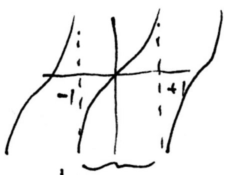
Problem 2. Find the Fourier Series (FS) of the periodic function $x(t)$.

2.1) $x(t) = \frac{\pi}{2} t^3; \quad -1 < t < 1$



$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad ; \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \;\Big|\; T=2$$

$$; \; a_k = \frac{1}{T} \int_{<T>} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$a_k = \frac{1}{2} \int_{t=-1}^{+1} \frac{\pi}{2} t^3\, e^{-jk\omega_0 t}\, dt$$

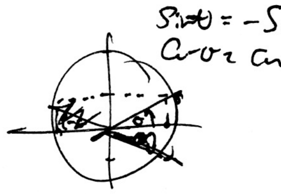
$$= \frac{\pi}{4} \int_{t=-1}^{1} t^3 e^{-jk\omega_0 t}\, dt$$

} $e^{j\theta} = \cos\theta + j\sin\theta$; Euler Formula , $e^{-j\theta} = \cos\theta - j\sin\theta$

$$= \frac{\pi}{4} \int t^3 \left(\cos(k\omega_0 t) - j\sin(k\omega_0 t) \right) dt$$

$$= \frac{\pi}{4} \left\{ \int_{t=-1}^{1} t^3 \cos(k\omega_0 t)\, dt - j \int_{t=-1}^{1} t^3 \sin(k\omega_0 t)\, dt \right\}$$

$\sin(-\theta) = -\sin\theta$.
$\cos(-\theta) = \cos\theta$.



~~$\sin\theta = \sin -\theta$~~
~~$\cos\theta = -\cos(-\theta)$~~
~~$\sin\theta - \sin(-\theta) = 0$~~
~~$\cos\theta - \cos(-\theta) = 2\cos\theta$~~

} Integration By parts: $\int u\,dv = uv - \int v\,du$

• $\int_{t=-1}^{1} t^3 \cos(k\omega_0 t)\, dt$

$u = t^3$, $dv = \cos(k\omega_0 t)\, dt$

$du = 3t^2 dt$, $v = \int \cos(k\omega_0 t)\, dt \times \frac{k\omega_0}{k\omega_0} = \frac{1}{k\omega_0} \sin(k\omega_0 t)$

$$\int_{t=-1}^{1} t^3 \cos(k\omega_0 t)\, dt = t^3 \cdot \frac{\sin(k\omega_0 t)}{k\omega_0} \Bigg|_{t=-1}^{+1} - \int_{t=-1}^{+1} \frac{\sin(k\omega_0 t)}{k\omega_0} \times 3t^2\, dt$$

$$\left[\int t^n \cos(k\omega_0 t)\, dt = t^n \frac{\sin(k\omega_0 t)}{k\omega_0} - \int \frac{\sin(k\omega_0 t)}{k\omega_0}\, n t^{n-1}\, dt \right]$$

$$\left[\int t^n \sin(k\omega_0 t)\, dt = t^n \frac{(-\cos(k\omega_0 t))}{k\omega_0} + \int \frac{\cos(k\omega_0 t)}{k\omega_0}\, n t^{n-1}\, dt \right]$$

$$\int \frac{\sin(k\omega_0 t)}{k\omega_0}\, 3t^2\, dt = \frac{3}{k\omega_0} \left(\frac{-t^2 \cos(k\omega_0 t)}{k\omega_0} + \frac{2}{k\omega_0} \int \cos(k\omega_0 t)\, t\, dt \right)$$

$$\int \cos(k\omega_0 t)\, t\, dt = \frac{t \sin(k\omega_0 t)}{k\omega_0} - \frac{1}{k\omega_0} \int \sin(k\omega_0 t)\, dt \times \frac{k\omega_0}{k\omega_0}$$

$$= \frac{t \sin(k\omega_0 t)}{k\omega_0} + \frac{1}{(k\omega_0)^2} \cos(k\omega_0 t)$$

