

## Problem 1.1 - DFT

Given the DFT spectrum  $X[k]$ , express the corresponding time-domain signal  $x[n]$  in term of its constituent real sinusoids.

$$\textcircled{1} \quad X[0] = 3, \quad X[1] = \frac{1}{\sqrt{2}} - j\left(\frac{1}{\sqrt{2}}\right), \quad X[2] = -2, \quad X[3] = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\left(\frac{2\pi}{N}\right)nk}$$



$$\begin{aligned} x[n] &= \frac{1}{4} \sum_{k=0}^3 X[k] e^{j\frac{\pi}{2}nk} \\ &= \frac{1}{4} \left\{ (3) e^{j\frac{\pi}{2}n \cdot 0} + \left( \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \right) e^{j\frac{\pi}{2}n \cdot 1} + (-2) e^{j\frac{\pi}{2}n \cdot 2} + \left( \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \right) e^{j\frac{\pi}{2}n \cdot 3} \right\} \\ &\stackrel{\text{ Euler formula}}{=} \frac{1}{4} \left\{ 3 + e^{j\left(\frac{\pi}{4}\right)} e^{j\frac{\pi}{2}n} - 2 e^{j\pi n} + e^{j\left(\frac{7}{4}\right)} e^{j\frac{3\pi}{2}n} \right\} \end{aligned}$$

$$x[n] = \frac{1}{4} \left\{ 3 + e^{j\left(\frac{\pi}{2}n - \frac{\pi}{4}\right)} - 2 e^{j\pi n} + e^{j\left(\frac{3\pi}{2}n + \frac{\pi}{4}\right)} \right\}$$

$$\therefore \Re(x[n]) = \frac{3}{4} + \frac{1}{4} \cos\left(\frac{\pi}{2}n - \frac{\pi}{4}\right) - \frac{1}{2} \cos(\pi n) + \frac{1}{4} \cos\left(\frac{3\pi}{2}n + \frac{\pi}{4}\right) \quad \text{**}$$

$$\textcircled{2} \quad X[0] = -2, \quad X[1] = 5_3 + j, \quad X[2] = 3, \quad X[3] = 5_3 - j$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\left(\frac{2\pi}{N}\right)nk}$$

$$\begin{aligned} x[n] &= \frac{1}{4} \sum_{k=0}^3 X[k] e^{j\frac{\pi}{2}nk} \\ &= \frac{1}{4} \left\{ (-2) e^{j\frac{\pi}{2}n \cdot 0} + \left( 5_3 + j \right) e^{j\frac{\pi}{2}n \cdot 1} + (3) e^{j\frac{\pi}{2}n \cdot 2} + \left( 5_3 - j \right) e^{j\frac{\pi}{2}n \cdot 3} \right\} \end{aligned}$$

$$\stackrel{\text{Euler formula}}{=} \frac{1}{4} \left\{ -2 + 2 \left( \frac{5_3 + j}{2} \right) e^{j\frac{\pi}{2}n} + 3 e^{j\pi n} + 2 \left( \frac{5_3 - j}{2} \right) e^{j\frac{3\pi}{2}n} \right\}$$

$$x[n] = \frac{1}{4} \left\{ -2 + 2e^{j\left(\frac{\pi}{2}n + \frac{\pi}{6}\right)} + 3e^{j\pi n} + 2e^{j\left(\frac{3\pi}{2}n - \frac{\pi}{6}\right)} \right\}$$

$$\therefore \Re(x[n]) = -\frac{1}{2} + \frac{1}{2} \cos\left(\frac{\pi}{2}n + \frac{\pi}{6}\right) + \frac{3}{4} \cos(\pi n) + \frac{1}{2} \cos\left(\frac{3\pi}{2}n - \frac{\pi}{6}\right) \quad \text{**}$$



$$③ X[0] = 1, \quad X[1] = 2 - j\sqrt{3}, \quad X[2] = -3, \quad X[3] = 2 + j\sqrt{3}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(\frac{2\pi}{N})nk}$$

$$x[n] = \frac{1}{4} \sum_{k=0}^3 X[k] e^{j\frac{\pi}{2}nk}$$

$$= \frac{1}{4} \left\{ (1) e^{j\frac{\pi}{2}n \cdot 0} + (2 - j\sqrt{3}) e^{j\frac{\pi}{2}n \cdot 1} + (-3) e^{j\frac{\pi}{2}n \cdot 2} + (2 + j\sqrt{3}) e^{j\frac{\pi}{2}n \cdot 3} \right\}$$

$$= \frac{1}{4} \left\{ 1 + 4 \left( \frac{1}{2} - j\frac{\sqrt{3}}{2} \right) e^{j\frac{\pi}{2}n} - 3 e^{j\pi n} + 4 \left( \frac{1}{2} + j\frac{\sqrt{3}}{2} \right) e^{j\frac{3\pi}{2}n} \right\}$$

$$x[n] = \frac{1}{4} \left\{ 1 + 4 e^{j(\frac{\pi}{2}n - \frac{\pi}{3})} - 3 e^{j\pi n} + 4 e^{j(\frac{3\pi}{2}n + \frac{\pi}{3})} \right\}$$

$$\therefore \operatorname{Re}(x[n]) = \frac{1}{4} + \cos\left(\frac{\pi}{2}n - \frac{\pi}{3}\right) - \frac{3}{4} \cos(\pi n) + \cos\left(\frac{3\pi}{2}n + \frac{\pi}{3}\right) *$$

