2.1) $x(t) = \frac{\pi t^3}{2}$; $-1<t<1$

Let $T=2$; $a_n = \frac{1}{2}\int_{-1}^{1} x(t)\, e^{-\pi j n t}\, dt$

$\omega = \pi$

$$a_n = \frac{1}{2}\int_{-1}^{1} \frac{\pi t^3}{2} e^{-\pi j n t}\, dt$$

$$a_n = \frac{\pi}{4}\int_{-1}^{1} t^3 e^{-\pi j n t}\, dt$$

$\therefore x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j\pi n t}$

where $a_n = \begin{cases} \dfrac{j e^{-\pi j n}(\pi^2 n^2 - 6)}{2\pi^2 n^3} & ; n \neq 0 \\ 0 & ; n = 0 \end{cases}$

Consider $\int t^3 e^{-\pi j n t} = \frac{t^3 e^{-\pi j n t}}{-\pi j n} - \int \frac{3t^2 e^{-\pi j n t}}{-\pi j n} dt$

$$= \frac{-t^3 e^{-\pi j n t}}{\pi j n} + \frac{3}{\pi j n}\int t^2 e^{-\pi j n t}\, dt$$

$$= \frac{-t^3 e^{-\pi j n t}}{\pi j n} + \frac{3}{\pi j n}\left(\frac{t^2 e^{-\pi j n t}}{-\pi j n} - \int \frac{2t e^{-\pi j n t}}{-\pi j n} dt\right)$$

$$= \frac{-t^3 e^{-\pi j n t}}{\pi j n} + \frac{3t^2 e^{-\pi j n t}}{\pi^2 n^2} - \frac{6}{\pi^2 n^2}\int t e^{-\pi j n t}\, dt$$

$$= \frac{-t^3 e^{-\pi j n t}}{\pi j n} + \frac{3t^2 e^{-\pi j n t}}{\pi^2 n^2} - \frac{6}{\pi^2 n^2}\left(\frac{t e^{-\pi j n t}}{-\pi j n} + \frac{e^{-\pi j n t}}{\pi^2 n^2}\right)$$

$$= \frac{j t^3 e^{-\pi j n t} \cdot \pi^3 n^3 + 3t^2 e^{-\pi j n t} \pi^2 n^2}{\pi^4 n^4} + \frac{6t e^{-\pi j n t}}{\pi^3 j n^3} - \frac{6e^{-\pi j n t}}{\pi^4 n^4}$$

$$= \frac{j\pi^3 n^3 t^3 e^{-\pi j n t} + 3\pi^2 n^2 t^2 e^{-\pi j n t} - 6j\pi n t e^{-\pi j n t} - 6e^{-\pi j n t}}{\pi^4 n^4}$$

So $a_n = \frac{\pi}{4}\int_{-1}^{1} t^3 e^{-\pi j n t}\, dt = \frac{\pi}{4}\left[\frac{j\pi^3 n^3 t^3 e^{-\pi j n t} + 3\pi^2 n^2 t^2 e^{-\pi j n t} - 6j\pi n t e^{-\pi j n t} - 6e^{-\pi j n t}}{\pi^4 n^4}\right]_{t=-1}^{1}$

$$= \frac{\pi}{4}\left(\frac{2j\pi^3 n^3 e^{-\pi j n} - 12 j\pi n e^{-\pi j n}}{\pi^4 n^4}\right)$$

$$= \frac{2j\pi^2 n^2 e^{-\pi j n} - 12 j e^{-\pi j n}}{4\pi^2 n^3}$$

$$= \frac{j e^{-\pi j n}(\pi^2 n^2 - 6)}{2\pi^2 n^3}$$

and $a_0 = \frac{\pi}{4}\int_{-1}^{1} t^3\, dt$

$$= \frac{\pi}{4}\left[\frac{t^4}{4}\right]_{-1}^{1}$$

$$= 0$$

2.2) $x(t) = \pi - t \;;\; -\pi \le t \le \pi$

$T = 2\pi \;;\; a_n = \frac{1}{T}\int_{-\pi}^{\pi} x(t)\, e^{-jn(\frac{2\pi}{T})t}\, dt$

