國立交通大學
資訊工程學系
Department of Computer Science

Matching Theory

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Li-Hsing Yen
CS, NCTU

Outline

- Introduction
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- One-to-one matching with one-sided preference
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- Many-to-one matching with two-sided preference
- Matching with Transfer
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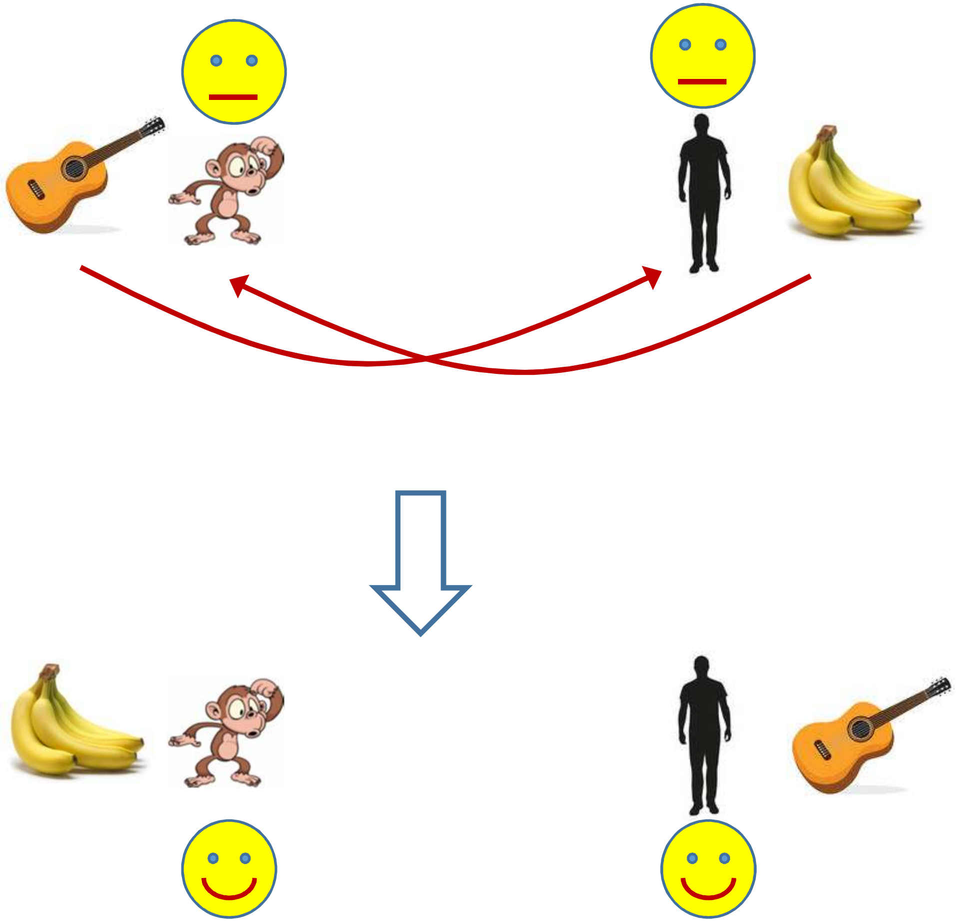
Introduction

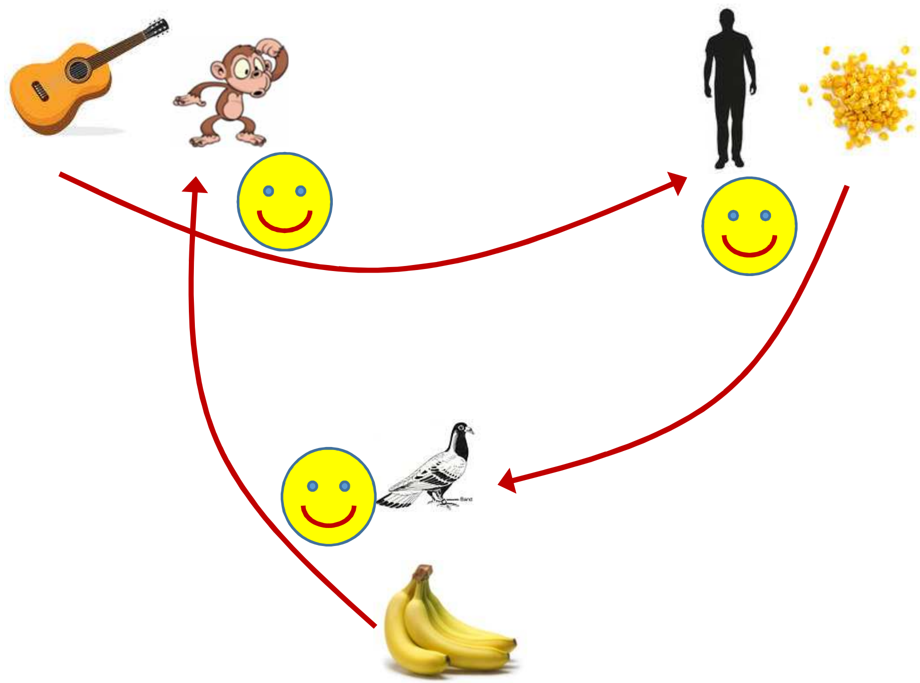
What this problem is about?

Pareto Improvement by Exchange



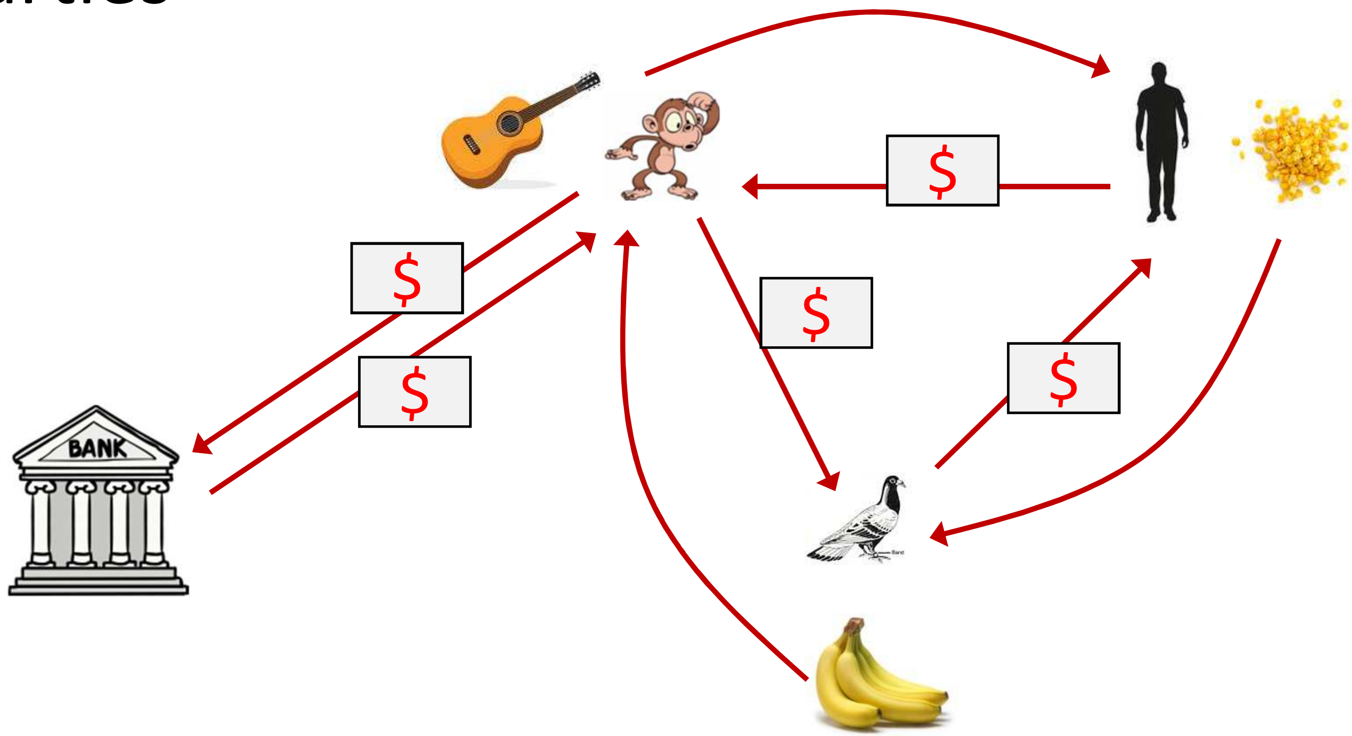
Pareto Improvement by Exchange Among Three or More Parties

- How to enable this type of exchange?



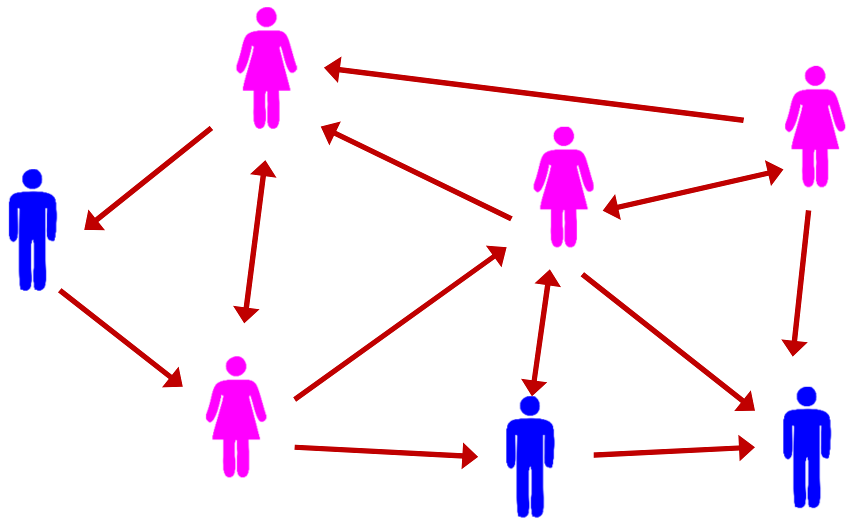
Introduction of Money

- enable item exchange among multiple parties



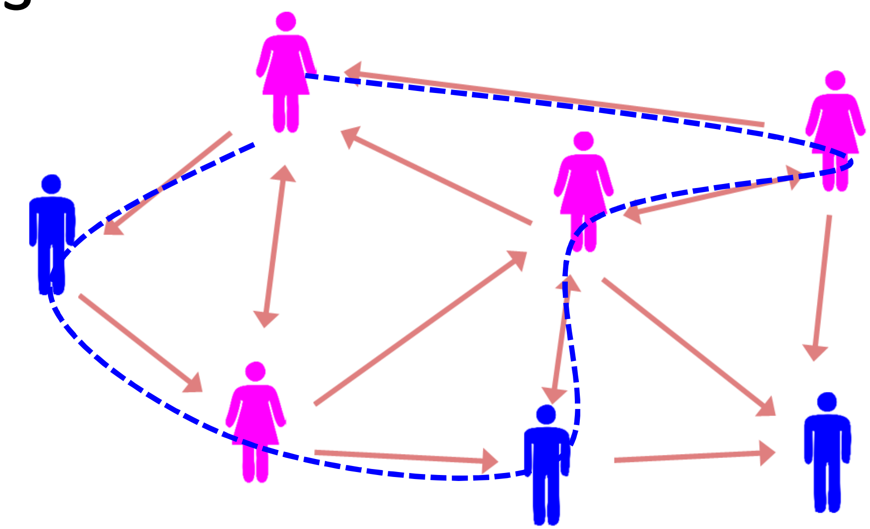
Exchange Without Money

- Some exchange does not allow the involvement of money
  - e.g., kidney exchange (for transplant)
- How to find possible Pareto improvement?



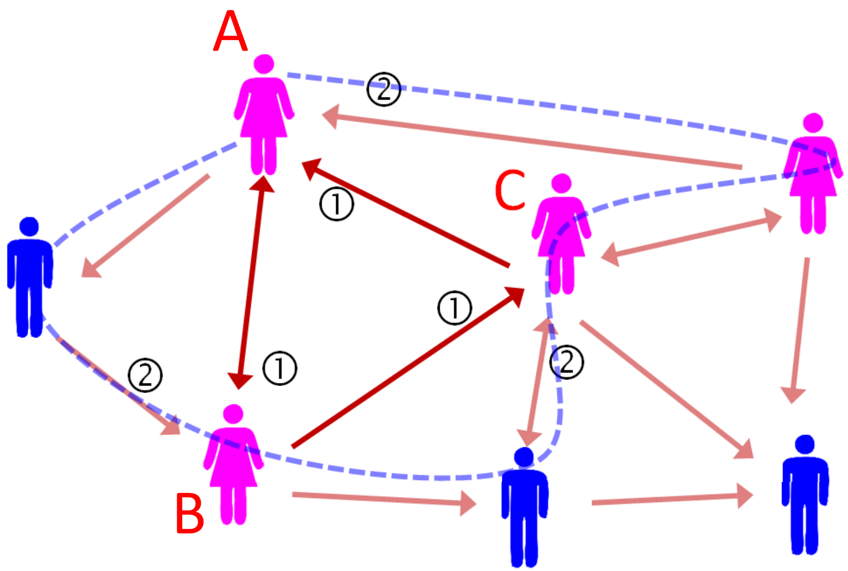
Pareto Improvement Involving Multiple Parties

- Find a (possibly longest) cycle in a directed graph
- All agents (parties) in the cycle get Pareto improvements



If Agents Have Preferences

- The cycle (A, B, C, A) gives A, B, C a better result
- A, B, C have the incentive to deviate from the result
- In this case, we say the result is not stable



Difference from Pareto optimality:

Stability concerns only a subset of (not all) agents. It doesn't care if other agent's results are worsened.

Exchange is a special case of Matching

- Exchange problem assumes that each agent holds something and wants something better by exchange
- In some cases, no item has been allocated to any agent initially
- matching problem is more general

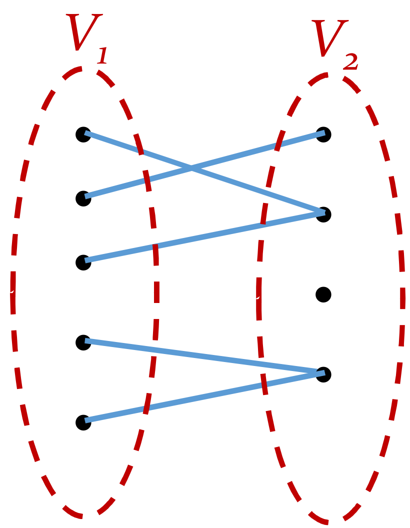
In CS: Bipartite Matching

Modeling matching on a bipartite graph

Bipartite Graphs

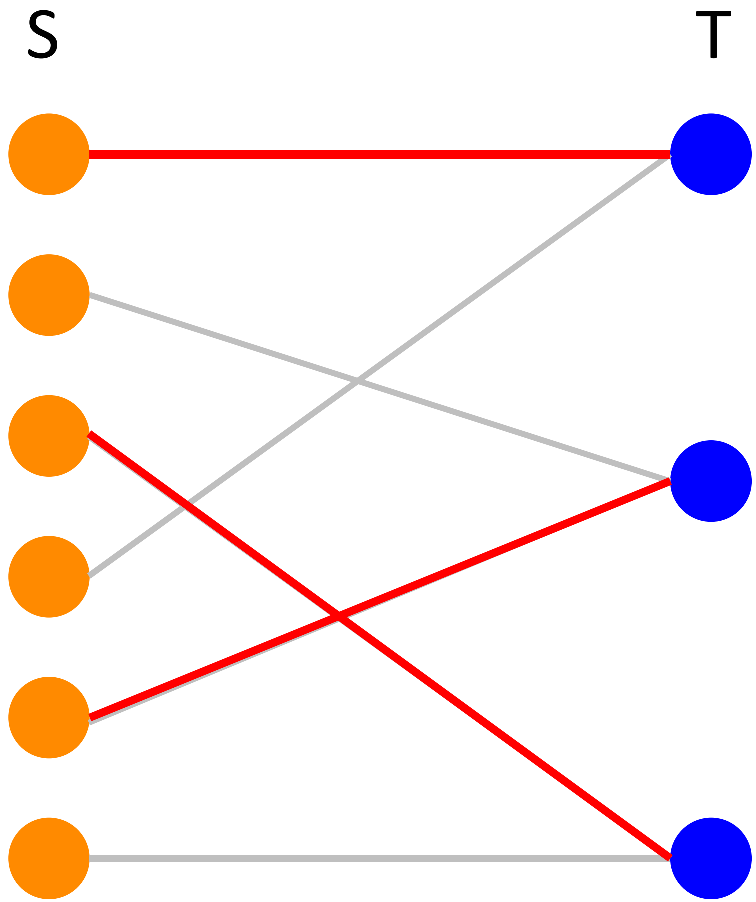
A simple graph $G=(V, E)$ is bipartite if we can find a way to partition V into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2.

In other words, there are no edges which connect two vertices in V_1 or in V_2.



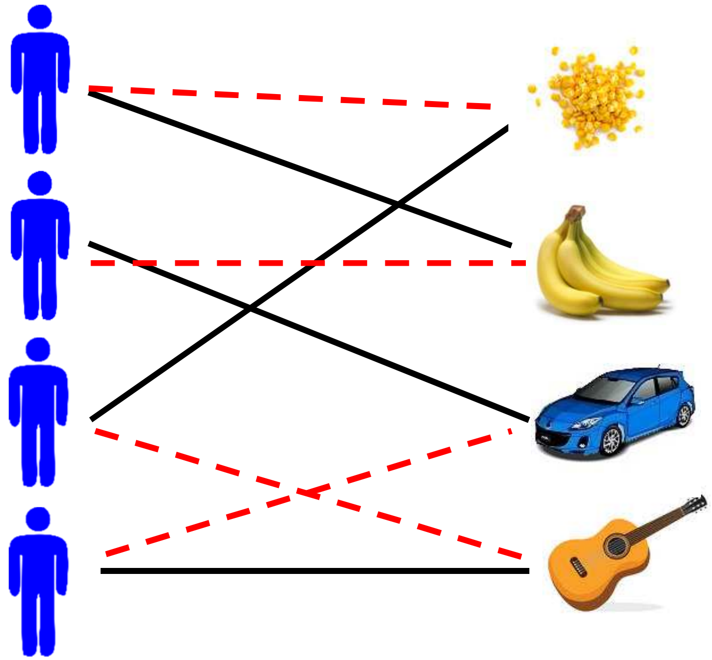
Bipartite Matching

- A vertex in one set can be matched with at most one vertex on the other set.
  - One-to-one matching
- Maximum Cardinality Bipartite Matching
  - Maximize the number of matchings



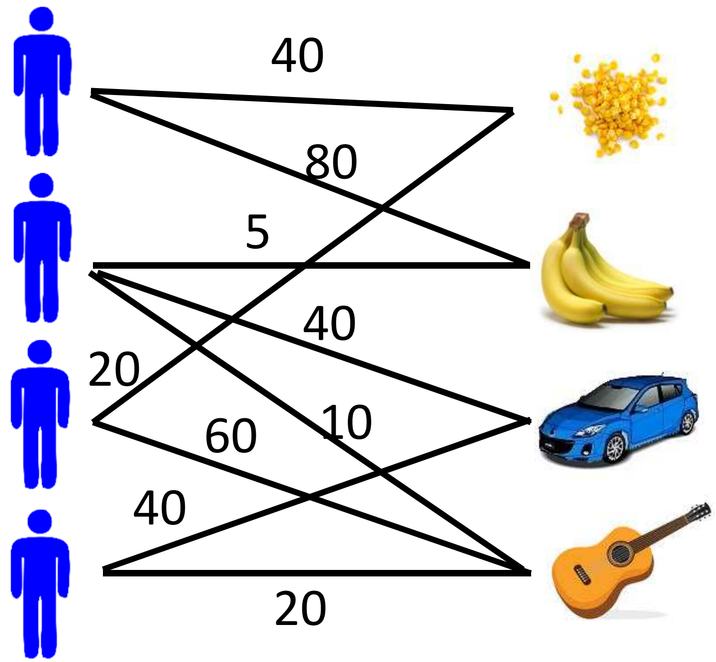
Matching Considering Preferences

- In max-cardinality matching, participants (agents) do not have preference over matchings
- In contrast, we assume that agents have preferences over matchings
- Any two matchings are equally good in terms of cardinality, but not if preferences are considered.



Representing Preference

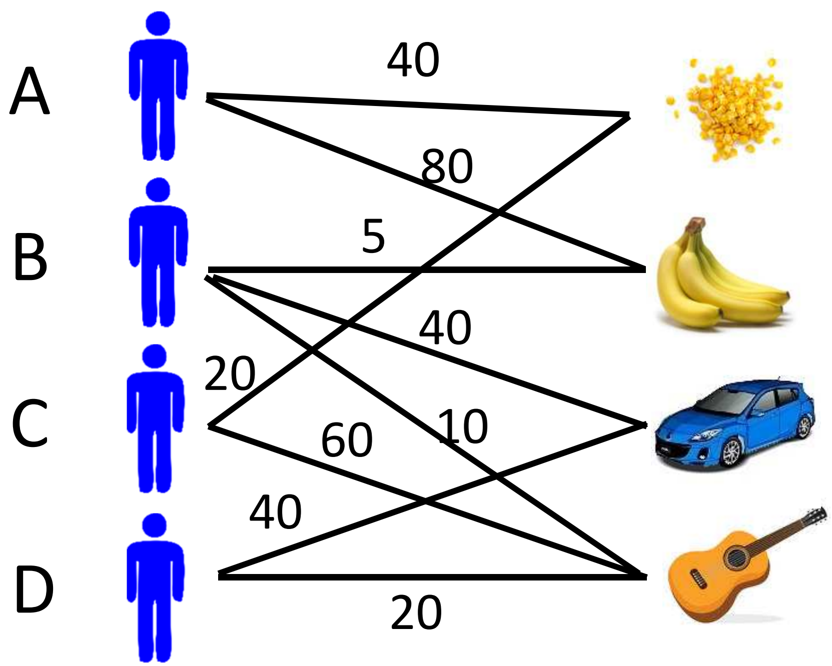
- We may label each edge with a weight to indicate the preference over that match
- Maximum Weight Bipartite Matching
  - Choose a set of one-to-one matchings that maximizes the total weight



Application: [LWZ17] Lei et al., "A semi-matching based load balancing scheme for dense IEEE 802.11 WLANs," *IEEE Access*, 5:15332-15339, July 2017.

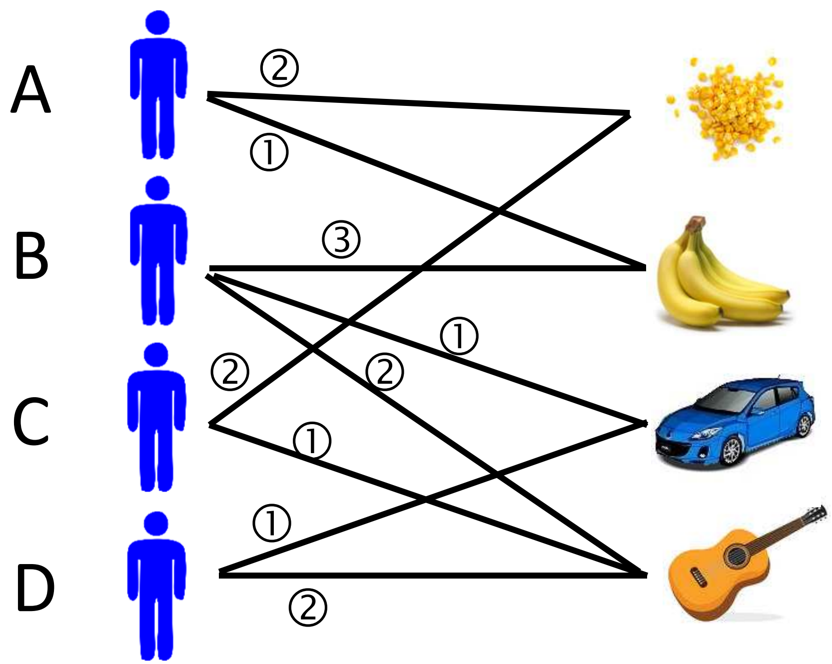
Problem with Weight-Based Preference

- Max-weight matching cares A's preference more than B's
- In most cases, we treat every agent equally



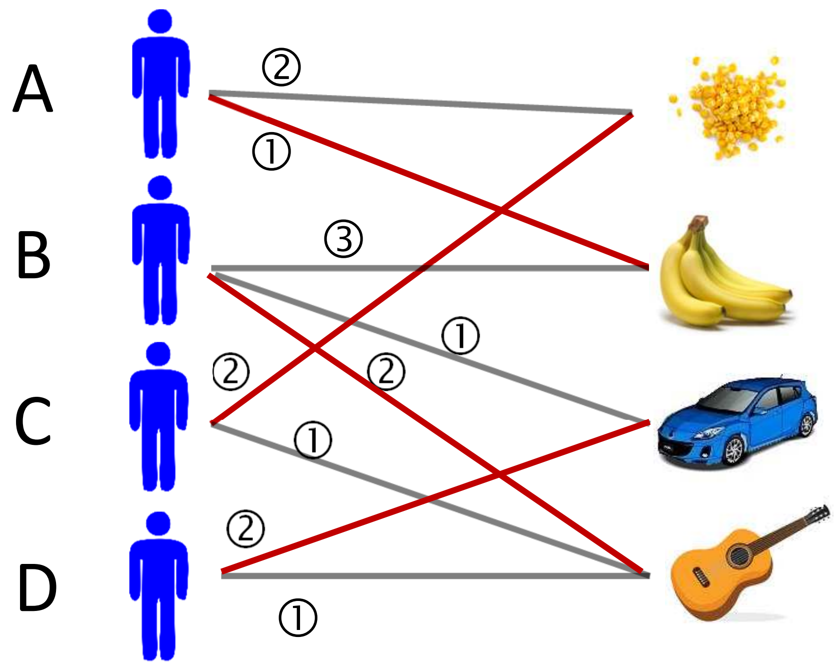
From Weights To Preference

- Hereafter, we change weights (cardinal) to preferences (ordinal)



Stability in Matching

- If B and D exchange their allocations, both can be better off
- B and D have the incentive to deviate from the matching result
- The matching result is not stable!



Stable Result May Not Exist

- Consider a group of students {A, B, C, D} to be matched to roommates, two in each room.
- Student’s preferences
  - A prefers B>C>D
  - B prefers C>A>D
  - C prefers A>B>D
  - No stable match exists: whoever is paired with D wants to change and can find a willing partner.
- So stability of matching may not exist, even if each match involves just two people.

One-Sided Preference

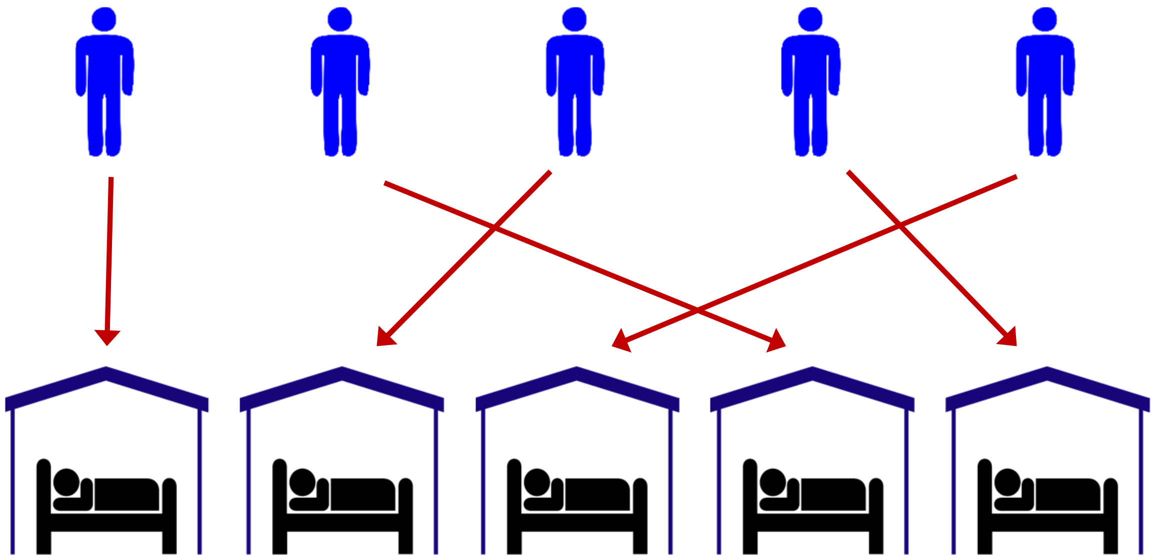
exchange, switch

One-Sided Preference

- Match people with items
- Or match people with another bunch of people that are totally fine with the matching result
- One-to-one matching
  - Housing Market Problem
  - House Allocation with Existing Tenants
- Many-to-one matching
  - Ca’pacitated House Allocation

House Allocation Problem

One-to-one matching with one-sided preference



House Allocation Problem $(A, H, \succ)$

- Assumption
  - a set of agents A
  - a set of individual objects (houses) H
  - a preference profile $(\succ_a)_{a \in A}$: a list of preference relations of agents over houses (strict total order)

Agent	1st prep.	2nd prep.	3rd prep.	4th prep.	
a_1	h_1	h_2	h_3	h_4	$h_1 \succ_{a_1} h_2$
a_2	h_1	h_3	h_2	h_4	
a_3	h_1	h_2	h_3	h_4	
a_4	h_1	h_3	h_2	h_4	$h_3 \succ_{a_4} h_2$
a_5	h_4	h_1	h_2	h_3	$h_4 \succ_{a_5} h_3$

Matching as a function

- The outcome of the housing allocation problem $(A, H, \succ)$ is a matching $\mu : A \to H$
  - Each agent a is allocated the house $\mu(a)$
- M: the set of all possible matchings
- Preference relations over matchings. Let $u, v \in M$

$$\left.\begin{aligned} &\mu \succ_a v \leftrightarrow \mu(a) \succ_a v(a) \\ &\mu \succcurlyeq_a v \leftrightarrow v \not\succ_a \mu \\ &\mu \sim_a v \leftrightarrow \mu(a) = v(a) \end{aligned}\right\}$$

The relation defines a weak total order on M

Pareto Improvement in Matching

	1st	2nd	3rd	4th
a_1	h_4	h_3	h_2	h_1
a_2	h_3	h_4	h_2	h_1
a_3	h_2	h_4	h_1	h_3
a_4	h_3	h_2	h_1	h_4

	1st	2nd	3rd	4th
a_1	h_4	h_3	h_2	h_1
a_2	h_3	h_4	h_2	h_1
a_3	h_2	h_4	h_1	h_3
a_4	h_3	h_2	h_1	h_4

a_1's and a_2's results are improved without degrading any other's result

$\Rightarrow$ a Pareto improvement

Pareto Domination & Pareto Efficiency

- Pareto domination
  - Suppose μ, ν are matchings. Then μ Pareto dominates ν if and only if

    (1) $\mu \succcurlyeq_a \nu$ for all $a \in A$,

    (2) $\mu \succ_a \nu$ for some $a \in A$.
- Pareto efficiency
  - a matching μ is Pareto efficient iff it is not Pareto dominated by any matching $\nu \in M$.

