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# Matching Theory

Ver. 1.0.3

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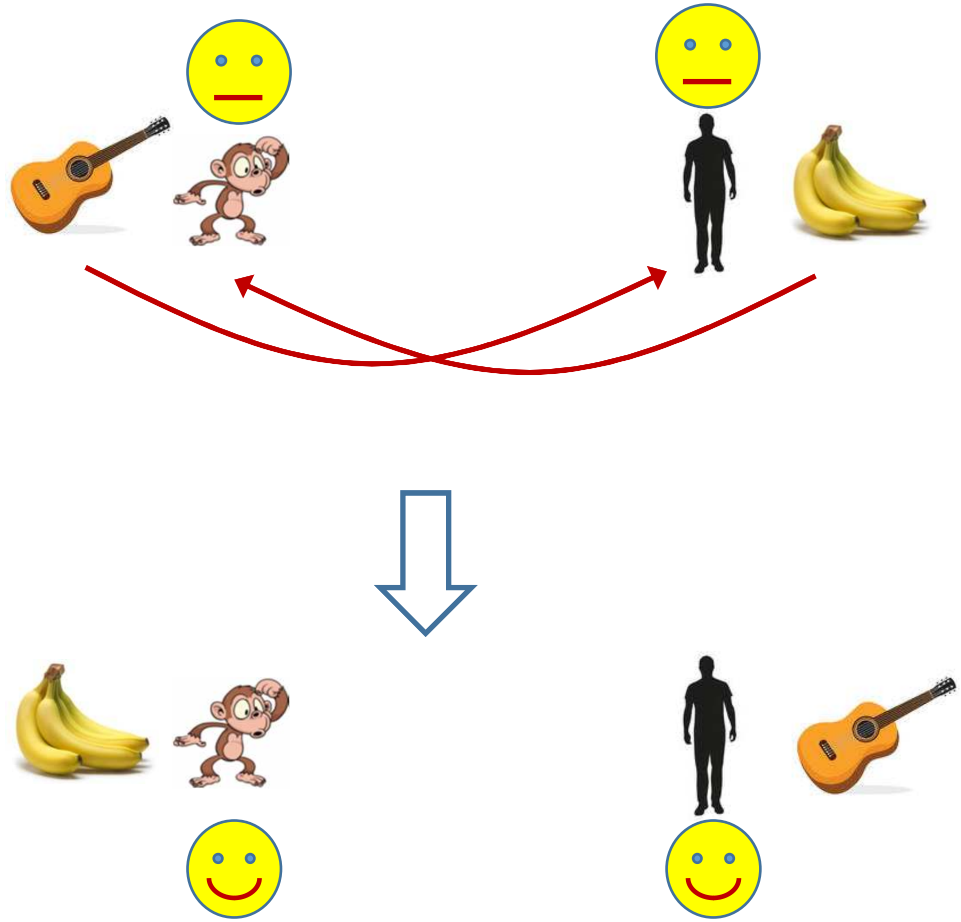
# Outline

- Introduction
- In CS: Bipartite Matching
- One-to-one matching with one-sided preference
- Many-to-one matching with one-sided preference
- One-to-one matching with two-sided preference
- Many-to-one matching with two-sided preference
- Matching with Transfer
- Applications
- Summary

# Introduction

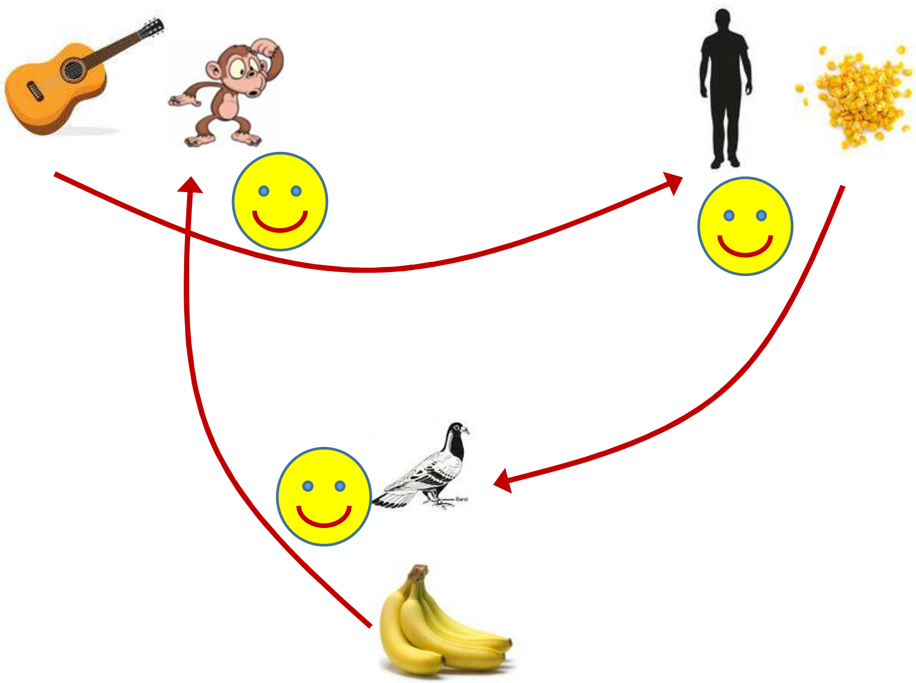
What this problem is about?

# Pareto Improvement by Exchange



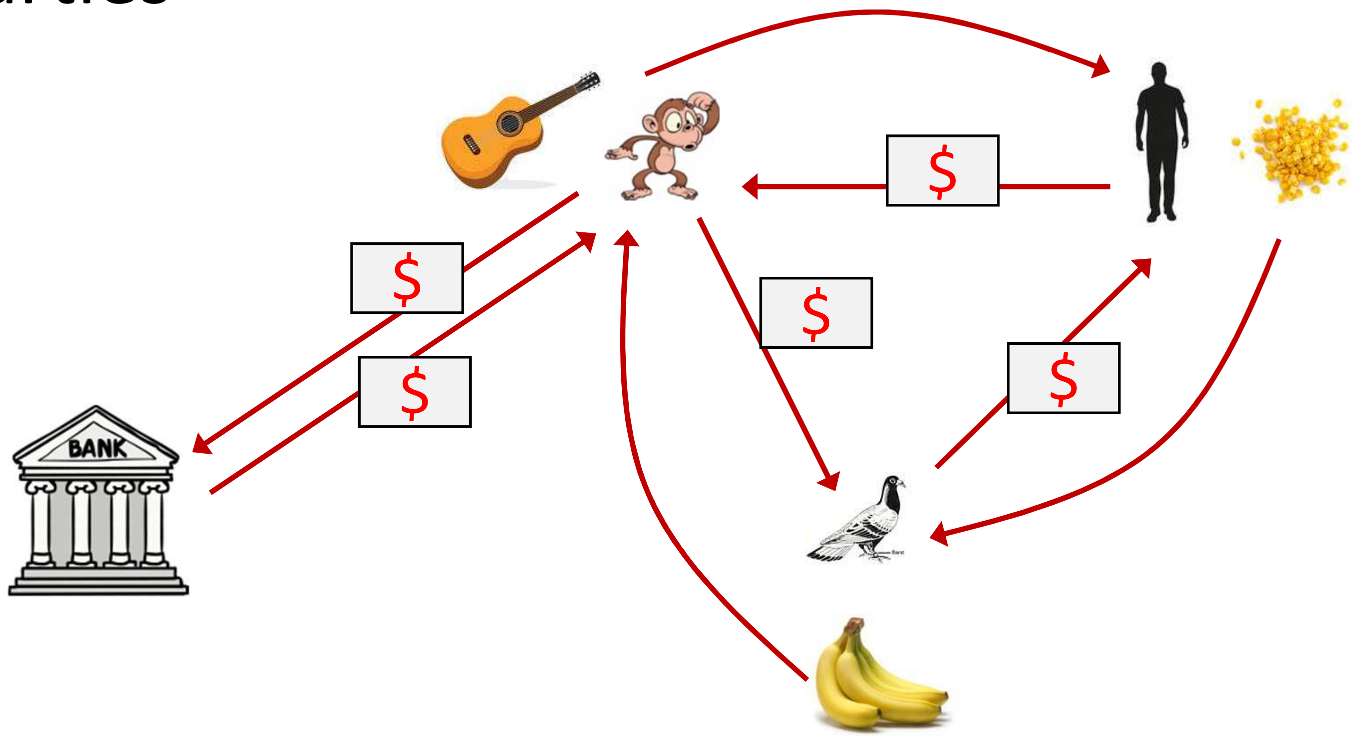
# Pareto Improvement by Exchange Among Three or More Parties

- How to enable this type of exchange?



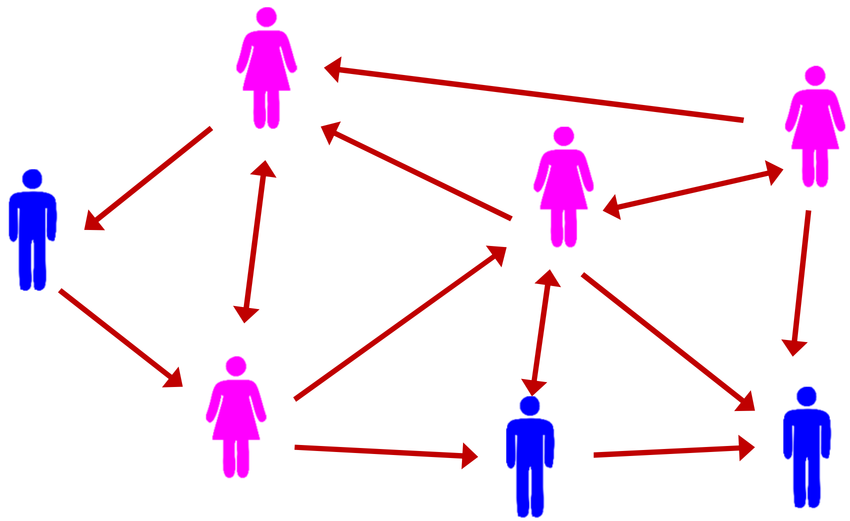
# Introduction of Money

- enable item exchange among multiple parties



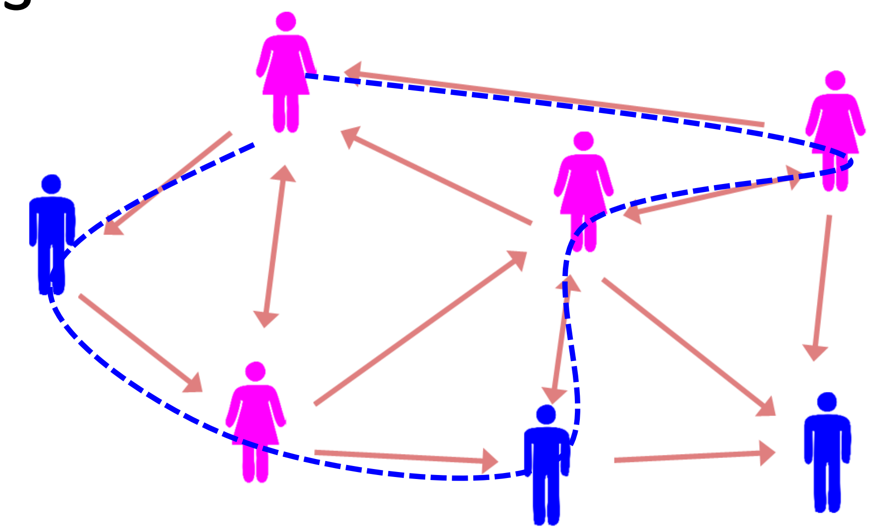
# Exchange Without Money

- Some exchange does not allow the involvement of money
  - e.g., kidney exchange (for transplant)
- How to find possible Pareto improvement?



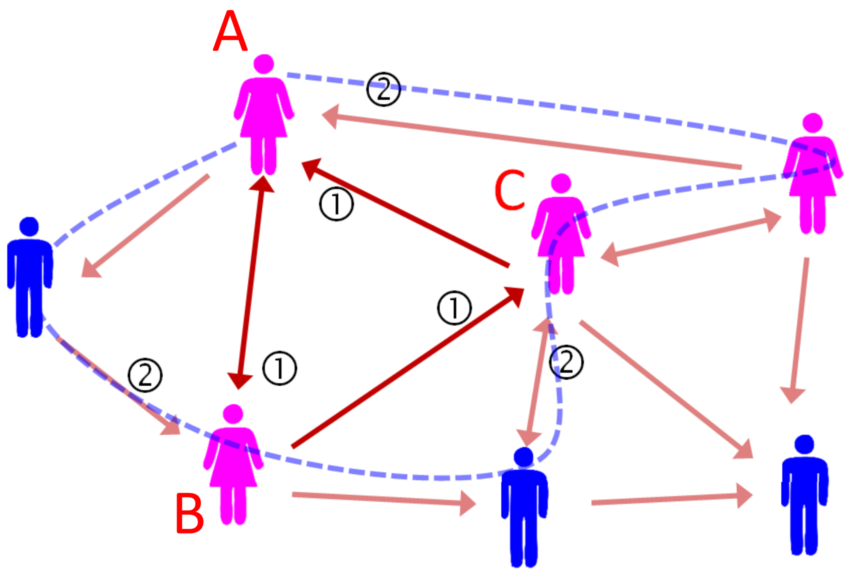
# Pareto Improvement Involving Multiple Parties

- Find a (possibly longest) cycle in a directed graph
- All agents (parties) in the cycle get Pareto improvements



# If Agents Have Preferences

- The cycle (A, B, C, A) gives A, B, C a better result
- A, B, C have the incentive to deviate from the result
- In this case, we say the result is not stable




Difference from Pareto optimality:

Stability concerns only a subset of (not all) agents. It doesn’t care if other agent’s results are worsened.

# Exchange is a special case of Matching

- Exchange problem assumes that each agent holds something and wants something better by exchange
- In some cases, no item has been allocated to any agent initially
- matching problem is more general

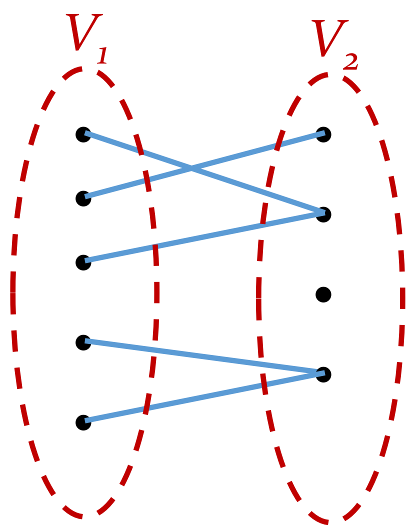
# In CS: Bipartite Matching

Modeling matching on a bipartite graph

# Bipartite Graphs

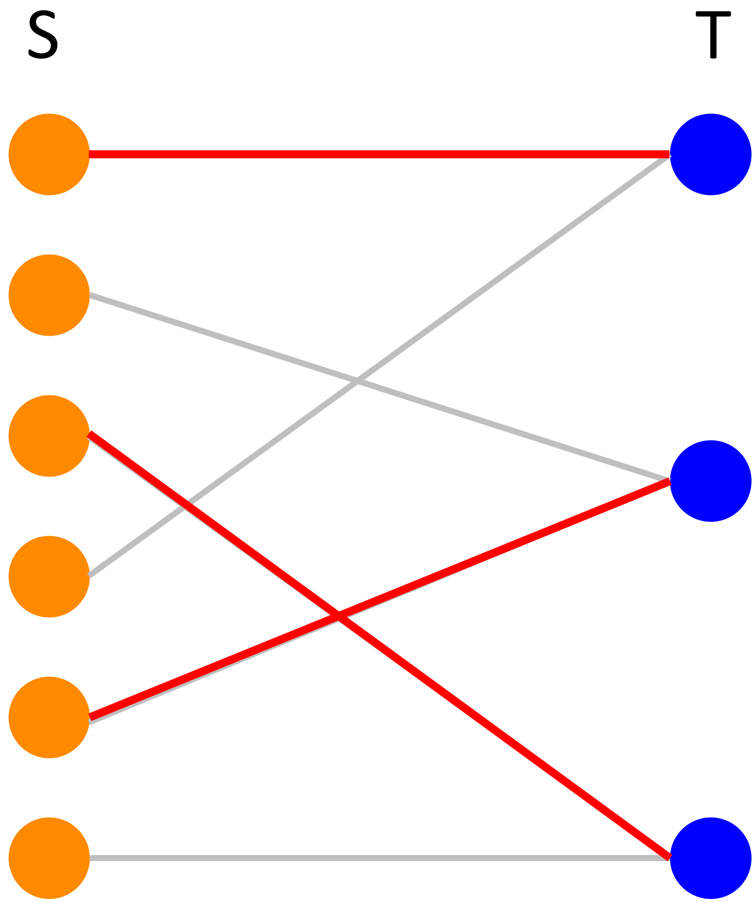
A simple graph $G$=($V$, $E$) is bipartite if we can find a way to partition $V$ into two disjoint subsets $V_1$ and $V_2$ such that every edge connects a vertex in $V_1$ and a vertex in $V_2$.

In other words, there are no edges which connect two vertices in $V_1$ or in $V_2$.



