Principal Fiber Bundles

Summer Term 2020

Michael Kunzinger

michael.kunzinger@univie.ac.at
Universität Wien
Fakultät für Mathematik
Oskar-Morgenstern-Platz 1
A-1090 Wien

Preface

Principal fiber bundles are a fundamental tool in differential geometry and global analysis, where they provide the language for understanding curvature and for studying the interplay between geometry, analysis and topology. In mathematical physics they set the stage for gauge theories and the standard model of particle physics. This course provides an introduction to the topic, closely following the excellent book [1] by Helga Baum, to which we also refer for further study. Occasional input also comes from [3, 4, 6]. The prerequisites for following the course are a working knowledge of Lie group theory and analysis on manifolds, as provided by [7, 9]. Some basics of (semi-)Riemannian geometry will be used from time to time, for which we refer to [10]. I would like to thank Roman Popovych for many helpful comments and corrections.

Michael Kunzinger, summer term 2020

Contents

Chapter 1

Transformation groups

1.1 Basics

Throughout these notes, we will always assume that smooth manifolds are Hausdorff and second countable (in particular this applies to Lie groups). Unless otherwise stated explicitly, neighborhoods will always be assumed to be open.

Recall from [9, Def. 16.1] the definition of a transformation group:

1.1.1 Definition. *A* transformation *of a manifold M is a diffeomorphism $M \to M$. A group G acts on M as a* transformation group *(on the left) if there exists a map $\phi : G \times M \to M$, $(g, x) \mapsto g \cdot x$ satisfying:*

(i) $\forall g \in G$*:* $l_g \equiv \phi_g := x \mapsto g \cdot x$ *is a transformation of M.*

(ii) $\forall g, h \in G$*:* $\phi_g \circ \phi_h = \phi_{gh}$*, i.e.,* $g \cdot (h \cdot x) = (g \cdot h) \cdot x$ *for all* $x \in M$*.*

In particular, $\phi_e = \mathrm{id}_M$, so $e \cdot x = x$ for all $x \in M$. G acts effectively *on M if $g \cdot x = x$ for all x implies $g = e$. It acts* transitively *if for all $x, y \in M$ there is a $g \in G$ with $y = g \cdot x$, and* simply transitively *if in addition this g is unique. It acts* freely *on M if $g \cdot x = x$ for some g and some x implies $g = e$. The set $G \cdot x \equiv \phi(G, x)$ is called the* orbit *of x under G. If G is a Lie group then the pair $[M, G]$ is called a* Lie transformation group*.*

Fixing $x \in M$, we write ϕ_x for the map $g \to \phi(g, x)$. Analogously, one defines right actions $\phi : M \times G \to M$. In this case we write $r_g \equiv \phi_g$ for the diffeomorphism $r_g : x \in M \mapsto x \cdot g := \phi(x, g)$.

1.1.2 Definition. *If a Lie group G acts transitively on a manifold M, then M is called a* homogeneous space*.*

Let H be a closed subgroup of a Lie group G. Two elements $g_1, g_2 \in G$ are called equivalent, $g_1 \sim_H g_2$, if $g_1 = g_2 \cdot h$ for some $h \in H$. Denote by $[g]$ the equivalence class of $g \in G$ and by G/H the set of all such equivalence classes. Also, let $\pi : G \to G/H$ be the quotient map. Then by [9, Th. 21.5], if H is open, G/H is discrete (in the quotient topology). Otherwise, we have:

1.1.3 Theorem. *Let H be a closed, non-open subgroup of a Lie group G. Then there exists a manifold structure on G/H such that:*

(i) The projection $\pi : G \to G/H$ is a submersion.

(ii) With respect to the left action $\phi : G \times G/H \to G/H$, $(g,[a]) \mapsto [ga]$, G/H is a homogeneous space.

(iii) There exist local sections *in $\pi : G \to G/H$, i.e., for any equivalence class $[a] \in G/H$ there exists a neighborhood $W([a]) \subseteq G/H$ of $[a]$ and a smooth map $s_{[a]} : W([a]) \to G$ with $\pi \circ s_{[a]} = \mathrm{id}_{W([a])}$.*

Proof. By [9, Th. 21.5], G/H can be endowed with a $\mathcal{C}^\infty$-structure as a quotient manifold of G, which just means that (i) is satisfied. Moreover, (iii) is a direct consequence of the general fact that submersions are precisely those maps that possess local sections ([9, Prop. 15.2]). So it only remains to verify (ii). Note first that ϕ is well-defined since $[a] = [b]$ implies that for some $h \in H$ we have $a = bh$, so that $[ga] = [gbh] = [gb]$. Also, with μ the multiplication on G we have

$$\phi \circ (\mathrm{id} \times \pi) = \pi \circ \mu,$$

where the right hand side is clearly smooth. Since $\mathrm{id} \times \pi$ is a surjective submersion, it follows that ϕ is smooth as well ([9, Rem. 13.2]). Finally, ϕ acts transitively: given $[a],[b] \in G/H$ we have $\phi(ba^{-1},[a]) = [b]$. □

Our next aim is to show that, conversely, if G is a second countable Lie group acting transitively on a manifold M, then the homogeneous space M is diffeomorphic to a quotient of G. From [9, Sec. 16] we know that any transformation group ϕ acting on a manifold M induces an equivalence relation on M: $x \sim x'$ iff there exists some $g \in G$ with $x' = gx$ (resp. $x' = xg$). The equivalence class of any $x \in M$ is precisely the orbit of x under G, i.e., it is the range of the map $\phi_x = g \mapsto \phi(g,x)$, $G \to M$.

1.1.4 Definition. *Let $\phi : G \times M \to M$ be a transformation group on a manifold M. For any $x \in M$, the subgroup $G_x := \phi_x^{-1}(x) = \{g \in G \mid g{\cdot}x = x\}$ (resp. $\{g \in G \mid x{\cdot}g = x\}$) is called the* isotropy group *(or* stabilizer*) of x.*

1.1.5 Remark. (i) Isotropy groups at equivalent points in M are conjugate subgroups of G. Indeed, suppose that $x' = gx$. Then $(gG_xg^{-1})x' = x'$, so $gG_xg^{-1} \subseteq G_{x'}$, and analogously $g^{-1}G_{x'}g \subseteq G_x$, so $gG_xg^{-1} = G_{x'}$.

(ii) For any $x \in M$, the map $\phi_x : G \to M$ projects to a map $\psi_x : G/G_x \to M$ defined by $gG_x \mapsto gx$. The range of ψ_x is the orbit of x. Also, ψ_x is injective: if $\psi_x(g_1G_x) = \psi_x(g_2G_x)$ then $g_1x = g_2x$, so $g_1^{-1}g_2 \in G_x$, i.e., $g_1G_x = g_2G_x$.

1.1.6 Remark. Let $\phi : G \times M \to M$ be a Lie transformation group on a manifold M. Then any G_x is a closed subgroup of G. Hence by [9, Th. 21.7], G_x is either discrete or it admits a unique structure as a (regular) submanifold of G. In the latter case it is also a Lie subgroup of G. If $x' = gx$ then by Remark 1.1.5 (i), G_x is mapped onto $G_{x'}$ by the diffeomorphism $L_g \circ R_{g^{-1}}$. Thus the isotropy groups at points of an orbit are either all discrete or are regular submanifolds and Lie subgroups of G that are pairwise diffeomorphic (since G_x, $G_{x'}$ are regular submanifolds, the restriction of $L_g \circ R_{g^{-1}}$ is also a diffeomorphism from G_x onto $G_{x'}$).

If G_x is open then it is closed and open, hence is a union of connected components of G, which, by [9, Prop. 2.4] are precisely the cosets of the normal subgroup G_e. If, for example, $G = g_1G_e \cup \dots \cup g_kG_e$ and $G_x = g_1G_e \cup \dots \cup g_lG_e$, then $G/G_x = \{g_1G_x, \dots, g_kG_x\}$, since for $g \in g_jG_e$ we have

$$\pi(g) = g \cdot \bigcup_{i=1}^{l} g_iG_e = \bigcup_{i=1}^{l} gG_eg_i = \bigcup_{i=1}^{l} g_jG_eg_i = g_jG_x.$$

Also, the orbit of x only consists of finitely many points (namely $\psi_x(G/G_x) = \{g_1 \cdot x, \dots, g_k \cdot x\}$). Otherwise, we have:

1.1.7 Theorem. *Let $\phi: G \times M \to M$ be a Lie transformation group on a manifold M and let $x \in M$. If the isotropy group G_x of x is not open in G then the map ψ_x from Remark 1.1.5 (ii) is an injective immersion of the quotient manifold G/G_x into M.*

Proof. Since $\phi_x = \psi_x \circ \pi$ (with $\pi : G \to G/G_x$), ψ_x is smooth by [9, Rem. 13.2]. Also, G/G_x is a quotient manifold of G by [9, Th. 21.5]. Moreover, ψ_x is injective by Remark 1.1.5 (ii), so it remains to show that its rank in any point equals the dimension of G/G_x. Since π is a submersion, this is the case if and only if the rank of ϕ_x is everywhere equal to $\dim(G/G_x)$. We begin by showing that this is true at e, for $\phi : M \times G \to M$ a right action, as this is the case we will need later on.

Let $X \in T_eG$ such that $T_e\phi_x(X) = 0$. Let $\tilde{X} : M \to TM$ be defined by $\tilde{X}(x) := T_e\phi_x(X)$. Then $\tilde{X}$ is smooth and $\tilde{X}(x) = T_e\phi_x(X) \in T_{\phi(x,e)}M = T_xM$, so $\tilde{X} \in \mathfrak{X}(M)$.[1] Moreover, we have $\mathrm{Fl}_t^{\tilde{X}}(x) = \phi(x, \exp(tX))$: this is clear for $t = 0$. Also,

$$\frac{d}{dt}\exp(tX) = \frac{d}{dt}\mathrm{Fl}_t^{L^X}(e) = L^X(\exp(tX)),$$

and since for a right action we have $\phi(\phi_x(g), h) = \phi_x(g)h = xgh = \phi_x(gh)$ we get $\phi_{\phi_x(\exp(tX))} = \phi(\phi(x, \exp(tX)), \,.\,) = \phi(x, \exp(tX)\,.\,) = \phi_x \circ L_{\exp(tX)}$. Thus

$$\begin{aligned}\frac{d}{dt}\phi(x, \exp(tX)) &= T_{\exp(tX)}\phi_x(L^X(\exp(tX))) = T_{\exp(tX)}\phi_x(T_eL_{\exp(tX)}(X)) \\ &= T_e(\phi_x \circ L_{\exp(tX)})(X) = T_e\phi_{\phi_x(\exp(tX))}(X) = \tilde{X}(\phi_x(\exp(tX))).\end{aligned}$$

Since $\tilde{X}(x) = 0$, it follows that $\mathrm{Fl}_t^{\tilde{X}}(x) = x \cdot \exp(tX) = x$ for all t, i.e., $\exp(tX) \in G_x$ for all t.

Now if G_x is discrete then the image of $t \mapsto \exp(tX)$, being connected, must consist solely of $e \in G$, so $X = \frac{d}{dt}|_0 \exp(tX) = 0$. In this case, then, $T_e\phi_x$ is injective, and so the rank of $T_e\phi_x$ equals the dimension of G, and thereby the dimension of G/G_x.

If G_x is non-discrete then by Remark 1.1.6 it is a regular submanifold of G. Hence $t \mapsto \exp(tX)$ is smooth as a map into G_x (see [7, 3.3.14]), and so $X = \frac{d}{dt}|_0 \exp(tX) \in T_eG_x$. Altogether, we obtain that $\ker(T_e\phi_x) \subseteq T_eG_x$. Conversely, ϕ_x is constant on G_x, so $T_e\phi_x|_{T_eG_x} \equiv 0$, hence in fact $\ker(T_e\phi_x) = T_eG_x$. Consequently, using [9, Th. 21.7] we obtain

$$\mathrm{rk}(T_e\phi_x) = \dim G - \dim G_x = \dim G/G_x.$$

Finally, if g is an arbitrary point in G then $\phi_x \circ L_g = \phi_{x'}$, where $x' = xg$. Then since L_g is a diffeomorphism we have

$$\mathrm{rk}_g(\phi_x) = \mathrm{rk}_e(\phi_{x'}) = \dim G/G_{x'}.$$

Now by Remark 1.1.6 G_x and $G_{x'}$ are either both discrete or they are diffeomorphic, so we conclude that the rank of $T_g\phi_x$ equals $\dim G/G_x$ for every $g \in G$. □

1.1.8 Corollary. *Under the assumptions of Theorem 1.1.7, if G_x is not open then the orbit $G \cdot x$ can be endowed with the structure of an immersive submanifold of M diffeomorphic to G/G_x.*

Proof. For clarity, we write $\tilde{\psi}_x$ for ψ_x, viewed as a (bijective) map from G/G_x to $G \cdot x$. Declaring $\tilde{\psi}_x$ to be a diffeomorphism provides $G \cdot x$ with a differentiable structure with respect to which the inclusion map $j : G \cdot x \hookrightarrow M$ is an immersion since $j \circ \tilde{\psi}_x = \psi_x : G/G_x \to M$ is an immersion. □

[1] $\tilde{X}$ is precisely the fundamental vector field corresponding to X, see Section 1.2.

1.1.9 Remark. Suppose that G is connected and let $x \in M$. By Corollary 1.1.8 the orbit $G \cdot x$ can be discrete only if G_x is open: if G_x is not open then G_x is non-discrete in its manifold topology hence also in the coarser trace topology. In this case, G_x is open and closed in G, so $G_x = G$ and therefore $G \cdot x = \{x\}$.

Recall that G is said to act transitively on M if for any x, $x' \in M$ there exists some $g \in G$ with $gx = x'$. Such a group action possesses only a single orbit, namely the manifold M itself. By Remark 1.1.5 (i) this means that any ψ_x is a bijection from G/G_x onto M. To further elaborate on this, we will need the following auxiliary result:

1.1.10 Lemma. *Let M^m and N^n be manifolds and suppose that M is second countable and that $m < n$. Then an immersion $\psi : M \to N$ cannot be onto any open subset W of N.*

Proof. See Appendix A. □

Using this, we can prove:

1.1.11 Theorem. *Let G be a second countable Lie group that acts transitively as a Lie transformation group on the manifold M. Then for any $x \in M$ the map $\psi_x : G/G_x \to M$ is a G-equivariant diffeomorphism.*

Proof. Fix any $x \in M$. If G_x is open, then so is any gG_x, hence any point in G/G_x (because $\pi^{-1}(\pi(g)) = gG_x$), which is therefore discrete (cf. [9, Rem. 20.1]). Otherwise, by [9, Th. 21.5] G/G_x possesses a differentiable structure as a quotient manifold of G. In both cases, the quotient map $\pi : G \to G/G_x$ is open and continuous, so also the topology of G/G_x is second countable. Hence if G/G_x were discrete it would be countable. But ψ_x is bijective, so this would imply that M was countable, which is impossible. Hence G/G_x is not discrete, and so it is a quotient manifold of G with a countable basis for its topology. Also, by Theorem 1.1.7 ψ_x is an injective immersion of G/G_x into M. Hence $\dim G/G_x \leq \dim M$. Since ψ_x is onto M, Lemma 1.1.10 implies that the dimensions in fact are equal. As $\psi_x : G/G_x \to M$ is an immersion, it follows that its tangent map is bijective at any point. Thus by the inverse function theorem it is a local diffeomorphism, hence a global diffeomorphism since it is bijective. Finally G-equivariance simply means $\psi_x(a \cdot (gG_x)) = \psi_x(agG_x) = agx = a\psi_x(gG_x)$, which is clear from the definition. □

1.1.12 Corollary. *Let G be a second countable Lie group that acts transitively and freely as a Lie transformation group on the manifold M. Then M is diffeomorphic to G.*

Proof. If G acts freely then $G_x = \{e\}$ for every $x \in M$. Hence $G/G_x = G$ and the result follows from Theorem 1.1.11. □

1.1.13 Corollary. *Let G be a compact Lie group that acts transitively as a Lie transformation group on the manifold M. Then M is compact.*

Proof. Since G is compact, it is second countable. By Theorem 1.1.11, M is diffeomorphic to G/G_x, for any $x \in M$. Let $\pi : G \to G/G_x$ be the quotient map. Then $\pi(G) = G/G_x$ is compact, hence so is M. □

1.2 Fundamental vector fields

In this section we introduce a fundamental tool for studying Lie group actions.

1.2.1 Definition. *Let $[M,G]$ be a left Lie transformation group and let $X \in \mathfrak{g}$. Then the vector field $\tilde{X} \in \mathfrak{X}(M)$ defined by*

$$\tilde{X}(x) := \frac{d}{dt}\Big|_0 (\exp(-tX)\cdot x) = -T_e\phi_x(X)$$

is called the fundamental vector field *corresponding to X. If G acts on the right then we set*

$$\tilde{X}(x) := \frac{d}{dt}\Big|_0 (x\cdot\exp(tX)) = T_e\phi_x(X).$$

Clearly, $\tilde{X}$ is smooth and since the curve $t \mapsto \exp(-tX)\cdot x$ has value x at $t=0$, $\tilde{X}$ is indeed a section of TM.

1.2.2 Theorem. *Let $[M,G]$ be a Lie transformation group.*

(i) The map

$$\begin{aligned} \mathfrak{g} &\to \mathfrak{X}(M) \\ X &\mapsto \tilde{X} \end{aligned}$$

is linear and $[X,Y]^\sim = [\tilde{X},\tilde{Y}]$, i.e., it is a Lie algebra homomorphism. In particular, the set of fundamental vector fields forms a Lie subalgebra of the Lie algebra $\mathfrak{X}(M)$.

(ii) If G_e (the connected component of e in G) acts effectively on M, then $X \mapsto \tilde{X}$, $\mathfrak{g} \to \{\tilde{X} \in \mathfrak{X}(M) \mid X \in \mathfrak{g}\}$ is a Lie algebra isomorphism.

(iii) If G acts on the right, then for all $g \in G$ and all $X \in \mathfrak{g}$, the push-forward $(r_g)_*\tilde{X}$ of $\tilde{X}$ under r_g is given by

$$(r_g)_*\tilde{X} = (\mathrm{Ad}(g^{-1})X)^\sim.$$

If G acts on the left,

$$(l_g)_*\tilde{X} = (\mathrm{Ad}(g)X)^\sim.$$

Proof. (i) Linearity of $X \mapsto \tilde{X}$ is immediate from that of $T_e\phi_x$. Suppose first that G acts on the right. If $X \in \mathfrak{g} \cong \mathfrak{X}_L(G)$ then

$$\begin{aligned} T_g\phi_x(X(g)) = T_g\phi_x(T_eL_g(X(e))) &= \frac{d}{dt}\Big|_0 (\phi_x(L_g(\exp(tX)))) \\ &= \frac{d}{dt}\Big|_0 (x\cdot g\cdot\exp(tX)) = \tilde{X}(x\cdot g) = \tilde{X}(\phi_x(g)). \end{aligned}$$

This shows that X and $\tilde{X}$ are ϕ_x-related for any $x \in M$, $X \sim_{\phi_x} \tilde{X}$. Thus also $[X,Y] \sim_{\phi_x} [X,Y]^\sim$, and by [9, Lemma 4.4] we get $[X,Y] \sim_{\phi_x} [\tilde{X},\tilde{Y}]$. Altogether,

$$[\tilde{X},\tilde{Y}](x) = [\tilde{X},\tilde{Y}](\phi_x(e)) = T\phi_x([X,Y](e)) = [X,Y]^\sim(\phi_x(e)) = [X,Y]^\sim(x).$$

If G acts on the left, let $\phi'_x : G \to M$, $\phi'_x(g) := g^{-1}\cdot x$. Then for $X \in \mathfrak{X}_L(G)$,

$$\begin{aligned} T_g\phi'_x(X(g)) = (T_g\phi'_x)(T_eL_g(X(e))) &= \frac{d}{dt}\Big|_0 \phi'_x(g\cdot\exp(tX)) \\ &= \frac{d}{dt}\Big|_0 (\exp(-tX)(g^{-1}\cdot x)) = \tilde{X}(g^{-1}x) = \tilde{X}(\phi'_x(g)), \end{aligned}$$

and the rest of the argument is as above.

For the remainder of the proof it will suffice to consider right actions.

(ii) Let G_e act effectively, then we have to show that $X \mapsto \tilde{X}$ is injective. Thus let $\tilde{X} = 0$, so that $\mathrm{Fl}_t^{\tilde{X}}(x) = x$ for all t and x. As we have seen in the proof of Theorem 1.1.7, $\mathrm{Fl}_t^{\tilde{X}}(x) = x \cdot \exp(tX)$, so $x \cdot \exp(tX) = x$ for all $x \in M$ and all $t \in \mathbb{R}$. Now for t_0 sufficiently small, $g_0 := \exp(t_0 X)$ lies in a normal neighborhood of $e \in G_e$ (i.e., a neighborhood onto which exp is a diffeomorphism). Since G_e acts effectively on M, we have $g_0 = e$ and thereby $X = 0$.

(iii) Let $x \in M$, $g \in G$ and $X \in \mathfrak{g}$. Then applying [9, Th. 9.2] (iii) we calculate

$$\begin{aligned}
(r_g)_* \tilde{X}(x) &= T_{xg^{-1}} r_g(\tilde{X}(xg^{-1})) = T_{xg^{-1}} r_g\Big(\frac{d}{dt}\Big|_0 (xg^{-1} \cdot \exp(tX))\Big) \\
&= \frac{d}{dt}\Big|_0 (x \cdot (g^{-1} \cdot \exp(tX) \cdot g)) = \frac{d}{dt}\Big|_0 (x \cdot (\mathrm{conj}_{g^{-1}}(\exp(tX)))) \\
&= \frac{d}{dt}\Big|_0 (x \cdot \exp(t\mathrm{Ad}(g^{-1})X)) = (\mathrm{Ad}(g^{-1})X)\tilde{}\,(x).
\end{aligned}$$

□

To conclude this section we prove a product rule that will be useful several times later on.

1.2.3 Lemma. *Let $[M, G]$ be a right Lie transformation group. Also, let $t \mapsto x(t)$ be a smooth curve in M with $x(0) = x$, and $t \mapsto g(t)$ a smooth curve in G with $g(0) = g$. Set $z(t) := x(t) \cdot g(t)$. Then*

$$\dot{z}(0) = T_x r_g(\dot{x}(0)) + (T_g L_{g^{-1}} \dot{g}(0))\tilde{}\,(x \cdot g). \tag{1.2.1}$$

Proof. As before, we denote the group action by $\phi : M \times G \to M$. Then using the identification $T_{(x,g)}(M \times G) \cong T_x M \oplus T_g G$ we calculate:

$$\begin{aligned}
\dot{z}(0) &= \frac{d}{dt}\Big|_0 \phi(x(t), g(t)) = T_{(x,g)}\phi(\dot{x}(0), \dot{g}(0)) = T_{(x,g)}\phi(\dot{x}(0), 0) + T_{(x,g)}\phi(0, \dot{g}(0)) \\
&= \frac{d}{dt}\Big|_0 \phi(x(t), g) + \frac{d}{dt}\Big|_0 \phi(x, g(t)) = \frac{d}{dt}\Big|_0 r_g(x(t)) + \frac{d}{dt}\Big|_0 \phi_x(g(t)) \\
&= T_x r_g(\dot{x}(0)) + T_g \phi_x(\dot{g}(0)).
\end{aligned}$$

Let $X := T_g L_{g^{-1}} \dot{g}(0) \in \mathfrak{g}$. Then viewed as a left-invariant vector field we have $X = h \mapsto L^{T_g L_{g^{-1}} \dot{g}(0)}(h) = T_e L_h(T_g L_{g^{-1}} \dot{g}(0))$, and in particular $X(g) = \dot{g}(0)$. By the calculation in the proof of (i) of Theorem 1.2.2 we know that

$$\tilde{X}(x \cdot g) = T_g \phi_x(X(g)) = T_g \phi_x(\dot{g}(0)),$$

so the claim follows. □

1.2.4 Corollary. *Let $[M, G]$ be a right Lie transformation group with group action $\phi : M \times G \to M$. Then*

$$\begin{aligned}
T_{(x,g)}\phi : T_x M \oplus T_g G \cong T_{(x,g)}(M \times G) &\to T_{xg} M \\
(X, Y) &\mapsto T_x r_g(X) + \mu_G(Y)\tilde{}_{xg},
\end{aligned}$$

where μ_G is the Maurer–Cartan form on G.

Proof. This is immediate from the previous result by letting $\dot{x}(0) = X$, $\dot{g}(0) = Y$ and recalling the definition of the Maurer–Cartan form from [9, Sec. 10]. □

Chapter 2

Principal fiber bundles

2.1 Fiber bundles

Our main objects of study throughout this course will be locally trivial fiber bundles (with certain additional structures).

2.1.1 Definition. *Let M, E, F be smooth manifolds and let $\pi : E \to M$ be a smooth map. The tuple (E, π, M, F) is called a smooth* fiber bundle *or* locally trivial fibration *of fiber type F if for any point x there exists a neighborhood $U \subseteq M$ and a diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times F$ such that $\mathrm{pr}_1 \circ \phi_U = \pi$:*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi_U"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ U & \end{tikzcd}$$

E is called the *total space*, M the *base space*, π the *projection*, and F the *fiber type* of the fibration. We will often abbreviate (E, π, M, F) by E.

Since $\pi = \mathrm{pr}_1 \circ \phi_U$, π is a surjective submersion. In particular, for any $x \in M$, $E_x := \pi^{-1}(x)$ is a regular submanifold of E (see [7, Cor. 3.3.23]), called the *fiber* over x. The pair (U, ϕ_U) is called a *bundle chart* or *local trivialization* of E over U. For any $V \subseteq M$ let $E_V := \pi^{-1}(V)$. If V is open, then also (E_V, π, V, F) is a fiber bundle in its own right, the *subbundle* over V.

For any $x \in U$ we have $\phi_U(E_x) = \{x\} \times F$ and the map

$$(\phi_U)_x := \mathrm{pr}_2 \circ \phi_U|_{E_x} : E_x \to F \tag{2.1.1}$$

is a diffeomorphism: taking any $p \in E_x$,

$$\dim E_x = \dim \ker T_p\pi = \dim E - \dim \mathrm{im}(T_p\pi) = \dim E - \dim M = \dim F$$

(because ϕ_U is a diffeomorphism). Thus $\mathrm{rk}_p((\phi_U)_x) = \mathrm{rk}\, T_p(\phi_U|_{E_x}) = \dim E_x = \dim F$, showing that $(\phi_U)_x$ is a local diffeomorphism. Since it is also bijective, it is indeed a diffeomorphism.

Let $\{U_i\}_{i \in I}$ be a covering of M and let (U_i, ϕ_i) be a bundle chart for each $i \in I$. Then the maps

$$\phi_i \circ \phi_k^{-1} : (U_i \cap U_k) \times F \to (U_i \cap U_k) \times F \tag{2.1.2}$$

are called *transition functions* between the bundle charts (U_k, ϕ_k) and (U_i, ϕ_i). Denoting by $\mathrm{Diff}(F)$ the diffeomorphism group over F we obtain maps

$$\begin{aligned}\phi_{ik} : U_i \cap U_k &\to \mathrm{Diff}(F)\\ x &\mapsto \phi_{ix} \circ \phi_{kx}^{-1} : F \to F\end{aligned}$$

that satisfy the so-called *cocycle conditions*

$$\phi_{ik}(x) \circ \phi_{kj}(x) = \phi_{ij}(x) \quad \text{and} \quad \phi_{ii}(x) = \mathrm{id}_F. \tag{2.1.3}$$

The collection of maps $\{\phi_{ik}\}_{i,k\in I}$ is called the *cocycle* of the bundle atlas $\{(U_i, \phi_i)\}_{i\in I}$.

2.1.2 Definition. *Two fiber bundles (E, π, M, F) and $(\tilde{E}, \tilde{\pi}, M, \tilde{F})$ over the same base manifold M are called* isomorphic*, $E \cong \tilde{E}$, if there exists a diffeomorphism $H : E \to \tilde{E}$ such that $\tilde{\pi} \circ H = \pi$.*

2.1.3 Examples. (i) Let M, F be manifolds and let $\mathrm{pr}_1 : M \times F \to M$ be the projection to the first factor. Then $\underline{F} := (M \times F, \mathrm{pr}_1, M, F)$ is a fiber bundle with bundle atlas consisting of the sole chart $\{(M \times F, \mathrm{id})\}$. Any fiber bundle isomorphic to such an $\underline{F}$ is called *trivial*.

(ii) For $\dim M = n$, the following are standard examples of fiber bundles over M: the tangent bundle $(TM, \pi, M, \mathbb{R}^n)$, the cotangent bundle $(T^*M, \pi, M, \mathbb{R}^n)$, the k-form bundle $(\Lambda^k T^*M, \pi, M, \mathbb{R}^{\lambda_k})$, with $\lambda_k := \binom{n}{k}$, and the (r,s)-tensor bundle $(T^r_s M, \pi, M, \mathbb{R}^{n^{r+s}})$.

To construct fiber bundles it is useful to know how to generate the manifold structure of the total space from knowledge of a bundle atlas. To this end we first introduce some notation:

2.1.4 Definition. *Let M, F be manifolds and let $\pi : E \to M$ be a surjective map from a set E onto M. If $U \subseteq M$ is open and $\phi_U : \pi^{-1}(U) \to U \times F$ is a bijective map with $\mathrm{pr}_1 \circ \phi_U = \pi|_{E_U}$, then (U, ϕ_U) is called a* formal bundle chart *for E. A family $\{(U_i, \phi_{U_i})\}_{i\in I}$ of formal bundle charts of E with respect to π is called a* formal bundle atlas *if $\{U_i\}_{i\in I}$ covers M.*

2.1.5 Theorem.

(i) Let M, F be manifolds and $\pi : E \to M$ be a surjective map from a set E onto M. Let $\{(U_i, \phi_i)\}_{i\in I}$ be a formal bundle atlas of E with respect to π such that all transition functions

$$\phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F \qquad (i, j \in I)$$

are smooth. Then there exist a uniquely determined topology and manifold structure on E such that (E, π, M, F) becomes a fiber bundle with bundle atlas $\{(U_i, \phi_i)\}_{i\in I}$.

(ii) Let (E, π, M, F) and $(\tilde{E}, \tilde{\pi}, M, \tilde{F})$ be fiber bundles over M and let $H : E \to \tilde{E}$ be a bijective map with $\tilde{\pi} \circ H = \pi$. Suppose that for all charts from two given bundle atlasses $\{(U_i, \phi_i)\}$ of E and $\{(U_i, \tilde{\phi}_i)\}$ of $\tilde{E}$ the chart representations

$$\tilde{\phi}_i \circ H \circ \phi_k^{-1} : (U_i \cap U_k) \times F \to (U_i \cap U_k) \times \tilde{F}$$

are diffeomorphisms. Then $H : E \to \tilde{E}$ is a bundle isomorphism.

Proof. (i) Define $O \subseteq E$ to be open if for every (U_i, ϕ_i) the set $\phi_i(O \cap \pi^{-1}(U_i))$ is open in $U_i \times F$. This defines a topology τ on E: clearly $\varnothing$ and E are open and so are arbitrary unions and finite intersections of open sets.

Then each $\pi^{-1}(U_i) = E_{U_i}$ is open because $\phi_j(E_{U_i} \cap E_{U_j}) = (U_i \cap U_j) \times F$ is open. Moreover, $\phi_i : E_{U_i} \to U_i \times F$ is a homeomorphism: Let $O \subseteq E_{U_i}$. Then, by definition of τ, O is open if and only if $\phi_j(O \cap E_{U_j})$ is open for each $j \in I$. Setting $j = i$ it follows that $\phi_i(O)$ is open in $U_i \times F$, so ϕ_i is open. To see that it is also continuous, let $W \subseteq U_i \times F$ be open. Then $\phi_i^{-1}(W)$ is open in E_{U_i} because

$$\begin{aligned}\phi_j(\phi_i^{-1}(W) \cap E_{U_j}) &= (\phi_j \circ \phi_i^{-1})(W \cap \phi_i(E_{U_i} \cap E_{U_j})) \\ &= (\phi_j \circ \phi_i^{-1})(W \cap ((U_i \cap U_j) \times F))\end{aligned}$$

is open in $U_j \times F$ for each $j \in I$.