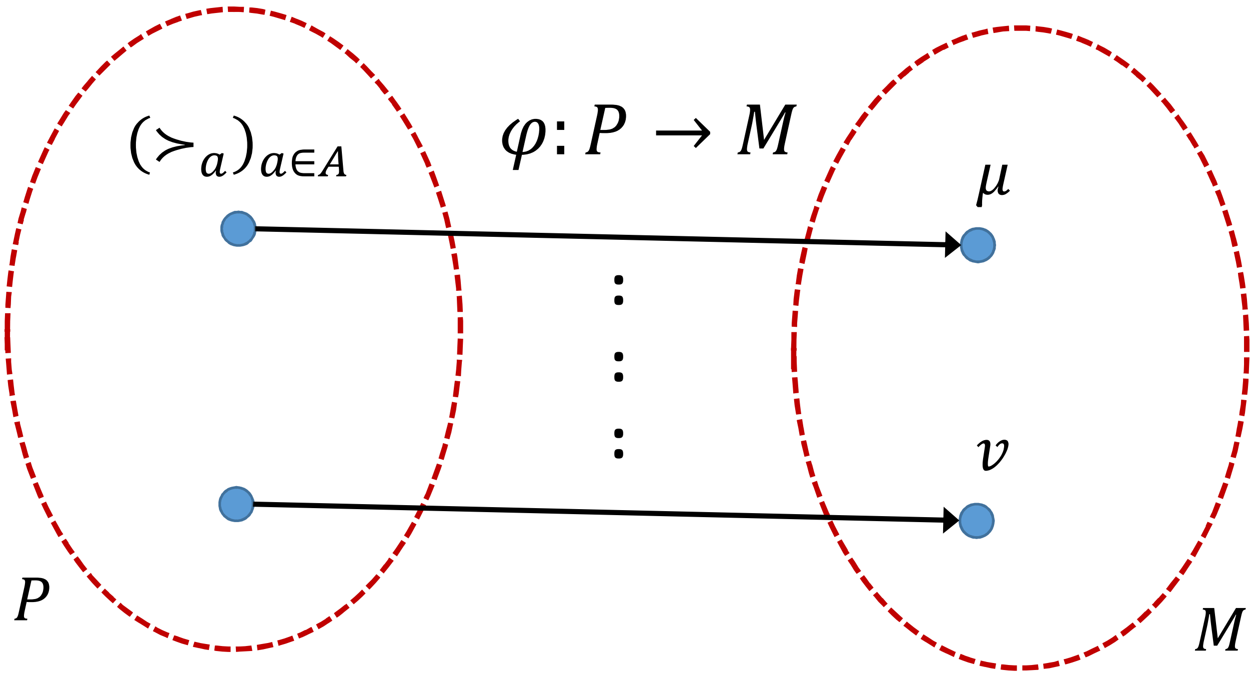
Matching Mechanism

- Let P denote the set of all preference profiles of all agents over houses ($P = \{(\succ_a)\}_{a \in A}$)
- Let M denote the set of all matchings of agents to houses
- A matching mechanism is a procedure for determining a matching given a housing allocation problem.
- Formally, matching mechanism is a function

$$\varphi: P \rightarrow M$$

Pareto Efficient Matching Mechanism

- A mechanism is Pareto efficient if it always produces a matching that is Pareto efficient on the announced preference profile.



Truthfulness May Not Be The Best Strategy Under Some Mechanism

	1st	2nd	3rd	4th
m_1	h_1	h_3	h_2	h_4
m_2	h_1	h_2	h_4	h_3
m_3	h_1	h_2	h_3	h_4
m_4	h_1	h_2	h_3	h_4

true preference

	1st	2nd	3rd	4th
m_1	h_1	h_3	h_2	h_4
m_2	h_1	h_2	h_4	h_3
m_3	h_1	h_2	h_3	h_4
m_4	h_2	h_1	h_3	h_4

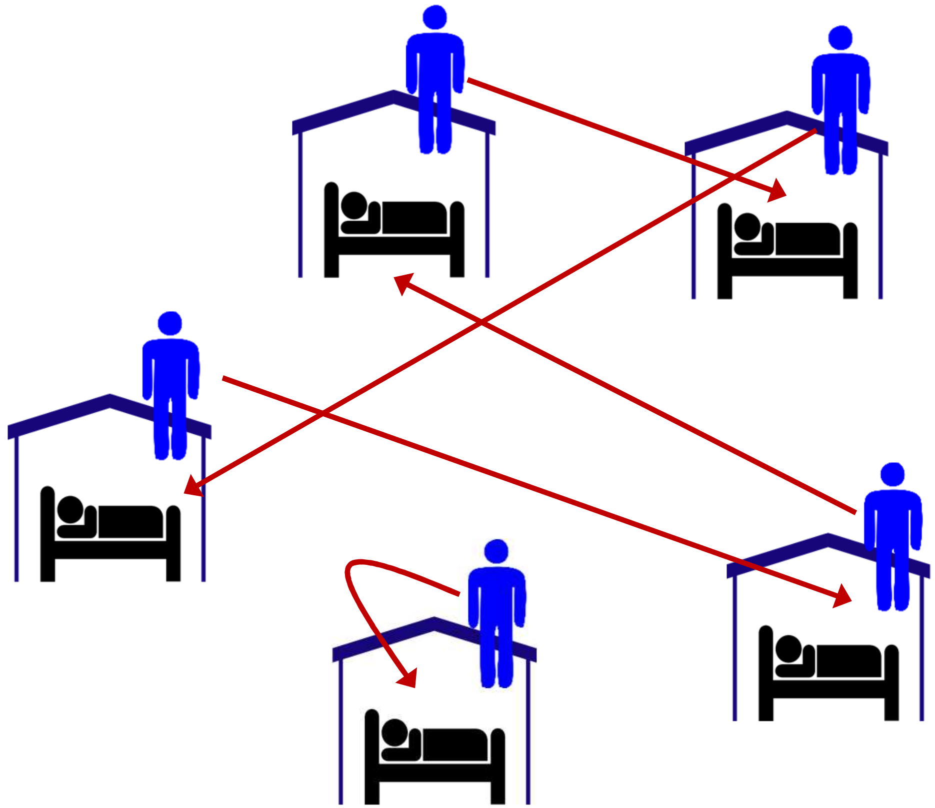
claimed preference

m_4 is better off by lying about its preference

Strategy-Proof Matching Mechanism

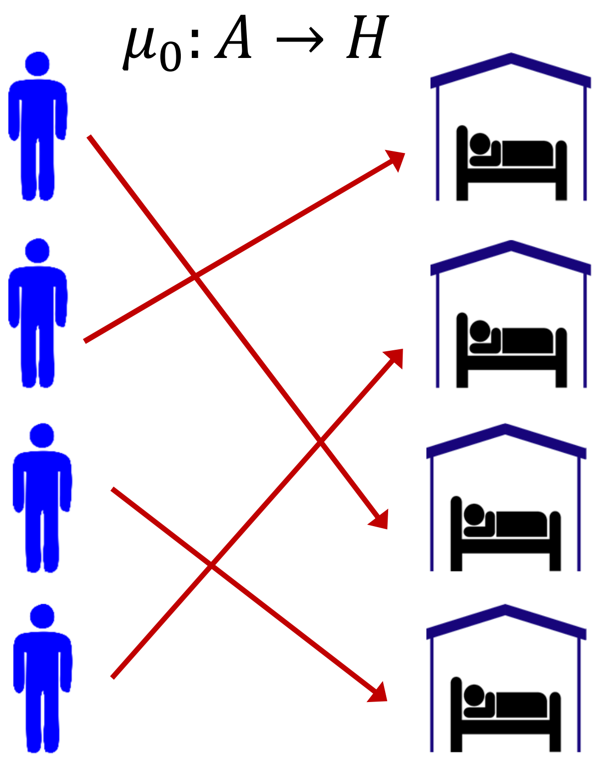
- Agents may lie about their preferences
- Suppose φ is a matching mechanism that induces agents to announce the preference profile $\rho \in P$.
- Let ρ_a be agent a's true preference over houses.
- Then φ is strategy proof if and only if every agent $a \in A$ weakly prefers its allocation (by φ) when a choose ρ_a over its allocation (by φ) when a chooses some other preference relation, regardless of the preference relations of all other agents in A.

Housing Market Problem



Housing Market Problem

- House allocation problem with initial allocation $\mu_0 : A \to H$, a bijection (we assume that $|A| = |H|$)



Individually Rational

- Suppose μ is a matching resulting from the housing market problem $(A, H, \succ, \mu_0)$.
- Then μ is individually rational if $\mu(a) \succcurlyeq_a \mu_0(a)$ for all $a \in A$.

No agent can be worse off by participating in the matching

Individually Rational: An Example

$\mu_0(a_i) = h_i$ for $i \in \{1, 2, 3,4\}$

	1st	2nd	3rd	4th
a_1	h_4	h_3	h_2	h_1
a_2	h_3	h_4	h_2	h_1
a_3	h_2	h_4	h_1	h_3
a_4	h_3	h_2	h_1	h_4

More Then One Matchings Can be Pareto Efficient

Matching μ

	1st	2nd	3rd	4th
a_1	h_4	h_3	h_2	h_1
a_2	h_3	h_4	h_2	h_1
a_3	h_2	h_4	h_1	h_3
a_4	h_3	h_2	h_1	h_4

Matching v

	1st	2nd	3rd	4th
a_1	h_4	h_3	h_2	h_1
a_2	h_3	h_4	h_2	h_1
a_3	h_2	h_4	h_1	h_3
a_4	h_3	h_2	h_1	h_4

Both are individually rational and Pareto efficient

Blocking Coalition

- In matching μ, if agent a_2 and a_3 do not participate in the matching and simply exchange their houses, agent a_3 can be better off (and a_2 gets the same room anyway)

Matching μ

	1st	**2nd**	**3rd**	**4th**
a_1	h_4	h_3	h_2	h_1
a_2	h_3	h_4	h_2	h_1
a_3	h_2	h_4	h_1	h_3
a_4	h_3	h_2	h_1	h_4

Such a "coalition" blocks the matching

A Pareto efficient matching can be unstable!

Blocking Coalition: Formal Definition

- In a housing market problem $(A, H, \succ, \mu_0)$
- A coalition $A' \subseteq A$ is said to block matching μ if there is a matching ν such that

$(1)\ v(a) \in \{\mu_0(b) | b \in A'\} \quad \forall a \in A'$

v allocates every $a \in A'$ a house initially owned by some $b \in A'$

$(2)\ v(a) \succcurlyeq_a \mu(a) \quad \forall a \in A'$

every $a \in A'$ weakly prefers its allocation by v to that by μ

$(3)\ \exists a \in A' \text{such that}\ v(a) \succ_a \mu(a)$

some $a \in A'$ strictly prefers its allocation by v to that by μ

Blocking Pairs

- It is difficult to know whether a matching has a blocking coalition
- If we are only concerned with blocking coalitions of size two, things will become much easier
- Blocking pairs are blocking coalitions of size two

The (Strong) Core in this problem

- The (strong) core of a housing market problem $(A, H, \succ, \mu_0)$ is a set of matchings C
- a matching $\mu \in M$ is in the (strong) core C if there exists no coalition $A' \subseteq A$ that can block μ

μ: not in the core

	1st	2nd	3rd	4th
a_1	h_4	h_3	h_2	h_1
a_2	h_3	h_4	h_2	h_1
a_3	h_2	h_4	h_1	h_3
a_4	h_3	h_2	h_1	h_4

v: in the core

	1st	2nd	3rd	4th
a_1	h_4	h_3	h_2	h_1
a_2	h_3	h_4	h_2	h_1
a_3	h_2	h_4	h_1	h_3
a_4	h_3	h_2	h_1	h_4

Is there any weak core?

- Yes.
- In a strong core, there exists no blocking coalition that could make all its members at least as good as and at least one member better off.
- In a weak core, there exists no coalition $A' \subseteq A$ that can redistribute the houses they own such that they all prefer the houses resulting from the reallocation.
  - In the previous example, matching μ is in the weak core.

The Implications of Strong Core

- Any matching in the strong core implies individual rationality
  - Because individual rationality is a special case of the strong core (when $|A'| = 1$)
- Any matching in the strong core implies Pareto optimality
  - Because Pareto optimality is a special case of the strong core (when $A' = A$)

Properties of the (Strong) Core

- The housing market problem as a non-empty (strong) core [SS74]
  - i.e., the (strong) core is not an empty set
- there is only one unique matching in the (strong) core [RP77]

[SS74] L. Shapley and H. Scarf, "On cores and indivisibility," *Journal of Mathematical Economics*, 1, pp. 23–37, 1974.

[RP77] A. E. Roth and A. Postlewaite, "Weak versus strong domination in a market with indivisible goods," *Journal of Mathematical Economics*, 4, pp. 131–137, 1977.

Gale's Top Trading Cycles (TTC) Algorithm [SS74]

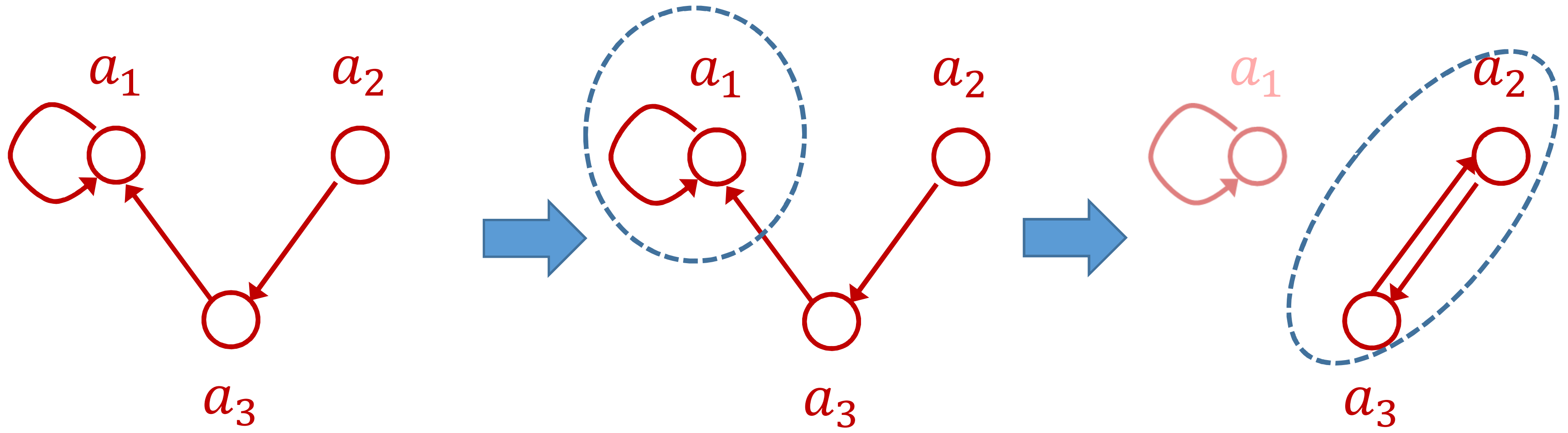
- Each agent points to the owner of its most preferred house.
- If a cycle of agents exists, then match all agents in the cycle with the house of the agent it points to.
- Remove the matched agents and houses from the problem
- each unmatched agent points to the owner of its most preferred remaining house and repeats the above procedure.

[SS74] L. Shapley and H. Scarf, "On cores and indivisibility," *Journal of Mathematical Economics*, 1, pp. 23–37, 1974.

Gale's TTC Algorithm: An Example

$a_1\colon h_1 \succ_{a_1} h_2 \succ_{a_1} h_3$
$a_2\colon h_3 \succ_{a_2} h_1 \succ_{a_2} h_2$
$a_3\colon h_1 \succ_{a_3} h_2 \succ_{a_3} h_3$

$\mu_0(a_i) = h_i$ for $i \in \{1, 2, 3\}$



Each agent points to the owner of its most preferred house.

a cycle of agents exists

points to the owner of the most preferred remaining house

Properties of Gale's TTC Algorithm

- Gale's TTC algorithm terminates with a matching
- The outcome of Gale's TTC algorithm is the unique matching in the core of each housing market.
- A mechanism that provides the matching in the core is the only mechanism that is Pareto efficient, individually rational, and strategy-proof. [Ma94]
    - Therefore, Gale's TTC algorithm is also strategy-proof.

[Ma94] J. Ma, "Strategy-proofness and the strict core in a market with indivisibilities," *International Journal of Game Theory*, 23(1):75-83, 1994.

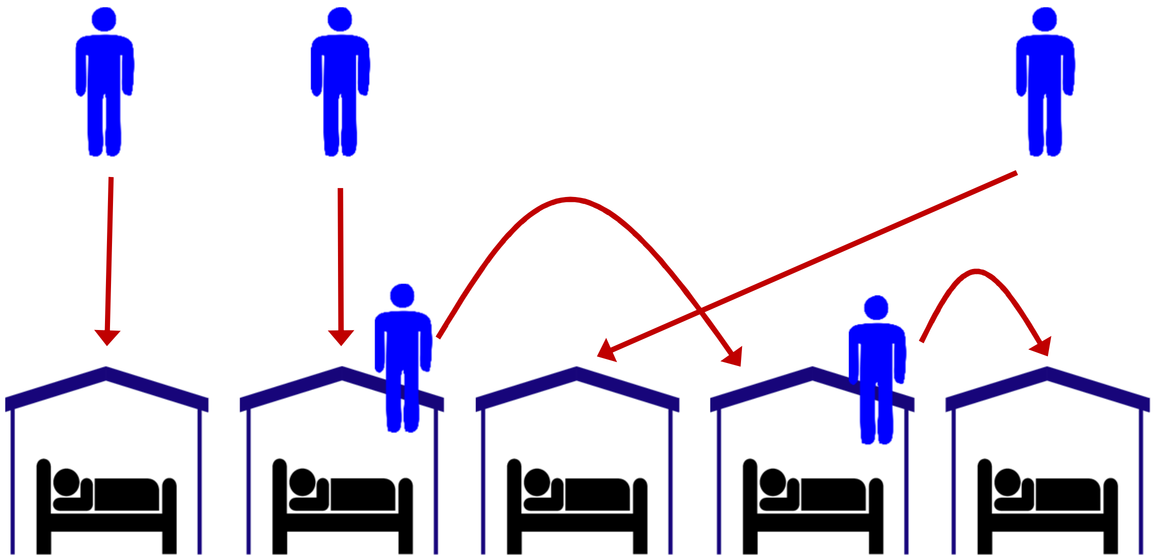
Practice: What is the Core?

$\mu_0(a_i) = h_i$ for $i \in \{1, 2, 3, 4, 5, 6, 7\}$

	1st	2nd	3rd	4th	5th	6th	7th
a_1	5	6	7	1	2	3	4
a_2	3	4	5	6	7	1	2
a_3	4	5	2	7	1	3	6
a_4	1	2	3	4	5	6	7
a_5	4	5	2	3	6	7	1
a_6	7	1	2	3	4	5	6
a_7	1	7	4	5	6	3	2

House Allocation With Existing Tenants

One-to-one, one-sided preference



House Allocation With Existing Tenants

- Some (but not all) houses already have tenants
- Consisting of a tuple $(A_e, A_n, H_o, H_v, \succ, \mu_0)$
  - A_e: the set of existing agents (who begins with a house)
  - A_n: the set of new agents (who begins without a house)
  - H_o: the set of occupied houses with $|H_o| = |A_e|$
  - H_v: the set of vacant houses
  - $\succ$: preference profile
  - $\mu_0: A_e \to H_o$ is a bijection
- We use $A = A_e \cup A_n$ and $H = H_o \cup H_v \cup \{h_0\}$
  - h_0: a *null* house for agents without real allocations

Agent Priority and Other Assumptions

- Agents have priorities
  - e.g., senior students have priorities over junior ones
  - can also be randomly determined
- Defined as a bijection function $f: \{1, 2, \cdots, |A|\} \rightarrow A$.
- f assigns a ranking to each agent
  - $f(1)$ has the highest priority
- every agent in A is assigned exactly one house
- only h_0 may be assigned to more than one agents

ψ_f: TTC for HAP with Existing Tenants (Step 1) [AS99]

- Each agent $a \in A$ points to its favorite house
- Each house $h \in H_o$ points to $\mu_0^{-1}(h)$ (the tenant)
- Each house $h \in H_v$ points to $f(1)$
- If a cycle (of alternating agents and houses) exists, then assign each agent the house that it points to.
- Remove the matched agents and houses from the problem
- If there are remaining agents and houses, then continue to the next step.

[AS99] A. Abdulkadiroğlu and T. Sönmez, "House allocation with existing tenants," *Journal of Economic Theory*, 88, pp. 233–260, 1999.

ψ_f: TTC for HAP with Existing Tenants (Step t)

- Each agent $a \in A$ points to its favorite remaining house
- Each house $h \in H_o$ points to $\mu_0^{-1}(h)$
- Each house $h \in H_v$ points to the remaining agent with the highest priority
- If a cycle (of alternating agents and houses) exists, then assign each agent the house that it points to.
- Remove the matched agents and houses from the problem
- If there are remaining agents and houses, then continue to the next step.

ψ_f: TTC for HAP with Existing Tenants (The Final Step)

- Assign the null house to any remaining agents.

An Example for ψ_f

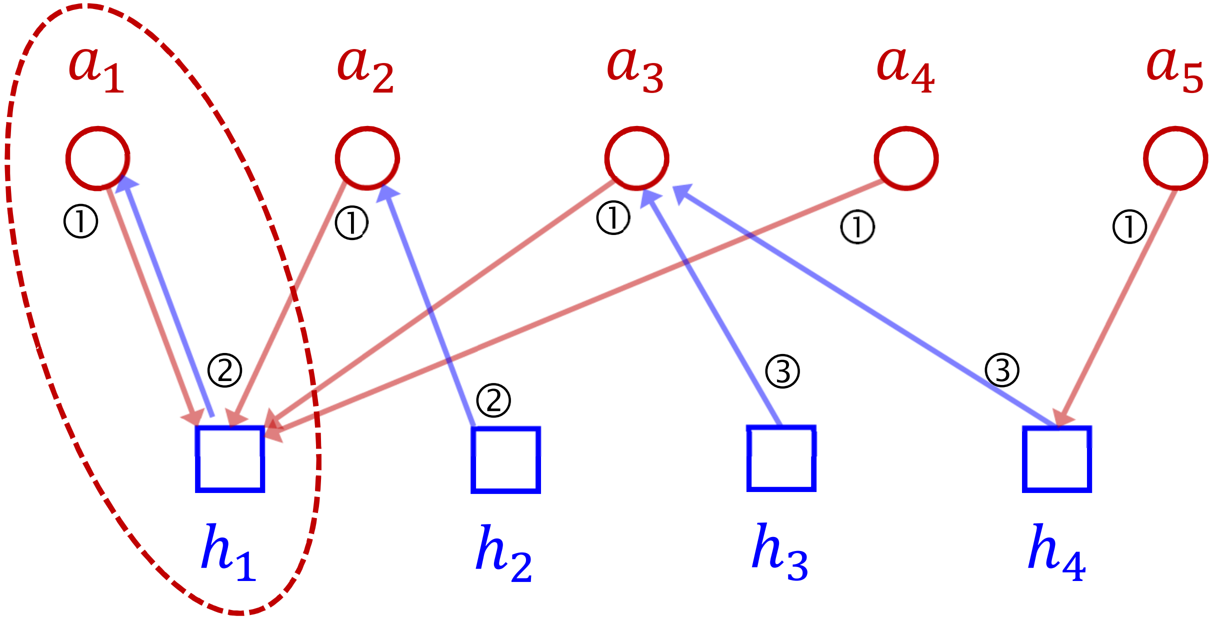
- $A_e = \{a_1, a_2\}$
- $A_n = \{a_3, a_4, a_5\}$
- $H_o = \{h_1, h_2\}$
- $H_v = \{h_3, h_4\}$
- $\mu_0(a_i) = h_i$ for $i \in \{1, 2\}$
- f defines the following priorities over agents

Agent	preference			
a_1	h_1	h_2	h_3	h_4
a_2	h_1	h_3	h_2	h_4
a_3	h_1	h_2	h_3	h_4
a_4	h_1	h_3	h_2	h_4
a_5	h_4	h_1	h_2	h_3

$$f: a_3, a_1, a_2, a_4, a_5$$

the highest priority

Step 1 of ψ_f



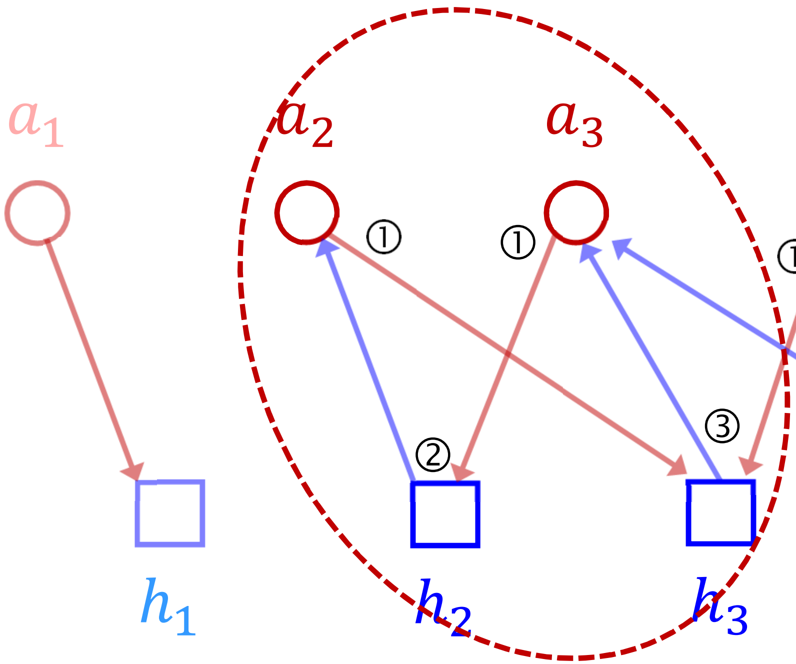
Agent	1st
a_1	h_1
a_2	h_1
a_3	h_1
a_4	h_1
a_5	h_4

① Each agent points to its favorite house

② Each house $h \in H_o$ points to $\mu_0^{-1}(h)$ $\quad$ $\mu_0(a_i) = h_i$ for $i \in \{1, 2\}$

③ Each house $h \in H_v$ points to $f(1) = a_3$

Step 2 of ψ_f



Agent	2nd
a_1	h_2
a_2	h_3
a_3	h_2
a_4	h_3
a_5	h_1

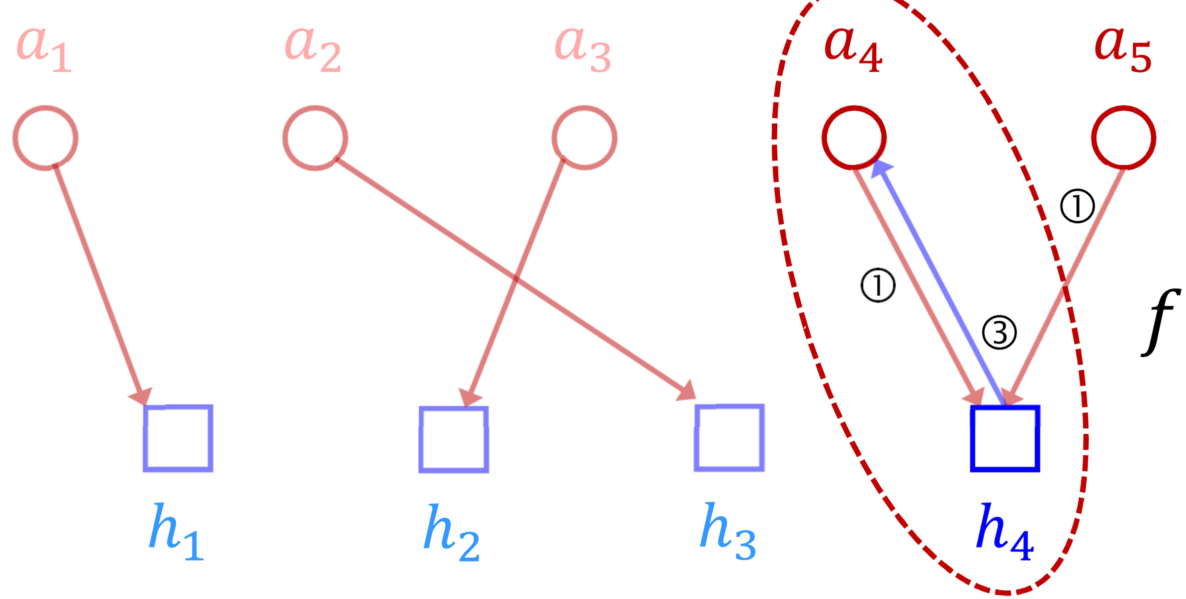
① Each agent points to its favorite remaining house

② Each house $h \in H_o$ points to $\mu_0^{-1}(h)$

③ Each house $h \in H_v$ points to the remaining agent with the highest priority $f(1) = a_3$

Step 3 of ψ_f

Agent	preference			
a_4	h_1	h_3	h_2	h_4
a_5	h_4	h_1	h_2	h_3



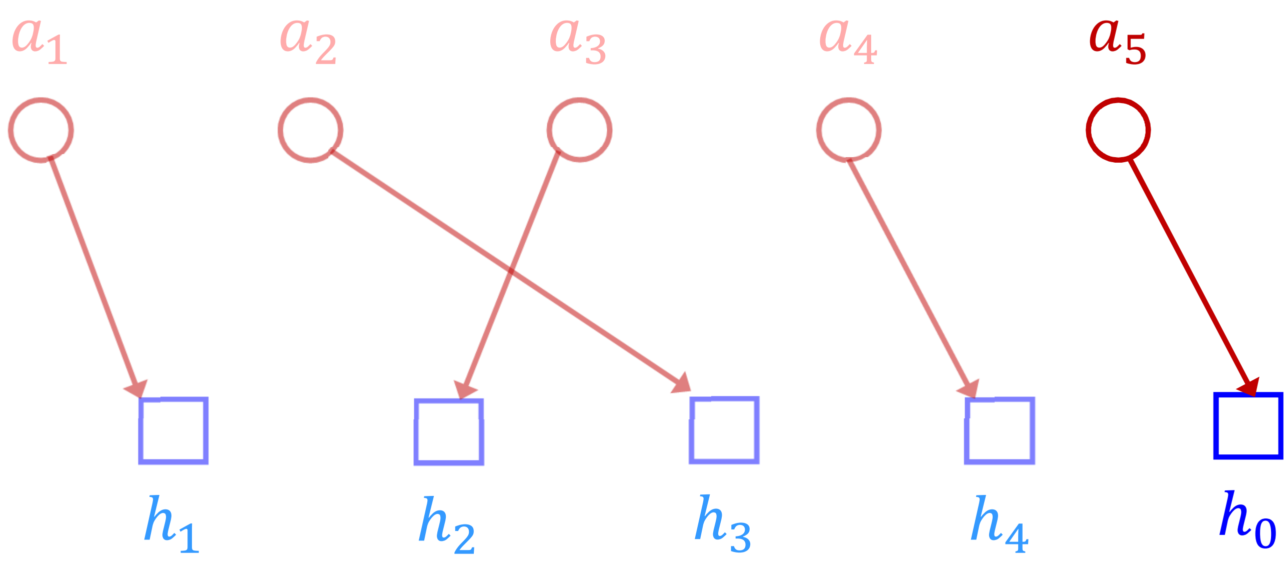
$f: a_3, a_1, a_2, a_4, a_5$

① Each agent points to its favorite remaining house

② Each house $h \in H_o$ points to $\mu_0^{-1}(h)$

③ Each house $h \in H_v$ points to the remaining agent with the highest priority $f(4) = a_4$

Termination of ψ_f



Assign the null house to any remaining agents.

