Homework 1
M1522.001300 Probabilistic Graphical Models (2017 Fall)

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1 Change of Random Variables [5 points]

If $a > 0$, then cdf $F_Y(y) = P\left[x \leq \frac{Y-b}{a}\right] = F_X\left(\frac{Y-b}{a}\right)$

On the other hand, if $a < 0$, then cdf $F_Y(y) = P\left[x \geq \frac{Y-b}{a}\right] = 1 - F_X\left(\frac{Y-b}{a}\right)$

We can get the pdf of Y by differentiating with repect to y

If $a > 1$, then $f_Y(y) = \frac{1}{a} f_x\left(\frac{y-b}{a}\right)$ and If $a < 1$, then $f_Y(y) = -\frac{1}{a} f_x\left(\frac{y-b}{a}\right)$
As a result,

$$f_Y(y) = \frac{1}{|a|} f_x\left(\frac{y-b}{a}\right) \rightarrow (1)$$

Gaussian pdf with mean m and standard deviation σ

$$f_X(x) = \frac{1}{\sqrt{2\pi\,|a\sigma|}} e^{\frac{-(x-m)^2}{2\sigma^2}} \rightarrow (2)$$

Substitution of Eq. (2) into Eq. (1) yields

$$f_Y(y) = \frac{1}{\sqrt{2\pi\,|a\sigma|}} e^{\frac{-(y-b-am)^2}{2a\sigma^2}}$$

Therfore $N(am + b; a^2\sigma^2)$

2 Conditional and Total Probability [10 points]

1.

$$P(Department\ A) = 0.62 \times \frac{825}{825 + 108} + 0.82 \times \frac{108}{825 + 108} = 0.64$$

$$P(Department\ C) = 0.33 \times \frac{417}{417 + 375} + 0.35 \times \frac{375}{417 + 375} = 0.33$$

2.

$$P(Men) = \frac{0.62 \times 825 + 0.63 \times 560 + 0.33 \times 417 + 0.06 \times 272}{825 + 108 + 560 + 25 + 417 + 375 + 272 + 341} = 0.34$$

$$P(Women) = \frac{0.82 \times 108 + 0.68 \times 25 + 0.35 \times 375 + 0.07 \times 341}{825 + 108 + 560 + 25 + 417 + 375 + 272 + 341} = 0.08$$

3. Because When Department C and D is given, the probability of Woman is low. In case of Woman Applicants of Department C, D is Higher than A, B. It means that the number of admitted students in Department C and D can affect to admission rate a lot

3 Independence Properties [20 points]

1. $(X, W \perp Y \mid Z) \xrightarrow{symmetry} (Y \perp X, W \mid Z) \xrightarrow{Decomposition} (Y \perp X \mid Z)$ and $(Y \perp W \mid Z) \xrightarrow{symmetry} (\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$ and $(W \perp Y \mid Z)$

2. $(X, W \perp Y \mid Z) \xrightarrow{symmetry} (Y \perp X, W \mid Z) \xrightarrow{WeakUnion} (Y \perp X \mid Z, W)$ and $(Y \perp W \mid Z) \xrightarrow{symmetry} (\mathbf{X} \perp \mathbf{Y} \mid \mathbf{W}, \mathbf{Z})$ and $(W \perp Y \mid Z)$

3.

4. $(X \perp Y \mid Z, W)$ and $(Y \perp W \mid Z, X) \xrightarrow{symmetry} (Y \perp X \mid Z, W)$ and $(Y \perp W \mid X, Z) \xrightarrow{Intersection} (Y \perp X, W \mid Z) \xrightarrow{symmetry} (\mathbf{X}, \mathbf{W} \perp \mathbf{Y} \mid \mathbf{Z})$

4 Naive Bayes Example [15 points]

1. By Chain Rules

$$
\begin{aligned}
P(C, X_1, ..., X_n) &= P(C)P(X_1, ..., X_n \mid C) \\
&= P(C)P(X_1 \mid C)P(X_2, ..., X_n \mid C, X_1) \\
&= P(C)P(X_1 \mid C)P(X_2 \mid C, X_1)P(X_3, ..., X_n \mid C, X_1, X_2) \\
&= P(C)P(X_1 \mid C)P(X_2 \mid C, X_1) \ ... \ P(X_n \mid C, X_1, X_2, \ ... \ X_{n-1})
\end{aligned}
$$