$\omega = 1$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi - t)\, e^{-jnt}\, dt \qquad u = t \quad dv = e^{-jnt}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi e^{-jnt} - te^{-jnt}\, dt \qquad du = dt \quad v = -\frac{1}{jn}e^{-jnt}$$

$$= \frac{1}{2}\left[-\frac{e^{-jnt}}{jn}\right]_{t=-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{-jnt}\, dt$$

consider $\int te^{-jnt}\, dt = \frac{-te^{-jnt}}{jn} - \int \frac{-e^{-jnt}}{jn}\, dt$

$$= \frac{-te^{-jnt}}{jn} + \frac{1}{jn}\left(-\frac{e^{-jnt}}{jn}\right)$$

$$= \frac{-te^{-jnt}}{jn} + \frac{e^{-jnt}}{n^2}$$

$$= \frac{jnte^{-jnt} + e^{-jnt}}{n^2}$$

So $a_n = \frac{1}{2}\left[-\frac{e^{-jnt}}{jn}\right]_{t=-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{-jnt}\, dt$

$$= \frac{1}{2}\left(-\frac{e^{-\pi jn}}{jn} + \frac{e^{\pi jn}}{jn}\right) - \frac{1}{2\pi}\left[\frac{jnte^{-jnt} + e^{-jnt}}{n^2}\right]_{t=-\pi}^{\pi}$$

$$= \cancelto{0}{\frac{e^{\pi jn} - e^{-\pi jn}}{2jn}} - \frac{1}{2\pi}\left(\frac{\pi jne^{-\pi jn} + e^{-\pi jn} + \pi jne^{\pi jn} - e^{\pi jn}}{n^2}\right)$$

$$= -\frac{1}{2\pi}\left(\frac{\pi j \cdot 2\cos(n\pi)}{n}\right)$$

$$= \frac{-j(-1)^n}{n}$$

when $n=0$ $\quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi - t)\, dt$

$$= \frac{1}{2\pi}\left[\pi t - \frac{t^2}{2}\right]_{t=-\pi}^{\pi}$$

$$= \frac{1}{2\pi}(2\pi^2)$$

$$= \pi$$

$$\boxed{\therefore x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jnt} \quad \text{where } a_n = \begin{cases} \pi & ; n=0 \\ \frac{-j(-1)^n}{n} & ; n \ne 0 \end{cases} \quad \#}$$

2.3) $x(t) = t^2 + \sin^3(\pi t)$; $-1 \le t \le 1$

Let $T = 2$; $a_n = \frac{1}{T}\int_{-1}^{1} x(t)\, e^{-jn(\frac{2\pi}{T})t}\, dt$

$\omega = 2$

$$a_n = \frac{1}{2}\int_{-1}^{1} (t^2 + \sin^3(\pi t))\, e^{-\pi j n t}\, dt$$

$$a_n = \frac{1}{2}\int_{-1}^{1} t^2 e^{-\pi j n t} dt + \frac{1}{2}\int_{-1}^{1} \sin^3(\pi t)\, e^{-\pi j n t}$$

consider $\int t^2 e^{-\pi j n t}\, dt = \frac{-t^2 e^{-\pi j n t}}{\pi j n} - \int \frac{-2t e^{-\pi j n t}}{\pi j n}\, dt$

$$= \frac{-t^2 e^{-\pi j n t}}{\pi j n} + \frac{2}{\pi j n}\int t e^{-\pi j n t}\, dt$$

$$= \frac{-t^2 e^{-\pi j n t}}{\pi j n} + \frac{2}{\pi j n}\left(\frac{-t e^{-\pi j n t}}{\pi j n} - \int \frac{-e^{-\pi j n t}}{\pi j n}\, dt\right)$$

$$= \frac{-t^2 e^{-\pi j n t}}{\pi j n} + \frac{2t e^{-\pi j n t}}{\pi^2 n^2} - \frac{2}{\pi^2 n^2}\left[\frac{-e^{-\pi j n t}}{\pi j n}\right]$$

$$= \frac{j t^2 e^{-\pi j n t}}{\pi n} + \frac{2t e^{-\pi j n t}}{\pi^2 n^2} - \frac{2j\, e^{-\pi j n t}}{\pi^3 n^3}$$

$$= \frac{j\pi^2 n^2 t^2 e^{-\pi j n t} + 2\pi n t e^{-\pi j n t} - 2j e^{-\pi j n t}}{\pi^3 n^3}$$

$$= \frac{e^{-\pi j n t}}{\pi^3 n^3}(j\pi^2 n^2 t^2 + 2\pi n t - 2j)$$

consider $\int \sin^3(\pi t)\, e^{-j\pi n t}\, dt = \int \left(\frac{e^{j\pi t} - e^{-j\pi t}}{2i}\right)^3 e^{-j\pi n t}\, dt$

$$= \int \left(\frac{e^{3j\pi t} - e^{-3j\pi t} - 3(e^{j\pi t} - e^{-j\pi t})}{-8j}\right) e^{-j\pi n t}\, dt$$