Properties of ψ_f

- Gale's TTC is a special case of this TTC
  - $A_n = H_v = \emptyset$. Thus no need for agent priority.
  - No need for $h \in H$ pointing to $a \in A$.
- It always terminates with a matching
- It is Pareto efficient, individually rational, and strategy proof. [AS99]
- it respects seniority [AS99]

[AS99] A. Abdulkadiroğlu and T. Sönmez, "House allocation with existing tenants," *Journal of Economic Theory*, 88, pp. 233–260, 1999.

Seniority

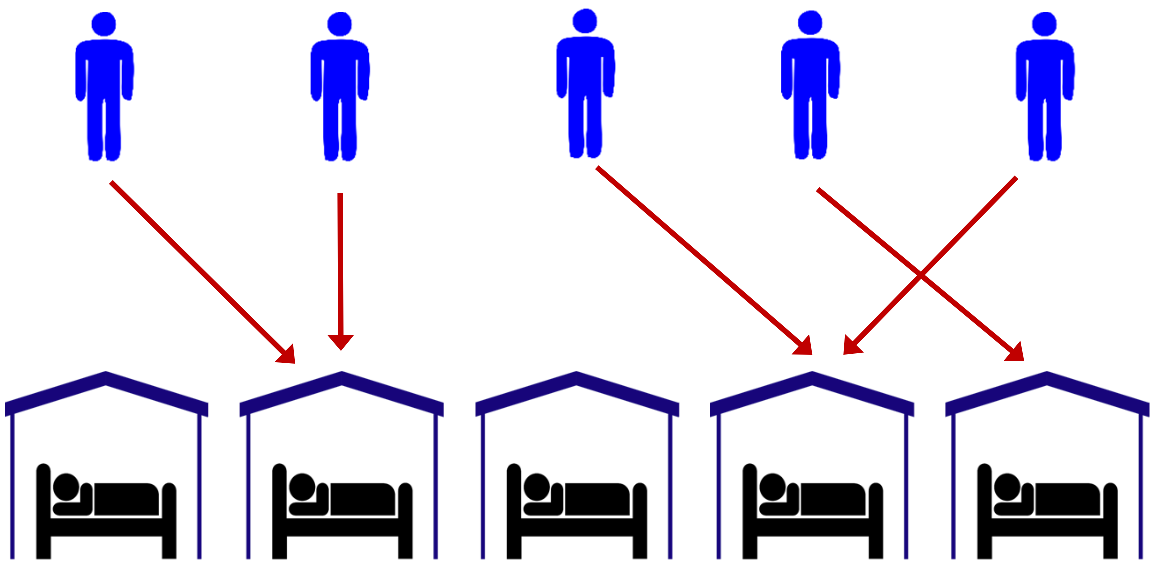
- A mechanism ψ_f that respects seniority meets the following criteria
  - ψ_f assigns $f(1)$ a house that is weakly preferred to the house assigned by any other mechanism that is Pareto efficient, individually rational and strategy proof.
  - out of all mechanisms that perform equally well for agent $f(1)$, ψ_f assigns $f(2)$ a house that is weakly preferred to the house assigned by any other mechanism that is Pareto efficient, individually rational and strategy proof
  - and so on, for all agents $f(3)$, $f(4)$, $\cdots$

What does this really mean?

- Compared with any other mechanism that is Pareto efficient, individually rational and strategy proof
  - either $f(1)$ prefers the house allocated by ψ_f
  - or both mechanisms allocate $f(1)$ the same house
- In the latter case,
  - either $f(2)$ prefers the house allocated by ψ_f
  - or both mechanisms allocate $f(2)$ the same house
- In the latter case,
  - either $f(3)$ prefers the house allocated by ψ_f
  - or both mechanisms allocate $f(3)$ the same house

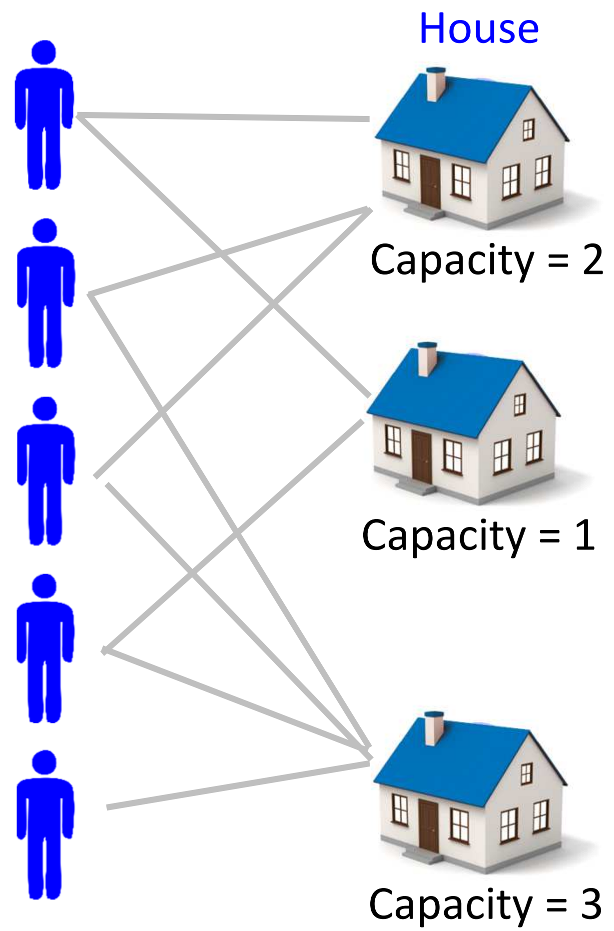
Capacitated House Allocation (CHA) Problem

Many-to-one, one-sided preference



Capacitated House Allocation (CHA) Problem

- Each house has a capacity value.
  - the number of the agents accommodated by the same house cannot exceed the capacity of the house
- How to determine the set of agents accommodated by a house?
- Depending on the objective



Objective 1: Maximum Cardinality

- When agents have neither weights nor preferences on houses
- Try to maximize the number of matchings subject to house capacity constraints
- Max-cardinality many-to-one bipartite matching

Objective 2: Maximum Cardinality Maximum Utility

- When agents have weights on houses
- Let $u_{a,h}$ be the weight (utility) of the allocation of house h to agent a
- Among all maximum cardinality matchings, find the one μ that maximizes

$$\sum_{(a,h)\in\mu} u_{a,h}$$

Objective 3: Weight

- When agents have weights but no preferences on houses
- Try to maximize the total weight of matched agents subject to house capacity constraints
- Max-weight many-to-one bipartite matching

Objective 4: Pareto Efficient

- Each agent $a \in A$ has preference $\succ_a$ over houses but no weight
- a matching μ is Pareto efficient iff there is no matching $v \neq \mu$ such that

  (1) $v \succcurlyeq_a \mu$ for all $a \in A$, and

  (2) $v \succ_a \mu$ for some $a \in A$.

[AS98] A. Abdulkadiroğlu and T. Sönmez, “Random serial dictatorship and the core from random endowments in house allocation problems,” *Econometrica*, 66(3):689–701, 1998.
[ACM+04] D. J. Abraham et al., “Pareto optimality in house allocation problems,” in *Proc. ISAAC 2004, LNCS v.3341*, pp. 3–15, 2004.

Objective 5: Rank Maximal

- Agents have preferences over houses but no weight
- A matching μ is rank maximal if, compared with any other matching,
  1. it assigns the maximum number of agents to their first-choice houses
  2. subject to 1, it assigns the maximum number of agents to their second-choice houses
  3. and so on.

[IKM+04] R.W. Irving et al., "Rank-maximal matchings," in *Proc. SODA '04*, pp. 68–75, 2004.

Objective 6: Popularity

- Agents have preferences over houses but no weight
- Let μ, v be two matchings.
- Let $P(\mu, v) = \{a \in A | \mu \succ_a v\}$
- Let $P(v, \mu) = \{a \in A | v \succ_a \mu\}$
- μ is more popular than v if $|P(\mu, v)| > |P(v, \mu)|$
- A matching μ is popular if there is no other matching that is more popular than μ

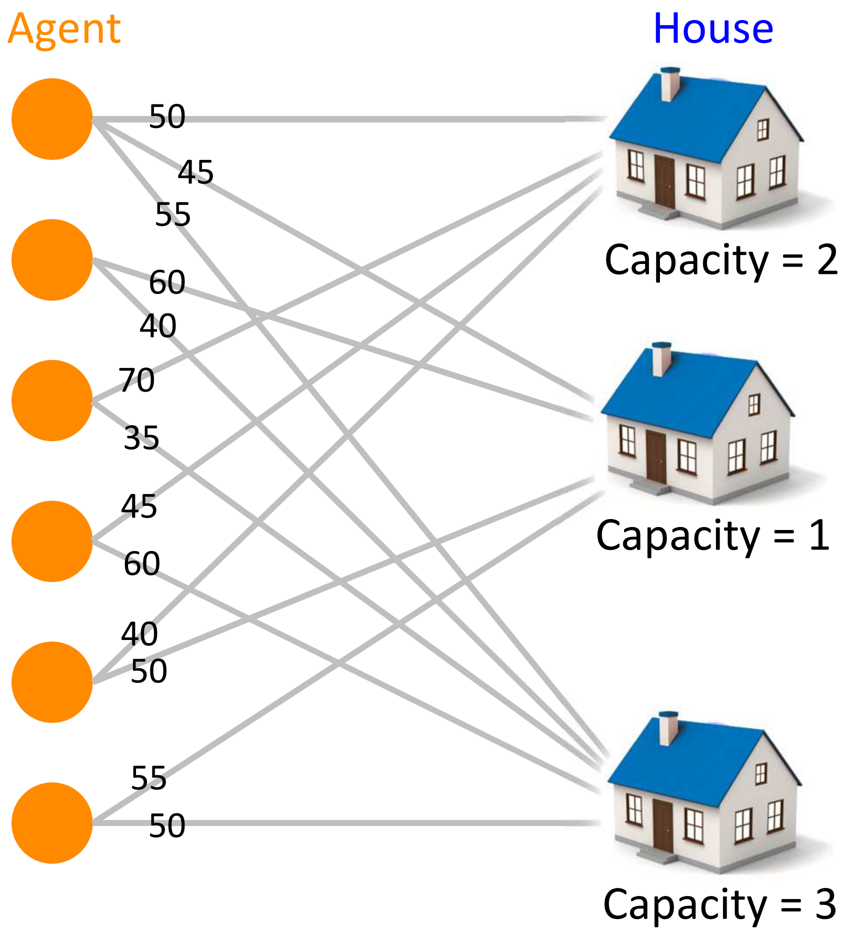
[MS06] D.F. Manlove and C.T.S. Sng, "Popular matchings in the capacitated house allocation problem," *LNCS v.4168*, pp. 492-503, 2006.

Objective 7: Weighted Popularity

- Agents have preferences over houses
- Every agent a also has a positive weight $w(a)$ indicating a's priority
- The satisfaction of a matching μ with respect to v is $sat(\mu, v) = \sum_{a \in P(\mu,v)} w(a) - \sum_{a \in P(v,\mu)} w(a)$
- μ is more popular than v if $sat(\mu, v) > 0$
- A matching μ is popular if there is no other matching that is more popular than μ

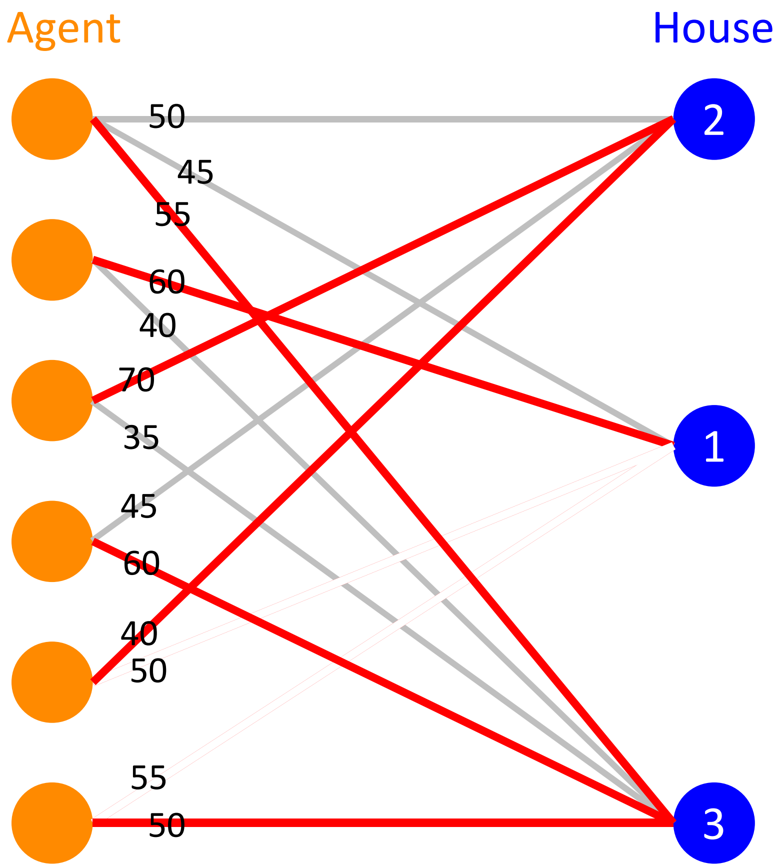
[SM10] C.T.S. Sng and D.F. Manlove, "Popular matchings in the weighted capacitated house allocation problem," Journal of Discrete Algorithms, 8: 102–116, 2010.

Example: Maximum Cardinality Maximum Utility



A Greedy Approach (Not Optimal)

1. Each agent chooses the edge that has the highest weight.
2. Houses for which demand exceeds capacity delete the edges that have lower weights.
3. Each rejected agent then chooses the second highest edge.
4. Go to Step 2



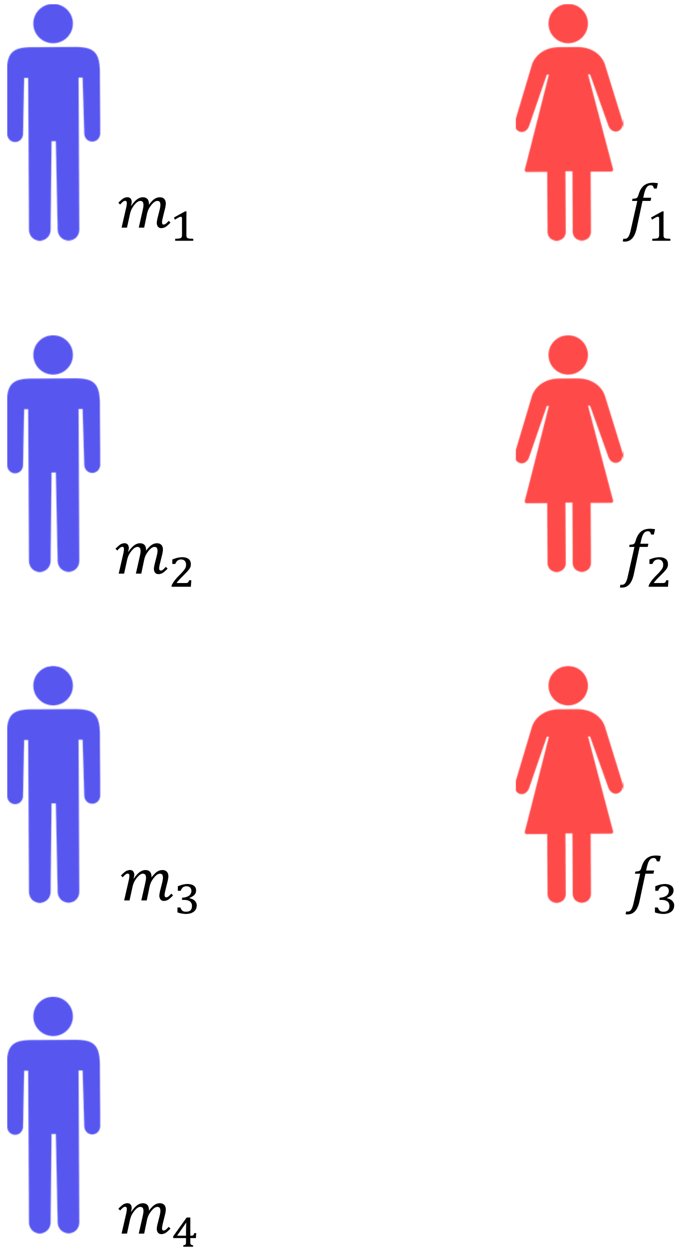
Two-Sided Preference

Stable Marriage Problem

- first described by Gale and Shapley in 1962.
  - [GS62]: College Admissions and the Stability of Marriage
- One-to-one matching
- Two sided-preference

Marriage Problem: Formal Definition

- Sets of men $M = \{m_i\}_{i=1}^{|M|}$ and women $F = \{f_j\}_{j=1}^{|F|}$
- Preference relation: $\succ$
  - $f_j \succ_{m_i} f_k$: m_i prefers f_j to f_k
  - $f_j \succ_{m_i} m_i$: f_j is acceptable to m_i
- Matching μ: $M \cup F \rightarrow M \cup F$
  - $\forall m_i \in M,\ \mu(m_i) \in F \cup \{m_i\}$
  - $\forall f_i \in F,\ \mu(f_i) \in M \cup \{f_i\}$
  - $\forall m_i \in M, \forall f_j \in F,\ \mu(m_i) = f_j \leftrightarrow \mu(f_j) = m_i$



A male set $M = \{m_1, m_2, \dots, m_{|M|}\}$

A female set $F = \{f_1, f_2, \dots, f_{|F|}\}$

Each male m_i has a complete and transitive preference on $F \cup \{m_i\}$, and so does each female.

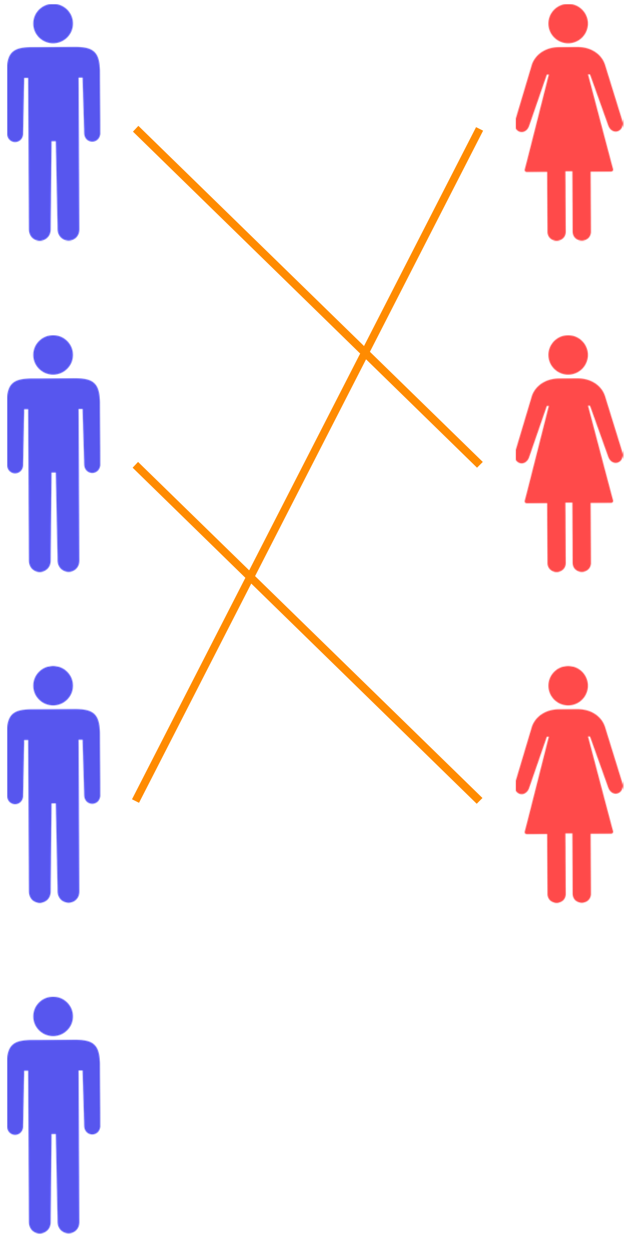
One man can be matched to one woman or to himself.

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$

Acceptable Pair

- If one man m_i is matched with one female f_j, we called it a pair (m_i, f_j).
  - $\mu(m_i) = f_j$ iff $\mu(f_j) = m_i$
- (m_i, f_j) is an acceptable pair if
  - m_i finds f_j acceptable: $f_j \succ_{m_i} m_i$ and
  - f_j finds m_i acceptable: $m_i \succ_{f_j} f_j$.



An example of matching is shown on the left.

Considering the two-sided preferences, how can we find a stable matching?

Stable Matching

- A blocking pair:
  - If they prefer each other than the current matching result.
- A blocking individual:
  - If he or she prefers being single to being matched.
- A stable matching:
  - A matching is stable if there is no blocking pairs or blocking individuals in the matching.

Matching Algorithm: Boston

① Every man proposes to his most preferred woman

② If a woman receives multiple proposals, she accepts the most-preferred one

③ All men with proposals rejected propose to their second-preferred women

④ The process repeats until all men’s proposals are either accepted or rejected and no more proposals are possible

Boston: Step 1

$\boxed{m_1 \rightarrow f_1}$

$m_2 \rightarrow f_1$

$\boxed{m_3 \rightarrow f_2}$

$m_4 \rightarrow f_2$

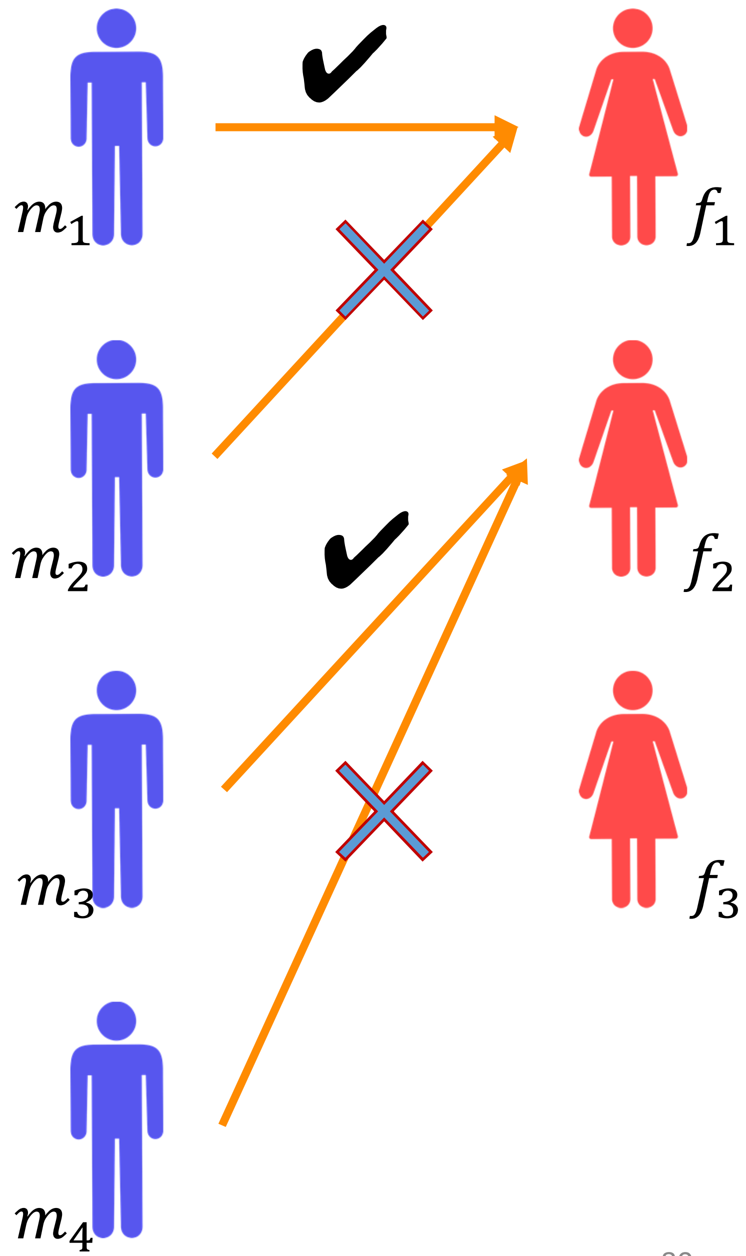
① Every man proposes to his most preferred woman

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

② Each woman accepts the most-preferred one

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$

multiple proposals received



Boston: Step 2

$\boxed{m_1 \to f_1}$

$m_2 \to f_1$ $\quad$ $m_2 \to f_2$

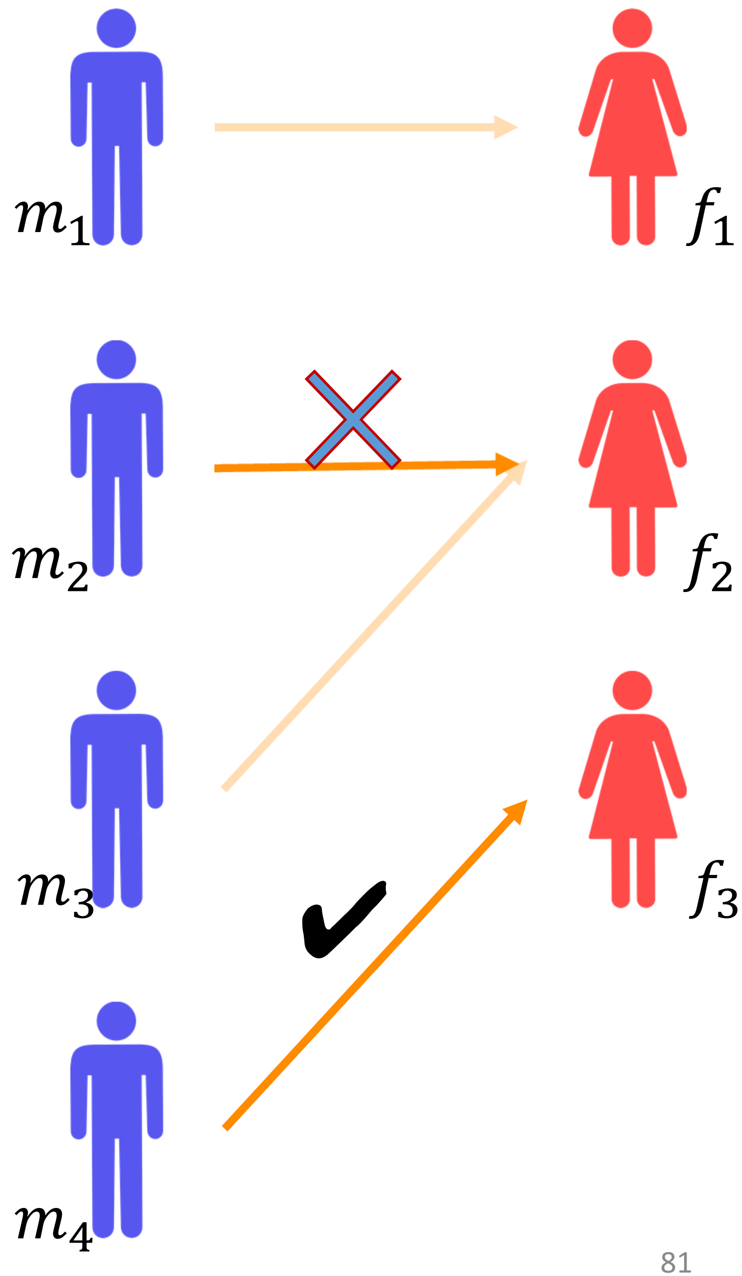
$\boxed{m_3 \to f_2}$

$m_4 \to f_2$ $\quad$ $\boxed{m_4 \to f_3}$

① m_2 and m_4 propose to their 2nd most-preferred woman

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$



Boston: Step 3

$m_1 \rightarrow f_1$

$m_2 \rightarrow f_1$ $m_2 \rightarrow f_2$ $m_2 \rightarrow f_3$

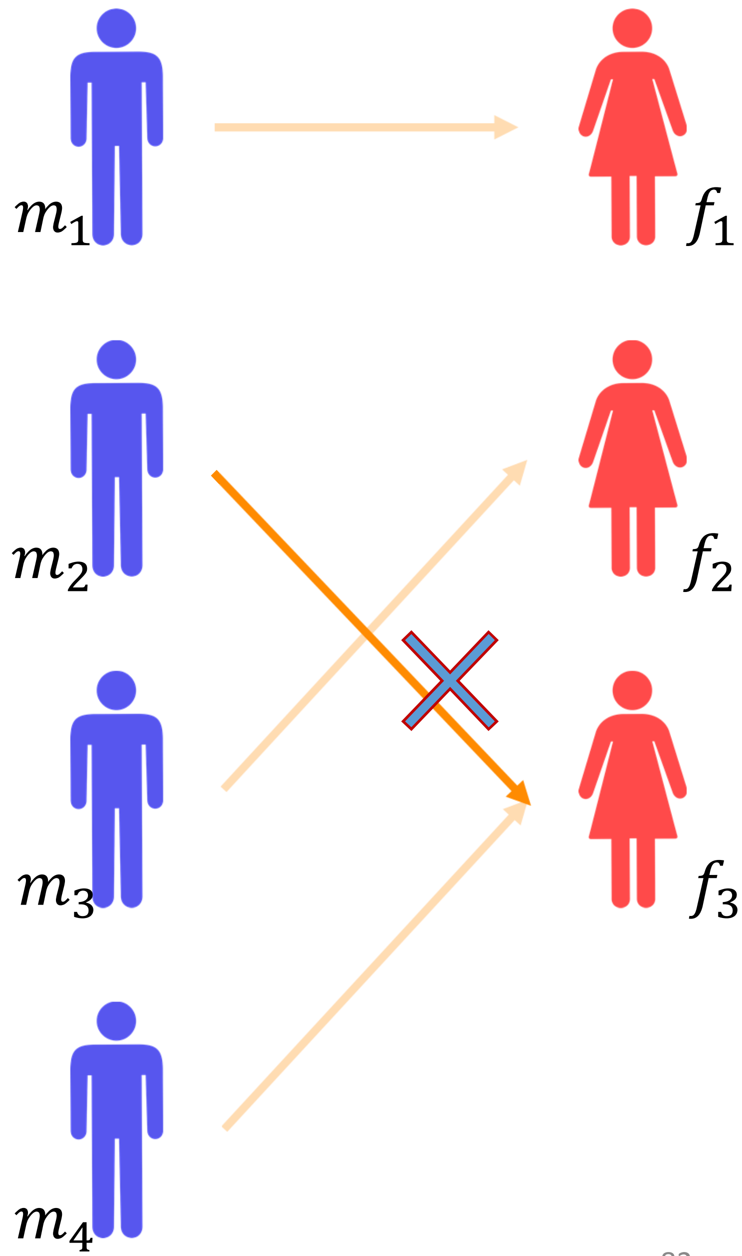
$m_3 \rightarrow f_2$

$m_4 \rightarrow f_2$ $m_4 \rightarrow f_3$

① m_2 proposes to his 3rd most-preferred woman

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$



Boston: Step 4

$\boxed{m_1 \rightarrow f_1}$

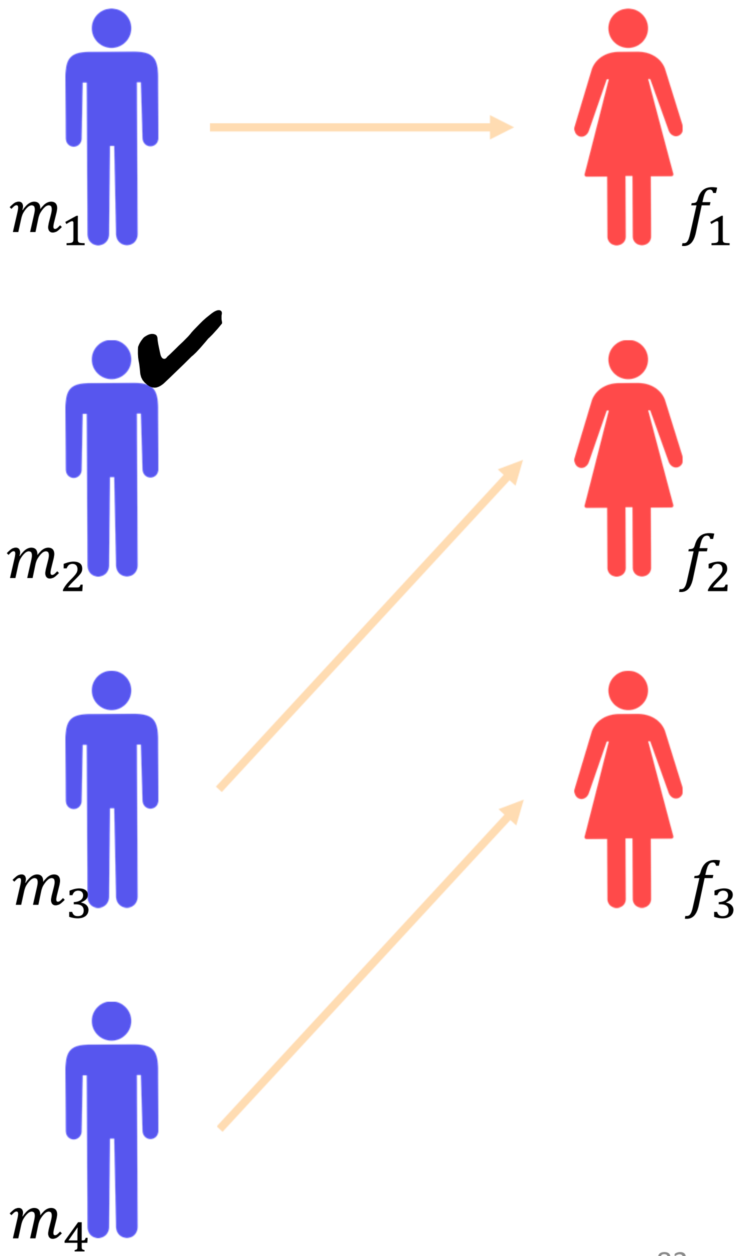
$m_2 \rightarrow f_1$ $m_2 \rightarrow f_2$ $m_2 \rightarrow f_3$ $\boxed{m_2 \rightarrow m_2}$