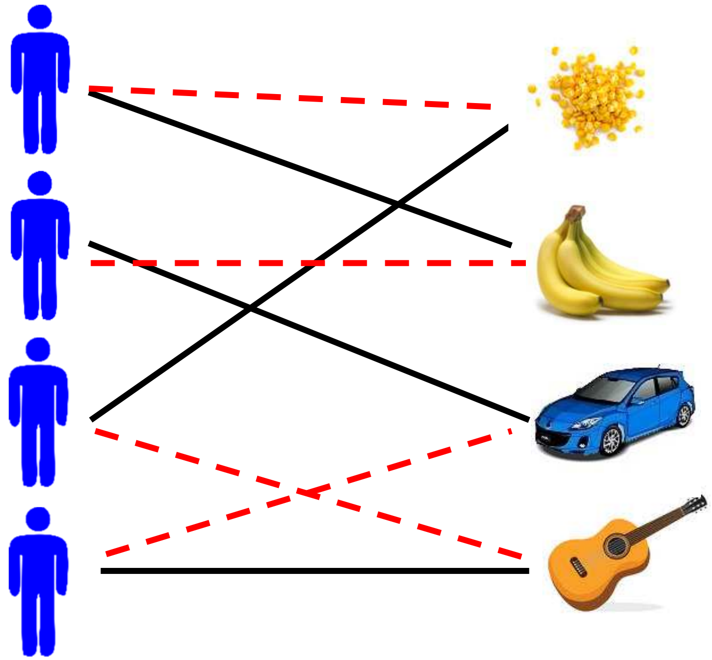
# Bipartite Matching

- A vertex in one set can be matched with at most one vertex on the other set.
  - One-to-one matching
- Maximum Cardinality Bipartite Matching
  - Maximize the number of matchings



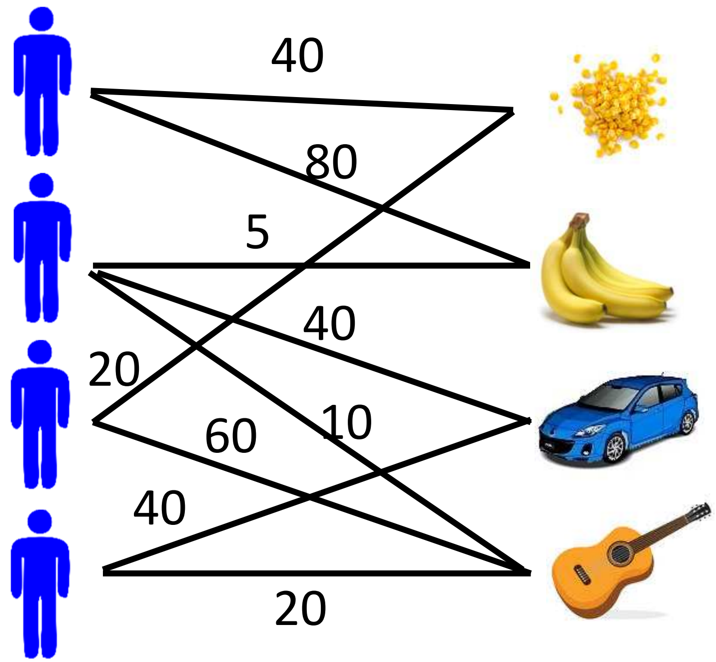
# Matching Considering Preferences

- In max-cardinality matching, participants (agents) do not have preference over matchings
- In contrast, we assume that agents have preferences over matchings
- Any two matchings are equally good in terms of cardinality, but not if preferences are considered.



# Representing Preference

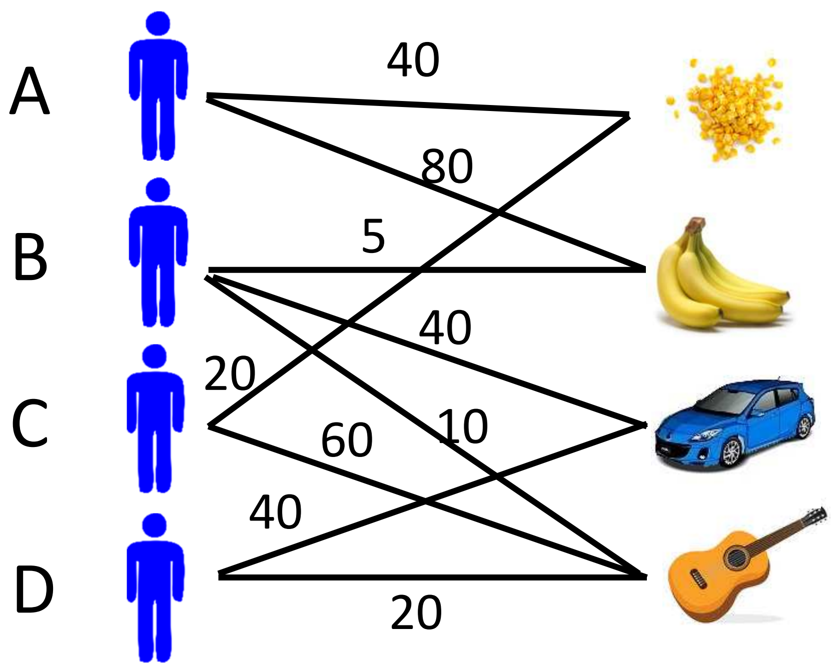
- We may label each edge with a weight to indicate the preference over that match
- Maximum Weight Bipartite Matching
  - Choose a set of one-to-one matchings that maximizes the total weight




Application: [LWZ17] Lei et al., “A semi-matching based load balancing scheme for dense IEEE 802.11 WLANs,” *IEEE Access*, 5:15332-15339, July 2017.

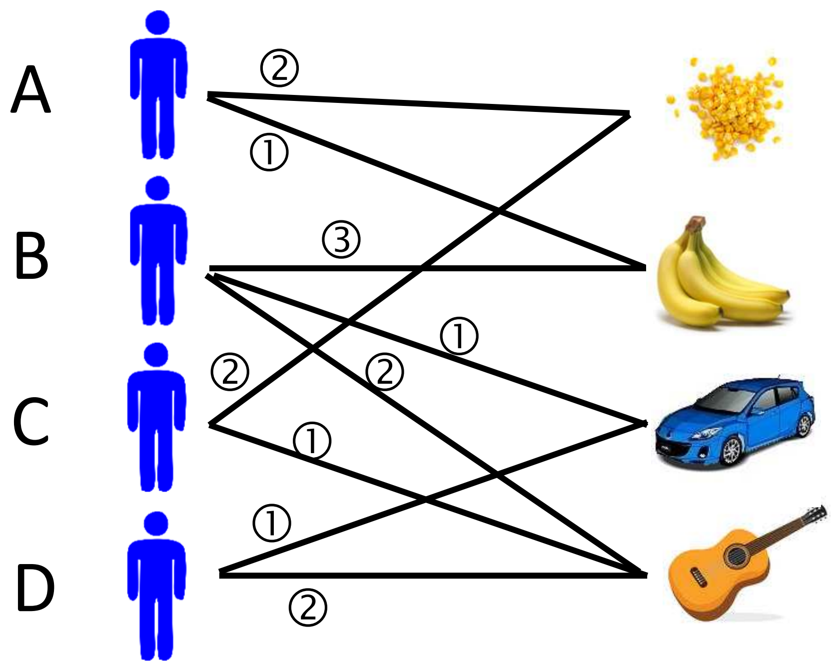
# Problem with Weight-Based Preference

- Max-weight matching cares A's preference more than B's
- In most cases, we treat every agent equally



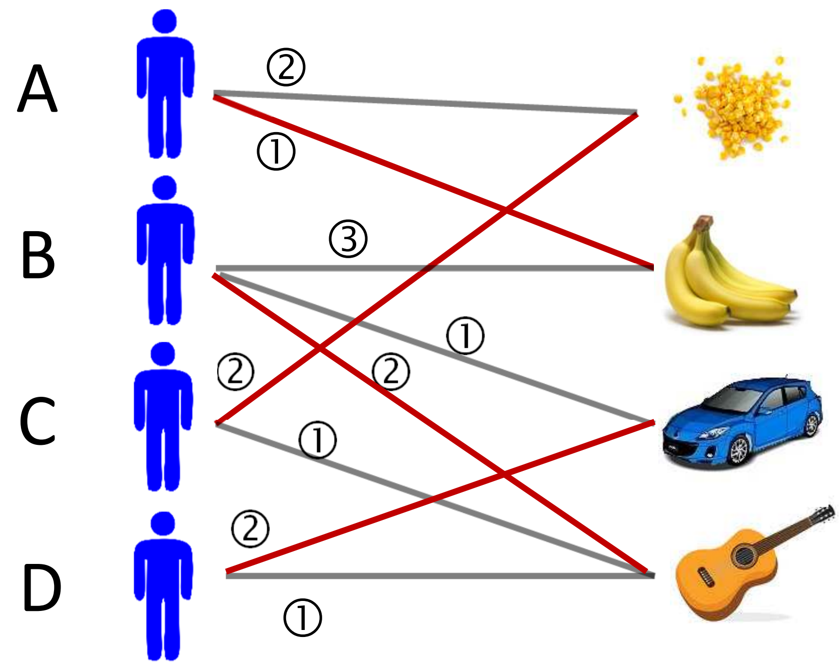
# From Weights To Preference

- Hereafter, we change weights (cardinal) to preferences (ordinal)



# Stability in Matching

- If B and D exchange their allocations, both can be better off
- B and D have the incentive to deviate from the matching result
- The matching result is not stable!



# Stable Result May Not Exist

- Consider a group of students {A, B, C, D} to be matched to roommates, two in each room.
- Student’s preferences
  - A prefers B>C>D
  - B prefers C>A>D
  - C prefers A>B>D
  - No stable match exists: whoever is paired with D wants to change and can find a willing partner.
- So stability of matching may not exist, even if each match involves just two people.

# One-Sided Preference

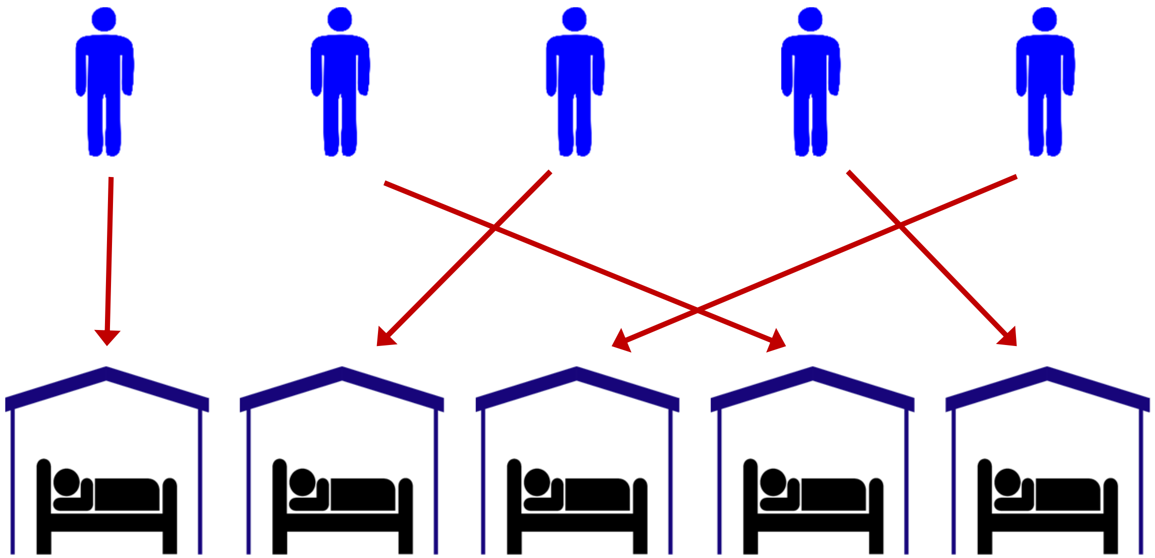
exchange, switch

# One-Sided Preference

- Match people with items
- Or match people with another bunch of people that are totally fine with the matching result
- One-to-one matching
  - Housing Market Problem
  - House Allocation with Existing Tenants
- Many-to-one matching
  - Ca’pacitated House Allocation

# House Allocation Problem

One-to-one matching with one-sided preference



# House Allocation Problem $(A, H, \succ)$

- Assumption
  - a set of agents $A$
  - a set of individual objects (houses) $H$
  - a preference profile $(\succ_a)_{a \in A}$: a list of preference relations of agents over houses (strict total order)

| Agent | 1st prep. | 2nd prep. | 3rd prep. | 4th prep. |
|---|---|---|---|---|
| $a_1$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ |
| $a_2$ | $h_1$ | $h_3$ | $h_2$ | $h_4$ |
| $a_3$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ |
| $a_4$ | $h_1$ | $h_3$ | $h_2$ | $h_4$ |
| $a_5$ | $h_4$ | $h_1$ | $h_2$ | $h_3$ |

$h_1 \succ_{a_1} h_2$

$h_3 \succ_{a_4} h_2$

$h_4 \succ_{a_5} h_3$

# Matching as a function

- The outcome of the housing allocation problem $(A, H, \succ)$ is a matching $\mu : A \to H$
  - Each agent $a$ is allocated the house $\mu(a)$
- $M$: the set of all possible matchings
- Preference relations over matchings. Let $u, v \in M$

$$
\left.\begin{array}{l}
\mu \succ_a v \leftrightarrow \mu(a) \succ_a v(a) \\
\mu \succcurlyeq_a v \leftrightarrow v \not\succ_a \mu \\
\mu \sim_a v \leftrightarrow \mu(a) = v(a)
\end{array}\right\}
$$

The relation defines a weak total order on $M$

# Pareto Improvement in Matching

| | **1st** | **2nd** | **3rd** | **4th** |
|---|---|---|---|---|
| $a_1$ | $h_4$ | $h_3$ | $h_2$ | $h_1$ |
| $a_2$ | $h_3$ | $h_4$ | $h_2$ | $h_1$ |
| $a_3$ | $h_2$ | $h_4$ | $h_1$ | $h_3$ |
| $a_4$ | $h_3$ | $h_2$ | $h_1$ | $h_4$ |

| | **1st** | **2nd** | **3rd** | **4th** |
|---|---|---|---|---|
| $a_1$ | $h_4$ | $h_3$ | $h_2$ | $h_1$ |
| $a_2$ | $h_3$ | $h_4$ | $h_2$ | $h_1$ |
| $a_3$ | $h_2$ | $h_4$ | $h_1$ | $h_3$ |
| $a_4$ | $h_3$ | $h_2$ | $h_1$ | $h_4$ |

$a_1$'s and $a_2$'s results are improved without degrading any other's result

$\Rightarrow$ a Pareto improvement

# Pareto Domination & Pareto Efficiency

- Pareto domination
  - Suppose $\mu, \nu$ are matchings. Then $\mu$ Pareto dominates $\nu$ if and only if

    (1) $\mu \succcurlyeq_a v$ for all $a \in A$,

    (2) $\mu \succ_a v$ for some $a \in A$.

- Pareto efficiency
  - a matching $\mu$ is Pareto efficient iff it is not Pareto dominated by any matching $v \in M$.

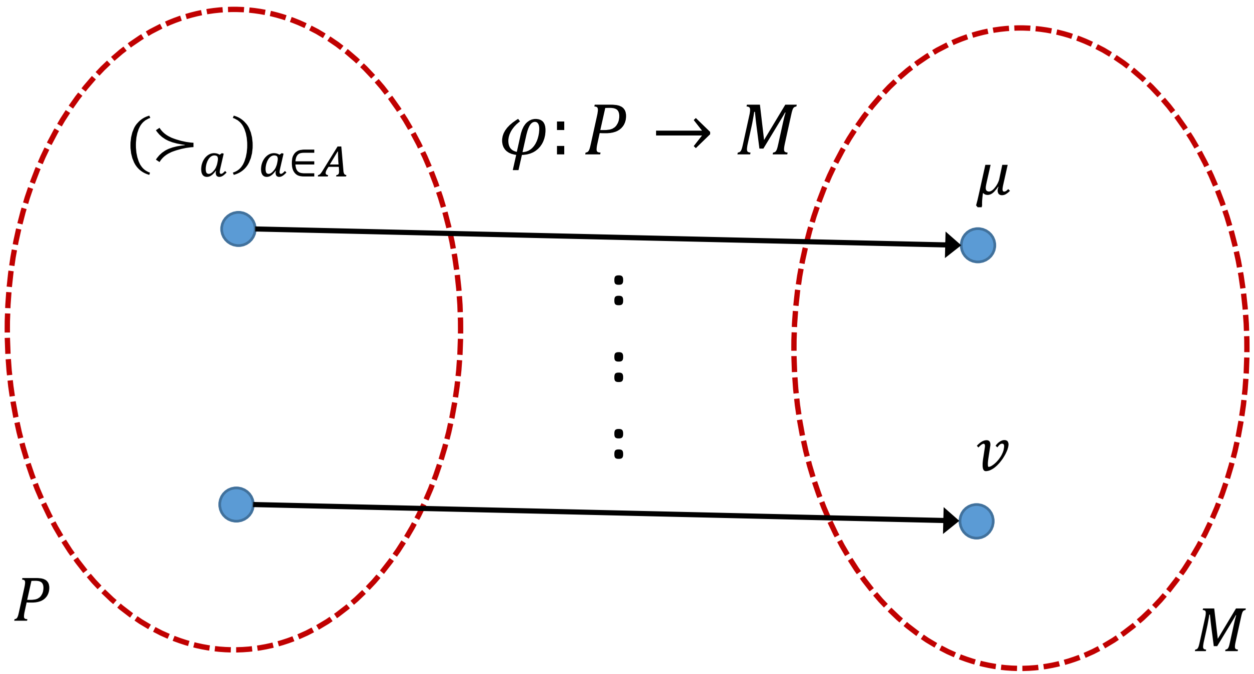
# Matching Mechanism

- Let $P$ denote the set of all preference profiles of all agents over houses ($P = \{(\succ_a)\}_{a \in A}$)
- Let $M$ denote the set of all matchings of agents to houses
- A matching mechanism is a procedure for determining a matching given a housing allocation problem.
- Formally, matching mechanism is a function

$$\varphi: P \to M$$

# Pareto Efficient Matching Mechanism

- A mechanism is Pareto efficient if it always produces a matching that is Pareto efficient on the announced preference profile.