$$= \int \frac{j e^{j\pi t(3-n)} - j e^{-j\pi t(n+3)} - 3j e^{j\pi t(1-n)} + 3j e^{-j\pi t(n+1)}}{8}\, dt$$

$$= \frac{1}{8}\left(\frac{\cancel{j} e^{j\pi t(3-n)}}{\cancel{j}\pi(3-n)} - \frac{\cancel{j} e^{-j\pi t(n+3)}}{-\cancel{j}\pi(n+3)} - \frac{3\cancel{j} e^{j\pi t(1-n)}}{\cancel{j}\pi(1-n)} + \frac{3\cancel{j} e^{-j\pi t(n+1)}}{-\cancel{j}\pi(n+1)}\right)$$

$$= \frac{1}{8\pi}\left(\frac{-e^{j\pi t(3-n)}}{n-3} + \frac{e^{-j\pi t(n+3)}}{n+3} + \frac{3e^{j\pi t(1-n)}}{n-1} - \frac{3e^{-j\pi t(n+1)}}{n+1}\right)$$

$$= \frac{e^{-j\pi t n}}{8\pi}\left(\frac{e^{3j\pi t}}{3-n} + \frac{e^{-3j\pi t}}{n+3} - \frac{3e^{j\pi t}}{1-n} - \frac{3e^{-j\pi t}}{n+1}\right)$$

so $a_n = \frac{1}{2}\int_{-1}^{1} t^2 e^{-\pi j n t}\,dt + \frac{1}{2}\int_{-1}^{1} \sin^3(\pi t)\, e^{-\pi j n t}\,dt$

$$= \frac{1}{2}\left[\frac{e^{-\pi j n t}}{\pi^3 n^3}\left(j\pi^2 n^2 t^2 + 2\pi n t - 2j\right)\right]_{t=-1}^{1} + \frac{1}{2}\left[\frac{1}{8\pi}\left(\frac{e^{j\pi t(3-n)}}{3-n} + \frac{e^{-j\pi t(n+3)}}{n+3} - \frac{3e^{j\pi t(1-n)}}{1-n} - \frac{3e^{-j\pi t(n+1)}}{n+1}\right)\right]_{t=-1}^{1}$$

$$= \frac{e^{-\pi j n}}{2\pi^3 n^3}\left(j\pi^2 n^2 + 2\pi n - 2j\right) - \frac{e^{\pi j n}}{2\pi^3 n^3}\left(j\pi^2 n^2 - 2\pi n - 2j\right) + \frac{1}{16\pi}\Bigg[\frac{e^{j\pi(3-n)}}{3-n} + \frac{e^{-j\pi(n+3)}}{n+3} - \frac{3e^{j\pi(1-n)}}{1-n} - \frac{3e^{-j\pi(n+1)}}{n+1}$$
$$- \left(\frac{e^{-j\pi(3-n)}}{3-n} + \frac{e^{j\pi(n+3)}}{n+3} - \frac{3e^{-j\pi(1-n)}}{1-n} - \frac{3e^{j\pi(n+1)}}{n+1}\right)\Bigg]$$

$$= \frac{e^{-\pi j n}}{2\pi^3 n^3}\left(j\pi^2 n^2 + 2\pi n - 2j\right) - \frac{e^{\pi j n}}{2\pi^3 n^3}\left(j\pi^2 n^2 - 2\pi n - 2j\right) + \frac{1}{16\pi}\Bigg(\frac{\cancelto{-1}{e^{3\pi j}}\, e^{-n\pi j}}{3-n} + \frac{e^{-n\pi j}\,\cancelto{-1}{e^{-3\pi j}}}{n+3} - \frac{3\cancelto{-1}{e^{\pi j}}\, e^{-n\pi j}}{1-n} - \frac{3e^{-n\pi j}\,\cancelto{-1}{e^{-\pi j}}}{n+1}$$
$$- \frac{\cancelto{-1}{e^{-3\pi j}}\, e^{n\pi j}}{3-n} - \frac{e^{n\pi j}\,\cancelto{-1}{e^{3\pi j}}}{n+3} + \frac{3\cancelto{-1}{e^{-\pi j}}\, e^{n\pi j}}{1-n} + \frac{3e^{n\pi j}\,\cancelto{-1}{e^{\pi j}}}{n+1}\Bigg)$$