$\boxed{m_3 \rightarrow f_2}$

$m_4 \rightarrow f_2$ $\boxed{m_4 \rightarrow f_3}$

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$



Boston does not guarantee stability

- The result

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$

(m_2, f_2) is a blocking pair

- m_2 and f_2 have the incentive to deviate from the matching result

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$

Boston is not strategy-proof

	1st	2nd	3rd	4th
m_1	f_1	f_3	f_2	f_4
m_2	f_1	f_2	f_4	f_3
m_3	f_1	f_2	f_3	f_4
m_4	f_1	f_2	f_3	f_4

true preference

	1st	2nd	3rd	4th
m_1	f_1	f_3	f_2	f_4
m_2	f_1	f_2	f_4	f_3
m_3	f_1	f_2	f_3	f_4
m_4	f_2	f_1	f_3	f_4

claimed preference

m_4 is better off by lying about its preference

Deferred Acceptance (DA) Algorithm

- In [GS62], they developed deferred acceptance algorithm to solve the marriage problem.
- It ensures a stable matching.
- Each one that receives a proposal only “tentatively” accepts.
- That is, some proposer may be rejected later.

DA: Step 1/4

$\boxed{m_1 \rightarrow f_1}$

$m_2 \rightarrow f_1$

$\boxed{m_3 \rightarrow f_2}$

$m_4 \rightarrow f_2$

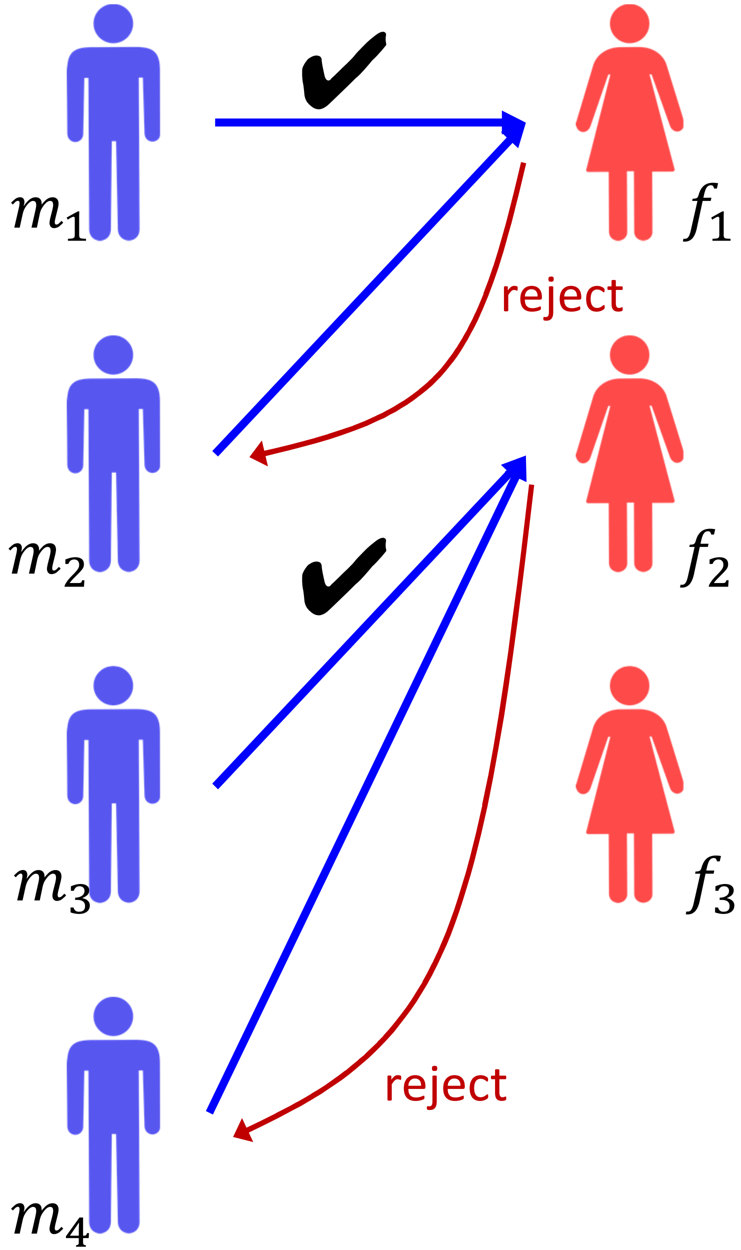
① Every man proposes to his most preferred woman

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

② Each woman accepts the most-preferred one

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$

multiple proposals received



DA: Step 2/4

$\boxed{m_1 \rightarrow f_1}$ ③ f_2 prefers m_2 to m_3

$m_2 \rightarrow f_1$ $\boxed{m_2 \rightarrow f_2}$

$\boxed{m_3 \rightarrow f_2}$

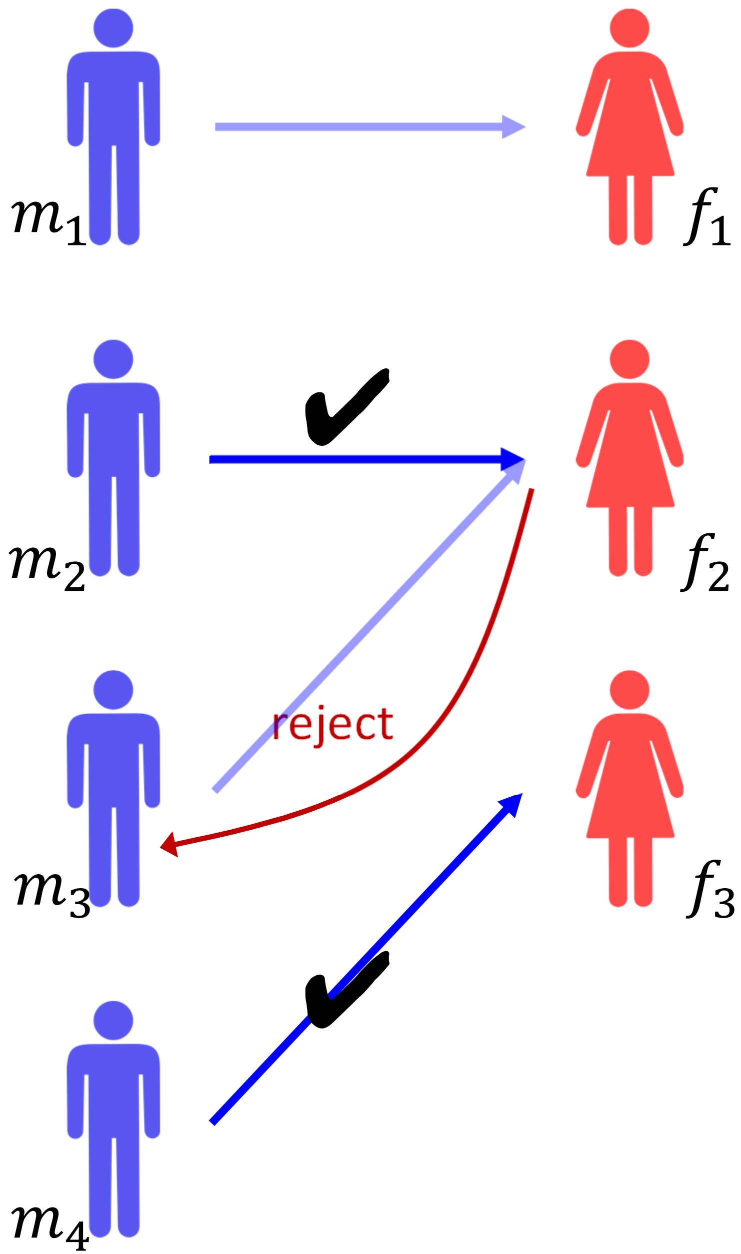
$m_4 \rightarrow f_2$ $\boxed{m_4 \rightarrow f_3}$

① m_2 and m_4 propose to their 2nd most-preferred woman

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$

② f_2 receives a new proposal



DA: Step 3/4

$\boxed{m_1 \rightarrow f_1}$

$m_2 \rightarrow f_1$ $\boxed{m_2 \rightarrow f_2}$

$m_3 \rightarrow f_2$ $m_3 \rightarrow f_1$

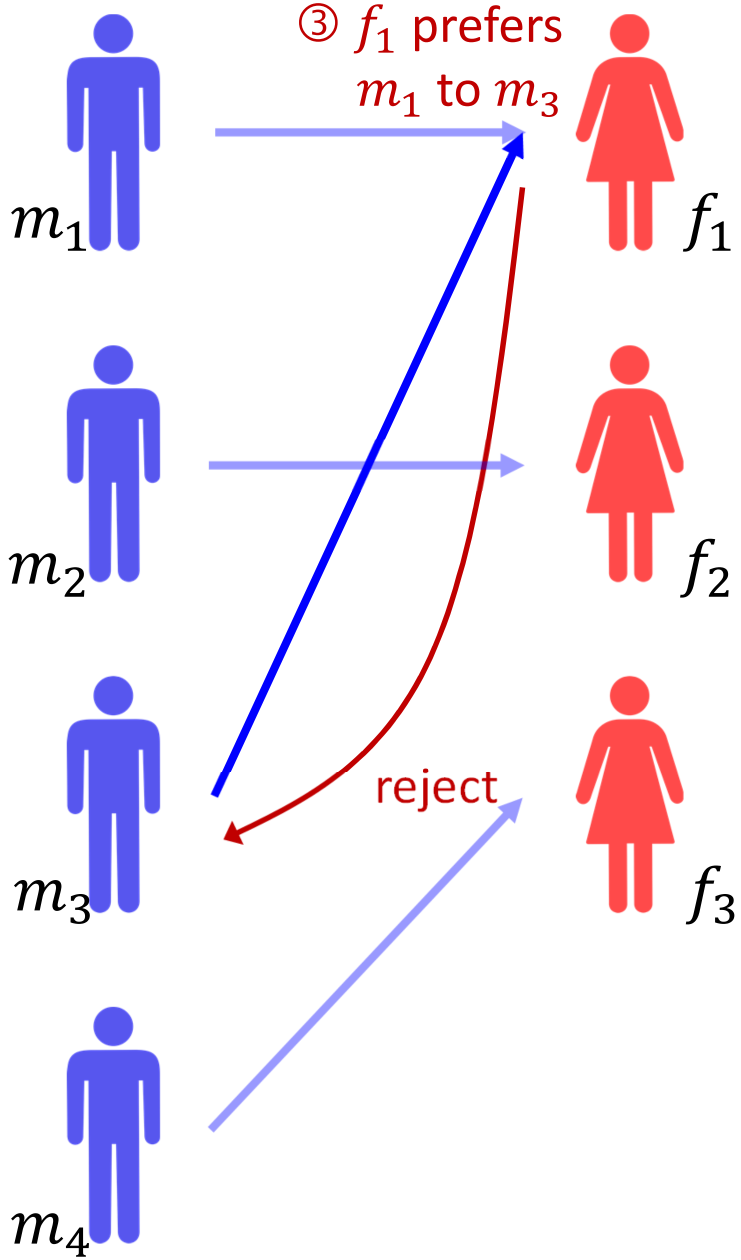
$m_4 \rightarrow f_2$ $\boxed{m_4 \rightarrow f_3}$

① m_3 proposes to his 2nd most-preferred woman

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$

② f_1 receives a new proposal



DA: Step 4/4

$\boxed{m_1 \rightarrow f_1}$

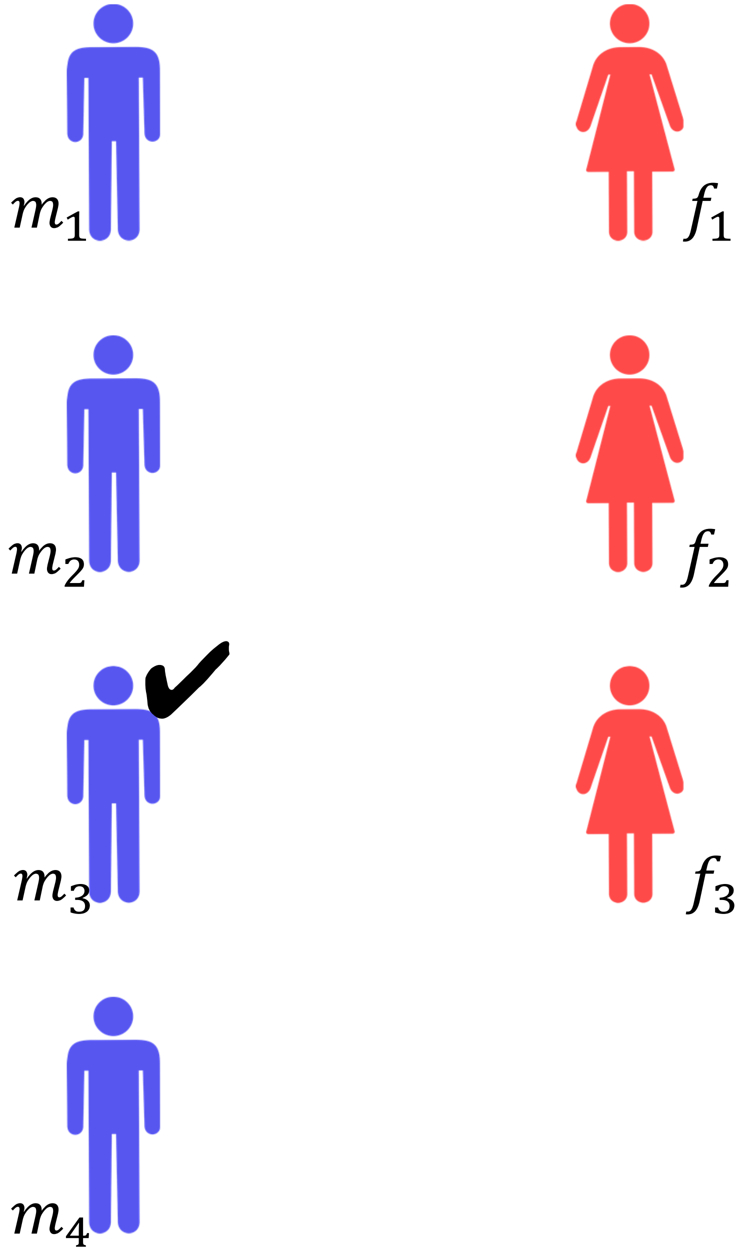
$m_2 \rightarrow f_1$ $\boxed{m_2 \rightarrow f_2}$

$m_3 \rightarrow f_2$ $m_3 \rightarrow f_1$ $\boxed{m_3 \rightarrow m_3}$

$m_4 \rightarrow f_2$ $\boxed{m_4 \rightarrow f_3}$

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$



Proof: Outcome of DA is Stable

- Algorithm must end in a finite number of rounds.
- Suppose *m, f* are matched, but *m* prefers *f'*.
    - At some point, *m* proposed to *f'* and was rejected.
    - At that point, *f'* preferred her tentative match to *m*.
    - As algorithm goes forward, *f'* can only do better.
    - So *f'* prefers her final match to *m.*
- Therefore, there are *NO BLOCKING PAIRS.*

Optimal stable matchings

- A stable matching is *male-optimal* if every male prefers his partner to any partner he could possibly have in a stable matching.
- **Theorem.** The male-proposing DA algorithm results in a male-optimal stable matching.
  - It’s impossible to improve any male’s result without impairing the results of all other males (and the matching is still stable)

Male-Optimal Stable Matching

- The above example is the DA algorithm proposed by male.
- every male prefers his partner to any partner he could possibly have in a stable matching.

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$

Stability vs. Pareto Efficiency

- A male-optimal stable matching is best for males given stability, but may not be Pareto efficient for the males.
- Example: $M = \{m_1, m_2, m_3\}; F = \{f_1, f_2\}$

$m_1: f_1 \succ f_2$ $m_2: f_2 \succ f_1$ $m_3: f_1 \succ f_2$	$f_1: m_2 \succ m_3 \succ m_1$ $f_2: m_1 \succ m_3 \succ m_2$	Stable but not Pareto efficient for males
$m_1: f_1 \succ f_2$ $m_2: f_2 \succ f_1$ $m_3: f_1 \succ f_2$	$f_1: m_2 \succ m_3 \succ m_1$ $f_2: m_1 \succ m_3 \succ m_2$	Pareto efficient for males but <u>not stable</u> (m_3, f_1) is a blocking pair

Male-optimal & Female-optimal

- If the algorithm starts from men proposing, then it will achieve male-optimal stable.
  - Pareto optimal for males in all stable matchings
  - It’s also female-pessimal (each woman gets worst outcome in any stable matching)
- If the algorithm starts from female proposing, then it will achieve female-optimal stable.
  - Pareto optimal for females in all stable matchings
  - It’s also male-pessimal (each male gets worst outcome in any stable matching)

DA (proposed by females): 1/2

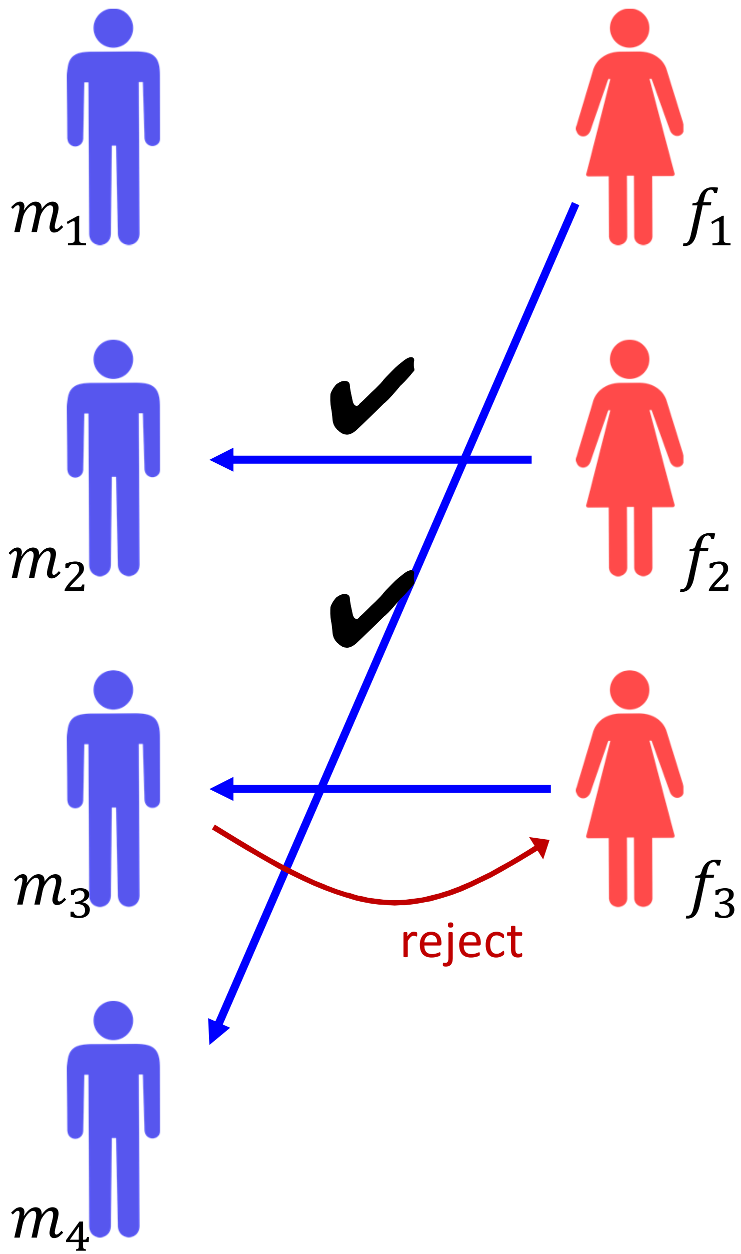
$\boxed{f_1 \rightarrow m_4}$

$\boxed{f_2 \rightarrow m_2}$

$f_3 \rightarrow m_3$

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$



DA (proposed by females): 2/2

$f_1 \rightarrow m_4$

$f_2 \rightarrow m_2$

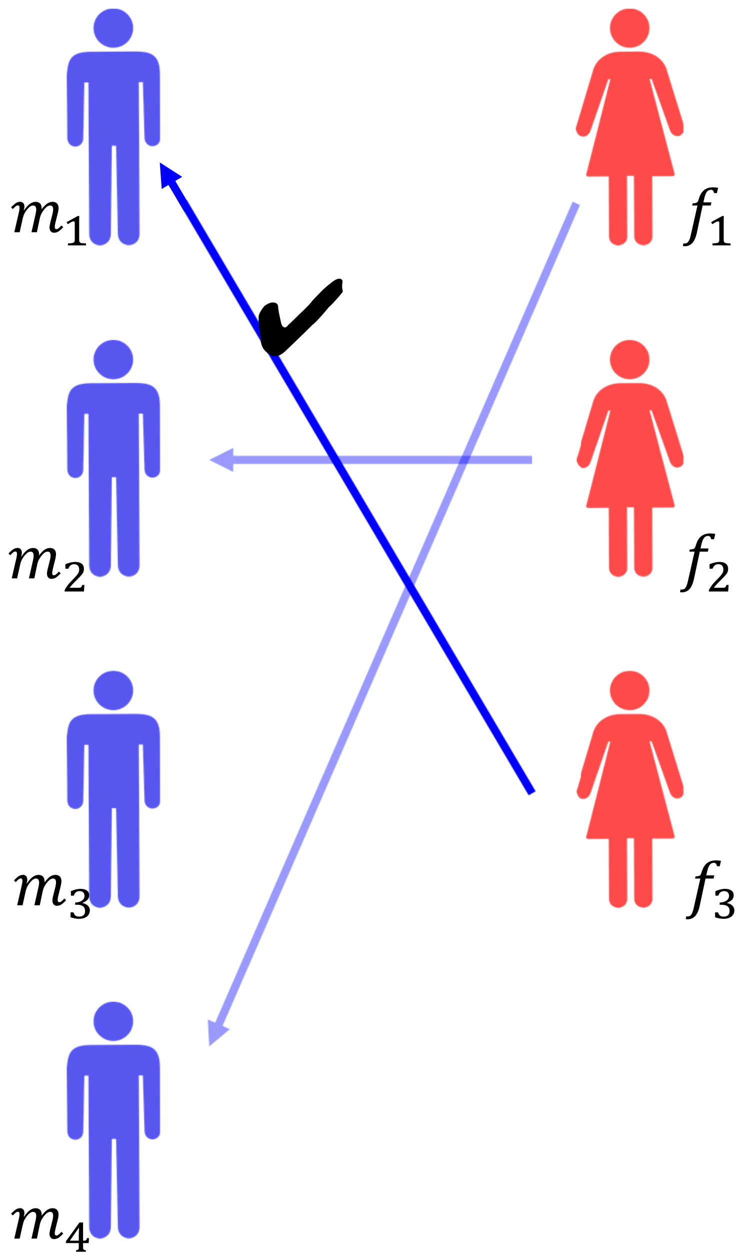
$f_3 \rightarrow m_3$ $\boxed{f_3 \rightarrow m_1}$

① f_3 proposes to her 2nd most-preferred woman

not Pareto optimal for males

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$



Female-Optimal Stable Matching

- The outcome is female-optimal stable, but not Pareto efficient for all females

Female-optimal stable matching

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$

Pareto optimal for all females

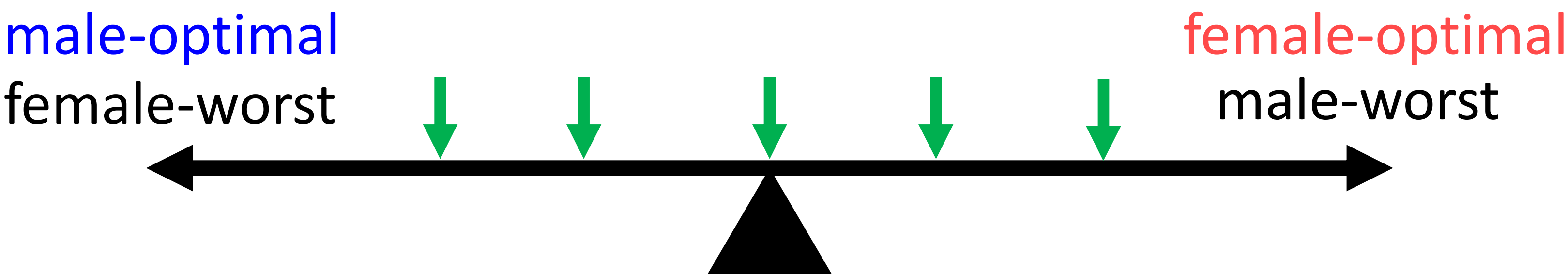
Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Not stable

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$

How many stable matchings?

- Besides the male-optimal and female-optimal matchings, there exist other matchings that are also stable.



Some other papers find the “egalitarian” stable matching.

Truthfulness

- We may ask if we can lie about our preferences to get a better matching result.
- ~~Some research has shown the following theorem:~~
  - ~~No stable matching exists when it is the dominant strategy for every agent revealing its true preference.~~
- Also, some other research has shown that:
  - When the matching is induced by male-proposing DA, it is a dominant strategy for every male to reveal his true preference.
  - But how about women?

If one woman lies (man proposing)

- If female f_3 lies about her preference as:
  - $m_3 \succ_{f_3} m_1 \succ_{f_3} m_2 \succ_{f_3} f_3 \succ_{f_3} m_4$
  - Then both f_1 and f_3 can get a better matching result.

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ m_4 \succ f_3$

Male	Preference
m_1	$f_1 \succ f_2 \succ f_3 \succ m_1$
m_2	$f_1 \succ f_2 \succ f_3 \succ m_2$
m_3	$f_2 \succ f_1 \succ m_3 \succ f_3$
m_4	$f_2 \succ f_3 \succ f_1 \succ m_4$

Female	Preference
f_1	$m_4 \succ m_1 \succ m_2 \succ m_3 \succ f_1$
f_2	$m_2 \succ m_1 \succ m_3 \succ m_4 \succ f_2$
f_3	$m_3 \succ m_1 \succ m_2 \succ f_3 \succ m_4$

Matching between Hospitals and Medical Interns/Residents

- Many-to-one matching
- Two sided-preference

[Rot84] A.E. Roth, "The evolution of the labor market for medical interns and residents: a case study in game theory," *Journal of Political Economy*, 92: 991-1016, 1984.