# Truthfulness May Not Be The Best Strategy Under Some Mechanism

| | 1st | 2nd | 3rd | 4th |
|---|---|---|---|---|
| $m_1$ | $h_1$ | $h_3$ | $h_2$ | $h_4$ |
| $m_2$ | $h_1$ | $h_2$ | $h_4$ | $h_3$ |
| $m_3$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ |
| $m_4$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ |

true preference

| | 1st | 2nd | 3rd | 4th |
|---|---|---|---|---|
| $m_1$ | $h_1$ | $h_3$ | $h_2$ | $h_4$ |
| $m_2$ | $h_1$ | $h_2$ | $h_4$ | $h_3$ |
| $m_3$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ |
| $m_4$ | $h_2$ | $h_1$ | $h_3$ | $h_4$ |

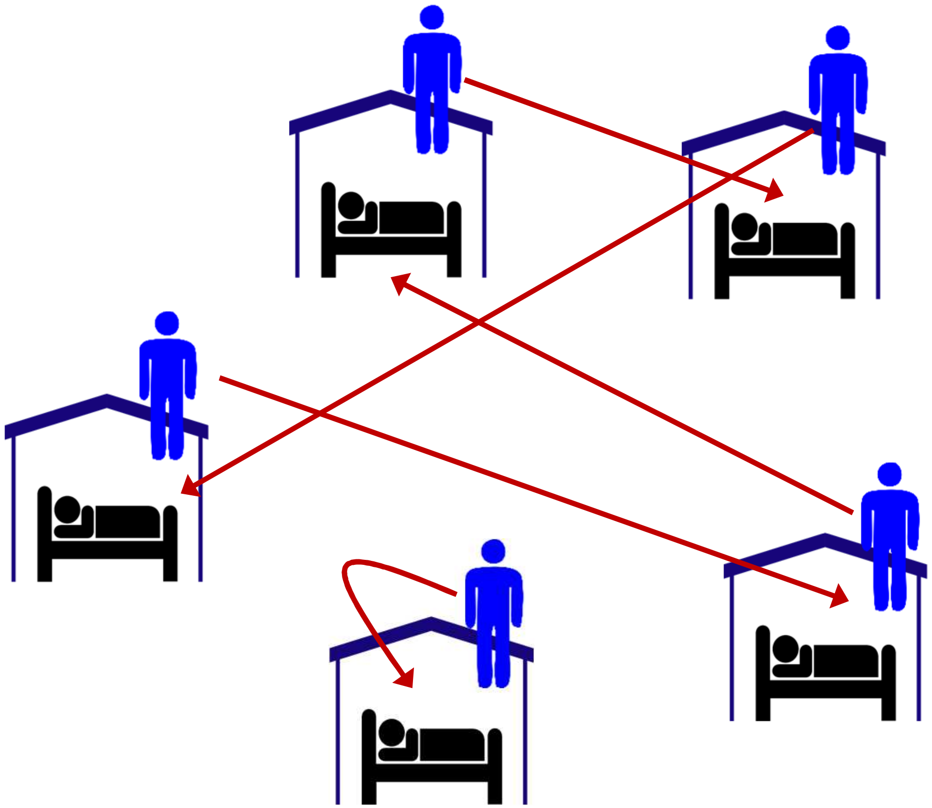
claimed preference

$m_4$ is better off by lying about its preference

# Strategy-Proof Matching Mechanism

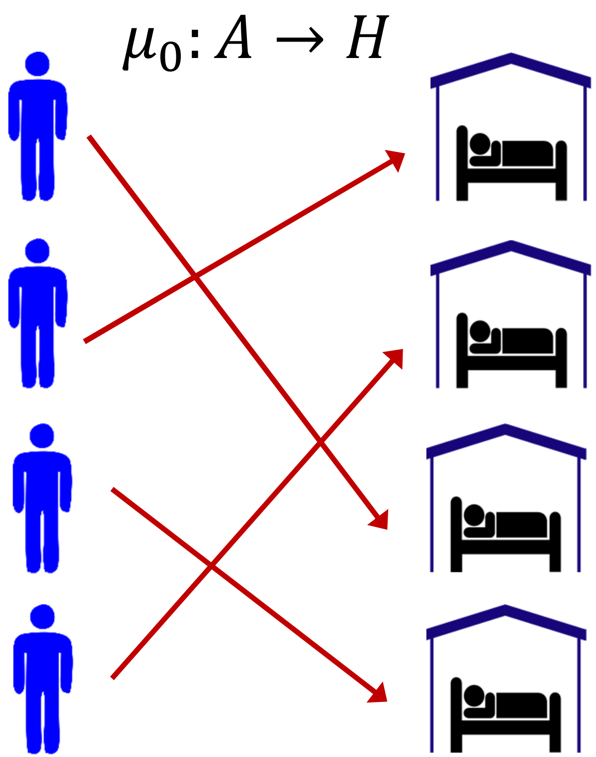
- Agents may lie about their preferences
- Suppose $\varphi$ is a matching mechanism that induces agents to announce the preference profile $\rho \in P$.
- Let $\rho_a$ be agent $a$'s true preference over houses.
- Then $\varphi$ is strategy proof if and only if every agent $a \in A$ weakly prefers its allocation (by $\varphi$) when $a$ choose $\rho_a$ over its allocation (by $\varphi$) when $a$ chooses some other preference relation, regardless of the preference relations of all other agents in $A$.

# Housing Market Problem



# Housing Market Problem

- House allocation problem with initial allocation $\mu_0 : A \to H$, a bijection (we assume that $|A| = |H|$)



# Individually Rational

- Suppose $\mu$ is a matching resulting from the housing market problem $(A, H, \succ, \mu_0)$.
- Then $\mu$ is individually rational if $\mu(a) \succcurlyeq_a \mu_0(a)$ for all $a \in A$.

No agent can be worse off by participating in the matching

# Individually Rational: An Example

$\mu_0(a_i) = h_i$ for $i \in \{1, 2, 3, 4\}$

| | **1st** | **2nd** | **3rd** | **4th** |
|---|---|---|---|---|
| $a_1$ | $h_4$ | $h_3$ | $h_2$ | $h_1$ |
| $a_2$ | $h_3$ | $h_4$ | $h_2$ | $h_1$ |
| $a_3$ | $h_2$ | $h_4$ | $h_1$ | $h_3$ |
| $a_4$ | $h_3$ | $h_2$ | $h_1$ | $h_4$ |

# More Then One Matchings Can be Pareto Efficient

Matching $\mu$

| | 1st | 2nd | 3rd | 4th |
|---|---|---|---|---|
| $a_1$ | $h_4$ | $h_3$ | $h_2$ | $h_1$ |
| $a_2$ | $h_3$ | $h_4$ | $h_2$ | $h_1$ |
| $a_3$ | $h_2$ | $h_4$ | $h_1$ | $h_3$ |
| $a_4$ | $h_3$ | $h_2$ | $h_1$ | $h_4$ |

Matching $v$

| | 1st | 2nd | 3rd | 4th |
|---|---|---|---|---|
| $a_1$ | $h_4$ | $h_3$ | $h_2$ | $h_1$ |
| $a_2$ | $h_3$ | $h_4$ | $h_2$ | $h_1$ |
| $a_3$ | $h_2$ | $h_4$ | $h_1$ | $h_3$ |
| $a_4$ | $h_3$ | $h_2$ | $h_1$ | $h_4$ |

Both are individually rational and Pareto efficient

# Blocking Coalition

- In matching $\mu$, if agent $a_2$ and $a_3$ do not participate in the matching and simply exchange their houses, agent $a_3$ can be better off (and $a_2$ gets the same room anyway)

Matching $\mu$

| | 1st | 2nd | 3rd | 4th |
|---|---|---|---|---|
| $a_1$ | $h_4$ | $h_3$ | $h_2$ | $h_1$ |
| $a_2$ | $h_3$ | $h_4$ | $h_2$ | $h_1$ |
| $a_3$ | $h_2$ | $h_4$ | $h_1$ | $h_3$ |
| $a_4$ | $h_3$ | $h_2$ | $h_1$ | $h_4$ |

Such a "coalition" blocks the matching

A Pareto efficient matching can be unstable!

# Blocking Coalition: Formal Definition

- In a housing market problem $(A, H, \succ, \mu_0)$
- A coalition $A' \subseteq A$ is said to block matching $\mu$ if there is a matching $v$ such that

$$(1)\ v(a) \in \{\mu_0(b) | b \in A'\} \quad \forall a \in A'$$

$v$ allocates every $a \in A'$ a house initially owned by some $b \in A'$

$$(2)\ v(a) \succcurlyeq_a \mu(a) \quad \forall a \in A'$$

every $a \in A'$ weakly prefers its allocation by $v$ to that by $\mu$

$$(3)\ \exists a \in A' \text{such that } v(a) \succ_a \mu(a)$$

some $a \in A'$ strictly prefers its allocation by $v$ to that by $\mu$

# Blocking Pairs

- It is difficult to know whether a matching has a blocking coalition
- If we are only concerned with blocking coalitions of size two, things will become much easier
- Blocking pairs are blocking coalitions of size two

# The (Strong) Core in this problem

- The (strong) core of a housing market problem $(A, H, \succ, \mu_0)$ is a set of matchings $C$
- a matching $\mu \in M$ is in the (strong) core $C$ if there exists no coalition $A' \subseteq A$ that can block $\mu$

$\mu$: not in the core

| | **1st** | **2nd** | **3rd** | **4th** |
|---|---|---|---|---|
| $a_1$ | $h_4$ | $h_3$ | $h_2$ | $h_1$ |
| $a_2$ | $h_3$ | $h_4$ | $h_2$ | $h_1$ |
| $a_3$ | $h_2$ | $h_4$ | $h_1$ | $h_3$ |
| $a_4$ | $h_3$ | $h_2$ | $h_1$ | $h_4$ |

$v$: in the core

| | **1st** | **2nd** | **3rd** | **4th** |
|---|---|---|---|---|
| $a_1$ | $h_4$ | $h_3$ | $h_2$ | $h_1$ |
| $a_2$ | $h_3$ | $h_4$ | $h_2$ | $h_1$ |
| $a_3$ | $h_2$ | $h_4$ | $h_1$ | $h_3$ |
| $a_4$ | $h_3$ | $h_2$ | $h_1$ | $h_4$ |

# Is there any weak core?

- Yes.
- In a strong core, there exists no blocking coalition that could make all its members at least as good as and at least one member better off.
- In a weak core, there exists no coalition $A' \subseteq A$ that can redistribute the houses they own such that they all prefer the houses resulting from the reallocation.
  - In the previous example, matching $\mu$ is in the weak core.

# The Implications of Strong Core

- Any matching in the strong core implies individual rationality
  - Because individual rationality is a special case of the strong core (when $|A'| = 1$)
- Any matching in the strong core implies Pareto optimality
  - Because Pareto optimality is a special case of the strong core (when $A' = A$)

# Properties of the (Strong) Core

- The housing market problem as a non-empty (strong) core [SS74]
  - i.e., the (strong) core is not an empty set
- there is only one unique matching in the (strong) core [RP77]

[SS74] L. Shapley and H. Scarf, "On cores and indivisibility," *Journal of Mathematical Economics*, 1, pp. 23–37, 1974.

[RP77] A. E. Roth and A. Postlewaite, "Weak versus strong domination in a market with indivisible goods," *Journal of Mathematical Economics*, 4, pp. 131–137, 1977.

# Gale's Top Trading Cycles (TTC) Algorithm [SS74]

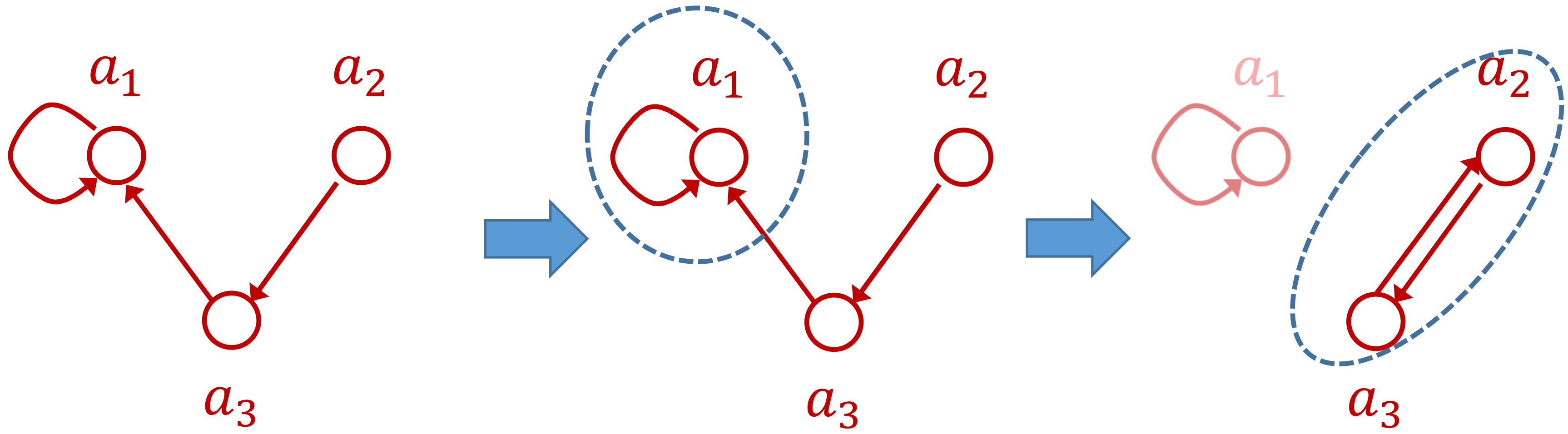
- Each agent points to the owner of its most preferred house.
- If a cycle of agents exists, then match all agents in the cycle with the house of the agent it points to.
- Remove the matched agents and houses from the problem
- each unmatched agent points to the owner of its most preferred remaining house and repeats the above procedure.

[SS74] L. Shapley and H. Scarf, "On cores and indivisibility," *Journal of Mathematical Economics*, 1, pp. 23–37, 1974.

# Gale's TTC Algorithm: An Example

$a_1 : h_1 \succ_{a_1} h_2 \succ_{a_1} h_3$
$a_2 : h_3 \succ_{a_2} h_1 \succ_{a_2} h_2$
$a_3 : h_1 \succ_{a_3} h_2 \succ_{a_3} h_3$

$\mu_0(a_i) = h_i$ for $i \in \{1, 2, 3\}$




Each agent points to the owner of its most preferred house.

a cycle of agents exists

points to the owner of the most preferred remaining house

# Properties of Gale's TTC Algorithm

- Gale's TTC algorithm terminates with a matching
- The outcome of Gale's TTC algorithm is the unique matching in the core of each housing market.
- A mechanism that provides the matching in the core is the only mechanism that is Pareto efficient, individually rational, and strategy-proof. [Ma94]
  - Therefore, Gale's TTC algorithm is also strategy-proof.

[Ma94] J. Ma, "Strategy-proofness and the strict core in a market with indivisibilities," *International Journal of Game Theory*, 23(1):75-83, 1994.

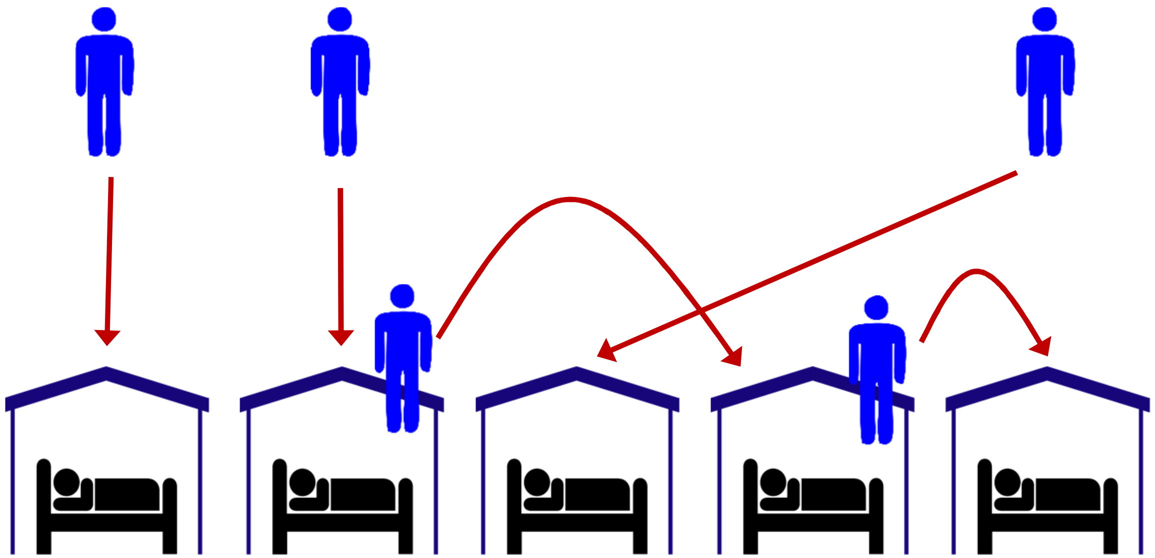
# Practice: What is the Core?

$\mu_0(a_i) = h_i$ for $i \in \{1, 2, 3, 4, 5, 6, 7\}$

| | $1^{st}$ | $2^{nd}$ | $3^{rd}$ | $4^{th}$ | $5^{th}$ | $6^{th}$ | $7^{th}$ |
|---|---|---|---|---|---|---|---|
| $a_1$ | 5 | 6 | 7 | 1 | 2 | 3 | 4 |
| $a_2$ | 3 | 4 | 5 | 6 | 7 | 1 | 2 |
| $a_3$ | 4 | 5 | 2 | 7 | 1 | 3 | 6 |
| $a_4$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $a_5$ | 4 | 5 | 2 | 3 | 6 | 7 | 1 |
| $a_6$ | 7 | 1 | 2 | 3 | 4 | 5 | 6 |
| $a_7$ | 1 | 7 | 4 | 5 | 6 | 3 | 2 |

# House Allocation With Existing Tenants

One-to-one, one-sided preference



# House Allocation With Existing Tenants

- Some (but not all) houses already have tenants
- Consisting of a tuple $(A_e, A_n, H_o, H_v, \succ, \mu_0)$
  - $A_e$: the set of existing agents (who begins with a house)
  - $A_n$: the set of new agents (who begins without a house)
  - $H_o$: the set of occupied houses with $|H_o| = |A_e|$
  - $H_v$: the set of vacant houses
  - $\succ$: preference profile
  - $\mu_0: A_e \to H_o$ is a bijection
- We use $A = A_e \cup A_n$ and $H = H_o \cup H_v \cup \{h_0\}$
  - $h_0$: a *null* house for agents without real allocations

# Agent Priority and Other Assumptions

- Agents have priorities
  - e.g., senior students have priorities over junior ones
  - can also be randomly determined
- Defined as a bijection function $f: \{1, 2, \cdots, |A|\} \rightarrow A$.
- $f$ assigns a ranking to each agent
  - $f(1)$ has the highest priority
- every agent in $A$ is assigned exactly one house
- only $h_0$ may be assigned to more than one agents

# $\psi_f$: TTC for HAP with Existing Tenants (Step 1) [AS99]

- Each agent $a \in A$ points to its favorite house
- Each house $h \in H_o$ points to $\mu_0^{-1}(h)$ (the tenant)
- Each house $h \in H_v$ points to $f(1)$
- If a cycle (of alternating agents and houses) exists, then assign each agent the house that it points to.
- Remove the matched agents and houses from the problem
- If there are remaining agents and houses, then continue to the next step.

[AS99] A. Abdulkadiroğlu and T. Sönmez, "House allocation with existing tenants," *Journal of Economic Theory*, 88, pp. 233–260, 1999.

# $\psi_f$: TTC for HAP with Existing Tenants (Step $t$)

- Each agent $a \in A$ points to its favorite remaining house
- Each house $h \in H_o$ points to $\mu_0^{-1}(h)$
- Each house $h \in H_v$ points to the remaining agent with the highest priority
- If a cycle (of alternating agents and houses) exists, then assign each agent the house that it points to.
- Remove the matched agents and houses from the problem
- If there are remaining agents and houses, then continue to the next step.

# $\psi_f$: TTC for HAP with Existing Tenants (The Final Step)

- Assign the null house to any remaining agents.