$$\int \frac{\sin(k\omega_0 t)}{k\omega_0}\, 3t^2\, dt = \frac{-3t^2 \cos(k\omega_0 t)}{(k\omega_0)^2} + \frac{6}{(k\omega_0)^2} \left(\frac{t \sin(k\omega_0 t)}{k\omega_0} + \frac{\cos(k\omega_0 t)}{(k\omega_0)^2} \right)$$

$$\int \frac{\sin(k\omega_0 t)}{k\omega_0}\, 3t^2\, dt = -\frac{3t^2}{(k\omega_0)^2}\cos(k\omega_0 t) + \frac{6t\sin(k\omega_0 t)}{(k\omega_0)^3} + \frac{6\cos(k\omega_0 t)}{(k\omega_0)^4}$$

$$\therefore \int_t t^3\cos(k\omega_0 t)\,dt = \frac{t^3}{k\omega_0}\sin(k\omega_0 t) + \frac{3t^2}{(k\omega_0)^2}\cos(k\omega_0 t) - \frac{6t}{(k\omega_0)^3}\sin(k\omega_0 t) - \frac{6}{(k\omega_0)^4}\cos(k\omega_0 t)$$

$\cos(\theta+\frac{\pi}{2}) = -\sin\theta$

$\cancel{\int \cos\theta = \sin(\theta+\frac{\pi}{2})}$ [illegible] Integral [illegible] $dt = d(t+\text{const.})$

$$\therefore \int t^3\sin(k\omega_0 t)\,dt = -\frac{t^3}{k\omega_0}\cos(k\omega_0 t) + \frac{3t^2}{(k\omega_0)^2}\sin(k\omega_0 t) + \frac{6t}{(k\omega_0)^3}\cos(k\omega_0 t) - \frac{6}{(k\omega_0)^4}\sin(k\omega_0 t)$$

$$\therefore \int_{t=-1}^{+1} t^3\cos(k\omega_0 t)\,dt = \overset{0}{\frac{2}{k\omega_0}\sin(k\omega_0)} + \cancel{\frac{6}{(k\omega_0)^2}\cos(k\omega_0)} - \cancel{\frac{6\cos(k\omega_0)}{(k\omega_0)^4}} = \cancel{\frac{2\sin(k\omega_0)}{k\omega_0}} - \cancel{\frac{12\sin(k\omega_0 t)}{(k\omega_0)^3}} = 0.$$

$$\cancel{\int_{t=-1}^{+1} t^3\sin(k\omega_0 t)\,dt = -\frac{2}{k\omega_0}\cos(k\omega_0 t) + \frac{12}{(k\omega_0)^3}\cos(k\omega_0 t) - \frac{6}{(k\omega_0)^2}\sin(k\omega_0) - \frac{12}{(k\omega_0)^4}\sin(k\omega_0)}$$

$$\cancel{\therefore a_k = \frac{\pi}{4}\left\{\int_{t=-1}^{+1} t^3\cos(k\omega_0 t)\,dt - j\int_{t=-1}^{+1} t^3\sin(k\omega_0 t)\,dt\right\}}$$

$$\cancel{= \frac{\pi}{4}\left\{\frac{6}{(k\omega_0)^2}\cos(k\omega_0) - \frac{6\cos(k\omega_0)}{(k\omega_0)^4} + j\frac{2}{k\omega_0}\cos(k\omega_0) - j\frac{12\cos(k\omega_0)}{(k\omega_0)^3}\right\}}$$

$$\cancel{\therefore a_k = \frac{3\pi}{2}\frac{\cos(k\omega_0)}{(k\omega_0)^2}\left(1-\frac{1}{(k\omega_0)^2}\right) + j\frac{\pi\cos(k\omega_0)}{2k\omega_0}\left(1-\frac{6}{(k\omega_0)^2}\right)}$$

$$\cancel{= \frac{\pi}{4}\left\{6\cos(k\omega_0)(k\omega_0)^2 - 6\cos(k\omega_0) + j2\cos(k\omega_0)(k\omega_0)^3 - j12\cos(k\omega_0)(k\omega_0) \Big/ (k\omega_0)^4\right\}}$$

