Projecting 3D color codes onto 3D toric codes

Arun B. Aloshious and Pradeep Kiran Sarvepalli
Department of Electrical Engineering, Indian Institute of Technology Madras, Chennai 600 036, India
(Dated: June 3, 2016)

Toric codes and color codes are two important classes of topological codes. Kubica, Yoshida, and Pastawski showed that any D-dimensional color code can be mapped to a finite number of toric codes in D-dimensions. In this paper we propose an alternate map of 3D color codes to 3D toric codes with a view to decoding 3D color codes. Our approach builds on Delfosse's result for 2D color codes and exploits the topological properties of these codes. Our result reduces the decoding of 3D color codes to that of 3D toric codes. Bit flip errors are decoded by projecting on one set of 3D toric codes while phase flip errors are decoded by projecting onto another set of 3D toric codes.

I. INTRODUCTION

Three dimensional toric codes [1, 2] and color codes [3] are topological quantum codes defined on 3D lattices. They are generalizations of the surface codes [5] to three dimensions. Like the 2D surface codes they are also Calderbank-Shor-Steane (CSS) codes [6] where the bit flip and phase errors can be corrected independently. However, unlike the 2D surface codes they are inherently asymmetrical in their error correcting capabilities for the bit flip errors and the phase flip errors. Perhaps for this reason they have not received as much attenton as their 2D counterparts. Nonetheless the growing interest in asymmetric error models [7–9] motivates us to study them in closer detail.

Another reason for studying 3D (color) codes comes from the fact that in some ways they are also richer than 2D codes. Certain 3D color codes also possess a transversal non-Clifford gate [3]. This is not possible for 2D codes or 3D toric codes. While the 3D toric code on a cubic lattice has been studied in [1, 2, 4], the general case of an arbitrary lattice has not been investigated as much.

For these codes to be useful for fault tolerant quantum computing it is necessary to develop efficient decoding algorithms. However, there appears to be no previous work on the decoding of 3D color codes. While we do not solve this problem in this paper, we make some progress in the decoding of 3D color codes by reducing it to the decoding of the 3D toric codes. Errors corresponding to chains in the lattice can be easily decoded on a 3D toric code. But efficient decoders are not known for errors corresponding to surfaces except in the case of cubic lattice. In this case a decoder similar to the decoder for the 4D toric code in [10] can be used.

The central result of our paper is a mapping from 3D color codes to 3D toric codes. By exploiting the topological properties of color codes, we establish a mapping between 3D color codes and 3D toric codes. The work most similar to ours is that of Kubica, Yoshida, and Pastawski [11] who showed that the 3D color code can be mapped to three copies of 3D toric codes. Our results give a different mapping from the color code to the toric codes. Our map also preserves the CSS nature of the color code. We project the X errors and Z errors onto different sets of toric codes unlike [11] which employs just one set of toric codes. Their map also implies that a 3D color code can be decoded via 3D toric codes. The question of which map is better for decoding is not yet known. This will be investigated in a later work.

Another work related to ours is that of Delfosse who showed that 2D color code can be projected onto surface codes [12]. Our results generalize his approach to 3D. We take a somewhat simpler approach and do not explicitly make use of chain complexes based on hypergraphs, although it is possible to use similar machinery. In passing we mention that similar mappings were known in 2D [13–15]. Their approaches can also be generalized to obtain similar results in 3D.

The paper is structured as follows. In Section II we give a brief review of 3D toric codes and color codes. In the subsequent section we present the central result of the paper showing how to project a color code onto a collection of 3D toric codes and propose a novel decoding scheme for color codes. We then conclude with a brief discussion and outlook for further research. We assume that the reader is familiar with stabilizer codes [16, 17].

II. PRELIMINARIES

A. 3D toric codes

We briefly review 3D topological codes. A 3D toric code is defined over a cell complex (denoted Γ) in 3D. We assume that the qubits are placed on the edges of the complex. For each vertex v and face f, we define stabilizer generators as follows:

$$A_v = \prod_{e\in\delta(v)} X_e \text{ and } B_f = \prod_{e\in\partial(f)} Z_e, \tag{1}$$

where $\delta(v)$ is the set of edges incident on v and $\partial(f)$ is the set of edges that constitute the boundary of f. When there are periodic boundary conditions, the stabilizer generators A_v and B_f are constrained as follows:

$$\prod_v A_v = \prod_f B_f = \prod_{f\in\partial(\nu)} B_f = I, \tag{2}$$

where ν is any 3-cell. If Γ has boundaries, then the constraints have to be modified accordingly. Additional constraints could be present depending on the cell complex.

Sometimes it is useful to define the (3D) toric codes using the dual complex. In this case the qubits are placed on the faces of the dual complex. The stabilizer generators for a 3-cell ν and an edge e in the dual complex are defined as

$$A_\nu = \prod_{f\in\partial(\nu)} X_f \text{ and } B_e = \prod_{f:e\in\partial(f)} Z_f, \tag{3}$$

where $\partial(\nu)$ is the boundary of ν and $\partial(f)$ is boundary of f.

Suppose there is a Z-error on a qubit. Then this can be detected by the operators A_v. Phase flip errors are visualized as paths or strings on the lattice. The nonzero syndromes always occur in an even number. The bit flip errors on the other hand are detected by the operators B_e. They are better visualized in the dual complex. In the dual complex, qubits are associated to faces, therefore errors can be viewed as (disjoint union of) surfaces and the (nonzero) syndrome as the boundary of these surfaces.

B. 3D color codes

Consider a complex with 4-valent vertices and 3-cells that are 4-colorable. Such colored complexes are called 3-colexes, [3]. A 3D color code is a topological stabilizer code constructed from a 3-colex. The stabilizer generators of the color code are given as

$$B_\nu^X = \prod_{v\in\nu} X_v \text{ and } B_f^Z = \prod_{v\in f} Z_v \tag{4}$$

where ν is a 3-cell and face f a face. It turns out that for each 3-cell ν we can define a (dependent) Z stabilizer as $B_\nu^Z = \prod_{v\in\nu} Z_v$. A 3-colex complex defines a stabilizer code with the parameters $[[v, 3h_1]]$ where h_1 is the first Betti number of the complex, [3, 18]. We can also define the color code in terms of the dual complex. Now qubits correspond to 3-cells, X-stabilizer generators to vertices and Z-stabilizer generators to edges of Γ^*.

$$B_v^X = \prod_{\nu:v\in\nu} X_\nu \text{ and } B_e^Z = \prod_{\nu:e\in\nu} Z_\nu \tag{5}$$

We quickly review some relevant colorability properties of 3-colexes. The edges of such a 3-colex can also be 4-colored: the outgoing edges of every 3-cell can be colored with the same color as the 3-cell. We can color the faces based on the colors of the 3-cells. A face is adjacent to exactly two 3-cells. A face adjacent to 3-cells colored c and c' is colored cc'. This means that the 3-colex is 6-face-colorable. In view of the colorability of the 3-colex we refer to a c-colored 3-cell as c-cell without explicitly mentioning that it is a 3-cell. Likewise we can unambiguously refer to the cc'-colored faces as cc'-cells or cc'-faces, c-colored edges as c-edges and c-colored vertices as c-vertices.