It now follows that τ is second countable: We may assume without loss of generality that I is countable. Then picking a countable base $\mathcal{B}_i$ in each U_i and $\mathcal{B}$ in F,

$$\{\phi_i^{-1}(B_i \times B) \mid i \in I,\ B_i \in \mathcal{B}_i,\ B \in \mathcal{B}\}$$

is a countable basis for τ. To see that τ is also Hausdorff, let $p \neq q$ be points in E. If $\pi(p) \neq \pi(q)$ then there exist disjoint U_i, U_j with $\pi(p) \in U_i$ and $\pi(q) \in U_j$, hence E_{U_i} and E_{U_j} are disjoint neighborhoods of p and q. Otherwise, $p, q \in E_x$ for some $x \in M$. Then for $x \in U_i$ and we can openly separate $\phi_i(p)$ and $\phi_i(q)$ in $U_i \times F$ and the claim follows because ϕ_i is a homeomorphism by the above.

To define the smooth structure on E, we declare the homeomorphisms $\phi_i : E_{U_i} \to U_i \times F$ to be diffeomorphisms. This provides a smooth atlas for E because the chart transition functions

$$\phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F \qquad (i, j \in I)$$

are smooth by assumption. For this $\mathcal{C}^\infty$-structure π is smooth because $\pi|_{E_{U_i}} = \mathrm{pr}_1 \circ \phi_{U_i}$. (E, π, M, F) then is a fiber bundle.

Uniqueness of both the topology τ and the smooth structure on E is immediate because the requirement that $\{(U_i, \phi_i)\}_{i \in I}$ be a bundle atlas uniquely determines both these structures.

(ii) This is clear because H is bijective and each local representation is a diffeomorphism, so H itself is a diffeomorphism as well. □

While the previous result gives a bottom-up method to construct fiber bundles, there are also ways to obtain new bundles from given ones. We next look at the pullback of fiber bundles. Let $f : N \to M$ be smooth and let $\xi = (E, \pi, M, F)$ be a fiber bundle over M. Then we define the pullback bundle $f^*\xi := (f^*E, \bar{\pi}, N, F)$ as follows:

$$\begin{aligned}f^*E &:= \{(y, e) \in N \times E \mid f(y) = \pi(e)\} \subseteq N \times E \\ \bar{\pi}(y, e) &:= y.\end{aligned}$$

Then the following diagram commutes:

$$\begin{tikzcd} f^*E \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\bar{\pi}"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

2.1.6 Theorem. *The pullback bundle $f^*\xi = (f^*E, \bar{\pi}, N, F)$ is a fiber bundle over N.*

Proof. Let $\{(U_i, \phi_i)\}$ be a bundle atlas for ξ and set $V_i := f^{-1}(U_i) \subseteq N$ and

$$\begin{aligned} \psi_i : \bar{\pi}^{-1}(V_i) = (f^*E)_{V_i} &\to V_i \times F \\ (y, e) &\mapsto (y, \mathrm{pr}_2 \circ \phi_i(e)). \end{aligned}$$

Then ψ_i is injective: Note that $\bar{\pi}^{-1}(V_i) = \{(y, e) \mid \pi(e) = f(y) \in U_i\}$. If $\psi_i(y_1, e_1) = \psi_i(y_2, e_2)$ then $y_1 = y_2$ and $\mathrm{pr}_2 \circ \phi_i(e_1) = \mathrm{pr}_2 \circ \phi_i(e_2)$, hence also

$$\mathrm{pr}_1 \circ \phi_i(e_1) = \pi(e_1) = f(y_1) = f(y_2) = \pi(e_2) = \mathrm{pr}_1 \circ \phi_i(e_2),$$

so $e_1 = e_2$. Moreover, ψ_i is also surjective: Let $(y, b) \in V_i \times F = f^{-1}(U_i) \times F$. Then there is a unique $e \in \pi^{-1}(U_i)$ with $\phi_i(e) = (f(y), b)$. Thus $(y, e) \in \bar{\pi}^{-1}(V_i)$ (since $U_i \ni f(y) = \mathrm{pr}_1 \circ \phi_i(e) = \pi(e)$) and $\psi_i(y, e) = (y, \mathrm{pr}_2 \circ \phi_i(e)) = (y, b)$.

For any $(y, v) \in (V_i \cap V_k) \times F$ we have $\psi_k^{-1}(y, v) = (y, \phi_k^{-1}(f(y), v))$: Since $\pi \circ \phi_k^{-1}(f(y), v) = \mathrm{pr}_1(f(y), v) = f(y)$, the right hand side is in $\bar{\pi}^{-1}(V_k)$, and

$$\psi_k(y, \phi_k^{-1}(f(y), v)) = (y, \mathrm{pr}_2 \circ \phi_k \circ \phi_k^{-1}(f(y), v)) = (y, v).$$

Consequently,

$$\begin{aligned} \psi_i \circ \psi_k^{-1} : (V_i \cap V_k) \times F &\to (V_i \cap V_k) \times F \\ (y, v) &\mapsto (y, \mathrm{pr}_2 \circ \phi_i \circ \phi_k^{-1}(f(y), v)) \end{aligned}$$

is smooth for any i, k. Therefore Theorem 2.1.5 shows that f^*E has a manifold structure that turns $f^*\xi$ into a fiber bundle with bundle atlas $\{(V_i, \psi_i)\}$. □

2.1.7 Corollary. *If $N \subseteq M$ is a regular submanifold and $\xi = (E, \pi, M, F)$ is a fiber bundle over M, then the restriction $(E|_N, \pi, N, F)$ is a fiber bundle over N.*

Proof. Apply Theorem 2.1.6 to the embedding $f := N \hookrightarrow M$. □

2.1.8 Definition. *A (smooth) section of a fiber bundle (E, π, M, F) is a smooth map $s : M \to E$ with $\pi \circ s = \mathrm{id}_M$. If $U \subseteq M$ is open then sections of the subbundle E_U are called* local sections *of E over U. The set of all smooth sections of E is denoted by $\Gamma(E)$, and $\Gamma(U, E) := \Gamma(E_U)$.*

For a trivial bundle $\underline{F} = (M \times F, \mathrm{pr}_1, M, F)$ we obviously have $\Gamma(\underline{F}) = \mathcal{C}^\infty(M, F)$. Sections of TM are smooth vector fields, those of $T^r_s M$ are smooth tensor fields.

If the fiber type of a fiber bundle is diffeomorphic to a real vector space then there always exist global sections:

2.1.9 Theorem. *Let (E, π, M, F) be a fiber bundle over M with F diffeomorphic to $\mathbb{R}^m$ and let $A \subseteq M$ be closed. Then any smooth section $s : A \to E$ can be extended to a global smooth section on M. In particular there always exists a global smooth section of E, so $\Gamma(E) \neq \emptyset$.*

Proof. That s is smooth on the closed set A means that for any $x \in A$ there exists a neighborhood U of x in M and a smooth map $S : U \to E$ with $S|_{U \cap A} = s|_{U \cap A}$. We first show that given a bundle chart (U, ϕ_U), the section $s|_{U \cap A}$ can be smoothly extended to U. Also, whenever we say that two smooth sections coincide on a closed set we mean that all derivatives (in any chart) coincide on that set. Without loss of generality, let $F = \mathbb{R}^m$. Then we can write

$$\begin{aligned} \phi_U \circ s : A \cap U &\to U \times \mathbb{R}^m \\ x &\mapsto (x, f_1(x), \ldots, f_m(x)), \end{aligned}$$

where $f_i : A \cap U \to \mathbb{R}$ is smooth for $i = 1, \dots, m$. Any real valued function f that is smooth on $A \cap U$ can be smoothly extended to all of U: Take an open neighborhood W_1 of $A \cap U$ in U to which f can be smoothly extended and set $W_2 := U \setminus A$. Then if $\{\chi_1, \chi_2\}$ is a partition of unity subordinate to $\{W_1, W_2\}$, $\hat{f} := \chi_1 f$ (extended by 0 outside of W_1) does the job. So let $\hat{f}_1, \dots, \hat{f}_m : U \to \mathbb{R}$ be smooth extensions and set $s_U : U \to E$,

$$s_U(x) := \phi_U^{-1}(x, \hat{f}_1(x), \dots, \hat{f}_m(x)).$$

Now let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ be a covering of E by bundle charts with $\overline{U_\alpha} \Subset M$ and let $\{\chi_\alpha\}_{\alpha \in \Lambda}$ be a partition of unity subordinate to this covering. Let $V_\alpha := \{x \in M \mid \chi_\alpha(x) > 0\}$. Then also $\{V_\alpha\}_{\alpha \in \Lambda}$ is an open covering of M and $\overline{V_\alpha} = \operatorname{supp} \chi_\alpha \Subset U_\alpha$ for each α.

For any $J \subseteq \Lambda$ we set $M_J := \bigcup_{\alpha \in J} \overline{V_\alpha}$. Then by the above $M = M_\Lambda$. Let

$$\mathcal{T} := \{(\tau, J) \mid J \subseteq \Lambda,\ \tau : M_J \to E \ \text{ smooth section},\ \tau|_{M_J \cap A} = s|_{M_J \cap A}\}.$$

This set is nonempty because $(s_{U_\alpha}, \{\alpha\}) \in \mathcal{T}$ for each $\alpha \in \Lambda$. We define a partial order on $\mathcal{T}$ by

$$(\tau', J') \leq (\tau'', J''). \Leftrightarrow J' \subseteq J'',\ \tau' = \tau''|_{M_{J'}}.$$

Then any chain (totally ordered subset) in $\mathcal{T}$ has an upper bound (namely the pair (τ, J) with J the union of the index sets in the chain and τ such that the restriction of τ to each index set is the given section). By Zorn's Lemma there is a maximal element $(\hat{s}, \hat{J})$ in $\mathcal{T}$.

To conclude the proof we show that $\hat{J} = \Lambda$. Suppose, to the contrary, that there exists some $\alpha_0 \in \Lambda \setminus \hat{J}$. By definition of $\mathcal{T}$ we have $\hat{s}|_{M_{\hat{J}} \cap A} = s|_{M_{\hat{J}} \cap A}$, so in particular

$$\hat{s}|_{M_{\hat{J}} \cap A \cap \overline{V_{\alpha_0}}} = s|_{M_{\hat{J}} \cap A \cap \overline{V_{\alpha_0}}} \tag{2.1.4}$$

Since $\{V_\alpha\}_{\alpha \in \Lambda}$ is locally finite, $(A \cup M_{\hat{J}}) \cap \overline{V_{\alpha_0}} \subseteq U_{\alpha_0}$ is closed and $\tilde{\tau}_{\alpha_0} : (A \cup M_{\hat{J}}) \cap \overline{V_{\alpha_0}} \to E$,

$$\begin{aligned} \tilde{\tau}_{\alpha_0}|_{A \cap \overline{V_{\alpha_0}}} &:= s|_{A \cap \overline{V_{\alpha_0}}} \\ \tilde{\tau}_{\alpha_0}|_{M_{\hat{J}} \cap \overline{V_{\alpha_0}}} &:= \hat{s}|_{M_{\hat{J}} \cap \overline{V_{\alpha_0}}} \end{aligned}$$

is well-defined due to (2.1.4) and smooth since all derivatives of s and $\hat{s}$ coincide on $M_{\hat{J}} \cap A \cap \overline{V_{\alpha_0}}$.[1] As above it therefore follows that there exists a smooth extension $\tau_{\alpha_0} : U_{\alpha_0} \to E$ of $\tilde{\tau}_{\alpha_0}$. Now set $\hat{J}' := \hat{J} \cup \{\alpha_0\}$ and $\hat{s}' : M_{\hat{J}'} \to E$,

$$\hat{s}' := \begin{cases} \hat{s} & \text{on } M_{\hat{J}} \\ \tau_{\alpha_0} & \text{on } \overline{V_{\alpha_0}}. \end{cases}$$

Then $(\hat{s}', \hat{J}') \in \mathcal{T}$, contradicting the maximality of $(\hat{s}, \hat{J})$. Thus $M = M_{\hat{J}}$ and $\hat{s} : M \to E$ is a section with $\hat{s}|_A = s$, as claimed. □

2.2 Principal fiber bundles

Our main objects of interest for the remainder of these lecture notes are certain fiber bundles whose typical fiber is a Lie group:

[1] Strictly speaking the existence of a smooth extension (both here and in the case of the upper bound of a chain above) as required in our definition of smoothness on closed sets follows from Whitney's extension theorem.

2.2.1 Definition. *Let G be a Lie group, M, P manifolds, and $\pi : P \to M$ smooth. The tuple (P, π, M, G) is called a G-principal fiber bundle over M if*

(i) G acts on P from the right as a Lie transformation group. The action is free and simply transitive on the fibers.

(ii) There exists a bundle atlas $\{(U_i, \phi_i)\}$ consisting of G-equivariant bundle charts, i.e.,

(a) $\phi_i : \pi^{-1}(U_i) \to U_i \times G$ is a diffeomorphism.

(b) $\mathrm{pr}_1 \circ \phi_i = \pi$.

(c) $\phi_i(p \cdot g) = \phi_i(p) \cdot g$ for all $p \in \pi^{-1}(U_i)$ and $g \in G$, where G acts on $U_i \times G$ via $(x, a) \cdot g = (x, ag)$.

2.2.2 Remark. Let (P, π, M, G) be a principal fiber bundle. Then M can be viewed as the quotient manifold of P by the group action. Indeed, denote by ρ the map that assigns to any $p \in P$ its orbit under the right action by G, i.e., the canonical quotient map. Now consider the map $i : P/G \to M$, $[p]_\rho \mapsto \pi(p)$:

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, "\rho"'] & M \\ P/G \arrow[ur, "i"'] & \end{tikzcd}$$

Since $\pi(pg) = \pi(p)$, i is well-defined. It is injective, since $\pi(p) = \pi(q)$ means that $p = qg$ for some g, so $[p]_\rho = [q]_\rho$. Indeed it is also surjective: for $x \in M$, take any $p \in \pi^{-1}(x)$. Then $i([p]_\rho) = \pi(p) = x$. We may now declare the bijection i to be a diffeomorphism, thereby inducing a manifold structure on P/G. With this structure, P/G (and thereby M) is a quotient manifold of P because $i \circ \rho = \pi$, implying that ρ is a submersion. This structure is unique by [9, Rem. 15.8]. Some sources (e.g., [3, 6]) define principal fiber bundles via P/G.

2.2.3 Definition. *Let (P, π_P, M, G) and (Q, π_Q, M', H) be principal fiber bundles. A* bundle morphism *$Q \to P$ is a pair (f, λ), where $f : Q \to P$ is a smooth map and $\lambda : H \to G$ is a Lie group homomorphism such that f is a λ-equivariant bundle map, i.e.,*

$$f(q \cdot h) = f(q) \cdot \lambda(h) \qquad \forall q \in Q,\ h \in H.$$

Any bundle morphism f induces a map $\underline{f} : M' \to M$ on the base manifolds such that

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d, "\pi_Q"'] & P \arrow[d, "\pi_P"] \\ M' \arrow[r, "\underline{f}"] & M \end{tikzcd} \tag{2.2.1}$$

commutes. Indeed, if $\pi_Q(q_1) = \pi_Q(q_2)$, then $q_1 = q_2 h$ for some $h \in H$, so $f(q_1) = f(q_2)\lambda(h)$, hence $\pi_P(f(q_1)) = \pi_P(f(q_2))$. Since π_Q is a surjective submersion, $\underline{f}$ is smooth.

A bundle morphism (f, λ) is called a bundle embedding if $f, \underline{f}$ and λ are embeddings of manifolds. If $M = M'$ and $\underline{f} = \mathrm{id}_M$, then Q, together with f is called a λ-reduction of P. If, in addition, H is a Lie subgroup of G and λ is the inclusion map then f is called an *H-reduction* of P, and the image of f is called a *principal G-subbundle* of P. We will study reductions in more detail in Section 2.5. Finally, if $G = H$, $\lambda = \mathrm{id}_G$, and f is a diffeomorphism, then f is called a *bundle isomorphism*.

In this case, $\underline{f}$ is itself a diffeomorphism: $\underline{f}$ is surjective by (2.2.1) since π_P and f are. Now suppose that $\underline{f}(\pi_Q(q_1)) = \underline{f}(\pi_Q(q_2))$. Then again by (2.2.1) $\pi_P(f(q_1)) = \pi_P(f(q_2))$, so there is some $g \in G$ with $f(q_1) = f(q_2)g = f(q_2 g)$. Thus $q_1 = q_2 g$ and $\pi_Q(q_1) = \pi_Q(g_2)$. In addition, $\underline{f}$ is a submersion by (2.2.1), so our claim follows from the following auxiliary result:

2.2.4 Lemma. *Let $f : M \to N$ be a bijective submersion. Then f is a local (hence global) diffeomorphism.*

Proof. For any $x \in M$ there exist charts φ of M around x and ψ around $f(x)$ of N such that $\psi \circ f \circ \varphi^{-1} = \mathrm{pr}_1 : U \times V \to U$ (cf. [7, Th. 3.3.3]). Since f is bijective, so is $\psi \circ f \circ \varphi^{-1}$, but this is only possible if $V = \varnothing$ and $\psi \circ f \circ \varphi^{-1} = \mathrm{id} : U \to U$. □

Next we give two equivalent descriptions of principal fiber bundles, starting by replacing local trivializations by local sections.

2.2.5 Theorem. *Let G be a Lie group and $\pi : P \to M$ a smooth map. Then (P, π, M, G) is a principal fiber bundle if and only if*

(i) G acts on P from the right as a Lie transformation group. The action is free and simply transitive on the fibers.

(ii) There exists an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of M and local sections $s_i : U_i \to P$ for each $i \in I$.

Proof. Given a local section $s : U \to P$, define ψ_s by

$$\begin{aligned} \psi_s : U \times G &\to P_U \\ (x, g) &\mapsto s(x) \cdot g. \end{aligned} \tag{2.2.2}$$

Since s is a section ψ_s takes values in P_U. Also, it is smooth and G-equivariant: $\psi_s(x, a) \cdot g = (s(x) \cdot a) \cdot g = s(x) \cdot (ag) = \psi_s(x, ag)$. Since G acts simply transitively on the fibers, ψ_s is bijective: it is clearly surjective and if $\psi_s(x, g) = \psi_s(y, h)$ then applying π we get $x = y$, and so $s(x) = s(x)hg^{-1}$, implying that $g = h$ as well.

Since π has local sections it is a submersion (cf. [9, Prop. 15.2]). Thus for any $x \in M$, $P_x = \pi^{-1}(x)$ is a regular submanifold, and for any $p \in P_x$ we have $T_p P_x = \ker T_p \pi$ ([7, 3.3.23, 3.3.25]). According to (i), G acts freely and transitively on P_x, so by Corollary 1.1.12 P_x is diffeomorphic to G. Indeed the proof of that result (cf. the proof of Theorem 1.1.7) shows that, denoting the group action by $\Phi : P \times G \to P$, for any $p \in P_x$ the map $\Phi_p : G \to P_x$, $\Phi_p(g) = p \cdot g$ ($x = \pi(p)$) is a diffeomorphism.

To show that ψ_s is a diffeomorphism, by Lemma 2.2.4 it suffices to check that it is a submersion. Let $(x, g) \in U \times G$, $X \in T_x M$ and $Z \in T_g G$. Pick smooth curves $t \mapsto x(t)$ in U and $t \mapsto g(t)$ in G with $x(0) = x$, $\dot{x}(0) = X$, $g(0) = g$ and $\dot{g}(0) = Z$. Also, let $y(t) := s(x(t))$ and $w(t) := y(t)g(t) = \psi_s(x(t), g(t))$. Then by Lemma 1.2.3

$$\begin{aligned} T_{(x,g)}\psi_s(X, Z) &= \dot{w}(0) = T_{s(x)} R_g(\dot{y}(0)) + (T_g L_{g^{-1}} \dot{g}(0))^{\tilde{}}(y(0) \cdot g) \\ &= T_{s(x)} R_g(T_x s X) + (T_g L_{g^{-1}} Z)^{\tilde{}}(s(x) \cdot g). \end{aligned}$$

Now given any $\hat{W} \in T_{s(x)g} P$ we need to find (X, Z) as above with $T_{(x,g)}\psi_s(X, Z) = \hat{W}$. Set $X := T\pi(\hat{W}) \in T_{\pi(s(x)g)} M = T_x M$ and $\hat{X} := TR_g(Ts(X))$. Then since $\pi \circ R_g = \pi$ and $\pi \circ s = \mathrm{id}$,

$$T\pi(\hat{W} - \hat{X}) = X - T\pi(TR_g(Ts(X))) = X - X = 0,$$

meaning that $\hat{W} - \hat{X} \in \ker T_{s(x)g}\pi = T_{s(x)g} P_x$. By what was shown above, $T_e \Phi_{s(x)g} : \mathfrak{g} \to T_{s(x)g} P_x$ is bijective, so there exists a unique $A \in \mathfrak{g}$ with (recall Definition 1.2.1)

$$\tilde{A}(s(x)g) = T_e \Phi_{s(x)g}(A) = \hat{W} - \hat{X}.$$

Now setting $Z := T_e L_g(A) \in T_g G$ we finally obtain

$$\begin{aligned} T_{(x,g)}\psi_s(X,Z) &= T_{s(x)}R_g(T_x sX) + (T_g L_{g^{-1}} Z)^{\tilde{}}(s(x)\cdot g) \\ &= \hat{X} + \tilde{A}(s(x)g) = \hat{X} + \hat{W} - \hat{X} = \hat{W}. \end{aligned}$$

Having established that ψ_s is a diffeomorphism we now claim that $\phi_s := \psi_s^{-1} : P_U \to U \times G$ is a G-equivariant bundle chart. Note first that $\pi \circ \psi_s(x,g) = x$ because G is fiber preserving and s is a section. Thus $\pi \circ \psi_s = \mathrm{pr}_1$, so $\mathrm{pr}_1 \circ \phi_s = \pi$, showing that ϕ_s is a bundle chart. It is also G-equivariant because any $p \in P_U$ can uniquely be written as $p = s(x)g$, so for $h \in G$ we have

$$\phi_s(p\cdot h) = \phi_s(s(x)\cdot(gh)) = (x,gh) = (x,g)\cdot h = \phi_s(p)\cdot h.$$

Conversely, suppose that (U,ϕ_U) is a G-equivariant bundle chart for P. Then the map $s : U \to P$, $s(x) := \phi_U^{-1}(x,e)$ is a smooth section of $P|_U$. □

For our second characterization of principal fiber bundles we need the following notion:

2.2.6 Definition. *Let M be a manifold and let G be a Lie group. Suppose that $\mathcal{U} := \{U_i\}_{i\in I}$ is an open covering of M and that $\{g_{ij}\}_{i,j\in I}$ is a family of smooth functions $g_{ij} : U_i \cap U_j \to G$ with*

$$\begin{aligned} g_{ij}(x)\cdot g_{jk}(x) &= g_{ik}(x) \qquad \forall x \in U_i \cap U_j \cap U_k \\ g_{ii}(x) &= e \qquad \forall x \in U_i \end{aligned} \tag{2.2.3}$$

$(i,j,k \in I)$. Then the family $\{g_{ij}\}_{i,j\in I}$ is called a G-cocycle on M.

2.2.7 Theorem. *Let M be a manifold, G a Lie group, $\mathcal{U} = \{U_i\}_{i\in I}$ an open covering of M and $\{g_{ij}\}_{i,j\in I}$ a G-cocycle on M. Then there exists a smooth principal fiber bundle (P,π,M,G) with a bundle atlas $\{(U_i,\phi_i)\}$ whose transition functions are given by the left translations by g_{ij}:*

$$\phi_{ij}(x) = L_{g_{ij}(x)} : G \to G \qquad \forall x \in U_i \cap U_j.$$

Moreover, (P,π,M,G) is unique up to bundle isomorphism.

Proof. For each $i \in I$, set $X_i := U_i \times G$ and let $X := \dot{\bigcup}_{i\in I} X_i$ be the topological sum of the X_i. Each element of X can be written in the form (i,x,a) with $i \in I$, $x \in X_i$, and $a \in G$. Since each X_i is a smooth manifold, so is X.

Now call (i,x,a) equivalent to (j,y,b) if $x = y \in U_i \cap U_j$ and $a = g_{ij}(x)b$. This defines an equivalence relation ρ on X, and we note that $(i,x,a) \sim_\rho (i,y,b)$ if and only if $x = y$ and $a = b$. $(*)$ Denote by $P := X/\sim_\rho$ the quotient of X with respect to ρ. We claim that G acts freely on P on the right.

Indeed, by definition $c \in G$ maps the ρ-equivalence class $[(i,x,a)]$ to $[(i,x,ac)]$. This is independent of the representative: if $(i,x,a) \sim (j,y,b)$, then $x = y$ and $b = g_{ji}(x)a$, hence $bc = g_{ji}(x)ac$, giving $(i,x,ac) \sim (j,y,bc)$. The action is free because $[(i,x,ac)] = [(i,x,c)]$ implies $ac = g_{ii}(x)a = a$, hence $c = e$.

The map $\pi : P = X/\sim_\rho \to M$, $[(i,x,a)] \mapsto x$ is well-defined, and for $p,q \in P$ we have $\pi(p) = \pi(q)$ if and only if $q = pc$ for some $c \in G$: Let $p = [(i,x,a)]$, $q = [(j,y,b)]$. Then if $q = pc$ we get $y = x$, so $\pi(p) = \pi(q)$. Conversely, let $\pi(p) = x = y = \pi(q) \in U_i \cap U_j$. Then $q = pc$, where $c = a^{-1}g_{ij}(x)b \in G$.

To define a smooth structure on P we first note that the quotient map $\chi : X \to P = X/\sim_\rho$, $(i,x,a) \mapsto [(i,x,a)]$ (for $x \in U_i$) maps each $X_i = U_i \times G$ bijectively onto $\pi^{-1}(U_i)$: surjectivity is clear and injectivity follows from $(*)$ above. Call the inverse of this map $\phi_i : \pi^{-1}(U_i) \to X_i$. Now we define the $\pi^{-1}(U_i)$ $(i \in I)$ to be open submanifolds of P and the ϕ_i to be diffeomorphisms. This indeed defines a smooth structure on P because the $\pi^{-1}(U_i)$ cover P and the transition functions are the smooth maps

$$\phi_j \circ \phi_i^{-1} : (x, a) \mapsto \phi_j([(i, x, a)]) = \phi_j([j, x, g_{ji}(x)a]) = (x, g_{ji}(x)a). \tag{2.2.4}$$

G acts smoothly on P because on $\pi^{-1}(U_i)$, the right multiplication r_c by $c \in G$ satisfies

$$\phi_i \circ r_c \circ \phi_i^{-1}(x, a) = \phi_i([(i, x, ac)]) = (x, ac). \tag{2.2.5}$$

Thus G acts as a Lie transformation group on P. Moreover, $\pi \circ \phi_i^{-1} = \mathrm{pr}_1 : (x, a) \mapsto x$, so π is smooth and the action of G clearly is fiber preserving and simply transitive on the fibers. It follows from (2.2.4) that $\{(U_i, \phi_i)\}_{i \in I}$ is a bundle atlas for P whose transition functions are given by the left translations by g_{ij}. Finally, the ϕ_i are G-equivariant due to (2.2.5).

It remains to show uniqueness up to bundle isomorphism. Suppose that (P', π', M, G) is another principal fiber bundle over M with bundle charts (U_i, ϕ_i') that give rise to the same cocycle. Then define the map $F : P' \to P$ on any U_i as follows:

$$\begin{tikzcd}
P'|_{U_i} \arrow[rr, "F"] \arrow[dr, "\phi_i'"] \arrow[ddr, "\pi'"'] & & P|_{U_i} \arrow[dl, "\phi_i"'] \arrow[ddl, "\pi"] \\
& U_i \times G \arrow[d, "\mathrm{pr}_1"] & \\
& U_i &
\end{tikzcd}$$

Thus $F(p') := \phi_i{}^{-1} \circ \phi_i'(p')$. This map is well-defined, i.e., independent of the i with $p' \in P'|_{U_i}$: we have $\pi \circ F(p') = \pi'(p') =: x \in U_i$, so $F(p') = \phi_{ix}^{-1} \circ \phi_{ix}'(p')$. Now if also $\pi'(p') \in U_k$, then $\phi_{ix} \circ \phi_{kx}^{-1} = L_{g_{ik}(x)}$, so $\phi_{ix}^{-1} = \phi_{kx}^{-1} \circ L_{g_{ik}(x)}^{-1}$, and analogously for the ϕ_{ix}'. Therefore,

$$F(p') = \phi_{ix}^{-1} \circ \phi_{ix}'(p') = \phi_{kx}^{-1} \circ L_{g_{ik}(x)}^{-1} \circ L_{g_{ik}(x)} \circ \phi_{kx}'(p') = \phi_{kx}^{-1} \circ \phi_{kx}'(p').$$

F is surjective since $\{(U_i, \phi_i)\}$ and $\{(U_i, \phi_i')\}$ are bundle atlasses. It is also injective: Let $F(p_1') = F(p_2')$, then $\pi'(p_1') = \pi(F(p_1')) = \pi(F(p_2')) = \pi'(p_2') \in U_i$ for some $i \in I$, and $F : P'|_{U_i} \to P|_{U_i}$ is bijective by definition. It also follows from the local representation that F and F^{-1} are smooth and G-equivariant, hence F is a bundle isomorphism. □

2.2.8 Theorem. *Let M, P be manifolds and let $\pi : P \to M$ be a smooth map. Then the following are equivalent:*

(i) (P, π, M, G) *is a principal fiber bundle.*

(ii) There exists a bundle atlas $\{(U_i, \phi_i)\}_{i \in I}$ *and a G-cocycle* $\{g_{ij}\}_{i,j \in I}$ *for the open cover* $\mathcal{U} = \{U_i\}_{i \in I}$ *of M such that the cocycle* $\{\phi_{ij}\}_{i,j \in I}$ *of P is given by the left translations* $\phi_{ij}(x) = L_{g_{ij}(x)} : G \to G$.

Moreover, (P, π, M, G) *satisfying (ii) is unique up to bundle isomorphism.*

Proof. (i)⇒(ii): Let $\{(U_i, \phi_i)\}_{i \in I}$ be a G-equivariant bundle atlas for P. Then

$$\begin{aligned} g_{ik} &: U_i \cap U_k \to G \\ x \mapsto g_{ik}(x) &:= \phi_{ix}(\phi_{kx}^{-1}(e)) = \phi_{ik}(x)(e) = \mathrm{pr}_2 \circ \phi_i \circ \phi_k^{-1}(x, e) \end{aligned}$$

is smooth. Also, $\phi_i(\phi_k^{-1}(x,e))\cdot g = \phi_i(\phi_k^{-1}(x,e)\cdot g) = \phi_i(\phi_k^{-1}(x,g))$, so

$$\begin{aligned} g_{ik}(x)\cdot g &= \mathrm{pr}_2 \circ \phi_i \circ \phi_k^{-1}(x,e)\cdot g = \mathrm{pr}_2(\phi_i \circ \phi_k^{-1}(x,e)\cdot g) \\ &= \mathrm{pr}_2(\phi_i \circ \phi_k^{-1}(x,g)) = \phi_{ik}(x)(g). \end{aligned}$$

(ii)⇒(i): Define the right action of G on P as follows: for $p \in P_x$ and $x \in U_i$ set

$$p\cdot g := \phi_{ix}^{-1}(\phi_{ix}(p)\cdot g). \tag{2.2.6}$$

This definition is independent of the chosen chart: let $x \in U_i \cap U_j$. Then by (ii)

$$\phi_{ix}\circ\phi_{jx}^{-1}(\phi_{jx}(p)\cdot g) = \phi_{ij}(x)(\phi_{jx}(p)\cdot g) = g_{ij}(x)\cdot\phi_{jx}(p)\cdot g = \phi_{ix}(p)\cdot g.$$

Then G acts as a Lie transformation group on P: $(p,g)\mapsto p\cdot g$ is smooth since writing $\phi_i(p) = (x,a)$ we have $p\cdot g = \phi_{ix}^{-1}(a\cdot g) = \phi_i^{-1}(x, a\cdot g)$. Also, it is easily seen that $(p\cdot g)\cdot h = p\cdot(g\cdot h)$. By definition, the action of G is fiber preserving. It is also simply transitive on the fibers because

$$q = p\cdot g \Leftrightarrow q = \phi_{ix}^{-1}(\phi_{ix}(p)\cdot g) \Leftrightarrow \phi_{ix}(q) = \phi_{ix}(p)\cdot g,$$

which is uniquely solvable. It only remains to verify (ii) (c) from Definition 2.2.1. Again let $\phi_i(p) = (x,a)$. Then

$$\begin{aligned} \phi_i(pg) &= \phi_i(\phi_{ix}^{-1}(\phi_{ix}(p)\cdot g)) = \phi_i(\phi_{ix}^{-1}(a\cdot g)) = \phi_i(\phi_i^{-1}(x,ag)) = (x,ag) \\ &= (x,a)\cdot g = \phi_i(p)\cdot g. \end{aligned}$$

Finally, the uniqueness claim follows from Theorem 2.2.7. □

2.2.9 Example. The trivial bundle $\underline{G} := (M\times G, \mathrm{pr}_1, M, G)$ is a principal fiber bundle with the single bundle chart $\phi = \mathrm{id} : M\times G \to M\times G$.

