Problem 2. Find the Fourier Series (FS) of the periodic function $x(t)$.

2.1) $x(t) = \frac{\pi}{2} t^3; \quad -1 < t < 1$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}; \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \quad \Big| \; T=2$$

$$; \; a_k = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$a_k = \frac{1}{2}\int_{t=-1}^{+1} \frac{\pi}{2} t^3 e^{-jk\omega_0 t}\, dt$$

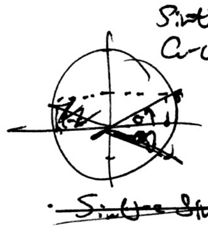
$$= \frac{\pi}{4}\int_{t=-1}^{1} t^3 e^{-jk\omega_0 t}\, dt$$

} $e^{j\theta} = \cos\theta + j\sin\theta$; Euler Formula, $e^{-j\theta} = \cos\theta - j\sin\theta$

$$= \frac{\pi}{4}\int t^3\left(\cos(k\omega_0 t) - j\sin(k\omega_0 t)\right) dt$$

$$= \frac{\pi}{4}\left\{\int_{t=-1}^{1} t^3\cos(k\omega_0 t)\, dt - j\int_{t=-1}^{1} t^3\sin(k\omega_0 t)\, dt\right\}$$

$\sin(-\theta) = -\sin\theta$.
$\cos(-\theta) = \cos\theta$.



$\cancel{\sin\theta = \sin(-\theta)}$
$\cancel{\cos\theta = -\cos(-\theta)}$
$\cancel{\sin\theta - \sin(-\theta) = 0}$
$\cancel{\cos\theta - \cos(-\theta) = 2\cos\theta}$

} Integration By parts: $\int u\,dv = uv - \int v\,du$

• $\int_{t=-1}^{1} t^3\cos(k\omega_0 t)\, dt$

$u = t^3$, $dv = \cos(k\omega_0 t)\, dt$

$du = 3t^2\, dt$, $v = \int\cos(k\omega_0 t)\, dt \times \frac{k\omega_0}{k\omega_0} = \frac{1}{k\omega_0}\sin(k\omega_0 t)$

$$\int_{t=-1}^{1} t^3\cos(k\omega_0 t)\, dt = t^3 \cdot \frac{\sin(k\omega_0 t)}{k\omega_0}\Big|_{t=-1}^{+1} - \int_{t=-1}^{+1} \frac{\sin(k\omega_0 t)}{k\omega_0} \times 3t^2\, dt$$

$$\left[\int t^n\cos(k\omega_0 t)\, dt = t^n\frac{\sin(k\omega_0 t)}{k\omega_0} - \int\frac{\sin(k\omega_0 t)}{k\omega_0}\, n t^{n-1}\, dt\right]$$

$$\left[\int t^n\sin(k\omega_0 t)\, dt = t^n\frac{(-\cos(k\omega_0 t))}{k\omega_0} + \int\frac{\cos(k\omega_0 t)}{k\omega_0}\, n t^{n-1}\, dt\right]$$

$$\int\frac{\sin(k\omega_0 t)}{k\omega_0}\, 3t^2\, dt = \frac{3}{k\omega_0}\left(\frac{-t^2\cos(k\omega_0 t)}{k\omega_0} + \frac{2}{k\omega_0}\int\cos(k\omega_0 t)\, t\, dt\right)$$

$$\int\cos(k\omega_0 t)\, t\, dt = t\,\frac{\sin(k\omega_0 t)}{k\omega_0} - \frac{1}{k\omega_0}\int\sin(k\omega_0 t)\, dt \times \frac{k\omega_0}{k\omega_0}$$

$$= \frac{t\sin(k\omega_0 t)}{k\omega_0} + \frac{1}{(k\omega_0)^2}\cos(k\omega_0 t)$$

$$\int\frac{\sin(k\omega_0 t)}{k\omega_0}\, 3t^2\, dt = \frac{-3t^2\cos(k\omega_0 t)}{(k\omega_0)^2} + \frac{6}{(k\omega_0)^2}\left(\frac{t\sin(k\omega_0 t)}{k\omega_0} + \frac{\cos(k\omega_0 t)}{(k\omega_0)^2}\right)$$

$$\int \frac{\sin(h\omega_0 t)}{h\omega_0} 3t^2 dt = -\frac{3t^2}{(h\omega_0)^2}\cos(h\omega_0 t) + \frac{6t \sin(h\omega_0 t)}{(h\omega_0)^3} + \frac{6\cos(h\omega_0 t)}{(h\omega_0)^4}$$

$$\therefore \int_t t^3 \cos(h\omega_0 t)\, dt = \frac{t^3}{h\omega_0}\sin(h\omega_0 t) + \frac{3t^2}{(h\omega_0)^2}\cos(h\omega_0 t) - \frac{6t}{(h\omega_0)^3}\sin(h\omega_0 t) - \frac{6}{(h\omega_0)^4}\cos(h\omega_0 t)$$