III. PROJECTING A 3D COLOR CODE ONTO 3D TORIC CODES

In this section we state and prove the central result of the paper, namely, 3D color codes can be projected onto finite collection of 3D toric codes. A more precise statement will be given later. We first give an intuitive explanation and then proceed to prove it rigorously.

A. Intuitive explanation through the decoding problem

The main intuition behind the projection of color codes onto toric codes is that it should preserve the error correcting capabilities and enable decoding. From the point of view of a decoder, the information available to it is simply the syndrome information. In a topological code this syndrome information can be represented by the cell complex. Our main goal is to preserve the syndrome information on the 3-colex while translating it into a different cell complex.

In a 3-colex, qubits reside on the vertices while the checks correspond to faces and volumes. If we look at the dual complex, the qubits correspond to 3-cells which are tetrahedrons; the X-type checks to vertices and the Z-type checks to edges. Due to this correspondence we often refer to the boundary of a qubit or a collection of qubits wherein we mean the boundary of the cells which correspond to those qubits.

We address the bit flip and phase flip errors separately. Suppose that an X error occurs. Since the qubits correspond to volumes, error correction is equivalent to identifying (i) the surface of the volumes which enclose the erroneous qubits and (ii) the *interior* of that surface. In other words we seek to identify the boundary of the qubits i.e. 3-cells in error and which cells lie *inside* that boundary . In case of bit flip errors, the syndrome information is present on the edges; this is clearly not the boundary of a volume. The question then arises how do we recover the boundary of the erroneous qubits when we appear to be in possession of some partial information about the boundary.

To see how we might solve this problem, let us assume that there is just one bit flip error. This causes all the edges of the tetrahedron to carry nonzero syndromes. While these edges are contained in the boundary of the tetrahedron it is still not the surface we are looking for. One way to recover the boundary of the tetrahedron is as follows. Imagine we deleted a corner of the tetrahedron we would end with just a face whose edges are carrying the nonzero syndrome. These edges are precisely the boundary of that face. Similarly deleting other corners of the tetrahedron we would be able to recover all the faces of the tetrahedron. Since the union of these faces constitutes the boundary of the erroneous qubit we have achieved what we set out. There remains the question of whether we choose the interior or the exterior of that boundary. This can be easily decided to be the volume which contains fewer number of qubits. Let us identify the key ideas in the previous procedure:

(i) We construct a collection of complexes obtained by deleting one set of corners of tetrahedrons.
(ii) Then using the edges we recover part of the boundary of the erroneous qubits.
(iii) Then we combine the pieces found in (ii) to recover the boundary of the erroneous qubits.
(iv) Finally we decide whether the interior or the exterior set

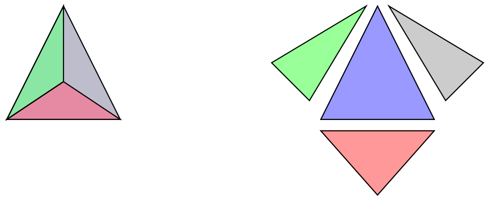

Figure 1: We can recover the boundary of a tetrahedron from the edges by reconstructing the faces and then combining the faces.

qubits are in error.

Step (ii) is key to making the connection with the 3D toric codes. This step is identical to the correction of the X-type errors in 3D toric codes.

A similar idea will lead us to the procedure for decoding the phase flip errors. In this case the syndromes that detect the phase flip errors correspond to the vertices in the dual complex. Suppose now that there is a single phase flip error. The vertices of the erroneous tetrahedron will carry the information about the error. Now we seem to have even lesser information about the boundary of the erroneous tetrahedron than before. However we can recover the boundary by the following procedure. Delete any pair of vertices of the tetrahedron. We will be left with one edge and two vertices. We can first identify the edge as piece of the boundary we are looking for. Deleting all the six possible pairs of vertices, we are able to recover the six edges which are in the boundary of the tetrahedron. Now the problem is identical to the one we solved for correcting bit flip errors. We summarize the key steps:

(i) We construct new complexes from the original complex by deleting pairs of vertices of each tetrahedron.
(ii) Then we recover the edges which are in the boundary of the erroneous qubits.
(iii) At this point the problem is same as the problem of decoding bit flip errors which can be solved using the previous procedure.

In correcting the Z-type errors the connection to the toric codes happens in (ii). This is precisely the process used to decode Z-type errors in 3D toric codes. Of course the procedures we outlined are heuristic and somewhat imprecise; they need a rigorous justification as to correctness and efficiency. We now turn to address these issues in the next section.

B. 3-colexes, duals and minors

As we saw in the previous section, our approach to decoding color codes leads us to duals and minors of complexes. So we begin by studying the properties of the 3-colexes and their minors. We first state some properties of the dual of a 3-colex. These are immediate from the properties of the 3-colex and we omit the proof. We denote the i-dimensional cells as C_i and the i-dimensional cells of color c as C_i^c.

Lemma 1. *Let Γ be a 3-colex. Then the dual complex Γ^* is 4-vertex-colorable, 6-edge-colorable, 4-face-colorable.*

In the 3-colex every qubit is identified with a tetravalent vertex, therefore in the dual complex, every qubit corresponds to a tetrahedron whose vertices are of different colors. Similarly the four faces of each tetrahedron are also of different colors. (Follows from Lemma 1.) Let us denote the minor of Γ^* obtained by deleting all the vertices of color c by $\Gamma^{\setminus c}$. We denote this operation as π_c so that

$$\pi_c(\Gamma^*) = \Gamma^{*\setminus c} \tag{6}$$

We are interested in knowing how π_c affects (i) the qubits, (ii) the errors on them and (iii) the associated syndrome. So we extend the action of π_c to the individual cells of Γ^*. A vertex that is not colored c will be mapped to a vertex in $\Gamma^{*\setminus c}$. An edge that is not incident on a c-vertex will be mapped to an edge in $\Gamma^{*\setminus c}$. Edges that are incident on a c-vertex can be thought as being mapped to the empty set. A 3-cell has exactly 4 faces and on the deletion of a c-vertex just the c-face in its boundary will be left in $\Gamma^{*\setminus c}$. A 3-cell in Γ^* corresponds to a qubit, so we can interpret the c-face in its boundary as the qubit in $\Gamma^{*\setminus c}$. Since every face is shared between two 3-cells, there exist two distinct cells ν_1 and ν_2 such that $\pi_c(\nu_1) = \pi_c(\nu_2)$. The following equations summarize this discussion.