College Admissions Problem

- Many-to-one matching
- Two sided-preference

College Admission Problem

- Many-to-one matching
  - a college can accept more than one students
  - The number of students that can be accepted is limited by its capacity
  - a student can only be accepted by one college
- Two-sided preference
  - Students have preference over colleges
  - Students may prefer not accepting some colleges
  - Colleges have preference over students
  - Colleges may prefer not accepting some students

Blocking Individual

- For a particular matching,
  - a student is a blocking individual if she/he is matched with some college that she/he prefers not accepting
  - a college is a blocking individual if it is matched with some student that it prefers not accepting
- The existence of a blocking individual makes the matching unstable

Blocking Pair

- For a particular matching μ,
  - If there is another matching v such that some student s prefers $v(s)$ to $\mu(s)$, $(s, v(s))$ is an acceptable pair, and
    - $v(s)$ does not yet accept the max number of students in μ, or
    - (if $v(s)$ already accepts the max number of students in μ) $v(s)$ prefers s to some student matched by μ
  - Then $(s, v(s))$ is a blocking pair
- The existence of a blocking individual makes the matching unstable

Blocking Pair Example

- Suppose each college can admit two students

Student	Preference
s_1	$c_2 \succ c_1$
s_2	$c_1 \succ c_2 \succ c_3$
s_3	$c_1 \succ c_2 \succ c_3$
s_4	$c_1 \succ c_2 \succ c_3$

College	Preference
c_1	$s_1 \succ s_3 \succ s_2 \succ s_4$
c_2	$s_3 \succ s_2 \succ s_1 \succ s_4$
c_3	$s_3 \succ s_2 \succ s_1 \succ s_4$

(s_2, c_1) is a blocking pair

Note: there is no Pareto improvement

(Pairwise) Stability

- A matching μ is stable if it is not blocked by any individual or any pair.

Student	Preference
s_1	$c_2 \succ c_1$
s_2	$c_1 \succ c_2 \succ c_3$
s_3	$c_1 \succ c_2 \succ c_3$
s_4	$c_1 \succ c_2 \succ c_3$

College	Preference
c_1	$s_1 \succ s_3 \succ s_2 \succ s_4$
c_2	$s_3 \succ s_2 \succ s_1 \succ s_4$
c_3	$s_3 \succ s_2 \succ s_1 \succ s_4$

This matching is stable

The Core

- Stable marriage problem is a special case of the stable college admission problem with capacity of each college equal to 1
- the core of this problem is non-empty [Rot84]
  - We can always find a result that is both individual rational and stable
- Particularly, an algorithm can find a core that is best for all the colleges and worst for all the students

Responsive Preference

- college may have preferences over groups of students (e.g., to build a football team)
- A college's preference list is responsive if its preference list is over the "individual" of the students in college admissions problem.
  - With responsive preference list, if two matchings differ only in one student in the college's matching, then the college prefers the matching containing the student with a higher preference.
- If we want to directly use DA, we should ensure that all college's preferences are responsive.

An Example Where Preferences Are Not Responsive

Outcome by DA

Student	Preference
s_1	$c_2 \succ c_1$
s_2	$c_1 \succ c_2$
s_3	c_1
s_4	c_1

College	Preference
c_1	$s_1 s_2 \succ s_3 s_4$
c_2	$s_3 s_4 \succ s_2$

Another matching μ'

Student	Preference
s_1	$c_2 \succ c_1$
s_2	$c_1 \succ c_2$
s_3	c_1
s_4	c_1

College	Preference
c_1	$s_1 s_2 \succ s_3 s_4$
c_2	$s_3 s_4 \succ s_2$

s_1, s_2, c_1 can be all better off

Stability When Preferences Are Not Responsive

- μ is not stable but there is no blocking pair in μ
- For responsive preferences, matching not blocked by any individual or any pair $\Rightarrow$ stable matching
- If preferences are not responsive, we should look into coalitions (subset of agents) instead of pairs

μ

Student	Preference
s_1	$c_2 \succ c_1$
s_2	$c_1 \succ c_2$
s_3	c_1
s_4	c_1

College	Preference
c_1	$s_1 s_2 \succ s_3 s_4$
c_2	$s_3 s_4 \succ s_2$

Blocking Coalition

- A blocking coalition of a matching μ is (C', S', μ'), where $C' \subseteq C$, $S' \subseteq S$, and $\mu' \neq \mu$ such that
  - $C' \cup S' \neq \emptyset$
  - $\mu'(s) \subseteq S'$ for all $s \in C'$
  - $\mu'(s) \in C' \cup \{s\}$ for all $s \in S'$
  - $\mu'(s) \succcurlyeq_s \mu(s)$ for all $s \in C' \cup S'$
  - $\mu'(s) \succ_s \mu(s)$ for some $s \in C' \cup S'$
- $(\{c_1\}, \{s_1, s_2\}, \mu')$ in our example is a blocking coalition
- μ is (setwise) stable if it is not blocked by any coalition

DA algorithm to college admissions

- When the colleges have responsive preferences, there may exist a matching that all colleges strictly prefer the college-optimal stable matching.

College	Preference
c_1	$s_1 \succ s_2 \succ s_3 \succ s_4 \succ c_1$
c_2	$s_1 \succ s_2 \succ s_3 \succ s_4 \succ c_2$
c_3	$s_3 \succ s_1 \succ s_2 \succ s_4 \succ c_3$

Student	Preference
s_1	$c_3 \succ c_1 \succ c_2 \succ s_1$
s_2	$c_2 \succ c_1 \succ c_3 \succ s_2$
s_3	$c_1 \succ c_3 \succ c_2 \succ s_3$
s_4	$c_1 \succ c_2 \succ c_3 \succ s_4$

Comparison

- Student-optimal & College-optimal
  - Student-optimal: No strictly preferred matching
  - College-optimal: Exists a preferred matching
- Truthfulness:
  - For student-optimal, it is a dominant strategy for every student to reveal its true preference.
  - However, for college-optimal, no stable matching algorithm for every college to reveal its true preference.

Many-to-Many Matching

- the number of allowable matches for the agents in both sides of the matching is unrestricted
- Consider a collection of firms and consultants.
  - Each firm wishes to hire a set of consultants, and each consultant wishes to work for a set of firms.
- Firms have preferences over the possible sets of consultants
- Consultants have preferences over the possible sets of firms

Example: Firms and Consultants

- Set of workers (consultants): $W = \{w_1, w_2, w_3\}$
- Set of firms: $F = \{f_1, f_2, f_3\}$
- Worker's preferences

$$\left.\begin{array}{l} w_1: f_3 \succ f_2 f_3 \succ f_1 f_3 \succ f_1 \succ f_2 \\ w_2: f_1 \succ f_1 f_3 \succ f_1 f_2 \succ f_2 \succ f_3 \\ w_3: f_2 \succ f_1 f_2 \succ f_2 f_3 \succ f_3 \succ f_1 \end{array}\right\} \text{Not responsive preference}$$

- Firm's preferences

$$\left.\begin{array}{l} f_1: w_3 \succ w_2 w_3 \succ w_1 w_3 \succ w_1 \succ w_2 \\ f_2: w_1 \succ w_1 w_3 \succ w_1 w_2 \succ w_2 \succ w_3 \\ f_3: w_2 \succ w_1 w_2 \succ w_2 w_3 \succ w_3 \succ w_1 \end{array}\right\} \text{Not responsive preference}$$

A Possible Matching of The Example

$w_1: f_3 \succ \boxed{f_2 f_3} \succ f_1 f_3 \succ f_1 \succ f_2$
$w_2: f_1 \succ \boxed{f_1 f_3} \succ f_1 f_2 \succ f_2 \succ f_3$
$w_3: f_2 \succ \boxed{f_1 f_2} \succ f_2 f_3 \succ f_3 \succ f_1$

Each worker is matched with 2 firms

$f_1: w_3 \succ \boxed{w_2 w_3} \succ w_1 w_3 \succ w_1 \succ w_2$
$f_2: w_1 \succ \boxed{w_1 w_3} \succ w_1 w_2 \succ w_2 \succ w_3$
$f_3: w_2 \succ \boxed{w_1 w_2} \succ w_2 w_3 \succ w_3 \succ w_1$

Each firm is matched with 2 workers

DA in Many-to-Many Matching

- Let $Ch(S, \succ_a)$ be agent a's most-preferred subset of S according to a's preference relation $\succ_a$
- Let $F = \{f_1, f_2, \cdots, f_n\}$, $W = \{w_1, w_2, \cdots, w_m\}$
- Suppose firms in F propose to workers in W
- In each round, each firm proposes to a set of workers that it prefers the most and can possibly hire (not rejected yet)
- Each worker selects a most-preferred set of proposals it has received to tentatively accept
- A proposal, once accepted, can be rejected later by the worker but cannot be dropped unilaterally by the firm

Procedure for Firm's Proposing

Each $f_i \in F$ performs the following actions:

1. $A_i \leftarrow \emptyset$; // accepted proposals
2. While $W \neq \emptyset$
3. $\quad P_i \leftarrow Ch(W, \succ_{f_i})$
4. $\quad$ If $A_i \subset P_i$ and $P_i \succ_{f_i} A_i$ then
5. $\qquad$ Proposes to each $w_j \in P_i \setminus A_i$
6. $\qquad R_i \leftarrow$ the set of workers that rejects f_i's proposal.
7. $\qquad A_i \leftarrow P_i \setminus R_i$;
8. $\qquad W \leftarrow W \setminus R_i$
9. $\quad$ Else if some f_i's previous proposal toward w_j is rejected
10. $\qquad$ remove all such w_j's from A_i and W
11. $\quad$ End If
12. End while

What's Wrong With Firm's Procedure?

- Assume f_1's preference

$$w_1 w_2 w_3 \succ_{f_1} w_1 w_4 \succ_{f_1} w_1 \succ_{f_1} w_1 w_3$$

- Suppose that f_1 proposes to w_1, w_2, and w_3.
- If w_2 rejects f_1's proposal, then f_1 is matched with $\{w_1, w_3\}$
- f_1 could also be better off if it could drop w_3
- Also, f_1 could have been matched with w_4 (if w_4 also prefers matching with f_1)

Individual Rationality of Many-to-X Matching

- For each agent a, let $\succ_a$ be a preference relation.
- A matching μ is individually rational if and only if

$$\mu(a) = Ch(\mu(a), \succ_a) \text{ for all } a \in F \cup W.$$

- In the previous example,

$$\mu(f_1) = \{w_1, w_3\} \neq Ch(\{w_1, w_3\}, \succ_a) = \{w_1\}.$$

- This means there is at least an agent a who prefers a proper subset $A \subset \mu(a)$ over $\mu(a)$
  - a could be better off by not matching with $\mu(a) \setminus A$

Pairwise Block

- Let $w \in W$, $f \in F$, and let μ be a matching.
- The pair (w, f) is a pairwise block of μ if
  $w \notin \mu(f), w \in Ch(\mu(f) \cup \{w\}, \succ_f))$, and
  $f \in Ch(\mu(w) \cup \{f\}, \succ_w)$
- Example: (f_1, w_4)

$$Ch\left(\{w_1, w_3\} \cup \{w_4\}, \succ_{f_1}\right) = \{w_1, w_4\}$$

$$f_1:\ w_1w_2w_3 \succ \boxed{w_1w_4} \succ w_1 \succ \boxed{w_1w_3}$$

$$w_4: \boxed{f_1f_2} \succ \boxed{f_2f_3}$$

$$Ch\left(\{f_2, f_3\} \cup \{f_1\}, \succ_{w_4}\right) = \{f_1, f_2\}$$

Pairwise Stability

- A matching μ is pairwise stable if it is individually rational and there is no pairwise block of μ.
- Even if a matching is pairwise stable, there may exist a blocking coalition of size 3 or larger
- Example: $\{f_1, w_4, w_5\}$

$$Ch(\{w_1, w_3\} \cup \{w_4, w_5\}, \succ_{f_1}) = \{w_4, w_5\}$$

$$f_1\colon\ w_1 w_2 w_3 \succ \boxed{w_4 w_5} \succ w_1 \succ \boxed{w_1 w_3}$$

$$w_4\colon \boxed{f_1} \succ \boxed{f_2 f_3}$$

$$w_5\colon \boxed{f_1} \succ \boxed{f_2 f_3}$$

$$Ch(\{f_2, f_3\} \cup \{f_1\}, \succ_{w_5}) = \{f_1\}$$

Procedure for Worker's Response

Each $w_j \in W$ performs the following actions:

$A_j \leftarrow \emptyset$; // accepted proposals

Let F_j be the set of firms that proposes to w_j.

While $F_j \neq \emptyset$

$P_j \leftarrow Ch(A_j \cup F_j, \succ_{w_j})$

If $P_j \succ_{w_j} A_j$ then

accept each $f_i \in P_j \setminus A_j$

reject each $f_i \in F_j \setminus P_j$ and each $f_i \in A_j \setminus P_j$

$A_j \leftarrow P_j$

Else

reject each $f_i \in F_j \setminus A_j$

End if

Let F_j be the set of firms that proposes to w_j.

End While

What's Wrong With Worker's Procedure?

- Assume w_1's preference

$$f_3 f_4 \succ_{w_1} f_1 f_2$$

- Suppose that w_1 receives proposals from f_1, f_2, and f_3 in the first round.
- By the procedure w_1 accepts f_1, f_2 and rejects f_3
- Suppose that w_1 receives f_4's proposal later. w_1 will reject it.
- In this case, $w_1, f_3,$ and f_4 form a blocking coalition.

Blocking Coalition of Many-to-Many Matching

- A blocking coalition of a matching μ is (W', F', μ'), where $W' \subseteq W$, $F' \subseteq F$, and $\mu' \neq \mu$ such that
  - $F' \cup W' \neq \emptyset$
  - $\mu'(s) \subseteq F' \cup W'$ for all $s \in F' \cup W'$
  - $\mu'(s) \succcurlyeq_s \mu(s)$ for all $s \in F' \cup W'$
  - $\mu'(s) \succ_s \mu(s)$ for some $s \in F' \cup W'$
- μ' is another matching among agents in $F' \cup W'$ so that every agent in $F' \cup W'$ is weakly better off and at least one of them is strictly better off
- We say that (W', F', μ') blocks μ

An Example of Blocking Coalition

- $W = \{\bar{w}, w_1, w_2, w_3, w_4\}, F = \{f_1, f_2, \bar{f}\}$
- Preferences and mapping μ

$\bar{w}: f_1\bar{f} \succ \bar{f} \succ f_1$

$w_1: f_1 \succ f_2 \succ \bar{f}$

$w_2: f_1 \succ f_2 \succ \bar{f}$

$w_3: f_1 \succ f_2 \succ \bar{f}$

$w_4: f_1 \succ f_2 \succ \bar{f}$

$f_1: \bar{w}w_1 \succ w_1w_2$

$f_2: w_2w_3 \succ w_3w_4 \succ \bar{w}w_4$

$\bar{f}: \bar{w} \succ w_1 \succ w_2 \succ w_3 \succ w_4$

There exists a mapping μ' such that $(\{\bar{w}, w_1\}, \{f_1, \bar{f}\}, \mu')$ blocks μ. Can you find it?

Corewise Stability

- A matching μ is in the strong core (strong corewise-stable) if there is no blocking coalition
- A matching μ is in the core (corewise-stable) if there is no (W', F', μ'), where $W' \subseteq W$, $F' \subseteq F$, and $\mu' \neq \mu$ such that
  - $F' \cup W' \neq \emptyset$
  - $\mu'(s) \subseteq F' \cup W'$ for all $s \in F' \cup W'$
  - $\mu'(s) \succ_s \mu(s)$ for all $s \in F' \cup W'$

An Example of Stable Matching

w_1: $f_3 \succ \boxed{f_2 f_3} \succ f_1 f_3 \succ f_1 \succ f_2$ $\quad$ f_1: $w_3 \succ \boxed{w_2 w_3} \succ w_1 w_3 \succ w_1 \succ w_2$
w_2: $f_1 \succ \boxed{f_1 f_3} \succ f_1 f_2 \succ f_2 \succ f_3$ $\quad$ f_2: $w_1 \succ \boxed{w_1 w_3} \succ w_1 w_2 \succ w_2 \succ w_3$
w_3: $f_2 \succ \boxed{f_1 f_2} \succ f_2 f_3 \succ f_3 \succ f_1$ $\quad$ f_3: $w_2 \succ \boxed{w_1 w_2} \succ w_2 w_3 \succ w_3 \succ w_1$

- To make f_1 better off, f_1 should hire only $w_3 \Rightarrow w_2$ is hired only by $f_3 \Rightarrow w_2$ is worse off
- If f_1 is in a coalition C, w_3 must be in C. Then f_2 must be in C, or w_3 would only be hired by f_1 and thus worse off.
- But f_2 in C implies that w_1 must be in C. Then f_3 must be in C, so w_2 must be in C, a contradiction.

Substitutability

- Let $Ch(S, \succ_a)$ be a's most-preferred subset of S according to a's preference relation $\succ_a$
- An agent a's preference relation $\succ_a$ satisfies substitutability if, for any sets S and S' of partners of a with $S' \subseteq S$,

$$b \in Ch(S \cup \{b\}, \succ_a) \text{ implies } b \in Ch(S' \cup \{b\}, \succ_a)$$

- e.g.,

$f: w_1w_2w_3w_4 \succ_f w_1w_2 \succ_f w_1w_4$ is not substitutable

Because $w_4 \in Ch(\{w_1, w_2, w_3, w_4\}, \succ_f)$ but $w_4 \notin Ch(\{w_1, w_2, w_4\}, \succ_f)$

Example of Substitutable Preference

can switch

$$f: w_1w_2w_3 \succ \underbrace{\boxed{w_1w_2 \succ w_1w_3 \succ w_2w_3}}_{\text{order can be different}} \succ \underbrace{\boxed{w_1 \succ w_2 \succ w_3}}_{\text{order can be different}}$$

- no rejected proposal becomes desirable when some other proposal becomes available.
  - It’s no regret to reject a proposal and substitute it with a better proposal
- substitutability is necessary for the existence of stable outcomes

Questions

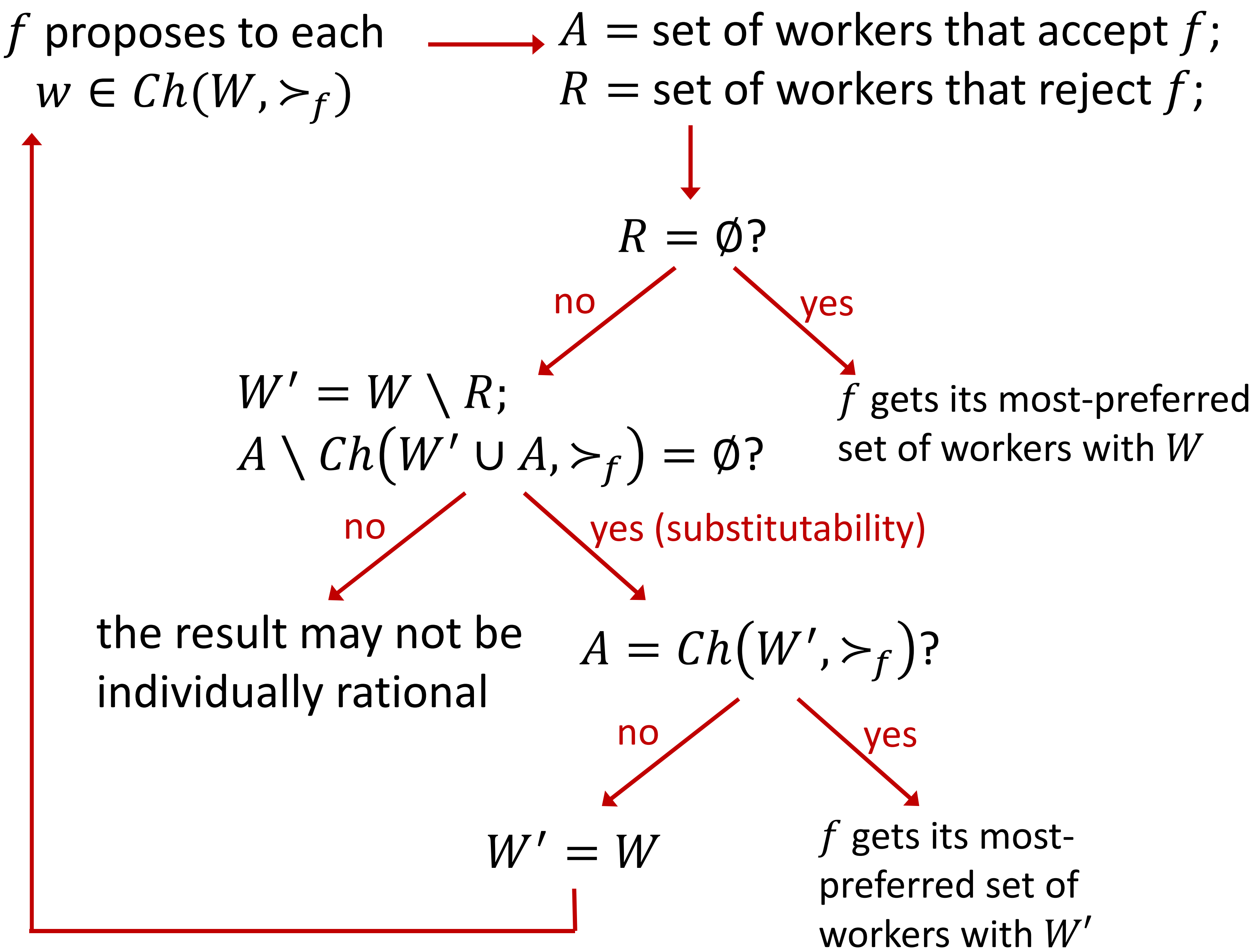
- If every agent’s preference is substitutable, does DA ensure pairwise stability?
- If every agent’s preference is substitutable, does DA ensure general stability (no blocking coalition)?

Offers Remain Good if Firm's Preferences are Substitutable

- Substitutability: for any sets S and S' of partners of a with $S' \subseteq S$, $b \in Ch(S \cup \{b\}, \succ_a) \to b \in Ch(S' \cup \{b\}, \succ_a)$
- Initially, a firm f considers all workers W in finding its best-preferred set of workers (i.e., $Ch(W, \succ_f)$)
- Let A be the set of workers in $Ch(W, \succ_f)$ that accept f's proposal. Clearly, $A \subseteq Ch\left(W, \succ_f\right) \subseteq W$.
- If there is some $w \in A$, f's offer to w remain good because $w \in Ch\left(W' \cup \{w\}, \succ_f\right)$ for any future set of workers W' that f may consider. ($W' \subseteq W$)

Substitutability for the Proposing Side (the Firm)

- Let $R = Ch(W, \succ_f) \setminus A$ be the set of workers in $Ch(W, \succ_f)$ that reject f's proposal.
- If $R = \emptyset$, f gets its most-preferred set of workers.
- Otherwise, let $W' = W \setminus R$.
- Because $W' \subseteq W$, substitutability ensures that $w \in A \rightarrow w \in Ch(W' \cup \{w\}, \succ_f)$
- If $A \neq Ch(W', \succ_f)$, f can make its 2nd–round proposal to each $w \in Ch(W', \succ_f) \setminus A$.


f proposes to each $w \in Ch(W, \succ_f)$
A = set of workers that accept f;
R = set of workers that reject f;
$R = \emptyset$?
no
yes
$W' = W \setminus R$;
$A \setminus Ch(W' \cup A, \succ_f) = \emptyset$?
f gets its most-preferred set of workers with W
no
yes (substitutability)
the result may not be individually rational
$A = Ch(W', \succ_f)$?
no
yes
$W' = W$
f gets its most-preferred set of workers with W'

Rejections are Final if Worker's Preferences are Substitutable

- Substitutability: for any sets S and S' of partners of a with $S' \subseteq S$, $b \in Ch(S \cup \{b\}, \succ_a) \rightarrow b \in Ch(S' \cup \{b\}, \succ_a)$
- This is logically equivalent to $b \notin Ch(S' \cup \{b\}, \succ_a) \rightarrow b \notin Ch(S \cup \{b\}, \succ_a)$
- Let $F' \subseteq F$ be the set of firms propose to worker w.
- If there is some $f \in F'$ but $f \notin Ch(F', \succ_w)$, w's rejection to f's proposal is final because $f \notin Ch(F' \cup F'', \succ_w)$ for any future set of proposals F'' that w may receive.

Matching with Transfer (Matching Market)

Firms and Workers Problem

[CK81] V. P. Crawford and E. M. Knoer, “Job matching with heterogeneous firms and workers,” *Econometrica*, vol. 49, no. 2, pp. 437–450, Mar. 1981.

Transfer

- transfer indicates any type of transaction between two different agents
  - can be real money, fictitious money or credit, etc.
- All previous examples do not consider transfer.
- What if we consider transfer (e.g., monetary)?
  - Matching with transfer
- Can mechanisms like DA still be used?

Matching with Transfer

- It was first described in [SS71] by Shapley and Shubik.
- In [CK81], Crawford and Knoer used the deferred acceptance algorithm to solve the firms and workers problem.
  - One-to-one matching with transfer
- In [KC82], Kelso and Crawford modified the algorithm that can suitable for many-to-one matching with transfer.

[KC82] A. S. Kelso, Jr. and V. P. Crawford, "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, vol. 50, no. 6, pp. 1483-1504, Nov. 1982.

The Firms-Workers Market

- One-to-one matching with transfer
- Set of firms $P = \{p_1, p_2, \cdots, p_m\}$ as buyers
- p_0: virtual firm
- Set of workers $S = \{s_1, s_2, \cdots, s_n\}$ as sellers
- s_0: virtual worker
- Given a matching μ, define

$$x_{ij} = \begin{cases} 1, & \text{if } \mu(s_i) = p_j \\ 0, & \text{otherwise} \end{cases}$$

Utility of Each Player

- each worker s_j has a reservation price (minimum selling price) c_j for working in a firm
- each firm p_i has a reservation price (maximum buying price) r_{ij} for the service by worker s_j (private information)
- if firm p_i pays for service provided from worker s_j at a salary β_{ij}
  - The utility of firm p_i is $u_i = r_{ij} - \beta_{ij}$
  - The utility of worker s_j is $v_j = \beta_{ij} - c_j$

$r_{ij} - c_j$ in total

Optimal Assignment Problem

- Let $\alpha_{ij} = \max(0, r_{ij} - c_j)$ be the gain of a matching $(p_i, s_j) \in P \times S$
- Objective:

social welfare

$$\max \boxed{\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} x_{ij}}$$

has an optimal solution called optimal matching but is NP-Hard

- such that

$$\left.\begin{aligned} \sum_{i=1}^{m} x_{ij} &\leq 1, \qquad \forall j \in \{1,2,\cdots,n\} \\ \sum_{j=1}^{n} x_{ij} &\leq 1, \qquad \forall i \in \{1,2,\cdots,m\} \end{aligned}\right\} \text{One-to-one}$$

$$x_{ij} \in \{0,1\}, \qquad \forall i \in \{1,2,\cdots,m\} \text{ and } \forall j \in \{1,2,\cdots,n\}$$

Optimal Assignment: An Example

workers (sellers)	s_1	s_2	s_3
c_j	18	15	19

firms (buyers)	p_1	p_2	p_3
r_{ij}	23,26,20	22,24,21	21,22,17

matching	Gain of matching						$\sum\sum\alpha_{ij}$
	$\mu(p_1)$	α_{1j}	$\mu(p_2)$	α_{2j}	$\mu(p_3)$	α_{3j}	
1	s_1	5	s_2	9	s_3	0	14
2	s_1	5	s_3	2	s_2	7	14
3	s_2	11	s_1	4	s_3	0	15
4	s_2	11	s_3	2	s_1	3	16
5	s_3	1	s_1	4	s_2	7	12
6	s_3	1	s_2	9	s_1	3	13

Stable Payoff Vectors

- A dual problem of the optimal assignment problem
- Finding vectors $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$ (stable payoff vectors) which form a solution of

$$\min\left(\sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j\right)$$

The set of stable payoff vectors is nonempty

such that

$u_i + v_j \geq \alpha_{ij}$ for each $(p_i, s_j) \in P \times S$

$u_i \geq 0, v_j \geq 0$ (individual rationality)

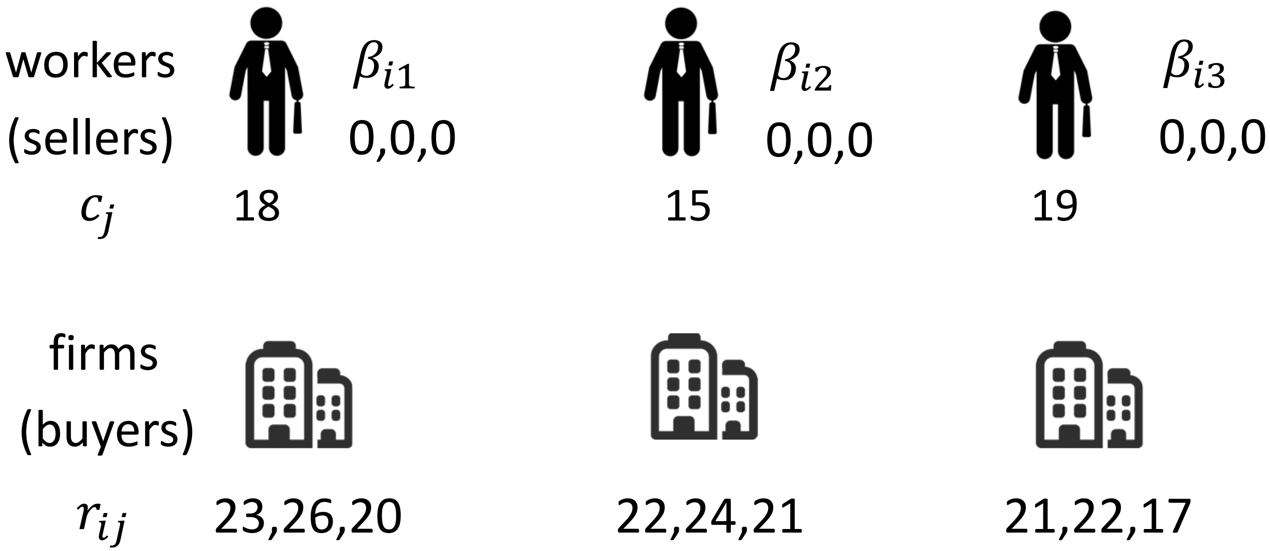
Stable Outcome $(\mathbf{x}; \mathbf{u}, \mathbf{v})$

- An outcome $(\mathbf{x}; \mathbf{u}, \mathbf{v})$ is a stable outcome if
  - $\mathbf{x} = \{x_{i,j}\}$ is an optimal matching and
  - $(\mathbf{u}, \mathbf{v})$ is a stable payoff.
- Every player with a positive payoff at a stable outcome is matched at every stable outcome
- If $x_{i,j} = 1$ at some optimal matching, and if $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{u}', \mathbf{v}')$ are stable payoff vectors, then $u_i' > u_i \Leftrightarrow v_j' < v_j$ for each j.

General Assignment Algorithm (1/27)

V. P. Crawford and E. M. Knoer, “Job matching with heterogeneous firms and workers,” *Econometrica*, 49(2): 437–450, 1981.

