

Non-commuting two-local Hamiltonians for quantum error suppression

Zhang Jiang^{1,2,*} and Eleanor G. Rieffel^{1,†}

¹*Quantum Artificial Intelligence Laboratory (QuAIL),*

NASA Ames Research Center, Moffett Field, California 94035, USA

²*Stinger Ghaffarian Technologies Inc., 7701 Greenbelt Rd., Suite 400, Greenbelt, MD 20770*

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Physical constraints make it challenging to implement and control multi-body interactions. For this reason, designing quantum information processes with Hamiltonians consisting of only one- and two-local terms is a worthwhile challenge. A common approach to robust storage of quantum information is to encode it in the ground subspace of a Hamiltonian. Even allowing particles with high Hilbert-space dimension, it is impossible to protect quantum information from single-site errors by encoding in the ground subspace of any Hamiltonian containing only commuting two-local terms [1]. Here, we demonstrate how to get around this no-go result by encoding in the ground subspace of a Hamiltonian consisting of non-commuting two-local terms arising from the gauge operators of a subsystem code. Specifically, we show how to protect stored quantum information against single-qubit errors using a Hamiltonian consisting of sums of the gauge generators from Bacon-Shor codes [2] and generalized-Bacon-Shor code [3]. Our results imply that non-commuting two-local Hamiltonians have more error-suppressing power than commuting two-local Hamiltonians. Finally, we comment briefly on the robustness of the whole scheme.

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Introduction. In order to realize the benefits of quantum computation [4, 5], machines capable of robust quantum computation must be built. An active area of research explores what resources are needed to achieve physically implementable robust quantum computing. Even determining the resources required to support the robust storage of quantum information is challenging [6]. While the ultimate goal is to achieve a fully fault-tolerant implementation, it is also important to determine what resources enable more robust, if not fully fault tolerant, quantum computations, so as to enable a richer set of experiments to be performed on hardware implementable in the near term.

One important question is when are many-body interactions required or when do two-local interactions suffice. Two-local interactions have been realized in the leading candidates for performing near-term quantum computation, while 3-local and higher locality interactions remain an implementation challenge. Within adiabatic quantum computation, a common approach to increase robustness is error suppression through energy gap protection (EGP) [7, 8] in which quantum information is stored in the ground subspace of a Hamiltonian derived from an error detecting code, and an error-suppressing term is added to the total Hamiltonian to penalize states outside that subspace. For quantum storage, which we consider here, there are no other terms in the Hamiltonian other than the error-suppressing term.

Unfortunately, an error-suppressing Hamiltonian constructed from a stabilizer code must contain interactions involving four or more qubits to protect against single-qubit errors. In [1], Marvian and Lidar proved a stronger negative result, namely that energy gap protection using an error suppressing Hamiltonian that is the sum

of two-local commuting terms cannot suppress single-site errors even with single particles of arbitrarily large Hilbert space dimension. Their proof uses a powerful theorem due to Bravyi and Vyalii [9], and generalized by Aharonov and Eldar [10], which says that ground states of commuting two-local Hamiltonians can only have short range two-body entanglement. This no go result implies that, in the words of Marvian and Lidar, “there is no advantage to using such codes.” Here, we answer positively the specific question of whether a Hamiltonian consisting of non-commuting two-local terms can be useful in combating single-qubit errors. We are not making broader claims about fault tolerance or quantum memories.

As Marvian and Lidar recognized, one potential way around their no-go result is to consider subsystem codes. However, unlike the commuting case in which the Hamiltonians are automatically gapped [9], there is no known general theorem on the gaps for non-commuting Hamiltonians. Here, we show that indeed one can obtain single-qubit error suppression by encoding in the ground subspace of a Hamiltonian consisting of a weighted sum of non-commuting two-local terms from the gauge generators of Bacon-Shor code [2] and Bravyi’s generalized-Bacon-Shor code [3]. Both of the Bacon-Shor and the generalized-Bacon-Shor codes satisfy the Knill-Laflamme condition exactly, and therefore there is no induced degeneracy splitting that leads to the difficulty identified by Marvian and Lidar in the commuting case. In particular, these codes do not come up against their generalization of the no-hiding theorem.

We explore three specific examples in detail, the $[[4, 1, 2]]$ Bacon-Shor code and the $[[6, 2, 2]]$ and $[[16, 2, 3]]$ generalized-Bacon-Shor codes. In each case, we compute the separation in energy between the ground subspace

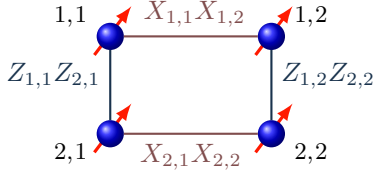


FIG. 1: The gauge generators of the Bacon-Shor $[[4, 1, 2]]$ code consist of two-qubit operators of type XX (ZZ) that couple qubits in a row (column).

and the orthogonal subspaces, making use of a technique to reduce the dimension of the Hilbert space that needs to be considered. Further, for the first two codes, we perform a numerical analysis of the open system dynamics with the spin-boson error model, confirming exponential suppression of single-qubit errors. We conclude with brief remarks on the robustness of this approach to control errors.

Suppression with the Bacon-Shor $[[4, 1, 2]]$ code.

The most familiar subsystem code is the Bacon-Shor $[[9, 1, 3]]$ code, the smallest Bacon-Shor code that corrects single-qubit errors. For error suppression, codes that only *detect* errors can be used, allowing us to consider the smallest error-detecting Bacon-Shor code, the $[[4, 1, 2]]$ -code only *detects* single-qubit errors. This code has been used before, by Brell *et al.* [11], as a gadget to obtain the Hamiltonians for the toric code and Kitaev's quantum double models as the low-energy limits of two-body Hamiltonians.

The gauge group for the $[[4, 1, 2]]$ -code is

$$\mathcal{G} = \langle G_1^X, G_2^X, G_1^Z, G_2^Z \rangle, \quad (1)$$

which is generated by the gauge generators

$$\begin{aligned} G_1^X &= X_{1,1}X_{1,2}, & G_2^X &= X_{2,1}X_{2,2}, \\ G_1^Z &= Z_{1,1}Z_{2,1}, & G_2^Z &= Z_{1,2}Z_{2,2}, \end{aligned} \quad (2)$$

shown in Fig. 1. The stabilizer subgroup is generated by

$$\begin{aligned} S^X &= X_{1,1}X_{1,2}X_{2,1}X_{2,2}, \\ S^Z &= Z_{1,1}Z_{2,1}Z_{1,2}Z_{2,2}. \end{aligned} \quad (3)$$

These two stabilizers define a $[[4, 2, 2]]$ -stabilizer code. Subsystem stabilizer codes [2, 12] can be viewed as stabilizer codes in which only some of the logical qubits are used to encode quantum information. The $[[4, 1, 2]]$ -subsystem code encodes information in only one of the logical qubits of the $[[4, 2, 2]]$ -stabilizer code. One choice of logical operators is

$$X_L = X_{1,1}X_{2,1}, \quad Z_L = Z_{1,1}Z_{1,2}. \quad (4)$$

A convenient choice of auxiliary operators that define the unused qubit are the elements of the gauge group

$$X_A = G_1^X = X_{1,1}X_{1,2}, \quad Z_A = G_1^Z = Z_{1,1}Z_{2,1}. \quad (5)$$

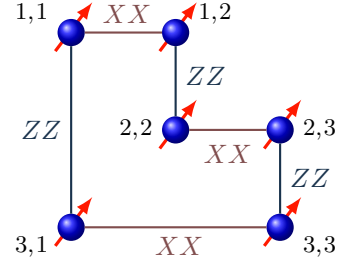


FIG. 2: The gauge generators of Bravyi's $[[6, 2, 2]]$ code formed using 6 physical qubits placed within a 3×3 lattice. The gauge generators consist two-qubit operators of type XX (ZZ) coupling qubits in a row (column).