$$\pi_c(v) = v \text{ if } v \in \mathsf{C}_0(\Gamma^*) \setminus \mathsf{C}_0^c(\Gamma^*) \tag{7a}$$
$$\pi_c(e) = e \text{ if } e \in \mathsf{C}_1(\Gamma^*) \setminus \mathsf{C}_1^{cc'}(\Gamma^*) \tag{7b}$$
$$\pi_c(\nu) = \partial\nu \cap \mathsf{C}_2^c(\Gamma^*); \nu \in \mathsf{C}_3(\Gamma^*) \tag{7c}$$

Denote the minor complex of Γ^* obtained by deleting all vertices colored c and c' as $\Gamma^{*\setminus cc'}$. Denote this operation as $\pi_{cc'}$. Then we have

$$\pi_{cc'}(\Gamma^*) = \Gamma^{*\setminus cc'}. \tag{8}$$

We can also extend $\pi_{cc'}$ to the cells of Γ^* as we did for π_c. Then we can write

$$\pi_{cc'}(v) = v \text{ if } v \in \mathsf{C}_0(\Gamma^*) \setminus (\mathsf{C}_0^c(\Gamma^*) \cup \mathsf{C}_0^{c'}(\Gamma^*)) \tag{9a}$$
$$\pi_{cc'}(e) = e \text{ if } e \in \mathsf{C}_1^{dd'}(\Gamma^*); d, d' \notin \{c, c'\} \tag{9b}$$
$$\pi_{cc'}(\nu) = \partial(\pi_c(\nu)) \cap \mathsf{C}_1^{dd'}(\Gamma^*); \nu \in \mathsf{C}_3(\Gamma^*) \tag{9c}$$

We assume $d \neq d'$ in the above equations. A simple example of 3-colex and related complexes are shown in Fig. 2. We can relate the various complexes and the objects of interest for us as follows.

	Γ	Γ^*	$\Gamma^{*\setminus c}$	$\Gamma^{*\setminus cc'}$
Qubit	vertex	tetrahedron	triangle	edge
Z-check	face	edge	edge	—
X-check	3-cell	vertex	vertex	vertex

Some relevant properties of the minor complexes are considered next. They concern both the structure and coloring properties of the minor complexes.

Lemma 2. *The 3-cells of $\Gamma^{*\setminus c}$ can be indexed by the c-vertices of Γ^* and the 3-cell $\nu_v \in \mathsf{C}_3(\Gamma^{*\setminus c})$ is the sum of the 3-cells which are incident on v in Γ^*. Further, the boundary of ν_v is the sum of c-faces of tetrahedrons incident on v.*

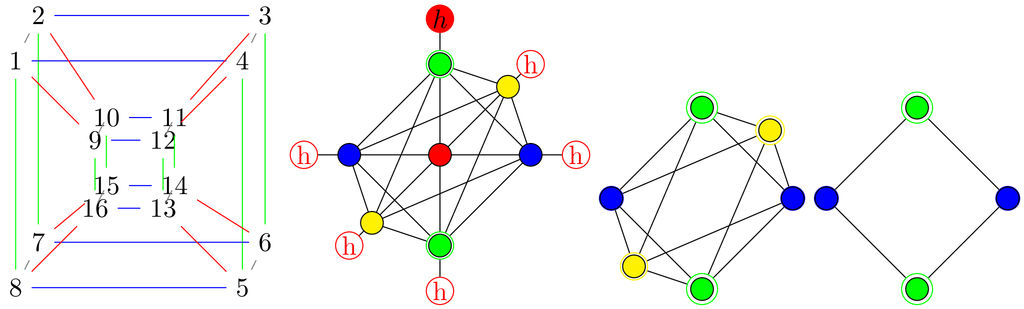

Figure 2: A 3-colex Γ and its dual Γ^*; i-cells of Γ^* correspond to the $3-i$ cells of Γ. The minor complexes $\Gamma^{*\backslash r}$ and $\Gamma^{*\backslash ry}$ are also shown.

Proof. The deletion of a c-vertex causes all the tetrahedrons incident on it to be combined into one single 3-cell in $\Gamma^{*\backslash c}$. Since every tetrahedron is incident on some c-vertex, none of them can remain in $\Gamma^{*\backslash c}$ and the 3-cells in $\Gamma^{*\backslash c}$ must be some combination of the tetrahedrons of Γ^*. Since the four vertices of each tetrahedron are different colors, two tetrahedrons can be merged only if they are incident on the same c-vertex. Thus two distinct c-vertices lead to distinct 3-cells. The deletion of the c-vertex v, leaves behind a c-face for every tetrahedron incident on v. The disjointness of two such 3-cells implies that these c-faces must form the boundary of each 3-cell ν_v. □

Lemma 3. *Let Γ be a 3-colex and $c \in \{r, b, g, y\}$. Then the minor complex $\Gamma^{*\backslash c}$ has only c-colored faces and edges which are colored dd' where $d, d' \in \{r, b, g, y\} \setminus \{c\}$.*

Proof. Suppose that we delete the vertices colored c in $\Gamma^{*\backslash c}$; this leads to the deletion of the edges and faces that are incident on these vertices. On any c-colored vertex only the c-colored faces are not incident. Any face colored with $c' \in \{r, b, g, y\} \setminus c$ is incident on some c-vertex. This leads to deletion of all but one face of each tetrahedron incident on any c-colored vertex. The remaining face is colored c. Thus $\Gamma^{*\backslash c}$ contains only c-colored faces. Since only $\{r, b, g, y\} \setminus \{c\}$ are present in $\Gamma^{*\backslash c}$, the edges connecting them are colored dd'. □

Lemma 4. *Let Γ be a 3-colex and $c, c' \in \{r, b, g, y\}$. Then the minor complex $\Gamma^{*\backslash cc'}$ has only dd'-colored edges where $\{d, d'\} = \{r, b, g, y\} \setminus \{c, c'\}$.*

Proof. Suppose that we delete all the vertices colored c, c' in Γ^*, then all edges incident on c-vertices and c'-vertices will be deleted. Thus only edges between d and d' colored vertices will remain. These edges are colored dd'. □

The faces of $\Gamma^{*\backslash cc'}$ are not a subset of the faces of Γ^*. Indeed none of the original faces of Γ^* will be left in this new complex. The preceding lemmas lead to the following corollary.