By Conditional independence property of Naive Bayes $i \neq j, k, l$

$$
\begin{aligned}
P(x_i \mid C, x_j) &= P(x_i \mid C), \\
P(x_i \mid C, x_j, x_k) &= P(x_i \mid C), \\
P(x_i \mid C, x_j, x_k, x_l) &= P(x_i \mid C),
\end{aligned}
$$

So we can get following result

$$
\begin{aligned}
P(C, X_1, ..., X_n) &= P(C)P(X_1, ..., X_n \mid C) \\
&= P(C)P(x_1 \mid C)P(x_3 \mid C)P(x_2 \mid C) \ ... \\
&= P(C)\prod_{i=1}^{n} P(x_i \mid C)
\end{aligned}
\tag{1}
$$

2.

$$
\begin{aligned}
\frac{P(C=1 \mid x_1,...,x_n)}{P(C=0 \mid x_1,...,x_n)} &= \frac{\frac{P(C=1,x_1,...,x_n)}{P(x_1,...,x_n)}}{\frac{P(C=0,x_1,...,x_n)}{P(x_1,...,x_n)}} \\
&= \frac{P(C=1,x_1,...,x_n)}{P(C=0,x_1,...,x_n)} \\
&= \frac{P(C=1)\prod_{i=1}^n P(x_i \mid C=1)}{P(C=0)\prod_{i=1}^n P(x_i \mid C=0)} = \frac{P(C=1)}{P(C=0)}\prod_{i=1}^n \frac{P(x_i \mid C=1)}{P(x_i \mid C=0)}
\end{aligned}
$$

3.

$$
\begin{aligned}
\frac{P(C=1 \mid x_1,...,x_n)}{P(C=0 \mid x_1,...,x_n)} &= \frac{P(C=1)}{P(C=0)}\prod_{i=1}^n \frac{P(x_i \mid C=1)}{P(x_i \mid C=0)} \\
&= \sum_{i=1}^n \log \frac{P(x_i \mid C=1)}{P(x_i \mid C=0)} + \log P(C=1) - \log P(C=0)
\end{aligned}
$$

5 Graphical Model and Independence [10 points]

1. TRUE
2. TRUE
3. FALSE
4. TRUE
5. FALSE
6. FALSE
7. FALSE
8. TRUE
9. TRUE
10. TRUE

6 V-structure [15 points]

1.

$$
\begin{aligned}
P(C_1=H) &= \frac{1}{2}, \quad P(C_1=T) = \frac{1}{2} \\
P(C_2=H) &= \frac{1}{2}, \quad P(C_2=T) = \frac{1}{2}
\end{aligned} \tag{2}
$$

$P(C_1 = H, C_2 = H) = \frac{1}{4} = P(C_1 = H)P(C_2 = H) = \frac{1}{2} \times \frac{1}{2}$
$P(C_1 = H, C_2 = T) = \frac{1}{4} = P(C_1 = H)P(C_2 = T) = \frac{1}{2} \times \frac{1}{2}$
$P(C_1 = T, C_2 = H) = \frac{1}{4} = P(C_1 = T)P(C_2 = H) = \frac{1}{2} \times \frac{1}{2}$
$P(C_1 = T, C_2 = T) = \frac{1}{4} = P(C_1 = T)P(C_2 = T) = \frac{1}{2} \times \frac{1}{2}$
Therefore, $P(C_1) \perp P(C_2)$

2.
$$P(C_2 = H|B = F) = \frac{P(B = F|C_2 = H)P(C_2 = H)}{P(B = F)} = \frac{0.5 \times 0.5}{0.75} = \frac{1}{3}$$

3.
$$P(C_2 = H \mid B = F, C_1 = T) = \frac{1}{2}$$

$P(C_1 = H, C_2 = H \mid B = T) = 1 \quad P(C_1 = H \mid B = T)P(C_2 = H \mid B = T) = 1 \cdot 1 = 1$

$P(C_1 = H, C_2 = T \mid B = F) = \dfrac{1}{3} \quad P(C_1 = H \mid B = F)P(C_2 = T \mid B = F) = 1 \cdot \dfrac{1}{3} = \dfrac{1}{3}$

Therefore, $P(C_1, C_2 \mid B) = P(C_1 \mid B)P(C_2 \mid B)$
$C_1 \mid B$ and $C_2 \mid B$ are independent

7 I-Equivalence [10 points]

8 Factorization Theorem [15 points]

References