$$\cancel{= -3\cos(k\omega_0)\pi - j6\pi k\omega_0\cos(k\omega_0) + 3\pi(k\omega_0)^2\cos(k\omega_0) + j\pi(k\omega_0)^3\cos(k\omega_0) \Big/ 2(k\omega_0)^4}$$

$$\therefore \int_{t=-1}^{+1} t^3\sin(k\omega_0 t)\,dt = -\frac{2}{k\omega_0}\cos(k\omega_0) + \frac{6}{(k\omega_0)^2}\sin(k\omega_0) + \frac{12}{(k\omega_0)^3}\cos(k\omega_0) - \frac{12}{(k\omega_0)^4}\sin(k\omega_0)$$

$$\therefore a_k = \frac{\pi}{4}\{0 - j(\ldots)\} = j\left\{\frac{\pi}{2}\frac{\cos(k\omega_0)}{k\omega_0} - \frac{3\pi}{2}\frac{\sin(k\omega_0)}{(k\omega_0)^2} - \frac{3\pi\cos(k\omega_0)}{(k\omega_0)^3} + \frac{3\pi\sin(k\omega_0)}{(k\omega_0)^4}\right\}$$

$$= j\frac{1}{2(k\omega_0)^4}\left\{\pi\cos(k\omega_0)k^3\omega_0^3 - 3\pi\sin(k\omega_0)k^2\omega_0^2 - 6\pi\cos(k\omega_0)k\omega_0 + 6\pi\sin(k\omega_0)\right\}$$

$$\therefore a_k = \frac{j}{2\pi^4k^4}\left\{\pi^4k^3\cos(k\omega_0) - 3\pi^3k^2\sin(k\omega_0) - 6\pi^2k\cos(k\omega_0) + 6\pi\sin(k\omega_0)\right\}$$

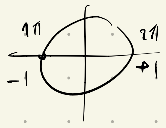
Con: $k=0$: $a_{k=0} = \frac{\pi}{4}\int t^3 \cancel{e^0}\,dt = \frac{\pi}{4}\frac{t^4}{4}\Big|_{-1}^{1} = 0$

Answer

$$\therefore x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\pi t}\ ; \quad a_k = \begin{cases} 0 & ; k=0 \\ j\left(\dfrac{\pi^4k^3\cos(k\omega_0) - 3\pi^3k^2\sin(k\omega_0) - 6\pi^2k\cos(k\omega_0) + 6\pi\sin(k\omega_0)}{2\pi^4k^4}\right) & \end{cases}$$

$;\ \omega_0 = \pi\ ;\ k \neq 0$

For k is integer: $a_k = \begin{cases} 0 & ; k = 0 \\ j\frac{1}{2k}\left(\frac{6}{\pi^2 k^2} - 1\right) & ; k \text{ is odd} \\ j\frac{1}{2k}\left(1 - \frac{6}{\pi^2 k^2}\right) & ; k \text{ is even} \end{cases}$ #



2.2)

$x(t) = \pi - t \quad ; \quad -\pi \le t \le \pi$

Fourier Ser: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad ; \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{(2\pi)} = 1 \;, \; T = 2\pi$

$$a_k = \frac{1}{T}\int_{\langle T\rangle} x(t) e^{-jk\omega_0 t}\,dt$$

$$a_k = \frac{1}{2\pi}\int_{t=-\pi}^{\pi} (\pi - t) e^{-jk\omega_0 t}\,dt$$

Case: $k=0$

$$a_{k=0} = \frac{1}{2\pi}\int_{t=-\pi}^{\pi} (\pi-t)\cancel{e^{0}}\,dt = \frac{-1}{2\pi}\int_{t=-\pi}^{+\pi}(\pi-t)\,d(\pi-t) = \frac{-1}{2\pi}\left.\frac{(\pi-t)^2}{2}\right|_{t=-\pi}^{+\pi}$$

$$= \frac{-1}{2\pi}\left(0 - \frac{\cancel{4}^{2}\pi^2}{\cancel{2}}\right) = \pi \qquad \therefore a_0 = \pi \quad \#$$