Corollary 5. *Let Γ be a 3-colex with v vertices, $e = 2v$ edges, $f_{cc'}$ cc'-faces, ν_c c-cells. Let the total number of faces be $f = \sum_{cc'} f_{cc'}$ and volumes be $\nu = \sum_c \nu_c$. The following table summarizes the number of cells in $\Gamma^{*\backslash c}$ and $\Gamma^{*\backslash cc'}$ where $c, c', d, d' \in \{r, b, g, y\}$ are distinct.*

	Γ	Γ^*	$\Gamma^{*\backslash c}$	$\Gamma^{*\backslash cc'}$
3-cells	ν	v	ν_c	—
Faces	f	$2v$	$v/2$	—
Edges	$2v$	f	$f_{dc'} + f_{c'd'} + f_{dd'}$	$f_{dd'}$
Vertices	v	ν	$\nu_{c'} + \nu_d + \nu_{d'}$	$\nu_d + \nu_{d'}$

Proof. The i-cells in dual complex Γ^* are in one to one correspondence with the $(3-i)$-cells of Γ. Now suppose that Γ^* is modified so that all vertices colored i are deleted. Then all 3-cells incident on it will be merged to form a new 3-cell. Since all the 3-cells in Γ^* are incident on some i-vertex, they will be part of some new 3-cell and $\Gamma^{*\backslash c}$ will contain c_i 3-cells. On deleting the i-vertices, exactly one face will remain from each 3-cell in Γ^*. Since each of these faces will be shared between two 3-cells there will be $v/2$ faces. The edges in $\Gamma^{*\backslash i}$ are those that are incident on vertices other than i-vertices. Similarly in $\Gamma^{*\backslash cc'}$, only the d-vertices, d'-vertices and the dd'-edges will survive. These are precisely $\nu_d + \nu_{d'}$ vertices and $f_{dd'}$ edges. □

C. X type errors on 3D color codes

Let us now see how to perform error correction in a color code. It is helpful to see the (topological) structure of the errors in the dual of the 3-colex. We analyze the bit flip and phase flip errors separately. Suppose that we have X errors on some set of qubits. In the dual colex the erroneous qubits correspond to a volume. Through π_c we can associate qubits to faces of the minor complex $\Gamma^{*\backslash c}$. Thus we can project errors from Γ^* to $\Gamma^{*\backslash c}$.

If a qubit ν is in error, then we place an error in the image of ν in $\Gamma^{*\backslash c}$. In other words,

$$\pi_c(X_\nu) = X_{\pi_c(\nu)}, \tag{10}$$

where $\pi_c(X_\nu)$ gives an X error is acting on the qubits in $\Gamma^{*\backslash c}$.

A consequence of Eq. (10), together with linearity of π_c, is that for two adjacent qubits ν_1 and ν_2 sharing a c-colored face f, we have $\pi_c(X_{\nu_1} X_{\nu_2}) = X_{\pi_c(\nu_1)} X_{\pi_c(\nu_2)} = I$; where we used the fact that $\pi_c(\nu_1) = \pi_c(\nu_2) = f$. In other words, a c-face common to two qubits in error corresponds to an error free qubit in $\Gamma^{*\backslash c}$.

The syndrome corresponding to bit flip errors is associated to edges of Γ^*. In $\Gamma^{*\backslash c}$ not all edges are present. But if an edge is present, we associate to that edge the same syndrome as in Γ^*. In other words syndromes in the minor complex are essentially the restriction of the syndromes in Γ^*. Let s_e be the syndrome on edge e, then

$$\pi_c(s_e) = s_{\pi_c(e)} = s_e. \tag{11}$$

This is consistent with Eq. (7b).

At this point we have qubits living on the faces and syndromes on the edges of $\Gamma^{*\backslash c}$ just as we would have in a 3D toric code. But it needs to be shown that indeed that we truly have the structure of a 3D toric code and not merely the appearance of it. This we shall take up next.

First let us consider the edge type checks on $\Gamma^{*\backslash c}$. Consider an edge e in $\Gamma^{*\backslash c}$. Then e is also present in Γ^*. For each qubit incident on e there is an c-colored face incident on e. These c-faces exhaust all the faces incident on e in $\Gamma^{*\backslash c}$. Thus in the 3D toric code associated to $\Gamma^{*\backslash c}$, every face incident on e participates in that check on e as required for the edge type checks in the 3D toric code. Next we look at the projected syndromes on the minor complex.

Lemma 6 (Validity of X syndrome restriction)**.** *Let s be the syndrome for an X error on the 3D color code defined on a 3-colex Γ and $\pi_c(s)$ the restriction of s on $\Gamma^{*\backslash c}$. Then $\pi_c(s)$ is a valid syndrome for (an X error on) the toric code on $\Gamma^{*\backslash c}$.*

Proof. We will focus our attention only on support of the syndrome i.e. the checks with nonzero syndrome. The syndrome for an X-error on a 3D toric code is nonzero on the boundary of a collection of faces. Therefore it is a union of cycles. So it suffices to show that the support of $\pi_c(s)$ is a union of cycles.

Denote by σ the collection of edges in $\Gamma^{*\backslash c}$ which have nonzero syndrome. We need to show that these edges form a union of cycles. This can be done by showing that every vertex v in $\Gamma^{*\backslash c}$ has an even degree with respect to the edges in σ. Recall that a qubit corresponds to a tetrahedron in Γ^* and its image is a triangle in $\Gamma^{*\backslash c}$. A qubit in error in Γ^* flips the syndrome on exactly six edges in Γ^*, equivalently three edges in $\Gamma^{*\backslash c}$. The key observation is that with respect to a single vertex v, a single qubit can flip exactly two edges in $\Gamma^{*\backslash c}$. In other words, the syndrome on exactly two edges of incident on v can be changed by an error on a qubit incident on v.

Suppose no qubit incident on v is in error. Then no edge incident on v has a nonzero syndrome and v has degree zero with respect to edges of σ. Suppose exactly one qubit incident on v is in error, then exactly two edges incident in v are in σ. Each additional qubit in error can flip the syndrome on two edges. If the additional qubit has no edges in common with σ, then it will add two edges to σ that are incident on v. If the additional qubit flips just one edge in σ, then with respect to v that particular edge will no longer be incident on v, but an additional edge which is is not already incident on it will be added to σ. If these two edges are already in σ, flipping them will change their syndrome to zero and they will be removed from σ. In all these cases the number of edges in σ that are incident on v are even, see Fig. 3. Thus σ must be a union of cycles and it is a valid syndrome on the 3D toric code. $\square$

Theorem 7 (Projection onto toric codes)**.** *Assume the same notation as in Lemma 6. Then the error $\pi_c(E)$ in $\Gamma^{*\backslash c}$ produces the syndrome $\pi_c(s)$ in the toric code associated to $\Gamma^{*\backslash c}$.*

Proof. We already know from Lemma 6, that $\pi_c(s)$ is a valid syndrome. We now will show that in $\Gamma^{*\backslash c}$ the syndrome produced by $\pi_c(E)$ is same as $\pi_c(s)$. Consider any edge in $\Gamma^{*\backslash c}$; by definition $\pi_c(s_e) = s_e$ where s_e is the syndrome on e with respect to Γ^*. Since Γ is 3-colex, an even number of qubits are incident on e, say $2m$. Then $s_e = \oplus_{i=1}^{2m} q_i$ where $q_i = 1$ if there is an X error on the ith qubit and zero otherwise. Each of these qubits (tetrahedrons) incident on e are projected to a qubit in $\Gamma^{*\backslash c}$. But note that two qubits which share a face are mapped to the same qubit in $\Gamma^{*\backslash c}$. Thus there are m qubits (triangles) incident on e with respect to $\Gamma^{*\backslash c}$. These (projected) qubits are in error if and only if one of the parent qubits in Γ^* are in error. Let $r_j = 1$ if there is an an error on the projected qubit and zero otherwise. Then $r_j = q_{2j-1} \oplus q_{2j}$, where $2j-1$ and $2j$ are the qubits which are projected onto the $j$the qubit in $\Gamma^{*\backslash c}$. The syndrome on e as computed in the 3D toric code is $\oplus_{j=1}^{m} r_j = \oplus_{j=1}^{m}(q_{2j-1} \oplus q_{2j}) = s_e$. Thus the projected error $\pi_c(E)$ produces the same syndrome as the projected syndrome $\pi_c(s)$. $\square$