2.2.10 Example. Let $\xi = (P,\pi,M,G)$ be a principal fiber bundle and let $f : N \to M$ be smooth. Then the pullback bundle $f^*\xi$ (cf. Theorem 2.1.6) is a principal G-fiber bundle over N. Recall that

$$f^*\xi = \{(n,p) \in N\times P \mid f(n) = \pi(p)\} \subseteq N\times P$$

and we already know that it is a fiber bundle with bundle charts derived from those of P $((U_i,\phi_i))$ via

$$\begin{aligned} \psi_i : \bar\pi^{-1}(V_i) = (f^*\xi)_{V_i} &\to V_i \times G \\ (n,p) &\mapsto (n, \mathrm{pr}_2\phi_i(p)). \end{aligned}$$

(and $V_i := f^{-1}(U_i) \subseteq N$). The action of G on $f^*\xi$, $(n,p)\cdot g := (n, p\cdot g)$ is well-defined since $\pi(p\cdot g) = \pi(p) = f(n)$, so $(n, p\cdot g) \in f^*\xi$. It is clearly fiber preserving, as well as smooth since

$$\psi_i \circ r_g \circ \psi_i^{-1}(n,a) = (n, a\cdot g).$$

Simple transitivity on the fibers follows since for $u, v \in f^*\xi$ with $\bar\pi(u) = n = \bar\pi(v)$ we get $u = (n,p)$, $v = (n,q)$ and there is a unique $g \in G$ with $p\cdot g = q$, and thereby with $(n,p)\cdot g = (n,q)$. It only remains to verify (ii) (c) from Definition 2.2.1:

$$\psi_i((n,p)\cdot g) = \psi_i(n, p\cdot g) = (n, \mathrm{pr}_2\circ\phi_i(p\cdot g)) = (n, \mathrm{pr}_2\circ\phi_i(p)\cdot g) = \psi_i(n,p)\cdot g.$$

2.2.11 Example. (Homogeneous bundle) Let H be a closed, non-open subgroup of the Lie group G and let G/H be the corresponding homogeneous space. By Theorem 1.1.3 G/H is a quotient manifold of G. Denote by $\pi : G \to G/H$ the canonical projection. Then $(G, \pi, G/H, H)$ is a principal fiber bundle with structure group H: The action of H on G is given by $(g,h) \mapsto g\cdot h$, which is clearly smooth. It preserves fibers since if $g_1 \in \pi^{-1}(gH)$ then $g_1 = gh_1$ for some $h_1 \in H$, so $g_1 h = gh_1 h \in gH$, and $g_1 h \in \pi^{-1}(gH)$. To see transitivity on the fibers, let $g_1, g_2 \in \pi^{-1}(gH)$. Then $g_i = gh_i$ $(i = 1,2)$, so $g_1 = g_2(h_2^{-1}h_1)$, and $h = g_2^{-1}g_1$ is the unique element of H with $g_1 = g_2 h$. Finally, by Theorem 1.1.3 there exist local sections of π, so Theorem 2.2.5 gives the claim.

2.2.12 Example. (The frame bundle of a manifold) Let M be an n-dimensional manifold and set

$$\mathrm{GL}(M)_x := \{\nu_x = (\nu_1, \dots, \nu_n) \mid \nu_x \text{ is a basis of } T_x M\},$$

and

$$\mathrm{GL}(M) := \dot{\bigcup_{x\in M}} \mathrm{GL}(M)_x.$$

Define $\pi : \mathrm{GL}(M) \to M$ by $\pi(\nu_x) := x$. The group $\mathrm{GL}(n,\mathbb{R})$ acts on the set $\mathrm{GL}(M)$ on the right by

$$(\nu_1, \dots, \nu_n)\cdot A := \Big(\sum_i \nu_i A_{i1}, \dots, \sum_i \nu_i A_{in}\Big), \tag{2.2.7}$$

where $A = (A_{ij})$. Obviously, this action is fiber preserving and acts transitively on the fibers.

To introduce a manifold structure on $\mathrm{GL}(M)$, let $(U_i, \varphi_i = (x^1, \dots, x^n))$ be a chart of M. Then we define $\phi_i : \mathrm{GL}(M)|_{U_i} := \pi^{-1}(U_i) \to U_i \times \mathrm{GL}(n,\mathbb{R})$ as follows: for $\nu_x \in \mathrm{GL}(M)|_{U_i}$ there is a unique $A(x) \in \mathrm{GL}(n,\mathbb{R})$ such that

$$\nu_x = \Big(\frac{\partial}{\partial x^1}\Big|_x, \dots, \frac{\partial}{\partial x^n}\Big|_x\Big)\cdot A(x). \tag{2.2.8}$$

Set $\phi_i(\nu_x) := (x, A(x))$. Then ϕ_i is bijective and the following diagram commutes:

$$\begin{tikzcd}
\mathrm{GL}(M)|_{U_i} \arrow[r, "\phi_i"] \arrow[d, "\pi"'] & U_i \times \mathrm{GL}(n,\mathbb{R}) \arrow[dl, "\mathrm{pr}_1"] \\
U_i &
\end{tikzcd}$$

Hence ϕ_i is a formal bundle chart. If $(U_k, \varphi_k = (y^1, \dots, y^n))$ is another chart in M then in the notation of (2.2.7) we have

$$\Big(\frac{\partial}{\partial y^1}\Big|_x, \dots, \frac{\partial}{\partial y^n}\Big|_x\Big) = \Big(\frac{\partial}{\partial x^1}\Big|_x, \dots, \frac{\partial}{\partial x^n}\Big|_x\Big)\cdot\Big(\frac{\partial x^i}{\partial y^j}\Big)_{i,j}.$$

Thus

$$\phi_i \circ \phi_k^{-1} : (U_i \cap U_k) \times \mathrm{GL}(n,\mathbb{R}) \to (U_i \cap U_k) \times \mathrm{GL}(n,\mathbb{R})$$
$$\phi_i \circ \phi_k^{-1}(x,B) = \phi_i\Big(\Big(\frac{\partial}{\partial y^1}\Big|_x, \dots, \frac{\partial}{\partial y^n}\Big|_x\Big)\cdot B\Big) = \Big(x, \Big(\frac{\partial x^i}{\partial y^j}\Big)\cdot B\Big)$$

is smooth, and so by Theorem 2.1.5 we obtain a fiber bundle structure on $\mathrm{GL}(M)$. $\mathrm{GL}(n,\mathbb{R})$ acts on the right and it is evident from the definition that this action is fiber preserving and simply transitive on the fibers. It is also smooth, because if $\phi_{ix}(\nu_x) = A(x)$, i.e., if (2.2.8) holds, then $\phi_{ix}(\nu_x \cdot B) = A(x)\cdot B = \phi_{ix}(\nu_x)\cdot B$, meaning that locally the action is given by $((x,A),B) \mapsto (x, A\cdot B)$. This also proves the last point of Definition 2.2.1, concluding the verification that $(\mathrm{GL}(M), \pi, M, \mathrm{GL}(n,\mathbb{R}))$ is a principal fiber bundle.

For any chart $(U_i, \varphi_i = (x^1, \dots, x^n))$, the map

$$s := \Big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\Big) : U \to \mathrm{GL}(M).$$

defines a local section, which is smooth because $\phi_i(s(x)) = (x, I_n)$. Note that the bundle charts constructed above are precisely the ones constructed from these sections according to the proof of Theorem 2.2.5.

2.2.13 Example. Analogously to Example 2.2.12, one can construct subbundles of the frame bundle if the manifold M is endowed with some additional geometric structure:

(i) Let $(M, \mathcal{O}_M)$ be oriented. Then let

$$\mathrm{GL}(M)^+_x := \{\nu_x \in \mathrm{GL}(M)_x \mid \nu_x \text{ positively oriented}\}.$$

This leads to the $\mathrm{GL}(n,\mathbb{R})^+$-principal fiber bundle of all positively oriented frames $(\mathrm{GL}(M)^+, \pi, M, \mathrm{GL}(n,\mathbb{R})^+)$.

(ii) Let $(M^{p,q}, g)$ be a semi-Riemannian manifold of signature (p,q) and set

$$\mathrm{O}(M,g)_x = \left\{\nu_x = (\nu_1, \dots, \nu_n) \in \mathrm{GL}(M)_x \mid (g_x(\nu_i, \nu_j)) = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}.\right\}$$

This gives the $\mathrm{O}(p,q)$-principal fiber bundle $(\mathrm{O}(M,g), \pi, M, \mathrm{O}(p,q))$ of orthonormal frames.

2.2.14 Theorem. *A principal fiber bundle (P, π, M, G) is trivial if and only if it possesses a global section.*

Proof. Let $s : M \to P$ be a global section. Then the map

$$\begin{aligned} \Phi : M \times G &\to P \\ (x,g) &\mapsto s(x)\cdot g \end{aligned}$$

is an isomorphism of principal fiber bundles. This follows exactly as in the proof of Theorem 2.2.5.

Conversely, if $\Phi : M \times G \to P$ is a bundle isomorphism, then $s(x) := \Phi(x,e)$ defines a global section of P. □

2.3 Associated fiber bundles

Given a principal fiber bundle (P, π, M, G) and a manifold F such that $[F, G]$ is a left Lie transformation group, one can construct a new fiber bundle by 'replacing the fiber'. Note first that G acts on the right on the product $P \times F$ via

$$(p,v)\cdot g := (p\cdot g, g^{-1}\cdot v). \tag{2.3.1}$$

Denote by $E := (P \times F)/G =: P \times_G F$ the corresponding quotient space, by $[p,v]$ the equivalence class of (p,v), and by

$$\begin{aligned} \hat{\pi} : E &\to M \\ [(p,v)] &\mapsto \pi(p) \end{aligned}$$

the projection (which is well-defined). Then we have:

2.3.1 Theorem. *$(E, \hat{\pi}, M, F)$ is a fiber bundle over M. It is called the fiber bundle associated to P and $[F, G]$.*

Proof. Let $x \in M$ and let (U, ϕ_U) be a bundle chart of P,

$$\begin{aligned} \phi_U : P_U &\to U \times G \\ p &\mapsto (\pi(p), \varphi_U(p)). \end{aligned}$$

Then $\varphi_U(p \cdot g) = \varphi_U(p) \cdot g$. Now set $E_U := \hat{\pi}^{-1}(U)$ and define

$$\begin{aligned} \psi_U : E_U &\to U \times F \\ [p, v] &\mapsto (\pi(p), \varphi_U(p) \cdot v). \end{aligned} \tag{2.3.2}$$

This map is well-defined: Let $[p_1, v_1] = [p_2, v_2]$, so $(p_1, v_1) = (p_2 g, g^{-1} v_2)$ for some $g \in G$. Then $\pi(p_1) = \pi(p_2)$ and

$$\varphi_U(p_1) v_1 = \varphi_U(p_2 g) \cdot (g^{-1} v_2) = (\varphi_U(p_2 g) \cdot g^{-1}) v_2 = \varphi_U(p_2) v_2.$$

Let $\psi_U([p_1, v_1]) = \psi_U([p_2, v_2])$, then $\pi(p_1) = \pi(p_2) =: x$, so there exists some $g \in G$ with $p_1 = p_2 g$. Also, $\varphi_U(p_2) \cdot g \cdot v_1 = \varphi_U(p_1) \cdot v_1 = \varphi_U(p_2) \cdot v_2$, so $g \cdot v_1 = v_2$, implying $[p_1, v_1] = [p_2, v_2]$ and thereby injectivity of ψ_U. To see surjectivity, let $(x, v) \in U \times F$ and set $p := \phi_U^{-1}(x, e)$. Then

$$\psi_U([p, v]) = (\pi(p), \varphi_U(p) \cdot v) = (x, v).$$

Thus (U, ψ_U) is a formal bundle chart of E. The above also shows that $\psi_U^{-1} : U \times F \to E_U$ is given by

$$\psi_U^{-1}(x, v) = [\phi_U^{-1}(x, e), v]. \tag{2.3.3}$$

We have

$$\begin{tikzcd} E_U \arrow[r, "\psi_U"] \arrow[d, "\hat{\pi}"'] & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ U & \end{tikzcd}$$

Now $E_U \cap E_V = E_{U \cap V}$ and $\psi_U(E_{U \cap V}) = \psi_U(\hat{\pi}^{-1}(U \cap V)) = \mathrm{pr}_1^{-1}(U \cap V) = (U \cap V) \times F$, so $\psi_V \circ \psi_U^{-1} : (U \cap V) \times F \to (U \cap V) \times F$, and setting $p := \phi_U^{-1}(x, e)$ we calculate

$$\psi_V \circ \psi_U^{-1}(x, v) = \psi_V([p, v]) = (x, \varphi_V(p) \cdot v) = (x, \varphi_V(\phi_U^{-1}(x, e)) \cdot v),$$

which shows that the chart transition maps are smooth, giving the claim. □

Let us also derive alternative representations for some of the above maps. Given $x \in U \cap V$, pick any $p \in \pi^{-1}(x)$. Then $\psi_U([p, \varphi_U(p)^{-1} \cdot v]) = (x, v)$, showing that

$$\psi_U^{-1}(x, v) = [p, \varphi_U(p)^{-1} \cdot v], \tag{2.3.4}$$

and consequently $\psi_V \circ \psi_U^{-1}(x, v) = (x, \varphi_V(p) \cdot \varphi_U(p)^{-1} \cdot v)$ for any $p \in \pi^{-1}(x)$. Also, we note that the map $(p, v) \mapsto [p, v]$, $P \times F \to E$ is smooth because

$$[p, v] = \psi_U^{-1}(\pi(p), \varphi_U(p) \cdot v). \tag{2.3.5}$$

Next we want to compare the chart transition functions in P and in E. Let $g_{ik} : U_i \cap U_k \to G$ be the G-cocycle induced by a bundle atlas $\{(U_i, \phi_i)\}_{i \in I}$ of P according to Theorem 2.2.8. In our current setting we have, writing $\phi_i = \phi_{U_i}$, $\psi_i = \psi_{U_i}$, cf. (2.1.1):

$$\begin{aligned} \psi_{ix} : E_x &\to F \\ [p, v] &\mapsto \mathrm{pr}_2 \circ \psi_i|_{E_x}([p, v]) = \varphi_i(p) \cdot v, \end{aligned}$$

where p is any point in $P_x = \pi^{-1}(x)$. We have

$$\psi_{ik}(x)(v) = \psi_{ix} \circ \psi_{kx}^{-1}(v) = \psi_{ix}(\psi_k^{-1}(x,v)) = \mathrm{pr}_2 \circ \psi_i \circ \psi_k^{-1}(x,v).$$

Here, $\psi_k^{-1}(x,v) = [p, \varphi_k(p)^{-1} \cdot v]$, so $\psi_i \circ \psi_k^{-1}(x,v) = (\pi(p), \varphi_i(p) \cdot \varphi_k(p)^{-1} \cdot v)$, hence

$$\psi_{ik}(x)(v) = \varphi_i(p) \cdot \varphi_k(p)^{-1} \cdot v.$$

Since $\phi_{kx} : P_x \to G$ is bijective, there is a unique $p_0 \in P_x$ with $\varphi_k(p_0) = \phi_{kx}(p_0) = e$. Then

$$\psi_{ik}(x)v = \varphi_i(p_0) \cdot \varphi_k(p_0)^{-1} \cdot v = \varphi_i(p_0) \cdot v = \phi_{ix}(\phi_{kx}^{-1}(e)) \cdot v = g_{ik}(x) \cdot v$$

(cf. Theorem 2.2.8). This means that the cocycle ψ_{ik} of the fiber bundle E is given by the left action $l_{g_{ik}(x)} \in \mathrm{Diff}(F)$ of the cocycle g_{ik} of P. It motivates the following general construction principle for fiber bundles:

2.3.2 Theorem. *Let M, F be manifolds and $[F, G]$ a left transformation group. Let $\{U_i\}_{i \in I}$ be an open covering of M and let $g_{ik} : U_i \cap U_k \to G$ $(i, k \in I)$ be a cocycle. Then there exists a unique (up to isomorphism) fiber bundle $(E, \hat{\pi}, M, F)$ whose cocycle is given by the left action $l_{g_{ik}(x)} \in \mathrm{Diff}(F)$. This fiber bundle is associated to the unique G-principal bundle whose cocycle is given by the left translations $L_{g_{ik}(x)} \in \mathrm{Diff}(G)$.*

Proof. We construct the fiber bundle similarly to the proof of Theorem 2.2.7. Let

$$\hat{E} := \dot{\bigcup_{i \in I}} U_i \times F,$$

and call elements $(x_i, v_i) \in U_i \times F$ and $(x_k, v_k) \in U_k \times F$ equivalent if $x_i = x_k = x \in U_i \cap U_k$ and $v_k = g_{ki}(x) \cdot v_i$. By the cocycle condition (2.2.3) this indeed gives an equivalence relation on $\hat{E}$. Denote the equivalence class of (x_i, v_i) by $[x_i, v_i]$ and set

$$E := \hat{E}/\sim \qquad \pi([x,v]) := x.$$

To obtain a bundle atlas, define

$$\begin{aligned} \psi_i : E_{U_i} &\to U_i \times F \\ [x_i, v] &\mapsto (x_i, v). \end{aligned} \tag{2.3.6}$$

This is well-defined, for if $[x, v] = [y, w]$, $x, y \in U_i$, then $x = y$ and $v = g_{ii}(x)w = w$. Also it is clearly bijective. Moreover,

$$\psi_{ix}(\psi_{kx}^{-1}(v)) = \psi_{ix}([x,v]) = \psi_{ix}([x, g_{ik}(x) \cdot v]) = g_{ik}(x) \cdot v = l_{g_{ik}(x)}(v),$$

so

$$\begin{aligned} \psi_i \circ \psi_k^{-1} : (U_i \cap U_k) \times F &\to (U_i \cap U_k) \times F \\ (x, v) &\mapsto (x, g_{ik}(x) \cdot v) \end{aligned}$$

is smooth. According to Theorem 2.1.5 we therefore obtain a manifold structure on E such that (E, π, M, F) becomes a fiber bundle. Uniqueness up to fiber bundle isomorphism follows exactly as in the proof of Theorem 2.2.7.

Now carrying out the same construction with $[G, G]$, by Theorem 2.2.8 we obtain a (unique, up to isomorphism) principal fiber bundle P with the given G-cocycle. Then the map

$$\begin{aligned} A : P \times_G F &\to E \\ [[x, g], v] &\mapsto [x, g \cdot v] \end{aligned}$$

is well-defined: if $[[x,g_1],v_1] = [[y,g_2],v_2]$ in $P \times_G F$, $x \in U_i, y \in U_j$ then there exists some $h \in G$ such that $[x,g_1] \cdot h = [y,g_2]$ and $h^{-1}v_1 = v_2$. Here, denoting the analogues of the ψ_i from (2.3.6) by ϕ_i and $p := [x,g_1]$ we have by (2.2.6):

$$[x,g_1] \cdot h = \phi_{ix}^{-1}(\phi_{ix}(p) \cdot h) = \phi_{ix}^{-1}(g_1 \cdot h) = [x, g_1 \cdot h].$$

Hence $[x, g_1 \cdot h] = [y, g_2]$, so $x = y$ and $g_1 \cdot h = g_{ij}(x) \cdot g_2$. It follows that

$$g_1 \cdot v_1 = g_1 \cdot h \cdot v_2 = g_{ij}(x) \cdot g_2 \cdot v_2,$$

giving $[x, g_1 \cdot v_1] = [y, g_2 \cdot v_2]$.

A is surjective as $[x_i, v] = A([[x_i, e], v])$. It is also injective: Let $A([[x_i,g_1],v_1]) = A([[x_j,g_2],v_2])$, i.e., $[x_i, g_1 \cdot v_1] = [x_j, g_2 \cdot v_2]$. Then $x_i = x_j =: x$ and $g_1 \cdot v_1 = g_{ij}(x) \cdot g_2 \cdot v_2$. Thus $v_1 = h \cdot v_2$ with $h := g_1^{-1} \cdot g_{ij}(x) \cdot g_2$, and $[x, g_1 \cdot h] = [x, g_2]$. Altogether,

$$[[x_i,g_1],v_1] = [[x_i,g_1] \cdot h, h^{-1} \cdot v_1] = [[x_j,g_2],v_2],$$

as desired.

To show smoothness of A, denote by χ_i the $P \times_G F$-bundle chart corresponding to ϕ_i as in (2.3.2). Then for any $[[x_i,g],v] \in P \times_G F$ we have $\chi_i([[x_i,g],v]) = (x_i, \varphi_i([x_i,g]) \cdot v) = (x_i, g \cdot v)$. But also $\psi_i \circ A([[x_i,g],v]) = \psi_i([x_i, g \cdot v]) = (x_i, g \cdot v)$, so $\psi_i \circ A \circ \chi_i^{-1} = \mathrm{id}$. This implies that A is a diffeomorphism.

Finally,

$$\pi \circ A([[x_i,g],v]) = \pi([x_i, g \cdot v]) = x_i = \pi_P([x_i,g]) = \hat{\pi}([[x_i,g],v]).$$

concluding the proof that A is a fiber bundle isomorphism. □

2.3.3 Definition. *Let (P,π,M,G) be a principal fiber bundle, F a manifold, and $[F,G]$ a left transformation group. Let $E = P \times_G F$ be the associated fiber bundle. For any $p \in P_x$ the map*

$$\begin{aligned} [p] : F &\to P_x \times_G F = E_x \\ v &\mapsto [p,v] \end{aligned} \tag{2.3.7}$$

is called the fiber diffeomorphism *defined by p.*

Using (2.3.2), note first that $[p]$ is smooth because

$$\psi_U \circ [p](v) = \psi_U([p,v]) = (\pi(p), \varphi_U(p) \cdot v)$$

(note that $E_x = \hat{\pi}^{-1}(x)$ is a regular submanifold of E). It is bijective with smooth inverse

$$[p,v] \xrightarrow{\psi_U} (x, \varphi_U(p) \cdot v) \xrightarrow{\mathrm{pr}_2} \varphi_U(p) \cdot v \xrightarrow{\cdot \varphi_U(p)^{-1}} v,$$

hence is indeed a diffeomorphism. Also

$$[p \cdot g](v) = [p \cdot g, v] = [p, g \cdot v] = [p](g \cdot v) = [p] \circ l_g(v). \tag{2.3.8}$$

To conclude this section, we consider smooth sections in associated fiber bundles. To this end we introduce the space of G-equivariant maps from P to F:

$$\mathcal{C}^\infty(P,F)^G := \{\bar{s} \in \mathcal{C}^\infty(P,F) \mid \bar{s}(p \cdot g) = g^{-1} \cdot \bar{s}(p) \ \forall p \in P, \ g \in G\}. \tag{2.3.9}$$

Then we have:

2.3.4 Theorem. *Let (P, π, M, G) be a principal fiber bundle, F a manifold, $[F, G]$ a left transformation group, and $E = P \times_G F$ the associated fiber bundle. Then the space of smooth sections of E can be identified with $\mathcal{C}^\infty(P, F)^G$:*

$$\Gamma(E) \cong \mathcal{C}^\infty(P, F)^G.$$

Proof. Given $\bar{s} \in \mathcal{C}^\infty(P, F)^G$, define $s : M \to E$, $s(x) := [p, \bar{s}(p)] \in E_x$, where $p \in P_x$ is arbitrary. This is well-defined, for if also $q \in P_x$ then $q = p \cdot g$ for some $g \in G$, so that

$$[q, \bar{s}(q)] = [pg, \bar{s}(pg)] = [pg, g^{-1}\bar{s}(p)] = [p, \bar{s}(p)].$$

Also, s is a section because $\hat{\pi}(s(x)) = \hat{\pi}([p, \bar{s}(p)]) = \pi(p) = x$. To see that s is smooth, let $f : U \to P$ be a smooth local section of P around $x_0 \in M$. Then locally we have $s(x) = [f(x), \bar{s}(f(x))]$, so $\psi_U \circ s(x) = (x, \varphi_U(f(x)) \cdot \bar{s}(f(x)))$.

Conversely, let $s \in \Gamma(E)$ and define $\bar{s} : P \to F$ by $\bar{s}(p) := [p]^{-1} \circ s(\pi(p))$. Let $\psi_U \circ s(x) = (x, \tilde{s}(x)) \in U \times F$ be the local representation of s. Then with $x = \pi(p)$,

$$\begin{aligned} \bar{s}(p) &= [p]^{-1} \circ \psi_U^{-1} \circ \psi_U \circ s(\pi(p)) = [p]^{-1} \circ \psi_U^{-1}(x, \tilde{s}(x)) \\ &\underset{(2.3.4)}{=} [p]^{-1}([p, \varphi_U(p)^{-1} \cdot \tilde{s}(x)]) = \varphi_U(p)^{-1} \cdot \tilde{s}(\pi(p)), \end{aligned}$$

showing smoothness. Also, $\bar{s} \in \mathcal{C}^\infty(P, F)^G$ since by (2.3.8),

$$\bar{s}(pg) = [pg]^{-1} \circ s(\pi(p)) = l_{g^{-1}} \circ [p]^{-1} \circ s(x) = g^{-1}\bar{s}(p).$$

Finally, we note that the maps $A : \bar{s} \mapsto s$ and $B : s \mapsto \bar{s}$ are inverses of each other:

$$\begin{aligned} &B(A(\bar{s}))(p) = [p]^{-1}(A(\bar{s})(\pi(p))) = [p]^{-1}([p, \bar{s}(p)]) = \bar{s}(p), \\ &A(B(s))(x) = [p, B(s)(p)] = [p, [p]^{-1}(s(x))] = [p]([p]^{-1}(s(x))) = s(x). \end{aligned}$$

□

2.4 Vector bundles

Fiber bundles whose typical fiber is a vector space play a prominent role in differential geometry:

2.4.1 Definition. *A fiber bundle (E, π, M, V) is called a* $\mathbb{K}$-vector bundle *of rank $m < \infty$ if*

(i) The typical fiber V is an m-dimensional vector space over $\mathbb{K}$.

(ii) Every fiber E_x is a $\mathbb{K}$-vector space.

(iii) There exists a bundle atlas $\{(U_i, \phi_i)\}_{i \in I}$ such that the fiber diffeomorphisms

$$\phi_{ix} : E_x \to V$$

are linear isomorphisms.

If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ then E is called real or complex vector bundle, respectively. If $m = 1$ then E is called a line bundle.

2.4.2 Definition. *Let E and $\tilde{E}$ be vector bundles over the same base manifold M. A map $L : E \to \tilde{E}$ is called a* vector bundle homomorphism *if it is smooth and fiber preserving and if $L|_{E_x} : E_x \to \tilde{E}_x$ is linear for each $x \in M$. If both L and L^{-1} are vector bundle homomorphisms, then L is called a* vector bundle isomorphism.

For E a vector bundle, $\Gamma(E)$ is a module over $\mathcal{C}^\infty(M)$, with multiplication defined pointwise: $(fs)(x) := f(x)s(x)$. Any vector bundle homomorphism $L : E \to \tilde{E}$ induces a linear map on the corresponding spaces of sections, denoted by the same letter:

$$\begin{aligned} L : \Gamma(E) &\to \Gamma(\tilde{E}) \\ s &\mapsto Ls, \quad (Ls)(x) := L(s(x)). \end{aligned}$$

Well known notions from (multi-)linear algebra can readily be extended to the vector bundle setting:

2.4.3 Remark. Constructing new vector bundles from given ones:

(i) **Whitney sum**
Let $E, \tilde{E}$ be vector bundles over M with typical fibers V, $\tilde{V}$. Then let

$$E \oplus \tilde{E} := \dot{\bigcup_{x \in M}} E_x \oplus \tilde{E}_x$$

with projection $\pi_\oplus : E_x \oplus \tilde{E}_x \ni (e_x, \tilde{e}_x) \mapsto x \in M$.

To turn $E \oplus \tilde{E}$ into a vector bundle we employ Theorem 2.1.5: let $(U, \phi_U = (\pi, \varphi_U))$, $(U, \tilde{\phi}_U = (\tilde{\pi}, \tilde{\varphi}_U))$ be bundle charts in E and $\tilde{E}$ over the same open set $U \subseteq M$. Then

$$\begin{aligned} \phi_U^\oplus : (E \oplus \tilde{E})_U &\to U \times (V \oplus \tilde{V}) \\ (e, \tilde{e}) &\mapsto (\pi(e), \varphi_U(e) \oplus \tilde{\varphi}_U(\tilde{e})) \end{aligned}$$

a formal bundle chart, with smooth transition functions (namely the direct sums of the individual transition functions of E and $\tilde{E}$). The vector bundle $(E \oplus \tilde{E}, \pi_\oplus, M, V \oplus \tilde{V})$ is called the *Whitney sum* of E and $\tilde{E}$.

(ii) **Tensor product**
With $E, \tilde{E}$ as in (i), set

$$E \otimes \tilde{E} := \dot{\bigcup_{x \in M}} E_x \otimes \tilde{E}_x$$

with projection $\pi_\otimes : E_x \otimes \tilde{E}_x \ni (e_x \otimes \tilde{e}_x) \mapsto x \in M$.

In this case, the formal bundle charts are given by

$$\begin{aligned} \phi_U^\otimes : (E \otimes \tilde{E})_U &\to U \times (V \otimes \tilde{V}) \\ (e \otimes \tilde{e}) &\mapsto (\pi(e), \varphi_U(e) \otimes \tilde{\varphi}_U(\tilde{e})) \end{aligned}$$

The resulting vector bundle $E \otimes \tilde{E}$ is called the *tensor product* of E and $\tilde{E}$.

(iii) **Dual vector bundle**
Given a vector bundle (E, π, M, V), and denoting by V^*, E_x^* the dual spaces, set

$$E^* := \dot{\bigcup_{x \in M}} E_x^*,$$

with projection $\pi^* : E_x^* \ni L_x \mapsto x$. The formal bundle charts then are

$$\begin{aligned} \phi_U^* : E_U^* &\to U \times V^* \\ E_x^* \ni L &\mapsto (\pi^*(L), \varphi_U^*(L)), \quad \text{where} \quad \varphi_U^*(L)(v) := L(\phi_{Ux}^{-1}(v)). \end{aligned}$$

(E^*, π^*, M, V^*) is called the dual bundle of E.