Case: $k \neq 0$

$$a_k = \frac{1}{\cancel{2\pi}}\left(\cancel{\int_{t=-\pi}^{\pi} \pi e^{-jkt}\,dt - \int_{t=-\pi} t\, e^{-jkt}\,dt}\right) = \frac{1}{2\pi}\int_{t=-\pi}^{\pi}(\pi-t)\left(\cos(kt) - j\sin(kt)\right)dt$$

$$= \frac{1}{2\pi}\left\{\int_t \pi\cos(kt)\,dt - j\pi\int \sin(kt)\,dt - \int t\cos(kt)\,dt + j\int t\sin(kt)\,dt\right\}$$

Bypass

$$= \frac{1}{2\pi}\left\{\frac{\pi}{k}\sin(kt) + j\frac{\pi}{k}\cos(kt) - t\frac{\sin(kt)}{k} + \frac{1}{k}\frac{(-\cos(kt))}{k} + j\,t\frac{(-\cos(kt))}{k} + j\frac{1}{k}\frac{\sin(kt)}{k}\right\}\Bigg|_{t=-\pi}^{t=+\pi}$$

$$= \frac{1}{2\pi}\left\{\frac{2\pi}{k}\sin(k\pi) \mp j\frac{2\pi}{k}\cos(k\pi) + j\frac{2}{k^2}\sin(k\pi)\right\}$$

$$= \frac{\sin(k\pi)}{k} - j\frac{\cos(k\pi)}{k} + j\frac{\sin(k\pi)}{k^2\pi}$$

$$\therefore \text{Fourier Series } x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkt} \;;\; a_k = \begin{cases} \pi & ; k=0 \\ \frac{\pi\sin(k\pi)}{k\pi} + j\left(\frac{\sin(k\pi)}{k^2\pi} - \frac{\cos(k\pi)}{k}\right) & ; k\neq 0. \end{cases}$$

$$= \pi\,\mathrm{Sinc}(k\pi) + \frac{j}{k}\left\{\mathrm{Sinc}(k\pi) - \cos(k\pi)\right\}$$

#

$$\text{for } k \text{ is integer}: \quad a_k = \begin{cases} 0 & ; k=0 \\ j/k & ; k \text{ is odd} \\ -j/k & ; k \text{ is even} \end{cases} \quad \#$$

Problem 2-3

$$x(t) = t^2 + \sin^3(\pi t) \quad ; \quad -1 \leq t \leq 1$$

Fourier Series: $\quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad ; \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$

$$a_k = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\pi t}\, dt$$

$$a_k = \frac{1}{2}\int_{t=-1}^{+1} \left(t^2 + \sin^3(\pi t)\right) e^{-jk\pi t}\, dt$$

$$= \frac{1}{2}\int_{t=-1}^{+1} t^2 e^{-jk\pi t}\, dt + \frac{1}{2}\int_{t=-1}^{+1} \sin^3(\pi t)\, e^{-jk\pi t}\, dt$$

$\}\ \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

$$= \frac{1}{2}\int t^2 e^{-jk\pi t}\, dt + \frac{1}{2}\int \left(\frac{3}{4}\sin(\pi t) - \frac{1}{4}\sin(3\pi t)\right) e^{-jk\pi t}\, dt$$

$\}\ \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

$$= \frac{1}{2}\int t^2 e^{-jk\pi t}\, dt + \frac{1}{8}\int 3\left(\frac{e^{j\pi t} - e^{-j\pi t}}{2j}\right) e^{-jk\pi t} - \left(\frac{e^{j3\pi t} - e^{-j3\pi t}}{2j}\right) e^{-jk\pi t}\, dt$$

$$= \frac{1}{2}\int t^2 e^{-jk\pi t}\, dt + \frac{1}{16j}\int \left(3e^{j\pi t(1-k)} - 3e^{-j\pi t(1+k)}\right) - \left(e^{j\pi t(3-k)} - e^{-j\pi t(3+k)}\right) dt$$

$$= \frac{1}{2}\int t^2 e^{-jk\pi t}\, dt + \frac{3}{16j}\int e^{j\pi(1-k)t}\, dt - \frac{3}{16j}\int e^{-j\pi(1+k)t}\, dt - \frac{1}{16j}\int e^{j\pi(3-k)t}\, dt + \frac{1}{16j}\int e^{-j\pi(3+k)t}\, dt$$