- Workers are sellers and firms are buyers
- Each seller s_j has a minimal selling price (called reservation value) c_j not known by any buyer



General Assignment Algorithm (2/27)

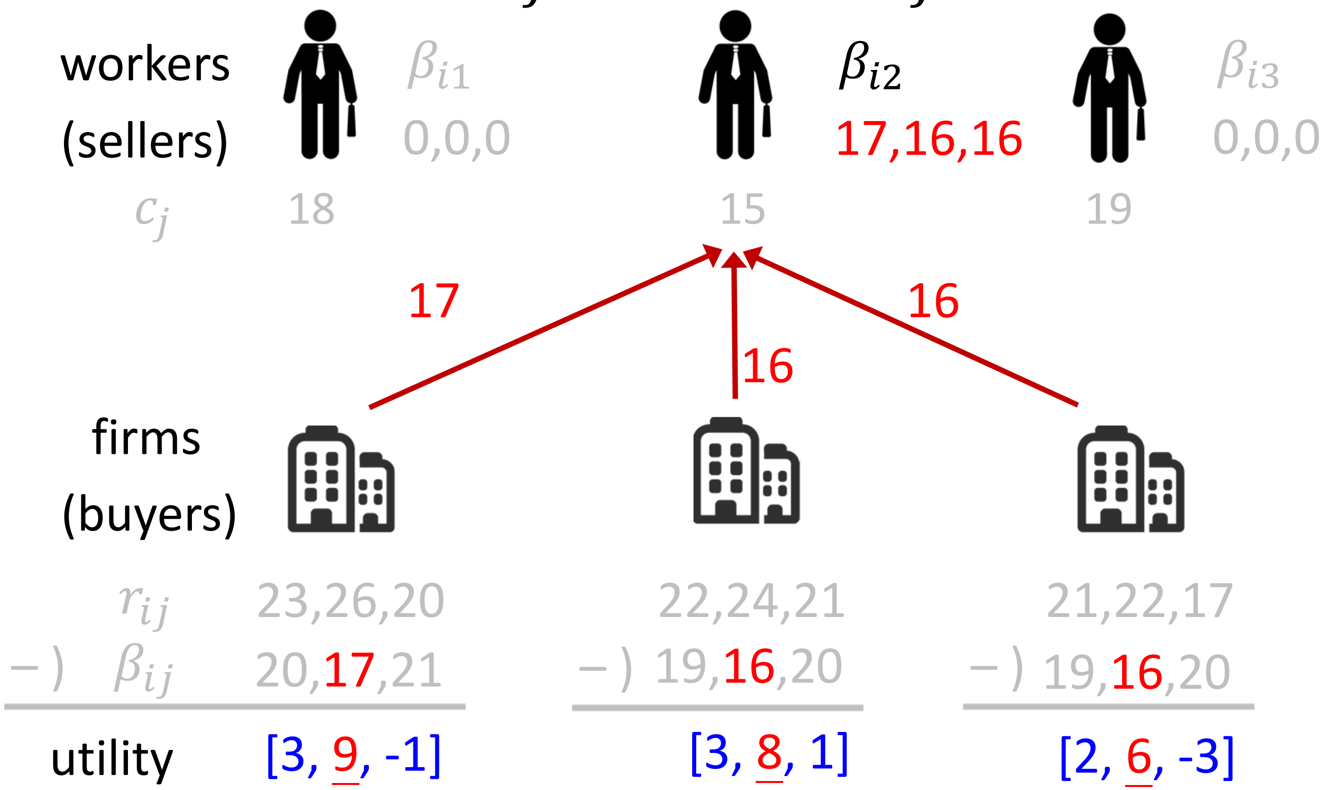
- Each buyer p_i makes an offer $c_j + \epsilon_i$ for each seller s_j (Here we assume $\epsilon_1 = 2$ and $\epsilon_2 = \epsilon_3 = 1$)

workers (sellers)			
	β_{i1} 0,0,0	β_{i2} 0,0,0	β_{i3} 0,0,0
c_j	18	15	19

firms (buyers)			
r_{ij}	23,26,20	22,24,21	21,22,17
$-)$ β_{ij}	20,17,21	$-)$ 19,16,20	$-)$ 19,16,20
utility	[3, 9, -1]	[3, 8, 1]	[2, 6, -3]

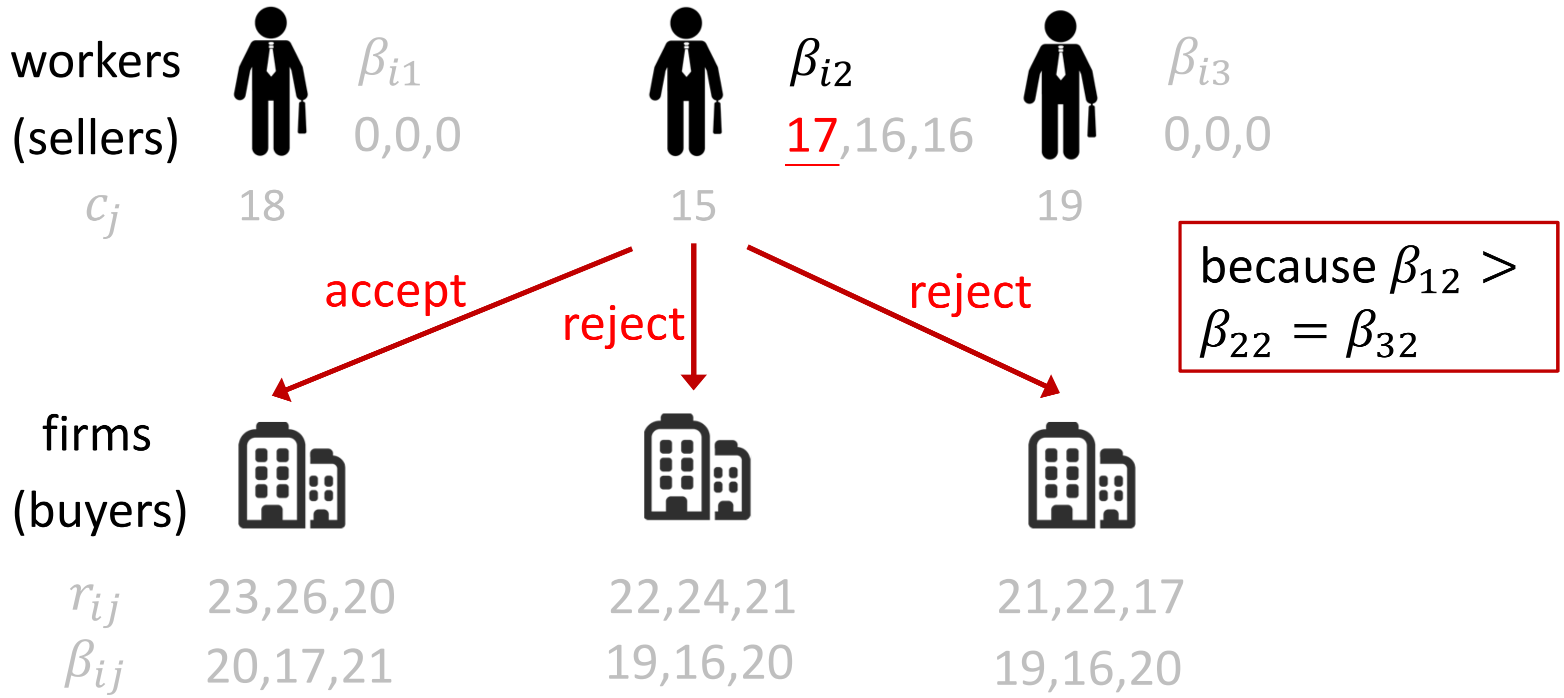
General Assignment Algorithm (3/27)

- Each buyer p_i identifies a seller s_j with the highest utility and sends s_j the offer β_{ij}



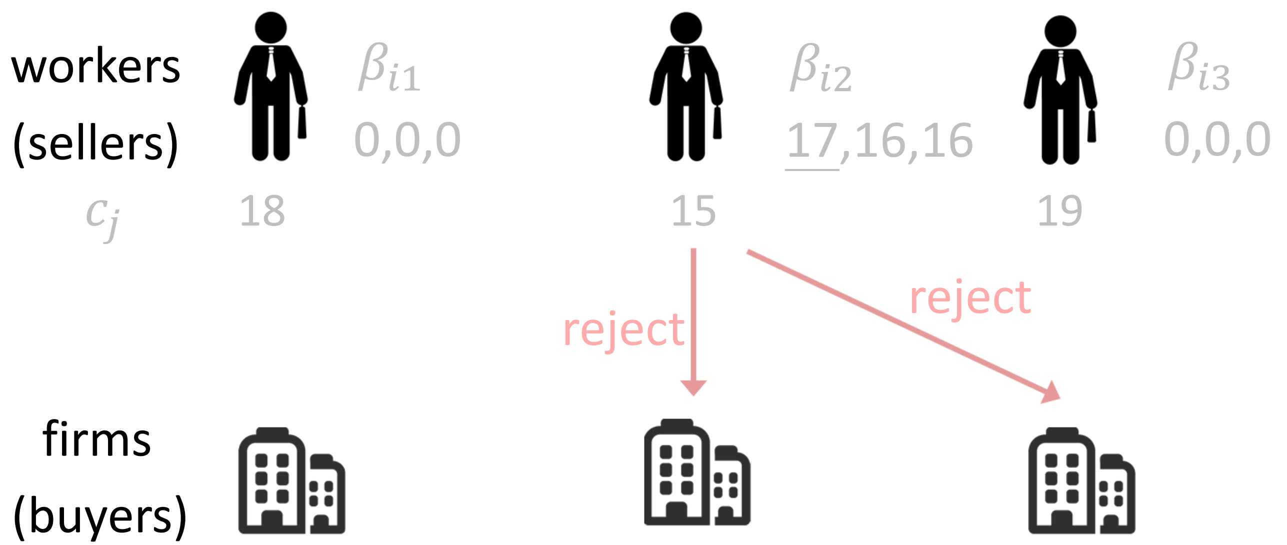
General Assignment Algorithm (4/27)

- a seller receiving multiple offers tentatively accepts the highest one and rejects all the others



General Assignment Algorithm (5/27)

- A buyer p_i receiving a reject considers raising her/his offer by ϵ_i (price-step number)

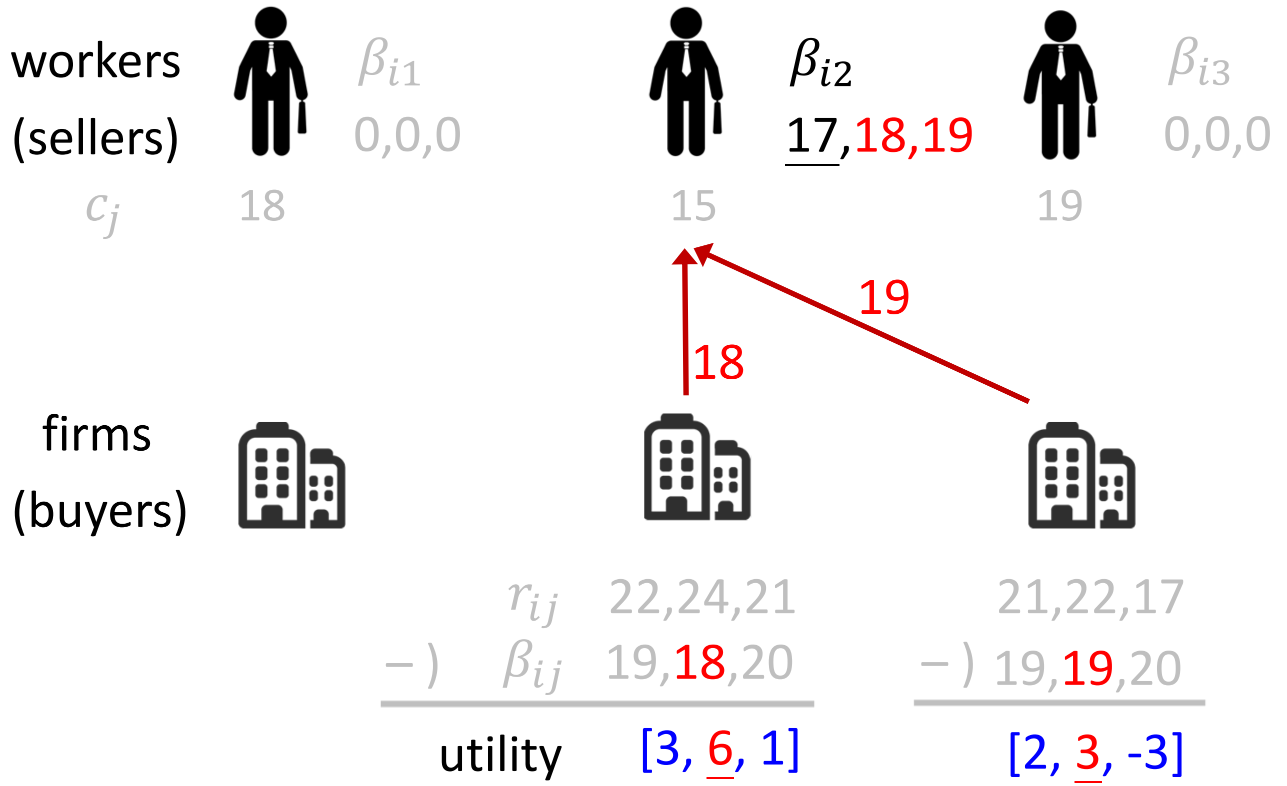


Here we assume $\epsilon_2 = 2$ & $\epsilon_3 = 3$

r_{ij}	22,24,21	21,22,17
−) β_{ij}	19,18,20	19,19,20
utility	[3, 6 ,1]	[2, 3, -3]

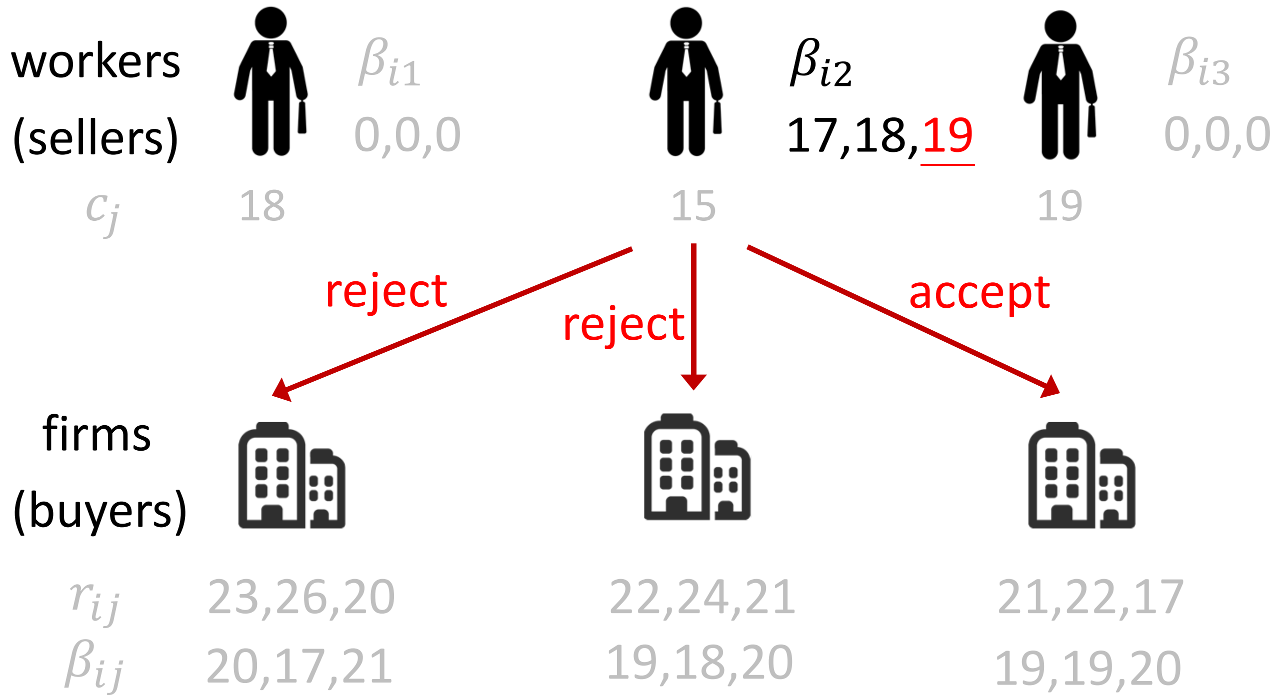
General Assignment Algorithm (6/27)

- Each unmatched buyer sends her/his offer to a seller with the highest utility (arbitrary tie breaking)



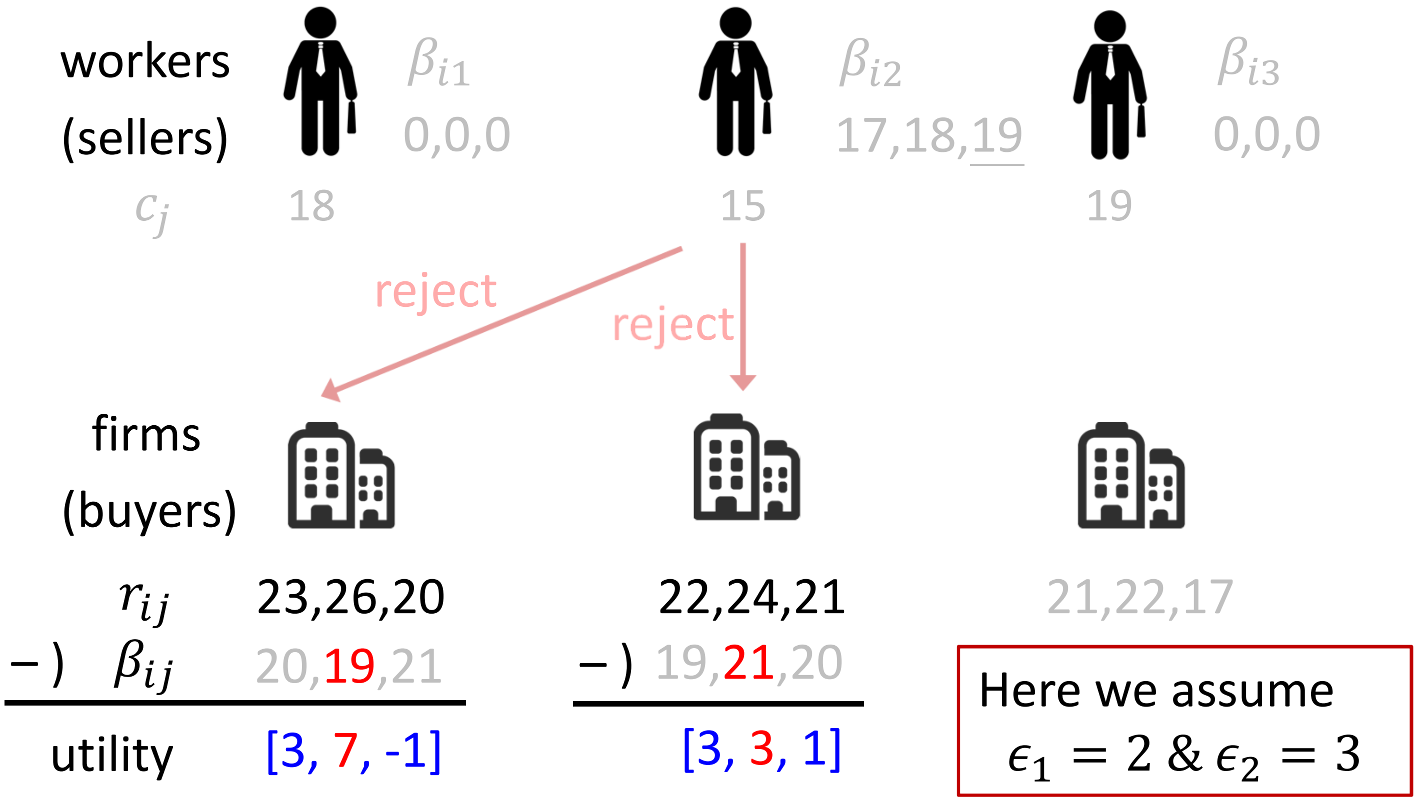
General Assignment Algorithm (7/27)

- The new offer may make a seller changing its mind. The seller then updates her/his decision.



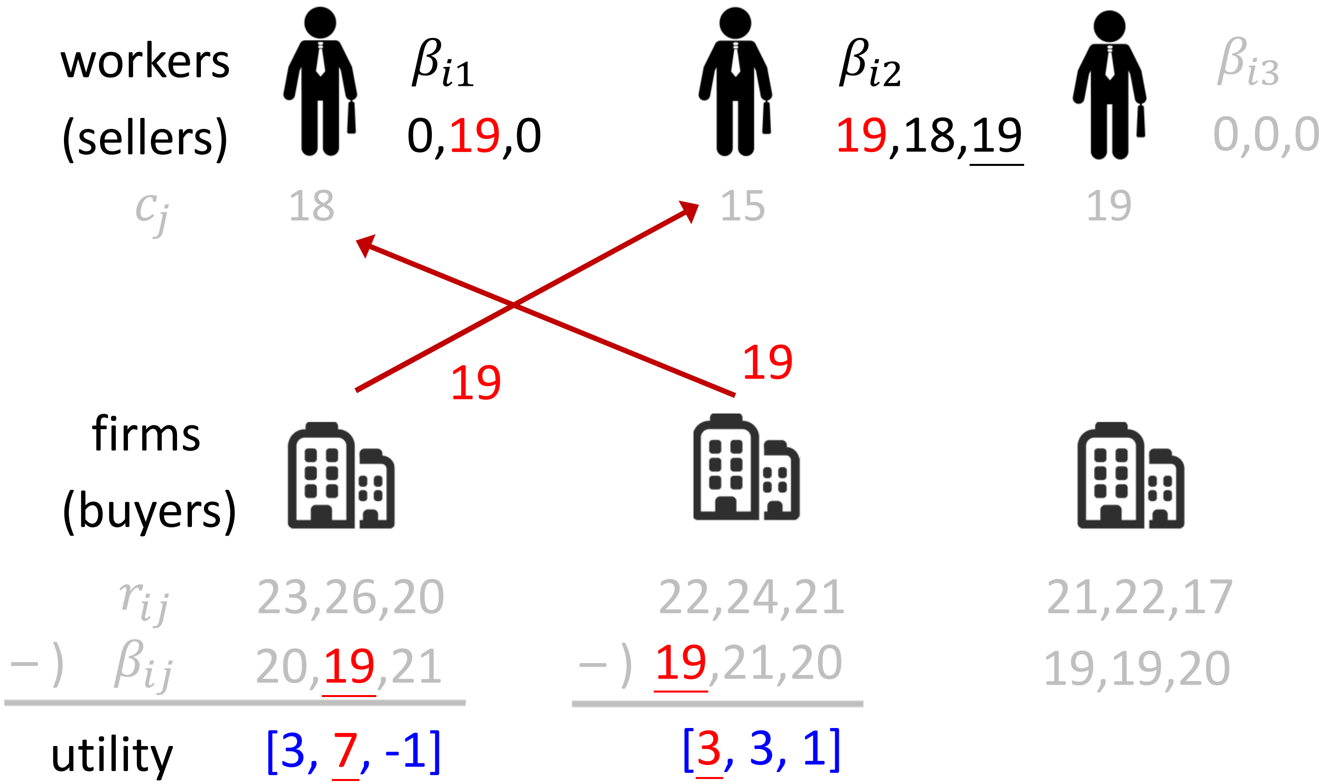
General Assignment Algorithm (8/27)

- Again, each buyer p_i receiving a reject considers raising her/his offer by ϵ_i (price-step number)



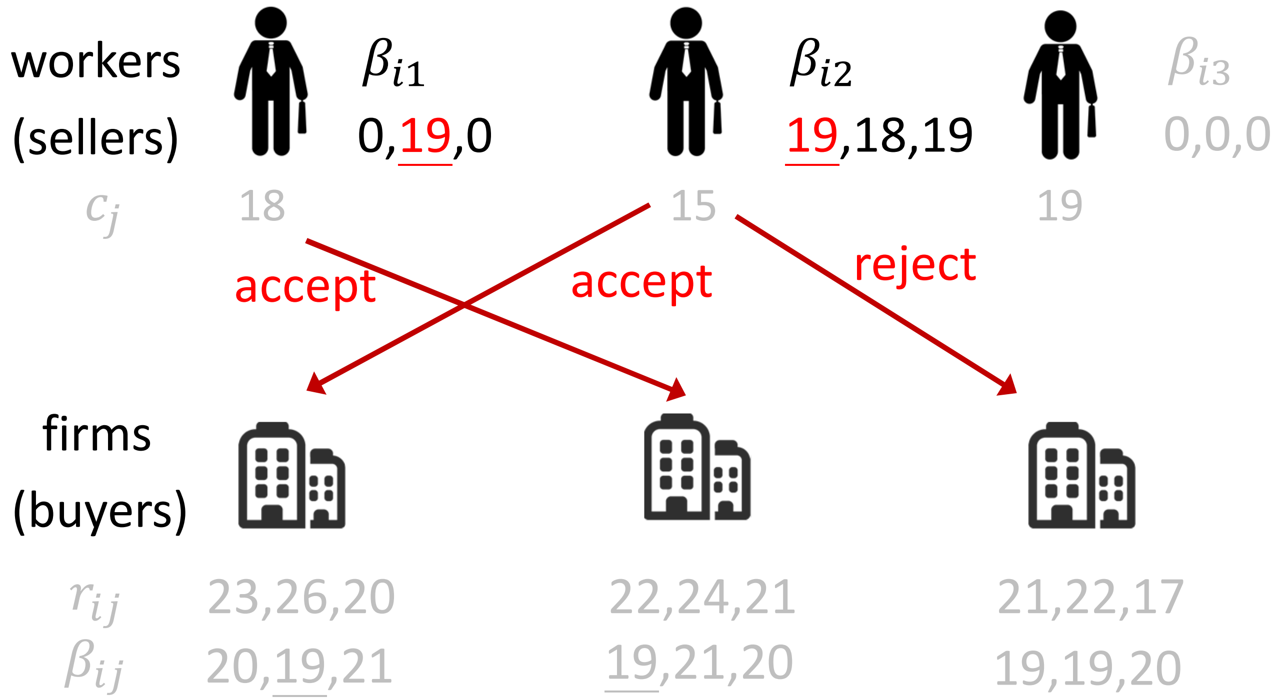
General Assignment Algorithm (9/27)

- And sends an offer to a seller with the new highest utility (arbitrary tie breaking)



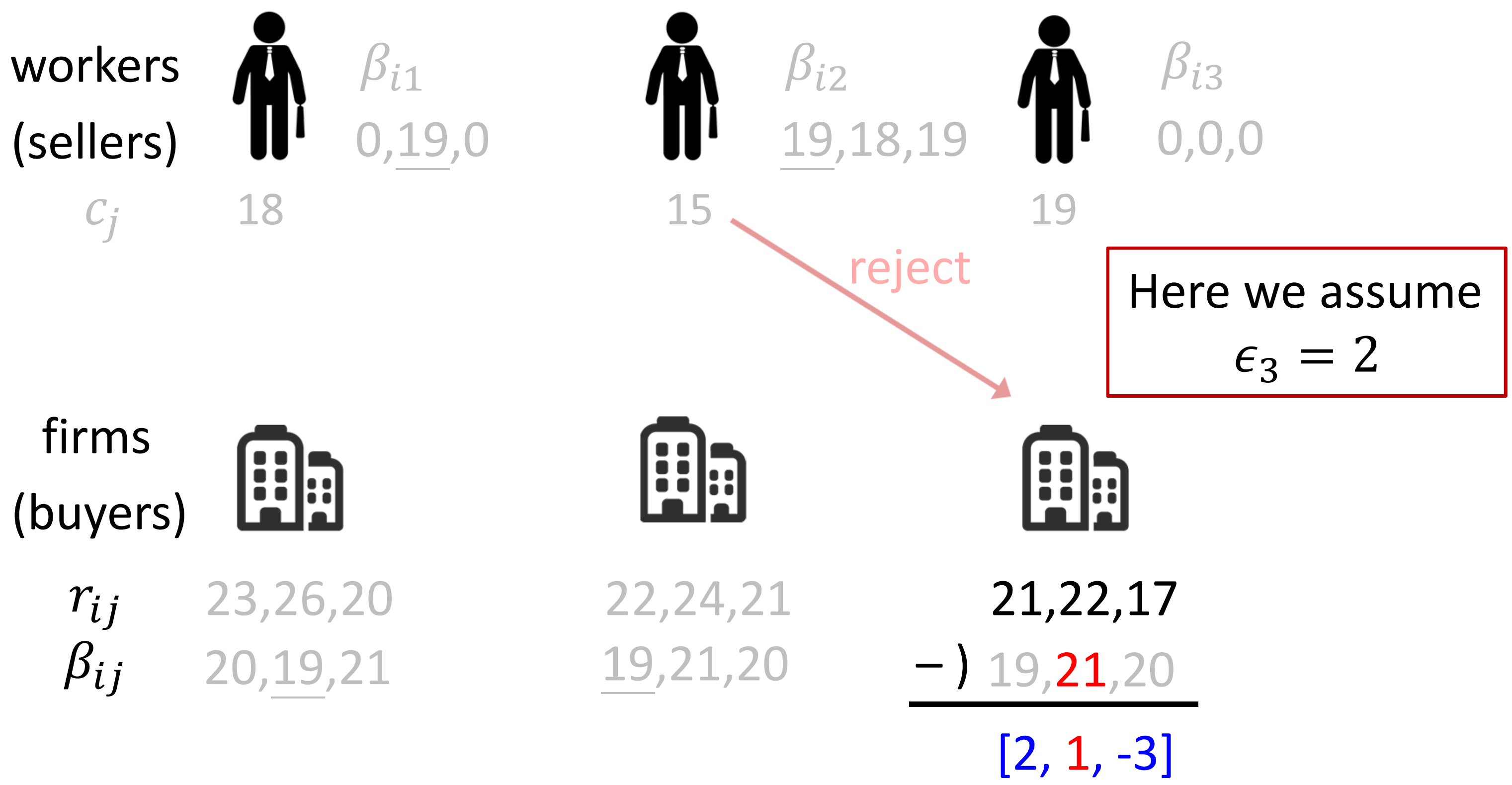
General Assignment Algorithm (10/27)

- a seller accepts the highest offer (arbitrary tie breaking) and rejects all the others