# An Example for $\psi_f$

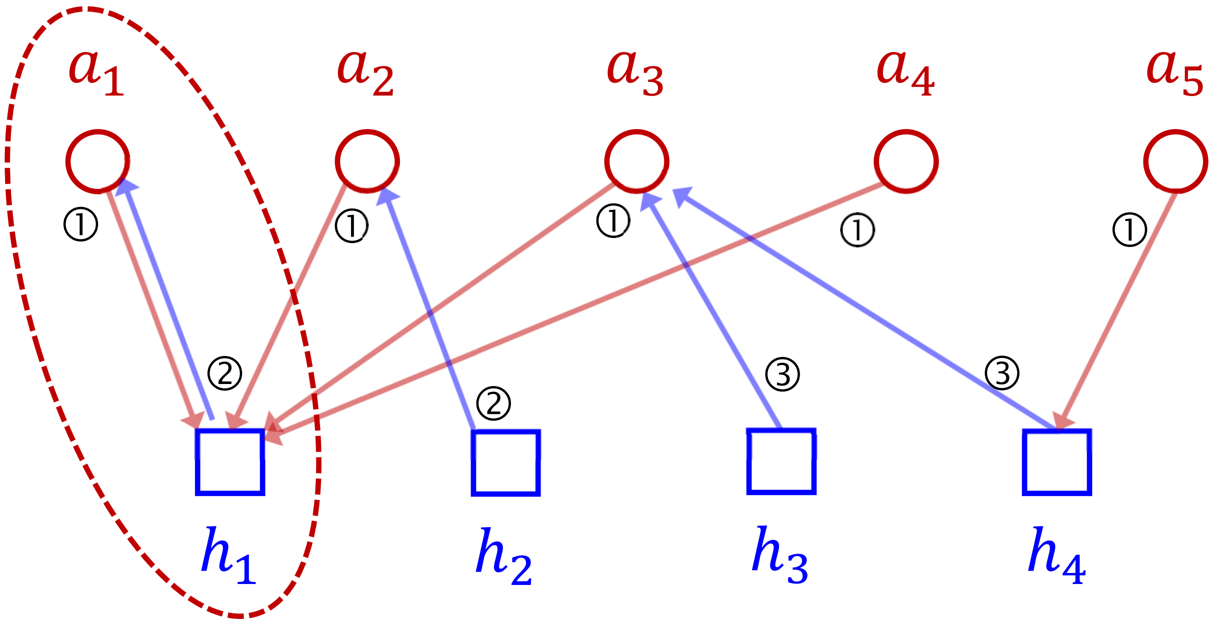
- $A_e = \{a_1, a_2\}$
- $A_n = \{a_3, a_4, a_5\}$
- $H_o = \{h_1, h_2\}$
- $H_v = \{h_3, h_4\}$
- $\mu_0(a_i) = h_i$ for $i \in \{1, 2\}$
- $f$ defines the following priorities over agents

| Agent | preference | | | |
|---|---|---|---|---|
| $a_1$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ |
| $a_2$ | $h_1$ | $h_3$ | $h_2$ | $h_4$ |
| $a_3$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ |
| $a_4$ | $h_1$ | $h_3$ | $h_2$ | $h_4$ |
| $a_5$ | $h_4$ | $h_1$ | $h_2$ | $h_3$ |

$$f: a_3, a_1, a_2, a_4, a_5$$

the highest priority

# Step 1 of $\psi_f$



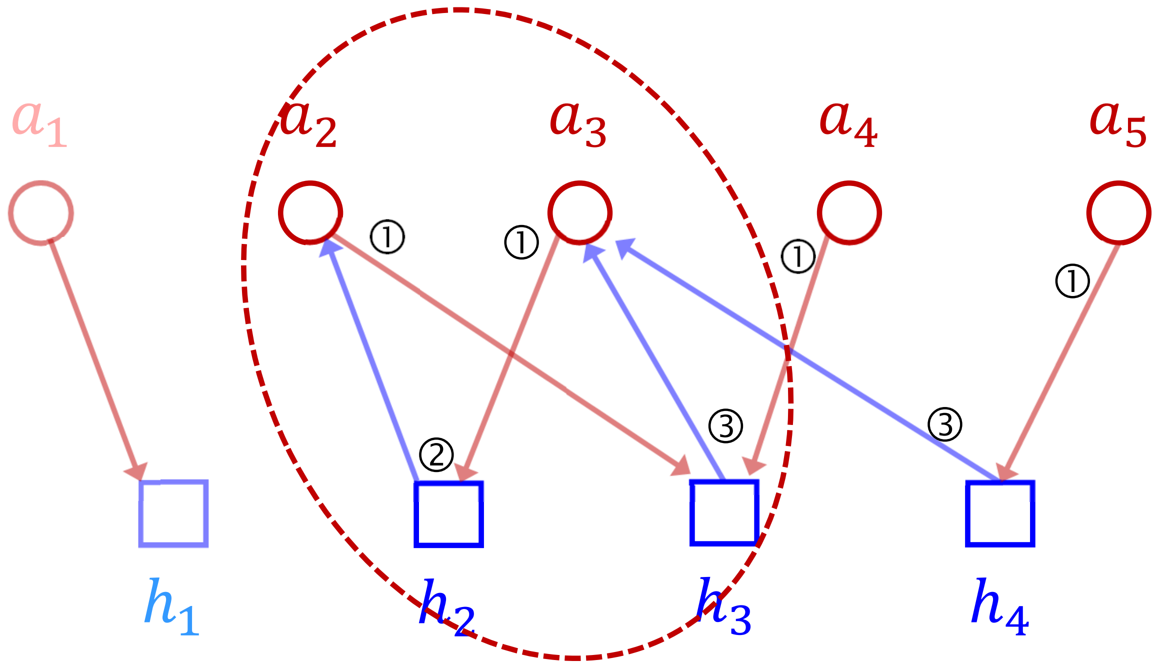

| Agent | 1st |
|---|---|
| $a_1$ | $h_1$ |
| $a_2$ | $h_1$ |
| $a_3$ | $h_1$ |
| $a_4$ | $h_1$ |
| $a_5$ | $h_4$ |

① Each agent points to its favorite house

② Each house $h \in H_o$ points to $\mu_0^{-1}(h)$ $\quad$ $\mu_0(a_i) = h_i$ for $i \in \{1, 2\}$

③ Each house $h \in H_v$ points to $f(1) = a_3$

# Step 2 of $\psi_f$




| Agent | 2nd |
|---|---|
| $a_1$ | $h_2$ |
| $a_2$ | $h_3$ |
| $a_3$ | $h_2$ |
| $a_4$ | $h_3$ |
| $a_5$ | $h_1$ |

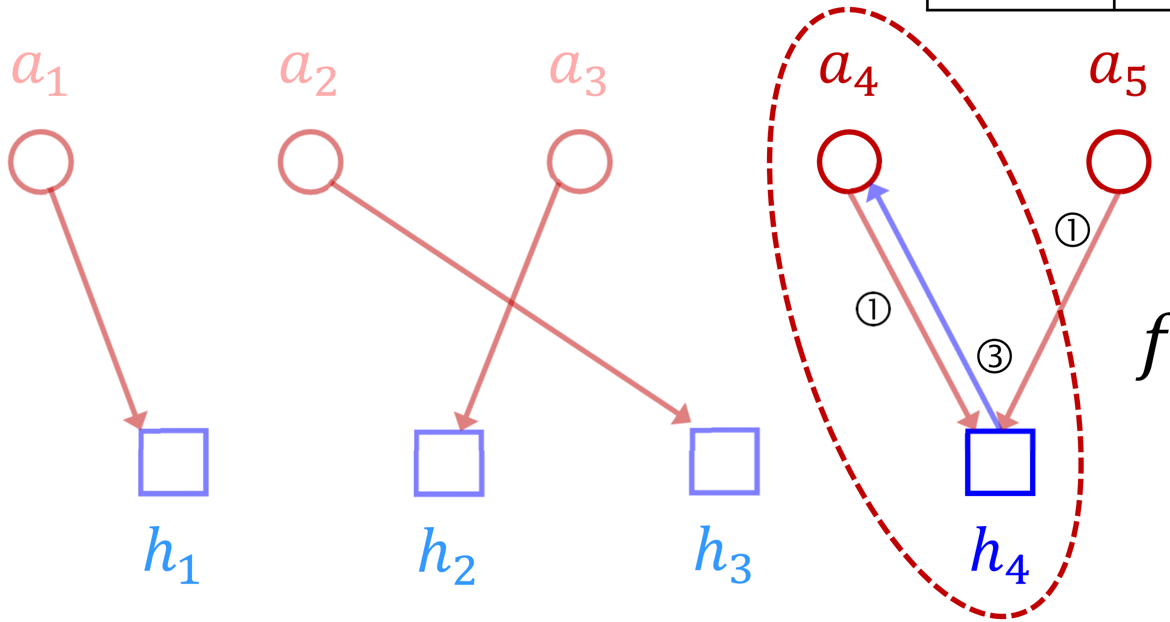
① Each agent points to its favorite remaining house

② Each house $h \in H_o$ points to $\mu_0^{-1}(h)$

③ Each house $h \in H_v$ points to the remaining agent with the highest priority $f(1) = a_3$

# Step 3 of $\psi_f$

| Agent | preference | | | |
|---|---|---|---|---|
| $a_4$ | $h_1$ | $h_3$ | $h_2$ | $h_4$ |
| $a_5$ | $h_4$ | $h_1$ | $h_2$ | $h_3$ |



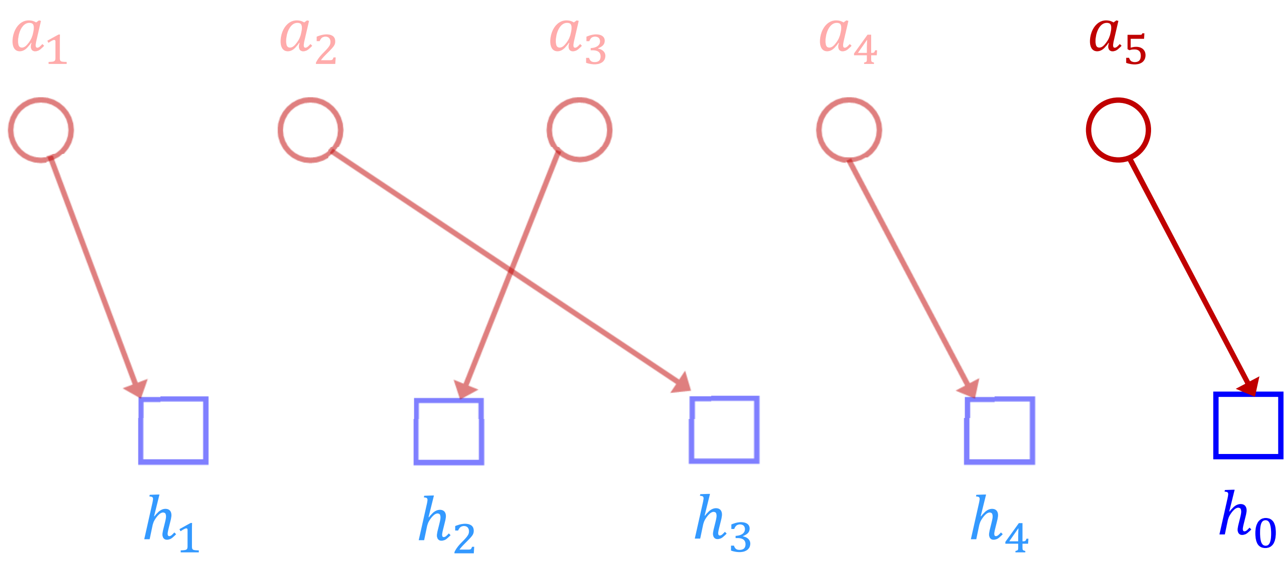

$f: a_3, a_1, a_2, a_4, a_5$

① Each agent points to its favorite remaining house

② Each house $h \in H_o$ points to $\mu_0^{-1}(h)$

③ Each house $h \in H_v$ points to the remaining agent with the highest priority $f(4) = a_4$

# Termination of $\psi_f$




Assign the null house to any remaining agents.

# Properties of $\psi_f$

- Gale's TTC is a special case of this TTC
  - $A_n = H_v = \emptyset$. Thus no need for agent priority.
  - No need for $h \in H$ pointing to $a \in A$.
- It always terminates with a matching
- It is Pareto efficient, individually rational, and strategy proof. [AS99]
- it respects seniority [AS99]

[AS99] A. Abdulkadiroğlu and T. Sönmez, "House allocation with existing tenants," *Journal of Economic Theory*, 88, pp. 233–260, 1999.

# Seniority

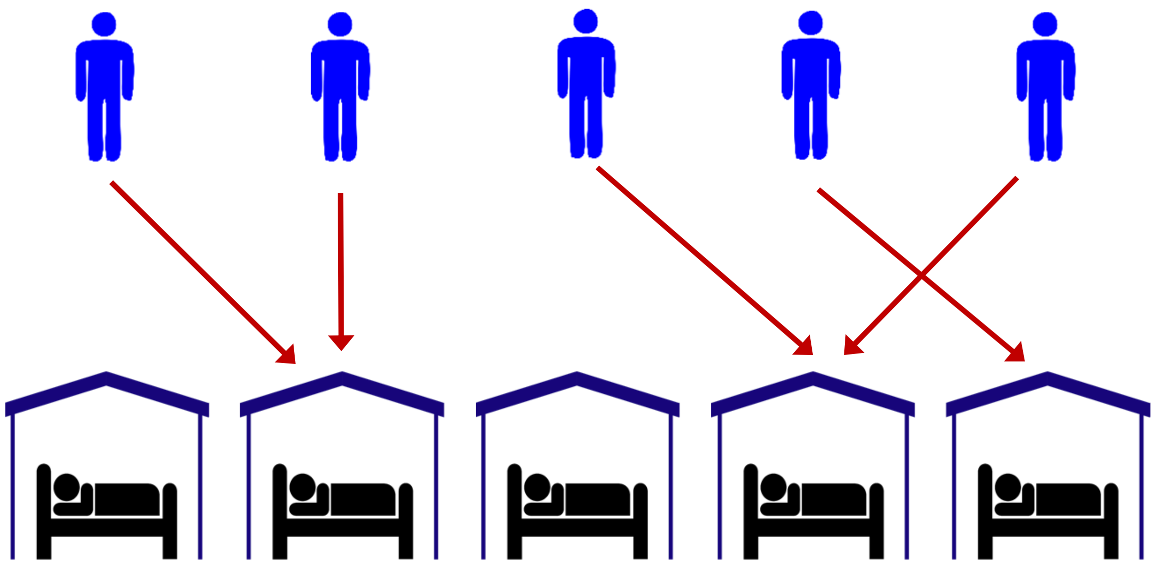
- A mechanism $\psi_f$ that respects seniority meets the following criteria
  - $\psi_f$ assigns $f(1)$ a house that is weakly preferred to the house assigned by any other mechanism that is Pareto efficient, individually rational and strategy proof.
  - out of all mechanisms that perform equally well for agent $f(1)$, $\psi_f$ assigns $f(2)$ a house that is weakly preferred to the house assigned by any other mechanism that is Pareto efficient, individually rational and strategy proof
  - and so on, for all agents $f(3)$, $f(4)$, $\cdots$

# What does this really mean?

- Compared with any other mechanism that is Pareto efficient, individually rational and strategy proof
    - either $f(1)$ prefers the house allocated by $\psi_f$
    - or both mechanisms allocate $f(1)$ the same house
- In the latter case,
    - either $f(2)$ prefers the house allocated by $\psi_f$
    - or both mechanisms allocate $f(2)$ the same house
- In the latter case,
    - either $f(3)$ prefers the house allocated by $\psi_f$
    - or both mechanisms allocate $f(3)$ the same house

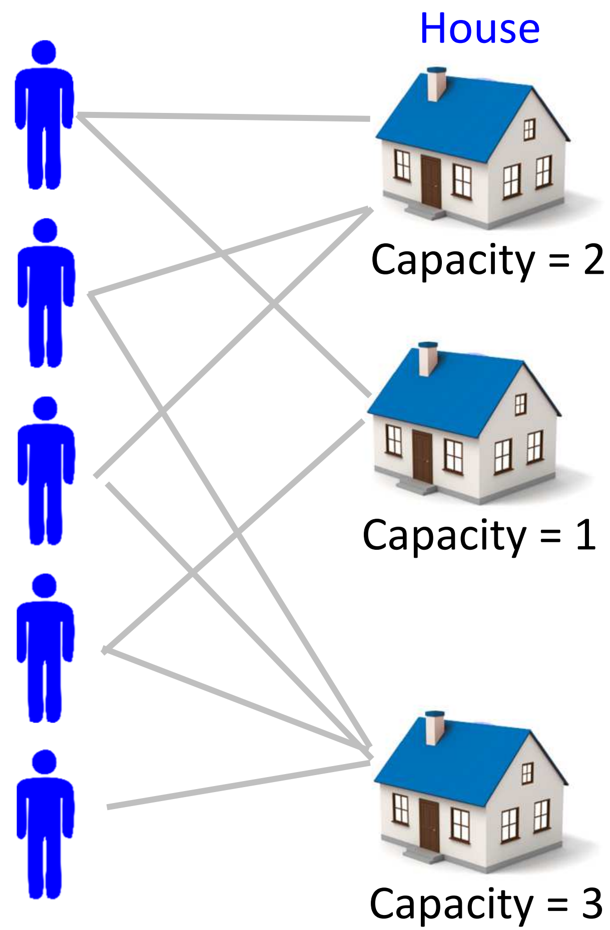
# Capacitated House Allocation (CHA) Problem

Many-to-one, one-sided preference



# Capacitated House Allocation (CHA) Problem

- Each house has a capacity value.
  - the number of the agents accommodated by the same house cannot exceed the capacity of the house
- How to determine the set of agents accommodated by a house?
- Depending on the objective



# Objective 1: Maximum Cardinality

- When agents have neither weights nor preferences on houses
- Try to maximize the number of matchings subject to house capacity constraints
- Max-cardinality many-to-one bipartite matching

# Objective 2: Maximum Cardinality Maximum Utility

- When agents have weights on houses
- Let $u_{a,h}$ be the weight (utility) of the allocation of house $h$ to agent $a$
- Among all maximum cardinality matchings, find the one $\mu$ that maximizes

$$\sum_{(a,h)\in\mu} u_{a,h}$$

# Objective 3: Weight

- When agents have weights but no preferences on houses
- Try to maximize the total weight of matched agents subject to house capacity constraints
- Max-weight many-to-one bipartite matching

# Objective 4: Pareto Efficient

- Each agent $a \in A$ has preference $\succ_a$ over houses but no weight
- a matching $\mu$ is Pareto efficient iff there is no matching $v \neq \mu$ such that

  (1) $v \succcurlyeq_a \mu$ for all $a \in A$, and

  (2) $v \succ_a \mu$ for some $a \in A$.

