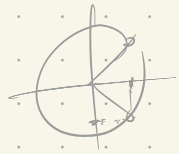
Problem 1.1 – DFT

Given the DFT spectrum $X[k]$, express the corresponding time-domain signal $x[n]$ in terms of its constituent real sinusoids.

① $X[0] = 3$, $X[1] = \frac{1}{\sqrt{2}} - j\left(\frac{1}{\sqrt{2}}\right)$, $X[2] = -2$, $X[3] = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\left(\frac{2\pi}{N}\right)nk}$$



$$x[n] = \frac{1}{4}\sum_{k=0}^{3} X[k]\, e^{j\frac{\pi}{2}nk}$$

$$= \frac{1}{4}\left\{ (3)\cancel{e^{j\frac{\pi}{2}n\cdot 0}} + \left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right) e^{j\frac{\pi}{2}n\cdot 1} + (-2)\, e^{j\frac{\pi}{2}n\cdot 2} + \left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) e^{j\frac{\pi}{2}n\cdot 3} \right\}$$

Euler formula

$$= \frac{1}{4}\left\{ 3 + e^{j\left(-\frac{\pi}{4}\right)} \cdot e^{j\frac{\pi}{2}n} - 2e^{j\pi n} + e^{j\left(\frac{\pi}{4}\right)} e^{j\frac{3\pi}{2}n} \right\}$$

$$x[n] = \frac{1}{4}\left\{ 3 + e^{j\left(\frac{\pi}{2}n - \frac{\pi}{4}\right)} - 2e^{j(\pi n)} + e^{j\left(\frac{3\pi}{2}n + \frac{\pi}{4}\right)} \right\}$$

$$\therefore \mathrm{Re}(x[n]) = \frac{3}{4} + \frac{1}{4}\cos\left(\frac{\pi}{2}n - \frac{\pi}{4}\right) - \frac{1}{2}\cos(\pi n) + \frac{1}{2}\cos\left(\frac{3\pi}{2}n + \frac{\pi}{4}\right) \quad \#$$

② $X[0] = -2$, $X[1] = \sqrt{3} + j$, $X[2] = 3$, $X[3] = \sqrt{3} - j$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\left(\frac{2\pi}{N}\right)nk}$$

$$x[n] = \frac{1}{4}\sum_{k=0}^{3} X[k]\, e^{j\frac{\pi}{2}nk}$$

$$= \frac{1}{4}\left\{ (-2)\cancel{e^{j\frac{\pi}{2}n\cdot 0}} + \left(\sqrt{3} + j\right) e^{j\frac{\pi}{2}n\cdot 1} + (3)\, e^{j\frac{\pi}{2}n\cdot 2} + \left(\sqrt{3} - j\right) e^{j\frac{\pi}{2}n\cdot 3} \right\}$$

$$= \frac{1}{4}\left\{ -2 + 2\left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right) e^{j\frac{\pi}{2}n} + 3e^{j\pi n} + 2\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) e^{j\frac{3\pi}{2}n} \right\}$$

$$x[n] = \frac{1}{4}\left\{ -2 + 2e^{j\left(\frac{\pi}{2}n + \frac{\pi}{6}\right)} + 3e^{j\pi n} + 2e^{j\left(\frac{3\pi}{2}n - \frac{\pi}{6}\right)} \right\}$$

$$\therefore \mathrm{Re}(x[n]) = \frac{-1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2}n + \frac{\pi}{6}\right) + \frac{3}{4}\cos(\pi n) + \frac{1}{2}\cos\left(\frac{3\pi}{2}n - \frac{\pi}{6}\right) \quad \#$$

③ $X[0] = 1$, $X[1] = 2 - j2\sqrt{3}$, $X[2] = -3$, $X[3] = 2 + j2\sqrt{3}$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\left(\frac{2\pi}{N}\right)nk}$$

$$x[n] = \frac{1}{4} \sum_{k=0}^{3} X[k]\, e^{j\frac{\pi}{2}nk}$$

$$= \frac{1}{4}\left\{ (1)\cancel{e^{j\frac{\pi}{2}n\cdot 0}} + \left(2 - j2\sqrt{3}\right) e^{j\frac{\pi}{2}n\cdot 1} + (-3)\, e^{j\frac{\pi}{2}n\cdot 2} + \left(2 + j2\sqrt{3}\right) e^{j\frac{\pi}{2}n\cdot 3} \right\}$$

$$= \frac{1}{4}\left\{ 1 + 4\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) e^{j\frac{\pi}{2}n} - 3e^{j\pi n} + 4\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) e^{j\frac{3\pi}{2}n} \right\}$$

$$x[n] = \frac{1}{4}\left\{ 1 + 4\, e^{j\left(\frac{\pi}{2}n - \frac{\pi}{3}\right)} - 3e^{j\pi n} + 4\, e^{j\left(\frac{3\pi}{2}n + \frac{\pi}{3}\right)} \right\}$$

$$\therefore \mathrm{Re}(x[n]) = \frac{1}{4} + \cos\left(\frac{\pi}{2}n - \frac{\pi}{3}\right) - \frac{3}{4}\cos(\pi n) + \cos\left(\frac{3\pi}{2}n + \frac{\pi}{3}\right)$$ ✱

Problem 2.1 - DTFT

Use the properties of the discrete-time transform to determine $X(e^{j\omega})$ for the following sequences.