(iv) **Conjugate vector bundle**

Let E be a complex vector bundle and denote by $\bar{V}$ the conjugate vector space of V, where scalar multiplication is defined by $(\lambda, v) \mapsto \bar{\lambda} \cdot v$. Given a bundle chart (U, ϕ_U) of E, let $\bar{\phi}_{Ux} : \bar{E}_x \to \bar{V}$ be the linear isomorphism induced by ϕ_{Ux}. Then set

$$\bar{E} := \dot{\bigcup_{x \in M}} \bar{E}_x,$$

and $\bar{\pi} : \bar{E}_x \ni \bar{e}_x \mapsto x \in M$. The formal bundle charts are given by

$$\begin{aligned} \bar{\phi}_U : \bar{E}_U &\to U \times \bar{V} \\ \bar{E}_x \ni \bar{e} &\mapsto (x, \bar{\phi}_{Ux}(\bar{e})). \end{aligned}$$

The resulting vector bundle $(\bar{E}, \bar{\pi}, M, \bar{V})$ is called *conjugate* to E.

(v) **Homomorphism bundle**

Given two vector bundles $E, \tilde{E}$ as before, denote by $\mathrm{Hom}(V, \tilde{V})$ the space of linear maps from V to $\tilde{V}$ and set

$$\mathrm{Hom}(E, \tilde{E}) := \dot{\bigcup_{x \in M}} \mathrm{Hom}(E_x, \tilde{E}_x)$$

with projection $\hat{\pi} : \mathrm{Hom}(E_x, \tilde{E}_x) \ni L_x \mapsto x$. The formal bundle charts now are

$$\begin{aligned} \hat{\phi}_U : \mathrm{Hom}(E, \tilde{E})|_U &\to U \times \mathrm{Hom}(V, \tilde{V}) \\ L_x &\mapsto (x, T), \quad \text{where } T(v) := (\tilde{\phi}_{Ux} \circ L_x \circ \phi_{Ux}^{-1})(v). \end{aligned}$$

The resulting vector bundle $(\mathrm{Hom}(E, \tilde{E}), \hat{\pi}, M, \mathrm{Hom}(V, \tilde{V}))$ is called the *homomorphism bundle* from E to $\tilde{E}$. Any vector bundle homomorphism $L : E \to \tilde{E}$ corresponds to a smooth section $s_L := x \mapsto L_x = L|_{E_x} \in \mathrm{Hom}(E_x, \tilde{E}_x)$ and, vice versa, $s \in \Gamma(\mathrm{Hom}(E, \tilde{E}))$ corresponds to the vector bundle homomorphism $L_s := E \ni v \mapsto s_{\pi(v)}(v)$.

2.4.4 Remark. Here we show that any vector bundle is associated to a principal fiber bundle with linear structure group (i.e., whose structure group is a Lie subgroup of some $\mathrm{GL}(n, \mathbb{K})$). Let (E, π, M, V) be a vector bundle. Since the transition functions $\psi_{ik}(x) := \psi_{ix} \circ \psi_{kx}^{-1} : V \to V$ are linear, they define a cocycle

$$g_{ik} := \psi_{ik} : U_i \cap U_k \to \mathrm{GL}(V).$$

Also, $[V, \mathrm{GL}(V)]$ is a left Lie transformation group, with the action given by matrix multiplication. Hence Theorem 2.3.2 applies and shows that E is associated to the $\mathrm{GL}(V)$-principal fiber bundle over M defined by the cocycle $\{g_{ik}\}$.

2.4.5 Remark. Let (P, π, M, G) be a G-principal fiber bundle and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G on the vector space V. Then ρ induces a left action on V,

$$G \times V \ni (g, v) \mapsto g \cdot v := \rho(g) v \in V.$$

According to Section 2.3 we therefore obtain a fiber bundle $E := P \times_{(G,\rho)} V$, and in fact E is a vector bundle with linear structure on the fibers $E_x = P_x \times_{(G,\rho)} V$ given by

$$[p, v] + \lambda [p, w] := [p, v + \lambda w] \qquad (p \in P_x,\ v, w \in V,\ \lambda \in \mathbb{K}).$$

Thus declaring $[p]$ from Definition 2.3.3 to be a linear isomorphism transfers the vector space structure from V to E_x. To verify (iii) from Definition 2.4.1 we use (2.3.2):

$$\begin{aligned}\psi_{Ux}([p,v]+\lambda[p,w]) &= \psi_{Ux}([p,v+\lambda w]) = (x,\varphi_U(p)\cdot(v+\lambda w))\\ &= (x,\rho(\varphi_U(p))(v+\lambda w)) = (x,\varphi_U(p)\cdot v)+(x,\lambda\varphi_U(p)\cdot w)\\ &= \psi_{Ux}([p,v])+\lambda\psi_{Ux}([p,w]).\end{aligned}$$

All the functorial operations considered in Remark 2.4.3 can be applied to given representations (cf. [9, Def. 23.3]). E.g., if $E = P\times_{(G,\rho)} V$ and $\tilde E = P\times_{(G,\tilde\rho)}\tilde V$, then

$$E\otimes\tilde E = P\times_{(G,\rho\otimes\tilde\rho)}(V\otimes\tilde V).$$

2.4.6 Example. Tensor bundles on a manifold

Let M be an m-dimensional manifold, then there are natural isomorphisms between the standard tensor bundles over M and vector bundles associated to suitable representations of $\mathrm{GL}(n,\mathbb{R})$. Denote by $\rho:\mathrm{GL}(n,\mathbb{R})\to\mathrm{GL}(\mathbb{R}^n)$ the representation given by matrix multiplication ($\rho: A\mapsto(v\mapsto A\cdot v)$). Then according to [9, Def. 23.3], ρ gives rise to the dual representation $\rho^*:\mathrm{GL}(n,\mathbb{R})\to\mathrm{GL}(\mathbb{R}^{n*})$, $\rho_k:\mathrm{GL}(n,\mathbb{R})\to\mathrm{GL}(\Lambda^k(\mathbb{R}^{n*}))$, and $\rho_{(r,s)}:\mathrm{GL}(n,\mathbb{R})\to\mathrm{GL}(T^r_s(\mathbb{R}^n))$. Then

$$\begin{aligned}TM &\cong \mathrm{GL}(M)\times_{(\mathrm{GL}(n,\mathbb{R}),\rho)}\mathbb{R}^n\\ T^*M &\cong \mathrm{GL}(M)\times_{(\mathrm{GL}(n,\mathbb{R}),\rho^*)}\mathbb{R}^{n*}\\ \Lambda^kT^*M &\cong \mathrm{GL}(M)\times_{(\mathrm{GL}(n,\mathbb{R}),\rho_k)}\Lambda^k\mathbb{R}^{n*}\\ T^r_sM &\cong \mathrm{GL}(M)\times_{(\mathrm{GL}(n,\mathbb{R}),\rho_{(r,s)})}T^r_s\mathbb{R}^n\end{aligned}$$

To check this explicitly in the case of the tangent bundle TM, consider the map

$$\begin{aligned}\Phi:\mathrm{GL}(M)\times_\rho\mathbb{R}^n&\to TM\\ [(s_1,\dots,s_n),(x_1,\dots,x_n)^t]&\mapsto\sum_{i=1}^n x_is_i.\end{aligned}$$

Then Φ is well-defined: let $[(s_1,\dots,s_n),(x_1,\dots,x_n)^t] = [(\tilde s_1,\dots,\tilde s_n),(\tilde x_1,\dots,\tilde x_n)^t]$, so for some $A\in\mathrm{GL}(n,\mathbb{R})$ we have $(s_1,\dots,s_n)=(\tilde s_1,\dots,\tilde s_n)\cdot A$ and $(x_1,\dots,x_n)^t = A^{-1}\cdot(\tilde x_1,\dots,\tilde x_n)^t$. Thus

$$\sum_{i=1}^n x_is_i \underset{(2.2.7)}{=} \sum_{i,j,k}(A^{-1})_{ij}\tilde x_jA_{ki}\tilde s_k = \sum_{j,k}\delta_{kj}\tilde x_j\tilde s_k = \sum_{k=1}^n\tilde x_k\tilde s_k.$$

Φ is surjective: Let $v_p\in T_pM$, y a local chart around p, then $v_p = \sum v_i\partial_{y^i}|_p = \Phi([(\partial_{y^1}|_p,\dots,\partial_{y^n}|_p),(v_1,\dots,v_n)^t])$.
Φ is injective: Let

$$[(s_1,\dots,s_n),(x_1,\dots,x_n)^t],[(\tilde s_1,\dots,\tilde s_n),(\tilde x_1,\dots,\tilde x_n)^t]\in\mathrm{GL}(M)\times_\rho\mathbb{R}^n$$

with $\sum_{i=1}^n x_is_i = \sum_{i=1}^n\tilde x_i\tilde s_i$. Since both $s:=(s_1,\dots,s_n)$ and $\tilde s:=(\tilde s_1,\dots,\tilde s_n)$ are bases of $\mathbb{R}^n$ there exists a unique $A\in\mathrm{GL}(n,\mathbb{R})$ such that $s=\tilde s\cdot A$. Then

$$\sum x_is_i = s\cdot x = \tilde s\cdot A\cdot x = \sum\tilde x_i\tilde s_i = \tilde s\tilde x,$$

so $A\cdot x=\tilde x$, or $x = A^{-1}\tilde x$, implying $[s,x]=[\tilde s,\tilde x]$.
To show smoothness of Φ, take $\psi_U:(\mathrm{GL}(M)\times_\rho\mathbb{R}^n)_U\to U\times\mathbb{R}^n$ as in (2.3.2), i.e.,

$$\psi_U = [(s_1,\dots,s_n),(v_1,\dots,v_n)^t]\mapsto(\pi((s_1,\dots,s_n)),\varphi_U(s_1,\dots,s_n)\cdot(v_1,\dots,v_n)^t).$$

Here, $\phi_U(s_1,\dots,s_n) = (\pi(s_1,\dots,s_n), \varphi_U(s_1,\dots,s_n))$ is a bundle chart of $\mathrm{GL}(M)$. According to Example 2.2.12, this map in turn arises from a chart $(U, \varphi = (x^1,\dots,x^n))$ of M as $\phi_U(s_1,\dots,s_n) = (x, A(x))$, where

$$(s_1,\dots,s_n) = (\partial_{x^1}|_x,\dots,\partial_{x^n}|_x)\cdot A(x).$$

Therefore, $\psi_U([(s_1,\dots,s_n),(v_1,\dots,v_n)^t]) = (x, A(x)\cdot(v_1,\dots,v_n)^t)$. In particular, for $s_i = \partial_{x^i}|_x$ we have $A(x) = I_n$, giving

$$\psi_U^{-1} : U \times \mathbb{R}^n \ni (x,w) \mapsto [(\partial_{x^1}|_x,\dots,\partial_{x^n}|_x),(w_1,\dots,w_n)^t],$$

so that

$$\begin{aligned} T\varphi\circ\Phi\circ\psi_U^{-1}(x,w) &= T\varphi\circ\Phi([(\partial_{x^1}|_x,\dots,\partial_{x^n}|_x),(w_1,\dots,w_n)^t]) \\ &= T\varphi\Big(\sum w_i \partial_{x^i}|_x\Big) = (\varphi(x), w). \end{aligned}$$

This shows, on the one hand, that Φ is a local, hence (since it is bijective) a global diffeomorphism, and on the other that $T\varphi\circ\Phi\circ\psi_U^{-1}$ is a local vector bundle isomorphism. Altogether, Φ thereby is a global vector bundle isomorphism.

A similar calculation with the map

$$\begin{aligned} \Phi^* : \mathrm{GL}(M)\times_{\rho^*}\mathbb{R}^{n*} &\to T^*M \\ [(s_1,\dots,s_n),(v_1,\dots,v_n)^t] &\mapsto \sum_{i=1}^n v_i s_i^* \end{aligned}$$

(with $\{s_i^*\}$ the basis dual to $\{s_i\}$) implies the result for T^*M, and analogously for the remaining cases.

In the remainder of this section we want to show that any real (resp. complex) vector bundle of rank m is associated to an $\mathrm{O}(m)$ (resp. $\mathrm{U}(m)$) principal fiber bundle. To do this, we need the following notion:

2.4.7 Definition. *A* bundle metric *on a real or complex vector bundle E over M is a section $\langle\,.\,,\,.\,\rangle \in \Gamma(E^*\otimes\bar{E}^*)$ that assigns a nondegenerate bilinear form (for $\mathbb{K}=\mathbb{R}$) resp. nondegenerate hermitian form (for $\mathbb{K}=\mathbb{C}$)*

$$\langle\,.\,,\,.\,\rangle_{E_x} := \langle\,.\,,\,.\,\rangle(x) : E_x\times E_x \to \mathbb{K}$$

to any $x\in M$.

Semi-Riemannian metrics on TM are examples of bundle metrics. In general we have:

2.4.8 Theorem. *Any real or complex vector bundle possesses a positive definite bundle metric.*

Proof. Let (E,π,M,V) be a vector bundle over M with bundle atlas $\{(U_i,\phi_i)\}_{i\in I}$ and pick a partition of unity $\{\chi_i\}_{i\in I}$ subordinate to the covering $\{U_i\}_{i\in I}$. Let $(v_1,\dots,v_n)$ be a basis in V and define, for $\alpha = 1,\dots,n$,

$$\begin{aligned} s_{i\alpha} : U_i &\to E \\ x &\mapsto s_{i\alpha}(x) := \phi_i^{-1}(x, v_\alpha). \end{aligned}$$

This gives a local frame for $E|_{U_i}$, which allows us to define a bundle metric $\langle\,.\,,\,.\,\rangle$ on $E|_{U_i}$ by

$$\langle s_{i\alpha}(x), s_{i\beta}(x)\rangle_{ix} := \delta_{\alpha\beta} \qquad (x\in U_i).$$

Then

$$\langle\,.\,,\,.\,\rangle(x) := \sum_{i\in I}\chi_i(x)\langle\,.\,,\,.\,\rangle_{ix}$$

gives the desired positive definite bundle metric. □

Using this we can now prove:

2.4.9 Theorem. *Any real (resp. complex) vector bundle of rank m is associated to an* $\mathrm{O}(m)$ *(resp.* $\mathrm{U}(m)$*) principal fiber bundle.*

Proof. Let (E,π,M,V) be a vector bundle over M and fix a positive definite bundle metric $\langle\,.\,,\,.\,\rangle$ on E according to Theorem 2.4.8. Now consider the set of bases

$$P_x := \{s_x = (s_1,\dots,s_m) \mid s_x \text{ is a basis in } E_x \text{ with } \langle s_\alpha, s_\beta\rangle_x = \delta_{\alpha\beta}\}.$$

Then

$$\begin{aligned} P := \dot{\bigcup_{x\in M}} P_x &\xrightarrow{\hat\pi} M \\ s_x &\mapsto x \end{aligned}$$

defines an $\mathrm{O}(m)$ (resp. $\mathrm{U}(m)$) principal fiber bundle over M: To obtain formal bundle charts, let U_i be trivializing and use the Gram-Schmidt procedure to construct an orthonormal basis $(e^i_\alpha \mid \alpha = 1,\dots,m)$ in $(\Gamma(E|_U),\langle\,.\,,\,.\,\rangle)$. Then set $\phi_i : P|_{U_i} \to U_i \times \mathbb{R}^{m^2}$, $s_x = (s_1,\dots,s_m) \mapsto (x, A(x))$, where $A(x) \in \mathrm{O}(m)$ (resp. $\mathrm{U}(m)$) is the uniquely determined matrix with

$$s_x = (e^i_1|_x,\dots,e^i_m|_x)\cdot A(x).$$

It then follows exactly as in Example 2.2.12 that $(P,\hat\pi,M,\mathrm{O}(m))$ (respectively, $(P,\hat\pi,M,\mathrm{U}(m))$) becomes a principal fiber bundle. Moreover, in complete analogy to Example 2.4.6 it follows that

$$P\times_{\mathrm{O}(m)}\mathbb{R}^m \cong E \qquad \text{resp.} \quad P\times_{\mathrm{U}(m)}\mathbb{R}^m \cong E$$

via the vector bundle isomorphism $[(s_1,\dots,s_m),(x_1,\dots,x_m)] \mapsto \sum_{\alpha=1}^m x_\alpha s_\alpha$. □

For vector bundles that are associated to principal fiber bundles there is a canonical way of obtaining bundle metrics.

2.4.10 Theorem. *Let* (P,π,M,G) *be a principal fiber bundle,* $\rho : G \to \mathrm{GL}(V)$ *a representation on a finite dimensional vector space, and* $\langle\,.\,,\,.\,\rangle_V$ *a* ρ*-invariant scalar product (for* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*) on* V*. Then on the associated vector bundle* $E = P\times_{(G,\rho)}V$ *a bundle metric is given by*

$$\langle e,\hat e\rangle_{E_x} := \langle v,\hat v\rangle_V \qquad (e,\hat e\in E_x), \tag{2.4.1}$$

where $e = [p,v]$ *and* $\hat e = [p,\hat v]$ *for some* $p\in P_x$*. The scalar products* $\langle\,.\,,\,.\,\rangle_V$ *and* $\langle\,.\,,\,.\,\rangle_{E_x}$ *have the same signature.*

Proof. To see that $\langle\,.\,,\,.\,\rangle_{E_x}$ is well-defined, let $q\in P_x$ and let $g\in G$ be the unique element such that $q = p\cdot g$. Then

$$e = [p,v] = [p\cdot g, \rho(g^{-1})v] = [q,\rho(g^{-1})v],\ \hat e = [p,\hat v] = [q,\rho(g^{-1})\hat v],$$

and $\langle\rho(g^{-1})v,\rho(g^{-1})\hat v\rangle_V = \langle v,\hat v\rangle_V$ by assumption. Hence (2.4.1) is independent of the chosen $p\in P_x$. To see that the resulting bundle metric is smooth, we have to show that for any $s_1, s_2 \in \Gamma(E)$ also $x \mapsto \langle s_1(x), s_2(x)\rangle$ is smooth. To see this, pick any smooth section $\gamma \in \Gamma(P|_U)$, U a trivializing neighborhood. Then on U we have

$$s_i(x) = \psi_U^{-1}(x, v_i(x)) \underset{(2.3.4)}{=} [\gamma(x), \varphi_U(\gamma(x))^{-1} \cdot v_i(x)],$$

where $v_i \in \mathcal{C}^\infty(U, V)$ $(i = 1, 2)$. Therefore,

$$\langle s_1(x), s_2(x)\rangle = \langle \varphi_U(\gamma(x))^{-1} \cdot v_1(x), \varphi_U(\gamma(x))^{-1} \cdot v_2(x)\rangle = \langle v_1(x), v_2(x)\rangle,$$

which is smooth. □

2.5 Reduction and extension of principal fiber bundles

Let (P, π_P, M, G) and (Q, π_Q, M, H) be principal fiber bundles and suppose that (Q, f) is a λ-reduction of P, i.e. (cf. Definition 2.2.3) that $f : Q \to P$ is smooth, $\lambda : H \to G$ is a Lie group homomorphism, and

(i) $\pi_P \circ f = \pi_Q$.

(ii) $f(q \cdot h) = f(q) \cdot \lambda(h)$.

Thus we have the following situation:

$$\begin{tikzcd} Q \times H \arrow[r, "\cdot"] \arrow[d, "f\times\lambda"'] & Q \arrow[d, "f"] \arrow[dr, "\pi_Q"] & \\ P \times G \arrow[r, "\cdot"] & P \arrow[r, "\pi_P"] & M \end{tikzcd}$$

2.5.1 Definition. *Two λ-reductions (Q, f) and $(\tilde{Q}, \tilde{f})$ of the principal fiber bundle P are called* isomorphic *if there exists an H-fiber bundle isomorphism $\Phi : Q \to \tilde{Q}$ such that $\tilde{f} \circ \Phi = f$. By $\mathrm{Red}_\lambda(P)$ we denote the set of all isomorphism classes of λ-reductions of P.*

Let M be an n-dimensional manifold with frame bundle $\mathrm{GL}(M)$. Any additional geometric structure on M induces a reduction of the frame bundle to a subgroup of $\mathrm{GL}(n, \mathbb{R})$. For example, if g is a semi-Riemannian metric on M of signature (k, l), then the bundle $\mathrm{O}(M, g)$ of orthonormal bases is a reduction on $\mathrm{GL}(M)$ to the group $\mathrm{O}(k, l)$. In this case, $\lambda = \mathrm{O}(k, l) \hookrightarrow \mathrm{GL}(n, \mathbb{R})$, and $f = \mathrm{O}(M, g) \hookrightarrow \mathrm{GL}(M)$.

Conversely, any $\mathrm{O}(k, l)$-reduction $(Q, \pi_Q, M, \mathrm{O}(k, l))$ of $\mathrm{GL}(M)$ induces a semi-Riemannian metric g of signature (k, l) on M as follows: Let $x \in M$ and pick any $q \in Q_x$. Then $f(q) = (v_1, \ldots, v_n)$ is a basis of T_xM and we define g_x by setting $g_x(v_i, v_j) := \varepsilon_i \delta_{ij}$, with $\varepsilon_i = -1$ for $1 \le i \le k$ and $\varepsilon_i = +1$ for $k + 1 \le i \le n$. This definition is independent of the chosen q: Let $\tilde{q} \in Q_x$, $\tilde{q} = q \cdot A$, $A \in \mathrm{O}(k, l)$. Then $(\tilde{v}_1, \ldots, \tilde{v}_n) := f(\tilde{q}) = f(q) \cdot A$, so

$$g_x(\tilde{v}_i, \tilde{v}_j) = g_x\Big(\sum_k v_k A_{ki}, \sum_l v_l A_{lj}\Big) = \sum_{k,l} \varepsilon_k \delta_{kl} A_{ki} A_{lj} = \sum_k \varepsilon_k (A^t)_{ik} A_{kj} = \varepsilon_i \delta_{ij}.$$

Moreover, $x \mapsto g_x$, $M \to \Gamma(T^*M \otimes T^*M)$ is smooth: Let s be a local section of Q (cf. Theorem 2.2.5). Then $x \mapsto f \circ s(x) = (v_{1x}, \ldots, v_{nx})$ is a smooth local frame for TM. Since $g(v_{ix}, v_{jx}) = \varepsilon_i \delta_{ij}$ is smooth, it follows that indeed $g(X, Y)$ is smooth for any $X, Y \in \mathfrak{X}_{\mathrm{loc}}(M)$, giving the claim. Thus g is indeed a semi-Riemannian metric of signature (k, l) on M.

2.5.2 Theorem. *Let (P, π_P, M, G) be a principal fiber bundle and let $\lambda : H \to G$ be a Lie group homomorphism. The following are equivalent:*

(i) There exists a λ-reduction of P.

(ii) There exists a G-cocycle $\{g_{ik}\}_{i,k \in I}$ for P that is induced by an H-cocycle $h_{ik} : U_i \cap U_k \to H$ via

$$g_{ik}(x) = \lambda(h_{ik}(x)) \qquad \forall x \in U_i \cap U_k.$$

Proof. (i)⇒(ii): Let (Q, π_Q, M, H), $f : Q \to P$ be a λ-reduction of P and consider a bundle atlas $\{(U_i, \psi_i)\}_{i \in I}$ of Q, inducing the H-cocycle $h_{ik} : U_i \cap U_k \ni x \mapsto \psi_{ix}(\psi_{kx}^{-1}(e)) \in H$ (cf. Theorem 2.2.8). To construct a bundle chart for P over U_i, let $x \in U_i$ and $p \in P_x$. Let $q \in Q_x$, then $\pi_P \circ f(q) = \pi_Q(q) = x$, so $f(q) \in P_x$ and there is a unique $g \in G$ with $p = f(q) \cdot g$. Now let

$$\begin{aligned} \phi_i : P_{U_i} &\mapsto U_i \times G \\ p &\mapsto (\pi_P(p), \lambda(\psi_{ix}(q)) \cdot g). \end{aligned}$$

This map is well-defined: Let $p = f(q')g' = f(q)g$. There exists a unique $h \in H$ with $q' = q \cdot h$, so $\psi_{ix}(q') = \psi_{ix}(q) \cdot h$ and

$$\lambda(\psi_{ix}(q')) \cdot g' = \lambda(\psi_{ix}(q)) \cdot \lambda(h) \cdot g'.$$

Since $f(q) \cdot \lambda(h) = f(q \cdot h) = f(q') = f(q) \cdot g \cdot (g')^{-1}$, we have $\lambda(h) = g \cdot (g')^{-1}$, proving the claim. Note also that (setting $p := f(q) \cdot e$)

$$\phi_i \circ f|_{Q_{U_i}} = (\mathrm{id}_{U_i} \times \lambda) \circ \psi_i. \tag{2.5.1}$$

Next, ϕ_i is smooth: Let $s : U_i \to Q$ be a smooth section of Q and let $\alpha_i : \pi_P^{-1}(U_i) \to U_i \times G$ be a bundle chart for P (w.l.o.g. defined on some U_i). Then $\alpha_i(p) = (x, \alpha_{ix}(p))$, and $\alpha_i(f(s_x)) = (x, \alpha_{ix}(f(s_x)))$. Now setting $g_p := \alpha_{ix}(f(s_x))^{-1} \cdot \alpha_{ix}(p) \in G$ we have $\alpha_i(f(s_x) \cdot g_p) = \alpha_i(p)$, i.e., $f(s_x) \cdot g_p = p$. Consequently, on $P|_{U_i}$ we can write

$$\phi_i = p \mapsto (\pi_P(p), \lambda(\psi_{ix}(s_x)) \cdot g_p),$$

which is smooth.

To construct an inverse of ϕ_i, let $(x, \tilde{g}) \in U_i \times G$. We are looking for a $p \in P_x$ with $\phi_i(p) = (x, \lambda(\psi_{ix}(s_x)) \cdot g_p) = (x, \tilde{g})$. Thus we need

$$\alpha_{ix}(f(s_x))^{-1} \cdot \alpha_{ix}(p) = g_p = \lambda(\psi_{ix}(s_x))^{-1} \cdot \tilde{g} \Rightarrow \alpha_{ix}(p) = \alpha_{ix}(f(s_x)) \cdot \lambda(\psi_{ix}(s_x))^{-1} \cdot \tilde{g}.$$

It follows that

$$\phi_i^{-1}(x, \tilde{g}) = p = \alpha_i^{-1}(x, \alpha_{ix}(f(s_x)) \cdot \lambda(\psi_{ix}(s_x))^{-1} \cdot \tilde{g}),$$

which is smooth, so that ϕ_i is a diffeomorphism.

Furthermore, ϕ_i is G-equivariant: Let $\tilde{g} \in G$, $p = f(q) \cdot g$. Then

$$\phi_i(p \cdot \tilde{g}) = (\pi_P(p), \lambda(\psi_{ix}(q)) \cdot g \cdot \tilde{g}) = (\pi_P(p), \lambda(\psi_{ix}(q)) \cdot g) \cdot \tilde{g} = \phi_i(p) \cdot \tilde{g}.$$

It remains to show that

$$g_{ik}(x) = \phi_{ix}(\phi_{kx}^{-1}(e)) = \lambda(\psi_{ix}(\psi_{kx}^{-1}(e))),$$

or equivalently that $\phi_k^{-1}(x, e) = \phi_i^{-1}(x, \lambda(\psi_{ix}(\psi_{kx}^{-1}(e))) \cdot e)$. Setting $q := \psi_{kx}^{-1}(e)$ and $p := f(q) \cdot e$, we have $\phi_i(p) = (x, \lambda(\psi_{ix}(q)) \cdot e)$ and we are left with showing that $\phi_k(p) = (x, e)$. Indeed,

$$\phi_k(p) = (x, \lambda(\psi_{kx}(q)) \cdot e) = (x, \lambda(e) \cdot e) = (x, e).$$

(ii)⇒(i): By Theorem 2.2.7, the H-cocycle generates a principal fiber bundle (Q, π_Q, M, H) over M with bundle charts $\psi_i : Q_{U_i} \to U_i \times H$. To define a suitable $f : Q \to P$ we note that in the first part of the proof, cf. (2.5.1), we got that $f|_{Q_{U_i}} = \phi_i^{-1} \circ (\mathrm{id}_{U_i} \times \lambda) \circ \psi_i$. Conversely we therefore define maps $f_i : Q_{U_i} \to P_{U_i}$, $f_i := \phi_i^{-1} \circ (\mathrm{id}_{U_i} \times \lambda) \circ \psi_i$,

$$\begin{tikzcd} Q_{U_i} \arrow[r, "f_i"] \arrow[d, "\psi_i"'] & P_{U_i} \arrow[d, "\phi_i"] \\ U_i \times H \arrow[r, "\mathrm{id}\times\lambda"] & U_i \times G \end{tikzcd}$$

We show that $f_i = f_k$ on $Q_{U_i \cap U_k}$: This is the case if and only if for each $(x, h) \in (U_i \cap U_k) \times H$ we have

$$\begin{aligned} (\mathrm{id} \times \lambda)(\psi_i \circ \psi_k^{-1})(x, h) &= (\phi_i \circ \phi_k^{-1}) \circ (\mathrm{id} \times \lambda)(x, h) \\ &\Leftrightarrow (\mathrm{id} \times \lambda)(x, h_{ik}(x) \cdot h) = (x, g_{ik}(x) \cdot \lambda(h)), \end{aligned}$$

which indeed is the case by (ii). Hence we obtain a well-defined smooth map $f : Q \to P$ such that on Q_{U_i}

$$\pi_P \circ f = \pi_P \circ \phi_i^{-1} \circ (\mathrm{id} \times \lambda) \circ \psi_i = \mathrm{pr}_1 \circ \psi_i = \pi_Q.$$

Finally,

$$\begin{aligned} f_i(q \cdot h) &= \phi_i^{-1} \circ (\mathrm{id}_{U_i} \times \lambda) \circ \psi_i(q \cdot h) = \phi_i^{-1}(x, \lambda(\psi_{ix}(q \cdot h))) \\ &= \phi_i^{-1}(x, \lambda(\psi_{ix}(q) \cdot h)) = \phi_i^{-1}(x, \lambda(\psi_{ix}(q))) \cdot \lambda(h) = f_i(q) \cdot \lambda(h). \end{aligned}$$

□

2.5.3 Remark. This result shows that if (Q, f) is a λ-reduction of P then one can choose trivializations of P and Q over the same set U such that, using $Q_U \cong U \times H$, $P_U \cong U \times G$, the map f reduces to $f|_{Q_U} \cong \mathrm{id}_U \times \lambda$ (see (ii)⇒(i) in the previous proof). This makes it possible to transfer local properties from λ to f. E.g., if λ is a covering map, so is f. If $\lambda = H \hookrightarrow G$ for a Lie subgroup H of G, then also $f : Q \to P$ is an injective immersion, hence $f(Q)$ is an immersive submanifold of P. If H is closed in G (so that by [9, Cor. 21.9] it is a regular submanifold of G), then also $f : Q \to P$ is an embedding, hence $f(Q)$ is a regular submanifold of P.

2.5.4 Theorem. *Let H be a Lie subgroup of G, (P, π, M, G) a principal fiber bundle and $Q \subseteq P$ a subset such that*

- *(i) The right action R of H preserves Q: $R_h(Q) = Q$ for all $h \in H$.*
- *(ii) If $q, \tilde{q} \in Q_x := Q \cap P_x$ and $q = \tilde{q} \cdot g$, then $g \in H$.*
- *(iii) For any $x \in M$ there exists an open neighborhood $U(x)$ in M and a smooth section $s : U(x) \to P$ with $s(U(x)) \subseteq Q$.*

Then Q is an immersive submanifold of P, $(Q, \pi|_Q, M, H)$ is a principal fiber bundle and (Q, ι) is an H-reduction (i.e., a subbundle) of P, where $\iota = Q \hookrightarrow P$ is the inclusion map.