[AS98] A. Abdulkadiroğlu and T. Sönmez, "Random serial dictatorship and the core from random endowments in house allocation problems," *Econometrica*, 66(3):689–701, 1998.
[ACM+04] D. J. Abraham et al., "Pareto optimality in house allocation problems," in *Proc. ISAAC 2004, LNCS v.3341*, pp. 3–15, 2004.

# Objective 5: Rank Maximal

- Agents have preferences over houses but no weight
- A matching $\mu$ is rank maximal if, compared with any other matching,
  1. it assigns the maximum number of agents to their first-choice houses
  2. subject to 1, it assigns the maximum number of agents to their second-choice houses
  3. and so on.

[IKM+04] R.W. Irving et al., "Rank-maximal matchings," in *Proc. SODA '04*, pp. 68–75, 2004.

# Objective 6: Popularity

- Agents have preferences over houses but no weight
- Let $\mu, v$ be two matchings.
- Let $P(\mu, v) = \{a \in A | \mu \succ_a v\}$
- Let $P(v, \mu) = \{a \in A | v \succ_a \mu\}$
- $\mu$ is more popular than $v$ if $|P(\mu, v)| > |P(v, \mu)|$
- A matching $\mu$ is popular if there is no other matching that is more popular than $\mu$

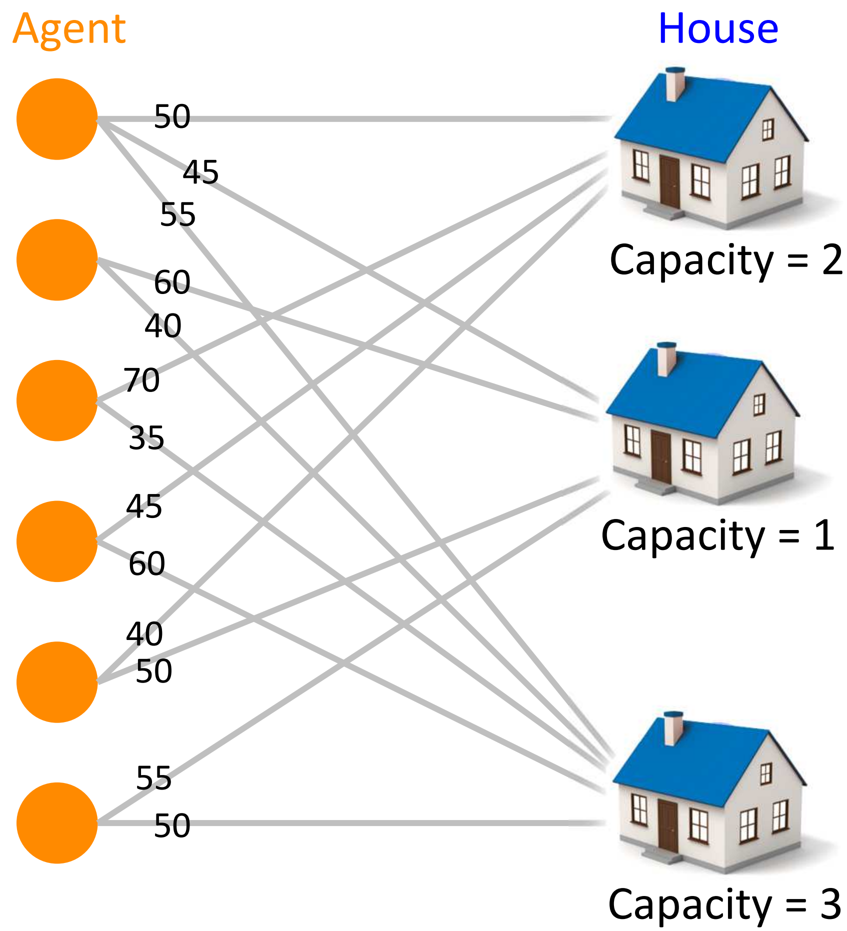
[MS06] D.F. Manlove and C.T.S. Sng, "Popular matchings in the capacitated house allocation problem," *LNCS v.4168*, pp. 492-503, 2006.

# Objective 7: Weighted Popularity

- Agents have preferences over houses
- Every agent $a$ also has a positive weight $w(a)$ indicating $a$'s priority
- The satisfaction of a matching $\mu$ with respect to $v$ is $sat(\mu, v) = \sum_{a \in P(\mu,v)} w(a) - \sum_{a \in P(v,\mu)} w(a)$
- $\mu$ is more popular than $v$ if $sat(\mu, v) > 0$
- A matching $\mu$ is popular if there is no other matching that is more popular than $\mu$

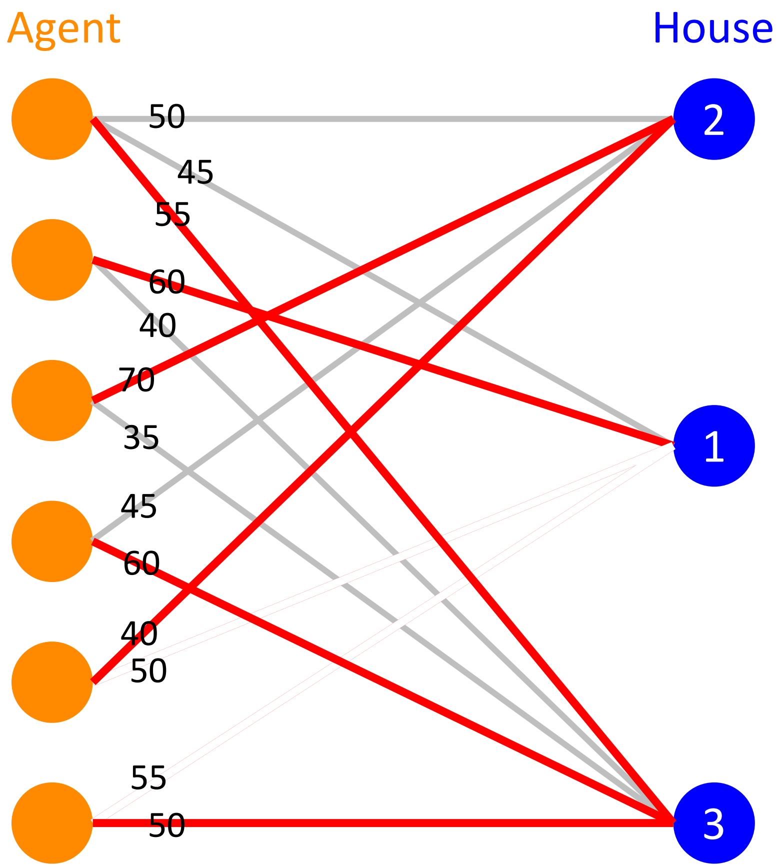
[SM10] C.T.S. Sng and D.F. Manlove, "Popular matchings in the weighted capacitated house allocation problem," Journal of Discrete Algorithms, 8: 102–116, 2010.

# Example: Maximum Cardinality Maximum Utility



# A Greedy Approach (Not Optimal)

1. Each agent chooses the edge that has the highest weight.
2. Houses for which demand exceeds capacity delete the edges that have lower weights.
3. Each rejected agent then chooses the second highest edge.
4. Go to Step 2



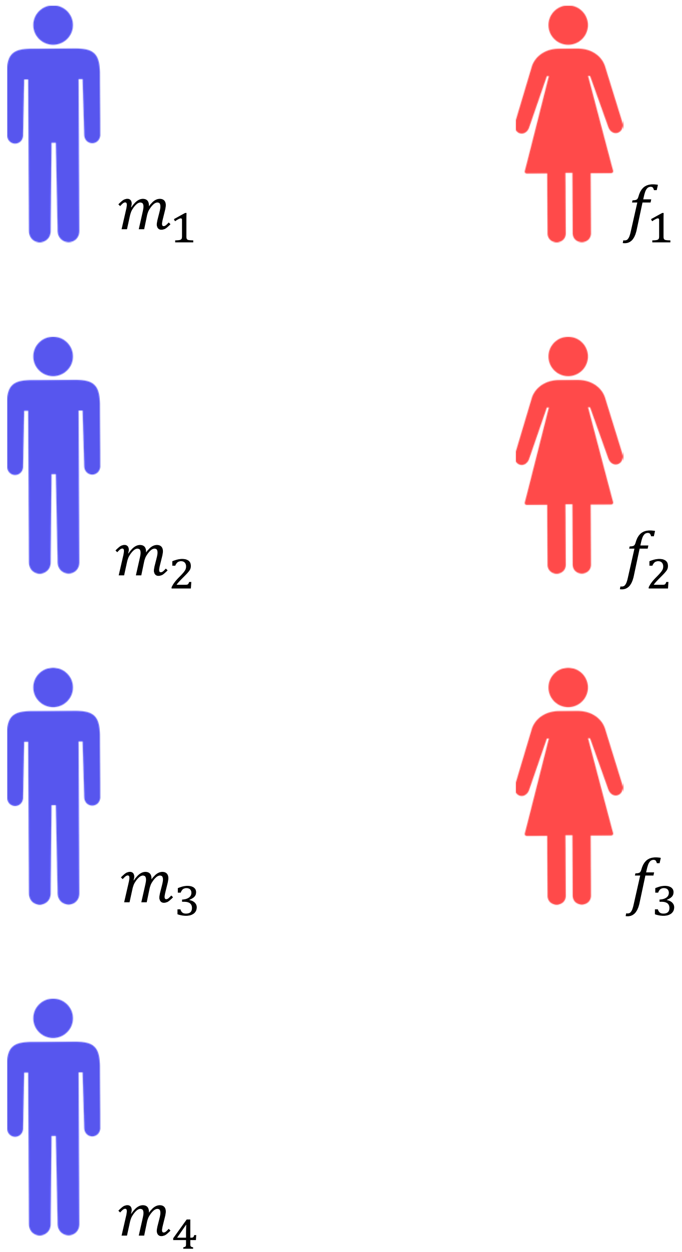
# Two-Sided Preference

# Stable Marriage Problem

- first described by Gale and Shapley in 1962.
  - [GS62]: College Admissions and the Stability of Marriage
- One-to-one matching
- Two sided-preference

# Marriage Problem: Formal Definition

- Sets of men $M = \{m_i\}_{i=1}^{|M|}$ and women $F = \{f_j\}_{j=1}^{|F|}$
- Preference relation: $\succ$
  - $f_j \succ_{m_i} f_k$: $m_i$ prefers $f_j$ to $f_k$
  - $f_j \succ_{m_i} m_i$: $f_j$ is acceptable to $m_i$
- Matching $\mu$: $M \cup F \rightarrow M \cup F$
  - $\forall m_i \in M,\ \mu(m_i) \in F \cup \{m_i\}$
  - $\forall f_i \in F,\ \mu(f_i) \in M \cup \{f_i\}$
  - $\forall m_i \in M, \forall f_j \in F,\ \mu(m_i) = f_j \leftrightarrow \mu(f_j) = m_i$




A male set $M = \{m_1, m_2, \dots, m_{|M|}\}$

A female set $F = \{f_1, f_2, \dots, f_{|F|}\}$

Each male $m_i$ has a complete and transitive preference on $F \cup \{m_i\}$, and so does each female.

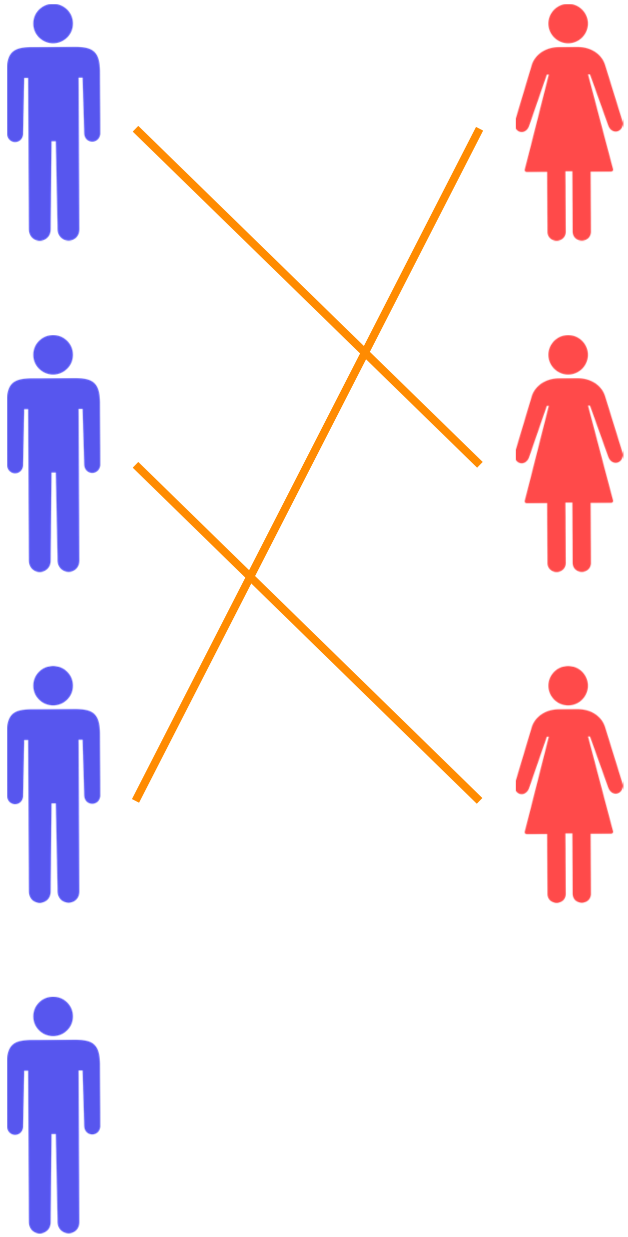
One man can be matched to one woman or to himself.

| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |

# Acceptable Pair

- If one man $m_i$ is matched with one female $f_j$, we called it a pair $(m_i, f_j)$.
  - $\mu(m_i) = f_j$ iff $\mu(f_j) = m_i$
- $(m_i, f_j)$ is an acceptable pair if
  - $m_i$ finds $f_j$ acceptable: $f_j \succ_{m_i} m_i$ and
  - $f_j$ finds $m_i$ acceptable: $m_i \succ_{f_j} f_j$.



An example of matching is shown on the left.

Considering the two-sided preferences, how can we find a stable matching?

# Stable Matching

- A blocking pair:
  - If they prefer each other than the current matching result.
- A blocking individual:
  - If he or she prefers being single to being matched.
- A stable matching:
  - A matching is stable if there is no blocking pairs or blocking individuals in the matching.

# Matching Algorithm: Boston

① Every man proposes to his most preferred woman

② If a woman receives multiple proposals, she accepts the most-preferred one

③ All men with proposals rejected propose to their second-preferred women

④ The process repeats until all men's proposals are either accepted or rejected and no more proposals are possible

# Boston: Step 1

$\boxed{m_1 \rightarrow f_1}$

$m_2 \rightarrow f_1$

$\boxed{m_3 \rightarrow f_2}$

$m_4 \rightarrow f_2$

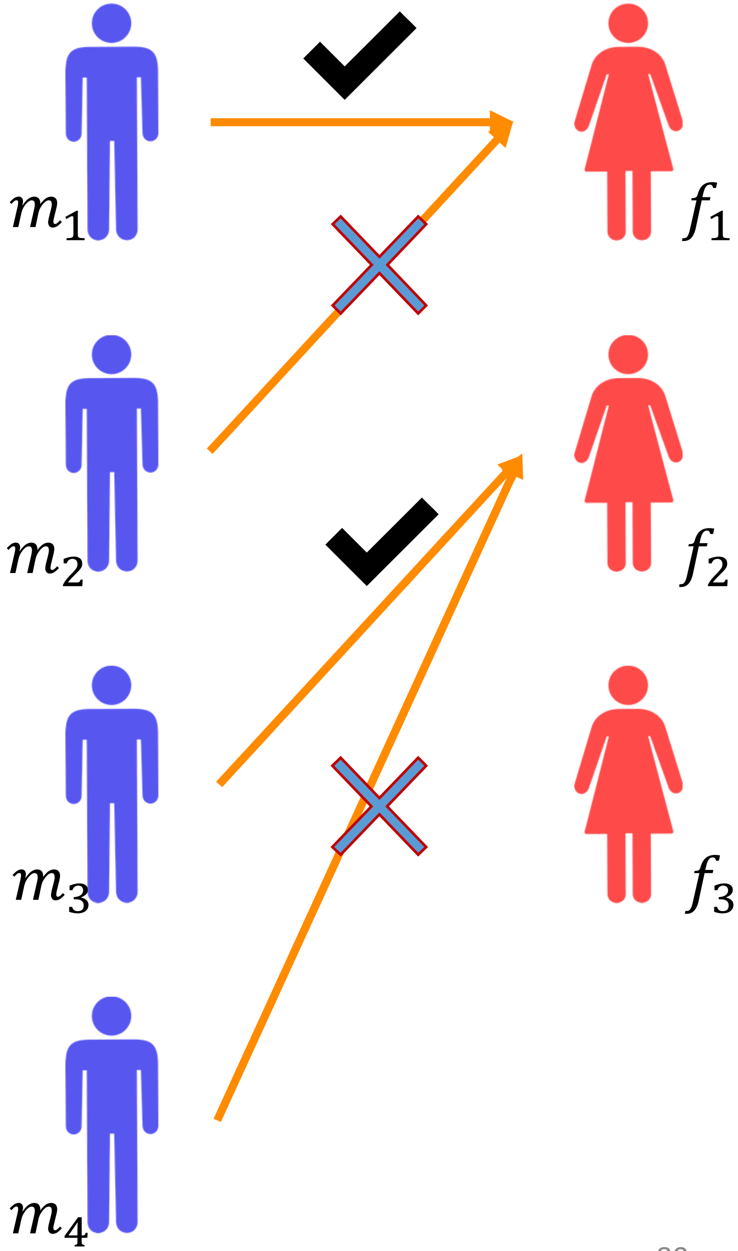
① Every man proposes to his most preferred woman

| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

② Each woman accepts the most-preferred one

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |

multiple proposals received



# Boston: Step 2

$\boxed{m_1 \to f_1}$

$m_2 \to f_1$ $\quad$ $m_2 \to f_2$

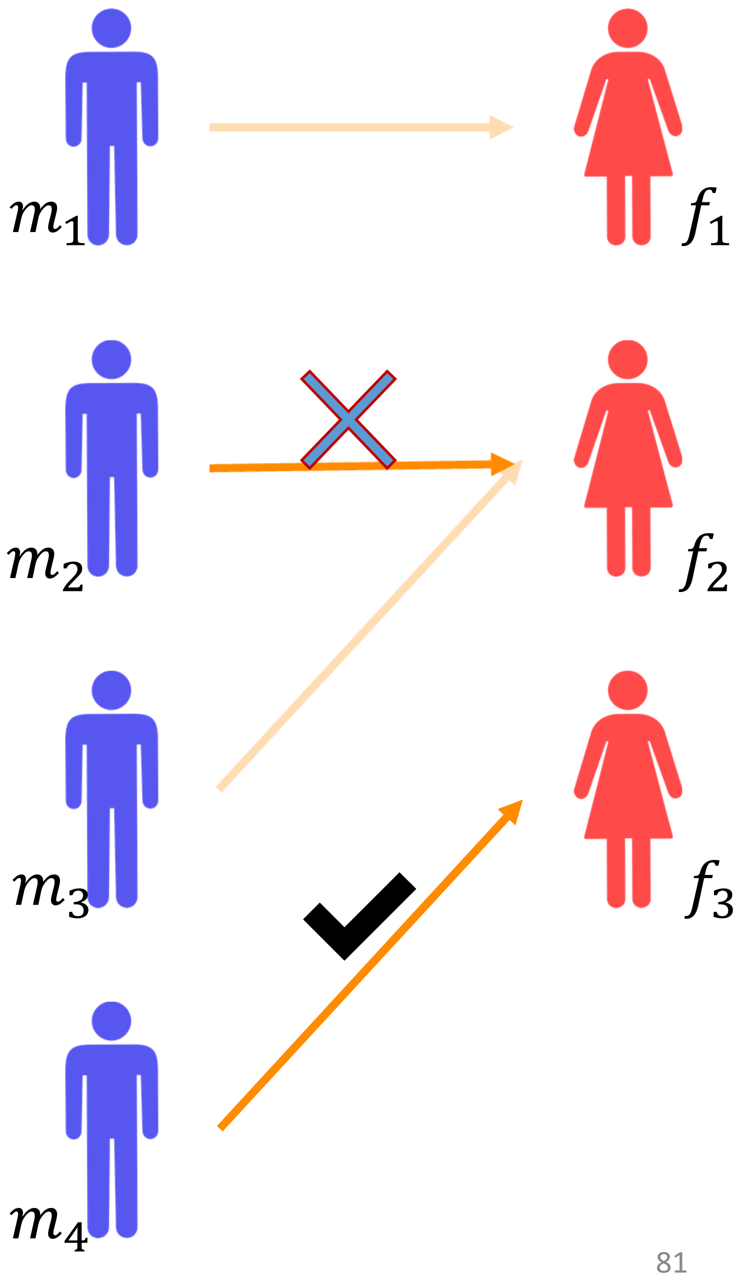
$\boxed{m_3 \to f_2}$

$m_4 \to f_2$ $\quad$ $\boxed{m_4 \to f_3}$

① $m_2$ and $m_4$ propose to their 2nd most-preferred woman

| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |



# Boston: Step 3

$m_1 \rightarrow f_1$

$m_2 \rightarrow f_1$ $m_2 \rightarrow f_2$ $m_2 \rightarrow f_3$

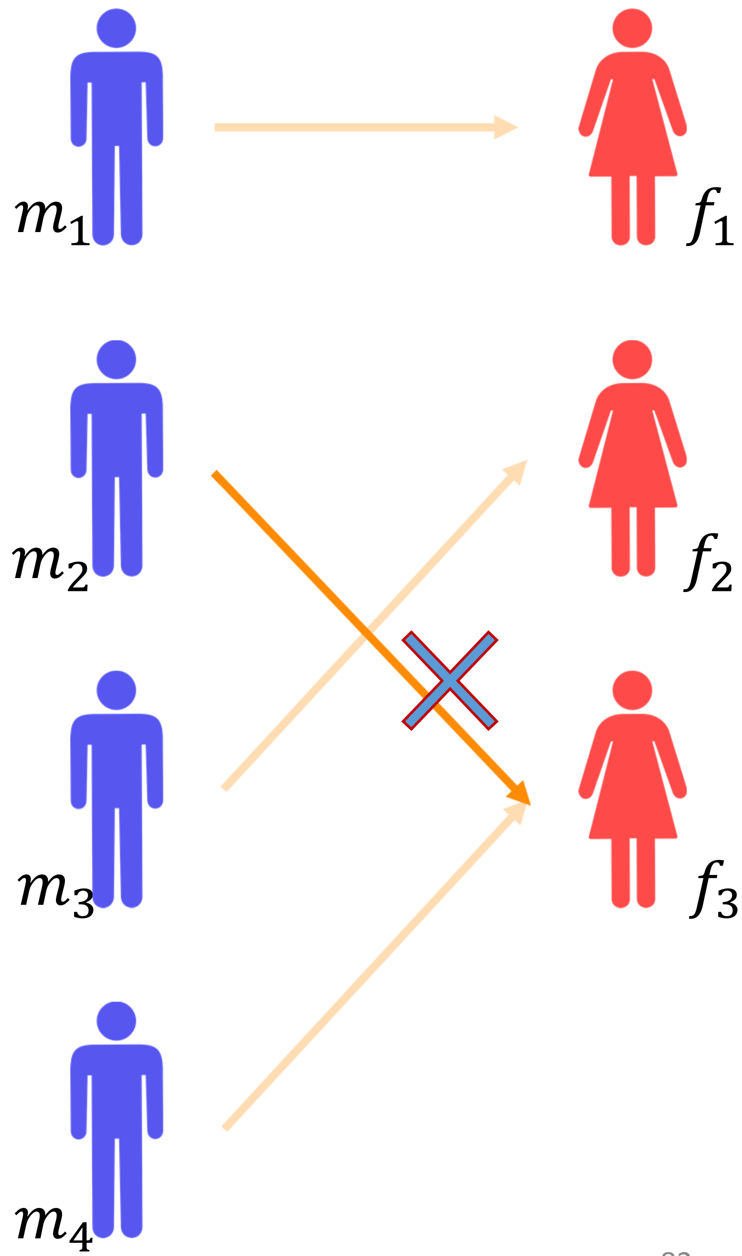
$m_3 \rightarrow f_2$

$m_4 \rightarrow f_2$ $m_4 \rightarrow f_3$

① $m_2$ proposes to his 3rd most-preferred woman

| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |



# Boston: Step 4

$m_1 \rightarrow f_1$

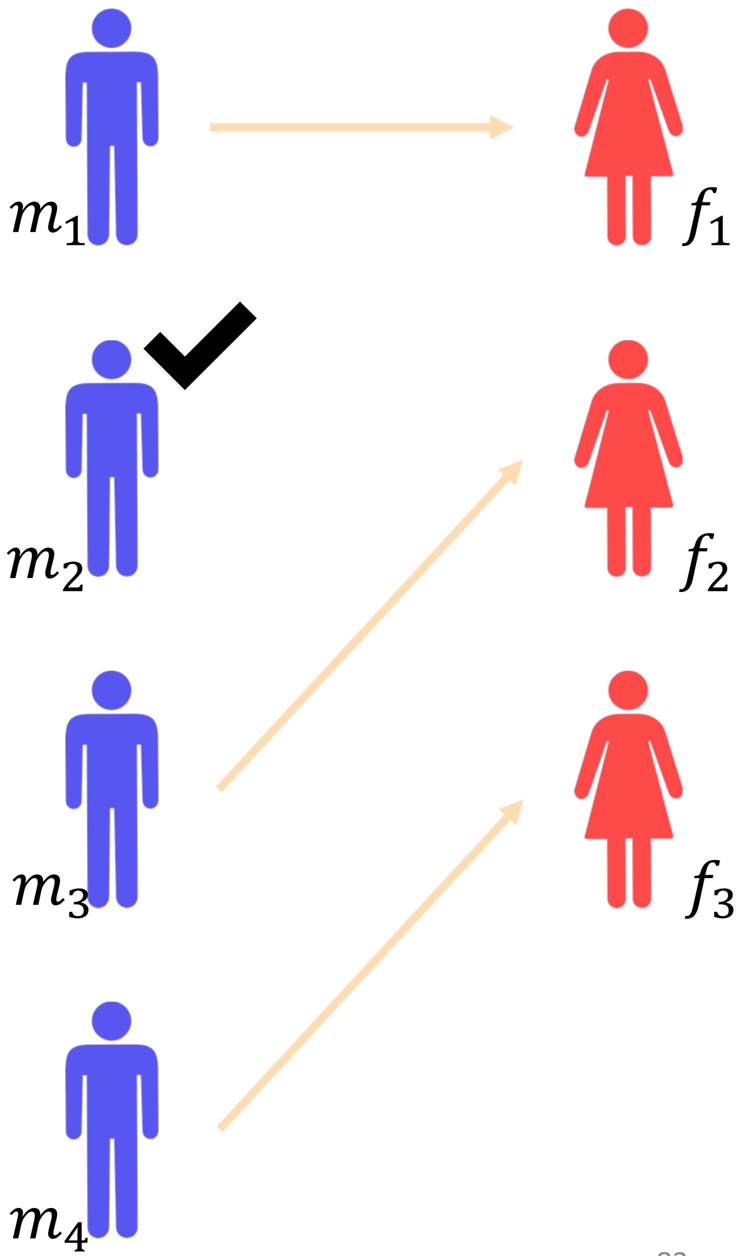
$m_2 \rightarrow f_1$ $m_2 \rightarrow f_2$ $m_2 \rightarrow f_3$ $m_2 \rightarrow m_2$

$m_3 \rightarrow f_2$

$m_4 \rightarrow f_2$ $m_4 \rightarrow f_3$

| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |



# Boston does not guarantee stability

- The result

| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |

$(m_2, f_2)$ is a blocking pair

- $m_2$ and $f_2$ have the incentive to deviate from the matching result

| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |

# Boston is not strategy-proof

| | 1st | 2nd | 3rd | 4th |
|---|---|---|---|---|
| $m_1$ | $f_1$ | $f_3$ | $f_2$ | $f_4$ |
| $m_2$ | $f_1$ | $f_2$ | $f_4$ | $f_3$ |
| $m_3$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ |
| $m_4$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ |

true preference

| | 1st | 2nd | 3rd | 4th |
|---|---|---|---|---|
| $m_1$ | $f_1$ | $f_3$ | $f_2$ | $f_4$ |
| $m_2$ | $f_1$ | $f_2$ | $f_4$ | $f_3$ |
| $m_3$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ |
| $m_4$ | $f_2$ | $f_1$ | $f_3$ | $f_4$ |

claimed preference

$m_4$ is better off by lying about its preference

# Deferred Acceptance (DA) Algorithm

- In [GS62], they developed deferred acceptance algorithm to solve the marriage problem.
- It ensures a stable matching.
- Each one that receives a proposal only "tentatively" accepts.
- That is, some proposer may be rejected later.

# DA: Step 1/4

$\boxed{m_1 \rightarrow f_1}$

$m_2 \rightarrow f_1$

$\boxed{m_3 \rightarrow f_2}$

$m_4 \rightarrow f_2$

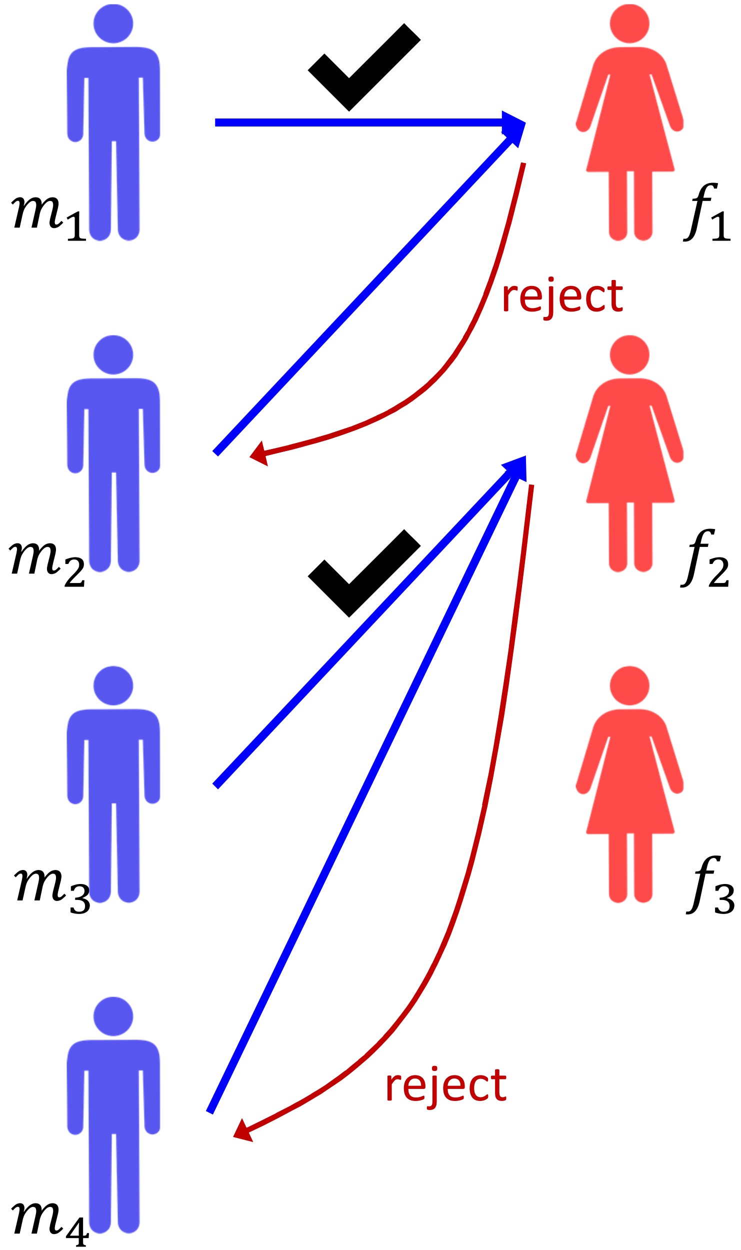
① Every man proposes to his most preferred woman

| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

② Each woman accepts the most-preferred one

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |

multiple proposals received



# DA: Step 2/4

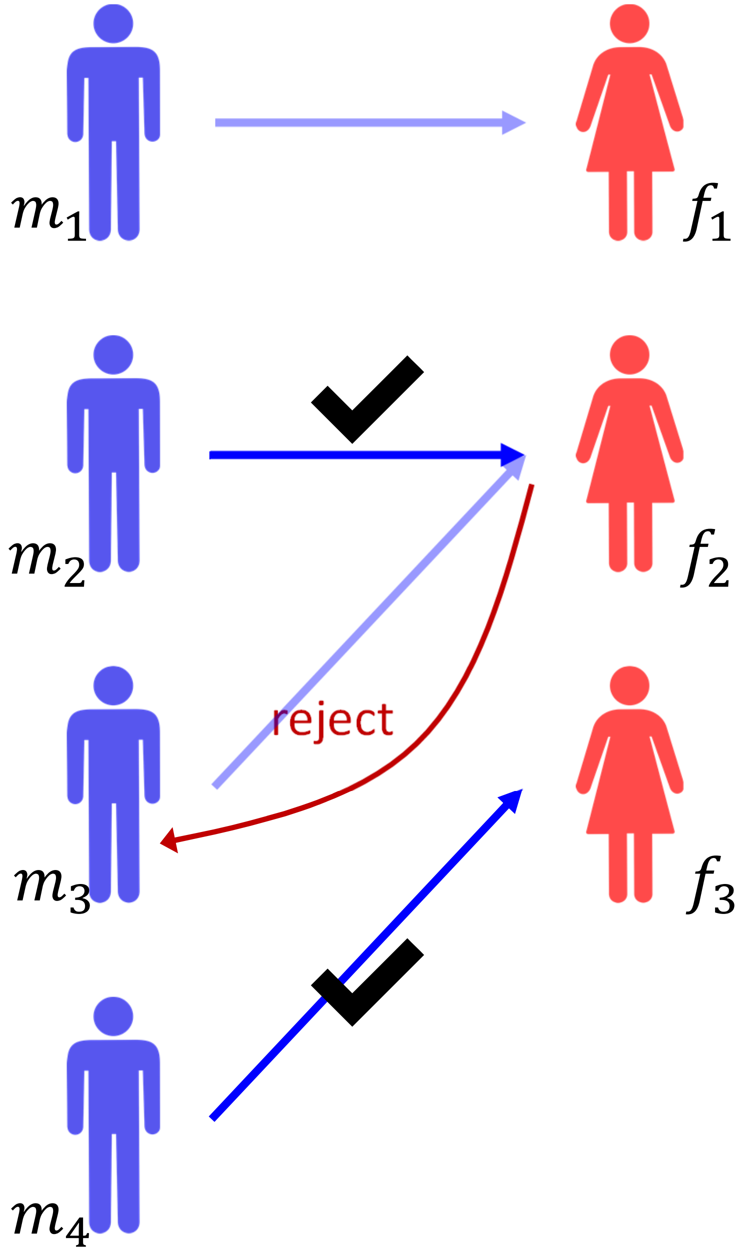
$\boxed{m_1 \to f_1}$ ③ $f_2$ prefers $m_2$ to $m_3$