$$= \frac{e^{-\pi j n} + e^{\pi j n}}{\pi^2 n^2} + \frac{(j\pi^2 n^2 - 2j)}{2\pi^3 n^3}\underbrace{\left(e^{-\pi j n} - e^{\pi j n}\right)}_{0} + 0$$

$$= \frac{2(-1)^n}{\pi^2 n^2}$$

when $n=0$; $a_0 = \frac{1}{2}\int_{-1}^{1}(t^2 + \sin^3(\pi t))\,dt$

$$= \frac{1}{2}\int_{-1}^{1}\left(t^2 + \sin(\pi t)(1-\cos^2(\pi t))\right)dt$$

$$= \frac{1}{2}\left[\frac{t^3}{3}\right]_{t=-1}^{1} + \frac{1}{2}\int_{-1}^{1}\left(\sin(\pi t) - \sin(\pi t)\cos^2(\pi t)\right)dt$$

$$= \frac{1}{3} + \frac{1}{2}\left[\frac{-\cos(\pi t)}{\pi} + \frac{\cos^3(\pi t)}{\pi}\right]_{t=-1}^{1}$$

$$= \frac{1}{3} + \frac{1}{2}\left(\frac{1}{\pi} - \frac{1}{\pi} - \frac{1}{\pi} + \frac{1}{\pi}\right)$$

$$= \frac{1}{3}$$

$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j\pi n t}$ where $a_n = \begin{cases} \frac{1}{3} & ; n=0 \\ \frac{2(-1)^n}{\pi^2 n^2} & ; n \neq 0 \end{cases}$

4) $X(j\omega) = \text{rect}[(\omega-1)/2]$

1. $x(-2t+4)$

$$\mathcal{F}\{x(t)\} \to \text{rect}[(\omega-1)/2]$$

time scaling

$$\mathcal{F}\{x(-2t)\} \to \frac{1}{2}\text{rect}[(-\frac{\omega}{4}-\frac{1}{2})]$$

time shifting

$$\mathcal{F}\{x(-2t+4)\} \to \frac{1}{2}\text{rect}[-\frac{\omega+2}{4}]e^{j\omega}$$

#

2. $(t-1)x(t-1)$

differentiation in frequency

$$\mathcal{F}\{t\,x(t)\} \to j\frac{d\,\text{rect}[\frac{\omega-1}{2}]}{d\omega} = j[\partial(\frac{\omega}{2})+\partial(\frac{\omega-2}{2})]$$

time shifting

$$\mathcal{F}\{(t-1)x(t-1)\} \to j(\partial[\frac{\omega}{2}]+\partial[\frac{\omega-2}{2}])e^{-j\omega}$$

3. $t\frac{dx(t)}{dt}$

differentiation in time

$$\mathcal{F}\{\frac{dx(t)}{dt}\} \to j\omega\,\text{rect}[\frac{\omega-1}{2}]$$

differentiation in frequency

$$\mathcal{F}\{t\frac{dx(t)}{dt}\} \to j\frac{d\,j\omega\,\text{rect}[\frac{\omega-1}{2}]}{d\omega} = -\omega(\delta[\frac{\omega}{2}]-\delta[\frac{(\omega-2)}{2}]) - \text{rect}[\frac{\omega-1}{2}]$$

#1

4. $x(2t-1)e^{-j2t}$

time scaling

$$\mathcal{F}\{x(2t)\} \to \frac{1}{2}\text{rect}[\frac{\omega}{4}-\frac{1}{2}]$$

frequency shifting

$$\mathcal{F}\{x(2t)e^{-j2t}\} \to \frac{1}{2}\text{rect}[\frac{\omega+1}{4}-\frac{1}{2}]$$

time shifting

$$\mathcal{F}\{x(2t-1)e^{-2j(t-\frac{1}{2})}\} \to \frac{1}{2}\text{rect}[\frac{\omega-1}{4}]e^{-\frac{1}{4}j(\omega+1)}$$

linerity

$$\mathcal{F}\{x(2t-1)e^{-2jt}\} \to \frac{1}{2}\text{rect}[\frac{\omega-1}{4}]e^{-\frac{1}{4}j(\omega+5)}$$

5. $x(t) * x(t-1)$

time shifting

$$\mathcal{F}\{x(t-1)\} \to \text{rect}[\frac{\omega-1}{2}]e^{-j\omega}$$

convolution

$$\mathcal{F}\{x(t) * x(t-1)\} \to \text{rect}[\frac{\omega-1}{2}]\cdot\text{rect}[\frac{\omega-1}{2}]e^{-j\omega} = \text{rect}[\frac{\omega-1}{2}]e^{-j\omega}$$