T1.

Likelihood ($L(\alpha)$)

find: MLE: $\underset{\alpha}{\arg\max}\ \overbrace{p(Y \mid \alpha)}^{\text{Likelihood } (L(\alpha))}$

$= \underset{\alpha}{\arg\max}\ p(y_2, y_1, y_0 \mid \alpha)$

$\cap$ intersect

probability density fn ... continuous random value.

} Conditional Probabilities

$p(\alpha \cap \beta) = p(\alpha)\, p(\beta \mid \alpha)$ in continuous RV., in discrete RV. $P(X\in C \mid Y \in D) = \frac{P(X \in C, Y \in D)}{P(Y \in D)}$

$p(\alpha\cap\beta \cap \gamma) = p(\alpha\cap\beta)\, p(\gamma \mid \alpha\cap\beta)$

$\vdots$

Chain Rule of Condition Probabilities

$p(\alpha_1 \cap \dots \cap \alpha_n) = p(\alpha_1)\, p(\alpha_2 \mid \alpha_1) \dots p(\alpha_n \mid \alpha_1 \cap \dots \cap \alpha_{n-1})$

$= \prod_{k=1}^{n} p\left(\alpha_k \mid \bigcap_{j=1}^{k-1} a_j\right)$

$= \underset{\alpha}{\arg\max}\ p(y_2 \mid y_1 \cap y_0 ; \alpha)\ p(y_1 \mid y_0 ; \alpha)\ p(y_0 ; \alpha)$

} from Q. : y_0 is independent ($\perp$) of the noise sequence (w_0, w_1)

$\therefore\ y_0 \perp w_0, w_1$

from $y_1 = \alpha y_0 + w_0$ implie $y_1 \perp y_2 ; y_0$

$y_2 = \alpha y_1 + w_1$ " $\therefore\ y_2 \perp y_0 ; y_1$

} from $y_2 \perp y_0 ; y_1$

, Markov process : $p(y_2 \mid y_1, y_0) = p(y_2 \mid y_1)$

$= \underset{\alpha}{\arg\max}\ p(y_2 \mid y_1 ; \alpha)\ p(y_1 \mid y_0 ; \alpha)\ p(y_0 ; \alpha)$

$$= \underset{\alpha}{\operatorname{argmax}}\ p(y_2 | y_1 ; \alpha)\ p(y_1 | y_0 ; \alpha)\ p(y_0 ; \alpha)$$

$$L(\alpha) = p(y_2 | y_1 ; \alpha)\ p(y_1 | y_0 ; \alpha)\ p(y_0 ; \alpha)$$

$$\ln L(\alpha) = \ln p(y_2 | y_1 ; \alpha) + \ln p(y_1 | y_0 ; \alpha) + \ln p(y_0 ; \alpha)$$

} in Normal Dist., probability density f^n :

- $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$\ln p(x) = \ln e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} - \ln (2\pi\sigma^2)^{\frac{1}{2}}$$
$$= -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 - \frac{1}{2}\ln(2\pi\sigma^2)$$

$$\ln p(x) = -\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2)$$

} from $w_0, w_1 \sim \mathcal{N}(0, \sigma^2)$

$y_0 \sim \mathcal{N}(0, \lambda)$

and $y_1 = \underbrace{\overset{\text{const.}}{\alpha}\ \underset{\text{const.}}{y_0}}_{\text{const.}} + \underset{\text{variable}}{w_0}$

} from Linear combination of normal random variables

$\therefore\ y_1 | y_0 \sim \mathcal{N}(\alpha y_0, \sigma^2)$

and $y_2 = \alpha y_1 + w_1$

$\therefore\ y_2 | y_1 \sim \mathcal{N}(\alpha y_1, \sigma^2)$

$$\ln L(\alpha) = \ln p(y_2 | y_1 ; \alpha) + \ln p(y_1 | y_0 ; \alpha) + \ln p(y_0 ; \alpha)$$

$$\ln L(\alpha) = -\frac{1}{2}\frac{(y_2 - \alpha y_1)^2}{\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2) + -\frac{1}{2}\frac{(y_1 - \alpha y_0)^2}{\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2) + -\frac{1}{2}\frac{y_0^2}{\lambda} - \frac{1}{2}\ln(2\pi\lambda)$$

$$\ln L(\alpha) = -\frac{1}{\cancel{2}}\frac{(y_2-\alpha y_1)^2}{\cancel{\sigma^2}} - \frac{1}{2}\cancel{\ln(2\pi\sigma^2)} + -\frac{1}{\cancel{2}}\frac{(y_1-\alpha y_0)^2}{\cancel{\sigma^2}} - \frac{1}{2}\cancel{\ln(2\pi\sigma^2)} + \cancel{-\frac{1}{2}\frac{y_0^2}{\lambda} - \frac{1}{2}\ln(2\pi\lambda)}$$

$$\frac{\partial}{\partial\alpha} L(\alpha) \equiv 0 = \cancel{2}(y_2-\alpha y_1)(-y_1) + \cancel{2}(y_1-\alpha y_0)(-y_0)$$

$$0 = -y_1 y_2 + \alpha y_1^2 - y_0 y_1 + \alpha y_0^2$$

$$\therefore \alpha = \frac{y_2 y_1 + y_1 y_0}{y_1^2 + y_0^2}$$

T1.

OT 1.

$$\frac{\partial}{\partial\alpha} L(\alpha) \equiv 0 = \sum_{i=0}^{n} (y_{i+1} - \alpha y_i)(-y_i)$$

; General Case

$y_{n+1} = \alpha y_n + w_n$; $n = 0, 1, 2, \ldots$

$$= \sum_{i=0}^{n} \left(\alpha y_i^2 - y_{i+1} y_i\right)$$

$$0 = \sum \alpha y_i^2 - \sum y_{i+1} y_i$$

$$\sum_{i=0}^{n} \alpha y_i^2 = \sum_{i=0}^{n} y_{i+1} y_i$$

$$\therefore \alpha = \frac{\sum_{i=0}^{n} y_{i+1} y_i}{\sum_{i=0}^{n} y_i^2}$$

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