$m_2 \to f_1$ $\boxed{m_2 \to f_2}$

$\boxed{m_3 \to f_2}$

$m_4 \to f_2$ $\boxed{m_4 \to f_3}$

① $m_2$ and $m_4$ propose to their 2nd most-preferred woman




| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |

② $f_2$ receives a new proposal

# DA: Step 3/4

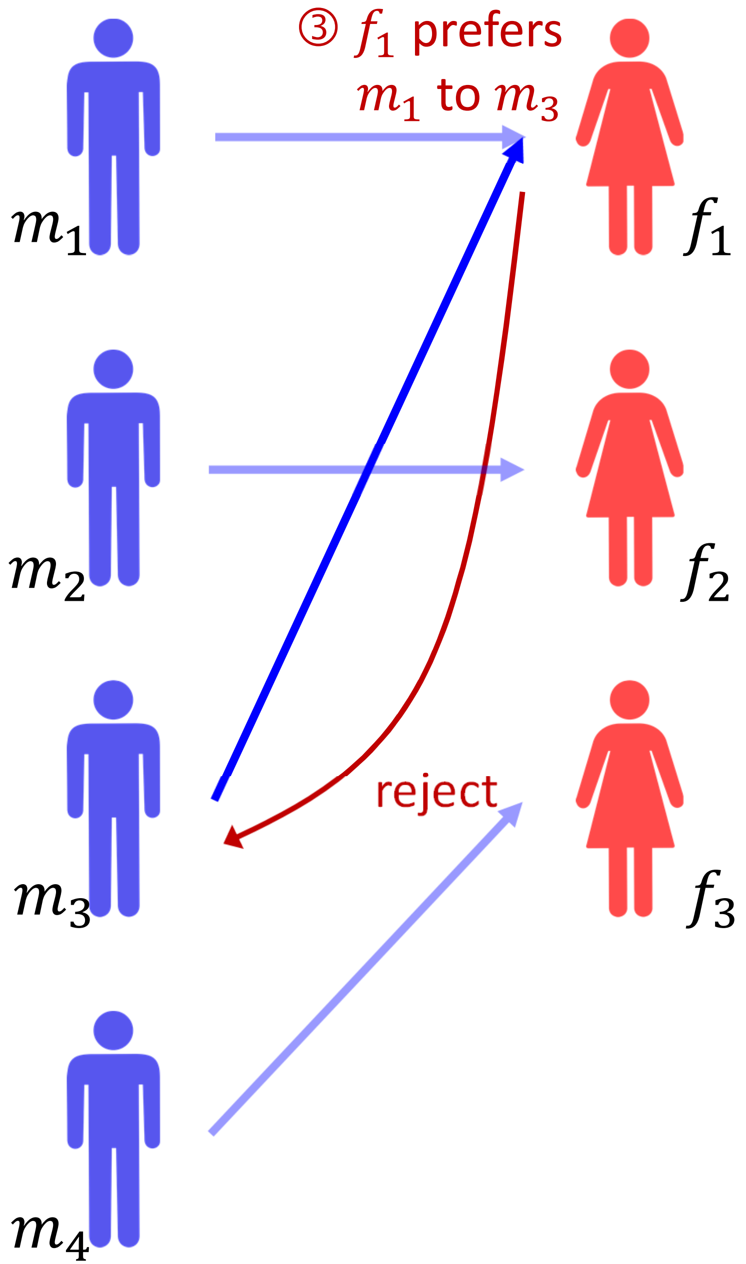
$\boxed{m_1 \rightarrow f_1}$

$m_2 \rightarrow f_1$ $\boxed{m_2 \rightarrow f_2}$

$m_3 \rightarrow f_2$ $m_3 \rightarrow f_1$

$m_4 \rightarrow f_2$ $\boxed{m_4 \rightarrow f_3}$

① $m_3$ proposes to his 2nd most-preferred woman




| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

② $f_1$ receives a new proposal

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |

# DA: Step 4/4

$\boxed{m_1 \rightarrow f_1}$

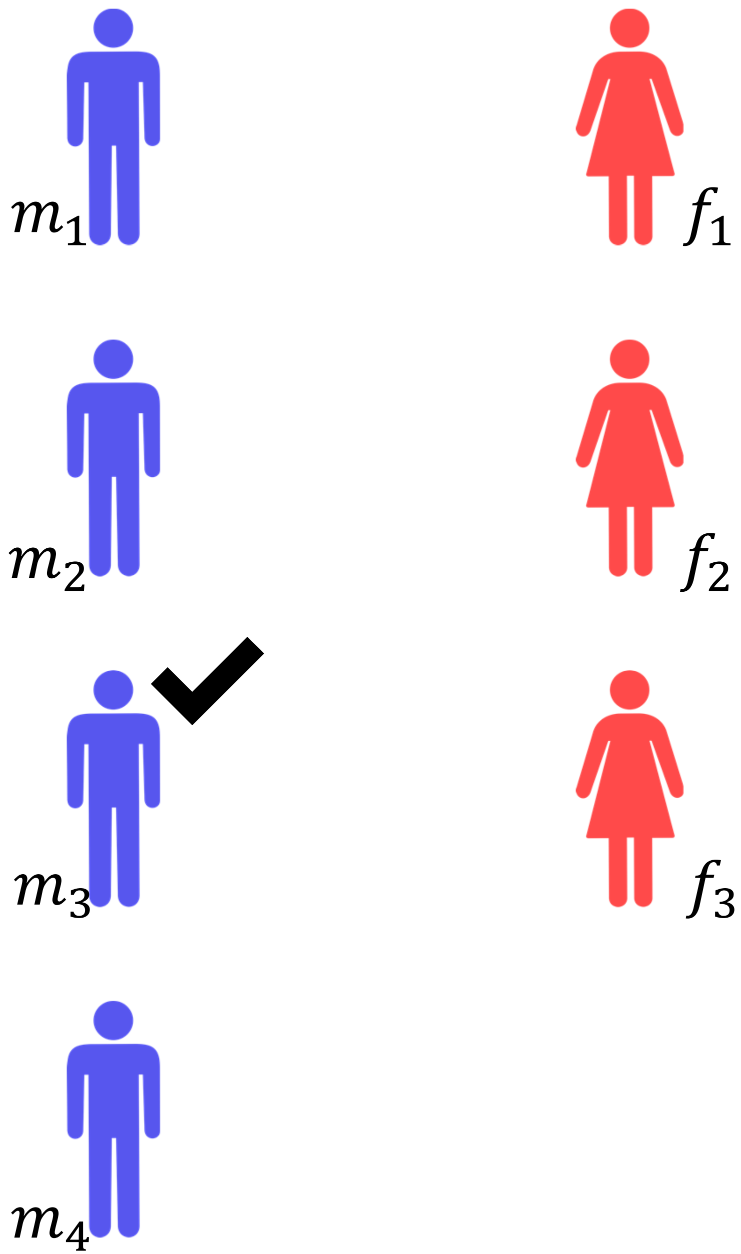
$m_2 \rightarrow f_1$ $\boxed{m_2 \rightarrow f_2}$

$m_3 \rightarrow f_2$ $m_3 \rightarrow f_1$ $\boxed{m_3 \rightarrow m_3}$

$m_4 \rightarrow f_2$ $\boxed{m_4 \rightarrow f_3}$

| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |



# Proof: Outcome of DA is Stable

- Algorithm must end in a finite number of rounds.
- Suppose *m, f* are matched, but *m* prefers *f'*.
  - At some point, *m* proposed to *f'* and was rejected.
  - At that point, *f'* preferred her tentative match to *m*.
  - As algorithm goes forward, *f'* can only do better.
  - So *f'* prefers her final match to *m.*
- Therefore, there are *NO BLOCKING PAIRS.*

# Optimal stable matchings

- A stable matching is *male-optimal* if every male prefers his partner to any partner he could possibly have in a stable matching.
- **Theorem.** The male-proposing DA algorithm results in a male-optimal stable matching.
  - It's impossible to improve any male's result without impairing the results of all other males (and the matching is still stable)

# Male-Optimal Stable Matching

- The above example is the DA algorithm proposed by male.
- every male prefers his partner to any partner he could possibly have in a stable matching.

| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |

# Stability vs. Pareto Efficiency

- A male-optimal stable matching is best for males given stability, but may not be Pareto efficient for the males.
- Example: $M = \{m_1, m_2, m_3\}; F = \{f_1, f_2\}$

$m_1: f_1 \succ f_2$ $\qquad$ $f_1: m_2 \succ m_3 \succ m_1$

$m_2: f_2 \succ f_1$ $\qquad$ $f_2: m_1 \succ m_3 \succ m_2$

$m_3: f_1 \succ f_2$

Stable but not Pareto efficient for males

$m_1: f_1 \succ f_2$ $\qquad$ $f_1: m_2 \succ m_3 \succ m_1$

$m_2: f_2 \succ f_1$ $\qquad$ $f_2: m_1 \succ m_3 \succ m_2$

$m_3: f_1 \succ f_2$

Pareto efficient for males but <u>not stable</u>

$(m_3, f_1)$ is a blocking pair

# Male-optimal & Female-optimal

- If the algorithm starts from men proposing, then it will achieve male-optimal stable.
  - Pareto optimal for males in all stable matchings
  - It's also female-pessimal (each woman gets worst outcome in any stable matching)
- If the algorithm starts from female proposing, then it will achieve female-optimal stable.
  - Pareto optimal for females in all stable matchings
  - It's also male-pessimal (each male gets worst outcome in any stable matching)

# DA (proposed by females): 1/2

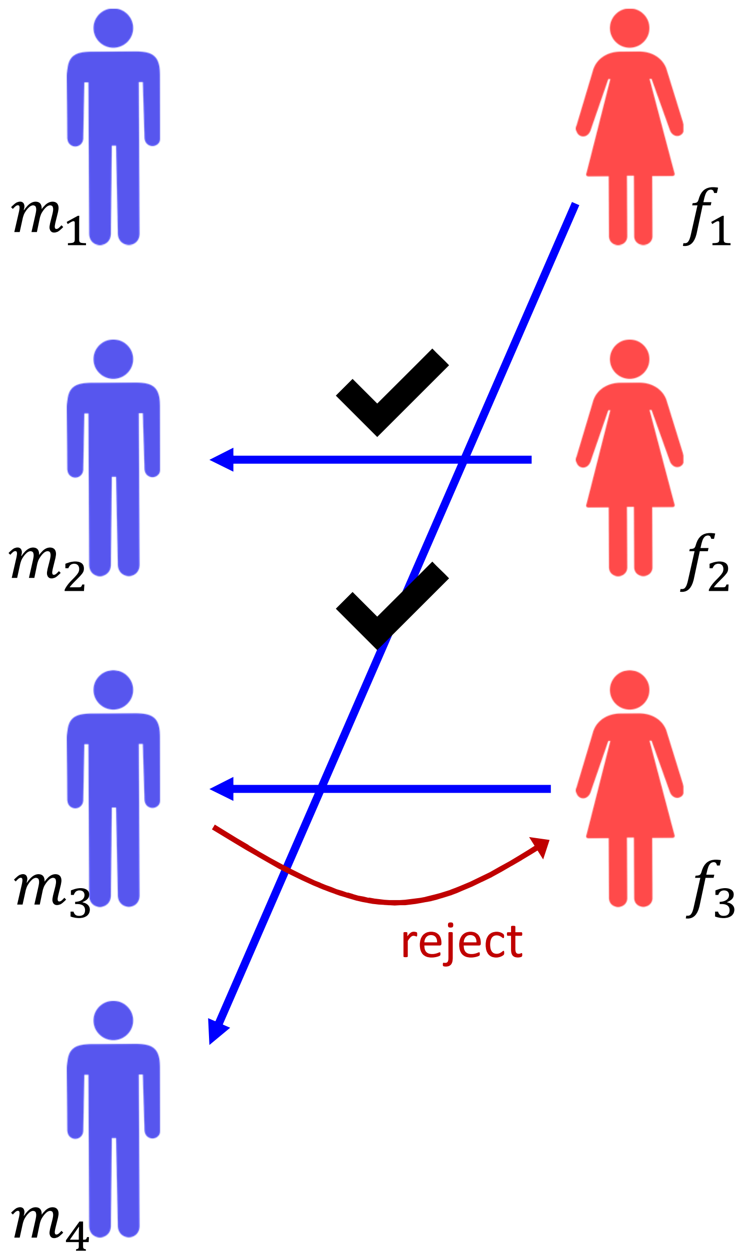
$\boxed{f_1 \rightarrow m_4}$

$\boxed{f_2 \rightarrow m_2}$

$f_3 \rightarrow m_3$

| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |



# DA (proposed by females): 2/2

$\boxed{f_1 \rightarrow m_4}$

$\boxed{f_2 \rightarrow m_2}$

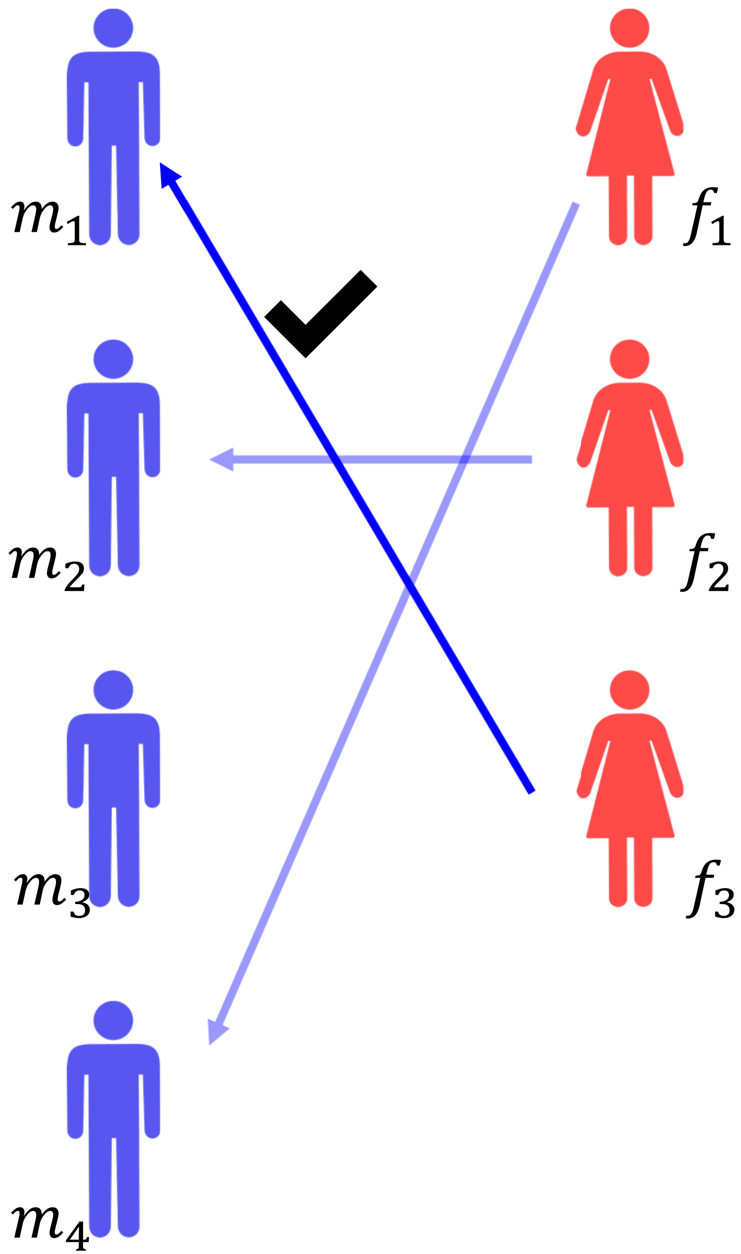
$f_3 \rightarrow m_3$ $\boxed{f_3 \rightarrow m_1}$

① $f_3$ proposes to her 2nd most-preferred woman

not Pareto optimal for males

| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |



# Female-Optimal Stable Matching

- The outcome is female-optimal stable, but not Pareto efficient for all females

Female-optimal stable matching

| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |

Pareto optimal for all females

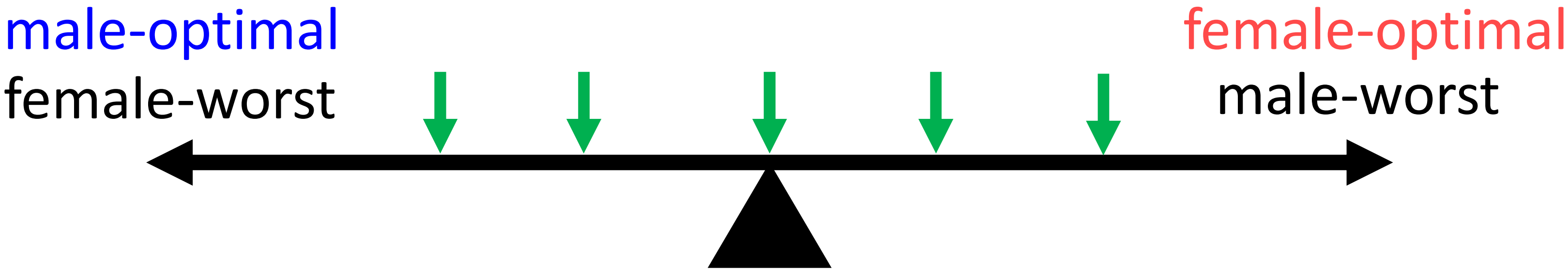
| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

Not stable

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |

# How many stable matchings?

- Besides the male-optimal and female-optimal matchings, there exist other matchings that are also stable.




Some other papers find the “egalitarian” stable matching.

# Truthfulness

- We may ask if we can lie about our preferences to get a better matching result.
- ~~Some research has shown the following theorem:~~
  - ~~No stable matching exists when it is the dominant strategy for every agent revealing its true preference.~~
- Also, some other research has shown that:
  - When the matching is induced by male-proposing DA, it is a dominant strategy for every male to reveal his true preference.
  - But how about women?

# If one woman lies (man proposing)

- If female $f_3$ lies about her preference as:
  - $m_3 \succ_{f_3} m_1 \succ_{f_3} m_2 \succ_{f_3} f_3 \succ_{f_3} m_4$
  - Then both $f_1$ and $f_3$ can get a better matching result.

| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$ |

| Male | Preference |
|---|---|
| $m_1$ | $f_1 \succ f_2 \succ f_3 \succ m_1$ |
| $m_2$ | $f_1 \succ f_2 \succ f_3 \succ m_2$ |
| $m_3$ | $f_2 \succ f_1 \succ m_3 \succ f_3$ |
| $m_4$ | $f_2 \succ f_3 \succ f_1 \succ m_4$ |

| Female | Preference |
|---|---|
| $f_1$ | $m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$ |
| $f_2$ | $m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$ |
| $f_3$ | $m_3 \succ m_1 \succ m_2 \succ f_3 \succ m_4$ |

# Matching between Hospitals and Medical Interns/Residents

- Many-to-one matching
- Two sided-preference

[Rot84] A.E. Roth, "The evolution of the labor market for medical interns and residents: a case study in game theory," *Journal of Political Economy*, 92: 991-1016, 1984.