Proof. Let $s : U \to P$ be a local section as in (iii) and consider the corresponding bundle chart (cf. Theorem 2.2.5)

$$\begin{aligned} \phi_U &: P_U \to U \times G \\ p = s(\pi(p)) \cdot g &\mapsto (\pi(p), g). \end{aligned}$$

By (i) and (ii) the restriction

$$\psi_U := \phi_U|_{Q_U} : Q_U := P_U \cap Q \to U \times H$$

is bijective, hence is a formal bundle chart for Q. By (iii), Q can be covered by a corresponding formal bundle atlas $\{(U_i, \psi_{U_i})\}_{i \in I}$. Since H is a Lie subgroup, hence an integral manifold of an integrable distribution ([9, 19.3]), the same is true for $(U_i \cap U_j) \times H \subseteq (U_i \cap U_j) \times G$. Therefore, smoothness of $\phi_{U_j} \circ \phi_{U_i}^{-1} : (U_i \cap U_j) \times G \to (U_i \cap U_j) \times G$ implies smoothness of

$$\psi_{U_j} \circ \psi_{U_i}^{-1} = \phi_{U_j} \circ \phi_{U_i}^{-1}|_{(U_i \cap U_j) \times H} : (U_i \cap U_j) \times H \to (U_i \cap U_j) \times H.$$

([9, 17.27,14.7]). By Theorem 2.1.5 this induces the structure of a fiber bundle on Q. By assumption, the right action of G on P restricts to a fiber preserving action on Q that is simply transitive on the fibers (by (i) and (ii)). As above, this action is smooth as a map $R : Q \times H \to Q$. This turns $(Q, \pi|_Q, M, H)$ into a principal fiber bundle, and the inclusion $\iota : Q \hookrightarrow P$ is an immersion ($\psi_U \circ \iota \circ \phi_U^{-1} = U \times H \hookrightarrow U \times G$). We conclude that (Q, ι) is an H-reduction of P. □

Next we want to derive a criterion for the reducibility of a principal fiber bundle (P, π, M, G) to a closed (and non-open) subgroup H of G. Consider the action of G on the homogeneous space G/H (cf. Theorem 1.1.3):

$$G \times G/H \ni (g, [a]) \mapsto [ga] \in G/H.$$

By Theorem 2.3.1 we obtain an associated fiber bundle $E := P \times_G G/H$. Now consider the map

$$\begin{aligned} f : E &\to P/H := \{p \cdot H \mid p \in P\} \\ [p, g \cdot H] &\mapsto (p \cdot g) \cdot H. \end{aligned}$$

It is readily verified that f is well-defined an bijective, so we may use it to transfer the fiber bundle structure from E to P/H, and we shall henceforth identify these two spaces.

2.5.5 Theorem. *Let (P, π, M, G) be a principal fiber bundle and let H be a closed (non-open) subgroup of G. Then the following are equivalent:*

(i) P is reducible to H.

(ii) The associated fiber bundle $(E, \pi_E, M, G/H)$ possesses a smooth global section.

Proof. (ii)⇒(i) Let $s \in \Gamma(E)$, then by Theorem 2.3.4 there exists a corresponding $\bar{s} \in \mathcal{C}^\infty(P, G/H)^G$. Now set

$$Q := \{p \in P \mid \bar{s}(p) = eH\}.$$

We first show that Q is a regular submanifold of P. Let $\mu : P \to E = P/H$, $\mu : p \mapsto [p](eH) = [p, eH] = p \cdot H$. Then μ is smooth, and $\pi_E(\mu(p)) = \pi_E([p, eH]) = \pi_P(p)$, so μ is a submersion. By Theorem 2.3.4 we have

$$Q = \{p \in P \mid [p]^{-1} \circ s(\pi(p)) = eH\} = \{p \in P \mid s(\pi(p)) = \mu(p)\}.$$

Therefore, $Q = \mu^{-1}(s(M))$: one direction is immediate from the above. For the other, let $\mu(p) = s(x)$. Then $\pi_P(p) = \pi_E \circ \mu(p) = \pi_E(s(x)) = x$, and again by the above we have $p \in Q$.

Since $s(M)$ is a regular submanifold of E (it is locally a graph) and μ is a submersion, it follows that Q is a regular submanifold as well (using a local trivialization of $s(M)$, locally $s(M) = \nu^{-1}(0)$ for some submersion ν, so $Q = (\nu \circ \mu)^{-1}(0)$ locally).

Next we claim that Q, together with $\pi_Q := \pi_P|_Q$, is a principal H-bundle. First, by (2.3.9),

$$\bar{s}(ph) = h^{-1}\bar{s}(p) = h^{-1}eH = eH \quad \forall h \in H,$$

showing that $ph \in Q$ if $p \in Q$, so H acts on Q on the right. Now let $q, \tilde{q} \in Q \cap P_x$. Then there is a unique $g \in G$ with $q = \tilde{q} \cdot g$, and again by (2.3.9)

$$\bar{s}(q) = eH = \bar{s}(\tilde{q}g) = g^{-1}\bar{s}(\tilde{q}) = g^{-1}eH = g^{-1}H,$$

so $g \in H$, implying that H acts fiber preserving and simply transitive on Q.

Let $\{(U_i, s_i)\}_{i \in I}$ be a covering of P by local sections (corresponding via Theorem 2.2.5 to a bundle atlas of P). By Theorem 1.1.3 there exist local sections $\sigma_i : W_i \to G$ in G/H, and by continuity we can arrange that $\bar{s} \circ s_i(U_i) \subseteq W_i$. Then $g_i := \sigma_i \circ \bar{s} \circ s_i : U_i \to G$ is smooth and we consider the section $\tilde{s}_i : U_i \to P$ defined by

$$\tilde{s}_i := s_i(x) \cdot g_i(x).$$

Since σ_i is a section in G/H, $g_i(x) \cdot H = \bar{s} \circ s_i(x)$, which together with (2.3.9) gives

$$\bar{s}(\tilde{s}_i(x)) = g_i(x)^{-1} \cdot \bar{s}(s_i(x)) = g_i(x)^{-1} \cdot g_i(x) \cdot H = eH.$$

We conclude that $\tilde{s}_i : U_i \to Q$ is a smooth local section. It now follows from Theorem 2.2.5 that (Q, π_Q, M, H) is a principal fiber bundle. Together with the inclusion maps $Q \hookrightarrow P$ and $H \hookrightarrow G$ it is a reduction of P since all operations on Q were defined by restriction of those on P.

(i)⇒(ii): Let (Q, π_Q, M, H) be a principal fiber bundle and $f : Q \to P$ an H-reduction of P. Then by Remark 2.5.3, $f : Q \to P$ is an embedding. The closed subgroup H acts on the left on G, inducing a fiber bundle structure on $Q \times_H G$ by Theorem 2.3.1. G acts on the right on $Q \times_H G$ by $[q, g] \cdot \tilde{g} := [q, g \cdot \tilde{g}]$, which is readily seen to be well-defined. It is also smooth: Using a bundle chart ψ_U as in (2.3.2), $\psi_U([q, g]) = (\pi(q), \varphi_U(q) \cdot g)$, so in terms of this chart the action is given by $((x, g), \tilde{g}) \mapsto (x, g \cdot \tilde{g})$.

It is immediate that the action is fiber preserving and simply transitive on the fibers. Finally, the bundle charts ψ_U from above satisfy (i)–(iii) from Definition 2.2.1, so $Q \times_H G$ is a G-principal fiber bundle. Now let

$$\begin{aligned} F : Q \times_H G &\to P \\ [q, g] &\mapsto f(q) \cdot g. \end{aligned}$$

Again it easily follows that F is well-defined. It is surjective, since given $p \in P$ we may pick any $q \in Q$ with $\pi(p) = \pi(q)$. Then there is a unique $g \in G$ with $p = f(q) \cdot g$, so $p = F([q, g])$. To see injectivity, suppose that $f(q_1)g_1 = f(q_2)g_2$. Since Q is a principal fiber bundle, there is a unique $h \in H$ with $q_1 = q_2 h$, so $f(q_1) = f(q_2) \cdot h$, implying that $h = g_2 g_1^{-1}$. Since f is injective (by Remark 2.5.3), $f(q_1) = f(q_2)h = f(q_2 h)$ implies $q_1 = q_2 h$, so altogether $[q_1, g_1] = [q_2, g_2]$. Smoothness of F follows since, using (2.3.3),

$$F \circ \psi_U^{-1} = (x, g) \mapsto F([\phi_U^{-1}(x, e), g]) = f(\phi_U^{-1}(x, e)) \cdot g.$$

Indeed F is a diffeomorphism: we already know it is bijective, so it suffices to show that its Jacobian is invertible everywhere. So let $\tilde{\phi}_U : P|_U \to U \times G$, $\tilde{\phi}_U(p) = (\pi(p), \tilde{\varphi}_U(p))$ be a bundle chart for P. Then

$$\tilde{\phi}_U \circ F \circ \psi_U^{-1}(x, g) = (x, \tilde{\varphi}_U(f(\phi_U^{-1}(x, e))) \cdot g) = (x, L_{a(x)}(g)),$$

with $a(x) := \tilde{\varphi}_U(f(\phi_U^{-1}(x,e)))$. Consequently,

$$T_{(x,g)}(\tilde{\phi}_U \circ F \circ \psi_U^{-1}) = \begin{pmatrix} I & 0 \\ * & T_g L_{a(x)} \end{pmatrix},$$

which, as desired, is bijective. Indeed F is even an isomorphism of principal fiber bundles, since

$$\begin{aligned} \pi_P \circ F([q,g]) &= \pi_P(f(q)\cdot g) = \pi_P(f(q)) = \pi_Q(q) = \hat{\pi}([q,g]), \\ F([q,g]\cdot \tilde{g}) &= F([q, g\cdot \tilde{g}]) = f(q)\cdot g \cdot \tilde{g} = F([q,g])\cdot \tilde{g}. \end{aligned}$$

Identifying P with $Q \times_H G$ via F, the desired section $\bar{s} : P \to G/H$ can now be defined by

$$\begin{aligned} \bar{s} : P \cong Q \times_H G &\to G/H \\ [q,g] &\mapsto g^{-1}H, \end{aligned}$$

which is clearly well-defined. Then $\bar{s} \circ \psi_U^{-1}(x,g) = \bar{s}([\phi_U^{-1}(x,e), g]) = g^{-1}H$, so $\bar{s}$ is smooth. Finally,

$$\bar{s}([q,g]\cdot\tilde{g}) = \bar{s}([q, g\cdot\tilde{g}]) = \tilde{g}^{-1}g^{-1}H = \tilde{g}^{-1}\bar{s}([q,g])$$

shows that $\bar{s} \in \mathcal{C}^\infty(P, G/H)^G$, concluding the proof via Theorem 2.3.4. □

We now want to use this criterion to show that any principal fiber bundle with non-compact structure group possesses a reduction to a compact group. To achieve this, we will make use of a result from the structure theory of Lie groups: A compact subgroup K of a Lie group G is called *maximally compact* if there does not exist another compact subgroup strictly containing K. Then the following holds (see e.g. [5] for a proof):

2.5.6 Theorem.

(i) Any connected Lie group G contains a maximally compact subgroup K. For any other compact subgroup $\hat{K}$ of G there exists some $g \in G$ with $g\hat{K}g^{-1} \subseteq K$.

(ii) Let K be a maximally compact subgroup of a connected Lie group G. Then there exists a submanifold N of G that is diffeomorphic to some $\mathbb{R}^r$ such that the map

$$\begin{aligned} N \times K &\to G \\ (n,k) &\mapsto n\cdot k \end{aligned}$$

is a diffeomorphism. Then the map $f : N \to G/K$, $n \mapsto n\cdot K$ is a diffeomorphism, so $G/K \cong \mathbb{R}^r$.

Using this result we can now show:

2.5.7 Theorem. *Let G be a connected, non-compact Lie group and let (P, π, M, G) be a principal fiber bundle. Then P can be reduced to any maximally compact subgroup K of G.*

Proof. Let K be a maximally compact subgroup of G. Then by Theorem 2.5.6, the homogeneous space G/K is diffeomorphic to $\mathbb{R}^r$. By Theorem 2.5.5 it suffices to show the existence of a global section in the associated fiber bundle $E = P \times_G G/K$. This follows directly from Theorem 2.1.9. □

Next we determine how associated vector bundles react to reduction of the structure group.

2.5.8 Theorem. *Let $\lambda : H \to G$ be a Lie group homomorphism and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. Also, let (P, π, M, G) be a principal fiber bundle and let (Q, f) be a λ-reduction of P. Then the associated vector bundles $P \times_{(G,\rho)} V$ and $Q \times_{(H,\rho\lambda)} V$ are isomorphic.*

Proof. The map

$$\begin{aligned} \Psi : Q \times_{(H,\rho\lambda)} V &\to P \times_{(G,\rho)} V \\ [q, v] &\to [f(q), v] \end{aligned}$$

is well-defined because

$$\Psi([qh, \rho \circ \lambda(h^{-1})v]) = [f(q)\lambda(h), \rho(\lambda(h)^{-1})v] = [f(q), v].$$

Moreover, it is fiber linear and fiber preserving. Suppose that $\Psi([q, v]) = \Psi([\tilde{q}, \tilde{v}])$, where $q, \tilde{q} \in Q_x$ and $v, \tilde{v} \in V$. Then there exists a unique $h \in H$ with $\tilde{q} = q \cdot h$, so $f(\tilde{q}) = f(q)\lambda(h)$ and therefore

$$[f(q), v] = [f(q)\lambda(h), \rho(\lambda(h)^{-1})v] = [f(\tilde{q}), \rho(\lambda(h)^{-1})v] = [f(\tilde{q}), \tilde{v}].$$

Thus $\tilde{v} = \rho(\lambda(h^{-1}))v$, implying $[q, v] = [qh, \rho(\lambda(h^{-1}))v] = [\tilde{q}, \tilde{v}]$, so Ψ is injective. Now let $[p, v] \in P \times_{(G,\rho)} V$ with $p \in P_x$ and pick any $q \in Q_x$. Then there exists a unique $g \in G$ with $f(q) = p \cdot g$. Consequently,

$$\Psi([q, \rho(g^{-1})v]) = [f(q), \rho(g^{-1})v] = [p, v],$$

showing surjectivity of Ψ. Finally, to show that Ψ is a diffeomorphism we proceed analogously to the proof of Theorem 2.5.5. With $\tilde{\psi}_U$ and ψ_U bundle charts for $P \times_{(G,\rho)} V$ and $Q \times_{(H,\rho\lambda)} V$ as in (2.3.2), we obtain using (2.3.3):

$$\begin{aligned} \tilde{\psi}_U \circ \Psi \circ \psi_U^{-1}(x, v) &= \tilde{\psi}_U \circ \Psi([\phi_U^{-1}(x, e), v]) = \tilde{\psi}_U([f(\phi_U^{-1}(x, e)), v]) \\ &= (x, \tilde{\varphi}_U(f(\phi_U^{-1}(x, e))) \cdot v) =: (x, a(x) \cdot v) \end{aligned}$$

(where $a(x) \in \mathrm{GL}(V)$), which shows smoothness of Ψ. Moreover,

$$T_{(x,v)}(\tilde{\psi}_U \circ \Psi \circ \psi_U^{-1}) = \begin{pmatrix} I & 0 \\ * & a(x) \end{pmatrix},$$

which is bijective. This shows that Ψ is a local, hence (being bijective) a global diffeomorphism and thereby a vector bundle isomorphism. □

We now turn to the operation that is inverse to the reduction of principal fiber bundles, the *extension* of principal fiber bundles. Let $\lambda : H \to G$ be a Lie group homomorphism. Then λ defines an action of H on G by

$$\begin{aligned} &H \times G \to G \\ &(h, g) \mapsto h \cdot g := \lambda(h) \cdot g. \end{aligned} \tag{2.5.2}$$

This action allows us to associate to any H-principal fiber bundle (Q, π_Q, M, H) a fiber bundle $P = Q \times_H G$.

2.5.9 Definition. *The fiber bundle $P = Q \times_H G$ is called the λ-extension of Q.*

2.5.10 Theorem. *Let $\lambda : H \to G$ be a Lie group homomorphism and (Q, π_Q, M, H) a principal fiber bundle.*

(i) The λ-extension $P = Q \times_H G$ of Q is a G-principal bundle over M.

(ii) *Let $f : Q \to P = Q \times_H G$ be the map $f(q) := [q, e]$, with e the unit element in G. Then (Q, f) is a λ-reduction of P.*

(iii) *Let P be a G-principal fiber bundle over M and let (Q, f) be a λ-reduction of P. Then P is isomorphic to the λ-extension of Q.*

Proof. (i) We define the action of G on P by

$$\begin{aligned} \mu : (Q \times_H G) \times G &\mapsto Q \times_H G \\ ([q, a], g) &\mapsto [q, ag]. \end{aligned}$$

It is easy to check that μ is well-defined and fiber preserving. It is also simply transitive on the fibers: if $[q, a_1], [q, a_2] \in \hat{\pi}^{-1}(x)$ (note that we can always arrange to have the same first component), then $g := a_1^{-1} a_2$ is the unique element of G with $[q, a_1] \cdot g = [q, a_2]$.

Let $\phi_U : Q_U \to U \times H$ be an H-equivariant bundle chart for Q with $\phi_U(q) = (\pi(q), \varphi_U(q))$. Then by Theorem 2.3.1 the map

$$\begin{aligned} \psi_U &: P_U \to U \times G \\ \psi_U([q, g]) &:= (\pi(q), \varphi_U(q) \cdot g) \end{aligned}$$

is a fiber bundle chart for P that satisfies (a) and (b) from Definition 2.2.1 (ii) due to Theorem 2.3.1. Also (c) holds because

$$\psi_U([q, a] \cdot g) = \psi_U([q, ag]) = (\pi(q), \varphi_U(q) \cdot a \cdot g) = \psi_U([q, a]) \cdot g.$$

To show smoothness of μ we calculate, using (2.3.3):

$$\begin{aligned} \psi_U \circ \mu \circ (\psi_U \times \mathrm{id}_G)^{-1}((x, a), g) &= \psi_U \circ \mu([\phi_U^{-1}(x, e), a], g) = \psi_U([\phi_U^{-1}(x, e), a \cdot g]) \\ &= (x, \varphi_U(\phi_U^{-1}(x, e)) \cdot a \cdot g) = (x, e \cdot a \cdot g) = (x, ag). \end{aligned}$$

Consequently, P is indeed a G-principal bundle over M.

(ii) f is obviously fiber preserving, and we have

$$f(qh) = [qh, e] = [q, \lambda(h)e] = [q, e] \cdot \lambda(h) = f(q)\lambda(h).$$

It is smooth because

$$\psi_U \circ f(q) = \psi_U([q, e]) = (\pi(q), \varphi_U(q) \cdot e).$$

(iii) Consider the map

$$\begin{aligned} \Psi : Q \times_H G &\to P \\ [q, g] &\mapsto f(q)g. \end{aligned}$$

Again it is easy to see that Ψ is well-defined. It is fiber preserving since

$$\pi_P \circ \Psi([q, g]) = \pi_P(f(q) \cdot g) = \pi_P \circ f(q) = \pi_Q(q) = \hat{\pi}([q, g]).$$

Smoothness follows since

$$\Psi \circ \psi_U^{-1}(x, a) = \Psi([\phi_U^{-1}(x, e), a]) = f(\phi_U^{-1}(x, e)) \cdot a.$$

Also, it commutes with the action of G:

$$\Psi([q, g] \cdot g_1) = \Psi([q, gg_1]) = f(q)gg_1 = \Psi([q, g]) \cdot g_1.$$

Let $\tilde{\phi}_U : \pi_P^{-1}(U) \to U \times G$ be a G-equivariant bundle chart for P, $\tilde{\phi}_U(p) = (\pi_P(p), \tilde{\varphi}_U(p))$ and let $s : U \to Q$ be a local section in Q. Then $f \circ s : U \to P$ is a local section in P and we define a map

$$g : P_U \to G,\ p = f(s(\pi(p))) \cdot g(p).$$

In terms of the bundle chart $\tilde{\phi}_U$ this means

$$\tilde{\phi}_U(f \circ s \circ \pi(p) \cdot g(p)) = (\pi(p), \tilde{\varphi}_U(f \circ s \circ \pi(p)) \cdot g(p)) \stackrel{!}{=} \tilde{\phi}_U(p) = (\pi(p), \tilde{\varphi}_U(p)),$$

so $g(p) = \tilde{\varphi}_U(f \circ s \circ \pi(p))^{-1}\tilde{\varphi}_U(p)$, showing smoothness of g.
Next we note that Ψ is injective: Let $f(q_1)g_1 = f(q_2)g_2$, then

$$\pi_Q(q_1) = \pi_P(f(q_1)g_1) = \pi_P(f(q_2)g_2) = \pi_Q(q_2),$$

so $q_1 \cdot h = q_2$ for some $h \in H$. Thus

$$\begin{aligned} f(q_1)g_1 = f(q_2)g_2 = f(q_1 h)g_2 = f(q_1)\lambda(h)g_2 &\Rightarrow g_1 = \lambda(h)g_2 \\ &\Rightarrow [q_1, g_1] = [q_1, \lambda(h)g_2] = [q_1 h, g_2] = [q_2, g_2]. \end{aligned}$$

Then since $\Psi([s(\pi(p)), g(p)]) = f(s(\pi(p))) \cdot g(p) = p$ on $P|_U$ it follows that Ψ is also surjective, hence bijective, and locally we have

$$\Psi^{-1} = p \mapsto [s(\pi(p)), g(p)].$$

Since

$$\psi_U \circ \Psi^{-1}(p) = \psi_U([s(\pi(p)), g(p)]) = (\pi(p), \varphi_U(s(\pi(p))) \cdot g(p))$$

is smooth, so is Ψ^{-1}, establishing that Ψ is a diffeomorphism and thereby an isomorphism of principal fiber bundles. □

As an application we prove a criterion for the existence of pseudo-Riemannian metrics. While any manifold can be endowed with a Riemannian metric, the same need no longer be true in the pseudo-Riemannian setting.

2.5.11 Theorem. *Let M be a manifold of dimension n and let $k, l \in \mathbb{N}_0$ such that $k + l = n$. Then the following are equivalent:*

(i) There exists a pseudo-Riemannian metric of signature (k, l) on M.

(ii) There exist real vector bundles ξ, η of rank k resp. l over M such that $TM = \xi \oplus \eta$.

Proof. (ii)⇒(i): Choose any Riemannian metric r on M and set

$$g|_{\xi\times\xi} := -r|_{\xi\times\xi},\ g|_{\eta\times\eta} := r|_{\eta\times\eta},\ g|_{\xi\times\eta} := 0$$

to obtain a pseudo-Riemannian metric of signature (k, l) on M.
(i)⇒(ii): Let g be a pseudo-Riemannian metric of signature (k, l) on M and consider the bundle $\mathrm{O}(M, g)$ of g-orthonormal frames on M. Then the structure group of $\mathrm{O}(M, g)$ is the pseudo-orthogonal group $\mathrm{O}(k, l)$, which is not compact (cf. Example 2.2.13).
We now consider the product $\mathrm{O}(k) \times \mathrm{O}(l)$ of the corresponding orthogonal groups as a subgroup of $\mathrm{O}(k, l)$:

$$\mathrm{O}(k) \times \mathrm{O}(l) \ni (A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathrm{O}(k, l).$$

Then $\mathrm{O}(k) \times \mathrm{O}(l)$ is a maximally compact subgroup of $\mathrm{O}(k,l)$, so by Theorem 2.5.7 we can reduce the $\mathrm{O}(k,l)$-principal fiber bundle $\mathrm{O}(M,g)$ to the compact group $\mathrm{O}(k) \times \mathrm{O}(l)$. Let (Q,f) be such a reduction. Then by Example 2.4.6, the Remark following Definition 2.5.1, and Theorem 2.5.8 we obtain

$$\begin{aligned} TM &\cong \mathrm{GL}(M) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n \cong \mathrm{O}(M,g) \times_{\mathrm{O}(k,l)} \mathbb{R}^n \cong Q \times_{\mathrm{O}(k)\times\mathrm{O}(l)} (\mathbb{R}^k \oplus \mathbb{R}^l) \\ &\cong (Q \times_{\mathrm{O}(k)} \mathbb{R}^k) \oplus (Q \times_{\mathrm{O}(l)} \mathbb{R}^l) =: \xi \oplus \eta. \end{aligned} \tag{2.5.3}$$

□

Chapter 3

Connections in principal fiber bundles

Having set the stage in the previous chapters, we now turn to the topic of geometric analysis on principal fiber bundles and associated bundles. The central notion on which everything else (e.g., curvature, parallel transport, holonomy) is grounded is that of a connection.

3.1 Basic notions

Recall from [9, Sec. 17] that a (geometric) distribution $\mathcal{E}$ (Ω in [9]) on a manifold N is a map

$$\mathcal{E} : N \ni x \mapsto E_x \subseteq T_x N$$

that assigns to any $x \in N$ an r-dimensional subspace E_x in a smooth way. The latter means that any $x \in N$ has a neighborhood U on which there are smooth vector fields $X_1, \dots, X_r$ that span $\mathcal{E}$, i.e.,

$$E_y = \operatorname{span}(X_1(y), \dots, X_r(y)) \qquad \forall y \in U.$$

Let (P, π, M, G) be a principal fiber bundle. Henceforth we shall always denote the right action of G on P by R_g:

$$\begin{aligned} P \times G &\to P \\ (u, g) &\mapsto R_g(u) := u \cdot g. \end{aligned}$$

On P there always exists a canonical geometric distribution given by the tangent spaces of the fibers of P. Indeed, since π is a submersion, any fiber $P_x = \pi^{-1}(x)$ is a regular submanifold, which thereby is an integral manifold for this distribution. We denote the tangent space to P_x in the point $u \in P_x$ by

$$Tv_u P := T_u(P_x) \subseteq T_u P.$$

$Tv_u P$ is called the *vertical tangent space* in u.

3.1.1 Theorem. *(Properties of the vertical tangent space)*

(i) $Tv_u P = \ker T_u \pi$.

(ii) The map

$$\Phi_u : \mathfrak{g} \ni X \mapsto \tilde{X}(u) := \frac{d}{dt}\Big|_0 (u \cdot \exp(tX)) \in Tv_uP,$$

assigning to each element of the Lie algebra the value at u of the fundamental vector field generated by it, is a linear isomorphism. Therefore,

$$Tv_uP = \{\tilde{X}(u) \mid X \in \mathfrak{g}\}.$$

(iii) For any $X \in \mathfrak{g}$, the flow of $\tilde{X}$ is given by $R_{\exp(tX)}$:

$$\mathrm{Fl}_t^{\tilde{X}}(u) = u \cdot \exp(tX) = R_{\exp(tX)}(u).$$

Proof. (i) follows from the fact that π is a submersion, cf. [7, 3.3.23,3.3.25].
(iii) This was shown in the proof of Theorem 1.1.7.
(ii) By Theorem 1.2.2 the map $X \mapsto \tilde{X}(u)$ is linear. It takes values in Tv_uP since $\pi(u \cdot \exp(tX)) \equiv \pi(u)$, so $T_u\pi(\tilde{X}(u)) = 0$. By (i) we have

$$\dim Tv_uP = \dim P - \dim \mathrm{im}(T_u\pi) = \dim P - \dim M = \dim G = \dim \mathfrak{g}.$$

It therefore suffices to show that $X \mapsto \tilde{X}(u)$ is injective. So let $\tilde{X}(u) = 0$. Then the integral curve of $\tilde{X}$ through u is constant, i.e. (by (iii)), $u = u \cdot \exp(tX)$ for all $t \in \mathbb{R}$. Thus since G acts simply transitively on the fibers, $\exp(tX) = e$ for all t. Choosing t so small that tX lies in a neighborhood where exp is injective it follows that $X = 0$. □

Point (ii) of Theorem 3.1.1 shows that for any basis $(X_1, \dots, X_r)$ of $\mathfrak{g}$, the corresponding fundamental vector fields $(\tilde{X}_1, \dots, \tilde{X}_r)$ span the distribution

$$Tv : P \ni u \mapsto Tv_uP \subseteq T_uP,$$

which is therefore smooth. Moreover, this distribution is right invariant, i.e., $T_uR_g(Tv_uP) = Tv_{u\cdot g}P$. Indeed, for $w \in \ker T_u\pi$,

$$T_{u\cdot g}\pi(T_uR_g(w)) = T_u(\pi \circ R_g)(w) = T_u\pi(w) = 0$$

so the claim follows from Theorem 3.1.1 (i), equality of the dimensions, and the fact that

$$T_uR_g : Tv_uP \to Tv_{u\cdot g}P \tag{3.1.1}$$

is linear and injective (hence a linear isomorphism).

Any subspace of T_uP that is complementary to Tv_uP is called a *horizontal* tangent space to P in u. A *connection* on a principal fiber bundle (P, π, M, G) is a smooth selection of horizontal tangent spaces that is compatible with the action of G in the following sense:

3.1.2 Definition. *A* connection *on a principal fiber bundle (P, π, M, G) is a geometric distribution of horizontal tangent spaces*

$$Th : P \ni u \mapsto Th_uP \subseteq T_uP$$

that is right invariant, i.e.,

$$T_uR_g(Th_uP) = Th_{u\cdot g}P. \tag{3.1.2}$$

Then $ThP = \bigcup_{u\in P} Th_uP \subseteq TP$ is called the horizontal tangent bundle, and the projections $\mathrm{pr}_v : TP \to TvP$, $\mathrm{pr}_h : TP \to ThP$ are smooth, as maps $TP \to TP$: In terms of local bases $(X_1, \dots, X_n)$, $(X_{n+1}, \dots, X_{n+m})$ of the distributions Th, Tv we have

$$\mathrm{pr}_h : \sum_{i=1}^{n+m} a_i X_i \mapsto \sum_{i=1}^{n} a_i X_i,$$

which when expressed in local charts is obviously smooth (and analogously for pr_v). By Theorem 3.1.1 (i)

$$T_u\pi : Th_uP \to T_{\pi(u)}M \tag{3.1.3}$$

is a linear isomorphism. Given $X_u \in T_uP$ and $g \in G$, we have $X_u = \mathrm{pr}_h(X_u) + \mathrm{pr}_v(X_u)$, and $T_uR_g(X_u) = \mathrm{pr}_h(T_uR_g(X_u)) + \mathrm{pr}_v(T_uR_g(X_u))$. On the other hand,

$$T_uR_g(X_u) = T_uR_g(\mathrm{pr}_h(X_u)) + T_uR_g(\mathrm{pr}_v(X_u)) \in Th_{ug}P \oplus Tv_{ug}P$$

by (3.1.2) and (3.1.1). We conclude that

$$\mathrm{pr}_h \circ TR_g = TR_g \circ \mathrm{pr}_h, \qquad \mathrm{pr}_v \circ TR_g = TR_g \circ \mathrm{pr}_v. \tag{3.1.4}$$

Note also that for any X, $T\pi(X) = T\pi(\mathrm{pr}_h(X)) + T\pi(\mathrm{pr}_v(X)) = T\pi(\mathrm{pr}_h(X))$, i.e.,

$$T\pi \circ \mathrm{pr}_v = 0, \quad T\pi \circ \mathrm{pr}_h = T\pi. \tag{3.1.5}$$

Next we want to examine alternative ways of introducing connections on principal fiber bundles. In what follows we will make use of vector valued k-forms, referring to [9, Sec. 10] for definitions and basic properties.

3.1.3 Definition. *A connection form (or connection* 1*-form) on a principal fiber bundle* (P, π, M, G) *is a* 1*-form* $A \in \Omega^1(P, \mathfrak{g})$ *that satisfies*

(i) $R_g^* A = \mathrm{Ad}(g^{-1}) \circ A$ *for all* $g \in G$, *and*

(ii) $A(\tilde{X}) = X$ *for all* $X \in \mathfrak{g}$.

The set of all connection forms on P *is denoted by* $\mathcal{C}(P)$.

3.1.4 Theorem. *Connections and connection forms on a principal fiber bundle* (P, π, M, G) *are in bijective correspondence:*

(i) If $Th : P \ni u \mapsto Th_uP$ *is a connection on* P*, then*

$$A_u(\tilde{X}(u) \oplus Y_h) := X \qquad \forall u \in P,\ X \in \mathfrak{g},\ Y_h \in Th_uP$$

defines a connection form on P.

(ii) If $A \in \Omega^1(P, \mathfrak{g})$ *is a connection form on* P*, then*

$$Th : P \ni u \mapsto Th_uP := \ker A_u$$

defines a connection on P.

