e}_2^*) &= Z^1_{0,1} Z^2_{-1,0}, \\
\mathcal{S}_g(1) &= \mathcal{S}_g^{\hexagon} - T^{(2)}\left(S^{\hexagon}\right), & g &\in \mathrm{Pro}(\epsilon_1[\mathbf{f}^*]), \\
\mathcal{G}_g(1) &= \mathcal{G}_g \cup T^{(2)}\left(\{gG^Y, gG^Z\}\right) - T^{(2)}\left(\{G^X, G^Y, G^Z\}\right),
\end{aligned} \tag{172}$$

- Topological subsystem color code ($L = 3$)

$$\begin{aligned}
\epsilon_1(\mathbf{e}_1) &:= Z^2_{1,0} X^3_{1,0} Y^1_{1,0} X^5_{1,0} X^2_{2,0} Y^4_{2,0} X^6_{2,0} Z^5_{2,0} Y^4_{3,0} Y^2_{3,0} X^6_{3,0} X^3_{3,0} Y^5_{3,0} Y^1_{3,0}, \\
\epsilon_1(\mathbf{e}_2) &:= Z^1_{0,0} Z^6_{0,1} Y^5_{0,1} Y^3_{0,1} X^1_{0,1} X^4_{0,2} Y^6_{0,2} Y^2_{0,2} Z^3_{0,2} X^4_{0,3} Y^2_{0,3} X^6_{0,3} X^3_{0,3} Y^5_{0,3} Y^1_{0,3}, \\
\epsilon_1(\mathbf{e}_1^*) &:= T_{2,-2}\left(\epsilon_1(\mathbf{e}_2)\right), \qquad \epsilon_1(\mathbf{e}_2^*) := T_{-1,1}\left(\epsilon_1(\mathbf{e}_1)\right), \\
G_g(1) &:= \mathcal{G}_g^{\mathrm{col}} - T^{(3)}\left(\{G^1_{0,0}, G^1_{2,1}\}\right), \\
S_g(1) &:= \mathcal{S}_g^{\mathrm{col}} \cup T^{(3)}\left(\{S^1_{0,0} S^2_{0,0}, S^2_{2,0} S^1_{2,1}, S^1_{2,1} S^2_{2,1}, S^2_{0,2} S^1_{3,0}\}\right) - \\
&\quad - T^{(3)}\left(\{S^1_{0,0}, S^2_{0,0}, S^2_{2,0}, S^1_{2,1}, S^2_{2,1}, S^2_{0,2}\}\right),
\end{aligned} \tag{173}$$

8.4 Local equivalence

Among the examples above, the three variants of the toric code have a particularly simple structure, suggesting the following definition.

Definition 43 *A TSSG $\mathcal{S}$ is elementary if it admits the structure of theorem 40 with $\mathcal{P}_\Gamma = \mathcal{P}$.*

From our examples it follows that there exist elementary TSSGs of any non-chiral characteristic. When $\beta = 0$, an elementary TSSG is a TSG.

Proposition 44 *An elementary TSSG is locally equivalent to any TSSG with the same characteristic.*

Proof — If $\mathcal{S}' \subset \mathcal{P}'$ is an elementary TSSG, it admits ϵ'_k as in theorem 40 with $\mathcal{P}_\Gamma = \mathcal{P}'$. Let $\mathcal{S} \subset \mathcal{P}$ be a TSSG with the same characteristic. Using the same extension of the topological spin as for ϵ'_k, choose ϵ_k and $\mathcal{S}_g$ according theorem 40. Since we are allowed to coarse grain, w.l.o.g. we assume that $L = 1$ in both cases. We can construct a LPG morphism G from $\mathcal{S}' \subset \mathcal{P}'$ to $\mathcal{S} \cap \mathcal{P}_\Gamma \subset \mathcal{P}$ and with image in $\mathcal{P}_\Gamma$. Namely, set $G[\epsilon'_k(e)] = \psi(e,k)\sigma(e,k)\epsilon_k(e)$ for any $k \in K_c \cup K_e$ and $e \in \Gamma_{\mathrm{edg}}$, where $\psi(e,k) = 1, i$ and $\sigma(e,k) = \pm 1$ are translationally invariant and as follows. There is a unique choice of $\psi(k,e)$ to preserve self-adjointness. We set $\sigma(e,k) = 1$ unless e is horizontal and belongs to an even row, and then it is easy to check that there is a unique choice for σ that maps stabilizers to stabilizers, without sign flips. Now choose $F : \mathcal{P}_{\mathrm{T}}^{\otimes n} \longrightarrow \mathcal{P}$ according to proposition 41. Consider the LPG morphism $H : \mathcal{P}' \otimes \mathcal{P}_{\mathrm{T}}^{\otimes n} \longrightarrow \mathcal{P}$ from $\mathcal{S}' \otimes \mathcal{S}_{\mathrm{T}}^{\otimes n}$ to $\mathcal{S}$ defined by $H(p \otimes q) = F(p)G(q)$. According to lemma 2 we can extend H to an isomorphism $\hat{H} : \mathcal{P}' \otimes \mathcal{P}_{\mathrm{T}}^{\otimes n+m} \longrightarrow \mathcal{P}$ that gives the local equivalence. ∎