$\}$ Integration By parts of $\int t^n e^{-jk\pi t}\, dt$

$u = t^n \quad , \quad dv = e^{-jk\pi t}\, dt$

$du = nt^{n-1}\, dt$

$$v = \int e^{-jk\pi t}\, dt\, \frac{(-jk\pi)}{(-jk\pi)} = \frac{e^{-jk\pi t}}{-jk\pi}$$

$$\therefore \int t^n e^{-jk\pi t}\, dt = t^n \frac{e^{-jk\pi t}}{-jk\pi} - \int \frac{e^{-jk\pi t}}{-jk\pi} \cdot nt^{n-1}\, dt$$

$$= \frac{1}{2}\, t^2 \frac{e^{-jk\pi t}}{-jk\pi} - \frac{1}{\cancel{2}}\int \frac{e^{-jk\pi t}}{-jk\pi} \cdot \cancel{2}t\, dt + \ldots$$

$$= \frac{t^2}{2}\frac{e^{-jk\pi t}}{-jk\pi} - \frac{1}{-jk\pi}\int \underbrace{e^{-jk\pi t} \cdot t\, dt}_{\text{By parts}} + \ldots$$

$$= \frac{t^2}{2}\frac{e^{-jh\pi t}}{-jh\pi} - \frac{1}{-jh\pi}\left(\frac{t e^{-jh\pi t}}{-jh\pi} - \frac{1}{-jh\pi}\int e^{-jh\pi t}\,dt \right) + \dots$$

$$= \frac{t^2}{2}\frac{e^{-jh\pi t}}{(-jh\pi)} - \frac{1}{(-jh\pi)^2} t e^{-jh\pi t} + \frac{e^{-jh\pi t}}{(-jh\pi)^3} + \dots$$

$$= \frac{t^2}{2}\frac{e^{-jh\pi t}}{(-jh\pi)} - t\frac{e^{-jh\pi t}}{(-jh\pi)^2} + \frac{e^{-jh\pi t}}{(-jh\pi)^3} + \frac{3}{16j}\int e^{j\pi(1-h)t}\,dt - \frac{3}{16j}\int e^{-j\pi(1+h)t}\,dt - \frac{1}{16j}\int e^{j\pi(3-h)t}\,dt$$
$$+ \frac{1}{16j}\int e^{-j\pi(3+h)t}\,dt$$

$$= \frac{t^2}{2}\frac{e^{-jh\pi t}}{(-jh\pi)}\Bigg|_{t=-1}^{+1} - t\frac{e^{-jh\pi t}}{(-jh\pi)^2}\Bigg|_{t=-1}^{+1} + \frac{e^{-jh\pi t}}{(-jh\pi)^3}\Bigg|_{t=-1}^{+1} + \frac{3}{16j}\frac{e^{j\pi(1-h)t}}{j\pi(1-h)}\Bigg|_{t=-1}^{+1} - \frac{3}{16j}\frac{e^{-j\pi(1+h)t}}{(-j\pi(1+h))}\Bigg|_{t=-1}^{+1} - \frac{1}{16j}\frac{e^{j\pi(3-h)t}}{j\pi(3-h)}\Bigg|_{t=-1}^{+1}$$
$$+ \frac{1}{16j}\frac{e^{-j\pi(3+h)t}}{(-j\pi(3+h))}\Bigg|_{t=-1}^{+1}$$

$$= \frac{1}{h\pi}\left(\frac{e^{jh\pi} - e^{-jh\pi}}{2j}\right) + \frac{2}{h^2\pi^2}\left(\frac{e^{-jh\pi} + e^{jh\pi}}{2}\right) - \frac{2}{h^2\pi^3}\left(\frac{e^{jh\pi} - e^{-jh\pi}}{2j}\right)$$