Consider an error-suppressing Hamiltonian that is a weighted sum of the gauge generators, where the weights λ_i and η_i are real:

$$\begin{aligned} H_{\text{supp}} &= -\lambda_1 G_1^X - \eta_1 G_1^Z - \lambda_2 G_2^X - \eta_2 G_2^Z \\ &= -\lambda_1 X_A - \eta_1 Z_A - \lambda_2 S^X X_A - \eta_2 S^Z Z_A \\ &= -(\lambda_1 + \lambda_2 S^X) X_A - (\eta_1 + \eta_2 S^Z) Z_A, \end{aligned} \quad (6)$$

where the last line uses the relation $G_2^X = S^X X_A$ and $G_2^Z = S^Z Z_A$. This Hamiltonian, being a linear combination of the gauge generators, commutes with the stabilizers and logical operators. It takes a block diagonal form in the eigenbasis of the stabilizers. For fixed values of the stabilizers, we have

$$H_{\text{supp}}^{(x,z)} = -(\lambda_1 + \lambda_2 x) X_A - (\eta_1 + \eta_2 z) Z_A \quad (7)$$

where $x, z = \pm 1$ are the values of S^X and S^Z , respectively. The eigenvalues of this Hamiltonian are

$$E_{\text{supp}}^{(x,z)} = \pm \sqrt{(\lambda_1 + x\lambda_2)^2 + (\eta_1 + z\eta_2)^2}. \quad (8)$$

For the case where $\lambda_1, \lambda_2 > 0$ and $\eta_1, \eta_2 > 0$, the ground subspace of $H_{\text{supp}}^{(x^*, z^*)}$ with $x^* = z^* = 1$ has energy strictly smaller than that of $E_{\text{supp}}^{(x,z)}$ with $(x, z) \neq (1, 1)$. Any single-qubit error on logical information encoded in the ground state of $H_{\text{supp}}^{(x^*, z^*)}$ is suppressed, because it raises the energy by changing the value of x^* or z^* . For $\lambda_1 = \lambda_2 = \eta_1 = \eta_2 = 1$, the eigenvalues are $\pm\sqrt{8}, \pm 2, 0$, so the gap is $2(\sqrt{2} - 1)$. Furthermore, since any set of positive coefficients works, the ground subspace of H_{supp} is robust to small implementation errors in the weights of the terms in H_{supp} .

Suppression with Bravyi's $[[6, 2, 2]]$ code. We now turn to Bravyi's $[[6, 2, 2]]$ generalized-Bacon-Shor code [3]. It has the advantage of being more compact, encoding two logical qubits in six physical qubits, and has the further advantage, as we will see, that certain two-qubit logical operators can be implemented using operations involving only two physical qubits.

The following matrix defines Bravyi's $[[6, 2, 2]]$ code

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}. \quad (9)$$

The gauge group of Bravyi's code is

$$\mathcal{G} = \langle G_1^X, G_2^X, G_3^X, G_1^Z, G_2^Z, G_3^Z \rangle, \quad (10)$$

generated by the gauge generators as shown in Fig. 2:

$$\begin{aligned} G_1^X &= X_{1,1}X_{1,2}, & G_2^X &= X_{2,2}X_{2,3}, & G_3^X &= X_{3,1}X_{3,3}, \\ G_1^Z &= Z_{1,1}Z_{3,1}, & G_2^Z &= Z_{1,2}Z_{2,2}, & G_3^Z &= Z_{2,3}Z_{3,3}. \end{aligned} \quad (11)$$

The center of \mathcal{G} is generated by the stabilizers

$$S^X = G_1^X G_2^X G_3^X, \quad S^Z = G_1^Z G_2^Z G_3^Z. \quad (12)$$

One choice for the logical operators is

$$\begin{aligned} X_{L1} &= X_{2,3}X_{3,3}, & Z_{L1} &= Z_{3,1}Z_{3,3}, \\ X_{L2} &= X_{1,2}X_{2,2}, & Z_{L2} &= Z_{1,1}Z_{1,2}. \end{aligned} \quad (13)$$

An advantage of this choice is that the logical operators are all two-local. Further, we have the following representations

$$X_{L1}X_{L2} = S^X X_{1,1}X_{3,1}, \quad Z_{L1}Z_{L2} = S^Z Z_{2,2}Z_{2,3}, \quad (14)$$

so certain two-qubit logical operators, those acting on two logical qubits encoded together, can be implemented with two-local interactions only. When multiple qubits are encoded using multiple copies of the code, two-qubit logical operators acting on logical qubits encoded by different copies will still need four-body interactions. Nevertheless, this code has an advantage over the $[[4, 1, 2]]$ code in which every logical qubit is encoded separated so one needs four-body interactions to implement every two-qubit logical operation.

The auxiliary logical operators can be chosen to be the gauge elements

$$\begin{aligned} X_{A1} &= X_{3,1}X_{3,3} = G_3^X, & Z_{A1} &= Z_{2,3}Z_{3,3} = G_3^Z, \\ X_{A2} &= X_{1,1}X_{1,2} = G_1^X, & Z_{A2} &= Z_{1,2}Z_{2,2} = G_2^Z. \end{aligned} \quad (15)$$

The original gauge generators can be expressed using the stabilizers and the unused logical operators

$$\begin{aligned} G_1^X &= X_{A2}, & G_2^X &= S^X X_{A1}X_{A2}, & G_3^X &= X_{A1}, \\ G_1^Z &= S^Z Z_{A1}Z_{A2}, & G_2^Z &= Z_{A2}, & G_3^Z &= Z_{A1}. \end{aligned} \quad (16)$$

Consider an error-suppressing Hamiltonian that is a weighted sum of the gauge generators

$$\begin{aligned} H_{\text{supp}} &= -\lambda_1 G_3^X - \lambda_2 G_1^X - \lambda'_1 G_3^Z - \lambda'_2 G_2^Z \\ &\quad - \eta G_2^X - \eta' G_1^Z, \end{aligned} \quad (17)$$

where all coefficients are positive real numbers. For fixed values of the stabilizers S^X and S^Z , the Hamiltonian (17) can be written in terms of the auxiliary logical operators

$$\begin{aligned} H_{\text{supp}}^{(x,z)} &= -\lambda_1 X_{A1} - \lambda_2 X_{A2} - \lambda'_1 Z_{A1} - \lambda'_2 Z_{A2} \\ &\quad - \eta x X_{A1}X_{A2} - \eta' z Z_{A1}Z_{A2}. \end{aligned} \quad (18)$$

To suppress errors detectable by the code, the ground states of H_{supp} should also be eigenstates of the stabilizers with fixed eigenvalues $x = x^*$ and $z = z^*$. Consequently, the Hamiltonian $H_{\text{supp}}^{(x,z)}$ must depend on the values of x or z , so we must have $\eta \neq 0$ and $\eta' \neq 0$. In general, to achieve error suppression, none of the coefficients in Eq. (18) can be zero. For example, we must have $\lambda_1 \neq 0$; otherwise, the eigenvalues of $H_{\text{supp}}^{(x,z)}$ would not depend on the value of x . We can show this by simply flipping the sign of the term with coefficient $-\eta x$ by applying the unitary transformation $Z_{A1} H_{\text{supp}}^{(x,z)} Z_{A1}^\dagger$, without changing anything else.