Corollary 8. *Let $\overline{S}$ be an X-type stabilizer on the color code. Then $\pi_c(\overline{S})$ is an X-stabilizer on the toric code defined by $\pi_c(\Gamma^*)$. Conversely for every stabilizer S on $\Gamma^{*\backslash c}$, there exists a stabilizer generator $\overline{S}$ on Γ^* such that $\pi_c(\overline{S}) = S$.*

Proof. It suffices to prove this for the generators of X-type stabilizers, namely, operators of the form $\overline{S} = \prod_{\nu:v\in\nu} X_\nu$, where v is a c-vertex in Γ^*. First observe that $\overline{S}$ being a stabilizer has zero syndrome in Γ^*, therefore by Theorem 7, $\pi_c(\overline{S})$ also has zero syndrome in $\Gamma^{*\backslash c}$ and it must be a stabilizer or a logical operator on $\Gamma^{*\backslash c}$. Observe that $\pi_c(\overline{S}) = \prod_{\nu:v\in\nu} X_{\pi_c(\nu)}$ and the support of $\pi_c(\overline{S})$ is precisely the boundary of the 3-cell formed by the union of all the tetrahedrons (qubits) incident on v. In other words, it is an X-type stabilizer of the toric code on $\Gamma^{*\backslash c}$.

We know from Lemma 2, the 3-cells of $\Gamma^{*\backslash c}$ can be indexed by vertices of Γ^*. Any stabilizer on $\Gamma^{*\backslash c}$ is generated by some combination of operators A_v, see Eq. (3), where v runs over the 3-cells of $\Gamma^{*\backslash c}$. The preceding argument shows that this stabilizer is the image of the stabilizer generator associated to $v \in \Gamma^*$. $\square$

We define the boundary of an error in Γ^* to be the boundary of the volume that corresponds to the collection of qubits on which the error acts nontrivially. In other words,

$$\partial E = \partial(\mathrm{supp}(E)). \tag{12}$$

We use the same notation ∂ for boundary of cells as well as operators. Note that $\partial(EE') = \partial E + \partial E'$.

Lemma 9 (X error boundary)**.** *Using the same notation as in Lemma 6, the boundary of an error E in Γ^* is $\partial E = \sum_c \mathrm{supp}(\pi_c(E))$.*

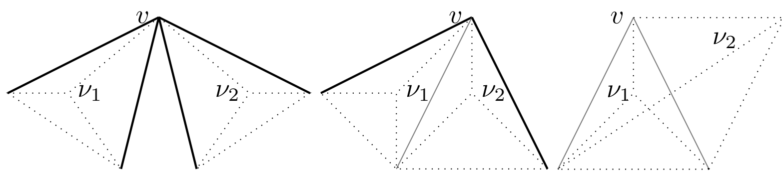

Figure 3: Edges with nonzero syndrome (shown in bold) for two qubits ν_1, ν_2 with X errors depending on the edges shared in common. In all cases v has even degree with respect to the edges with nonzero syndrome.

Proof. Let E_i denote the error on the ith qubit and ν_i the 3-cell corresponding to the ith qubit. Then we can write

$$\partial E \overset{(a)}{=} \partial(\mathrm{supp}(E)) = \sum_{i:E_i\neq I} \partial \nu_i \tag{13}$$

$$\overset{(b)}{=} \sum_{i:E_i\neq I} \sum_c \pi_c(\nu_i) = \sum_c \sum_{i:E_i\neq I} \pi_c(\nu_i) \tag{14}$$

$$\overset{(c)}{=} \sum_c \sum_{i:E_i\neq I} \mathrm{supp}(\pi_c(E_i)) \overset{(d)}{=} \sum_c \mathrm{supp}(\pi_c(E)) \tag{15}$$

where (a) follows from the definition of the boundary of an error; (b) follows from the fact that the boundary of a single qubit is the collection of four faces of the tetrahedron that correspond to the qubit, and rearranging the order of summation; (c) follows from the observation that $\pi_c(\nu_i)$, see Eq. (7c), is the c-face in the boundary of ν_i and if $E_i \neq I$, then this is the same as the support of $\pi_c(E_i)$; (d) is immediate and completes the proof. □

Suppose that we have an error E and its boundary ∂E. By Lemma 9, this boundary can be broken down into four pieces of surfaces each lying in $\Gamma^{*\setminus c}$. If these surfaces were modified by the support of a stabilizer in the minor complexes, then these modified surfaces form the boundary of an error E' which is equivalent to E up to a stabilizer i.e. $E' = ES$ for some X stabilizer S on the color code.

Lemma 10 (X error boundary modulo stabilizer)**.** *Let E be an error on Γ^*. Let S_c be X-stabilizer generators on $\Gamma^{*\setminus c}$. Then $\partial E + \sum_c \mathrm{supp}(S_c)$ is the boundary of ES, for some X-stabilizer S on Γ^* i.e.*

$$\partial E + \sum_c \mathrm{supp}(S_c) = \partial(ES) \tag{16}$$

Proof. Since S_c is a stabilizer generator, by corollary 8, there exists an X-stabilizer $\overline{S}_c$ on Γ^* such that $\partial(\overline{S}_c) = \mathrm{supp}(S_c)$. Then using $\partial EF = \partial E + \partial F$, we obtain $\partial E\overline{S}_c = \partial E + \partial \overline{S}_c$. Repeating this for all c we have $\partial(E \prod_c \overline{S}_c) = \partial E + \sum_c \partial \overline{S}_c$. Letting $\prod_c \overline{S}_c = S$, we can write this as $\partial(ES) = \partial E + \sum_c \mathrm{supp}(S_c)$ as claimed. □

With these results in hand we can now claim the following:

Theorem 11 (Estimating face boundary of X errors)**.** *The boundary an X-error E on the dual of color code can be estimated by Algorithm 1. The algorithm estimates ∂E up to the boundary of an X stabilizer of the color code, provided $\pi_c(E)$ is estimated up to a stabilizer on $\Gamma^{*\setminus c}$, where $c \in \{r, g, b, y\}$.*

