Homework 1
M1522.001300 Probabilistic Graphical Models (2017 Fall)
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1 Change of Random Variables [5 points]

If $a > 0$, then cdf $F_Y(y) = P\left[x \leq \frac{Y-b}{a}\right] = F_X\left(\frac{Y-b}{a}\right)$

On the other hand, if $a < 0$, then cdf $F_Y(y) = P\left[x \geq \frac{Y-b}{a}\right] = 1 - F_X\left(\frac{Y-b}{a}\right)$

We can get the pdf of Y by differentiating with repect to y

If $a > 1$, then $f_Y(y) = \frac{1}{a} f_x\left(\frac{y-b}{a}\right)$ and If $a < 1$, then $f_Y(y) = -\frac{1}{a} f_x\left(\frac{y-b}{a}\right)$
As a result,

$$f_Y(y) = \frac{1}{|a|} f_x\left(\frac{y-b}{a}\right) \rightarrow (1)$$

Gaussian pdf with mean m and standard deviation σ

$$f_X(x) = \frac{1}{\sqrt{2\pi\,|a\sigma|}} e^{\frac{-(x-m)^2}{2\sigma^2}} \rightarrow (2)$$

Substitution of Eq. (2) into Eq. (1) yields

$$f_Y(y) = \frac{1}{\sqrt{2\pi\,|a\sigma|}} e^{\frac{-(y-b-am)^2}{2a\sigma^2}}$$

Therfore $N(am + b; a^2\sigma^2)$

2 Conditional and Total Probability [10 points]

1.

$$P(Department\ A) = 0.62 \times \frac{825}{825 + 108} + 0.82 \times \frac{108}{825 + 108} = 0.64$$

$$P(Department\ C) = 0.33 \times \frac{417}{417 + 375} + 0.35 \times \frac{375}{417 + 375} = 0.33$$

2.

$$P(Men) = \frac{0.62 \times 825 + 0.63 \times 560 + 0.33 \times 417 + 0.06 \times 272}{825 + 108 + 560 + 25 + 417 + 375 + 272 + 341} = 0.34$$

$$P(Women) = \frac{0.82 \times 108 + 0.68 \times 25 + 0.35 \times 375 + 0.07 \times 341}{825 + 108 + 560 + 25 + 417 + 375 + 272 + 341} = 0.08$$

3. Because When Department C and D is given, the probability of Woman is low. In case of Woman Applicants of Department C, D is Higher than A, B. It means that the number of admitted students in Department C and D can affect to admission rate a lot

3 Independence Properties [20 points]

1. $(X, W \perp Y \mid Z) \xrightarrow{symmetry} (Y \perp X, W \mid Z) \xrightarrow{Decomposition} (Y \perp X \mid Z)$ and $(Y \perp W \mid Z) \xrightarrow{symmetry} (\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$ and $(W \perp Y \mid Z)$

2. $(X, W \perp Y \mid Z) \xrightarrow{symmetry} (Y \perp X, W \mid Z) \xrightarrow{WeakUnion} (Y \perp X \mid Z, W)$ and $(Y \perp W \mid Z) \xrightarrow{symmetry} (\mathbf{X} \perp \mathbf{Y} \mid \mathbf{W}, \mathbf{Z})$ and $(W \perp Y \mid Z)$

3.

4. $(X \perp Y \mid Z, W)$ and $(Y \perp W \mid Z, X) \xrightarrow{symmetry} (Y \perp X \mid Z, W)$ and $(Y \perp W \mid X, Z) \xrightarrow{Intersection} (Y \perp X, W \mid Z) \xrightarrow{symmetry} (\mathbf{X}, \mathbf{W} \perp \mathbf{Y} \mid \mathbf{Z})$

4 Naive Bayes Example [15 points]

1. By Chain Rules

$$\begin{aligned} P(C, X_1, ..., X_n) &= P(C)P(X_1, ..., X_n \mid C) \\ &= P(C)P(X_1 \mid C)P(X_2, ..., X_n \mid C, X_1) \\ &= P(C)P(X_1 \mid C)P(X_2 \mid C, X_1)P(X_3, ..., X_n \mid C, X_1, X_2) \\ &= P(C)P(X_1 \mid C)P(X_2 \mid C, X_1) \ ... \ P(X_n \mid C, X_1, X_2, \ ... \ X_{n-1}) \end{aligned}$$