A particular choice of the error-suppressing Hamiltonian that can be solved easily is

$$\begin{aligned} H_{\text{supp}}^{(x,z)} &= -\lambda(X_{A1} + X_{A2} + Z_{A1} + Z_{A2}) \\ &\quad - \eta(xX_{A1}X_{A2} + zZ_{A1}Z_{A2}), \end{aligned} \quad (19)$$

where $\lambda, \eta > 0$. In the Bell basis of the two auxiliary logical qubits, we have

$$H_{\text{supp}}^{(s_\pm)} = -2 \begin{pmatrix} \eta s_+ & \lambda & \lambda & 0 \\ \lambda & \eta s_- & 0 & 0 \\ \lambda & 0 & -\eta s_- & 0 \\ 0 & 0 & 0 & -\eta s_+ \end{pmatrix}, \quad (20)$$

where $s_\pm = (x \pm z)/2 \in \{-1, 0, 1\}$, and the singlet state is always decoupled from the other states. The eigenvalues of this matrix are $\{-\eta s_+ \pm \sqrt{8\lambda^2 + \eta^2}, 2\eta s_+, 0\}$, for $s_+ = \pm 1$ and $s_- = 0$, and $\{\pm 2\sqrt{2\lambda^2 + \eta^2}, 0, 0\}$, for $s_+ = 0$ and $s_- = \pm 1$. Thus, the ground subspace for the case $x = z = 1$ has the lowest energy $-(\eta + \sqrt{8\lambda^2 + \eta^2})$, and the two ground states for $x = -z = \pm 1$ have the second-lowest energy $-2\sqrt{2\lambda^2 + \eta^2}$. The gap is $\eta + \sqrt{8\lambda^2 + \eta^2} - 2\sqrt{2\lambda^2 + \eta^2}$. The gap goes to η for $\lambda \rightarrow \infty$, and is zero for $\lambda \rightarrow 0$. It equals to $4 - 2\sqrt{3} \simeq 0.536$ for $\lambda = \eta = 1$.

Numerical analysis of suppression. We consider a spin-boson error model [13] that generalizes the one used in [14], in which individual system qubits are coupled to independent baths via all Pauli operators not just Z operators:

$$\begin{aligned} H &= H_{\text{supp}} + H_B \\ &\quad + \sum_{k=1}^n \left(X_k \otimes B_k^X + Y_k \otimes B_k^Y + Z_k \otimes B_k^Z \right), \end{aligned} \quad (21)$$

where H_B is the bath Hamiltonian, B_k^X is the bath operator coupling to the X operator of the k th qubit, and

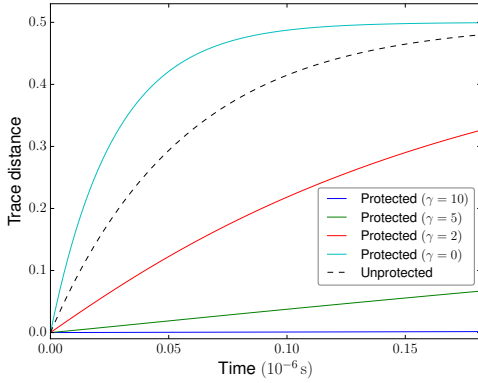


FIG. 3: The trace distance between the ideal state and evolved state under single-qubit noise for different penalty weights $\gamma = \lambda/k_B T$. The results are obtained from numerical simulation of the open-system dynamics for a single logical qubit encoded in the $[[4, 1, 2]]$ subsystem code. The initial state of the logical qubit is set to be $|+\rangle_L$, the $+1$ eigenstate of X_L . A spin-boson error model of Ohmic spectral is used.

similarly for B_k^Y and B_k^Z (see the Supplemental Material for more details). The error-suppressing Hamiltonian is proportional to the sum over a generating set \mathcal{R} for \mathcal{G} , $H_{\text{supp}} = -\lambda \sum_{G \in \mathcal{R}} G$.

We assume that the bath spectral functions $C_{\text{bath}}(\omega)$ are the same for all physical qubits. Its Fourier transformation takes the following form for the Ohmic case,

$$\tilde{C}_{\text{bath}}(\omega) = \frac{2\pi\hbar^2\chi\omega e^{-|\omega|/\omega_c}}{1 - e^{-\hbar\omega/k_B T}}, \quad (22)$$

where T is the temperature, ω_c is the cut-off frequency, and χ is a dimensionless parameter proportional to the product of the system-bath coupling strength and the bath spectral density. For $\omega < 0$ (a transition that raises the energy of the system) and $|\omega|$ sufficiently large, the term in the denominator becomes most salient, yielding exponential suppression of decoherence.

In our analysis, we use the parameters from [14]: $\chi = 3.18 \times 10^{-4}$, $\omega_c = 8\pi \times 10^9$ rad/s, and $k_B T/\hbar = 2.2 \times 10^9$ rad/s (at 17 mK). In Fig. 3 we show results for the $[[4, 1, 2]]$ and subsystem codes at various implementable values of $\gamma = \lambda/k_B T$. Error suppression is helpful once γ is larger than a threshold value of $\simeq 0.6$. Below that value, the system is more vulnerable to decoherence because the encoded logical state is an entangled state involving more physical qubits than the unprotected state, and thus is exposed to more noise. We obtained similar results for the $[[6, 2, 2]]$ code, and also examined the preservation of the entanglement of an encoded Bell state, using both the $[[4, 1, 2]]$ code and the $[[6, 2, 2]]$ code (see Supplemental Material).

The decoherence effects on qubits under noise mod-

eled by coupling to bosonic baths with Ohmic spectra are exponentially suppressed when the energy separation of H_{supp} becomes larger than $k_B T$. A physical interpretation for that condition is: the bath modes, which resonate with the transition frequencies of the system to higher-energy states, are close to the vacuum state at the temperature T and thus are not capable of driving such transitions.

Generalized construction. We have seen that it is useful to write the error-suppressing Hamiltonian in terms of the auxiliary operators and the stabilizers. Doing so removes the degeneracy in H_{supp} and reduces the size of the Hilbert space under consideration, enabling numerical calculation of the energy separation between the code subspace and orthogonal subspaces. Such a reduction is always possible for generalized-Bacon-Shor codes; in the Supplemental Material we give an algorithm that provides a systematic way of finding auxiliary operators satisfying the standard commutation relations that, together with the stabilizer operators, generate the gauge group.