Proof. We only sketch the proof as it is a straightforward consequence of the results we have shown thus far. By Lemma 9 the boundary of the error consists of support of $\pi_c(E)$ i.e. the projections of the error on the 3D toric codes on $\Gamma^{*\setminus c}$. By Lemma 7, the syndrome of $\pi_c(E)$ is the restriction of the syndrome on Γ^*. Therefore, $\pi_c(E)$ can be estimated by decoding on $\Gamma^{*\setminus c}$. By Lemma 10, if the estimate for $\pi_c(E)$ is equivalent up to a stabilizer on $\Gamma^{*\setminus c}$, we can obtain the boundary of E up to the boundary of a stabilizer on the color code. □

Algorithm 1: Estimating (face) boundary of X type error

Input: A 3-colex Γ, Syndromes on edges $s_e, e \in \Gamma^*$.
Output: $\mathsf{F} \subseteq \mathsf{C}_2(\Gamma^*)$
for each $c \in \{r, b, g, y\}$ **do**
  for each edge e in $\Gamma^{*\setminus c}$ **do**
    $s_{\pi_c(e)} = s_e$ // syndrome projection
  end for
  Estimate the error F_c using any 3D toric code decoder for X errors on $\Gamma^{*\setminus c}$
end for
Estimate $\mathsf{F} = \bigcup_c \mathsf{F}_c$

D. Z type errors on 3D color codes

One can prove similar results similar to previous section for the Z-type errors also. As we noted earlier, the syndrome information about Z errors resides on the vertices of Γ^* while the error corresponds to a volume. To recover the boundary of the error, the objects of interest are the minor complexes $\Gamma^{*\setminus cc'}$. Recall that the edges of $\Gamma^{*\setminus cc'}$ are associated with qubits. So we project the errors on cell ν to $\pi_{cc'}(\nu)$ which is an edge in $\Gamma^{*\setminus cc'}$. We define

$$\pi_{cc'}(Z_\nu) = Z_{\pi_{cc'}(\nu)} \tag{17}$$

Let syndrome on $v \in \mathsf{C}_0(\Gamma^*)$ be s_v. Then we define the syndrome on $v \in \Gamma^{*\setminus cc'}$ as

$$\pi_{cc'}(s_v) = s_{\pi_{cc'}(v)} = s_v \tag{18}$$

so the syndrome on $\pi_{cc'}(\Gamma^*)$ is simply the restriction of the syndrome on Γ^*. To project the color code onto toric codes, this syndrome must be a valid syndrome on the minor complex.

Lemma 12 (Validity of Z syndrome restriction)**.** *Let s be the syndrome for a Z error on Γ^*. Then $\pi_{cc'}(s)$ is a valid syndrome for a Z error on $\Gamma^{*\setminus cc'}$.*

Proof. A valid syndrome for a Z error on a toric code is nonzero for an even number of vertices. So we need to show that the $\pi_{cc'}(s)$ is nonzero on an even number of vertices in $\Gamma^{*\backslash cc'}$. A Z error on Γ^* for a single qubit ν creates a non zero syndrome on the vertices of the corresponding tetrahedron. Only two of these vertices will be present in $\Gamma^{*\backslash cc'}$. Clearly, $\pi_{cc'}(s)$ is nonzero on even number if there is only one error. A Z error on another qubit ν' will share at most two vertices of ν, (in $\Gamma^{*\backslash cc'}$). If they share no vertices then the syndrome is nonzero in two additional vertices. If they share one vertex, then the syndrome is zero on the shared vertex and nonzero on the unshared vertices. If they share both the vertices then the syndrome is zero on both the vertices. So in all the cases we end up with an even number of vertices with nonzero syndrome. This argument can be extend to the case of multiple errors. $\square$

Theorem 13 (Projection onto toric codes)**.** *Let E be a Z-type error on Γ^* and its associated syndrome s. Then the error $\pi_{cc'}(E)$ in $\Gamma^{*\backslash cc'}$ produces the syndrome $\pi_{cc'}(s)$.*

Proof. Let s_v be the syndrome on a vertex v. Then $s_v = \bigoplus_{i:v\in\nu_i} q_i$, where $q_i = 1$, if there is a Z error on ν_i and zero otherwise. Every qubit incident on v must have a dd'-edge incident on v. Two qubits incident on v can share at most one such edge. Label the dd' edges incident on v as e_j. Then we can partition the qubits incident on v depending on the dd' edge on which they are incident. So we can write

$$s_v = \bigoplus_{i:v\in\nu_i} q_i = \bigoplus_j \bigoplus_{i:v\in\nu_i, e_j\in\nu_i} q_i = \bigoplus_j r_j, \qquad (19)$$

where $r_j = \bigoplus_{i:v\in\nu_i, e_j\in\nu_i} q_i$. The qubits incident on e_j are projected onto the edge e_j in the minor complex and there is an error on e_j if and only if $\bigoplus_{i:v\in\nu_i, e_j\in\nu_i} q_i$ is one. Thus the syndrome on the vertex v as computed with respect to the toric code on $\Gamma^{*\backslash cc'}$ is $\pi_{cc'}(E) = \bigoplus_j r_j = s_v = \pi_{cc'}(s)$ as stated in the lemma. $\square$

Lemma 14. *Suppose S is a Z-stabilizer on $\Gamma^{*\backslash cc'}$, then there exists a Z-stabilizer $\overline{S}$ in Γ^* such that $\pi_{cc'}(\overline{S}) = S$.*

Proof. A stabilizer S on $\Gamma^{*\backslash cc'}$ is a collection of cycles of trivial homology. These cycles must consist of dd' edges only. We observe that dd' edges are present in $\Gamma^{*\backslash c}$ and $\Gamma^{*\backslash c'}$. Since σ is a cycle it is a valid syndrome for the 3D toric codes on $\Gamma^{*\backslash c}$ and $\Gamma^{*\backslash c'}$. Therefore, there exists an X error E such that it has nonzero syndrome on σ only. Further, E must have trivial homology because σ is trivial. Let $\Omega = \mathrm{supp}(E)$ and $\overline{S} = \prod_{\nu\in\Omega} Z_\nu$. We claim that $\overline{S}$ is a Z-type stabilizer of the color code. Since E has nonzero syndrome on σ, there must be an odd number of qubits in E that are incident on any edge in σ. Further no two qubits in E can be incident on two distinct dd' edges of σ. Since σ is a cycle any vertex v must have even degree with respect to σ. Every such edge has an odd number of edges of E incident on it. Therefore v has even degree with respect to edges in Ω. Now let v be a vertex not in σ, then it does not have any edges in σ incident on it. On every such edge E has zero syndrome. In other words, an even number of qubits of E are incident on every edge incident on v. Therefore $\overline{S}$ has zero syndrome on all vertices. It must be a stabilizer since we have shown that E has trivial homology. $\square$

Define the edge boundary of an error/volume as the collection of edges given by

$$\delta E = \bigcup_{cc'} \mathrm{supp}(\pi_{cc'}(E)) \qquad (20)$$

Theorem 15 (Estimating edge boundary of Z errors)**.** *The boundary a Z-error E on the dual of color code can be estimated by Algorithm 2. The algorithm estimates δE up to the boundary of an Z stabilizer of the color code, provided $\pi_{cc'}(E)$ is estimated up to a stabilizer on $\Gamma^{*\backslash cc'}$.*

