OTS.

$$\nabla_A \operatorname{tr}(AB) = B^T$$

$$\underset{m\times p}{\mathbf{C}} = \begin{pmatrix} a_{11}b_{11}+\cdots+a_{1n}b_{n1} & a_{11}b_{12}+\cdots+a_{1n}b_{n2} & \cdots & a_{11}b_{1p}+\cdots+a_{1n}b_{np} \\ a_{21}b_{11}+\cdots+a_{2n}b_{n1} & a_{21}b_{12}+\cdots+a_{2n}b_{n2} & \cdots & a_{21}b_{1p}+\cdots+a_{2n}b_{np} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{11}+\cdots+a_{mn}b_{n1} & a_{m1}b_{12}+\cdots+a_{mn}b_{n2} & \cdots & a_{m1}b_{1p}+\cdots+a_{mn}b_{np} \end{pmatrix} ; \quad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj},$$

Sum Matrix $\underset{\substack{m\times n \;\, n\times m \\ m\times m}}{AB} = \begin{bmatrix} \sum_{k=1}^{n} A_{1k}B_{k1} & \sum_{k=1}^{n} A_{1k}B_{k2} & \sum_{k=1}^{n} A_{1k}B_{km} \\ \sum_{k=1}^{n} A_{2k}B_{k1} & \sum_{k=1}^{n} A_{2k}B_{k2} & \sum_{k=1}^{n} A_{1k}B_{km} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} A_{mk}B_{k1} & & \sum_{k=1}^{n} A_{mk}B_{km} \end{bmatrix}_{m\times m}$

$$\operatorname{tr}(AB) = \sum_{k=1}^{n} A_{1k}B_{k1} + \sum_{k=1}^{n} A_{2k}B_{k2} + \cdots + \sum_{k=1}^{n} A_{mk}B_{km}$$

$$= \sum_{i=1}^{m}\sum_{k=1}^{n} A_{ik}B_{ki} = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ji}$$

$$\operatorname{tr}(AB) = \begin{bmatrix} A_{11}B_{11} & A_{12}B_{21} & \cdots & A_{1n}B_{n1} \\ A_{21}B_{12} & A_{22}B_{22} & \ddots & A_{2n}B_{n2} \\ \vdots & & & \vdots \\ A_{m1}B_{1m} & & & A_{mn}B_{nm} \end{bmatrix}_{m\times n}$$

$$\nabla_A \operatorname{tr}(AB) = \begin{bmatrix} \frac{\partial}{\partial A_{11}} A_{11}B_{11} & \frac{\partial}{\partial A_{12}} A_{12}B_{21} & \cdots & \frac{\partial}{\partial A_{1n}} A_{1n}B_{n1} \\ \frac{\partial}{\partial A_{21}} A_{21}B_{12} & \frac{\partial}{\partial A_{22}} A_{22}B_{22} & \ddots & \frac{\partial}{\partial A_{2n}} A_{2n}B_{n2} \\ \vdots & & & \vdots \\ \frac{\partial}{\partial A_{m1}} A_{m1}B_{1m} & & & \frac{\partial}{\partial A_{mn}} A_{mn}B_{nm} \end{bmatrix}_{m\times n}$$

$$= \begin{bmatrix} B_{11} & B_{21} & \cdots & B_{n1} \\ B_{12} & B_{22} & \ddots & B_{n2} \\ \vdots & & & \vdots \\ B_{1m} & & & B_{nm} \end{bmatrix}_{m\times n}$$

$$= \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & \ddots & & \\ \vdots & & \ddots & \\ B_{n1} & & & B_{mn} \end{bmatrix}^{T}_{n\times m}$$

$$\therefore \nabla_A \operatorname{tr}(AB) = B^T \quad \#$$

OT6.