(1) $x[n] = \left(\frac{1}{3}\right)^{|n|}$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} \left(\frac{1}{3}\right)^{|n|} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} \left(\frac{1}{3}\right)^{-n} e^{-j\omega n} + \sum_{n=0}^{+\infty} \left(\frac{1}{3}\right)^{n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} \left(\frac{e^{j\omega}}{3}\right)^{-n} + \sum_{n=0}^{\infty} \left(\frac{e^{-j\omega}}{3}\right)^{n}$$

$\}$ $m \equiv -n$

$$= \sum_{m=1}^{\infty} \left(\frac{e^{j\omega}}{3}\right)^{m} + \sum_{n=0}^{\infty} \left(\frac{e^{-j\omega}}{3}\right)^{n}$$

$\}$ Geometric Series: $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$; $|r| < 1$

$$\sum_{k=1}^{\infty} ar^k = \frac{ar}{1-r}$$

$$= \frac{\left(\frac{e^{j\omega}}{3}\right)}{1 - \left(\frac{e^{j\omega}}{3}\right)} + \frac{1}{1 - \left(\frac{e^{-j\omega}}{3}\right)}$$

$$= \frac{1}{3e^{-j\omega} - 1} + \frac{3}{3 - e^{-j\omega}}$$

$$= \frac{8e^{-j\omega}}{10e^{-j\omega} - 3e^{-2j\omega} - 3}$$

$$= \frac{8}{10 - 3e^{-j\omega} - 3e^{+j\omega}}$$

$$= \frac{8}{10 - 6\left(\frac{e^{-j\omega} + e^{+j\omega}}{2}\right)}$$

$$\therefore X(e^{j\omega}) = \frac{4}{5 - 3\cos\omega} \quad \#$$

(2) $x[n] = a^n \cos(\Omega_0 n) \cdot u[n]$; $|a| < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n \cos(\Omega_0 n) \cdot u[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n \cos(\Omega_0 n)\, e^{-j\omega n}$$

} Geometric Series : $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$; $|r| < 1$

$\sum_{k=1}^{\infty} ar^k = \frac{ar}{1-r}$

} $\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$

$$= \sum_{n=0}^{\infty} a^n \left(\frac{e^{j\Omega_0 n} + e^{-j\Omega_0 n}}{2} \right) e^{-j\omega n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(a e^{-j(\omega - \Omega_0)} \right)^n + \frac{1}{2} \sum_{n=0}^{\infty} \left(a e^{-j(\omega + \Omega_0)} \right)^n$$

$$= \frac{1}{2} \frac{1}{1 - a e^{-j(\omega - \Omega_0)}} + \frac{1}{2} \frac{1}{1 - a e^{-j(\omega + \Omega_0)}}$$

$$= \frac{1}{2} \left(\frac{2 - a\left(e^{-j(\omega + \Omega_0)} + e^{-j(\omega - \Omega_0)} \right)}{1 - a e^{-j(\omega + \Omega_0)} - a e^{-j(\omega - \Omega_0)} + a^2 e^{-j(\omega - \Omega_0)} e^{-j(\omega + \Omega_0)}} \right)$$

$$\therefore X(e^{j\omega}) = \frac{1 - a e^{-j\omega} (\cos \Omega_0)}{1 - a e^{-j\omega} (2\cos \Omega_0 - a e^{-j\omega})}$$ #

(3) $x[n] = (n+1)a^n \cdot u[n] \quad ; \quad |a| < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} (n+1) a^n u[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} n (a e^{-j\omega})^n + \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

; $\quad S \equiv \sum_{n=0}^{\infty} n (a e^{-j\omega})^n$

$$S = 1 a e^{-j\omega(1)} + 2a^2 e^{-j\omega(2)} + 3a^3 e^{-j\omega(3)} + \dots$$

$$(a e^{-j\omega}) S = 1 a^2 e^{-j\omega(2)} + 2a^3 e^{-j\omega(3)} + \dots$$

$$(1 - a e^{-j\omega}) S = 1 a e^{-j\omega(1)} + 1 a^2 e^{-j\omega(2)} + 1 a^3 e^{-j\omega(3)} + \dots$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$$(1 - a e^{-j\omega}) S = \frac{1}{1 - a e^{-j\omega}}$$

$$S = \frac{1}{(1 - a e^{-j\omega})^2}$$

$$\therefore X(e^{j\omega}) = \left(\frac{1}{1 - a e^{-j\omega}}\right)^2 + \left(\frac{1}{1 - a e^{-j\omega}}\right) \quad \#$$