1.1)

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j(2\pi/N)nk}$$

1) $X[0]=3,\ X[1]=1/\sqrt{2}-j/\sqrt{2},\ X[2]=-2,\ X[3]=1/\sqrt{2}+j/\sqrt{2}$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j(2\pi/N)nk}$$

$$N=4;\quad x[n]=\frac{1}{4}\sum_{k=0}^{3}X[k]e^{\frac{j\pi nk}{2}}$$

$$=\frac{1}{4}\left(X[0]e^{0}+X[1]e^{\frac{j\pi n}{2}}+X[2]e^{j\pi n}+X[3]e^{\frac{3}{2}j\pi n}\right)$$

$$=\frac{1}{4}\left(3+\left(\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}}\right)e^{\frac{j\pi n}{2}}-2e^{j\pi n}+\left(\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}\right)e^{\frac{3}{2}j\pi n}\right)$$

$$=\frac{1}{4}\left(3+\left(\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}}\right)\left(\cos\left(\frac{n\pi}{2}\right)+j\sin\left(\frac{n}{2}\pi\right)\right)-2\cos\pi n-2j\overset{0}{\cancel{\sin}}\pi n+\left(\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}}\right)\left(\cos\left(\frac{3}{2}\pi n\right)+j\sin\left(\frac{3}{2}\pi n\right)\right)\right)$$

$$=\frac{1}{4}\left(3+\frac{j}{\sqrt{2}}\sin\left(\frac{n\pi}{2}\right)+\frac{1}{\sqrt{2}}\sin\left(\frac{n\pi}{2}\right)-2\cos\pi n+\frac{j}{\sqrt{2}}\sin\left(\frac{3}{2}\pi n\right)-\frac{1}{\sqrt{2}}\sin\left(\frac{3}{2}\pi n\right)\right)$$

$$=\frac{3}{4}+\frac{\cos\left(\frac{n}{2}\pi\right)}{4\sqrt{2}}-\frac{j\cos\left(\frac{n}{2}\pi\right)}{4\sqrt{2}}+\frac{j}{4\sqrt{2}}\sin\left(\frac{n\pi}{2}\right)+\frac{1}{4\sqrt{2}}\sin\left(\frac{n\pi}{2}\right)-\frac{\cos\pi n}{2}+\frac{j}{4\sqrt{2}}\sin\left(\frac{3}{2}\pi n\right)-\frac{1}{4\sqrt{2}}\sin\left(\frac{3}{2}\pi n\right)+\frac{\cos\left(\frac{3}{2}\pi n\right)}{4\sqrt{2}}+\frac{j\cos\left(\frac{3}{2}\pi n\right)}{4\sqrt{2}}\ \#$$

$$=\frac{3}{4}+\frac{1}{4\sqrt{2}}\left(\cos\left(\frac{n}{2}\pi\right)+\cos\left(\frac{3\pi}{2}n\right)\right)-\frac{1}{4\sqrt{2}}\left(-\sin\left(\frac{n}{2}\pi\right)+\sin\left(\frac{3n}{2}\pi\right)\right)-\frac{\cos\pi n}{2}+\frac{j}{4\sqrt{2}}\left(\sin\left(\frac{n\pi}{2}\right)+\sin\left(\frac{3}{2}\pi n\right)\right)+\frac{j}{4\sqrt{2}}\left(-\cos\left(\frac{n\pi}{2}\right)+\cos\left(\frac{3n\pi}{2}\right)\right)$$

$$=\frac{3}{4}+\frac{1}{4\sqrt{2}}\left(\cos\left(\frac{n}{2}\pi\right)+\cos\left(\frac{3\pi}{2}n\right)\right)-\frac{1}{4\sqrt{2}}\left(-\sin\left(\frac{n}{2}\pi\right)+\sin\left(\frac{3n}{2}\pi\right)\right)-\frac{\cos\pi n}{2}+\frac{j}{4\sqrt{2}}\left(2\underbrace{\cancel{\sin(\pi n)}}_{0}\cos\left(\frac{\pi n}{2}\right)\right)+\frac{j}{4\sqrt{2}}\left(2\underbrace{\cancel{\sin(\pi n)}}_{0}\sin\left(\frac{\pi n}{2}\right)\right)$$

$$=\frac{3}{4}+\frac{1}{4\sqrt{2}}\left(\cos\left(\frac{n}{2}\pi\right)+\cos\left(\frac{3\pi}{2}n\right)\right)-\frac{1}{4\sqrt{2}}\left(-\sin\left(\frac{n}{2}\pi\right)+\sin\left(\frac{3n}{2}\pi\right)\right)-\frac{\cos\pi n}{2}\ \#$$