$$\nabla_{A^T} f(A) = \left(\nabla_A f(A) \right)^T$$

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & \\ \vdots & & \ddots & \\ A_{m1} & & & A_{mn} \end{bmatrix}_{m \times n} \qquad A^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{m1} \\ A_{12} & A_{22} & & \\ \vdots & & \ddots & \\ A_{1n} & & & A_{mn} \end{bmatrix}_{m \times n}$$

$$\nabla_{A^T} f(A) = \begin{bmatrix} \frac{\partial}{\partial A_{11}} f(A) & \frac{\partial}{\partial A_{21}} f(A) & \cdots & \frac{\partial}{\partial A_{m1}} f(A) \\ \frac{\partial}{\partial A_{12}} f(A) & \frac{\partial}{\partial A_{22}} f(A) & & \\ \vdots & & \ddots & \\ \frac{\partial}{\partial A_{1n}} f(A) & & & \frac{\partial}{\partial A_{mn}} f(A) \end{bmatrix}_{n \times m}$$

$$= \begin{bmatrix} \frac{\partial}{\partial A_{11}} f(A) & \frac{\partial}{\partial A_{12}} f(A) & \cdots & \frac{\partial}{\partial A_{1n}} f(A) \\ \frac{\partial}{\partial A_{21}} f(A) & \frac{\partial}{\partial A_{22}} f(A) & & \\ \vdots & & \ddots & \\ \frac{\partial}{\partial A_{m1}} f(A) & & & \frac{\partial}{\partial A_{mn}} f(A) \end{bmatrix}_{m \times n}^T$$

$$\therefore \nabla_{A^T} f(A) = \left(\nabla_A f(A) \right)^T \quad \#$$

OT 7.

$$\underset{mn}{\nabla_A} \operatorname{tr} \underset{mn}{A}\,\underset{nn}{B}\,\underset{nm}{A^T}\,\underset{mm}{C} = \underset{mm}{C}\,\underset{mn}{A}\,\underset{nn}{B} + C^T A B^T$$

$$\nabla_A \operatorname{tr} ABA^TC$$

$$= \nabla_A \sum_m \left(ABA^TC \right)_{mm}$$

$$= \nabla_A \sum_m \left(\sum_{nhl} A_{mn} B_{nh} (A^T)_{hl} C_{lm} \right)_{mm}$$

$$= \nabla_A \sum_{mnhl} A_{mn} B_{nh} A_{lh} C_{lm}$$

$$= \left(\partial_A \sum_{mnhl} A_{mn} B_{nh} A_{lh} C_{lm} \right)_{ij}$$

$$= \partial_{A_{ij}} \sum_{mnhl} A_{mn} B_{nh} A_{lh} C_{lm}$$

$$= \sum_{mnhl} \partial_{A_{ij}} \left(A_{mn} B_{nh} A_{lh} C_{lm} \right)$$

$$= \sum_{mnhl} \Big((\partial_{A_{ij}} A_{mn})(B_{nh} A_{lh} C_{lm}) + A_{mn} (\partial_{A_{ij}} B_{nh} A_{lh} C_{lm}) \Big) \quad \text{; product rule}$$

$$= \sum_{mnhl} (\cancelto{1}{\partial_{A_{ij}} A_{mn}})(B_{nh} A_{lh} C_{lm}) + \sum_{mnhl} A_{mn} (\partial_{A_{ij}} B_{nh} A_{lh} C_{lm})$$

$$= \sum_{hl} B_{jh} A_{lh} C_{li} + \sum_{mnhl} A_{mn} (\partial_{A_{ij}} B_{nh} A_{lh} C_{lm})$$

$$= \sum_{hl} B_{jh} A_{lh} C_{li} + \sum_{mnhl} A_{mn} \Big[(\partial_{A_{ij}} A_{lh})(B_{nh} C_{lm}) + (A_{lh})(\cancelto{0}{\partial_{A_{ij}} B_{nh} C_{lm}}) \Big] \quad \text{; product rule}$$

$$= \sum_{hl} B_{jh} A_{lh} C_{li} + \sum_{mnhl} A_{mn} (\partial_{A_{ij}} A_{lh}) B_{nh} C_{lm}$$

$$= \sum_{hl} B_{jh} A_{lh} C_{li} + \sum_{mn} A_{mn} B_{nj} C_{im}$$

$$= \sum_{hl} C_{li} A_{lh} B_{jh} + \sum_{mn} C_{im} A_{mn} B_{nj}$$

$$= \sum_{hl} (C^T)_{il} A_{lh} (B^T)_{hj} + \sum_{mn} C_{im} A_{mn} B_{nj}$$

$$= \left(C^TAB^T + CAB \right)_{ij}$$

$$= C^TAB^T + CAB$$

$$= CAB + C^TAB^T$$