$$- \frac{3j}{8\pi(1-h)}\left(\frac{e^{j\pi(1-h)} - e^{-j\pi(1-h)}}{2j}\right) + \frac{3j}{8\pi(1+h)}\left(\frac{e^{j\pi(1+h)} - e^{-j\pi(1+h)}}{2j}\right)$$

$$+ \frac{j}{8\pi(3-h)}\left(\frac{e^{j\pi(3-h)} - e^{-j\pi(3-h)}}{2j}\right) - \frac{j}{8\pi(3+h)}\left(\frac{e^{j\pi(3+h)} - e^{-j\pi(3+h)}}{2j}\right)$$

$$= \frac{\sin(h\pi)}{h\pi} + \frac{2}{h^2\pi^2}\cos(h\pi) - \frac{2}{h^2\pi^3}\sin(h\pi) - j\frac{3}{8\pi(1-h)}\sin(\pi(1-h)) + j\frac{3}{8\pi(1+h)}\sin(\pi(1+h))$$

$$+ j\frac{1}{8\pi(3-h)}\sin(\pi(3-h)) - j\frac{1}{8\pi(3+h)}\sin(\pi(3+h))$$

$$\therefore a_h = \mathrm{Sinc}(h\pi) + \frac{2}{h^2\pi^2}\left(\cos(h\pi) - \mathrm{Sinc}(h\pi)\right) - j\frac{3}{8}\mathrm{Sinc}(\pi(1-h)) + j\frac{3}{8}\mathrm{Sinc}(\pi(1+h))$$

$$+ j\frac{1}{8}\mathrm{Sinc}(\pi(3-h)) - j\frac{1}{8}\mathrm{Sinc}(\pi(3+h))$$

Case $k=0$;

$$a_0 = \frac{1}{2}\int_{t=-1}^{+1} (t^2 + \sin^3(\pi t))\, \cancel{e^{0}}\, dt = \frac{1}{2}\frac{t^3}{3}\Big|_{-1}^{+1} + \frac{1}{2}\int_{-1}^{+1} \sin^3 \pi t\, dt = \frac{1}{3} + \frac{1}{2\pi}\int_{-1}^{+1} \sin^3(\pi t)\, d(\pi t)$$

$$= \frac{1}{3} + \frac{1}{2\pi}\int_{-1}^{+1} \frac{3}{4}\sin(\pi t) - \frac{1}{4}\sin(3\pi t)\, dt$$

$$= \frac{1}{3} - \frac{1}{\pi^2}\frac{3}{8}\cancel{\cos(\pi t)}\Big|_{-1}^{1} + \frac{1}{24\pi^2}\cancel{\cos(3\pi t)}\Big|_{-1}^{1}$$

$\therefore a_0 = \frac{1}{3}$

$\therefore$ Four. Series: $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\pi t}$

$$a_k = \begin{cases} 1/3 & ; k=0 \\ \mathrm{Sinc}(k\pi) + \frac{2}{k^2\pi^2}(\cos(k\pi) - \mathrm{Sinc}(k\pi)) & ; k \neq 0 \\ \quad - j\frac{3}{8}\mathrm{Sinc}(\pi(1-k)) + j\frac{3}{8}\mathrm{Sinc}(\pi(1+k)) & \\ \quad + j\frac{1}{8}\mathrm{Sinc}(\pi(3-k)) - j\frac{1}{8}\mathrm{Sinc}(\pi(3+k)) & \end{cases}$$ #

for k is integer

$$\therefore a_k = \begin{cases} 0 & ; k=0 \\ \frac{-2}{k^2\pi^2} & ; k \text{ is odd} \\ +\frac{2}{k^2\pi^2} & ; k \text{ is even} \end{cases}$$ #

Problem 4

Let $\mathcal{F}\{x(t)\} = X(j\omega) = \text{rect}[(\omega-1)/2]$ Find Fourier transform of