$\cos(\theta + \frac{\pi}{2}) = -\sin\theta$

$\cancel{\cos\theta = \sin(\theta + \frac{\pi}{2})}$ [illegible] Integral [illegible] Bypass [illegible] $dt = d(t + \text{const.})$ $\cancel{\sin}$

$$\therefore \int t^3 \sin(h\omega_0 t)\, dt = -\frac{t^3}{h\omega_0}\cos(h\omega_0 t) + \frac{3t^2}{(h\omega_0)^2}\sin(h\omega_0 t) + \frac{6t}{(h\omega_0)^3}\cos(h\omega_0 t) - \frac{6}{(h\omega_0)^4}\sin(h\omega_0 t)$$

$$\therefore \int_{t=-1}^{+1} t^3 \cos(h\omega_0 t)\, dt = \cancel{\frac{2}{h\omega_0}\sin(h\omega_0)}^{0} + \cancel{\frac{6}{(h\omega_0)^2}\cos(h\omega_0)} - \cancel{\frac{6\cos(h\omega_0)}{(h\omega_0)^4}} = \cancel{\frac{2\sin(h\omega_0)}{h\omega_0}} - \cancel{\frac{12\sin(h\omega_0 t)}{(h\omega_0)^3}} = 0.$$

$$\cancel{\int_{t=-1}^{+1} t^3 \sin(h\omega_0 t)\, dt = -\frac{2}{h\omega_0}\cos(h\omega_0 t) + \frac{12\cos(h\omega_0 t)}{(h\omega_0)^3} = \frac{6t^2 \sin(h\omega_0)}{(h\omega_0)^2} - \frac{12}{(h\omega_0)^4}\sin(h\omega_0)}$$

$$\cancel{\therefore a_h = \frac{\pi}{4}\left\{ \int_{t=-1}^{+1} t^3 \cos(h\omega_0 t)\, dt - j\int_{t=-1}^{+1} t^3 \sin(h\omega_0 t)\, dt \right\}}$$

$$\cancel{= \frac{\pi}{4}\left\{ \frac{6}{(h\omega_0)^2}\cos(h\omega_0) - \frac{6\cos(h\omega_0)}{(h\omega_0)^4} + j\frac{2}{h\omega_0}\cos(h\omega_0) - j\frac{12\cos(h\omega_0)}{(h\omega_0)^3} \right\}}$$

$$\cancel{\therefore a_h = \frac{3\pi}{2}\frac{\cos(h\omega_0)}{(h\omega_0)^2}\left(1 - \frac{1}{(h\omega_0)^2}\right) + j\frac{\pi}{2h\omega_0}\cos(h\omega_0)\left(1 - \frac{6}{(h\omega_0)^2}\right)}$$

$$\cancel{= \frac{\pi}{4}\left\{ 6\cos(h\omega_0)(h\omega_0)^2 - 6\cos(h\omega_0) + j2\cos(h\omega_0)(h\omega_0)^3 - j12\cos(h\omega_0)(h\omega_0) \Big/ (h\omega_0)^4 \right\}}$$

$$\cancel{= -3\cos(h\omega_0)\pi - j6\pi h\omega_0 \cos(h\omega_0) + 3\pi(h\omega_0)^2\cos(h\omega_0) + j\pi(h\omega_0)^3\cos(h\omega_0) \Big/ 2(h\omega_0)^4}$$

$$\therefore \int_{t=-1}^{+1} t^3 \sin(h\omega_0 t)\, dt = -\frac{2}{h\omega_0}\cos(h\omega_0) + \frac{6}{(h\omega_0)^2}\sin(h\omega_0) + \frac{12}{(h\omega_0)^3}\cos(h\omega_0) - \frac{12}{(h\omega_0)^4}\sin(h\omega_0)$$