Proof. By Lemma 12, the restriction of the syndrome of E is a valid syndrome on $\Gamma^{*\backslash cc'}$. By Theorem 13, $\pi_{cc'}(E)$ has the same syndrome as the restriction and it can be estimated using a 3D toric decoder on $\Gamma^{*\backslash cc'}$. By linearity of $\pi_{cc'}$ and Lemma 14 if the estimates for $\pi_{cc'}(E)$ are upto a stabilizer on $\Gamma^{*\backslash cc'}$, the estimate for edge boundary of E will differ by the boundary of a Z-stabilizer on the color code. $\square$

Algorithm 2: Estimating (edge) boundary of Z type error

Input: A 3-colex Γ, syndromes on vertices s_v, where $v \in \Gamma^*$
Output: $\mathsf{E} \subseteq \mathsf{C}_1(\Gamma^*)$

```
for each c, c' ∈ {r, b, g, y} do
    for each vertex v in Γ*\cc' do
        s_{π_cc'(v)} = s_v                    //syndrome projection
    end for
    Estimate the error E_cc' using any 3D toric code decoder for Z
    errors on Γ*\cc'
end for
Estimate E = ⋃_{c,c'} E_cc'
```

We still have to work a little more before we can decode the Z error. At this point we have shown that we can estimate the edge boundary up to the boundary of a Z stabilizer. We show in Lemma 16 something more interesting, namely, that this edge boundary is a valid X syndrome i.e. there is some X error whose syndrome coincides with this boundary.

Lemma 16 (Edge boundary corresponds to an X syndrome)**.** *Let E be a Z-type error on the color code and $S_{cc'}$ stabilizers on $\Gamma^{*\backslash cc'}$. Let σ be the collection of edges given by*

$$\sigma = \bigcup_{cc'} \mathrm{supp}(\pi_{cc'}(E)) \cup \mathrm{supp}(S_{cc'}) \qquad (21)$$

Then $\pi_c(\sigma)$ is a cycle on $\Gamma^{*\backslash c}$ for any c.

Proof. First we note that $\mathrm{supp}(\pi_{cc'}(E))$ is a collection of paths on $\Gamma^{*\backslash cc'}$ and Γ^*. Each path is made up of dd' edges and terminating on two distinct vertices of color d or d'. Secondly $\mathrm{supp}(S_{cc'})$ is a collection of cycles of dd' edges. These paths and cycles will be present in the minor complexes $\Gamma^{*\backslash c}$ and $\Gamma^{*\backslash c'}$. Similarly $\Gamma^{*\backslash c}$ also contains paths and cycles from

$\Gamma^{*\setminus cd}$ and $\Gamma^{*\setminus cd'}$ which consist of $c'd$ and $c'd'$ edges respectively.

Now consider any vertex in $\Gamma^{*\setminus c}$. If v has a zero syndrome, then none of the paths can terminate on this vertex. So with respect to the edges of σ in $\Gamma^{*\setminus c}$ v has an even degree. If v has a nonzero syndrome, then some of the paths must terminate on v. Suppose that v is c', then it is present in exactly two minor complexes $\Gamma^{*\setminus cd}$ and $\Gamma^{*\setminus cd'}$. From both these complexes we estimate two paths that terminate on v. So with respect to $\Gamma^{*\setminus c}$, there are two edges incident on v. Thus it has even degree. Similarly we can show that any vertex of other colors is also of even degree for any minor complex $\Gamma^{*\setminus c}$. Then σ is a valid syndrome for an X type error on the color code. □

What we have achieved through Lemma 16 is that we have taken a Z error whose syndrome is on vertices and converted it to the syndrome of an X error. Since we know how to obtain the boundary of an X error, we can take this syndrome and obtain the boundary using Theorem 11. Before we can proceed like that we need to show that this X error has the same support as the original Z error up to a Z-stabilizer.

We next show that this syndrome exactly coincides with an X-error which has the same support as the Z error whose boundary we computed.

Lemma 17 (Error switching)**.** *Let $E_Z = \prod_{\nu\in\Omega} Z_\nu$. Then the syndrome of $E_X = \prod_{\nu\in\Omega} X_\nu$ is nonzero on the edges in $\sigma = \bigcup_{cc'} \operatorname{supp}(\pi_{cc'}(E_Z))$.*

Proof. Consider the syndrome of E_X on an edge e, say dd'-edge. Then

$$s_e \neq 0 \text{ iff. } |\{\nu \in \Omega \mid e \in \nu\}| \text{ is odd} \tag{22}$$

In other words, s_e is nonzero if and only if the number of qubits in Ω incident on e is odd. Similarly, $\pi_{cc'}(E_Z)$ has support on e if and only if odd number of qubits in Ω are incident e. Thus $e \in \sigma$ if and only if odd number of qubits in Ω are incident on e i.e.

$$e \in \operatorname{supp}(\pi_{cc'}(E_Z)) \text{ iff. } |\{\nu \in \Omega \mid e \in \nu\}| \text{ is odd} \tag{23}$$

Comparing Eqs. (22) & (23), we see that the syndrome of E_X is nonzero precisely on the support of $\pi_{cc'}(E_Z)$. □

Theorem 18 (Estimating face boundary of Z type errors)**.** *Let E be a Z-type error on Γ^* whose edge boundaries are estimated using Algorithm 2. If these edge boundaries are used to estimate a face boundary boundary using Algorithm 1, then we can estimate the face boundary of E up to a Z-stabilizer on the color code.*

Proof. By Theorem 15 we can estimate the edge boundary of E up to a Z-stabilizer boundary. By Lemma 16 these edges are precisely the syndrome for an X error. By Lemma 17, this X error has the same support as E up to a X-stabilizer. But in a color code every X-stabilizer there exists a Z-stabilizer with the same support. Thus the final boundary of E is estimated up to a Z-stabilizer provided all the intermediate estimates from Algorithms 1 & 2 are all up to stabilizer on the respective 3D toric code decoders. □

E. Application: Decoding 3D color codes

The projection onto the toric codes allow us to decode the 3D color code. Before we can give the complete decoding algorithm we need one more component. In both Algorithms 1 & 2 we only end up with the boundary of the error. We need to identify the qubits which are in error. The procedure for this lifting is given in Algorithm 3. The main idea behind this algorithm is the fact that the color code is connected and we can partition the qubits into two groups: those inside and those outside of the boundary. The following lemma justifies the procedure in Algorithm 3.