## Problem 2.1 - DFT

For the problem & the discrete-time domain transform to determine  $X(e^{j\omega})$  for the following signals.

$$\textcircled{1} \quad x[n] = \left(\frac{1}{3}\right)^{|n|}$$

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{+\infty} \left(\frac{1}{3}\right)^{|n|} e^{-j\omega n} \\ &= \sum_{n=-\infty}^{-1} \left(\frac{1}{3}\right)^{-n} e^{-j\omega n} + \sum_{n=0}^{+\infty} \left(\frac{1}{3}\right)^n e^{-j\omega n} \\ &\Rightarrow \sum_{n=-\infty}^{-1} \left(\frac{e^{j\omega}}{3}\right)^{-n} + \sum_{n=0}^{\infty} \left(\frac{e^{-j\omega}}{3}\right)^n \\ &\quad \{ m \equiv -n \} \\ &= \sum_{m=1}^{\infty} \left(\frac{e^{j\omega}}{3}\right)^m + \sum_{n=0}^{\infty} \left(\frac{e^{-j\omega}}{3}\right)^n \end{aligned}$$

$$\} \text{ Geometric Series: } \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}, |r| < 1$$

$$\sum_{k=1}^{\infty} ar^k = \frac{ar}{1-r}$$

$$2 \quad \frac{\left(\frac{e^{j\omega}}{3}\right)}{1 - \left(\frac{e^{j\omega}}{3}\right)} + \frac{1}{1 - \left(\frac{e^{-j\omega}}{3}\right)}$$

$$= \frac{1}{3e^{j\omega} - 1} + \frac{3}{3 - e^{-j\omega}}$$

$$= \frac{8e^{-j\omega}}{10e^{-j\omega} - 3e^{j\omega} - 3}$$

$$= \frac{8}{10 - 3e^{-j\omega} - 3e^{j\omega}}$$

$$= \frac{8}{10 - 6\left(\frac{e^{-j\omega} + e^{j\omega}}{2}\right)}$$

$$\therefore X(e^{j\omega}) = \frac{4}{5 - 3 \cos \omega} *$$

$$\textcircled{2} \quad x[n] = a^n c_n(\Omega_0 n) \cdot u[n] \quad ; \quad |a| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n c_n(\Omega_0 n) \cdot u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n c_n(\Omega_0 n) e^{-j\omega n}$$

? Geometrische Reihe:  $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} ; |r| < 1$

$$\left\{ \cos \theta = \frac{e^{j\theta} + \bar{e}^{j\theta}}{2} \right.$$

$$\left. \sum_{k=1}^{\infty} ar^k = \frac{ar}{1-r} \right.$$

$$= \sum_{n=0}^{\infty} a^n \left( \frac{e^{j\omega n} + e^{-j\omega n}}{2} \right) e^{-j\omega n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (ae^{-j(\omega - \Omega_0)})^n + \frac{1}{2} \sum_{n=0}^{\infty} (ae^{-j(\omega + \Omega_0)})^n$$

$$= \frac{1}{2} \frac{1}{1 - ae^{-j(\omega - \Omega_0)}} + \frac{1}{2} \frac{1}{1 - ae^{-j(\omega + \Omega_0)}}$$

$$= \frac{1}{2} \left( \frac{2 - a(e^{-j(\omega + \Omega_0)} + e^{-j(\omega - \Omega_0)})}{1 - ae^{-j(\omega + \Omega_0)} - ae^{-j(\omega - \Omega_0)} + ae^{-j(\omega - \Omega_0)} e^{-j(\omega + \Omega_0)}} \right)$$

$$\therefore X(e^{j\omega}) = \frac{1 - ae^{-j\omega} (c_0 \Omega_0)}{1 - ae^{-j\omega} (2c_0 \Omega_0 - a \cdot e^{-j\omega})} *$$

$$(3) \quad x[n] = (n+1)a^n \cdot n[n], \quad |a| < 1$$

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} (n+1)a^n n[n] e^{-j\omega n} \\ &= \underbrace{\sum_{n=0}^{\infty} n(ae^{-j\omega})^n}_{S} + \sum_{n=0}^{\infty} (ae^{-j\omega})^n \end{aligned}$$

$$\{ S = \sum_{n=0}^{\infty} n(ae^{-j\omega})^n \}$$

$$S = 1ae^{-j\omega(1)} + 2a^2 e^{-j\omega(2)} + 3a^3 e^{-j\omega(3)} + \dots$$

$$(ae^{-j\omega})S = 1a^2 e^{-j\omega(2)} + 2a^3 e^{-j\omega(3)} + \dots$$

$$(1 - ae^{-j\omega})S = 1ae^{-j\omega(1)} + 1a^2 e^{-j\omega(2)} + 1a^3 e^{-j\omega(3)} + \dots$$

$$= \sum_{n=1}^{\infty} (ae^{-j\omega})^n \quad \{ \text{Geometric Series: } \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}, \quad |r| < 1 \}$$

$$(1 - ae^{-j\omega})S = \frac{ae^{-j\omega}}{1 - ae^{-j\omega}}$$

$$\sum_{k=1}^{\infty} ar^k = \frac{ar}{1-r}$$

$$S = \frac{ae^{-j\omega}}{(1 - ae^{-j\omega})^2}$$

$$\therefore X(e^{j\omega}) = \frac{ae^{-j\omega}}{(1 - ae^{-j\omega})^2} + \left( \frac{1}{1 - ae^{-j\omega}} \right)$$

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