Corolary 45 *Two non-chiral TSSGs are locally equivalent iff they have the same characteristic.*

Proposition 46 *Given a TSG, there exists a elementary TSG with the same characteristic.*

Proof — Let $\mathcal{S} \subset \mathcal{P}$ be a TSG and $F : \mathcal{P}_{\mathrm{T}}^{\otimes n} \longrightarrow \mathcal{P}$ the LPG isomorphism of proposition 41, where we assume w.l.o.g. $L = 1$. Then $\mathcal{P}_1 = F[\mathcal{P}_{\mathrm{T}}^n]$ and $\mathcal{G}_1 = \langle i\mathbf{1} \rangle$ in corollary 42. According to lemma 2 there exists for some $m \in \mathbf{N}_0$ a lattice Pauli morphism $G : \mathcal{P}_{\mathrm{T}}^m \longrightarrow \mathcal{P}$ with $G[\mathcal{P}_{\mathrm{T}}^m] = \mathcal{Z}_{\mathcal{P}}(\mathcal{P}_1) = \mathcal{P}_\Gamma$ and period 1. The desired elementary TSG is then $\mathcal{S}_0 \subset \mathcal{P}_0$, with $\mathcal{S}_0 := G^{-1}[\mathcal{S} \cap \mathcal{P}_\Gamma] \subset \mathcal{P}_{\mathrm{T}}^m =: \mathcal{P}_0$. Indeed, G is a monomorphism and $\mathcal{Z}_{\mathcal{P}_\Gamma}(\mathcal{P}_\Gamma \cap \mathcal{S}) \propto \mathcal{P}_\Gamma \cap \mathcal{S}$. ∎

Corolary 47 *Two TSGs are locally equivalent iff they have the same characteristic.*

9 Conclusions and outlook

Under the sole assumption of translational invariance, we have provided a detailed study of the structure of two-dimensional TSCs. This structure can always be understood in terms of string operators that carry a topological charge, allowing to extend the insights from well known codes such as the toric code.

Codes can be classified in terms of their topological charges, which are invariant under local transformations. Both for subspace codes and non-chiral subsystem codes, we show that two codes with isomorphic topological charges can be related by a local transformation. The existence of chiral subspace codes remains open.

From a computational perspective, the results have interesting implications. The relevant charges in a code can be arranged into several copies of the toric code or subsystem color codes. But by means of code deformations [27], in these codes the whole Clifford group of gates can be implemented in a topologically protected way. In the case of subsystem color codes this only involves the introduction of twist defects to encode qubits [11]. For toric codes, single qubit Clifford gates can be recovered by encoding in twists [25] and CNot gates by encoding in hole pairs [26, 27]. But it is possible to switch between both encodings, so that also in the toric code deformations are enough to recover the whole Clifford group. As a result —since such techniques do not depend on details of the codes but rather on the charge content—, we can perform all Clifford gates by code deformation in any two-dimensional TSC.

We finish with a discussion of some natural extensions of the present work.

Boundaries — The same ideas that we have used to classify TSCs can be applied to classify boundaries between them. To model them, we can take the right and left half-plane to correspond to two possibly different TSSGs, allowing for arbitrary but translationally symmetric —along the axis direction— gauge and stabilizer generators in the axis between them. Of course, we have to impose the topological condition on $\mathcal{S}$, which still makes sense because we do not want to have localized degrees of freedom on small portions of the boundary.

Clearly there exist natural morphisms from the charge groups of the two TSSGs to the charge group of this boundary TSSG. These turn out to be enough to label all the charges —because no global constraints, and thus no charge, can be confined to the axis. For charges not confined to either half-plane, the values of κ and θ on each side must agree. This is in particular true for the trivial charge. Thus, those charges that form the kernel of each of the two natural morphisms must be bosons with trivial mutual interactions. Such charges are especially relevant because they 'disolve' in the boundary, giving rise to logical string operators with endpoints on the boundary [11].

General codes — It would be more interesting to investigate the structure of general two-dimensional topological quantum error correcting codes, defined in terms of LTI sets of commuting projectors. The expected result is that such codes would be describable in terms of anyon models with additional structure.

Higher dimensions — Even in the case of the relatively simple stabilizer formalism, the general structure of (translationally invariant) topological codes in three dimensions and above turns out to be quite rich. Already in the three-dimensional case there exist examples that do not fit on the standard homological picture [12, 14]. This is of great interest due to the possible thermal stability [32] of the corresponding quantum memories, protected by the local

Hamiltonian.

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