1.) $x(-2t+4)$

Time Scaling: $\mathcal{F}\{x(\alpha t)\} = \frac{1}{|\alpha|} X\left(\frac{j\omega}{\alpha}\right)$

$$\mathcal{F}\{x(-2t)\} = \frac{1}{2} X\left(j\frac{\omega}{-2}\right) = \frac{1}{2}\,\text{rect}\left[\frac{-\frac{\omega}{2}-1}{2}\right] = \frac{1}{2}\,\text{rect}\left[-\frac{\omega+2}{4}\right]$$

} rect is even f^n : $\text{rect}(x) = \text{rect}(-x)$

$$\therefore \mathcal{F}\{x(-2t)\} = \frac{1}{2}\,\text{rect}\left[\frac{\omega+2}{4}\right]$$

Time Shifting: $\mathcal{F}\{x(t-t_0)\} = X(j\omega)e^{-j\omega t_0}$

$$\therefore \mathcal{F}\{x(-2t+4)\} = \frac{1}{2}\,\text{rect}\left[\frac{\omega+2}{4}\right] e^{-j\omega(2)}$$ *

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2) $(t-1)\,x(t-1)$

Time Shifting: $\mathcal{F}\{x(t-1)\} = X(j\omega)\,e^{-j\omega(1)}$

$$= \text{rect}\left[\frac{\omega-1}{2}\right] e^{-j\omega}$$

D.A. in freq: $\frac{d}{d\omega} X(j\omega) = \frac{d}{d\omega}\int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$

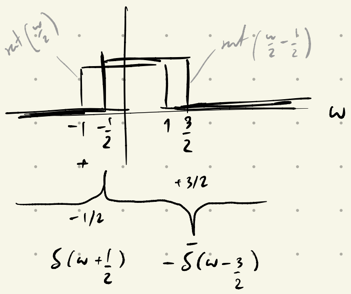
$$= \int x(t)\,(-jt)\,e^{-j\omega t}\,dt$$

$$= -j\int t\,x(t)\,e^{-j\omega t}\,dt$$

$$\therefore \frac{d}{d\omega}X(j\omega) = -j\,\mathcal{F}\{t\,x(t)\}\ ,\quad \mathcal{F}\{t\,x(t)\} = j\frac{d}{d\omega}X(j\omega) = j\frac{d}{d\omega}\mathcal{F}\{x(t)\}$$

$$\mathcal{F}\{(t-1)x(t-1)\} = j\frac{d}{d\omega}\mathcal{F}\{x(t-1)\}$$

$$= j\frac{d}{d\omega}\left(\text{rect}\left[\frac{\omega-1}{2}\right]e^{-j\omega}\right)$$



$$= j\left\{\text{rect}\left[\frac{\omega-1}{2}\right]e^{-j\omega}(-j) + e^{-j\omega}\frac{d}{d\omega}\text{rect}\left[\frac{\omega-1}{2}\right]\right\}$$

$$= \text{rect}\left[\frac{\omega-1}{2}\right]e^{-j\omega} + j\,e^{-j\omega}\left(\delta\left(\omega+\frac{1}{2}\right)-\delta\left(\omega-\frac{3}{2}\right)\right) \quad *$$

; δ is Dirac delta function

3.) $\mathcal{F}\{t\frac{d}{dt}x(t)\} = ?$

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int X(j\omega)\, e^{j\omega t}(j\omega)\, d\omega$$

$$\underbrace{\frac{d}{dt}x(t)}_{f(t)} = \frac{1}{2\pi}\int \underbrace{j\omega X(j\omega)}_{f(\omega)}\, e^{j\omega t}\, d\omega \quad ; \ \mathcal{F}(f(t)) = \frac{1}{2\pi}\int f(\omega)\, e^{j\omega t}\, d\omega$$

$$\therefore \mathcal{F}\left\{\frac{d}{dt}x(t)\right\} = j\omega X(j\omega)$$

$$\mathcal{F}\{t\frac{d}{dt}x(t)\} = j\frac{d}{d\omega}\mathcal{F}\left\{\frac{d}{dt}x(t)\right\}$$

$$= j\frac{d}{d\omega}\left(j\omega X(j\omega)\right)$$

$$= j^2\left(\omega\frac{d}{d\omega}X(j\omega) + X(j\omega)\right)$$

$$= -\omega\frac{d}{d\omega}\text{rect}\left[\frac{\omega-1}{2}\right] - \text{rect}\left[\frac{\omega-1}{2}\right]$$

$$= -\omega\left(\delta\left(\omega+\frac{1}{2}\right)-\delta\left(\omega-\frac{3}{2}\right)\right) - \text{rect}\left[\frac{\omega-1}{2}\right] \quad *$$