We made use of this algorithm to compute the energy separation for the $[[16, 2, 3]]$ generalized-Bacon-Shor code. Again, we may encode information in the ground subspace with all stabilizers taking value $+1$. Then, for $\lambda = 1$, when the error suppression term is precisely the (unweighted) sum of all gauge generators, the ground subspace energy is -13.83 and the energy separation between the code space and the orthogonal subspaces is 0.33 .

We can apply this construction to generalized-Bacon-Shor codes that encode a larger number of qubits at once, thus increasing the proportion of logical operators that can be implemented using only two-local interactions. However, for these larger codes, computing the energy separation becomes computationally prohibitive. An open question is whether it is possible to obtain a non-zero asymptotic separation for some family of generalized-Bacon-Shor codes.

Robustness to implementation errors. When the weight of any implementation error is less than the code distance, the implementation errors in H_{supp} are suppressed by H_{supp} itself. Even when the code distance is smaller or equal to the weight of H_{supp} , our scheme is still robust to simple errors which change the weight of the terms in H_{supp} . Such errors can change the ground state of the auxiliary qubits, and thus could affect the logical operators, depending on how they are implemented, but that is not a concern for the simple quantum storage case considered here.

Conclusion. We have introduced a scheme for energy gap protection through the encoding of quantum information in the ground subspace of an error-suppressing Hamiltonian H_{supp} consisting of non-commuting two-local terms. The Hamiltonian H_{supp} is a weighted sum of the gauge generators of a stabilizer subsystem code.

Subsystem codes offer great flexibility in designing mechanisms to support robust quantum computation, because how errors affect the unused logical qubits is not a concern [15, 16]. Subsystem codes support routines that make use of the subsystem structure to break stabilizer terms into lower-weight terms from gauge generators. We showed how to use Bacon-Shor codes and generalized-Bacon-Shor codes to design two-local Hamiltonians that suppress arbitrary single-qubit errors.

As examples, we explored the $[[4, 1, 2]]$ Bacon-Shor code and $[[6, 2, 2]]$ and $[[16, 2, 3]]$ generalized-Bacon-Shor code, computing the energy separation and, for the first two, simulating the open-system dynamics. One advantage of generalized-Bacon-Shor codes is the ability to implement certain two-qubit logical operators by involving only two physical qubits. Finally, we discussed the robustness of the whole scheme to implementation errors.

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[†] eleanor.rieffel@nasa.gov

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* zhang.jiang@nasa.gov

SUPPLEMENTARY MATERIAL

Rewriting the Hamiltonian in terms of auxiliary operators and stabilizers. A critical step in calculating the energy separation for a two-local error-suppressing Hamiltonian arising from subsystem codes is rewriting the Hamiltonian in terms of auxiliary operators and stabilizers. This procedure makes explicit the dependence of the error-suppressing Hamiltonian on the values of the stabilizers. It also makes easier the analytical and numerical calculations of the energy separation by reducing the size of the Hilbert space that needs to be considered and removes the degeneracy in the error-suppressing Hamiltonian.

Here, we describe a systematic method for finding the $m - k$ X -type stabilizers, $m - k$ Z -type stabilizers, and $n - 2(m - k) - k$ X -type auxiliary logical operators, and $n - 2(m - k) - k$ Z -type auxiliary logical operators, defining $n - 2(m - k) - k$ auxiliary qubits for a $[[n, k, d]]$ generalized-Bacon-Shor code defined by a $m \times m$ binary matrix M of rank k . We illustrate the application of this algorithm by using it to obtain a set of auxiliary operators and stabilizers for the $[[16, 2, 3]]$ generalized-Bacon-Shor code.

We first present pseudocode for the algorithm and then comment on its workings.

```

 $\mathcal{A} \leftarrow \emptyset$  ▷ Holds set of auxiliary operators
 $\mathcal{S} \leftarrow \emptyset$  ▷ Holds set of stabilizer generators
procedure ROW EXTRACTION( $M$ )
  Let the set  $\mathcal{R}$  hold all the rows of  $M$ 
   $\mathcal{R}_{rem} \leftarrow \mathcal{R}$  ▷ Holds remaining rows
   $\mathcal{R}_{cur} \leftarrow \emptyset$  ▷ Holds rows under consideration
  Add the top row of  $M$  to  $\mathcal{R}_{cur}$ 
  while  $\mathcal{R}_{rem}$  linearly dependent do
    if  $\mathcal{R}_{cur}$  linearly independent of  $\mathcal{R}_{rem} \setminus \mathcal{R}_{cur}$  then
      Move the rows in  $\mathcal{R}_{cur}$  to the bottom of the matrix and set  $\mathcal{R}_{cur} \leftarrow \emptyset$ 
      Add the top row to  $\mathcal{R}_{cur}$ 
    end if
    Add a minimally linearly independent set of rows in  $\mathcal{R}_{rem} \setminus \mathcal{R}_{cur}$  to  $\mathcal{R}_{cur}$  to make the top row linearly
    dependent on these rows and other rows in  $\mathcal{R}_{cur}$ 
    Move these rows to the top of the matrix, above rows already in  $\mathcal{R}_{cur}$ 
    Add to  $\mathcal{S}$  the product of  $Z$ -type operators for every qubit in the rows of  $\mathcal{R}_{cur}$ 
    Add  $X$ -type operators to  $\mathcal{A}$  corresponding to all pairs of qubits in the top row containing the first qubit
    For all qubits in the top row except the first qubit, add  $Z$ -type operators to  $\mathcal{A}$  that connect that qubit with
    the next qubit below it in its column
    Remove the top row from  $\mathcal{R}_{cur}$  and  $\mathcal{R}_{rem}$ 
  end while
  Form  $M'$  from  $\mathcal{R}_{rem}$ 
end procedure
procedure COLUMN EXTRACTION( $M'$ )
  Let the set  $\mathcal{C}'$  hold all the columns of  $M'$ 
   $\mathcal{C}_{rem} \leftarrow \mathcal{C}'$ 
   $\mathcal{C}_{cur} \leftarrow \emptyset$ 
  Add the far left column of  $M'$  to  $\mathcal{C}_{cur}$ 
  while  $\mathcal{C}_{rem}$  linearly dependent do
    if  $\mathcal{C}_{cur}$  linearly independent of  $\mathcal{C}_{rem} \setminus \mathcal{C}_{cur}$  then
      Move the columns in  $\mathcal{C}_{cur}$  to the right side of the matrix and set
       $\mathcal{C}_{cur} \leftarrow \emptyset$ 
      Add the far left column to  $\mathcal{C}_{cur}$ 
    end if
    Add a minimally linearly independent set of columns in  $\mathcal{C}_{rem} \setminus \mathcal{C}_{cur}$  to  $\mathcal{C}_{cur}$  to make it linearly dependent
    Move these columns to the left side of the matrix
    Add to  $\mathcal{S}$  the product of  $X$ -type operators for every qubit in the corresponding columns to  $\mathcal{C}_{cur}$  of the original
    matrix  $M$ 
    Add  $Z$ -type operators to  $\mathcal{A}$  corresponding to all pairs of qubits in the left-most column containing the first
    qubit
    for each qubit in the left-most column except the first qubit do