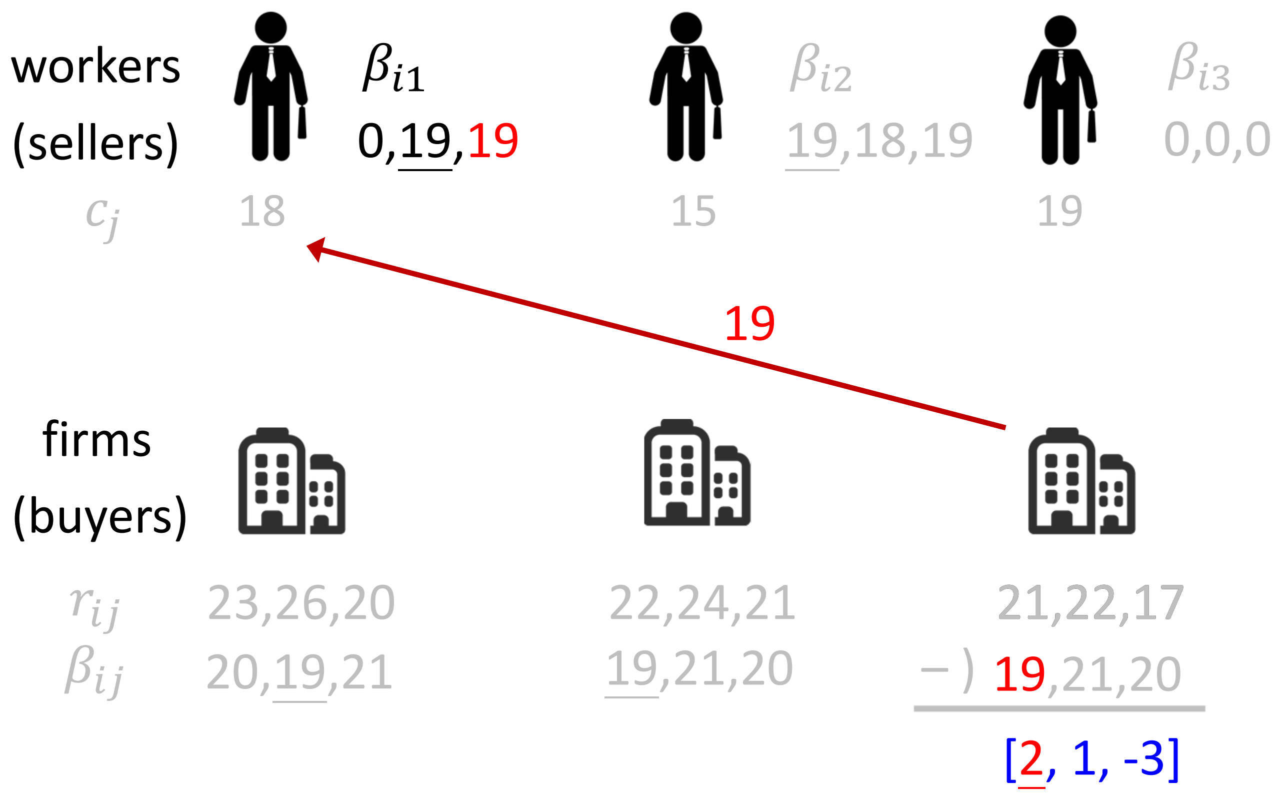
General Assignment Algorithm (11/27)

- Buyer p_3 attempts raising her/his offer to s_2



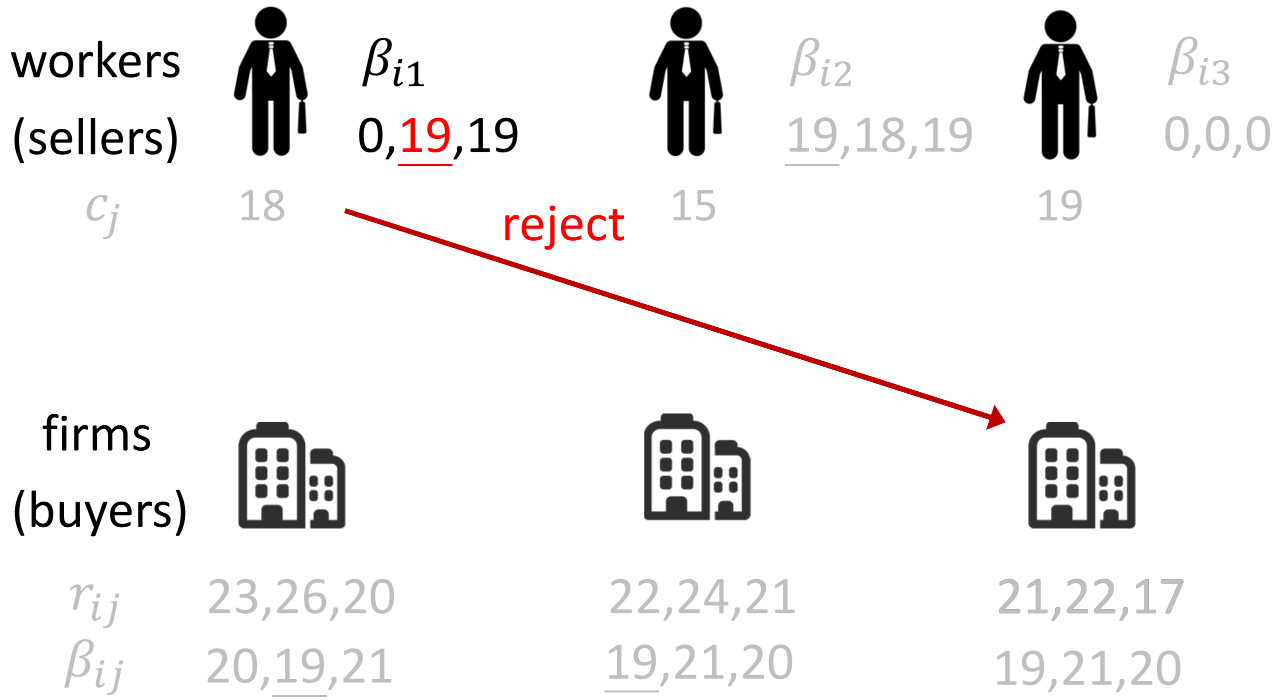
General Assignment Algorithm (12/27)

- but finds s_1 more attractive than s_2



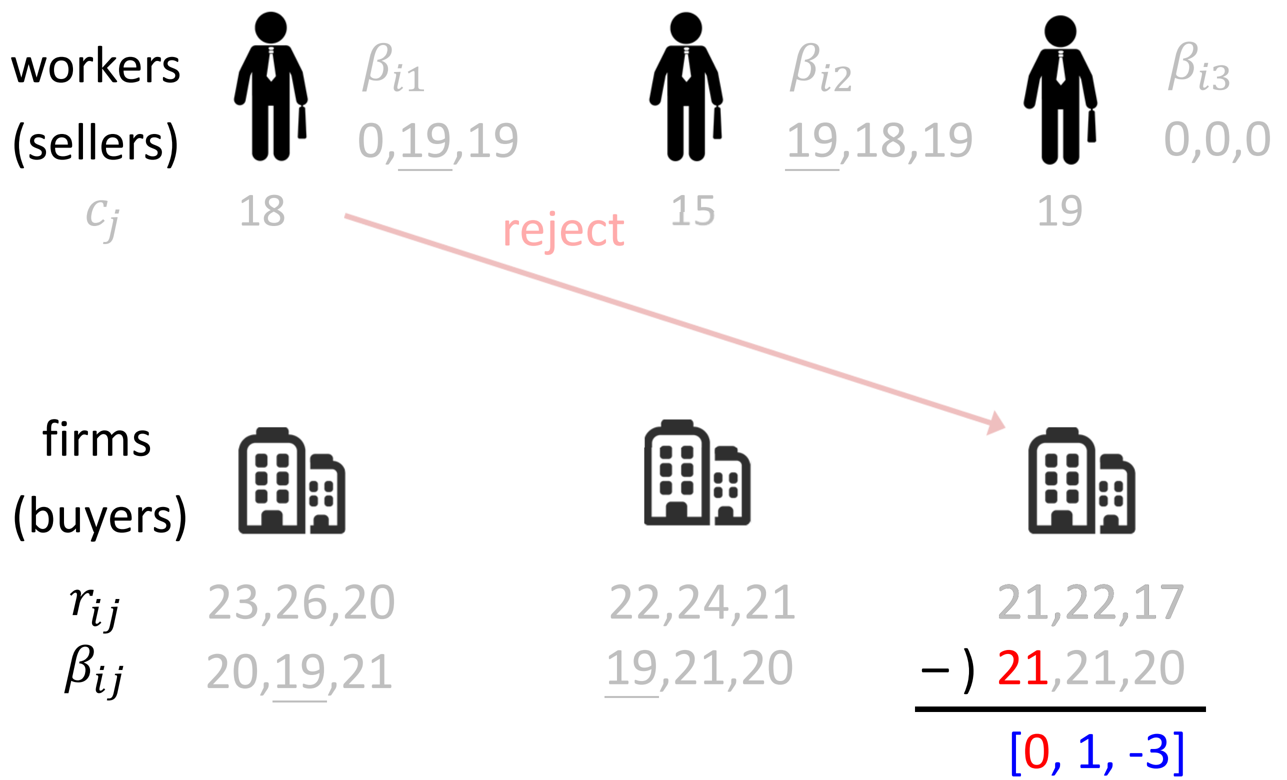
General Assignment Algorithm (12/27)

- p_3's new offer is as good as p_2's, s_1 selects to reject p_3's offer (just a possibility)



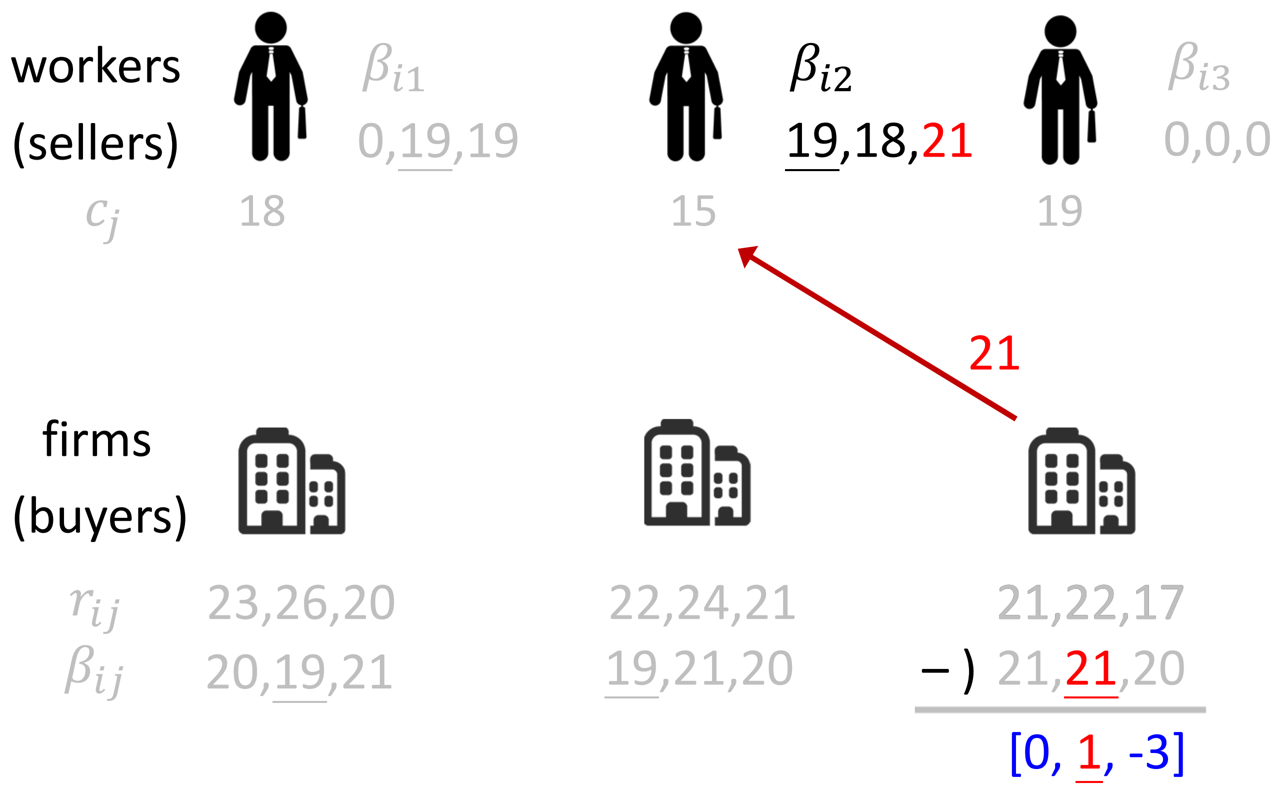
General Assignment Algorithm (13/27)

- When receiving the rejection from s_1, p_3 updates her/his offer β_{31} (assuming $\epsilon_3 = 2$)



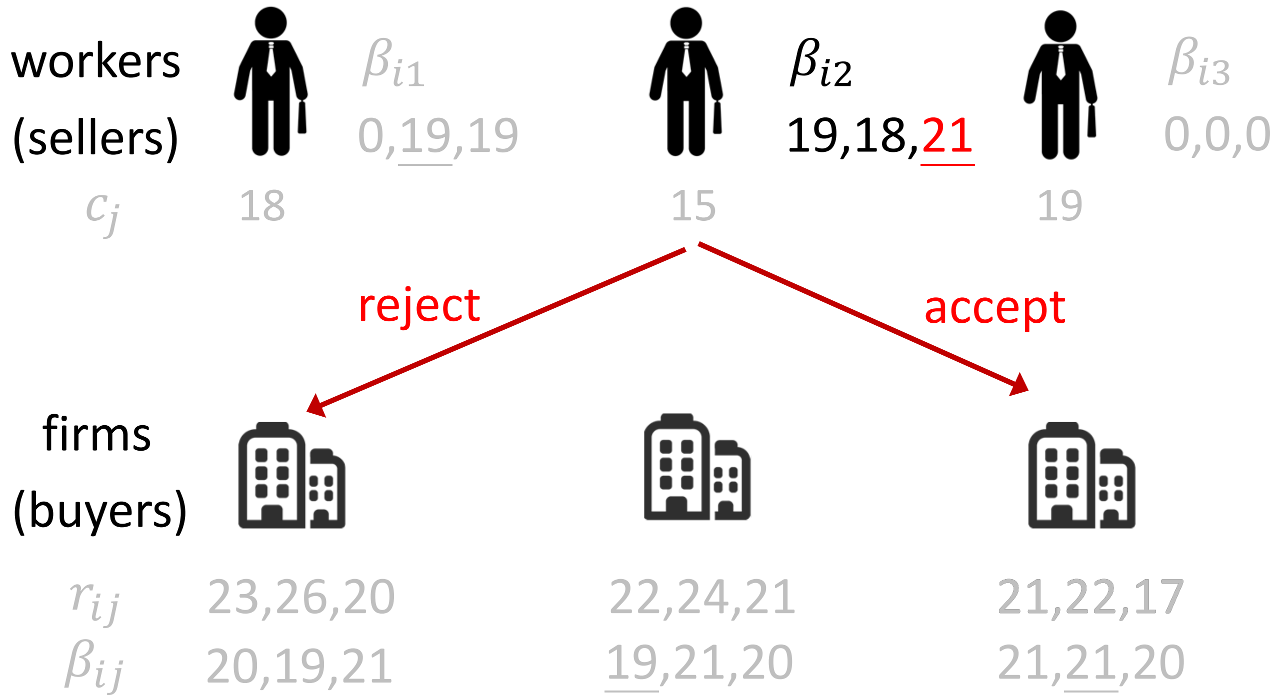
General Assignment Algorithm (14/27)

- Since the service from s_1 is no longer worthy, p_3 turns to s_2



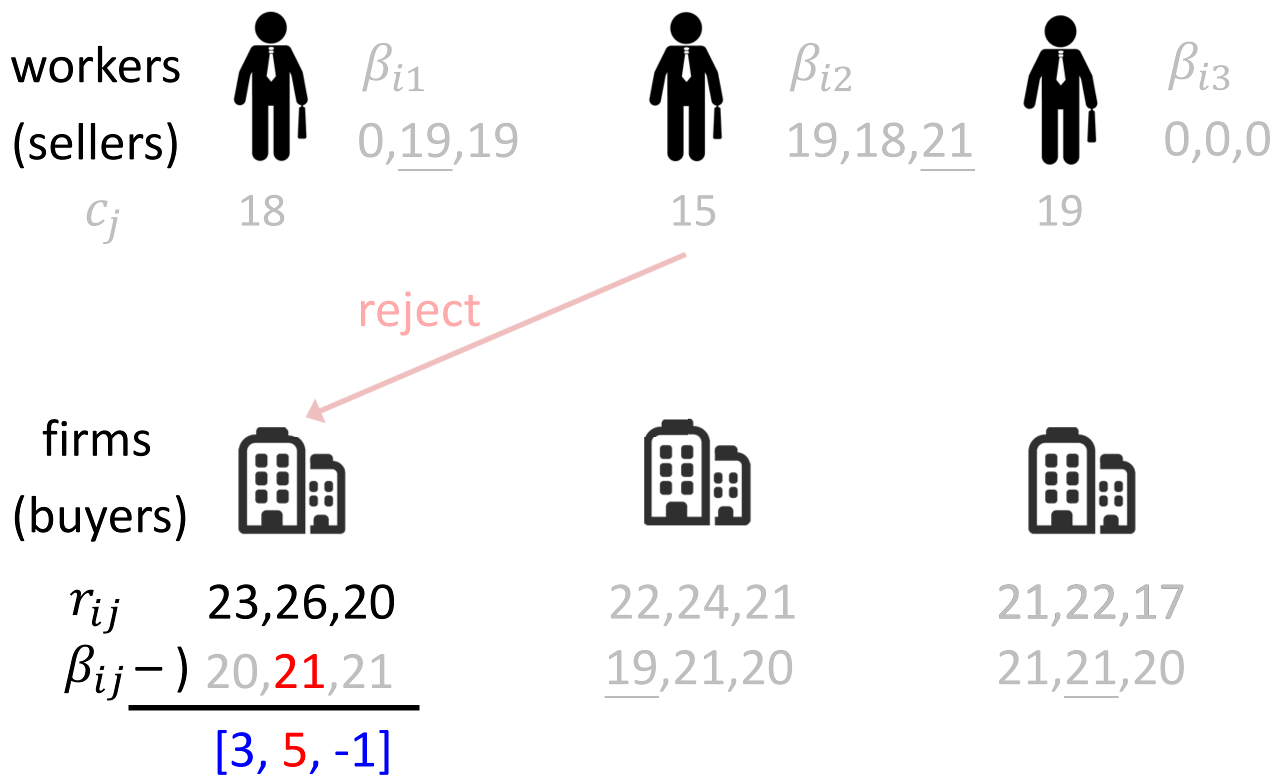
General Assignment Algorithm (15/27)

- s_2 finds p_3’s offer more profitable (than p_1’s) and thus accepts p_3’s offer and rejects p_1’s



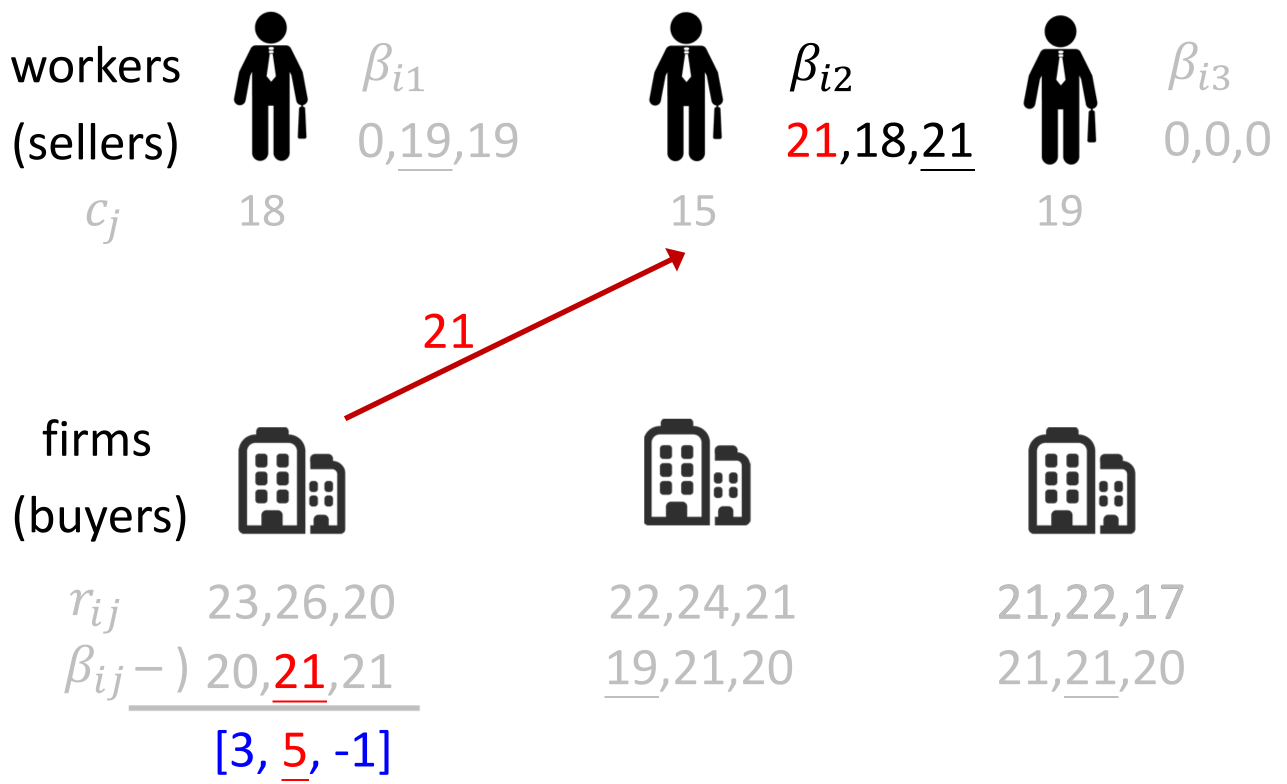
General Assignment Algorithm (16/27)

- When receiving the rejection from s_2, p_1 updates her/his offer β_{12} (assuming $\epsilon_1 = 2$)



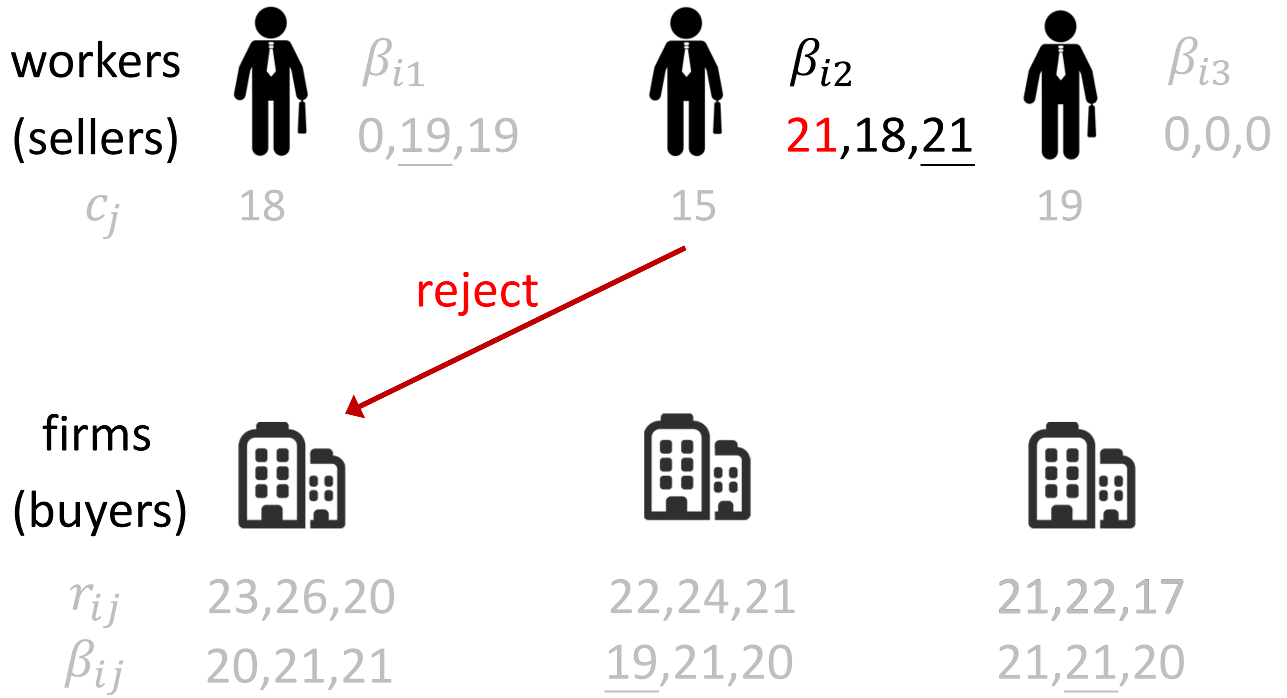
General Assignment Algorithm (17/27)

- and sends her/his updated offer back to s_2



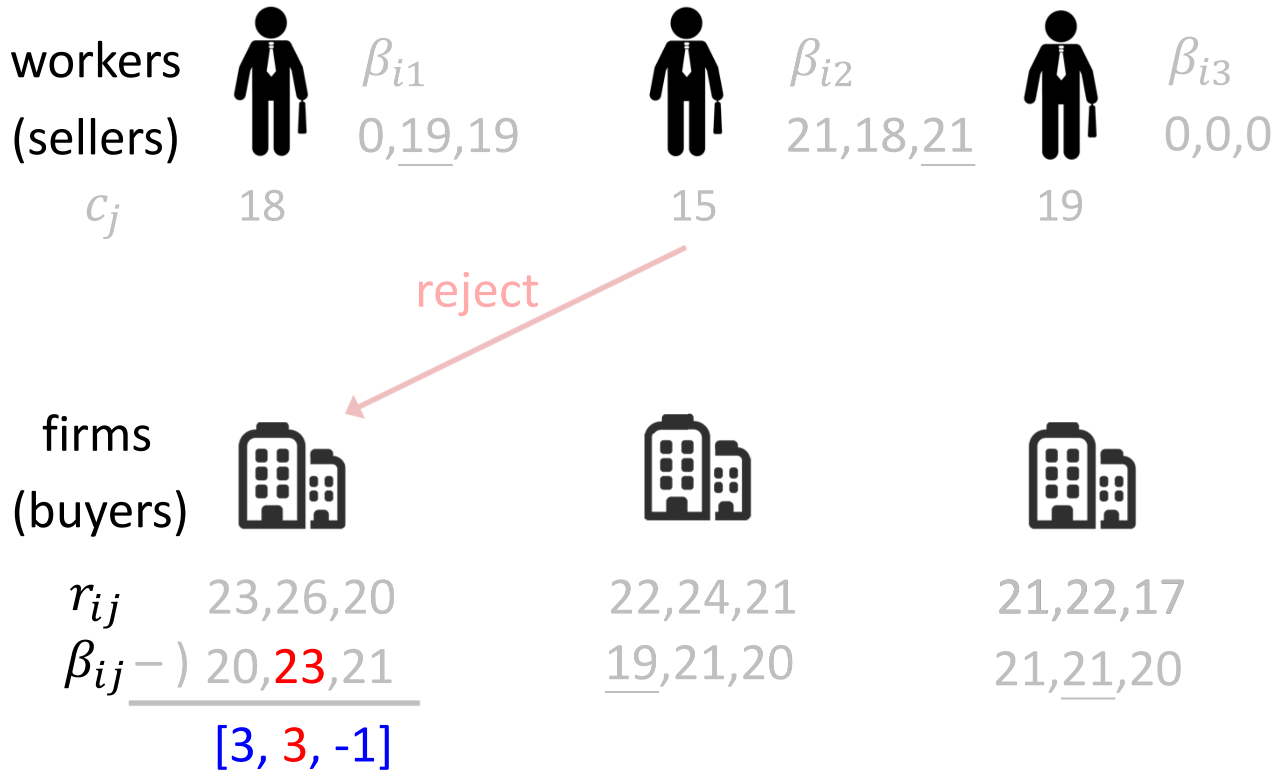
General Assignment Algorithm (18/27)

- This is a tie, so s_2 arbitrarily selects one offer (say, p_3) to accept and rejects the other (p_1 in this case)



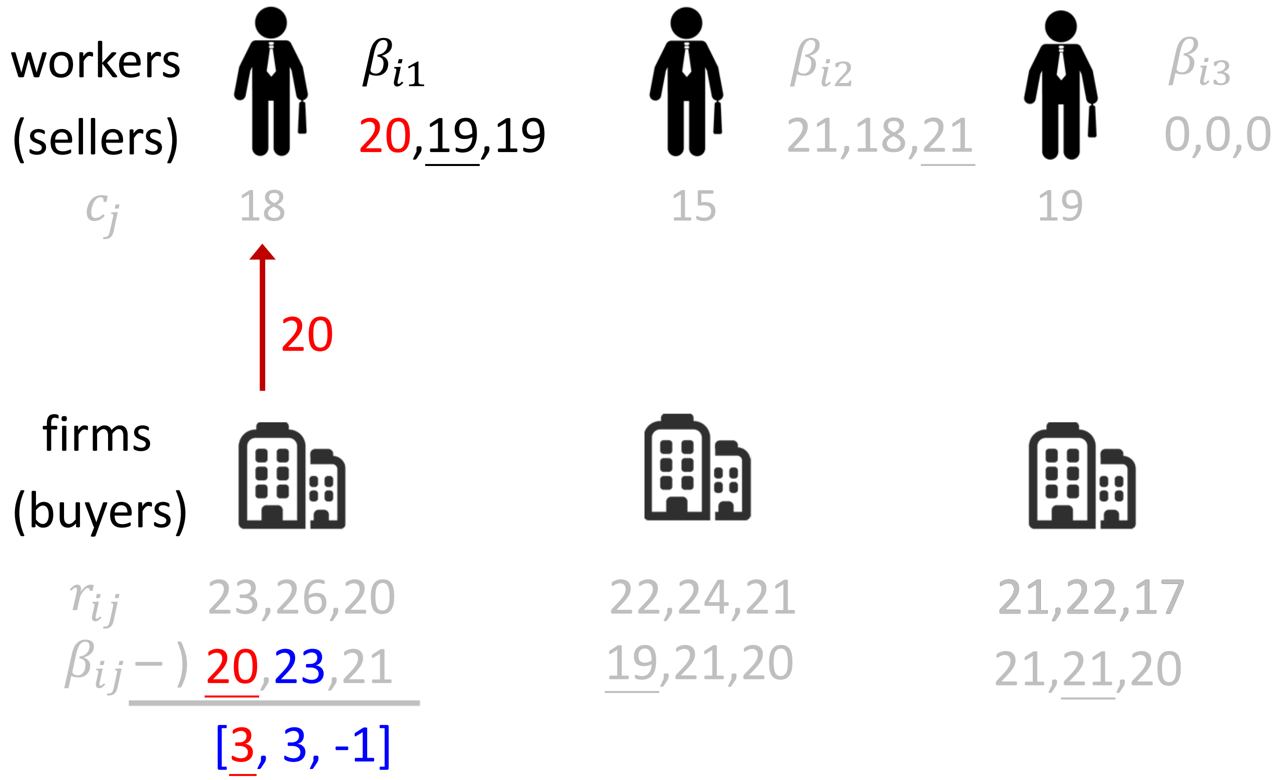
General Assignment Algorithm (19/27)