$$\therefore a_h = \frac{\pi}{4}\left\{ 0 - j(\ldots) \right\} = j\left\{ \frac{\pi}{2}\frac{\cos(h\omega_0)}{h\omega_0} - \frac{3\pi}{2}\frac{\sin(h\omega_0)}{(h\omega_0)^2} - \frac{3\pi\cos(h\omega_0)}{(h\omega_0)^3} + \frac{3\pi\sin(h\omega_0)}{(h\omega_0)^4} \right\}$$

$$= j\frac{1}{2(h\omega_0)^4}\left\{ \pi\cos(h\omega_0)h^3\omega_0^3 - 3\pi\sin(h\omega_0)h^2\omega_0^2 - 6\pi\cos(h\omega_0)h\omega_0 + 6\pi\sin(h\omega_0) \right\}$$

$$\therefore a_h = \frac{j}{2\pi^4 h^4}\left\{ \pi^4 h^3 \cos(h\omega_0) - 3\pi^3 h^2 \sin(h\omega_0) - 6\pi^2 h\cos(h\omega_0) + 6\pi\sin(h\omega_0) \right\}$$

Case: $h = 0$: $a_{h=0} = \frac{\pi}{4}\int t^3 e^0\, dt = \frac{\pi}{4}\frac{t^4}{4}\Big|_{-1}^{1} = 0$

Answer

$$\therefore x(t) = \sum_{h=-\infty}^{\infty} a_h e^{jh\pi t}\ ;\ a_h = \begin{cases} 0 & ;\ h = 0 \\ j\left(\dfrac{\pi^4 h^3 \cos(h\omega_0) - 3\pi^3 h^2 \sin(h\omega_0) - 6\pi^2 h\cos(h\omega_0) + 6\pi\sin(h\omega_0)}{2\pi^4 h^4}\right) & ;\ h \neq 0 \end{cases}$$

2.2)

$x(t) = \pi - t \;;\quad -\pi \le t \le \pi$

Fourier Series: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \;;\quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{(2\pi)} = 1 \;,\quad T = 2\pi$

$$a_k = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$a_k = \frac{1}{2\pi}\int_{t=-\pi}^{\pi} (\pi - t)\, e^{-jk\omega_0 t}\, dt$$

Case: $k = 0$

$$a_{k=0} = \frac{1}{2\pi}\int_{t=-\pi}^{\pi} (\pi - t)\, \cancel{e^{0}}^{\,1}\, dt = \frac{-1}{2\pi}\int_{t=-\pi}^{+\pi} (\pi - t)\, d(\pi - t) = \frac{-1}{2\pi}\, \frac{(\pi - t)^2}{2}\Bigg|_{t=-\pi}^{+\pi}$$

$$= \frac{-1}{2\pi}\left(0 - \frac{4\pi^2}{2}\right) = \pi \qquad \therefore\; a_0 = \pi$$

Case: $k \neq 0$

$$a_k = \cancel{\frac{1}{2\pi}\left(\int_{t=-\pi}^{\pi} \pi e^{-jkt}\, dt - \int_{t=-\pi} t\, e^{-jkt}\, dt\right)} = \frac{1}{2\pi}\int_{t=-\pi}^{\pi} (\pi - t)\big(\cos(kt) - j\sin(kt)\big)\, dt$$

$$= \frac{1}{2\pi}\left\{\int_t \pi\cos(kt)\, dt - j\pi\int \sin(kt)\, dt - \int t\cos(kt)\, dt + j\int t\sin(kt)\, dt\right\}$$

Bypass

$$= \frac{1}{2\pi}\left\{\frac{\pi}{k}\sin(kt) + j\frac{\pi}{k}\cos(kt) - \frac{t\sin(kt)}{k} + \frac{1}{k}\frac{(-\cos(kt))}{k} + j\frac{t(-\cos(kt))}{k} + j\frac{1}{k}\frac{\sin(kt)}{k}\right\}\Bigg|_{t=-\pi}^{t=+\pi}$$

$$= \frac{1}{2\pi}\left\{\frac{2\pi}{k}\sin(k\pi) - j\frac{2\pi}{k}\cos(k\pi) + j\frac{2}{k^2}\sin(k\pi)\right\}$$

$$= \frac{\sin(k\pi)}{k} - j\frac{\cos(k\pi)}{k} + j\frac{\sin(k\pi)}{k^2\pi}$$

$\therefore$ Fourier Series $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkt} \;;\quad a_k = \begin{cases} \pi & ; k = 0 \\ \dfrac{\sin(k\pi)}{k} + j\left(\dfrac{\sin(k\pi)}{k^2\pi} - \dfrac{\cos(k\pi)}{k}\right) & ; k \neq 0 \end{cases}$