```



```

    Let  $A_X$  be the  $X$ -type operator connecting that qubit with the next qubit in its row
    for  $Z$ -type operator  $A'_Z$  in  $\mathcal{A}$  do
        if  $A_X$  anti-commutes with  $A'_Z$  then
            Find the  $X$ -type operator  $A'_X$  that anticommutes with  $A'_Z$ 
             $A_X \leftarrow A_X A'_X$ 
        end if
    end for
end for
Remove the left-most column from  $\mathcal{C}_{cur}$  and  $\mathcal{C}_{rem}$ 
end while
Form  $M''$  from  $\mathcal{C}_{rem}$ 
end procedure
procedure CORE EXTRACTION( $M''$ )
    for each pair of adjacent qubits in same column of  $M''$  do
        Let  $A_Z$  be the  $Z$ -type operator for this pair
        for  $X$ -type operator  $A'_X$  in  $\mathcal{A}$  do
            if  $A_Z$  anti-commutes with  $A'_X$  then
                Find the  $Z$ -type operator  $A'_Z$  that anticommutes with  $A'_X$ 
                 $A_Z \leftarrow A_Z A'_Z$ 
            end if
        end for
        Add  $A_Z$  to  $\mathcal{A}$ .
    end for
    for each adjacent pair of qubits in same row of  $M''$  do
        Let  $A_X$  be the  $X$ -type operator for this pair
        for  $Z$ -type operator  $A'_Z$  in  $\mathcal{A}$  do
            if  $A_X$  anti-commutes with  $A'_Z$  then
                Find the  $X$ -type operator  $A'_X$  that anticommutes with  $A'_Z$ 
                 $A_X \leftarrow A_X A'_X$ 
            end if
        end for
        Add  $A_X$  to  $\mathcal{A}$ 
    end for
end procedure

```

Switching the order of the rows of a matrix defining a generalized-Bacon-Shor code does not change the gauge group, only the gauge generators. Moving rows added to \mathcal{R}_{cur} to the top of the matrix ensures that the auxiliary X -type operators we define commute with all previously defined auxiliary operators. Specifically, the Z -type operators previously defined do not involve this row. The Z -type operators we define are all within \mathcal{R}_{cur} , because the constraint of minimal linear dependence guarantees that there will be a non-zero entry in the column below each qubit in the top row. We exclude the left-most column Z -type operator because it is the product of the stabilizer and the other Z -type operators which we define. Because we will not be including that column, we choose X -type operators that include the first qubit since that enables us to satisfy easily the canonical commutation relation for X - and Z -type operators defining auxiliary qubits. Once we are considering columns, we must be more careful. While the Z -type operators we define automatically commute with all previously defined operators, because those rows were not used to extract X -type operators, this property is not guaranteed for the X -type operators, which is why the extraction of X -type operators at this state is more complicated than for Z -type operators.

Example: extracting auxiliary operators and stabilizers for the $[[16, 2, 3]]$ code. The following symmetric matrix defines a generalized-Bacon-Shor $[[16, 2, 3]]$ code,

$$M_{5|5} = \begin{matrix} & c_1 & c_2 & c_3 & c_4 & c_5 \\ \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{matrix}. \quad (23)$$

The subscripts in $M_{5|5}$ indicates the numbers of the rows and columns, respectively. We use c_j and r_k to label the

row and column indices, so that we can keep track of the qubits after row and column manipulations. The rank of the matrix $M_{5|5}$ is 2, and the distance of the code is 3 (since any linear combination of its rows or columns has either at least three 1s or none). We will use the algorithm to find the 3 X -type stabilizers, the 3 Z -type stabilizers, and the 8 auxiliary qubits defined by 8 pairs of auxiliary operators satisfying the canonical commutation relations.

Moving row r_3 to the top of the matrix yields

$$M'_{5|5} = \begin{matrix} & c_1 & c_2 & c_3 & c_4 & c_5 \\ \begin{matrix} \underline{r_3} \\ \underline{r_1} \\ r_2 \\ r_4 \\ r_5 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{matrix}, \quad (24)$$

where we underline the row labels to indicate the rows (columns) we are currently considering, rows that are in \mathcal{R}_{cur} . Since the top two rows in $M'_{5|5}$ are identical, and thus linearly dependent, we define a stabilizer $S_1^Z = R_3^Z R_1^Z$, where $R_3^Z = Z_{3,2}Z_{3,4}Z_{3,5}$ and $R_1^Z = Z_{1,2}Z_{1,4}Z_{1,5}$. We now extract auxiliary operators as we eliminate the top row. We define two auxiliary operators $X_{A1} = X_{3,2}X_{3,4}$ and $X_{A2} = X_{3,2}X_{3,5}$, and define the corresponding Z -type auxiliary operators to be $Z_{A1} = Z_{3,4}Z_{1,4}$ and $Z_{A2} = Z_{3,5}Z_{1,5}$. It is easy to check that two pairs of auxiliary operators satisfy the standard commutation relations. Note $Z_{3,2}Z_{1,2} = S_1^Z Z_{A1}Z_{A2}$. Having used the top row to obtain auxiliary operators and stabilizers, we may remove the top row. We consider the resulting matrix

$$M_{4|5} = \begin{matrix} & c_1 & c_2 & c_3 & c_4 & c_5 \\ \begin{matrix} \underline{r_1} \\ r_2 \\ r_4 \\ r_5 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{matrix}. \quad (25)$$

A minimally linearly independent set on which the top row is linearly dependent is $\{r_2, r_5\}$. We move these rows to the top of the matrix to obtain

$$M'_{4|5} = \begin{matrix} & c_1 & c_2 & c_3 & c_4 & c_5 \\ \begin{matrix} \underline{r_2} \\ \underline{r_5} \\ \underline{r_1} \\ r_4 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{matrix}. \quad (26)$$

We define a stabilizer $S_2^Z = R_2^Z R_5^Z R_1^Z$, where $R_2^Z = Z_{2,1}Z_{2,3}Z_{2,5}$, $R_5^Z = Z_{5,1}Z_{5,2}Z_{5,3}Z_{5,4}$, and $R_1^Z = Z_{1,2}Z_{1,4}Z_{1,5}$. We define X -type auxiliary operators $X_{A3} = X_{2,1}X_{2,3}$ and $X_{A4} = X_{2,1}X_{2,5}$, and Z -type auxiliary operators $Z_{A3} = Z_{2,3}Z_{5,3}$, and $Z_{A4} = Z_{2,5}Z_{1,5}$. We may now remove the top row to obtain

$$M_{3|5} = \begin{matrix} & c_1 & c_2 & c_3 & c_4 & c_5 \\ \begin{matrix} \underline{r_5} \\ \underline{r_1} \\ r_4 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{matrix}. \quad (27)$$

We move r_4 to the top of the matrix to obtain

$$M'_{3|5} = \begin{matrix} & c_1 & c_2 & c_3 & c_4 & c_5 \\ \begin{matrix} \underline{r_4} \\ \underline{r_5} \\ \underline{r_1} \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{matrix}. \quad (28)$$