# College Admissions Problem

- Many-to-one matching
- Two sided-preference

# College Admission Problem

- Many-to-one matching
  - a college can accept more than one students
  - The number of students that can be accepted is limited by its capacity
  - a student can only be accepted by one college
- Two-sided preference
  - Students have preference over colleges
  - Students may prefer not accepting some colleges
  - Colleges have preference over students
  - Colleges may prefer not accepting some students

# Blocking Individual

- For a particular matching,
  - a student is a blocking individual if she/he is matched with some college that she/he prefers not accepting
  - a college is a blocking individual if it is matched with some student that it prefers not accepting
- The existence of a blocking individual makes the matching unstable

# Blocking Pair

- For a particular matching $\mu$,
  - If there is another matching $v$ such that some student $s$ prefers $v(s)$ to $\mu(s)$, $(s, v(s))$ is an acceptable pair, and
    - $v(s)$ does not yet accept the max number of students in $\mu$, or
    - (if $v(s)$ already accepts the max number of students in $\mu$) $v(s)$ prefers $s$ to some student matched by $\mu$
  - Then $(s, v(s))$ is a blocking pair
- The existence of a blocking individual makes the matching unstable

# Blocking Pair Example

- Suppose each college can admit two students

| Student | Preference |
|---|---|
| $s_1$ | $c_2 \succ c_1$ |
| $s_2$ | $c_1 \succ c_2 \succ c_3$ |
| $s_3$ | $c_1 \succ c_2 \succ c_3$ |
| $s_4$ | $c_1 \succ c_2 \succ c_3$ |

| College | Preference |
|---|---|
| $c_1$ | $s_1 \succ s_3 \succ s_2 \succ s_4$ |
| $c_2$ | $s_3 \succ s_2 \succ s_1 \succ s_4$ |
| $c_3$ | $s_3 \succ s_2 \succ s_1 \succ s_4$ |

$(s_2, c_1)$ is a blocking pair

Note: there is no Pareto improvement

# (Pairwise) Stability

- A matching $\mu$ is stable if it is not blocked by any individual or any pair.

| Student | Preference |
|---|---|
| $s_1$ | $c_2 \succ c_1$ |
| $s_2$ | $c_1 \succ c_2 \succ c_3$ |
| $s_3$ | $c_1 \succ c_2 \succ c_3$ |
| $s_4$ | $c_1 \succ c_2 \succ c_3$ |

| College | Preference |
|---|---|
| $c_1$ | $s_1 \succ s_3 \succ s_2 \succ s_4$ |
| $c_2$ | $s_3 \succ s_2 \succ s_1 \succ s_4$ |
| $c_3$ | $s_3 \succ s_2 \succ s_1 \succ s_4$ |

This matching is stable

# The Core

- Stable marriage problem is a special case of the stable college admission problem with capacity of each college equal to 1
- the core of this problem is non-empty [Rot84]
  - We can always find a result that is both individual rational and stable
- Particularly, an algorithm can find a core that is best for all the colleges and worst for all the students

# Responsive Preference

- college may have preferences over groups of students (e.g., to build a football team)
- A college's preference list is responsive if its preference list is over the "individual" of the students in college admissions problem.
  - With responsive preference list, if two matchings differ only in one student in the college's matching, then the college prefers the matching containing the student with a higher preference.
- If we want to directly use DA, we should ensure that all college's preferences are responsive.

# An Example Where Preferences Are Not Responsive

Outcome by DA

| Student | Preference |
|---|---|
| $s_1$ | $c_2 \succ c_1$ |
| $s_2$ | $c_1 \succ c_2$ |
| $s_3$ | $c_1$ |
| $s_4$ | $c_1$ |

| College | Preference |
|---|---|
| $c_1$ | $s_1 s_2 \succ s_3 s_4$ |
| $c_2$ | $s_3 s_4 \succ s_2$ |

Another matching $\mu'$

| Student | Preference |
|---|---|
| $s_1$ | $c_2 \succ c_1$ |
| $s_2$ | $c_1 \succ c_2$ |
| $s_3$ | $c_1$ |
| $s_4$ | $c_1$ |

| College | Preference |
|---|---|
| $c_1$ | $s_1 s_2 \succ s_3 s_4$ |
| $c_2$ | $s_3 s_4 \succ s_2$ |

$s_1, s_2, c_1$ can be all better off

# Stability When Preferences Are Not Responsive

- $\mu$ is not stable but there is no blocking pair in $\mu$
- For responsive preferences, matching not blocked by any individual or any pair $\Rightarrow$ stable matching
- If preferences are not responsive, we should look into coalitions (subset of agents) instead of pairs

$\mu$

| Student | Preference |
|---|---|
| $s_1$ | $c_2 \succ c_1$ |
| $s_2$ | $c_1 \succ c_2$ |
| $s_3$ | $c_1$ |
| $s_4$ | $c_1$ |

| College | Preference |
|---|---|
| $c_1$ | $s_1 s_2 \succ s_3 s_4$ |
| $c_2$ | $s_3 s_4 \succ s_2$ |

# Blocking Coalition

- A blocking coalition of a matching $\mu$ is $(C', S', \mu')$, where $C' \subseteq C$, $S' \subseteq S$, and $\mu' \neq \mu$ such that
  - $C' \cup S' \neq \emptyset$
  - $\mu'(s) \subseteq S'$ for all $s \in C'$
  - $\mu'(s) \in C' \cup \{s\}$ for all $s \in S'$
  - $\mu'(s) \succcurlyeq_s \mu(s)$ for all $s \in C' \cup S'$
  - $\mu'(s) \succ_s \mu(s)$ for some $s \in C' \cup S'$
- $(\{c_1\}, \{s_1, s_2\}, \mu')$ in our example is a blocking coalition
- $\mu$ is (setwise) stable if it is not blocked by any coalition

# DA algorithm to college admissions

- When the colleges have responsive preferences, there may exist a matching that all colleges strictly prefer the college-optimal stable matching.

| College | Preference |
|---|---|
| $c_1$ | $s_1 \succ s_2 \succ s_3 \succ s_4 \succ c_1$ |
| $c_2$ | $s_1 \succ s_2 \succ s_3 \succ s_4 \succ c_2$ |
| $c_3$ | $s_3 \succ s_1 \succ s_2 \succ s_4 \succ c_3$ |

| Student | Preference |
|---|---|
| $s_1$ | $c_3 \succ c_1 \succ c_2 \succ s_1$ |
| $s_2$ | $c_2 \succ c_1 \succ c_3 \succ s_2$ |
| $s_3$ | $c_1 \succ c_3 \succ c_2 \succ s_3$ |
| $s_4$ | $c_1 \succ c_2 \succ c_3 \succ s_4$ |

# Comparison

- Student-optimal & College-optimal
  - Student-optimal: No strictly preferred matching
  - College-optimal: Exists a preferred matching
- Truthfulness:
  - For student-optimal, it is a dominant strategy for every student to reveal its true preference.
  - However, for college-optimal, no stable matching algorithm for every college to reveal its true preference.

# Many-to-Many Matching

- the number of allowable matches for the agents in both sides of the matching is unrestricted
- Consider a collection of firms and consultants.
  - Each firm wishes to hire a set of consultants, and each consultant wishes to work for a set of firms.
- Firms have preferences over the possible sets of consultants
- Consultants have preferences over the possible sets of firms

# Example: Firms and Consultants

- Set of workers (consultants): $W = \{w_1, w_2, w_3\}$
- Set of firms: $F = \{f_1, f_2, f_3\}$
- Worker's preferences

$$\left.\begin{aligned} &w_1: f_3 \succ f_2 f_3 \succ f_1 f_3 \succ f_1 \succ f_2 \\ &w_2: f_1 \succ f_1 f_3 \succ f_1 f_2 \succ f_2 \succ f_3 \\ &w_3: f_2 \succ f_1 f_2 \succ f_2 f_3 \succ f_3 \succ f_1 \end{aligned}\right\}$$

Not responsive preference

- Firm's preferences

$$\left.\begin{aligned} &f_1: w_3 \succ w_2 w_3 \succ w_1 w_3 \succ w_1 \succ w_2 \\ &f_2: w_1 \succ w_1 w_3 \succ w_1 w_2 \succ w_2 \succ w_3 \\ &f_3: w_2 \succ w_1 w_2 \succ w_2 w_3 \succ w_3 \succ w_1 \end{aligned}\right\}$$

Not responsive preference

# A Possible Matching of The Example

$w_1 : f_3 \succ \boxed{f_2 f_3} \succ f_1 f_3 \succ f_1 \succ f_2$
$w_2 : f_1 \succ \boxed{f_1 f_3} \succ f_1 f_2 \succ f_2 \succ f_3$
$w_3 : f_2 \succ \boxed{f_1 f_2} \succ f_2 f_3 \succ f_3 \succ f_1$

Each worker is matched with 2 firms

$f_1 : w_3 \succ \boxed{w_2 w_3} \succ w_1 w_3 \succ w_1 \succ w_2$
$f_2 : w_1 \succ \boxed{w_1 w_3} \succ w_1 w_2 \succ w_2 \succ w_3$
$f_3 : w_2 \succ \boxed{w_1 w_2} \succ w_2 w_3 \succ w_3 \succ w_1$

Each firm is matched with 2 workers

# DA in Many-to-Many Matching

- Let $Ch(S, \succ_a)$ be agent $a$'s most-preferred subset of $S$ according to $a$'s preference relation $\succ_a$
- Let $F = \{f_1, f_2, \cdots, f_n\}$, $W = \{w_1, w_2, \cdots, w_m\}$
- Suppose firms in $F$ propose to workers in $W$
- In each round, each firm proposes to a set of workers that it prefers the most and can possibly hire (not rejected yet)
- Each worker selects a most-preferred set of proposals it has received to tentatively accept
- A proposal, once accepted, can be rejected later by the worker but cannot be dropped unilaterally by the firm

# Procedure for Firm's Proposing

Each $f_i \in F$ performs the follows actions:

1. $A_i \leftarrow \emptyset$; // accepted proposals
2. While $W \neq \emptyset$
3. $\quad P_i \leftarrow Ch(W, \succ_{f_i})$
4. $\quad$ If $A_i \subset P_i$ and $P_i \succ_{f_i} A_i$ then
5. $\quad\quad$ Proposes to each $w_j \in P_i \setminus A_i$
6. $\quad\quad$ $R_i \leftarrow$ the set of workers that rejects $f_i$'s proposal.
7. $\quad\quad$ $A_i \leftarrow A_i \cup P_i \setminus R_i$;
8. $\quad\quad$ If $A_i \cap R_i \neq \emptyset$ then $A_i \leftarrow A_i \setminus R_i$
9. $\quad\quad$ $W \leftarrow W \setminus R_i$
10. $\quad$ Else if some $f_i$'s previous proposal toward $w_j$ is rejected
11. $\quad\quad$ remove all such $w_j$'s from $A_i$ and $W$
12. $\quad$ End If
13. End while

# What's Wrong With Firm's Procedure?

- Assume $f_1$'s preference

$$w_1 w_2 w_3 \succ_{f_1} w_1 w_4 \succ_{f_1} w_1 \succ_{f_1} w_1 w_3$$

- Suppose that $f_1$ proposes to $w_1$, $w_2$, and $w_3$.
- If $w_2$ rejects $f_1$'s proposal, then $f_1$ is matched with $\{w_1, w_3\}$
- $f_1$ could also be better off if it could drop $w_3$
- Also, $f_1$ could have been matched with $w_4$ (if $w_4$ also prefers matching with $f_1$)

# Individual Rationality of Many-to-X Matching

- For each agent $a$, let $\succ_a$ be a preference relation.
- A matching $\mu$ is individually rational if and only if

$$\mu(a) = Ch(\mu(a), \succ_a) \text{ for all } a \in F \cup W.$$

- In the previous example,

$$\mu(f_1) = \{w_1, w_3\} \neq Ch(\{w_1, w_3\}, \succ_a) = \{w_1\}.$$

- This means there is at least an agent $a$ who prefers a proper subset $A \subset \mu(a)$ over $\mu(a)$
  - $a$ could be better off by not matching with $\mu(a) \setminus A$

# Pairwise Block

- Let $w \in W, f \in F$, and let $\mu$ be a matching.
- The pair $(w, f)$ is a pairwise block of $\mu$ if
  $w \notin \mu(f), w \in Ch(\mu(f) \cup \{w\}, \succ_f))$, and
  $f \in Ch(\mu(w) \cup \{f\}, \succ_w)$
- Example: $(f_1, w_4)$

$Ch(\{w_1, w_3\} \cup \{w_4\}, \succ_{f_1}) = \{w_1, w_4\}$

$f_1$: $w_1 w_2 w_3 \succ \boxed{w_1 w_4} \succ w_1 \succ \boxed{w_1 w_3}$

$w_4$: $\boxed{f_1 f_2} \succ \boxed{f_2 f_3}$

$Ch(\{f_2, f_3\} \cup \{f_1\}, \succ_{w_4}) = \{f_1, f_2\}$

# Pairwise Stability

- A matching $\mu$ is pairwise stable if it is individually rational and there is no pairwise block of $\mu$.
- Even if a matching is pairwise stable, there may exist a blocking coalition of size 3 or larger
- Example: $\{f_1, w_4, w_5\}$

$$Ch\left(\{w_1, w_3\} \cup \{w_4, w_5\}, \succ_{f_1}\right) = \{w_4, w_5\}$$

$$f_1:\ w_1 w_2 w_3 \succ \boxed{w_4 w_5} \succ w_1 \succ \boxed{w_1 w_3}$$

$$w_4: \boxed{f_1} \succ \boxed{f_2 f_3}$$

$$w_5: \boxed{f_1} \succ \boxed{f_2 f_3}$$

$$Ch\left(\{f_2, f_3\} \cup \{f_1\}, \succ_{w_5}\right) = \{f_1\}$$

# Procedure for Worker's Response

Each $w_j \in W$ performs the follows actions:

1. $A_j \leftarrow \emptyset$; // accepted proposals
2. Let $F_j$ be the set of firms that proposes to $w_j$.
3. While $F_j \neq \emptyset$
4. $\quad P_j \leftarrow Ch(A_j \cup F_j, \succ_{w_j})$
5. $\quad$ If $P_j \succ_{w_j} A_j$ then
6. $\quad\quad$ accept each $f_i \in P_j \setminus A_j$
7. $\quad\quad$ reject each $f_i \in F_j \setminus P_j$ and each $f_i \in A_j \setminus P_j$
8. $\quad\quad A_j \leftarrow P_j$
9. $\quad$ Else
10. $\quad\quad$ reject each $f_i \in F_j \setminus A_j$
11. $\quad$ End if
12. $\quad$ Let $F_j$ be the set of firms that proposes to $w_j$.
13. End While

# What's Wrong With Worker's Procedure?

- Assume $w_1$'s preference
$$f_3 f_4 \succ_{w_1} f_1 f_2$$
- Suppose that $w_1$ receives proposals from $f_1$, $f_2$, and $f_3$ in the first round.
- By the procedure $w_1$ accepts $f_1$, $f_2$ and rejects $f_3$
- Suppose that $w_1$ receives $f_4$'s proposal later. $w_1$ will reject it.
- In this case, $w_1, f_3,$ and $f_4$ form a blocking coalition.

# Blocking Coalition of Many-to-Many Matching

- A blocking coalition of a matching $\mu$ is $(W', F', \mu')$, where $W' \subseteq W$, $F' \subseteq F$, and $\mu' \neq \mu$ such that
  - $F' \cup W' \neq \emptyset$
  - $\mu'(s) \subseteq F' \cup W'$ for all $s \in F' \cup W'$
  - $\mu'(s) \succcurlyeq_s \mu(s)$ for all $s \in F' \cup W'$
  - $\mu'(s) \succ_s \mu(s)$ for some $s \in F' \cup W'$
- $\mu'$ is another matching among agents in $F' \cup W'$ so that every agent in $F' \cup W'$ is weakly better off and at least one of them is strictly better off
- We say that $(W', F', \mu')$ blocks $\mu$

# An Example of Blocking Coalition

- $W = \{\bar{w}, w_1, w_2, w_3, w_4\}, F = \{f_1, f_2, \bar{f}\}$
- Preferences and mapping $\mu$

$\bar{w}: f_1\bar{f} \succ \bar{f} \succ f_1$

$w_1: f_1 \succ f_2 \succ \bar{f}$

$w_2: f_1 \succ f_2 \succ \bar{f}$

$w_3: f_1 \succ f_2 \succ \bar{f}$

$w_4: f_1 \succ f_2 \succ \bar{f}$

$f_1: \bar{w}w_1 \succ w_1w_2$

$f_2: w_2w_3 \succ w_3w_4 \succ \bar{w}w_4$

$\bar{f}: \bar{w} \succ w_1 \succ w_2 \succ w_3 \succ w_4$

There exists a mapping $\mu'$ such that $(\{\bar{w}, w_1\}, \{f_1, \bar{f}\}, \mu')$ blocks $\mu$. Can you find it?

# An Example of Stable Matching

$w_1$: $f_3 \succ \boxed{f_2 f_3} \succ f_1 f_3 \succ f_1 \succ f_2$ $\quad$ $f_1$: $w_3 \succ \boxed{w_2 w_3} \succ w_1 w_3 \succ w_1 \succ w_2$

$w_2$: $f_1 \succ \boxed{f_1 f_3} \succ f_1 f_2 \succ f_2 \succ f_3$ $\quad$ $f_2$: $w_1 \succ \boxed{w_1 w_3} \succ w_1 w_2 \succ w_2 \succ w_3$

$w_3$: $f_2 \succ \boxed{f_1 f_2} \succ f_2 f_3 \succ f_3 \succ f_1$ $\quad$ $f_3$: $w_2 \succ \boxed{w_1 w_2} \succ w_2 w_3 \succ w_3 \succ w_1$

- To make $f_1$ better off, $f_1$ should hire only $w_3 \Rightarrow w_2$ is hired only by $f_3 \Rightarrow w_2$ is worse off
- If $f_1$ is in a coalition $C$, $w_3$ must be in $C$. Then $f_2$ must be in $C$, or $w_3$ would only be hired by $f_1$ and thus worse off.
- But $f_2$ in $C$ implies that $w_1$ must be in $C$. Then $f_3$ must be in $C$, so $w_2$ must be in $C$, a contradiction.

# Substitutability

- Let $Ch(S, \succ_a)$ be $a$'s most-preferred subset of $S$ according to $a$'s preference relation $\succ_a$
- An agent $a$'s preference relation $\succ_a$ satisfies substitutability if, for any sets $S$ and $S'$ of partners of $a$ with $S \subseteq S'$,

$$b \in Ch(S' \cup b, \succ_a) \text{ implies } b \in Ch(S \cup b, \succ_a)$$

- e.g.,

$f: w_1 w_2 w_3 w_4 \succ_f w_1 w_2 \succ_f w_1 w_4$ is not substitutable

Because $w_4 \in Ch(\{w_1, w_2, w_3, w_4\}, \succ_f)$ but $w_4 \notin Ch(\{w_1, w_2, w_4\}, \succ_f)$

# Example of Substitutable Preference

can switch

$$f: w_1w_2w_3 \succ \underbrace{\boxed{w_1w_2 \succ w_1w_3 \succ w_2w_3}}_{\text{order can be different}} \succ \underbrace{\boxed{w_1 \succ w_2 \succ w_3}}_{\text{order can be different}}$$

- no rejected proposal becomes desirable when some other proposal becomes available.
  - It's no regret to reject a proposal and substitute it with a better proposal
- substitutability is necessary for the existence of stable outcomes

# Questions

- If every agent’s preference is substitutable, does DA ensure pairwise stability?
- If every agent’s preference is substitutable, does DA ensure general stability (no blocking coalition)?