2) $X[0]=-2,\ X[1]=\sqrt{3}+j,\ X[2]=3,\ X[3]=\sqrt{3}-j$

$$N=4;\quad x[n]=\frac{1}{4}\sum_{k=0}^{N-1}X[k]e^{j\left(\frac{2\pi}{N}\right)nk}$$

$$=\frac{1}{4}\sum_{k=0}^{3}X[k]e^{\frac{\pi}{2}jnk}$$

$$=\frac{1}{4}\left(X[0]e^{0}+X[1]e^{\frac{\pi}{2}jn}+X[2]e^{\pi jn}+X[3]e^{\frac{3\pi}{2}jn}\right)$$

$$=\frac{1}{4}\left(-2+(\sqrt{3}+j)\left(\cos\left(\frac{\pi}{2}n\right)+j\sin\left(\frac{\pi}{2}n\right)\right)+3\left(\cos(\pi n)+j\overset{0}{\cancel{\sin(\pi n)}}\right)+(\sqrt{3}-j)\left(\cos\left(\frac{3\pi}{2}n\right)+j\sin\left(\frac{3\pi n}{2}\right)\right)\right)$$

$$=\frac{1}{4}\left(-2+\sqrt{3}\cos\left(\frac{\pi}{2}n\right)+j\cos\left(\frac{\pi}{2}n\right)+\sqrt{3}j\sin\left(\frac{\pi}{2}n\right)-\sin\left(\frac{\pi}{2}n\right)+3\cos(\pi n)+\sqrt{3}\cos\left(\frac{3\pi}{2}n\right)+\sqrt{3}j\sin\left(\frac{3\pi}{2}n\right)-j\cos\left(\frac{3\pi}{2}n\right)+\sin\left(\frac{3\pi n}{2}\right)\right)$$

$$=-\frac{1}{2}+\frac{\sqrt{3}}{4}\cos\left(\frac{\pi}{2}n\right)+\frac{j}{4}\cos\left(\frac{\pi}{2}n\right)+\frac{\sqrt{3}}{4}j\sin\left(\frac{\pi}{2}n\right)-\frac{\sin\left(\frac{\pi}{2}n\right)}{4}+\frac{3}{4}\cos(\pi n)+\frac{\sqrt{3}}{4}\cos\left(\frac{3\pi}{2}n\right)+\frac{\sqrt{3}}{4}j\sin\left(\frac{3\pi}{2}n\right)-\frac{j}{4}\cos\left(\frac{3\pi}{2}n\right)+\frac{\sin\left(\frac{3\pi n}{2}\right)}{4}$$

$$=-\frac{1}{2}+\frac{\sqrt{3}}{4}\cos\left(\frac{\pi}{2}n\right)-\frac{1}{4}\sin\left(\frac{\pi}{2}n\right)+\frac{3}{4}\cos\pi n+\frac{\sqrt{3}}{4}\cos\left(\frac{3\pi}{2}n\right)+\frac{1}{4}\sin\left(\frac{3\pi}{2}n\right)+\frac{\sqrt{3}}{4}j\left(\sin\left(\frac{\pi}{2}n\right)\underbrace{+}_{0}\sin\left(\frac{3\pi}{2}n\right)\right)+\frac{j}{4}\left(\cos\frac{\pi n}{2}\underbrace{-}_{0}\cos\frac{3\pi}{2}n\right)$$

$$=-\frac{1}{2}+\frac{\sqrt{3}}{4}\cos\left(\frac{\pi}{2}n\right)-\frac{1}{4}\sin\left(\frac{\pi}{2}n\right)+\frac{3}{4}\cos\pi n+\frac{\sqrt{3}}{4}\cos\left(\frac{3\pi}{2}n\right)+\frac{1}{4}\sin\left(\frac{3\pi}{2}n\right)\ \#$$

3) $X[0] = 1,\ X[1] = 2-2j\sqrt{3},\ X[2] = -3,\ X[3] = 2+2j\sqrt{3}$

$$N=4;\quad x[n] = \frac{1}{4}\sum_{k=0}^{3} X[k]e^{\frac{\pi}{2}jnk}$$

$$= \frac{1}{4}\left(X[0]e^{0} + X[1]e^{j\frac{\pi}{2}n} + X[2]e^{j\pi n} + X[3]e^{\frac{3}{2}j\pi n}\right)$$

$$= \frac{1}{4}\left(1 + (2-2j\sqrt{3})\left(\cos\left(\tfrac{\pi}{2}n\right) + j\sin\left(\tfrac{\pi}{2}n\right)\right) - 3\left(\cos(\pi n) + j\overset{0}{\cancel{\sin(\pi n)}}\right) + (2+2j\sqrt{3})\left(\cos\left(\tfrac{3}{2}\pi n\right) + j\sin\left(\tfrac{3}{2}\pi n\right)\right)\right)$$