We define a stabilizer $S_3^Z = R_4^Z R_5^Z R_1^Z$, where $R_4^Z = Z_{4,1}Z_{4,3}Z_{4,5}$, $R_5^Z = Z_{5,1}Z_{5,2}Z_{5,3}Z_{5,4}$, and $R_1^Z = Z_{1,2}Z_{1,4}Z_{1,5}$. We define X -type auxiliary operators $X_{A5} = X_{4,1}X_{4,3}$ and $X_{A6} = X_{4,1}X_{4,5}$, and Z -type operators $Z_{A5} = Z_{4,3}Z_{5,3}$ and $Z_{A6} = Z_{4,5}Z_{1,5}$. We may now remove the top row to obtain

$$M_{2|5} = \begin{matrix} & c_1 & c_2 & c_3 & c_4 & c_5 \\ \begin{matrix} r_5 \\ r_1 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{matrix}. \quad (29)$$

Now that the rows are linearly independent, we can engage “column elimination” to extract further operators. We move c_3 to the far left to obtain

$$M'_{2|5} = \begin{matrix} & \frac{c_3}{r_5} & \frac{c_1}{r_5} & \frac{c_2}{r_5} & \frac{c_4}{r_5} & \frac{c_5}{r_5} \\ \begin{matrix} r_5 \\ r_1 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{matrix} . \quad (30)$$

Were $M'_{2|5}$ the starting matrix, the first X -type stabilizer would be $\tilde{S}_1^X = X_{5,3}X_{5,1}$. However, this X -type stabilizer does not commute with some of the Z -type auxiliary operators we introduced before. To obtain the correct stabilizer, we need to iteratively multiply \tilde{S}_1^X by X -type operators corresponding to (anti-commuting with) Z -type operators that do not commute with \tilde{S}_1^X . It does not commute with $Z_{A5} = Z_{4,3}Z_{5,3}$, so we need to multiply by $X_{A5} = X_{4,1}X_{4,3}$, which in turn does not commute with $Z_{A3} = Z_{2,3}Z_{5,3}$, so we need to multiply by $X_{A3} = X_{2,1}X_{2,3}$. The result is the stabilizer

$$\begin{aligned} S_1^X &= X_{2,3}X_{4,3}X_{5,3}X_{2,1}X_{4,1}X_{5,1} \\ &= C_3^X C_1^X . \end{aligned} \quad (31)$$

The left-most column contains no pairs of 1s, so we do not extract any auxiliary operators at this step. We may now remove the far left column to obtain

$$M_{2|4} = \begin{matrix} & \frac{c_1}{r_5} & \frac{c_2}{r_5} & \frac{c_4}{r_5} & \frac{c_5}{r_5} \\ \begin{matrix} r_5 \\ r_1 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{matrix} . \quad (32)$$

Columns c_2 and c_5 form a minimally linearly independent set on which the left-most column c_1 depends. We therefore move these columns to the far left to obtain

$$M'_{2|4} = \begin{matrix} & \frac{c_2}{r_5} & \frac{c_5}{r_5} & \frac{c_1}{r_5} & \frac{c_4}{r_5} \\ \begin{matrix} r_5 \\ r_1 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \end{matrix} . \quad (33)$$

A similar argument to the one above leads to defining the stabilizer

$$\begin{aligned} S_2^X &= X_{1,2}X_{3,2}X_{5,2}X_{1,5}X_{2,5}X_{3,5}X_{4,5}X_{2,1}X_{4,1}X_{5,1} \\ &= C_2^X C_5^X C_1^X . \end{aligned} \quad (34)$$

We also define the pair of operators $Z_{A7} = Z_{5,2}Z_{1,2}$ and $\tilde{X}_{A7} = X_{1,2}X_{1,5}$. Since \tilde{X}_{A7} anticommutes with Z_{A2} , Z_{A4} , and Z_{A6} , we have $X_{A7} = \tilde{X}_{A7}X_{A2}X_{A4}X_{A6}$. We now remove the left-most column to obtain

$$M_{2|3} = \begin{matrix} & \frac{c_5}{r_5} & \frac{c_1}{r_5} & \frac{c_4}{r_5} \\ \begin{matrix} r_5 \\ r_1 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \end{matrix} . \quad (35)$$

Moving the row c_4 to far left, we have

$$M'_{2|3} = \begin{matrix} & \frac{c_4}{r_5} & \frac{c_5}{r_5} & \frac{c_1}{r_5} \\ \begin{matrix} r_5 \\ r_1 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{matrix} . \quad (36)$$

These columns contribute a stabilizer

$$\begin{aligned} S_3^X &= C_4^X C_5^X C_1^X \\ &= X_{1,4}X_{3,4}X_{5,4}X_{1,5}X_{2,5}X_{3,5}X_{4,5}X_{2,1}X_{4,1}X_{5,1} . \end{aligned} \quad (37)$$

We also extract the final set of auxiliary operators $Z_{A8} = Z_{5,4}Z_{1,4}$ and $X_{A8} = \tilde{X}_{A8}X_{A1}X_{A2}X_{A4}X_{A6}$, where $\tilde{X}_{A8} = X_{1,4}X_{1,5}$. Removing the first column results in a matrix with linearly independent rows and columns and in which no row or column contains a pair of 1s, therefore we stop extracting operators. Indeed, we have already obtained 3 X -type and 3 Z -type stabilizers and 8 pairs of auxiliary operators as expected.

In conclusion, we have following auxiliary operators

$$X_{A1} = X_{3,2}X_{3,4}, \quad X_{A2} = X_{3,2}X_{3,5}, \quad X_{A3} = X_{2,1}X_{2,3}, \quad X_{A4} = X_{2,1}X_{2,5}, \quad X_{A5} = X_{4,1}X_{4,3}, \quad (38)$$

$$Z_{A1} = Z_{1,4}Z_{3,4}, \quad Z_{A2} = Z_{1,5}Z_{3,5}, \quad Z_{A3} = Z_{2,3}Z_{5,3}, \quad Z_{A4} = Z_{2,5}Z_{1,5}, \quad Z_{A5} = Z_{4,3}Z_{5,3}, \quad (39)$$

$$X_{A6} = X_{4,1}X_{4,5}, \quad X_{A7} = X_{1,2}X_{1,5}X_{A2}X_{A4}X_{A6}, \quad X_{A8} = X_{1,4}X_{1,5}X_{A1}X_{A2}X_{A4}X_{A6}, \quad (40)$$

$$Z_{A6} = Z_{4,5}Z_{1,5}, \quad Z_{A7} = Z_{5,2}Z_{1,2}, \quad Z_{A8} = Z_{5,4}Z_{1,4}, \quad (41)$$

and the stabilizers take the form

$$S_1^X = C_3^X C_1^X, \quad S_2^X = C_2^X C_5^X C_1^X, \quad S_3^X = C_5^X C_1^X C_4^X, \quad (42)$$

$$S_1^Z = R_1^Z R_3^Z, \quad S_2^Z = R_2^Z R_5^Z R_1^Z, \quad S_3^Z = R_5^Z R_1^Z R_4^Z. \quad (43)$$