- When receiving the rejection from s_2, p_1 updates her/his offer β_{12} (assuming $\epsilon_1 = 2$)



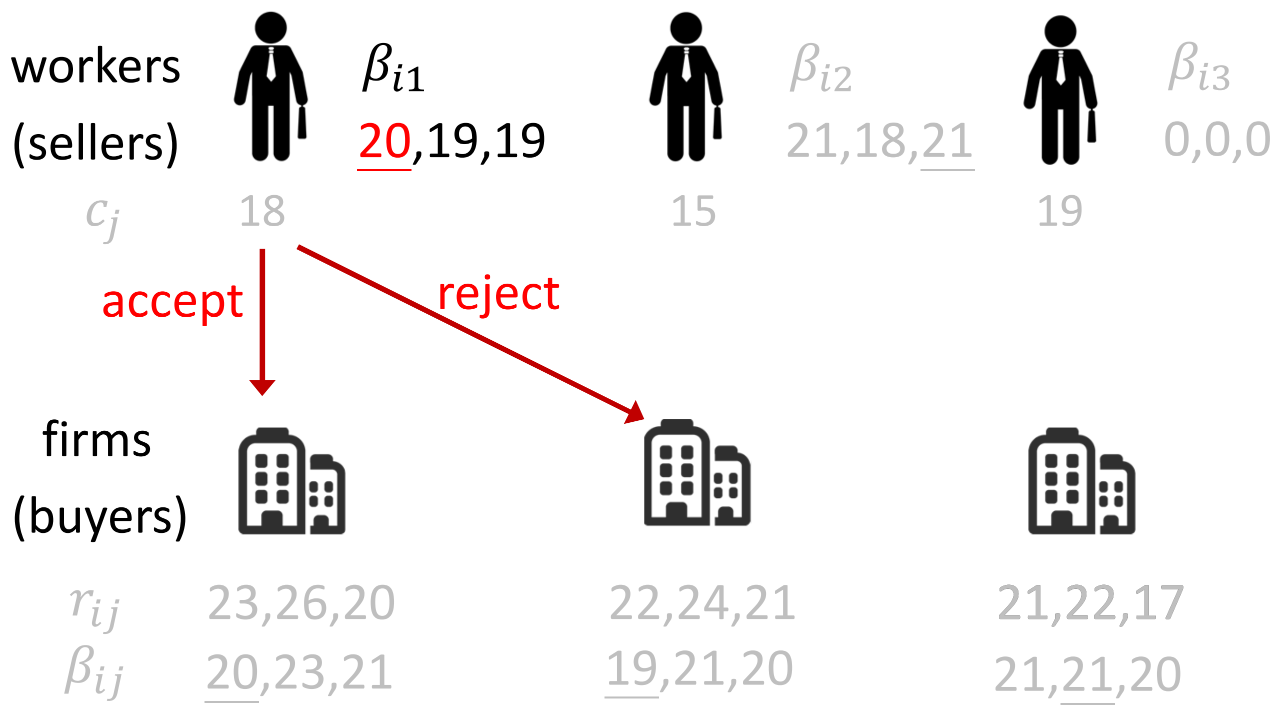
General Assignment Algorithm (20/27)

- There is a tie. p_1 arbitrarily selects one (say, s_1) to send her/his offer



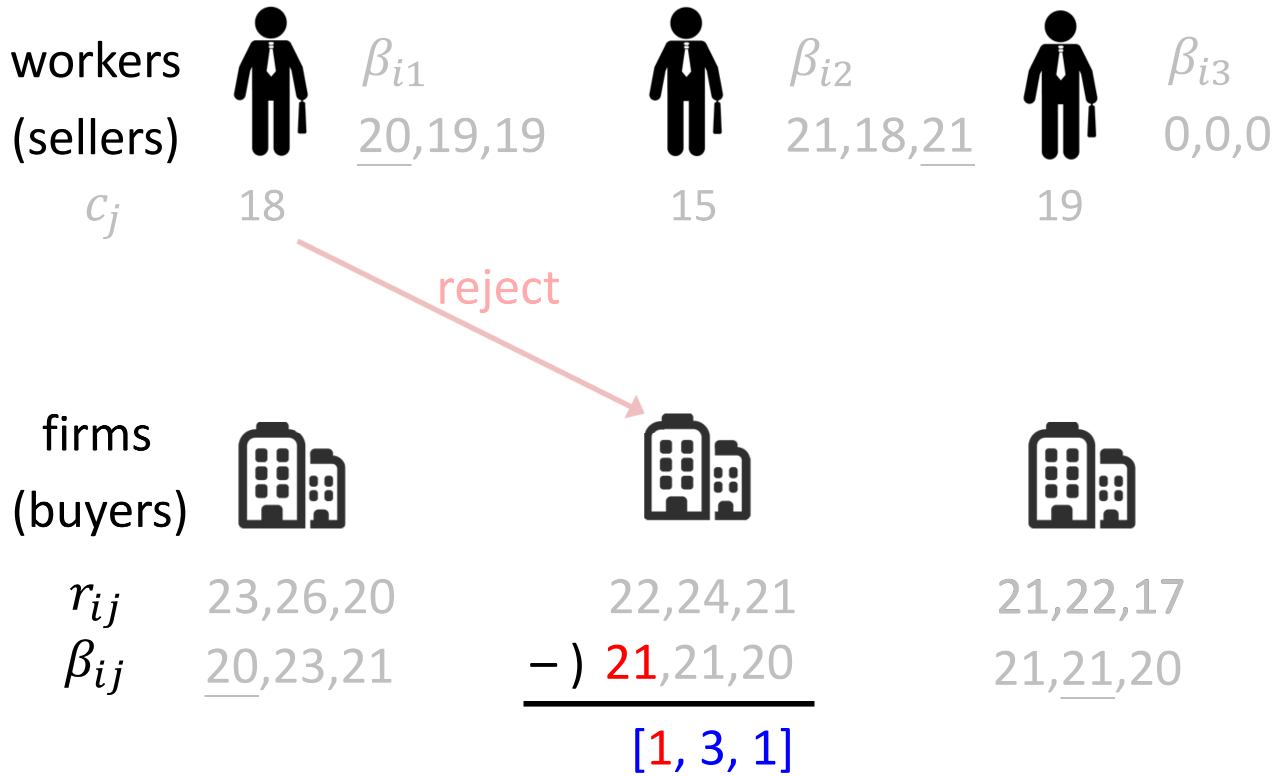
General Assignment Algorithm (21/27)

- p_1's offer is higher than p_2's, so she/he accepts the former and rejects the latter



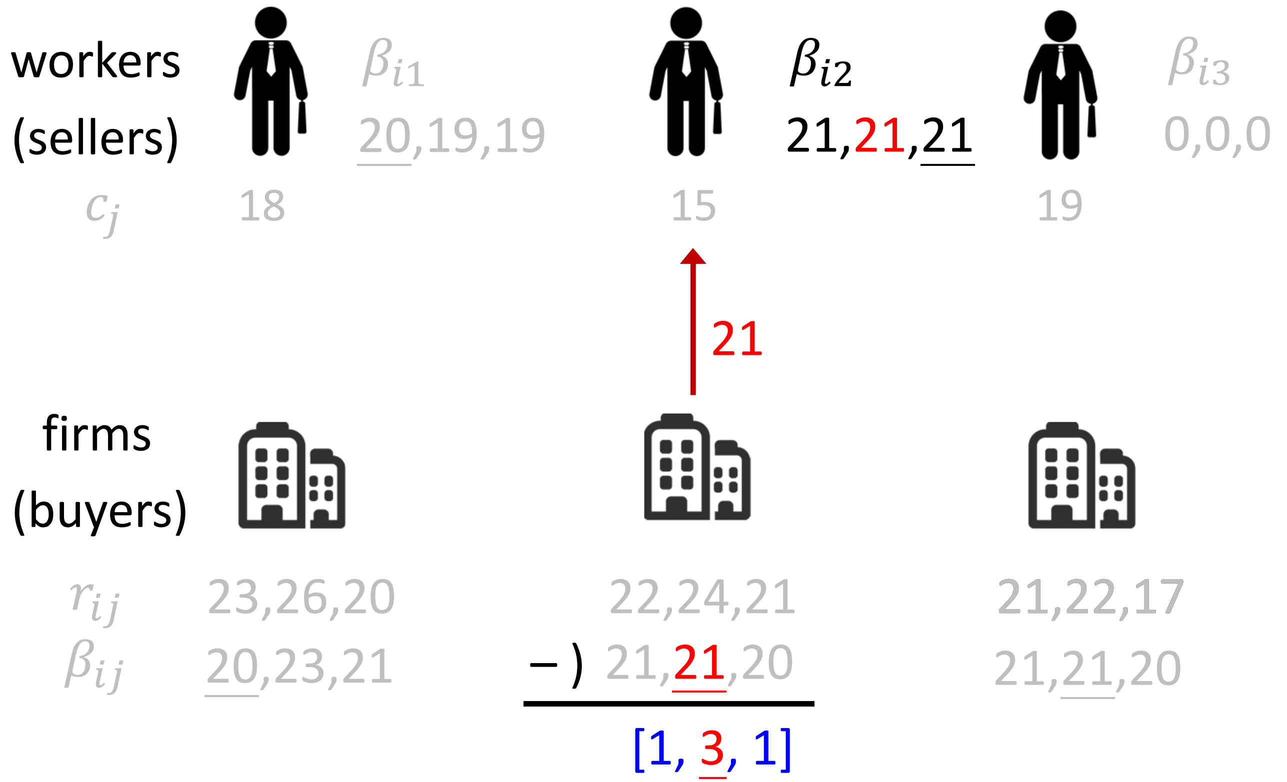
General Assignment Algorithm (22/27)

- After receiving the rejection from s_1, p_2 raises her/his offer β_{21} (assuming $\epsilon_2 = 2$)



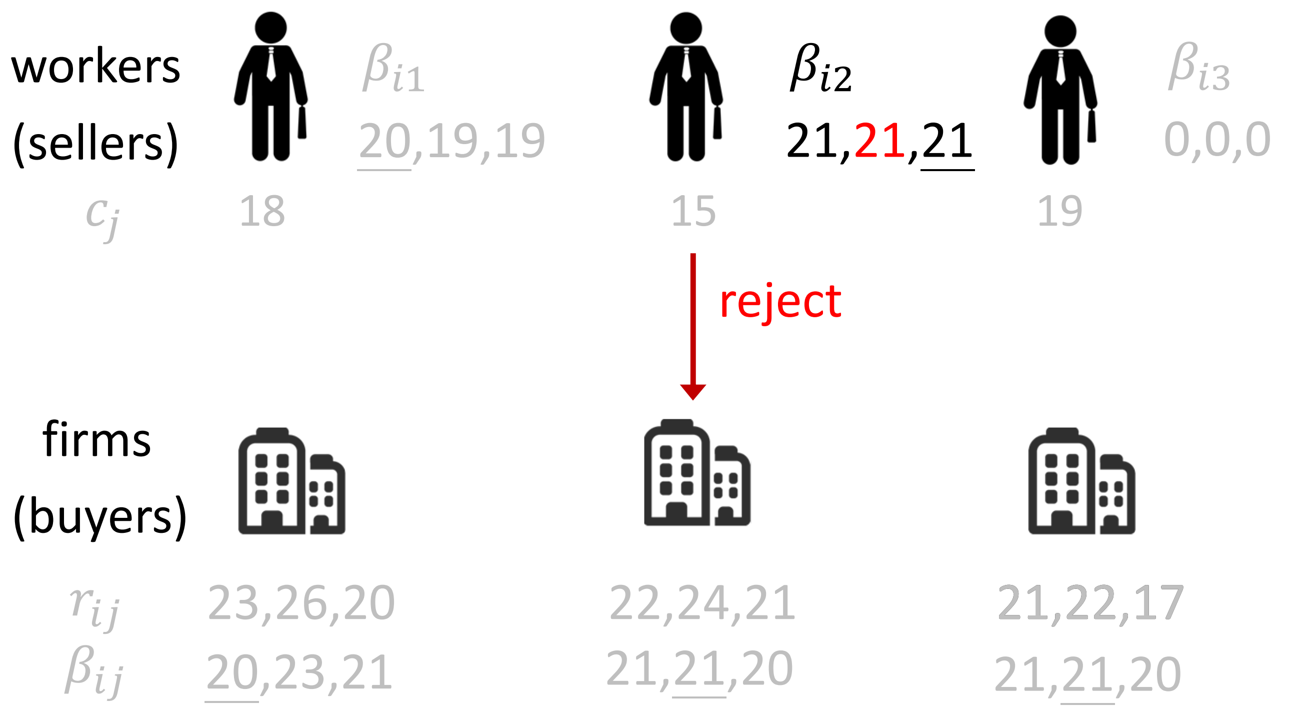
General Assignment Algorithm (23/27)

- p_2 selects s_2 to send her/his offer because of the highest utility with s_2



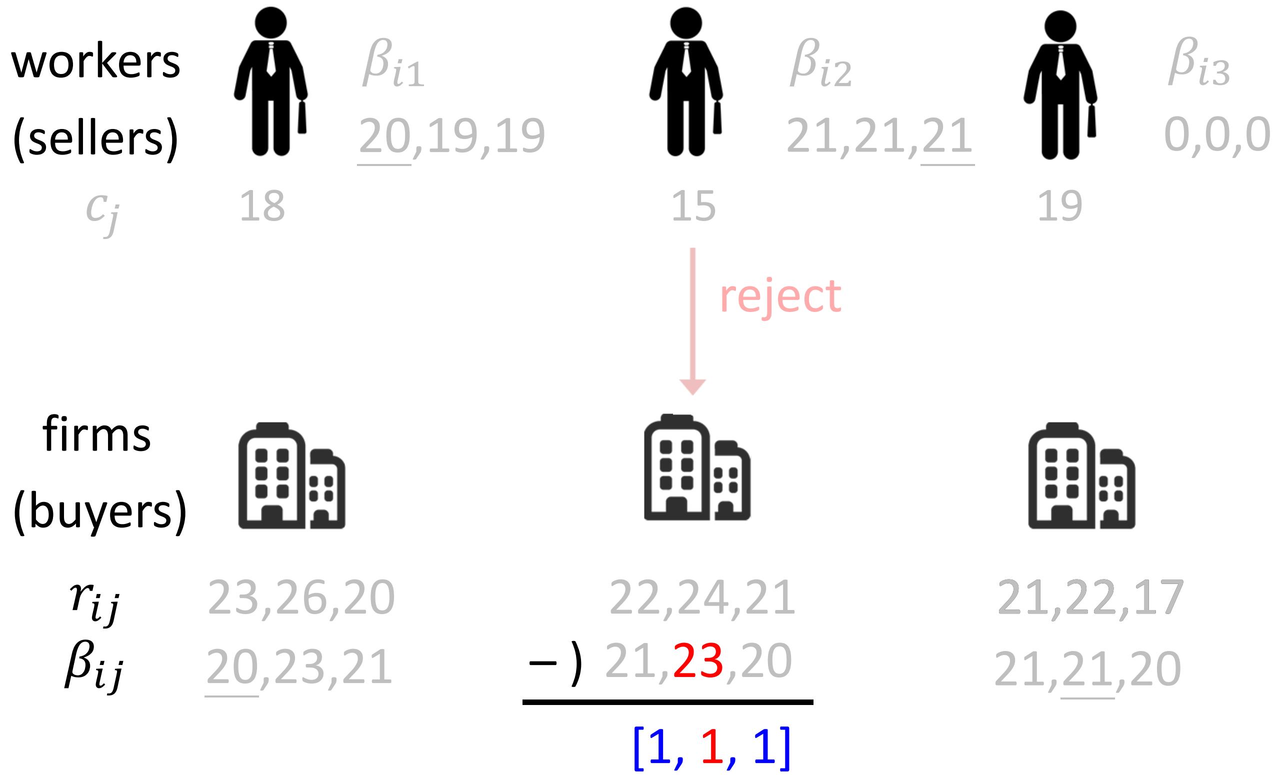
General Assignment Algorithm (24/27)

- s_2 rejects p_2's offer because this offer is not higher than what she/he already has



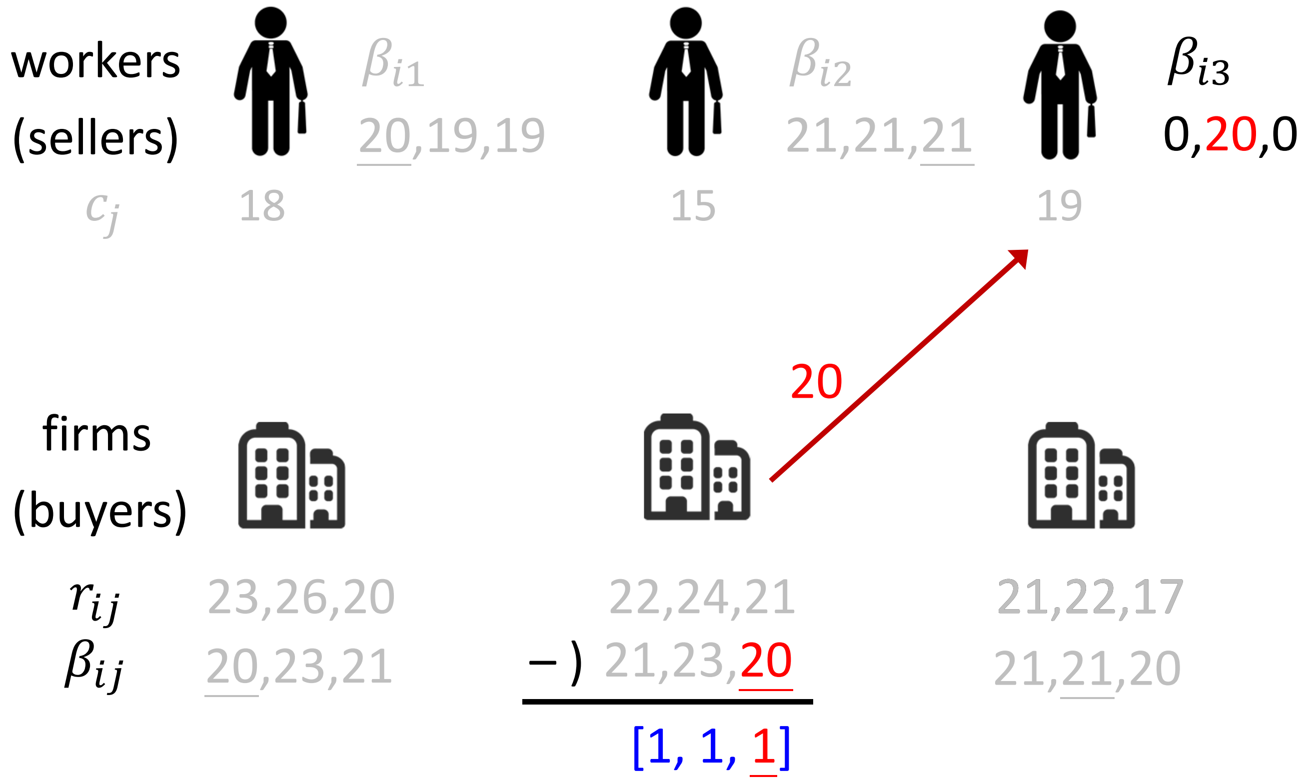
General Assignment Algorithm (25/27)

- After receiving the rejection from s_2, p_2 raises her/his offer β_{22} (assuming $\epsilon_2 = 2$)



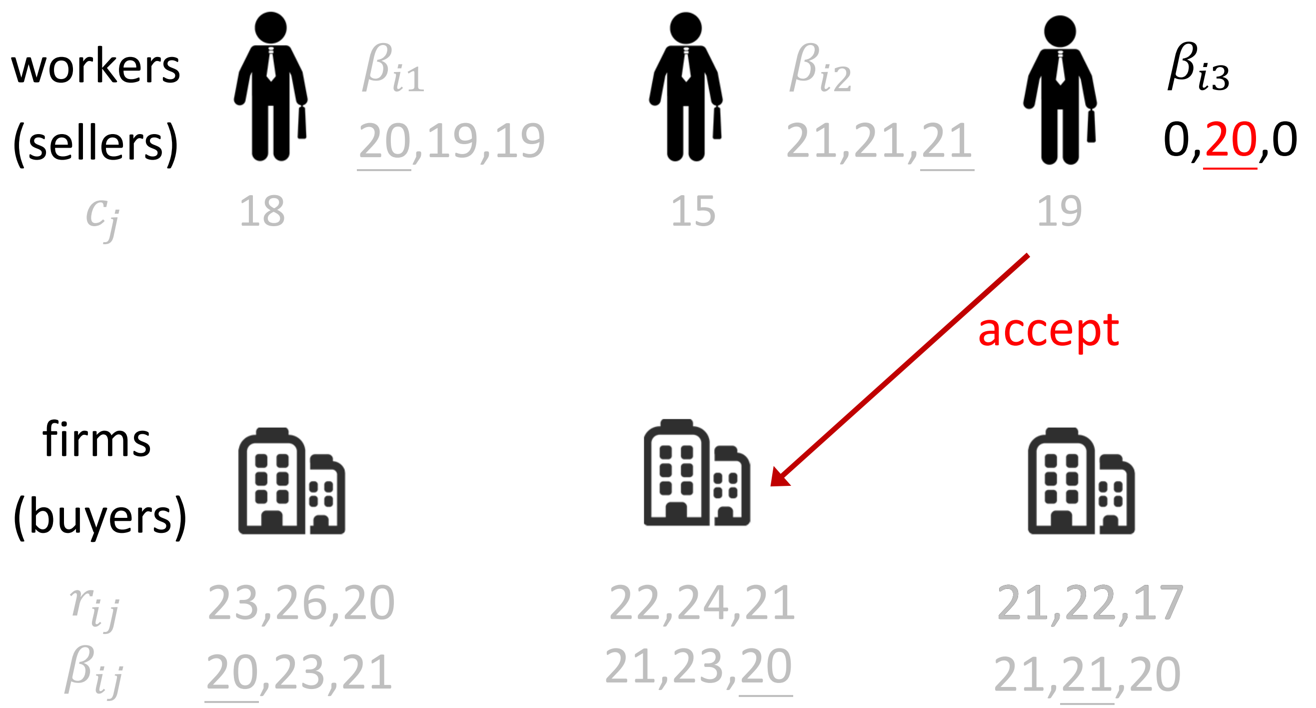
General Assignment Algorithm (26/27)

- As the utilities from all workers are the same, p_2 randomly selects s_3 to send her/his offer



General Assignment Algorithm (27/27)

- p_2's offer is the only one s_3 has, so s_3 accepts the offer



The Final Result

workers (sellers): s_1, s_2, s_3

c_j: 18, 15, 19

firms (buyers): p_1, p_2, p_3

	p_1	p_2	p_3
r_{ij}	23,26,20	22,24,21	21,22,17
β_{ij}	$\underline{20}$,23,21	21,23,$\underline{20}$	21,$\underline{21}$,20

matching	Gain of matching						$\sum\sum \alpha_{ij}$
	$\mu(p_1)$	α_{1j}	$\mu(p_2)$	α_{2j}	$\mu(p_3)$	α_{3j}	
2	s_1	5	s_3	2	s_2	7	14

Payoff vector $\mathbf{u}$ (for buyers)

u_1	u_2	u_3
3	1	1

Payoff vector $\mathbf{v}$ (for sellers)

v_1	v_2	v_3
2	6	1

Properties of the Algorithm

- The algorithms converges in finite time
- The algorithms converges to an individually rational allocation
- If all prices and offers are integers, and $\epsilon_i = 1$ for all p_i, then the algorithm converges to a core (stable outcome)

If $\epsilon_i = 1$ for all p_i (1/5)

Round	Activity	p_1	p_2	p_3
1	offer	$(s_1, 18)(s_2, 15)(s_3, 19)$		
	result	$(s_1, 18)$ $(s_2, 15)$ accepted	$(s_3, 19)$ accepted	All offers rejected
2	offer		$(s_2, 16)$	$(s_2, 16)$
	result	Rejected	Rejected	Accepted
3	offer	$(s_2, 16)$	$(s_2, 17)$	
	result	Rejected	Accepted	Rejected
4	offer	$(s_2, 17)$		$(s_2, 17)$
	result	Accepted	Rejected	Rejected
5	offer		$(s_2, 18)$	$(s_2, 18)$
	result	Rejected	Accepted	Rejected

If $\epsilon_i = 1$ for all p_i (2/5)

Round	Activity	p_1	p_2	p_3
6	offer	$(s_2, 18)$		$(s_2, 19)$
	result	Rejected	Rejected	Accepted
7	offer	$(s_2, 19)$	$(s_2, 19)$	
	result	Rejected	Rejected	
8	offer	$(s_2, 20)$	$(s_1, 18)$	
	result	Accepted	Accepted	Rejected
9	offer			$(s_1, 18)$
	result		Rejected	Accepted
10	offer		$(s_2, 20)$	
	result		Rejected	

If $\epsilon_i = 1$ for all p_i (3/5)

Round	Activity	p_1	p_2	p_3
11	offer		$(s_1, 19)$	
	result		Accepted	Rejected
12	offer			$(s_2, 20)$
	result	Rejected		Accepted
13	offer	$(s_2, 21)$		
	result	Accepted		Rejected
14	offer			$(s_1, 19)$
	result			Rejected
15	offer			$(s_2, 21)$
	result			Rejected

If $\epsilon_i = 1$ for all p_i (4/5)

Round	Activity	p_1	p_2	p_3
16	offer			$(s_1, 20)$
	result		Rejected	Accepted
17	offer		$(s_2, 21)$	
	result	Rejected	Accepted	
18	offer	$(s_1, 18)$		
	result	Rejected		
19	offer	$(s_2, 22)$		
	result	Accepted	Rejected	
20	offer		$(s_2, 22)$	
	result		Rejected	

If $\epsilon_i = 1$ for all p_i (5/5)

Round	Activity	p_1	p_2	p_3
21	offer		$(s_1, 20)$	
	result		Rejected	
22	offer		$(s_3, 19)$	
	result		Accepted	

matching	Gain of matching						$\sum\sum \alpha_{ij}$
	$\mu(p_1)$	α_{1j}	$\mu(p_2)$	α_{2j}	$\mu(p_3)$	α_{3j}	
4	s_2	11	s_3	2	s_1	3	16

Payoff vector $\mathbf{u}$ (for buyers)

u_1	u_2	u_3
4	2	1

Payoff vector $\mathbf{v}$ (for sellers)

v_1	v_2	v_3
2	7	0

Competitive Equilibrium

- Any stable outcome corresponds to a competitive equilibrium.
- A competitive equilibrium is a feasible matching of agents such that
  - there are no agents who could form a matching pair in such a way that would benefit both of them better than their current state, and
  - there is no matched agent who would prefer to be unmatched.

Competitive Equilibrium of The Example (Seller's Perspective)

- The competitive equilibrium of the above example is $\beta_{12} = 22, \beta_{23} = 19, \beta_{31} = 20$
  - With that payment $u_1 = 4, u_2 = 2, u_3 = 1$
- If a seller can be better off by matching with another buyer, the buyer is not better off
  - If s_2 gets a pay higher than 22, $u_2 < 2$ if $\beta_{22} > 22$ and $u_3 < 0$ if $\beta_{32} > 22$. Neither p_2 nor p_3 can be better off.
  - If s_3 gets a pay higher than 19, $u_1 < 1$ if $\beta_{13} > 19$ and $u_3 < -2$ if $\beta_{33} > 19$. Neither p_1 nor p_3 can be better off.
  - If s_1 gets a pay higher than 20, $u_1 < 3$ if $\beta_{11} > 20$ and $u_2 < 2$ if $\beta_{21} > 20$. Neither p_1 nor p_2 can be better off.

Competitive Equilibrium of The Example (Buyer's Perspective)

- The competitive equilibrium of the above example is $\beta_{12} = 22, \beta_{23} = 19, \beta_{31} = 20$
  - With that payment $u_1 = 4, u_2 = 2, u_3 = 1$
- No buyer can pay less because
  - if $\beta_{12} < 22$, p_2 can have $u_2 = 2$ by hiring s_2 with $\beta_{22} = 22$
  - if $\beta_{23} < 19$, s_3 would prefer to be unmatched (with p_2)
  - if $\beta_{31} < 20$, p_2 can have $u_2 = 2$ by hiring s_1 with $\beta_{21} = 20$

Many-to-One Firms-Workers Market

- Each firm p_i can hire no more than q_i workers and thus maintains up to q_i profitable offers
- Firms preferences over workers are separable across pairs
  - The utility of hiring s_i and s_j together is the utility of hiring s_i plus the utility of hiring s_j

Applications

Only a few

Resource Allocation Problems

- in cognitive radio (CR) networks [LZ10], [BLV+13]
- in heterogeneous cellular networks

[LZ10] Y. Leshem and E. Zehavi, "Stable matching for channel access control in cognitive radio systems," in *Proc. 2nd Int. Workshop Cognitive Information Processing* , pp. 470–475, 2010.
[BLV+13] S. Bayat et al., "Dynamic decentralized algorithms for cognitive radio relay networks with multiple primary and secondary users utilizing matching theory," *Trans. Emerg. Telecomms. Techs*., vol. 24, no. 5, pp. 486–502, Aug. 2013.
[BLH+14] S. Bayat et al., "Distributed user association and femtocell allocation in heterogeneous wireless networks," *IEEE Trans. Commun*., 62(8): 3027–3043, Aug. 2014.

Other examples

- In [TP17], they gave an example of 3D printing services.
  - Some designers need 3D printers to produce their designs to physical prototypes.
  - Different designers need different 3D printers that suit their requirements.

Recommend Readings

- Books:
  - [RS90]: Roth and Sotomayor, “Two-sided matching, A Study in Game-theoretic Modeling and Analysis”
  - 坂井豊貴, “如何設計市場機制”
- Papers:
  - [Rot07]: Alvin E. Roth, “Deferred Acceptance Algorithms: History, Theory, Practice, and Open Questions”
  - [TP17]: Thekinen and Panchal, “Resource allocation in cloud-based design and manufacturing: A mechanism design approach”

For Further Readings:

- [GS62]: Gale and Shapley, “College Admissions and the Stability of Marriage”
- [SS71]: Shapley and Shubik, “The Assignment Game I: The Core”
- [CK81]: Crawford and Knoer, “Job Matching with Heterogeneous Firms and Workers”
- [KC82]: Kelso and Crawford, “Job Matching, Coalition Formation, and Gross Substitutes”
- [Irv89]: Robert W. Irving, “An Efficient Algorithm for the “Optimal” Stable Marriage”
- [XL11]: Xu and Li, “Egalitarian Stable Matching for VM Migration in Cloud Computing”
- [BAM07]: Polynomial time algorithm for an optimal stable assignment with multiple partners”
- [BLS+16]: Bayat et al., “Matching Theory: Applications in wireless communications”