$$= \frac{1}{4} + \frac{\cos(\frac{\pi}{2}n)}{2} + \frac{j}{2}\sin\left(\tfrac{\pi}{2}n\right) - \frac{\sqrt{3}}{2}j\cos\left(\tfrac{\pi}{2}n\right) + \frac{\sqrt{3}}{2}\sin\left(\tfrac{\pi}{2}n\right) - \frac{3}{4}\cos(\pi n) + \frac{\cos(\frac{3}{2}\pi n)}{2} + \frac{j\sin(\frac{3}{2}\pi n)}{2} + \frac{j\sqrt{3}}{2}\cos\left(\tfrac{3}{2}\pi n\right) - \frac{\sqrt{3}}{2}\sin\left(\tfrac{3}{2}\pi n\right)$$

$$= \frac{1}{4} + \frac{1}{2}\cos\left(\tfrac{\pi}{2}n\right) + \frac{\sqrt{3}}{2}\sin\left(\tfrac{\pi}{2}n\right) - \frac{3}{4}\cos(\pi n) + \frac{1}{2}\cos\left(\tfrac{3}{2}\pi n\right) - \frac{\sqrt{3}}{2}\sin\left(\tfrac{3}{2}\pi n\right) + \frac{j}{2}\underbrace{\left(\sin\left(\tfrac{\pi n}{2}\right) + \sin\left(\tfrac{3\pi n}{2}\right)\right)}_{0} + \frac{j\sqrt{3}}{2}\underbrace{\left(\cos\tfrac{3\pi}{2}n - \cos\tfrac{\pi}{2}n\right)}_{0}$$

$$= \frac{1}{4} + \frac{1}{2}\cos\left(\tfrac{\pi}{2}n\right) + \frac{\sqrt{3}}{2}\sin\left(\tfrac{\pi}{2}n\right) - \frac{3}{4}\cos(\pi n) + \frac{1}{2}\cos\left(\tfrac{3}{2}\pi n\right) - \frac{\sqrt{3}}{2}\sin\left(\tfrac{3}{2}\pi n\right) \#$$

2.1) $x[n] = \left(\frac{1}{3}\right)^{|n|}$, find $X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\right)^{|n|}e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n}e^{-j\omega n} + \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n}e^{j\omega n}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{3}e^{-j\omega}\right)^{n} + \sum_{n=1}^{\infty}\left(\frac{1}{3}e^{j\omega}\right)^{n}$$

$$= \frac{1}{1-\frac{1}{3}e^{-j\omega}} + \left(\frac{1}{3}e^{j\omega}\right)\frac{1}{1-\frac{1}{3}e^{j\omega}}$$

$$= \frac{3e^{-j\omega}}{3e^{-j\omega}-1} + \frac{1}{3e^{j\omega}-1}$$

2.2) $x[n] = a^n\cos(\Omega_0 n)\cdot u[n],\ |a|<1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n\cos(\Omega_0 n)e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n\cdot\frac{1}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right)e^{-j\omega n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} a^n\cdot e^{j\Omega_0 n}\cdot e^{-j\omega n} + a^n\cdot e^{-j\Omega_0 n}e^{-j\omega n}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}\left(a\cdot e^{j\Omega_0 - j\omega}\right)^n + \sum_{n=0}^{\infty}\left(a\cdot e^{-j\Omega_0 - j\omega}\right)^n\right)$$

$$= \frac{1}{2}\left(\frac{1}{1-ae^{j\Omega_0 - j\omega}} + \frac{1}{1-ae^{-j\Omega_0 - j\omega}}\right)$$

2.3) $x[n] = (n+1)a^n\cdot u[n],\ |a|<1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty}(n+1)a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} na^n e^{-j\omega n} + a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} na^n e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

Consider $\displaystyle\sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1-ae^{-j\omega}}$

if $x[n] = a^n \longleftrightarrow X(e^{j\omega}) = \dfrac{1}{1-ae^{-j\omega}}$

then $n\,x[n] = na^n \longleftrightarrow j\dfrac{dX(e^{j\omega})}{d\omega} = j\dfrac{d(1-ae^{-j\omega})^{-1}}{d\omega} = j\cdot(-1)(aj\cdot e^{-j\omega})(1-ae^{-j\omega})^{-2}$

$$= \frac{ae^{-j\omega}}{(1-ae^{-j\omega})^2}$$

So $$X(e^{j\omega}) = \frac{a\cdot e^{-j\omega}}{(1-ae^{-j\omega})^2} + \frac{1}{1-ae^{-j\omega}}$$

$$= \frac{ae^{-j\omega} + 1 - ae^{-j\omega}}{(1-ae^{-j\omega})^2}$$

$$= \frac{1}{(1-ae^{-j\omega})^2}$$