By Conditional independence property of Naive Bayes $i \neq j, k, l$

$$\begin{aligned} P(x_i \mid C, x_j) &= P(x_i \mid C), \\ P(x_i \mid C, x_j, x_k) &= P(x_i \mid C), \\ P(x_i \mid C, x_j, x_k, x_l) &= P(x_i \mid C), \end{aligned}$$

So we can get following result

$$\begin{aligned} P(C, X_1, ..., X_n) &= P(C)P(X_1, ..., X_n \mid C) \\ &= P(C)P(x_1 \mid C)P(x_3 \mid C)P(x_2 \mid C) \ ... \\ &= P(C) \prod_{i=1}^{n} P(x_i \mid C) \end{aligned} \tag{1}$$

2.

$$\begin{aligned}
\frac{P(C=1 \mid x_1,...,x_n)}{P(C=0 \mid x_1,...,x_n)} &= \frac{\frac{P(C=1,x_1,...,x_n)}{P(x_1,...,x_n)}}{\frac{P(C=0,x_1,...,x_n)}{P(x_1,...,x_n)}} \\
&= \frac{P(C=1,x_1,...,x_n)}{P(C=0,x_1,...,x_n)} \\
&= \frac{P(C=1)\prod_{i=1}^{n} P(x_i \mid C=1)}{P(C=0)\prod_{i=1}^{n} P(x_i \mid C=0)} = \frac{P(C=1)}{P(C=0)} \prod_{i=1}^{n} \frac{P(x_i \mid C=1)}{P(x_i \mid C=0)}
\end{aligned}$$

3.

$$\begin{aligned}
\frac{P(C=1 \mid x_1,...,x_n)}{P(C=0 \mid x_1,...,x_n)} &= \frac{P(C=1)}{P(C=0)} \prod_{i=1}^{n} \frac{P(x_i \mid C=1)}{P(x_i \mid C=0)} \\
&= \sum_{i=1}^{n} \log \frac{P(x_i \mid C=1)}{P(x_i \mid C=0)} + \log P(C=1) - \log P(C=0)
\end{aligned}$$

5 Graphical Model and Independence [10 points]

1. TRUE
2. TRUE
3. FALSE
4. TRUE
5. FALSE
6. FALSE
7. FALSE
8. TRUE
9. TRUE
10. TRUE

6 V-structure [15 points]

You can insert your figure by using `\begin{figure}`. You can refer your figure by using `\ref{fig:example_figure}`.

You can enumerate sub questions like this.

1. Enumerate item 1.

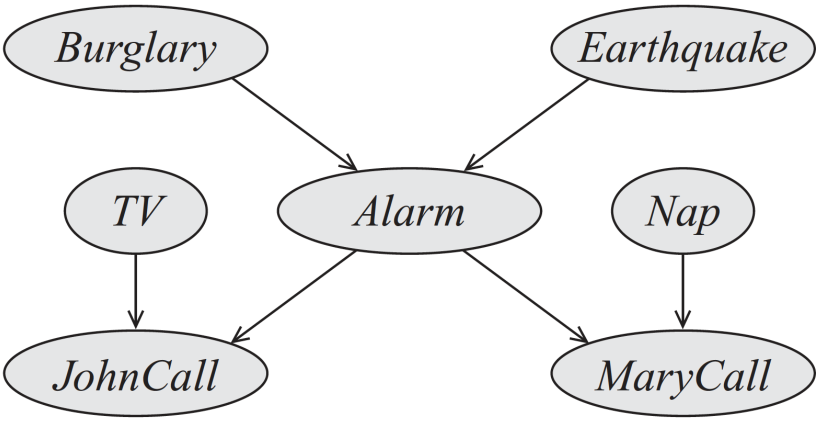

Figure 1: Example Figure

2. Enumerate item 2.
3. Enumerate item 3.

You can write your equations by using `\begin{aligned}`. We highly recommend you to use `\newcommand` to simplify your equation in LaTeX.

$$\begin{aligned} \boldsymbol{A} &= 12 && (2) \\ \alpha_{12}^{35} &= 1234 && (3) \\ \beta_1 &= 10 && (4) \end{aligned}$$

You can cite your reference by using `\cite{reference_name}`. For example, cite Koller's PGM book [1] like this.

7 Some materials for HW2 [5 points]

Step	Variable eliminated	Factors used	Variables involved	New factor
1				
2				
3				
4				
5				
6				
7				
8				

Table 1: A run of variable elimination for the query $P(J)$

References

[1] Daphne Koller and Nir Friedman. *Probabilistic Graphical Models: Principles and Techniques - Adaptive Computation and Machine Learning*. The MIT Press, 2009.