Noise Model. For our numerical analyses of error suppression, we consider the spin-boson Hamiltonian [13],

$$H(t) = H_S(t) + H_B + \sum_{k=1}^n \left(X_k \otimes B_k^X + Y_k \otimes B_k^Y + Z_k \otimes B_k^Z \right), \quad (44)$$

where $H_S = H_{\text{supp}} + H_{\text{logic}}$ is the system Hamiltonian, and X_k , Y_k , and Z_k are Pauli operators of the k th qubit. Since we are considering storage, $H_{\text{logic}} = 0$. The sum in Eq. (44) describes interactions between individual Pauli operators of the system qubits and independent bath modes, where

$$B_k^X = \sum_{\mu} g_{\mu}^X (a_{\mu,k} + a_{\mu,k}^{\dagger}), \quad B_k^Y = \sum_{\nu} g_{\nu}^Y (b_{\nu,k} + b_{\nu,k}^{\dagger}), \quad B_k^Z = \sum_{\tau} g_{\tau}^Z (c_{\tau,k} + c_{\tau,k}^{\dagger}), \quad (45)$$

with g_{μ}^X , g_{ν}^Y , and g_{τ}^Z being the coupling constants. We will be considering the case in which all of these coupling constants have the same value. The term H_B in Eq. (44) is the bath Hamiltonian,

$$H_B = \sum_{\mu,j} \hbar \omega_{\mu,j}^X a_{\mu,j}^{\dagger} a_{\mu,j} + \sum_{\nu,k} \hbar \omega_{\nu,k}^Y b_{\nu,k}^{\dagger} b_{\nu,k} + \sum_{\tau,l} \hbar \omega_{\tau,l}^Z c_{\tau,l}^{\dagger} c_{\tau,l}. \quad (46)$$

Going to the Heisenberg picture of the bath Hamiltonian, we have

$$B_k(t) = e^{itH_B/\hbar} B_k e^{-itH_B/\hbar} = \sum_{\mu} \left(g_{\mu} a_{\mu,k} e^{-i\omega_{\mu} t} + g_{\mu}^* a_{\mu,k}^{\dagger} e^{i\omega_{\mu} t} \right), \quad (47)$$

where the superscripts X , Y , and Z are neglected for abbreviation of notation. The bath correlation function then takes the form

$$C_{\text{bath}}(j, t; k, t') = \langle B_j(t) B_k(t') \rangle = \delta_{j,k} \sum_{\mu} |g_{\mu}|^2 \left(\langle a_{\mu,k}^{\dagger} a_{\mu,k} \rangle e^{-i\omega_{\mu,k}(t-t')} + \langle a_{\mu,k} a_{\mu,k}^{\dagger} \rangle e^{i\omega_{\mu,k}(t-t')} \right) = \delta_{j,k} C_{\text{bath}}(t - t'). \quad (48)$$

The expectation values in Eq. (48) satisfy the Planck condition for thermal baths,

$$\langle a_{\mu}^{\dagger} a_{\mu} \rangle = \frac{1}{e^{\hbar\omega_{\mu}/k_B T} - 1}, \quad \langle a_{\mu} a_{\mu}^{\dagger} \rangle = \langle a_{\mu}^{\dagger} a_{\mu} \rangle + 1 = \frac{1}{1 - e^{-\hbar\omega_{\mu}/k_B T}}, \quad (49)$$

where the qubit subscript k is omitted. The Fourier transformation of the bath correlation function is

$$\tilde{C}_{\text{bath}}(\omega) = \int dt e^{-i\omega t} C_{\text{bath}}(t) = \frac{2\pi J(|\omega|)}{|1 - e^{-\hbar\omega/k_B T}|}, \quad (50)$$

where $J(\omega)$ is the bath spectral function arising from the substitution of the sum in Eq. (48) with an integral,

$$\sum_{\mu} |g_{\mu}|^2 \simeq \int_0^{\infty} d\omega J(\omega). \quad (51)$$

The bath correlation function determines the transition rate from one system state $|\psi_{\alpha}\rangle$ to another state $|\psi_{\beta}\rangle$,

$$\Gamma_{|\psi_{\alpha}\rangle \rightarrow |\psi_{\beta}\rangle} = \frac{|\langle \psi_{\alpha} | Q | \psi_{\beta} \rangle|^2}{\hbar^2} \tilde{C}_{\text{bath}}((E_{\alpha} - E_{\beta})/\hbar), \quad (52)$$

where E_α and E_β are the energies of the two states and Q is the system noise operator (in this case X , Y , and Z). The ratio of the transition rates between any two states satisfy

$$\Gamma_{|\psi_\alpha\rangle\rightarrow|\psi_\beta\rangle}/\Gamma_{|\psi_\beta\rangle\rightarrow|\psi_\alpha\rangle} = e^{(E_\alpha-E_\beta)/k_B T}, \quad (53)$$

which gives the correct population ratio of $|\psi_\beta\rangle$ and $|\psi_\alpha\rangle$ at thermal equilibrium, i.e., the Boltzmann distribution. The function $\tilde{C}_{\text{bath}}(\omega)$ determines the transition rate from a lower-energy state to a higher energy state when $\omega < 0$, and the other way around when $\omega > 0$. While transitions to higher energy states are detrimental, transitions to lower energy states are beneficial for adiabatic quantum computation.

We further assume that the bath spectral function satisfy the Ohmic condition,

$$J(\omega) \simeq \hbar^2 \chi \omega e^{-\omega/\omega_c}, \quad \text{for } \omega \geq 0, \quad (54)$$

where χ is a dimensionless constant and ω_c is the cutoff frequency. The bath correlation function can thus be simplified to

$$\tilde{C}_{\text{bath}}(\omega) = \frac{2\pi\hbar^2\chi\omega e^{-|\omega|/\omega_c}}{1 - e^{-\omega/\omega_T}}, \quad (55)$$

where $\omega_T = k_B T/\hbar$. We plot this function with the parameters given in [14]: $\chi = 3.18 \times 10^{-4}$, $\omega_c = 8\pi \times 10^9$ rad/s, and $\omega_T = 2.2 \times 10^9$ rad/s (at 17 mK). At zero frequency, i.e., transitions between two states of the same energy,

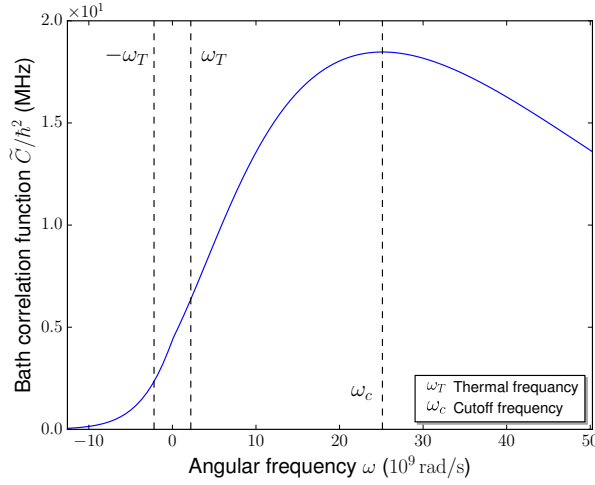


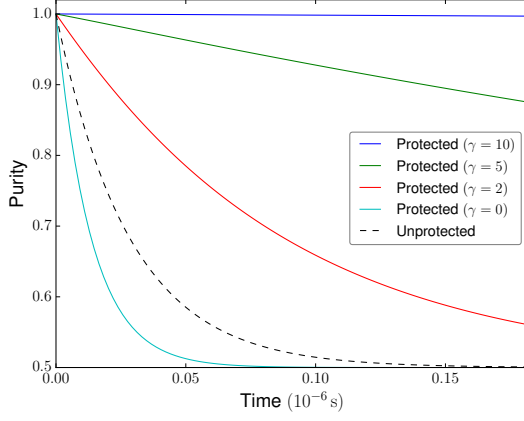
FIG. 4: Bath correlation function for Ohmic noise