; δ is Dirac delta function

4.) $\mathcal{F}\{x(2t-1)\,e^{-j2t}\} = ?$

Time Scaling : $\mathcal{F}\{x(2t)\} = \frac{1}{2} X(j\frac{\omega}{2}) = \frac{1}{2}\,\mathrm{rect}\left[\frac{\omega-2}{4}\right]$

Time Shifting : $\mathcal{F}\{x(\underbrace{2t-1}_{\text{ทำให้เหมือน } \frac{1}{2} \text{ ใน } t})\} = \frac{1}{2}\,\mathrm{rect}\left[\frac{\omega-2}{4}\right] e^{-j\omega(\frac{1}{2})}$

Frequency Shifting : $\mathcal{F}\{x(t)\,e^{j\omega_0 t}\} = X(j(\omega-\omega_0))$

$$\mathcal{F}\{x(2t-1)\,e^{j(-2)t}\} = X(j(\omega-(-2)))$$
$$= \frac{1}{2}\,\mathrm{rect}\left[\frac{\omega+2-2}{4}\right] e^{-j(\omega+2)\cdot\frac{1}{2}}$$
$$= \frac{1}{2}\,\mathrm{rect}\left[\frac{\omega}{4}\right] e^{-j(\frac{\omega+2}{2})} \quad \#$$

<u>Sol - II</u>

$$\mathcal{F}\{x(2t-1)\,e^{-j2t}\} = \int_{t=-\infty}^{+\infty} x(2t-1)\,e^{-j2t}\cdot e^{-j\omega t}\,dt$$
$$= \int_{t=-\infty}^{t=+\infty} x(2t-1)\,e^{-j(\omega+2)t}\,dt$$

$\}$ $t' \equiv 2t-1$, $t = \frac{t'+1}{2}$, $dt = \frac{1}{2}dt'$

$$= \int_{t=-\infty}^{+\infty} x(t')\,e^{-j(\omega+2)(\frac{t'+1}{2})}\,\frac{1}{2}dt'$$
$$= \frac{1}{2}\int_{t'=-\infty}^{+\infty} x(t')\,e^{-j(\omega+2)\frac{t'}{2}}\,e^{-j(\omega+2)\frac{1}{2}}\,dt'$$
$$= \frac{1}{2}\,e^{-j(\frac{\omega+2}{2})}\int x(t')\,e^{-j(\frac{\omega+2}{2})t'}\,dt'$$
$$= \frac{1}{2}\,e^{-j(\frac{\omega+2}{2})}\,X(j(\frac{\omega+2}{2}))$$
$$= \frac{1}{2}\,e^{-j(\frac{\omega+2}{2})}\,\mathrm{rect}\left[\frac{(\frac{\omega+2}{2})-1}{2}\right]$$

$$\therefore\ \mathcal{F}\{x(2t-1)\,e^{-j2t}\} = \frac{1}{2}\,e^{-j(\frac{\omega}{2}+1)}\,\mathrm{rect}\left[\frac{\omega}{4}\right] \quad \#$$

5.) $\mathcal{F}\{x(t) * x(t-1)\} = ?$

$\mathcal{F}\{x(t)\} = X(j\omega)$

$\mathcal{F}\{x(t-1)\} = X(j\omega)\, e^{-j\omega(1)}$

Convolution: $\mathcal{F}\{x(t) * x(t-1)\} = X(j\omega) \cdot X(j\omega)\, e^{-j\omega}$

$= \mathrm{rect}^2\left[\frac{\omega-1}{2}\right] e^{-j\omega}$ #