Proof. (i) To demonstrate that A is smooth we have to show that for any $Z \in \mathfrak{X}(P)$ we have $A(Z) \in \mathcal{C}^\infty(P, \mathfrak{g})$. We can write $Z = Z_v \oplus Z_h$, with $Z_h = \mathrm{pr}_h \circ Z$, $Z_v = \mathrm{pr}_v \circ Z$. Let $\Phi_u : \mathfrak{g} \to Tv_uP, X \mapsto \tilde{X}(u)$ be the linear isomorphism from Theorem 3.1.1 (ii). Then $A(Z)(u) = \Phi_u^{-1}(Z_v(u))$. Let $(X_1, \dots, X_r)$ be a basis of $\mathfrak{g}$. Then there exist $f_1, \dots, f_r \in \mathcal{C}^\infty(P)$ such that $Z_v = u \mapsto \sum_{i=1}^r f_i(u)\tilde{X}_i(u)$. Consequently,

$$A(Z)(u) = \Phi_u^{-1}\Big(\sum_{i=1}^r f_i(u)\tilde{X}_i(u)\Big) = \sum_{i=1}^r f_i(u) X_i$$

is smooth.

By Theorem 1.2.2 we have

$$TR_g \circ \tilde{X} \circ R_{g^{-1}} = (R_g)_* \tilde{X} = (\mathrm{Ad}(g^{-1})X)^{\sim},$$

so $TR_g(\tilde{X}(u)) = (\mathrm{Ad}(g^{-1})X)^{\sim}(ug)$. If $Y_h \in Th_uP$, then by (3.1.2) we have $TR_g(Y_h) \in Th_{u\cdot g}P$. Consequently,

$$\begin{aligned}(R_g^*A)_u(\tilde{X}(u) + Y_h) &= A_{ug}(TR_g(\tilde{X}(u)) + TR_gY_h) \\ &= A_{ug}((\mathrm{Ad}(g^{-1})X)^{\sim}(ug) + TR_gY_h) \\ &= \mathrm{Ad}(g^{-1})X = \mathrm{Ad}(g^{-1}) \circ A_u(\tilde{X}(u) + Y_h).\end{aligned}$$

Thus $R_g^*A = \mathrm{Ad}(g^{-1}) \circ A$.

(ii) We have to show that $u \mapsto \ker A_u$ is a smooth horizontal and right invariant distribution on P.

To see smoothness, let $(W, (x^1, \dots, x^m))$ be a chart for P around $u \in P$ and let $(X_1, \dots, X_r)$ be a basis of $\mathfrak{g}$. Let $Y \in T_uP$, $Y = \sum_i \xi^i \partial_{x^i}|_u$. Since A is smooth, $A(\partial_{x^i}) = \sum_j A_{ij}X_j$, with A_{ij} smooth on W. Now $Y \in \ker(A_u)$ if and only if

$$\sum_i \xi^i A_{ij}(u) = 0 \qquad j = 1, \dots, r.$$

The solutions of this system of linear equations depend smoothly on u, so we obtain a smooth local basis for $\ker A$.

Let $Y \in T_uP$ be an element of $\ker A_u$. Then

$$A_{ug}(TR_gY) = (R_g^*A)_u(Y) = \mathrm{Ad}(g^{-1})(A_u(Y)) = 0,$$

which shows that $T_uR_g(\ker A_u) \subseteq \ker A_{ug}$. Since T_uR_g is a linear isomorphism and the above also shows $T_{ug}R_{g^{-1}}(\ker A_{ug}) \subseteq \ker A_u$, we have equality, and right invariance of Th follows.

Finally, we show that $\ker A_u$ is horizontal. Let $Y \in \ker A_u \cap Tv_uP$. Then $Y = \tilde{X}(u)$ for some $X \in \mathfrak{g}$. Therefore, $0 = A_u(Y) = X$, so $Y = \tilde{X} = 0$, and $\ker A_u$ is transversal to Tv_uP. Since A_u is surjective by Definition 3.1.3 (ii),

$$\dim \ker A_u = \dim T_uP - \dim \mathfrak{g} = \dim T_uP - \dim Tv_uP,$$

implying that $T_uP = \ker A_u \oplus Tv_uP$. □

To give a local characterization of connections we use local 1-forms on the base manifold.

3.1.5 Definition. *Let $A \in \Omega^1(P, \mathfrak{g})$ be a connection form on the principal fiber bundle (P, π, M, G) and let $s : U \subseteq M \to P$ be a local section in P. Then the 1-form*

$$A^s := A \circ Ts \in \Omega^1(U, \mathfrak{g}), \quad T_xM \ni X \mapsto A^s(X) := A_{s(x)}(T_xs(X)) \in \mathfrak{g} \tag{3.1.6}$$

is called the local connection form induced by s.

If $(U, \varphi = (x^1, \dots, x^n))$ is a local chart of M, then let $A_\mu := A^s(\partial_\mu)$ $(\mu = 1, \dots, n)$. If, in addition, e_a $(a = 1, \dots, r)$ is a basis of $\mathfrak{g}$, we can expand

$$A_\mu = \sum_{a=1}^r A_\mu^a e_a, \quad A^s = \sum_{a=1}^r (A^s)^a e_a$$

Then the real valued fields $A_\mu^a \in \mathcal{C}^\infty(U, \mathbb{R})$, as well as the corresponding $(A^s)^a \in \Omega^1(U, \mathbb{R})$ are called (local) gauge boson fields.

Now let $s_i : U_i \to P$ and $s_j : U_j \to P$ be local sections with $U_i \cap U_j \neq \varnothing$. Then for each $x \in U_i \cap U_j$ there is a unique $g_{ij}(x) \in G$ such that

$$s_i(x) = s_j(x) \cdot g_{ij}(x). \tag{3.1.7}$$

Using the bundle chart ψ_{s_j} induced by s_j from (2.2.2), we have $g_{ij}(x) = \mathrm{pr}_2 \circ \psi_{s_j}^{-1}(s_i(x))$, so g_{ij} is smooth.

Denote by $\mu_G \in \Omega^1(G, \mathfrak{g})$ the Maurer–Cartan form of G (cf. [9, Sec. 10]),

$$\mu_G(Y_g) := TL_{g^{-1}}(Y_g), \quad Y_g \in T_g G$$

and let $\mu_{ij} := g_{ij}^* \mu_G \in \Omega^1(U_i \cap U_j, \mathfrak{g})$ be its pullback under g_{ij} to $U_i \cap U_j$:

$$\mu_{ij}(X) = TL_{g_{ij}^{-1}(x)}(Tg_{ij}(X)), \quad X \in T_x(U_i \cap U_j).$$

Then we have:

3.1.6 Theorem. *(Local characterization of connection forms)*

(i) Let $A \in \Omega^1(P, \mathfrak{g})$ be a connection form in the principal fiber bundle P and let (s_i, U_i), (s_j, U_j) be local sections in P with $U_i \cap U_j \neq \varnothing$. Then

$$A^{s_i} = \mathrm{Ad}(g_{ij}^{-1}) \circ A^{s_j} + \mu_{ij}.$$

(ii) Conversely, if $\{(s_i, U_i)\}_{i \in I}$ is a covering of M by local sections and if $\{A_i \in \Omega^1(U_i, \mathfrak{g})\}_{i \in I}$ is a family of local 1-forms such that, whenever $U_i \cap U_j \neq \varnothing$,

$$A_i = \mathrm{Ad}(g_{ij}^{-1}) \circ A_j + \mu_{ij} \quad \text{on} \quad U_i \cap U_j, \tag{3.1.8}$$

then there is a unique connection form $A \in \Omega^1(P, \mathfrak{g})$ on P with $A^{s_i} = A_i$ for each $i \in I$.

Proof. (i) Let $x \in U_i \cap U_j$, $X \in T_x M$ and γ a smooth curve in M with $\gamma(0) = x$ and $\gamma'(0) = X$. Then by Lemma 1.2.3 and (3.1.7) we have

$$\begin{aligned} T_x s_i(X) &= \frac{d}{dt}\Big|_0 (s_i(\gamma(t))) = \frac{d}{dt}\Big|_0 (s_j(\gamma(t)) \cdot g_{ij}(\gamma(t))) \\ &= TR_{g_{ij}(x)}(Ts_j(X)) + (\mu_{ij}(X))^{\sim}(s_i(x)). \end{aligned} \tag{3.1.9}$$

Therefore, using Definition 3.1.3,

$$\begin{aligned} A^{s_i}(X) &= A(T_x s_i(X)) = A(TR_{g_{ij}(x)}(Ts_j(X))) + \mu_{ij}(X) \\ &= A_{s_j(x) \cdot g_{ij}(x)}(TR_{g_{ij}(x)}(Ts_j(X))) + \mu_{ij}(X) = R^*_{g_{ij}(x)} A(Ts_j(X)) + \mu_{ij}(X) \\ &= \mathrm{Ad}(g_{ij}(x)^{-1})(A(Ts_j(X))) + \mu_{ij}(X) = \mathrm{Ad}(g_{ij}(x)^{-1}) A^{s_j}(X) + \mu_{ij}(X). \end{aligned}$$

(ii) We begin by showing that A_i is a connection form on the trivial subbundle P_{U_i}. Let $x \in U_i$ and set $u := s_i(x) \in P$. Then

$$T_u P = Tv_u P \oplus T_x s_i(T_x U_i). \tag{3.1.10}$$

To see this, note first that since s_i is a section, $T\pi \circ Ts_i = \mathrm{id}$, which implies that $\ker(T_{s_i(x)}\pi) \cap \mathrm{im}(T_x s_i) = \{0\}$. Also, $\dim P = \dim Tv_u P + \dim M = \dim Tv_u P + \dim T_x s_i(T_x U_i)$, giving the claim.

Now we define a smooth $\mathfrak{g}$-valued 1-form A on P_{U_i} by prescribing its action on any $Z \in \mathfrak{X}(P|_{U_i})$. Recall from the proof of Theorem 3.1.4 (i) that $u \mapsto \Phi_u^{-1} \circ \mathrm{pr}_v$ is a smooth 1-form. Taking into account (2.2.2) it then follows that

$$\begin{aligned} A(Z)(s_i(x) \cdot g) := \mathrm{Ad}(g^{-1}) \circ [\Phi^{-1}_{s_i(x)} \circ & \mathrm{pr}_v(TR_{g^{-1}}(Z_{s_i(x) \cdot g})) \\ & + A_i(T\pi(\mathrm{pr}_h(TR_{g^{-1}}(Z_{s_i(x) \cdot g}))))] \end{aligned} \tag{3.1.11}$$

defines an element of $\Omega^1(P|_{U_i}, \mathfrak{g})$. Note that in this equation, we currently are given pr_v and pr_h only on $s_i(U_i)$, due to the direct sum decomposition (3.1.10). Globally defined pr_h, pr_v would precisely amount to having a connection, which we are in the process of constructing.

In particular inserting $g = e$, $u = s_i(x)$ (and recalling that $T\pi$ is a left inverse of Ts_i) this implies

$$A_u(\tilde{Y}(u) \oplus T_x s_i(X)) = Y + A_i(X), \qquad Y \in \mathfrak{g},\ X \in T_x U_i. \tag{3.1.12}$$

In particular, $A^{s_i} = A_i$. Moreover, directly from (3.1.11) we read off the following relation between the values of A at $u = s_i(x)$ and at $u \cdot g = s_i(x) \cdot g$:

$$A_{ug} = \mathrm{Ad}(g^{-1}) \circ A_u(T_{ug} R_{g^{-1}}(\,.\,)). \tag{3.1.13}$$

Then for any $Y \in \mathfrak{g}$, by Theorem 1.2.2 (iii) we have:

$$\begin{aligned} A_{ug}(\tilde{Y}(ug)) &= \mathrm{Ad}(g^{-1})(A_u(T_{ug}R_{g^{-1}}(\tilde{Y}(R_g u))) = \mathrm{Ad}(g^{-1})A_u((\mathrm{Ad}(g)Y)^{\tilde{}}(u)) \\ &\underset{(3.1.12)}{=} \mathrm{Ad}(g^{-1})\mathrm{Ad}(g)Y = Y. \end{aligned}$$

Moreover, for any $Z \in T_{ug}P$, we get, using (3.1.13):

$$\begin{aligned} (R_a^* A)_{ug}(Z) &= A_{uga}(TR_a(Z)) = \mathrm{Ad}(a^{-1})\mathrm{Ad}(g^{-1})A_u(TR_{a^{-1}g^{-1}}(TR_a(Z))) \\ &= \mathrm{Ad}(a^{-1})A_{ug}(Z). \end{aligned}$$

This means that both conditions from Definition 3.1.3 are satisfied, showing that A is a connection form on P_{U_i}.

To conclude the proof it remains to show that the connection forms A and $\hat{A}$ induced in this way by (A_i, s_i), (A_j, s_j) coincide on the bundle $P_{U_i \cap U_j}$. Again looking at (3.1.12) and (3.1.13) we see that they coincide on the vertical tangent spaces and are uniquely determined by $A_{s_i(x)}$ resp. $\hat{A}_{s_j(x)}$. So we are left with showing that for any $x \in U_i \cap U_j$, in $u = s_i(x)$ we have

$$\hat{A}_u(T_x s_i(X)) = A_u(T_x s_i(X)) = A_i(X) \qquad \forall X \in T_x M.$$

As in (3.1.7), let $s_i(x) = s_j(x) \cdot g_{ij}(x)$, $X \in T_x M$, and let γ be a smooth curve in M with $\gamma'(0) = X$. Then by (3.1.9) we have

$$Ts_i(X) = \frac{d}{dt}\Big|_0 (s_j(\gamma(t)) \cdot g_{ij}(\gamma(t))) = TR_{g_{ij}(x)}(Ts_j(X)) + \mu_{ij}(X)^{\tilde{}}(s_i(x)).$$

By (3.1.12) and (3.1.13) we conclude that

$$\begin{aligned} \hat{A}(Ts_i(X)) &= \hat{A}(TR_{g_{ij}(x)}(Ts_j(X))) + \hat{A}(\mu_{ij}(X)^{\tilde{}}(s_i(x))) \\ &= \mathrm{Ad}(g_{ij}(x)^{-1})\hat{A}(Ts_j(X)) + \mu_{ij}(X) \\ &= \mathrm{Ad}(g_{ij}(x)^{-1})A_j(X) + \mu_{ij}(X) = A_i(X), \end{aligned}$$

where in the last equality we used (3.1.8). □

3.1.7 Remark. (i) An important special case occurs if $G \subseteq \mathrm{GL}(r, \mathbb{K})$ is a matrix group. Then by linearity we have $TL_g X = gX$ and $\mathrm{Ad}(g)X = \mathrm{conj}_g(X) = gXg^{-1}$ for all $g \in G$ and $X \in \mathfrak{g}$. Thus (3.1.8) reduces to

$$A_i = g_{ij}^{-1} \circ A_j(\,.\,) \circ g_{ij} + g_{ij}^{-1} T g_{ij}. \tag{3.1.14}$$

(ii) If P is trivial then it possesses a global section. A connection on P is then given by a 1-form on M with values in $\mathfrak{g}$ (namely by A^s for this global section s).

Next we look at some important examples of connections on principal fiber bundles.

3.1.8 Example. *The canonical flat connection*
Consider the trivial principal fiber bundle $(P = M \times G, \mathrm{pr}_1, M, G)$ over M. Then

$$Tv_{(x,g)}P = T_{(x,g)}(\{x\} \times G) \cong T_g G.$$

Using this identification, the fundamental vector fields on P coincide precisely with the left invariant vector fields on G:

$$\tilde{Y}(x,g) = \frac{d}{dt}\Big|_0 ((x,g) \cdot \exp(tY)) = \frac{d}{dt}\Big|_0 (x, g \cdot \exp(tY)) = 0 \oplus T_e L_g(Y) = 0 \oplus L^Y(g).$$

As horizontal tangent spaces we choose the tangent spaces to M:

$$Th_{(x,g)}P := T_{(x,g)}(M \times \{g\}) \cong T_x M.$$

The resulting connection on P is called the *canonical flat connection*. By the above and Theorem 3.1.4 (i), the corresponding connection form is given by the Maurer–Cartan form of G:

$$\begin{aligned} A : T_{(x,g)}(M \times G) \cong T_x M \oplus T_g G &\to \mathfrak{g} \\ X + Y &\mapsto T_g L_{g^{-1}}(Y) = \mu_G(Y) \end{aligned}$$

3.1.9 Example. *Left invariant connections on reductive homogeneous spaces*
Let H be a closed (non-open) subgroup of the Lie group G, and let $\mathfrak{h}$ be its Lie algebra. The homogeneous space $M := G/H$ (cf. Theorem 1.1.3) is called *reductive* if there exists a vector space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $\mathrm{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m}$. Now let G/H be a reductive homogeneous space and let $(G, \pi, G/H, H)$ be the homogeneous H-principal fiber bundle over G/H (see Example 2.2.11). Then for the fundamental vector field $\tilde{X} \in \mathfrak{X}(G)$ generated by any vector $X \in \mathfrak{h} = T_e H$ we have

$$\tilde{X}(g) = \frac{d}{dt}\Big|_0 (g \cdot \exp(tX)) = T_e L_g(X) = L^X(g),$$

so $\tilde{X}$ is precisely the left invariant vector field generated by X. Thus the vertical tangent space in $g \in G$ is $Tv_g G = T_e L_g(\mathfrak{h}) \subseteq T_g G$. Now $T_g G = T_e L_g(\mathfrak{h}) \oplus T_e L_g(\mathfrak{m})$, so the left invariant distribution

$$Th : G \ni g \mapsto Th_g G := T_e L_g(\mathfrak{m}) \subseteq T_g G$$

defines a connection on $(G, \pi, G/H, H)$: Smoothness is clear, so it remains to show right invariance. For $a \in H$ we get, recalling that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$,

$$\begin{aligned} TR_a(Th_g G) = TR_a(TL_g \mathfrak{m}) = TL_g TR_a(\mathfrak{m}) &= TL_g TL_a \mathrm{Ad}(a^{-1})\mathfrak{m} \\ &\subseteq TL_g TL_a \mathfrak{m} = TL_{ga}\mathfrak{m} = Th_{ga}G, \end{aligned}$$

and since the dimensions agree we in fact have equality. To calculate the corresponding connection form, by Theorem 3.1.4 we have to secure

$$A_g(\tilde{X}(g) \oplus Y_h) = X \qquad \forall g \in G\ X \in \mathfrak{h},\ Y_h \in Th_g G.$$

In the current situation, $\tilde{X}(g) = TL_g(X)$ and $\mu_G = X_g \mapsto TL_{g^{-1}}(X_g)$, so

$$\mathrm{pr}_{\mathfrak{h}}(\mu_G(\tilde{X}(g) \oplus Y_h)) = \mathrm{pr}_{\mathfrak{h}}(X \oplus TL_{g^{-1}}(Y_h)) = X$$

because $TL_{g^{-1}}(Y_h) \in \mathfrak{m}$. Consequently, $A = \mathrm{pr}_{\mathfrak{h}} \circ \mu_G \in \Omega^1(G, \mathfrak{h})$.

3.1.10 Example. *Connections on the frame bundle*

Let M be a smooth manifold of dimension n and let $\mathrm{GL}(M)$ be the frame bundle over M (cf. Example 2.2.12). We are going to show that the set of covariant derivatives on TM is in bijective correspondence to the set of connections on $\mathrm{GL}(M)$.

First, let $A \in \Omega^1(\mathrm{GL}(M), \mathfrak{gl}(n,\mathbb{R}))$ be a connection form on $\mathrm{GL}(M)$. Denote by (B_{ij}) the $n \times n$ matrix that has a 1 in the i-th row and j-th column and zeros otherwise. Then in terms of the basis $(B_{ij})_{i,j=1,\dots,n}$ of $\mathfrak{gl}(n,\mathbb{R})$ we can write

$$A = \sum_{i,j=1}^{n} \omega_{ij} B_{ij}, \tag{3.1.15}$$

where $\omega_{ij} \in \Omega^1(\mathrm{GL}(M), \mathbb{R})$. Now given a local section $s = (s_1, \dots, s_n) : U \to \mathrm{GL}(M)$ in the frame bundle, we define the covariant derivative corresponding to A by

$$\nabla_X s_k := \sum_{i=1}^{n} \omega_{ik}(Ts(X)) s_i, \quad X \in \mathfrak{X}(U),\ k = 1, \dots, n, \tag{3.1.16}$$

and by extending it linearly so as to satisfy the product rule

$$\nabla_X (f s_k) := X(f) s_k + f \nabla_X s_k \qquad \forall f \in \mathcal{C}^\infty(U). \tag{3.1.17}$$

To see that ∇_X is well-defined, let $\tilde{s} = (\tilde{s}_1, \dots, \tilde{s}_n)$ be another local section, also defined on U. Then by (3.1.7) there exists some smooth $C : U \to \mathrm{GL}(n,\mathbb{R})$ with $\tilde{s}(x) = s(x) \cdot C(x)$, which explicitly means (cf. (2.2.7))

$$\tilde{s}_i(x) = \sum_k s_k(x) C_{ki}(x).$$

According to (3.1.14) we have

$$A \circ T\tilde{s}(X) = C^{-1} \cdot (A(Ts(X))) \cdot C + C^{-1} \cdot TC(X),$$

so

$$\omega_{ik}(T\tilde{s}(X)) = \sum_{j,l} (C^{-1})_{ij} \omega_{jl}(Ts(X)) C_{lk} + \sum_j (C^{-1})_{ij} (TC)_{jk}(X) \tag{3.1.18}$$

Let $\tilde{\nabla}_X$ be defined as above, but with $\tilde{s}$ instead of s. Then we have to verify that $\tilde{\nabla}_X \tilde{s}_k = \nabla_X \tilde{s}_k$. Now

$$\begin{aligned}
\tilde{\nabla}_X \tilde{s}_k &= \sum_i \omega_{ik}(T\tilde{s}(X)) \tilde{s}_i = \sum_{i,j,k} (C^{-1})_{ij} \omega_{jl}(Ts(X)) C_{lk} \tilde{s}_i + \sum_{i,j} (C^{-1})_{ij} (TC)_{jk}(X) \tilde{s}_i \\
&= \sum_{i,j,l,r} \underbrace{C_{ri}(C^{-1})_{ij}}_{\sum_i = \delta_{rj}} \omega_{jl}(Ts(X)) C_{lk} s_r + \sum_{i,j,r} \underbrace{C_{ri}(C^{-1})_{ij}}_{\sum_i = \delta_{rj}} (TC)_{jk}(X) s_r \\
&= \sum_{j,l} \omega_{jl}(Ts(X)) C_{lk} s_j + \sum_j (TC)_{jk}(X) s_j,
\end{aligned}$$

and, on the other hand,

$$\begin{aligned}
\nabla_X(\tilde{s}_k) &= \nabla_X \Big(\sum_l s_l C_{lk} \Big) = \sum_l X(C_{lk}) s_l + \sum_l C_{lk} \nabla_X s_l \\
&= \sum_l (TC)_{lk}(X) s_l + \sum_{l,j} \omega_{jl}(Ts(X)) C_{lk} s_j,
\end{aligned}$$

giving the claim.

Conversely, let ∇ be a covariant derivative on TM and let $s = (s_1, \dots, s_n) : U \to \mathrm{GL}(M)$ be a local section in the frame bundle. Then for certain $\omega_{ji} \in \Omega^1(U)$ we can write

$$\nabla s_i = \sum_{j=1}^{n} \omega_{ji} \otimes s_j .$$

Now define $A_s \in \Omega^1(U, \mathfrak{gl}(n, \mathbb{R}))$ by

$$A_s := \sum_{i,j=1}^{n} \omega_{ij} B_{ij} .$$

To show that this family of local 1-forms (as s runs through the local sections of $\mathrm{GL}(M)$) defines a connection form on $\mathrm{GL}(M)$, by Theorem 3.1.1 we have to verify the transformation rule (3.1.14). Thus let $\tilde{s}$ be another section, and let

$$\nabla \tilde{s}_i = \sum_{j=1}^{n} \tilde{\omega}_{ji} \otimes \tilde{s}_j , \quad A_{\tilde{s}} := \sum_{i,j=1}^{n} \tilde{\omega}_{ij} B_{ij} .$$

Note that for any covariant derivative we have $\nabla_Y(fX) = Y(f)X + f\nabla_Y X$, so $\nabla(fX) = df \otimes X + f\nabla X$. Using this, with C as above we calculate:

$$\begin{aligned} \nabla \tilde{s}_i = \nabla\Big(\sum_k s_k C_{ki}\Big) &= \sum_k [dC_{ki} \otimes s_k + C_{ki} \nabla s_k] = \sum_k \Big[dC_{ki} \otimes s_k + C_{ki} \sum_j \omega_{jk} \otimes s_j\Big] \\ &= \sum_k \Big(\sum_j C_{ji}\omega_{kj} + dC_{ki} \Big) \otimes s_k \end{aligned}$$

On the other hand,

$$\nabla \tilde{s}_i = \sum_j \tilde{\omega}_{ji} \otimes \Big(\sum_k s_k C_{kj} \Big) = \sum_{j,k} C_{kj} \tilde{\omega}_{ji} \otimes s_k$$

Combining this, we get

$$\sum_j C_{ji}\omega_{kj} + TC_{ki} = \sum_j C_{kj}\tilde{\omega}_{ji} \Rightarrow (A_s \cdot C)_{ki} + (TC)_{ki} = (C \cdot A_{\tilde{s}})_{ki} .$$

In total, $A_s \cdot C + TC = C \cdot A_{\tilde{s}}$, or $A_{\tilde{s}} = C^{-1} A_s C + C^{-1} TC$, as claimed.

In the above situation, given a connection form A on $\mathrm{GL}(M)$, denote by ∇^A the corresponding covariant derivative and, conversely, given a covariant derivative ∇, let A^∇ be the connection form on $\mathrm{GL}(M)$ constructed in the previous example. Then by construction we have

$$\nabla^{A^\nabla} = \nabla \quad \text{and} \quad A^{\nabla^A} = A, \tag{3.1.19}$$

so we have proved:

3.1.11 Corollary. *The map $A \mapsto \nabla^A$ is a bijection between the set of connection forms on $\mathrm{GL}(M)$ and the set of covariant derivatives on M.*

3.1.12 Example. *The Levi-Civita connection on a semi-Riemannian manifold*

Let (M^n, g) be a semi-Riemannian manifold, where g has signature (k, l), with $n = k + l$. Then on (M, g) there exists a unique metric and torsion free covariant derivative

$$\nabla^{LC} : \Gamma(TM) \to \Gamma(T^*M \otimes TM),$$

the *Levi-Civita connection* of (M,g) (cf. [13]). We show that ∇^{LC} corresponds to a unique connection form A^{LC} on the principal fiber bundle $(\mathrm{O}(M,g),\pi,M,\mathrm{O}(k,l))$ of orthonormal frames on M. Let B_{ij} be as in Example 3.1.10 and let E_{ij} be the $n\times n$ matrix

$$E_{ij} \coloneqq \varepsilon_i B_{ji} - \varepsilon_j B_{ij}, \quad \varepsilon_i \coloneqq \begin{cases} -1, & i = 1,\dots,k \\ 1, & i = k+1,\dots,k+l. \end{cases}$$

Then for the Lie algebra $\mathfrak{o}(k,l)$ of $\mathrm{O}(k,l)$ we have

$$\mathfrak{o}(k,l) = \mathrm{span}\{E_{ij} \mid i<j\}.$$

Let $s = (s_1,\dots,s_n) : M \supseteq U \to \mathrm{O}(M,g)$ be a local section, so that $(s_1(x),\dots,s_n(x))$ is a g_x-orthonormal basis in (T_xM,g_x) for each $x\in U$. Then we define an element of $\Omega^1(U,\mathfrak{o}(k,l))$ by

$$A_s(X) \coloneqq \sum_{i<j} \varepsilon_i\varepsilon_j g(\nabla^{LC}_X s_i, s_j) E_{ij} \in \mathfrak{o}(k,l). \tag{3.1.20}$$

Again we have to verify that for these forms the transformation rule (3.1.14) holds. Let $\tilde{s}$ be another local section. Then by (3.1.7) we have $\tilde{s}(x) = s(x)C(x)$, with C smooth and $C(x)\in \mathrm{O}(k,l)$ for each x. Recall that

$$\mathrm{O}(k,l) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid A^tJA = J\},$$

where $J = \mathrm{diag}(-1,\dots,-1,1,\dots,1)$ (k minuses). Note that $J = J^{-1}$, and for any matrix $a = (a_{ij})$ we have $(\varepsilon_i a_{ij}) = J\cdot a$. In particular we have $C^tJC = J$. Also note that, since $\nabla^{LC}_X(g(s_i,s_j)) = g(\nabla_X s_i, s_j) + g(s_i,\nabla_X s_j) = 0$,

$$\begin{aligned} A_s(X) &= \sum_{i<j} \varepsilon_i\varepsilon_j g(\nabla^{LC}_X s_i, s_j)(\varepsilon_i B_{ji} - \varepsilon_j B_{ij}) \\ &= \sum_{j<i} \varepsilon_i g(\nabla^{LC}_X s_j, s_i) B_{ij} - \sum_{i<j} \varepsilon_i g(\nabla^{LC}_X s_i, s_j) B_{ij} \\ &= -\sum_{i\neq j} \varepsilon_i g(\nabla^{LC}_X s_i, s_j) B_{ij} = -\sum_{i,j} \varepsilon_i g(\nabla^{LC}_X s_i, s_j) B_{ij}, \end{aligned} \tag{3.1.21}$$

and analogously $A_{\tilde{s}}(X) = -\sum_{i,j} \varepsilon_i g(\nabla^{LC}_X \tilde{s}_i, \tilde{s}_j) B_{ij}$. Thus we have

$$\begin{aligned} (-A_{\tilde{s}}(X))_{ij} &= \varepsilon_i g\Big(\nabla^{LC}_X\Big(\sum_p s_p C_{pi}\Big), \sum_q s_q C_{qj}\Big) \\ &= \sum_{p,q} \varepsilon_i g(X(C_{pi})s_p, s_q C_{qj}) + \sum_{p,q} \varepsilon_i g(\nabla^{LC}_X s_p C_{pi}, s_q C_{qj}) \\ &= \sum_{p,q} \varepsilon_i (TC(X))_{pi} C_{qj} \underbrace{g(s_p,s_q)}_{=\varepsilon_p\delta_{pq} = J_{pq}} + \sum_{p,q} \varepsilon_i C_{pi} C_{qj} g(\nabla^{LC}_X s_p, s_q) \\ &= \sum_p (JTC(X)^t)_{ip}(JC)_{pj} + \sum_p (JC^t)_{ip} \underbrace{g(\nabla^{LC}_X s_p, s_q)}_{=\sum_r J_{pr}\cdot\varepsilon_r g(\nabla^{LC}_X s_r, s_q)} C_{qj} \\ &= (J(TC(X))^tJC)_{ij} + (JC^tJ(-A_s(X))C)_{ij} \end{aligned}$$

Now note that $0 = (TI)(X) = T(C^{-1}C)(X) = TC^{-1}(X)C + C^{-1}TC(X)$, so $T(C^{-1})(X) = -C^{-1}TC(X)C^{-1}$. Combining this with $JC^tJ = C^{-1}$ we obtain $J((TC)(X))^tJC = T(C^{-1})(X)C = -C^{-1}TC(X)C^{-1}C = -C^{-1}TC(X)$. Altogether,

$$A_{\tilde{s}}(X) = C^{-1}A_s(X)C + C^{-1}(TC)(X),$$

as desired. Thus the family A_s defines a connection form A^{LC} on the bundle of orthonormal frames.