$\tilde{C}_{\text{bath}}(0) = 2\pi\hbar^2\chi\omega_T$ is proportional to the temperature T . The derivative of $\tilde{C}_{\text{bath}}(\omega)$ is not continuous at $\omega = 0$ due to the finite cutoff frequency ω_c . The function $\tilde{C}_{\text{bath}}(\omega)$ decays quickly once ω is smaller than $-\omega_T$; the transition rate to higher-energy states is low for an energy difference that is several times larger than $k_B T$. Consequently, an energy gap as large as several times of $k_B T$ can keep the system in the ground state for a much longer time than the gapless case. The asymmetry of the function $\tilde{C}_{\text{bath}}(\omega)$ (see Fig. 4) can also be used to prepare the initial state when the energy gap is less than $\hbar\omega_c$ but larger than $\hbar\omega_T$, as the noise terms drive the system to its ground state while the opposite effect is suppressed.

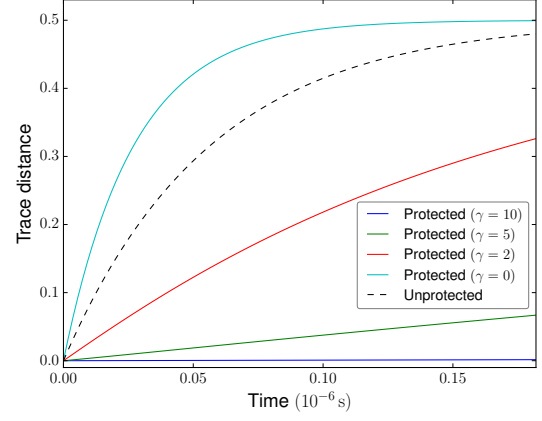
Additional numerical analysis of error suppression. In the main text, we showed results from a simulation of open system dynamics under the spin-boson noise model described above. There we showed the trace distance between the evolved and ideal states for error suppression based on the $[[4, 1, 2]]$ code. Fig. 5 shows the extent to which the purity of the state is maintained during evolution at various strengths γ of the error suppression term.

In addition, here we show, under the same noise model, and for various values of γ ,

- the purity of the state and the trace distance between the evolved and ideal states for two logical qubits, initially in a Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$, encoded together in a single block of the $[[6, 2, 2]]$ code (Fig. 6),
- the entanglement of formation of two logical qubits, initially in a Bell state, each encoded separately using the $[[4, 1, 2]]$ code (Fig. 7a), and

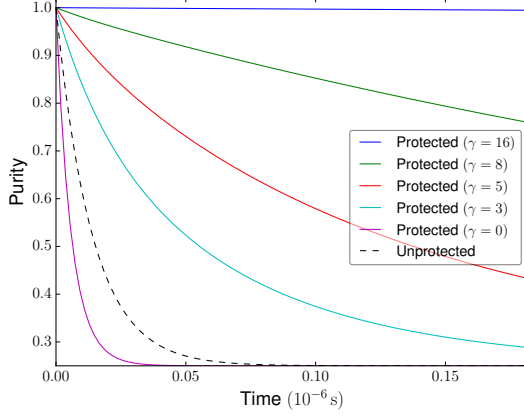


(a) Purity of the evolved state

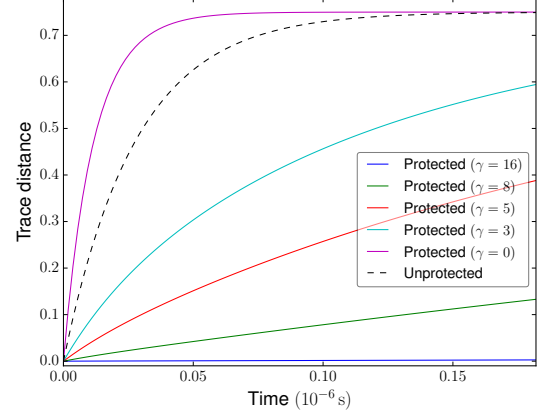


(b) Trace distance between the initial and evolved state

FIG. 5: Simulations of the open-system dynamics of a single logical qubit encoded in the $[[4, 1, 2]]$ subsystem code. The initial state of the logical qubit is set to be the plus state, i.e., the $+1$ eigenstate of X_L .



(a) Purity of the evolved state



(b) Trace distance between the initial and evolved state

FIG. 6: Simulations of the open-system dynamics of the two logical qubits encoded in the $[[6, 2, 2]]$ subsystem code. The initial state for the logical qubits is set to be the Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$.

- the entanglement of formation of two logical qubits, initially in a Bell state, encoded together using the $[[6, 2, 2]]$ code (Fig. 7b).

Entanglement is better preserved by the $[[4, 1, 2]]$ code due to the larger energy separation between its ground subspace and orthogonal subspaces.

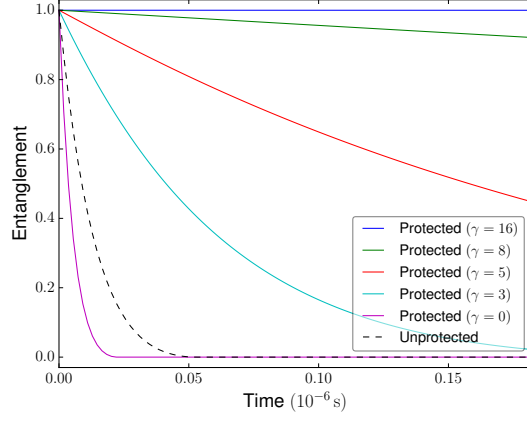
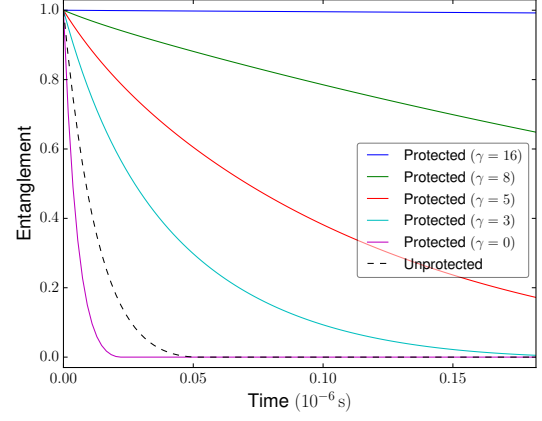
(a) Two qubits encoded separately using the $[[4, 1, 2]]$ code(b) Two qubits encoded together using the $[[6, 2, 2]]$ code

FIG. 7: Simulations of the time evolution of the entanglement between two logical qubits. The initial state is set to be the Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$ of the logical qubits.