Algorithm 3: Lifting a boundary to a volume

Input: Complex Γ^*, Set of faces $\mathsf{F} \subseteq \mathsf{C}_2(\Gamma^*)$
Output: $\Omega \subseteq \mathsf{C}_3(\Gamma^*)$
Set $\Omega = \emptyset; m_\nu = 0$ for all $\nu \in \mathsf{C}_3(\Gamma^*)$ // Initialization
Initialize $\Omega = \{\nu_o\}, m_{\nu_o} = 1$ // For some 3-cell ν_o
while $m_\mu = 0$ for some $\mu \in \mathcal{N}_\nu$ with $m_\nu \neq 0$ **do**
  for each $\mu \in \mathcal{N}_\nu$ **do**
    if $m_\mu = 0$ **then**
      if $\mu \cap \nu \in \mathsf{F}$ **then**
        $m_\mu = -m_\nu$ //Qubits on different side of error boundary
      else
        $m_\mu = m_\nu$ //Qubits on same side of error boundary
      end if
      if $m_\mu = 1$ **then**
        $\Omega = \Omega \cup \{\mu\}$
      end if
    else
      if $\mu \cap \nu \in \mathsf{F}$ and $m_\mu \neq -m_\nu$ **then**
        $\Omega = \emptyset$; Exit // Decoder failure
      end if
    end if
  end for
end while
if $|\Omega| > |\mathsf{C}_3 \setminus \Omega|$ **then**
  $\Omega = \mathsf{C}_3 \setminus \Omega$ //Pick the smaller volume
end if

Lemma 19 (Lifting the boundary of error)**.** *Algorithm 3 will give the smallest collection of* 3*-cells $\Omega \subseteq \mathsf{C}_3(\Gamma^*)$ such that $\partial\Omega = \mathsf{F}$. The algorithm returns an empty set if F is not a valid boundary.*

Proof. The algorithm takes as input a collection of faces supposed to enclose a volume. If the faces enclose a volume, we can label all the 3-cells inside and outside the boundary differently. Cells adjacent to each other and enclosed within the same boundary are labeled same. The algorithms proceeds by labeling a random choice of initial qubit and then proceeds to assign labels to all its adjacent qubits. If two qubits share a face that is not in the boundary F, then they must have the same label because one must cross the boundary to change the label. Two qubits, that are adjacent and share a face that is in the boundary must have different labels. The algorithm stops when there are no more qubits to be labeled or when a qubit is assigned contradicting labels, indicating that F is not a boundary. □

The running time of the algorithm is $O(|\mathsf{C}_3|)$. The algorithm assumes that all qubits have the same error probability. It can be modified so that it picks the most likely qubits if the error probabilities are not uniform. We now give the decoding procedure for color codes.

Theorem 20 (Decoding 3D color codes via 3D toric codes)**.** *An error E on a color code can be estimated using Algorithm 4. The estimate will be within a stabilizer on the color code provided the intermediate decoders also estimate within a stabilizer on the respective codes.*

Proof. The proof of this theorem is straightforward given our previous results. The decoding is performed separately for X and Z errors and it makes use of the fact that the color code is a CSS code. The algorithm proceeds by first estimating the boundary of the X-type errors and Z-errors separately. The correctness of these procedures is due to Theorems 11 and 18. Lemma 19 ensures that these boundaries can be lifted to find the qubits that are in error. $\square$

Algorithm 4: Decoding 3D color codes

Input: A 3-colex Γ and the syndrome
Output: Error estimate $E = E_X E_Z$
1: Measure the syndrome s_X for X type errors
2: Obtain the face boundary ∂E_X from Algorithm 1 with s_X as input
3: Lift the boundary ∂E_X by running Algorithm 3 and obtain Ω_X, the support of E_X
4: Measure the syndrome s_Z for Z type errors
5: Obtain the edge boundary δE_Z from Algorithm 2 with s_Z as input
6: Obtain the face boundary ∂E_Z from Algorithm 1 with δE_Z as input
7: Lift the boundary ∂E_Z by running Algorithm 3 and obtain Ω_Z, the support of E_Z
8: $E = \prod_{\nu \in \Omega_X} X_\nu \prod_{\nu \in \Omega_Z} Z_\nu$

Remark 21. *The decoder could fail if any of the intermediate decoders make a logical error.*

The overall running time depends on the running time of the 3D toric code decoders. We can run them independently or we can take advantage of the fact that the errors on different 3D toric codes are correlated.

IV. CONCLUSION

In this paper we have shown how to project 3D color codes onto 3D toric codes. The projection was motivated by the problem of decoding 3D color codes. The toric codes thus obtained are linearly related to the size of the parent color code. So if 3D toric codes on arbitrary lattices can be decoded efficiently then so can the 3D color codes by projecting them onto 3D toric codes using our map. Our work provides an alternative perspective to that of [11] who also proposed a map between color codes and toric codes. Our approach emphasizes the topological properties of color codes. At this point there is no data available for performance of the decoders based on our map as well as the map due to [11]. One difficulty to compare the performance of the decoders arising out of these maps, as we mentioned earlier, is that we do not have efficient decoders for 3D toric code on an arbitrary lattice. (Decoders are known only for the cubic lattice.) So an open question for further research is to study the decoding of 3D toric codes. Another avenue for further research is to study the possible use of this map for fault tolerant quantum computing protocols.

Acknowledgement This research was supported by a grant from Center for Industrial Consultancy and Sponsored Research.

[1] C. Castelnovo and C. Chamon, Phys. Rev. B **78**, 155120 (2008).
[2] A. Hamma, P. Zanardi, and X.-G. Wen, Phys. Rev. B **72**, 035307 (2005).
[3] H. Bombin and M. A. Martin-Delgado, Phys. Rev. Lett. **98**, 160502 (2007).
[4] S. Bravyi and B. Leemhius and B. Terhal, Annals. of Physics **326**, 839-866 (2011).
[5] A. Kitaev, Annals. of Physics **303**, 2 (2003).
[6] A. R. Calderbank and P. W. Shor, Phys. Rev. A **54**, 1098 (1996).
[7] L. Ioffe and M. Mézard, Phys. Rev. A **75**, 032345 (2007).
[8] P. Brooks and J. Preskill, Phys. Rev. A **87**, 032310 (2013).
[9] P. K. Sarvepalli, A. Klappenecker, and M. Rötteler, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **465**, 1645 (2009).
[10] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, Journal of Mathematical Physics **43**, 4452 (2002).
[11] A. Kubica, B. Yoshida, and F. Pastawski, New Journal of Physics **17**, 083026 (2015).
[12] N. Delfosse, Physical Review A **89**, 012317 (2014).
[13] B. Yoshida, Annals. of Physics **326**, 15 (2011).
[14] A. Bhagoji and P. Sarvepalli, 2015 IEEE International Symposium on Information Theory (ISIT) , 1109 (2015).
[15] H. Bombin, G. Duclos-Cianci, and D. Poulin, New Journal of Physics **14**, 073048 (2012).
[16] A. Calderbank, E. Rains, P. Shor, and N. Sloane, IEEE Trans. Inform. Theory **44**, 1369 (1998).
[17] D. Gottesman, "Stabilizer codes and quantum error correction," (1997), caltech Ph. D. Thesis, eprint: quant-ph/9705052.
[18] H. Bombin and M. A. Martin-Delgado, Phys. Rev. B **75** (2007).