Conversely, let $A \in \Omega^1(\mathrm{O}(M,g), \mathfrak{o}(k,l))$ be a connection form, $A = \sum_{i,j} \omega_{ij} B_{ij}$ (cf. (3.1.15)), with $\omega_{ij} \in \Omega^1(\mathrm{O}(M,g), \mathbb{R})$ and such that $(\omega_{ij})_{i,j} \in \mathfrak{o}(k,l)$, i.e., $J \cdot A + A^t \cdot J = 0$. Then by Example 3.1.10 A defines a covariant derivative ∇ on M. We claim that ∇ is metric, i.e.,

$$\nabla_Z \langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \tag{3.1.22}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. By expanding the involved vector fields in a local orthonormal frame $s = (s_1, \dots, s_n)$ it readily follows that to prove (3.1.22) it suffices to show that for all k, l we have $\langle \nabla_X s_k, s_l\rangle = -\langle s_k, \nabla_X s_l\rangle$ for any X. Using (3.1.16) we have $\nabla_X s_k = \sum_i \omega_{ik}(Ts(X)) s_i$, so indeed

$$\begin{aligned}\langle \nabla_X s_k, s_l\rangle &= \sum_i \omega_{ik}(Ts(X)) \underbrace{\langle s_i, s_l\rangle}_{=\varepsilon_i \delta_{il}} = \varepsilon_l \omega_{lk}(Ts(X)) = (J \cdot A_s(X))_{lk} = -(A_s^t(X) \cdot J)_{lk} \\ &= -\sum_r \omega_{rl}(Ts(X)) \varepsilon_r \delta_{rk} = -\varepsilon_k \omega_{kl}(Ts(X)) = \dots = -\langle s_k, \nabla_X s_l\rangle.\end{aligned}$$

Note also that this calculation gives

$$A_s(X) = \sum_{i,j} \omega_{ij}(Ts(X)) B_{ij} = -\sum_{i,j} \varepsilon_i \langle \nabla_X s_i, s_j\rangle B_{ij},$$

consistent with (3.1.21) and thereby with (3.1.20).

Analogously to Corollary 3.1.11 and with the analogous notation we have thereby shown (note that we did not use the fact that the Levi-Civita connection is torsion free in Example 3.1.12, so what we did applies in fact to any metric connection):

3.1.13 Corollary. *The map $A \mapsto \nabla^A$ is a bijection between the set of connection forms on $\mathrm{O}(M,g)$ and the set of metric covariant derivatives on M, and the analogue of* (3.1.19) *holds:*

$$\nabla^{A^\nabla} = \nabla \quad \textit{and} \quad A^{\nabla^A} = A.$$

To conclude this section we show that *any* principal fiber bundle possesses a connection:

3.1.14 Theorem. *On any principal fiber bundle there exists a connection.*

Proof. Let (P, π, M, G) be any principal fiber bundle and fix an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ of M consisting of trivializing neighborhoods for P, $P|_{U_\alpha} \cong U_\alpha \times G$. Let $\{\chi_\alpha\}$ be a partition of unity subordinate to $\mathcal{U}$. Denoting by $A_\alpha \in \Omega^1(P_{U_\alpha}, \mathfrak{g})$ the canonical flat connection on the trivial subbundle P_{U_α} (cf. Example 3.1.8) we define $A \in \Omega^1(P, \mathfrak{g})$ by

$$A := \sum_\alpha (\chi_\alpha \circ \pi) A_\alpha.$$

Then for any $X \in \mathfrak{g}$ and any $p \in P$ we have

$$A(\tilde{X}(p)) = \sum_\alpha \chi_\alpha(\pi(p)) A_\alpha(\tilde{X}(p)) = \sum_\alpha \chi_\alpha(\pi(p)) X = X.$$

Moreover,

$$(R_g^* A)_p(Y) = A_{pg}(TR_g Y) = \sum_\alpha \chi_\alpha(\pi(p)) A_\alpha(TR_g(Y)) = \mathrm{Ad}(g^{-1}) A_p(Y),$$

concluding the proof that A is a connection form on P □

3.2 The affine space of connections

Our aim here is to prove that the set of all connections on a principal fiber bundle forms an infinite dimensional affine space. To do this we need some preparations:

Let E be a vector bundle over a manifold M. A k-form with values in E is a smooth map

$$\omega : M \ni x \mapsto \omega_x \in L^k_{\mathrm{alt}}(T_x M, E_x),$$

where $L^k_{\mathrm{alt}}(T_x M, E_x)$ denotes the space of k-linear alternating maps $(T_x M)^k \to E_x$. Smoothness means that for any $X_1, \dots, X_k \in \mathfrak{X}(U)$ ($U \subseteq M$ open) the local section

$$s : U \ni x \mapsto \omega_x(X_1(x), \dots, X_k(x)) \in E_x \subseteq E_U$$

is smooth. Thus an E-valued k-form is precisely a smooth section of the vector bundle $\Lambda^k T^* M \otimes E$. For the space of these forms we write

$$\Omega^k(M, E) := \Gamma(\Lambda^k T^* M \otimes E).$$

As in [7, 4.1.19] it follows that $\Omega^k(M, E)$ can be identified with the $\mathcal{C}^\infty(M)$-module of $\mathcal{C}^\infty(M)$-multilinear and skew symmetric maps

$$\omega : \mathfrak{X}(M)^k \to \Gamma(E),$$

where $\omega(X_1, \dots, X_k)(x) := \omega_x(X_1(x), \dots, X_k(x)) \in E_x$.

If V is a finite dimensional vector space then the space of V-valued k-forms is a special case:

$$\Omega^k(M, V) = \Gamma(\Lambda^k T^* M \otimes \underline{V}),$$

where $\underline{V} := M \times V$ is the trivial vector bundle with fiber V.

The wedge product has the following extension to E-valued forms:

$$\begin{aligned} \wedge : \Omega^k(M) \times \Omega^l(M, E) &\to \Omega^{k+l}(M, E) \\ (\sigma, \omega) &\mapsto \sigma \wedge \omega, \end{aligned}$$

where, for $t_1, \dots, t_{k+l} \in T_x M$,

$$\begin{aligned} &(\sigma \wedge \omega)_x(t_1, \dots, t_{k+l}) \\ &\quad := \frac{1}{k!l!} \sum_{\tau \in \mathcal{S}_{k+l}} \operatorname{sgn}(\tau) \sigma_x(t_{\tau(1)}, \dots, t_{\tau(k)}) \cdot \omega_x(t_{\tau(k+1)}, \dots, t_{\tau(k+l)}). \end{aligned} \tag{3.2.1}$$

Analogously to Theorem 2.3.4 we now want to derive a characterization of k-forms that take values in a vector bundle associated to a principal fiber bundle. Thus let (P, π, M, G) be a principal fiber bundle, $\rho : G \to \mathrm{GL}(V)$ a representation and $E = P \times_{(G,\rho)} V$ the corresponding associated vector bundle. Then from Theorem 2.3.4 we know that

$$\Gamma(E) \cong \mathcal{C}^\infty(P, V)^{(G,\rho)} := \{ s \in \mathcal{C}^\infty(P, V) \mid s(p \cdot g) = \rho(g^{-1}) s(p) \}.$$

3.2.1 Definition. *A V-valued k-form $\omega \in \Omega^k(P, V)$ is called*

(i) horizontal *if $\omega_p(X_1, \dots, X_k) = 0$ whenever at least one $X_i \in T_p P$ is vertical.*

(ii) of type ρ *if $R_a^* \omega = \rho(a^{-1}) \circ \omega$ for all $a \in G$.*

The set of all horizontal k-forms of type ρ is denoted by $\Omega^k_{\mathrm{hor}}(P, V)^{(G,\rho)}$.

3.2.2 Remark. Denote by $\mathcal{C}(P)$ the set of all connection forms on P. Then if $A_1, A_2 \in \mathcal{C}(P)$, by Definition 3.1.3 their difference $A := A_1 - A_2$ satisfies $R_g^* A = \mathrm{Ad}(g^{-1}) \circ A$, hence is of type Ad. In addition, A is horizontal: By Theorem 3.1.1, any element of $Tv_u P$ is of the form $\tilde{X}(u)$ for some $X \in \mathfrak{g}$. Thus $A(\tilde{X}) = A_1(\tilde{X}) - A_2(\tilde{X}) = X - X = 0$. Conversely, if $A \in \mathcal{C}(P)$ and ω is a horizontal $\mathfrak{g}$-valued 1-form of type Ad, then also $A + \omega \in \mathcal{C}(P)$. Thus the set $\mathcal{C}(P)$ of all connection forms on P is an affine space over the vector space $\Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$.

3.2.3 Theorem. *With E as above, the vector space $\Omega^k(M, E)$ is canonically isomorphic to the space $\Omega^k_{\mathrm{hor}}(P, V)^{(G,\rho)}$.*

Proof. Let $p \in P_x$ and as in (2.3.7) let

$$[p] : V \ni v \mapsto [p, v] \in E_x$$

be the corresponding fiber diffeomorphism. We define a linear map

$$\Phi : \Omega^k_{\mathrm{hor}}(P, V)^{(G,\rho)} \to \Omega^k(M, E)$$

as follows: Given $\bar{\omega} \in \Omega^k_{\mathrm{hor}}(P, V)^{(G,\rho)}$, let $\omega := \Phi(\bar{\omega})$ be such that $\omega_x \in \Lambda^k(T_x^* M) \otimes E_x$ is given by

$$\omega_x(t_1, \dots, t_k) := [p](\bar{\omega}_p(X_1, \dots, X_k)) = [p, \bar{\omega}_p(X_1, \dots, X_k)], \tag{3.2.2}$$

where $\pi(p) = x$, $t_1, \dots, t_k \in T_x M$ and $X_1, \dots, X_k \in T_p P$ with $T\pi(X_j) = t_j$.[1]

To see that ω is well-defined suppose first that also Y_j are vectors with $T\pi(Y_j) = t_j$. Then $T\pi(Y_j - X_j) = 0$, meaning that $Y_j - X_j$ is vertical. Since $\bar{\omega}$ is horizontal, this implies $\bar{\omega}_p(\dots, Y_j - X_j, \dots) = 0$. Independence from the choice of $p \in P_x$ follows from the fact that $\bar{\omega}$ is of type ρ: Let $\tilde{p} = pg$ and let $Y_1, \dots, Y_k \in T_{\tilde{p}} P$ be vectors with $T\pi(Y_j) = t_j$. Then

$$\begin{aligned}
[\tilde{p}, \bar{\omega}_{\tilde{p}}(Y_1, \dots, Y_k)] &= [pg, \bar{\omega}_{pg}(Y_1, \dots, Y_k)] = [p, \rho(g)\bar{\omega}_{pg}(Y_1, \dots, Y_k)] \\
&= [p, (R^*_{g^{-1}}\bar{\omega})_{pg}(Y_1, \dots, Y_k)] = [p, \bar{\omega}_p(TR_{g^{-1}}Y_1, \dots, TR_{g^{-1}}Y_k)] \\
&= [p, \bar{\omega}_p(X_1, \dots, X_k)],
\end{aligned}$$

where the last step follows from what was shown before since $T\pi(TR_{g^{-1}}Y_j) = T\pi(Y_j) = t_j$.

To see smoothness of ω, let $s : M \supseteq U \to P$ be a local section of P and let $Z_1, \dots, Z_k$ be local vector fields on U. Then

$$\omega(Z_1, \dots, Z_k)|_U = [s, \bar{\omega}_{s(\,.\,)}(Ts(Z_1), \dots, Ts(Z_k))].$$

is smooth by (2.3.5). Hence $\omega = \Phi(\bar{\omega}) \in \Omega^k(M, E)$.

We claim that the inverse of Φ is given by $\Phi^{-1} : \Omega^k(M, E) \to \Omega^k_{\mathrm{hor}}(P, V)^{(G,\rho)}$, $\omega \mapsto \bar{\omega}$, where

$$\begin{aligned}
\bar{\omega}_p(X_1, \dots, X_k) &:= [p]^{-1}\omega_{\pi(p)}(T\pi(X_1), \dots, T\pi(X_k)) \\
&= [p]^{-1}\pi^*\omega(X_1, \dots, X_k) \in V.
\end{aligned} \tag{3.2.3}$$

Smoothness of $\bar{\omega}$ is equivalent to smoothness, for any local smooth vector fields X_j on P, of $p \mapsto (p, \bar{\omega}_p(X_1, \dots, X_k))$, hence by (2.3.5) to that of

$$p \mapsto [p, \bar{\omega}_p(X_1, \dots, X_k)] = \omega_{\pi(p)}(T\pi(X_1), \dots, T\pi(X_k)),$$

[1] Note that for $\Gamma(E) = \Omega^0(M, E)$ this reduces precisely to the map $\bar{s} \mapsto s$ from the proof of Theorem 2.3.4.

which clearly holds. If X_i is vertical, then $T\pi(X_i) = 0$ and thereby also $\bar{\omega}_p(X_1, \ldots, X_k) = 0$, so $\bar{\omega}$ is horizontal. Next, note that generally we have

$$[pa]^{-1}([p, v]) = [pa]^{-1}([pa, \rho(a^{-1})v]) = \rho(a^{-1})v,$$

so that

$$\begin{aligned}
R_a^* \bar{\omega}|_p(X_1, \ldots, X_k) &= \bar{\omega}_{pa}(TR_a(X_1), \ldots, TR_a(X_k)) \\
&= [pa]^{-1}\omega_{\pi(pa)}(T\pi(TR_a(X_1)), \ldots, T\pi(TR_a(X_k))) \\
&= [pa]^{-1}\omega_{\pi(p)}(T\pi(X_1), \ldots, T\pi(X_k)) \\
&= [pa]^{-1}([p, \bar{\omega}_p(X_1, \ldots, X_k)]) \\
&= \rho(a^{-1})\bar{\omega}_p(X_1, \ldots, X_k),
\end{aligned}$$

showing that $\bar{\omega}$ is of type ρ, hence $\bar{\omega} \in \Omega^k_{\mathrm{hor}}(P, V)^{(G,\rho)}$. Finally,

$$\begin{aligned}
\Phi(\Phi^{-1}(\omega))_x(t_1, \ldots, t_k) &= [p](\bar{\omega}_p(X_1, \ldots, X_k)) = \cancel{[p]}\cancel{[p]^{-1}}\omega_x(t_1, \ldots, t_k) \\
\Phi^{-1}(\Phi(\bar{\omega}))_p(X_1, \ldots, X_k) &= [p]^{-1}\Phi(\bar{\omega})_{\pi(p)}(T\pi(X_1), \ldots, T\pi(X_k)) \\
&= [p]^{-1}[p, \bar{\omega}_p(X_1, \ldots, X_k)] = \bar{\omega}_p(X_1, \ldots, X_k).
\end{aligned}$$

□

3.2.4 Remark. The above result allows us to give an alternative description of $\mathcal{C}(P)$. The Lie group G acts on its Lie algebra $\mathfrak{g}$ via the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, turning $[G, \mathfrak{g}]$ into a Lie transformation group. We call the corresponding associated vector bundle

$$\mathrm{Ad}(P) \coloneqq P \times_G \mathfrak{g}.$$

the *adjoint bundle*. Then combining Remark 3.2.2 with Theorem 3.2.3 we obtain that $\mathcal{C}(P)$ is an affine space over the vector space $\Omega^1(M, \mathrm{Ad}(P))$.

3.3 Parallel transport in principal fiber bundles

The choice of a connection on a principal fiber bundle allows one to define parallel transport of fibers both in the bundle itself and in associated vector bundles. Throughout this section, we fix a principal fiber bundle (P, π, M, G) and a connection form A on P.

3.3.1 Definition. *Let $X \in \mathfrak{X}(M)$. A vector field X^* on P is called a* horizontal lift *of X if for each $p \in P$ we have:*

(i) $X^*(p) \in Th_p(P)$, and

(ii) $T_p\pi(X^*(p)) = X(\pi(p))$.

3.3.2 Theorem.

(i) For any $X \in \mathfrak{X}(M)$ there exists a unique horizontal lift $X^* \in \mathfrak{X}(P)$. Moreover, X^* is right invariant.

(ii) Conversely, if $Z \in \mathfrak{X}(P)$ is horizontal and right invariant, then there is a unique vector field $X \in \mathfrak{X}(M)$ with $Z = X^*$.

(iii) Let $X, Y \in \mathfrak{X}(M)$, $f \in \mathcal{C}^\infty(M)$. Then

$$\begin{aligned} X^* + Y^* &= (X+Y)^* \\ (fX)^* &= (f \circ \pi) X^* \\ [X,Y]^* &= \mathrm{pr}_h([X^*, Y^*]). \end{aligned}$$

(iv) Let $Z \in \mathfrak{X}(P)$ be horizontal, $X \in \mathfrak{X}(M)$, and let $\tilde{B}$ ($B \in \mathfrak{g}$) be a fundamental vector field on P. Then $[\tilde{B}, Z]$ is horizontal and $[\tilde{B}, X^*] = 0$.

Proof. (i) By (3.1.3), $T_p\pi : Th_pP \to T_{\pi(p)}M$ is a linear isomorphism. Therefore our only choice for a horizontal lift is given by

$$X^*(p) := (T\pi|_{ThP})^{-1}(X(\pi(p))), \tag{3.3.1}$$

and we are left with proving that the resulting vector field is smooth and right invariant.

To see smoothness, let $\phi : P_U \to U \times G$ be a bundle chart around $\pi(p)$, and denote by Y the smooth vector field $Y := T\phi^{-1}(X \oplus 0)$ on P_U. Then $T\pi(\mathrm{pr}_h(Y)) = T\pi(Y) = X$, i.e., $X^* = \mathrm{pr}_h Y$. Since also pr_h is smooth (see the discussion following (3.1.2)), X^* is as well. By (3.1.2) we have

$$TR_g(X^*(p)) \in Th_{pg}P.$$

Also, $T\pi(TR_g X^*(p)) = T\pi(X^*(p)) = X(\pi(pg))$. Since we saw above that the horizontal lift is unique, we conclude that $TR_g(X^*(p)) = X^*(pg)$, so X^* is right invariant.

(ii) Define X by

$$X(x) := T_p\pi(Z(p)),$$

where $p \in P_x$ is arbitrary. This is well-defined because

$$T_{pg}\pi(Z(pg)) = T_{pg}\pi(T_pR_g(Z(p))) = T_p\pi(Z(p)) = X(p),$$

and smooth because we may set $p = s(x)$ for a local section s of P. $X^* = Z$ holds by definition.

(iii) The first two rules are immediate from (3.3.1). Moreover, since $X^* \sim_\pi X$, [9, 4.4] and (3.1.5) imply

$$T\pi(\mathrm{pr}_h[X^*, Y^*]_p) = T\pi([X^*, Y^*]_p) = [X,Y]_{\pi(p)} = T\pi([X,Y]^*_p),$$

so uniqueness of the horizontal lift implies the third equality.

(iv) Using [9, 17.7] and Theorem 3.1.1 (iii), we calculate

$$\begin{aligned} [\tilde{B}, Z](p) = (L_{\tilde{B}} Z)(p) &= \frac{d}{dt}\Big|_0 (T\mathrm{Fl}^{\tilde{B}}_{-t}(Z(\mathrm{Fl}^{\tilde{B}}_t(p)))) \\ &= \frac{d}{dt}\Big|_0 (TR_{\exp(-tB)}(Z(p \cdot \exp(tB)))). \end{aligned}$$

Here $Z(p \cdot \exp(tB)) \in Th_{p \cdot \exp(tB)}P$ since Z is horizontal, so by right invariance of the connection $TR_{\exp(-tB)}(Z(p \cdot \exp(tB))) \in Th_pP$ for all t. Consequently, so is the above limit, i.e., $[\tilde{B}, Z]$ is horizontal. If $Z = X^*$, then Z is right invariant by (i). In this case, $TR_{\exp(-tB)}(Z(p \cdot \exp(tB))) \equiv Z(p)$ for all t, so $[\tilde{B}, Z](p) = 0$. □

In what follows, by a *path* we will always mean a piecewise smooth map from some interval into a manifold.

3.3.3 Definition. *A path* $\gamma^* : I \to P$ *is called* horizontal lift *of a path* $\gamma : I \to M$ *if*

(i) $\pi(\gamma^*(t)) = \gamma(t)$ *for all* $t \in I$, *and*

(ii) $\dot{\gamma}^*(t)$ *is horizontal for each* $t \in I$.

For the proof of the next theorem we need the following auxiliary result:

3.3.4 Lemma. *Let* G *be a Lie group and* $v : [0,1] \to \mathfrak{g}$ *a continuous (resp. smooth) curve. Then there exist unique* C^1 *(resp.* $\mathcal{C}^\infty$*) curves* $g, a : [0,1] \to G$ *that solve the following ODEs:*

$$\dot{g}(t) = TL_{g(t)}(v(t)),\ \ g(0) = e$$
$$\dot{a}(t) = TR_{a(t)}(v(t)),\ \ a(0) = e.$$

Proof. See the Appendix. □

3.3.5 Theorem. *Let* $\gamma : I \to M$ *be a path in* M, $t_0 \in I$ *(*I *compact) and* $u \in P_{\gamma(t_0)}$. *Then there exists a unique horizontal lift* γ_u^* *of* γ *with* $\gamma_u^*(t_0) = u$.

Proof. We may without loss of generality assume that $I = [0,1]$ and $t_0 = 0$. Since P is locally trivial, there exists a path $\delta : I \to P$ with $\delta(0) = u$ and $\pi \circ \delta = \gamma$: for $P \overset{\phi}{\cong} U \times G$, $\phi(u) = (x_0, g_0)$ we may simply set $\delta(t) = \phi^{-1}(\gamma(t), g_0)$. Now cover P by such trivializing sets and patch δ together (which is unproblematic since we are only seeking a piecewise smooth path). Our task now is to modify δ in such a way as to make it horizontal. To do so we look for a path $g : I \to G$ such that $\gamma_u^*(t) := \delta(t) \cdot g(t)$ becomes horizontal, i.e., such that $A(\dot{\gamma}_u^*(t)) = 0$ for all t. According to Lemma 1.2.3 we therefore require

$$0 = A\big(TR_{g(t)}\dot{\delta}(t) + (TL_{g(t)^{-1}}\dot{g}(t))^{\sim}(\gamma_u^*(t))\big) = \mathrm{Ad}(g(t)^{-1})A(\dot{\delta}(t)) + TL_{g(t)^{-1}}\dot{g}(t).$$

Recalling that $\mathrm{Ad}(g^{-1}) = TL_{g^{-1}} \circ TR_g$, this is equivalent to

$$0 = TR_{g(t)}A(\dot{\delta}(t)) + \dot{g}(t).$$

Now consider the piecewise smooth curve $Y(t) := -A(\dot{\delta}(t)) : I \to \mathfrak{g}$. By Lemma 3.3.4 there exists a unique path $g : I \to G$ with $g(0) = e$ and $\dot{g}(t) = TR_{g(t)}(Y(t)) = -TR_{g(t)}A(\dot{\delta}(t))$, so we indeed obtain a horizontal lift γ^* of γ.

To show uniqueness, suppose that γ_u° is another horizontal lift of γ with $\gamma_u^\circ(0) = u$. Since $\gamma_u^*(t)$ and $\gamma_u^\circ(t)$ lie in the same fiber $P_{\gamma(t)}$, there is a unique path $t \mapsto g(t)$ in G such that $\gamma_u^*(t) = \gamma_u^\circ(t) \cdot g(t)$ for all t, and g is piecewise smooth since both $\gamma_u^*(t)$ and $\gamma_u^\circ(t)$ are. In particular, $g(0) = e$. Differentiating and using Lemma 1.2.3 we obtain

$$\dot{\gamma}_u^*(t) = TR_{g(t)}\dot{\gamma}_u^\circ(t) + (TL_{g(t)^{-1}}\dot{g}(t))^{\sim}(\gamma_u^*(t)).$$

Since both $\gamma_u^*(t)$ and $TR_{g(t)}\dot{\gamma}_u^\circ(t)$ are horizontal, while $(TL_{g(t)^{-1}}\dot{g}(t))^{\sim}(\gamma_u^*(t))$ is vertical, the latter must vanish, so in fact $\dot{g}(t) = 0$ for all t. This implies that $g(t) = g(0) = e$ for all t and hence that $\gamma_u^* = \gamma_u^\circ$. □

By using horizontal lifts we can now construct maps between the fibers in a bundle:

3.3.6 Definition. *Let* $\gamma : [a,b] \to M$ *be a path in* M. *The map*

$$\begin{aligned} \mathcal{P}_\gamma^A : P_{\gamma(a)} &\to P_{\gamma(b)} \\ u &\mapsto \gamma_u^*(b) \end{aligned}$$

is called parallel transport *in* P *along* γ *with respect to* A.

Uniqueness of horizontal lifts implies that the lift of a reparametrization of γ is given by the same reparametrization of the lift. Therefore parallel transport is independent of the chosen parametrization of γ.

Let us briefly recall the standard operations on paths known from homotopy theory. Given $\gamma : [a, b] \to M$ a path from x to y and $\mu : [c, d] \to M$ a path from y to z, the concatenation $\mu * \gamma : [0, 1] \to M$ is given by

$$(\mu * \gamma)(t) := \begin{cases} \gamma(a + 2t(b - a)), & t \in [0, 1/2] \\ \mu(c + (2t - 1)(d - c)) & t \in [1/2, 1]. \end{cases}$$

By $\gamma^- : [0, 1] \to M$ we denote the inverse path $\gamma^-(t) = \gamma(b - t(b - a))$.

3.3.7 Theorem.

(i) With γ, μ as above, $\mathcal{P}^A_{\mu*\gamma} = \mathcal{P}^A_\mu \circ \mathcal{P}^A_\gamma$.

(ii) Parallel transport $\mathcal{P}^A_\gamma$ is a diffeomorphism from $P_{\gamma(a)} \cong G$ to $P_{\gamma(b)} \cong G$ with inverse $(\mathcal{P}^A_\gamma)^{-1} = \mathcal{P}^A_{\gamma^-}$.

(iii) $\mathcal{P}^A_\gamma$ is G-equivariant: $\mathcal{P}^A_\gamma \circ R_g = R_g \circ \mathcal{P}^A_\gamma$ for all $g \in G$.

Proof. (i) is clear from the uniqueness part of Theorem 3.3.5.

(ii) In any trivializing neighborhood $U \subseteq M$ it follows from standard ODE results that since δ depends smoothly on $u \in P_{\gamma(a)}$, so does g, hence $\mathcal{P}^A_\gamma : P_{\gamma(a)} \to P_{\gamma(t_0)}$ is smooth when $\gamma([0, t_0])$ is contained in U. Together with (i), smoothness of $\mathcal{P}^A_\gamma$ follows by covering $\gamma([0, 1])$ by finitely many such neighborhoods. The form of the inverse is again clear by uniqueness of parallel transport, and also gives a smooth map.

(iii) We have to show that $R_g\gamma^*_u$ is the horizontal lift of γ through $R_g u$. Indeed, $\pi(R_g\gamma^*_u(t)) = \pi(\gamma^*_u(t)) = \gamma(t)$, $R_g\gamma^*_u(0) = R_g(u)$, and

$$(R_g\gamma^*_u)^{\cdot}(t) = TR_g(\dot\gamma^*_u(t)) \in Th_{\gamma^*_u(t)\cdot g}.$$

□

3.3.8 Example. Let $(P_0 = M \times G, \mathrm{pr}_1, M, G)$ be the trivial principal fiber bundle over M with the canonical flat connection $Th_{(x,g)}P_0 = T_xM$ and corresponding connection form A_0. Then if γ is a path emanating from $x \in M$, the horizontal lift of γ through (x, g) is given by $\gamma^*(t) = (\gamma(t), g)$. Hence parallel transport along γ from x to y is the map

$$\begin{aligned} \mathcal{P}^{A_0}_\gamma : \{x\} \times G &\to \{y\} \times G \\ (x, g) &\mapsto (y, g), \end{aligned}$$

which is in fact independent of the path γ.

In general, however, parallel transport will depend on the path, and in fact the following result shows that the previous example is the only case in which it doesn't:

3.3.9 Theorem. *Let A be a connection form on (P, π, M, G) and suppose that parallel transport with respect to A does not depend on the path. Then (P, A) is isomorphic to the trivial G principal fiber bundle P_0 with the canonical flat connection A_0, i.e., there is a principal fiber bundle isomorphism $\Phi : P_0 \to P$ such that $\Phi^*A = A_0$, i.e., $T\Phi(ThP_0) = Th_{\Phi(\,.\,)}P$.*

Proof. Let us first verify that the requirements on Φ are equivalent. We have $X \in \ker(\Phi^* A) \Leftrightarrow A(T\Phi(X)) = 0 \Leftrightarrow X \in (T\Phi)^{-1}(\ker(A))$, so (cf. Theorem 3.1.4)

$$T\Phi(ThP_0) = Th_{\Phi(\,.\,)}P \Leftrightarrow A_0 = \Phi^* A.$$

We know from Theorem 2.2.14 that P is trivial if and only if it possesses a global section. We call a section $s : M \to P$ horizontal if $T_x s(T_x M) = Th_{s(x)}P$ for each $x \in M$. Based on this notion, we carry out the proof in two steps:

1.) (P, A) is isomorphic to $(P_0, A_0) \Leftrightarrow$ there exists a global A-horizontal section in P.

$\Rightarrow$: Let $\Phi : P_0 = M \times G \to P$ be an isomorphism with $T\Phi(T_{(x,g)}(M \times \{g\})) = Th_{\Phi(x,g)}P$ for all (x, g). Consider the section

$$\begin{aligned} s : M &\to P \\ x &\mapsto s(x) := \Phi(x, e). \end{aligned}$$

Then s is horizontal because $Ts(T_x M) = T_{(x,e)}\Phi(T_{(x,e)}(M \times \{e\})) = Th_{s(x)}P$.

$\Leftarrow$: Let $s : M \to P$ be a global A-horizontal section and let Φ be the trivialization induced by s according to Theorem 2.2.14:

$$\begin{aligned} \Phi : P_0 = M \times G &\to P \\ (x, g) &\mapsto s(x) \cdot g = R_g(s(x)). \end{aligned}$$

Then Φ is an isomorphism and

$$T_{(x,g)}\Phi(T_{(x,g)}(M \times \{g\})) = TR_g \circ Ts(T_x M) = TR_g(Th_{s(x)}P) = Th_{s(x)\cdot g}P.$$

2.) Path-Independence of parallel transport implies the existence of a global A-horizontal section.

Fix a point $x_0 \in M$ as well as some $u \in P_{x_0}$ and define

$$s(x) := \mathcal{P}^A_\gamma(u) = \gamma^*_u(1),$$

where $\gamma : [0, 1] \to M$ is any path from x_0 to x. To see that s is smooth, consider first the case where x varies in a trivializing neighborhood of x_0. Then γ can be chosen to depend smoothly on x (e.g. by taking a straight line connnecting x_0 to x in a chart). Therefore the proof of Theorem 3.3.5 and ODE theory show that also $\gamma^*_u(1)$ depends smoothly on x. For general x one can find finitely many trivializing neighborhoods such that s can be written as the composition of the smooth maps so constructed. Also, s is horizontal because given $X \in T_x M$ we may pick a smooth curve $\gamma : [0, 1] \to M$ from x_0 to x with $\gamma'(1) = X$, and calculate:

$$Ts(X) = \frac{d}{dt}\Big|_{t=1}(s(\gamma(t))) = \frac{d}{dt}\Big|_{t=1}(\gamma^*_u(t)) \in Th_{s(x)}P.$$

Since $T_x s$ is injective (due to $\pi \circ s = \mathrm{id}_M$) and $\dim Th_{s(x)}P = \dim T_x M$, $T_x s(T_x M) = Th_{s(x)}P$. □

Parallel transport in a principal fiber bundle P induces a notion of parallel transport also in any fiber bundle associated to P: Let (P, π, M, G) be equipped with a connection form A, let $[F, G]$ be a Lie transformation group, and denote by $E := P \times_G F$ the corresponding associated fiber bundle. Let $\gamma : [a, b] \to M$ be a path in M. Then

$$\begin{aligned} \mathcal{P}^{E,A}_\gamma : E_{\gamma(a)} &\to E_{\gamma(b)} \\ [p, v] &\mapsto [\mathcal{P}^A_\gamma(p), v] \end{aligned} \tag{3.3.2}$$

is well-defined: By Theorem 3.3.7 (iii), $[\mathcal{P}^A_\gamma(pg), g^{-1}v] = [\mathcal{P}^A_\gamma(p)g, g^{-1}v] = [\mathcal{P}^A_\gamma(p), v]$. The map $\mathcal{P}^{E,A}_\gamma$ is called the parallel transport on E induced by A.

Using the notation from the proof of Theorem 2.3.1, in terms of bundle charts (U_1, ψ_{U_1}) around $\gamma(a)$ and (U_2, ψ_{U_2}) around $\gamma(b)$, and for an arbitrary $p \in P_{\gamma(a)}$, we have

$$\begin{aligned}(\gamma(a), v) &\xrightarrow{\psi_{U_1}^{-1}} [p, \varphi_{U_1}(p)^{-1} \cdot v] \xrightarrow{\mathcal{P}^{E,A}_\gamma} [\mathcal{P}^A_\gamma(p), \varphi_{U_1}(p)^{-1} \cdot v] \\ &\xrightarrow{\psi_{U_2}} (\underbrace{\pi(\mathcal{P}^A_\gamma(p))}_{=\gamma(b)}, \varphi_{U_2}(\mathcal{P}^A_\gamma(p)) \cdot \varphi_{U_1}(p)^{-1} \cdot v).\end{aligned}$$

So parallel transport is smooth (being expressible in terms of the action of G on F), indeed a diffeomorphism since its inverse is of the same type, and is even a linear isomorphism if E is a vector bundle. Furthermore, if γ^* is any horizontal lift of γ then using the fiber diffeomorphisms from (2.3.7) we have

$$\mathcal{P}^{E,A}_\gamma = [\gamma^*(b)] \circ [\gamma^*(a)]^{-1}. \tag{3.3.3}$$

Indeed, since $\mathcal{P}^A_\gamma(\gamma^*(a)) = \gamma^*(b)$ and any element of $E_{\gamma(a)}$ is of the form $[\gamma^*(a), v]$ for some v, we have

$$\begin{aligned}[\gamma^*(b)] \circ [\gamma^*(a)]^{-1}([\gamma^*(a), v]) &= [\gamma^*(b)](v) = [\gamma^*(b), v] = [\mathcal{P}^A_\gamma(\gamma^*(a)), v] \\ &= \mathcal{P}^{E,A}_\gamma([\gamma^*(a), v]).\end{aligned}$$

3.4 The absolute differential of a connection

Throughout this section let (P, π, M, G) be a principal fiber bundle with connection form A, let $\rho : G \to \mathrm{GL}(V)$ be a representation of G and let $E = P \times_G V$ be the corresponding associated vector bundle.

3.4.1 Definition. *A linear map*

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

is called a covariant derivative *on E if*

$$\nabla(fe) = df \otimes e + f \cdot \nabla e \qquad (f \in \mathcal{C}^\infty(M),\ e \in \Gamma(E)).$$

Our aim is to lay out how to employ connections on principal fiber bundles to define covariant derivatives on associated vector bundles. Indeed we will use A to define linear operators

$$d_A : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$$

for each $k \geq 0$ and then show that for $k = 0$ we obtain a covariant derivative.

According to Theorem 3.2.3, E-valued differential forms on M correspond precisely to the horizontal differential forms of type ρ on P with values in V. Thus we are looking for differential operators that preserve these properties. Recall from [9, Sec. 10] the definition of the exterior derivative of V-valued forms:

$$\begin{aligned}d : \Omega^k(P, V) &\to \Omega^{k+1}(P, V) \\ \omega &\to d\omega,\end{aligned}$$

where

$$
\begin{aligned}
d\omega(X_0,\dots,X_k) &:= \sum_{i=0}^{k}(-1)^i X_i(\omega(X_0,\dots,\hat{X}_i,\dots,X_k)) \\
&\quad + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots X_k),
\end{aligned}
\tag{3.4.1}
$$

where $X_0,\dots,X_k$ are vector fields on P. Then $d(h^*\omega) = h^*(d\omega)$ for $h : Q \to P$ smooth, and $d^2 = d \circ d = 0$.

3.4.2 Example. In general, the differential of a horizontal k-form need not be horizontal: Let $M = \mathbb{R}$, $G = (\mathbb{R},+)$, and P the trivial bundle $\mathbb{R}\times\mathbb{R}$ with the canonical flat connection. Given $f \in \mathcal{C}^\infty(P,\mathbb{R})$, $\omega_{(t,s)} = f(t,s)dt$ defines a horizontal 1-form: if $X = h(t,s)\partial_s$ is a vertical vector field then $\omega(X) = f \cdot h dt(\partial_s) = 0$. However, the differential $d\omega = \frac{\partial f}{\partial s} ds \wedge dt$ is horizontal only if $d\omega = 0$, i.e., if and only if ω is closed.

We therefore use the connection to modify d in the following way:

3.4.3 Definition. *The linear map*

$$
\begin{aligned}
D_A : \Omega^k(P,V) &\to \Omega^{k+1}(P,V) \\
(D_A\omega)_p(t_0,\dots,t_k) &:= d\omega_p(\mathrm{pr}_h t_0,\dots,\mathrm{pr}_h t_k) \qquad (t_i \in T_pP)
\end{aligned}
\tag{3.4.2}
$$

is called the absolute differential *defined by A.*

This modification indeed does what it should:

3.4.4 Theorem. *The absolute differential maps horizontal differential forms of type ρ into horizontal forms of type ρ again:*

$$
D_A : \Omega^k_{\mathrm{hor}}(P,V)^{(G,\rho)} \to \Omega^{k+1}_{\mathrm{hor}}(P,V)^{(G,\rho)}
$$

Moreover, for any $\omega \in \Omega^k_{\mathrm{hor}}(P,V)^{(G,\rho)}$,

$$
D_A\omega = d\omega + \rho_*(A) \wedge \omega, \tag{3.4.3}
$$

where

$$
(\rho_*(A) \wedge \omega)(t_0,\dots,t_k) := \sum_{i=0}^{k}(-1)^i \rho_*(A(t_i))(\omega(t_0,\dots,\widehat{t_i},\dots,t_k)) \tag{3.4.4}
$$

(with $\rho_* = T_e\rho : \mathfrak{g} \to L(V,V)$).[2]

Note: More generally, for $\sigma \in \Omega^k(P,\mathfrak{g})$, $\omega \in \Omega^l_{\mathrm{hor}}(P,V)^{(G,\rho)}$ one sets

$$
\begin{aligned}
&(\rho_*(\sigma) \wedge \omega)_x(t_1,\dots,t_{k+l}) \\
&\quad := \frac{1}{k!l!} \sum_{\tau \in \mathcal{S}_{k+l}} \mathrm{sign}(\tau)\rho_*(\sigma(t_{\tau(1)},\dots,t_{\tau(k)}))\big(\omega(t_{\tau(k+1)},\dots,t_{\tau(k+l)})\big).
\end{aligned}
\tag{3.4.5}
$$

An analogous formula is used to define $\alpha \wedge \omega$ for general $\alpha \in \Omega^k(P, L(V,V))$.

[2] In [9], $T_e\rho$ was denoted by ρ', but here we follow the convention from [1].

Proof. Let $\omega \in \Omega^k_{\mathrm{hor}}(P,V)^{(G,\rho)}$. Then $D_A\omega$ is horizontal because $\mathrm{pr}_h t = 0$ when t is vertical. Concerning G-invariance, we have

$$\begin{aligned}(R_g^* D_A\omega)(t_0,\dots,t_k) &= (D_A\omega)(TR_g t_0,\dots,TR_g t_k) = d\omega(\mathrm{pr}_h TR_g t_0,\dots,\mathrm{pr}_h TR_g t_k)\\ &\underset{(3.1.4)}{=} d\omega(TR_g \mathrm{pr}_h t_0,\dots,TR_g \mathrm{pr}_h t_k)\\ &= (R_g^* d\omega)(\mathrm{pr}_h t_0,\dots,\mathrm{pr}_h t_k) = d(R_g^*\omega)(\mathrm{pr}_h t_0,\dots,\mathrm{pr}_h t_k)\\ &= d(\rho(g^{-1})\circ\omega)(\mathrm{pr}_h t_0,\dots,\mathrm{pr}_h t_k)\\ &= \rho(g^{-1})\circ d\omega(\mathrm{pr}_h t_0,\dots,\mathrm{pr}_h t_k)\\ &= \rho(g^{-1})((D_A\omega)(t_0,\dots,t_k)).\end{aligned}$$

Thus $D_A\omega$ is of type ρ, and it only remains to prove (3.4.3). By linearity and the fact that any tangent vector can be decomposed into a horizontal and a vertical component, it suffices to examine the case where each t_i is either horizontal or vertical. We distinguish the following cases:

1.) All t_i are horizontal. In this case, $D_A\omega(t_0,\dots,t_k) = d\omega(t_0,\dots,t_k)$ and $A(t_i) = 0$ for all i.

2.) At least two t_i are vertical. Then since both ω and $D_A\omega$ are horizontal,

$$(D_A\omega)(t_0,\dots,t_k) = 0, \quad \text{and} \quad (\rho_*(A)\wedge\omega)(t_0,\dots,t_k) = 0,$$

so we have to show that also $d\omega(t_0,\dots,t_k) = 0$. For this, note that by Theorem 3.1.1, any $t \in Tv_pP$ can be written in the form $t = \tilde{X}(p)$ for some $X \in \mathfrak{g}$. Moreover, by Theorem 1.2.2, the commutator $[\tilde{X},\tilde{Y}] = [X,Y]^\sim$ of two fundamental vector fields is again vertical. Thus any summand in (3.4.1) has in the argument of ω at least one vertical vector and thereby vanishes.

3.) One t_i is vertical, all others are horizontal. Since both sides of (3.4.3) are skew symmetric, we may suppose that t_0 is vertical and $t_1,\dots,t_k$ are horizontal. Pick $X \in \mathfrak{g}$ with $t_0 = \tilde{X}(p)$ and $V_1,\dots,V_k \in \mathfrak{X}(M)$ with horizontal lifts $V_i^*(p) = t_i$. Then

$$\begin{aligned}(D_A\omega)(t_0,\dots,t_k) &= d\omega(\mathrm{pr}_h t_0, t_1,\dots,t_k) = 0,\\ (\rho_*(A)\wedge\omega)(t_0,\dots,t_k) &= \rho_*(A(t_0))(\omega(t_1,\dots,t_k)) = \rho_*(X)(\omega(t_1,\dots,t_k)),\end{aligned}$$

and

$$\begin{aligned}(d\omega)(t_0,\dots,t_k) &= \tilde{X}(\omega(V_1^*,\dots,V_k^*))(p)\\ &\qquad + \sum_{i=1}^k (-1)^i \omega([\tilde{X},V_i^*],V_1^*,\dots,\widehat{V_i^*},\dots,V_k^*)(p).\end{aligned}$$

Here, $[\tilde{X},V_i^*] = 0$ by Theorem 3.3.2, and applying Theorem 3.1.1 (iii) we obtain

$$\begin{aligned}d\omega(t_0,\dots,t_k) &= \tilde{X}(\omega(V_1^*,\dots,V_k^*))(p) = \frac{d}{dt}\Big|_0 (\mathrm{Fl}_t^{\tilde{X}})^*(\omega(V_1^*,\dots,V_k^*))(p)\\ &= \frac{d}{dt}\Big|_0 \big((R^*_{\exp(tX)}\omega)|_p((R^*_{\exp(tX)}V_1^*)(p),\dots,(R^*_{\exp(tX)}V_k^*)(p))\big)\\ &= \frac{d}{dt}\Big|_0 \big((R^*_{\exp(tX)}\omega)|_p(t_1,\dots,t_k)\big),\end{aligned}$$

where we used right-invariance of the V_i^* in the last step. Since ω is of type ρ, using [9, 8.8] we arrive at

$$\begin{aligned}d\omega(t_0,\dots,t_k) &= \frac{d}{dt}\Big|_0 \big((R^*_{\exp(tX)}\omega)|_p(t_1,\dots,t_k)\big) = \frac{d}{dt}\Big|_0 \big(\rho(\exp(-tX))\omega(t_1,\dots,t_k)\big)\\ &= \frac{d}{dt}\Big|_0 \big(e^{-t\rho_*(X)}\omega(t_1,\dots,t_k)\big) = -\rho_*(X)(\omega(t_1,\dots,t_k)).\end{aligned}$$

□

We now want to use the isomorphism $\Phi \equiv \Phi_k : \Omega^k_{\text{hor}}(P,V)^{(G,\rho)} \to \Omega^k(M,E)$ from Theorem 3.2.3 to define a corresponding map on $\Omega^k(M,E)$, as follows:

$$\begin{array}{ccc} \Omega^k(M,E) & \xrightarrow{d_A} & \Omega^{k+1}(M,E) \\ {\scriptstyle \omega\mapsto\bar\omega}\big\downarrow\big\uparrow{\scriptstyle \Phi_k} & & {\scriptstyle \sigma\mapsto\bar\sigma}\big\downarrow\big\uparrow{\scriptstyle \Phi_{k+1}} \\ \Omega^k_{\text{hor}}(P,V)^{(G,\rho)} & \xrightarrow{D_A} & \Omega^{k+1}_{\text{hor}}(P,V)^{(G,\rho)} \end{array}$$

Thus

$$\begin{aligned} d_A : \Omega^k(M,E) &\to \Omega^{k+1}(M,E) \\ \omega &\mapsto d_A\omega, \qquad \overline{d_A\omega} := D_A\bar\omega. \end{aligned} \tag{3.4.6}$$

Now let $p \in P_x$, $t_i \in T_xM$, and $t_i^* \in T_pP$ the horizontal lift of t_i. Then according to (3.2.3) we have

$$\begin{aligned} (d_A\omega)_x(t_0,\dots,t_k) &= [p, \overline{d_A\omega}_p(t_0^*,\dots,t_k^*)] = [p,(D_A\bar\omega)_p(t_0^*,\dots,t_k^*)] \\ &\underset{(3.4.2)}{=} [p, d\bar\omega_p(t_0^*,\dots,t_k^*)]. \end{aligned} \tag{3.4.7}$$

Moreover, if $s : U \to P$ is a local section around x, then (3.2.2) implies

$$(d_A\omega)_x(t_0,\dots,t_k) = [s(x),(D_A\bar\omega)_{s(x)}(Ts(t_0),\dots,Ts(t_k))]. \tag{3.4.8}$$

One may reasonably expect that in the case of the trivial representation $\rho : G \to \mathrm{GL}(V)$, d_A should reduce simply to d, for any connection form A. To actually prove this, we first show an auxiliary result:

3.4.5 Lemma. *Let $f \in \mathcal{C}^\infty(P)$ be right invariant, $f \circ R_g = f$ for all $g \in G$. Then $f = \tilde f \circ \pi$ for a unique $\tilde f \in \mathcal{C}^\infty(M)$, and if $X \in \mathfrak{X}(M)$ with horizontal lift X^* we have*

$$X^*(f) = X(\tilde f) \circ \pi.$$

Proof. $\tilde f$ is the projection of f under π, cf. Remark 2.2.2 and [9, 15.13]. Moreover, for $x = \pi(p)$ we have

$$X^*(f)|_p = df(X_p^*) = T_x\tilde f \circ T_p\pi(X_p^*) = T_x\tilde f(X_x) = X(\tilde f)|_x.$$

□

3.4.6 Remark. Let $\rho : G \to \mathrm{GL}(V)$ be the trivial representation $g \mapsto \mathrm{id}_V$ for all $g \in G$. Then $\rho_* = T_e\rho = 0$, and being of type ρ means being right-invariant, so

$$\Omega^k_{\text{hor}}(P,V)^{(G,\rho)} = \{\bar\omega \in \Omega^k(P,V) \mid \bar\omega \text{ right invariant and horizontal}\},$$

and $D_A = d$ by (3.4.3). A typical vector bundle chart for $E = P \times_G V$ is given by (2.3.2):

$$\psi_U : [p,v] \mapsto (\pi(p), \rho(\varphi_U(p)) \cdot v) = (\pi(p), v),$$

so in fact $[p,v] \mapsto (\pi(p),v)$ is a global vector bundle isomorphism $E \to \underline{V} = M \times V$. Thus we can identify differential forms with values in E with standard differential forms valued in V:

$$\Omega^k(M,E) \cong \Omega^k(M,V).$$

Using this identification, (3.2.3) becomes

$$\begin{aligned} \omega_x(t_1,\dots,t_k) &= [p,\bar\omega_p(X_1,\dots,X_k)] \\ &= (x,\bar\omega_p(X_1,\dots,X_k)) \cong \bar\omega_p(X_1,\dots,X_k), \end{aligned} \tag{3.4.9}$$

for $x = \pi(p)$ and $t_i \in T_xM$ with horizontal lifts X_i. Now extend the t_i to local vector fields on M and let $X_i = t_i^*$ $(i = 0, \dots, k)$ be their horizontal lifts. Then the smooth function $p \mapsto \bar{\omega}_p(X_1, \dots, X_k)$ is right invariant:

$$\begin{aligned}\bar{\omega}(X_1, \dots, X_k)|_{pg} &= \bar{\omega}_{pg}(X_1(pg), \dots, X_k(pg)) = \omega_x(t_1, \dots, t_k) \\ &= \bar{\omega}(X_1, \dots, X_k)|_p,\end{aligned} \tag{3.4.10}$$

since $T\pi(X_i(pg)) = T\pi(TR_g(X_i(p))) = t_i$ for all i. Now by (3.4.1) we have

$$\begin{aligned}d\bar{\omega}(X_0, \dots, X_k) = &\sum_{i=0}^{k}(-1)^i X_i(\bar{\omega}(X_0, \dots, \hat{X}_i, \dots, X_k)) \\ &+ \sum_{i<j}(-1)^{i+j}\bar{\omega}([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots X_k),\end{aligned}$$

In this expression, $X_i(\bar{\omega}(X_0, \dots, \hat{X}_i, \dots, X_k)) = t_i(\omega(t_0, \dots, \hat{t}_i, \dots, t_k))$ due to (3.4.10) and Lemma 3.4.5. Also, $[t_i, t_j]^* = \mathrm{pr}_h([X_i, X_j])$ by Theorem 3.3.2, so

$$\begin{aligned}\bar{\omega}([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots X_k) &= \bar{\omega}(\mathrm{pr}_h[X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots X_k) \\ &= \bar{\omega}([t_i, t_j]^*, t_0^*, \dots, \hat{t_i^*}, \dots, t_k^*) \underset{(3.4.9)}{=} \omega([t_i, t_j], t_0, \dots, \hat{t}_i, \dots, t_k).\end{aligned}$$

Inserting this above and again using (3.4.1) we conclude that

$$d\omega(t_0, \dots, t_k)|_x = d\bar{\omega}(X_0, \dots, X_k)|_p.$$

Combining this with (3.4.9) for $d\omega$ we get

$$\overline{d\omega}(X_0, \dots, X_k)|_p = d\omega(t_0, \dots, t_k)|_x = d\bar{\omega}(X_0, \dots, X_k)|_p,$$

i.e.,

$$d\bar{\omega} = \overline{d\omega}. \tag{3.4.11}$$

In particular, $D_A\bar{\omega} = d\bar{\omega} = \overline{d\omega}$, so by (3.4.8) for a local section s of P we can calculate

$$\begin{aligned}(d_A\omega)_x(t_0, \dots, t_k) &= [s(x), (D_A\bar{\omega})_{s(x)}(Ts(t_0), \dots, Ts(t_k))] \\ &= [s(x), \overline{d\omega}_{s(x)}(Ts(t_0), \dots, Ts(t_k))] \underset{(3.2.2)}{=} d\omega_x(t_0, \dots, t_k).\end{aligned}$$

We conclude that for the trivial representation ρ we have

$$d_A = d : \Omega^k(M, V) \to \Omega^{k+1}(M, V) \tag{3.4.12}$$

for *any* connection A on P.

Although, as we shall see, d_A no longer satisfies $d_A \circ d_A = 0$, we still have the usual product rule:

3.4.7 Theorem. *Let $d_A : \Omega^*(M, E) \to \Omega^{*+1}(M, E)$ be the differential induced by A. Then for $\sigma \in \Omega^k(M)$ and $\omega \in \Omega^l(M, E)$ $(k, l \geq 0)$ we have*

$$d_A(\sigma \wedge \omega) = d\sigma \wedge \omega + (-1)^k\sigma \wedge d_A\omega. \tag{3.4.13}$$

Proof. We have $\sigma \in \Omega^k(M) \cong \Omega^k(M, \underline{\mathbb{R}})$, with $\underline{\mathbb{R}} = M \times \mathbb{R}$ the trivial bundle. This corresponds to the situation from Remark 3.4.6 with $\tilde{\rho} : G \to \mathrm{GL}(\mathbb{R}) = \mathbb{R} \setminus \{0\}$, $g \mapsto 1$ the trivial representation. So there is a corresponding right invariant horizontal k-form $\bar{\sigma} \in \Omega^k(P, \mathbb{R})$. By (3.4.11), $d\bar{\sigma} = \overline{d\sigma}$ and $d\bar{\sigma}$ is horizontal and right invariant since $D_A = d$ preserves these properties by Theorem 3.4.4.

Moreover, by Theorem 3.2.3 $\omega \in \Omega^l(M, E)$ corresponds to $\bar{\omega} \in \Omega^l_{\mathrm{hor}}(P, V)^{(G,\rho)}$, and by (3.2.1) and (3.2.2) we have $\overline{\sigma \wedge \omega} = \bar{\sigma} \wedge \bar{\omega}$. Now (3.4.7) implies

$$\begin{aligned} d_A(\sigma \wedge \omega)_x(t_0, \dots, t_{k+l}) &= [p, d(\overline{\sigma \wedge \omega})_p(t_0^*, \dots, t_{k+l}^*)] = [p, d(\bar{\sigma} \wedge \bar{\omega})_p(t_0^*, \dots, t_{k+l}^*)] \\ &= [p, (d\bar{\sigma} \wedge \bar{\omega} + (-1)^k \bar{\sigma} \wedge d\bar{\omega})_p(t_0^*, \dots, t_{k+l}^*)] \\ &= (d\sigma \wedge \omega + (-1)^k \sigma \wedge d_A\omega)_x(t_0, \dots, t_{k+l}). \end{aligned}$$

□

Let us now focus on the properties of d_A on 0-forms. Since $\Gamma(E) = \Omega^0(M, E)$ and $\Omega^1(M, E) = \Gamma(T^*M \otimes E)$, d_A is a linear operator

$$d_A : \Gamma(E) \to \Gamma(T^*M \otimes E).$$

By Theorem 3.4.7 it satisfies the product rule

$$d_A(fe) = df \otimes e + f d_A e \qquad (f \in \mathcal{C}^\infty(M),\ e \in \Gamma(E)),$$

hence according to Definition 3.4.1 it is a covariant derivative on E.

3.4.8 Definition. *The map*

$$\nabla^A := d_A|_{\Omega^0(M,E)} : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

is called the covariant derivative induced by the connection form A.

Another consequence of Theorem 3.4.7 is that for $\sigma \in \Omega^k(M)$ and $e \in \Gamma(E)$ we have

$$d_A(\sigma \otimes e) = d\sigma \otimes e + (-1)^k \sigma \wedge \nabla^A e. \tag{3.4.14}$$

3.4.9 Theorem. *Let $e \in \Gamma(E)$, $X \in \mathfrak{X}(M)$, $s : U \to P_U$ a local section around $x \in M$, $v \in \mathcal{C}^\infty(U, V)$ such that $e|_U = [s, v]$, and $A^s = A \circ Ts$ the local connection form corresponding to s. Then*

$$(\nabla^A_X e)(x) = [s(x), T_x v(X_x) + \rho_*(A^s(X_x))v(x)] \in E_x. \tag{3.4.15}$$

Proof. By the proof of Theorem 2.3.4 we have $e(x) = [p, \bar{e}(p)]$, where $p \in P_x$ is arbitrary and $\bar{e} \in \mathcal{C}^\infty(P, V)^G$. Setting $p := s(x)$ we get $v(x) = \bar{e}(s(x))$, which is smooth and clearly unique with $e|_U = [s, v]$. Note also that by the footnote following (3.2.2), $\bar{e}$ is precisely the map corresponding to e in that equation. Thus

$$\begin{aligned} (\nabla^A_X e)(x) = (d_A e)_x(X_x) &\underset{(3.4.8)}{=} [s(x), (D_A\bar{e})(T_x s(X_x))] \\ &\underset{(3.4.3)}{=} [s(x), d\bar{e}(T_x s(X_x)) + \rho_*(A(T_x s(X_x)))\bar{e}(s(x))] \\ &= [s(x), T(\bar{e} \circ s)_x(X_x) + \rho_*(A^s(X_x))(\bar{e} \circ s)(x)] \\ &= [s(x), T_x v(X_x) + \rho_*(A^s(X_x))v(x)]. \end{aligned}$$

□

We saw in Theorem 2.4.10 that any G-invariant scalar product $\langle\, . \,,\, . \,\rangle_V$ on V induces a bundle metric $\langle\, . \,,\, . \,\rangle_E$ on $E = P \times_G V$ via

$$\langle e, \tilde{e} \rangle_E := \langle v, \tilde{v} \rangle_V \qquad \text{for } e = [p, v],\ \tilde{e} = [p, \tilde{v}].$$

Then the covariant derivative ∇^A is *metric* with respect to this bundle metric:

3.4.10 Theorem. *Let $\langle\,.\,,\,.\,\rangle_E$ be a bundle metric on $E = P \times_G V$ induced by a G-invariant scalar product $\langle\,.\,,\,.\,\rangle_V$ on V. Then for any $e, \tilde{e} \in \Gamma(E)$ and any $X \in \mathfrak{X}(M)$ we have*

$$X(\langle e, \tilde{e}\rangle_E) = \langle \nabla^A_X e, \tilde{e}\rangle_E + \langle e, \nabla^A_X \tilde{e}\rangle_E.$$

Proof. As in the proof of Theorem 3.4.9 we represent the sections in the form $e = [s, v]$, $\tilde{e} = [s, \tilde{v}]$. Then by (3.4.15) (and using [9, 23.9]) we get

$$\begin{aligned}\langle \nabla^A_X e, \tilde{e}\rangle_E &= \langle X(v) + \rho_*(A^s(X))v, \tilde{v}\rangle_V = \langle X(v), \tilde{v}\rangle_V - \langle v, \rho_*(A^s(X))\tilde{v}\rangle_V \\ &= X(\langle v, \tilde{v}\rangle_V) - \langle v, X(\tilde{v}) + \rho_*(A^s(X))\tilde{v}\rangle = X(\langle e, \tilde{e}\rangle_E) - \langle e, \nabla^A_X \tilde{e}\rangle_E.\end{aligned}$$

□

In the remainder of this section we want to clarify the relation between various notions of covariant derivatives. We begin by establishing a relationship between covariant derivatives and parallel transport. Let γ be a path in M starting at $\gamma(0) = x$ and let

$$\mathcal{P}^{E,A}_{t,0} : E_{\gamma(t)} \to E_{\gamma(0)}$$

be the parallel transport induced by A on E along the inverse curve γ^- (cf. (3.3.2)).

3.4.11 Theorem. *Let $e \in \Gamma(E)$, $X \in \mathfrak{X}(M)$ and γ a path in M with $\gamma(0) = x$ and $\dot{\gamma}(0) = X_x$. Then*

$$(\nabla^A_X e)(x) = \frac{d}{dt}\Big|_0 \big(\mathcal{P}^{E,A}_{t,0}(e(\gamma(t)))\big).$$

Proof. Let γ^* be a horizontal lift of γ. Then by (3.3.3) we have

$$\mathcal{P}^{E,A}_{t,0} = [\gamma^*(0)] \circ [\gamma^*(t)]^{-1}.$$

By Theorem 2.3.4, $e(x) = [p, \bar{e}(p)]$ for any $p \in P_x$, so $e(\gamma(t)) = [\gamma^*(t), \bar{e}(\gamma^*(t))]$, implying $[\gamma^*(t)]^{-1}(e(\gamma(t))) = \bar{e}(\gamma^*(t))$. Using this and the fact that $[\gamma^*(0)]$ is a linear isomorphism we get

$$\begin{aligned}\frac{d}{dt}\Big|_0 \big(\mathcal{P}^{E,A}_{t,0}(e(\gamma(t)))\big) &= \frac{d}{dt}\Big|_0 \big([\gamma^*(0)] \circ [\gamma^*(t)]^{-1}(e(\gamma(t)))\big) = [\gamma^*(0)](T\bar{e}(\dot{\gamma}^*(0))) \\ &= [\gamma^*(0), T\bar{e}(\dot{\gamma}^*(0))] = [p, T\bar{e}(X^*_p)] \quad (\text{with } p := \gamma^*(0)) \\ &\underset{(3.4.7)}{=} (d_A e)_x(X_x) = (\nabla^A_X e)(x).\end{aligned}$$

□

For a general vector bundle E over M with covariant derivative ∇ there is a 'standard' notion of parallel transport induced by ∇ as follows: Let $\gamma : [a, b] \to M$ be a path in M with $x = \gamma(a)$ and let $e \in E_x$. Then due to the standard existence and uniqueness result for linear ODEs there is a unique section φ_e of E over γ such that

$$\frac{\nabla \varphi_e}{dt} = 0, \qquad \varphi_e(a) = e. \tag{3.4.16}$$

Then we call the map

$$\begin{aligned}\mathcal{P}^{\nabla}_{\gamma} : E_{\gamma(a)} &\to E_{\gamma(b)} \\ e &\mapsto \varphi_e(b)\end{aligned} \tag{3.4.17}$$

the parallel transport induced by ∇ on E. From the theory of ODEs it follows that $\mathcal{P}^{\nabla}_{\gamma}$ is a linear isomorphism between the fibers. Moreover, we have the following compatibility result:

3.4.12 Theorem. *Let $E = P \times_G V$ be associated to P via a representation $\rho : G \to \mathrm{GL}(V)$. Let A be a connection form on P and let ∇^A be the covariant derivative induced on E by A according to Definition 3.4.8. Then the corresponding parallel transports coincide:*

$$\mathcal{P}_\gamma^{E,A} = \mathcal{P}_\gamma^{\nabla^A}$$

for each path $\gamma : [a,b] \to M$.

Proof. With φ_e as above, by Theorem 3.4.11 we have for any $t \in [a,b]$:

$$0 = (\nabla^A_{\dot\gamma(t)}\varphi_e)(t) = \frac{d}{ds}\Big|_0 \big((\mathcal{P}^{E,A}_{\gamma|_{[t,t+s]}})^{-1}(\varphi_e(t+s))\big).$$

Here, $(\mathcal{P}^{E,A}_{\gamma|_{[t,t+s]}})^{-1} = \mathcal{P}^{E,A}_{\gamma(t)} \circ \mathcal{P}^{E,A}_{\gamma^-(t+s)}$, so

$$0 = \mathcal{P}^{E,A}_{\gamma(t)} \frac{d}{ds}\Big|_0 \big(\mathcal{P}^{E,A}_{\gamma^-(t+s)}(\varphi_e(t+s))\big).$$

This means that the curve $t \mapsto \mathcal{P}^{E,A}_{\gamma^-(t)}(\varphi_e(t)) \in E_{\gamma(a)}$ is constant on $[a,b]$. Consequently,

$$\mathcal{P}^{E,A}_{\gamma^-(t)}(\varphi_e(t)) = \mathcal{P}^{E,A}_{\gamma^-(a)}(\varphi_e(a)) = \mathrm{id}_{E_{\gamma(a)}}(\varphi_e(a)) = \varphi_e(a) = e,$$

so that $\varphi_e(t) = \mathcal{P}^{E,A}_{\gamma(t)}(e)$, as claimed. □

3.4.13 Example. Let M be an n-dimensional manifold equipped with a covariant derivative ∇. This means that ∇ is a covariant derivative on the vector bundle TM over M, hence defines a notion of parallel transport as discussed before Theorem 3.4.12.

We know from Example 2.4.6 that TM is associated to the frame bundle, $TM = \mathrm{GL}(M) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n$. Moreover, by Example 3.1.10, the covariant derivative ∇ induces a connection form A^∇ on $\mathrm{GL}(M)$ and (3.1.19) shows that $\nabla = \nabla^{A^\nabla}$. Note, however, that we do not (yet) know that the covariant derivative that was denoted by ∇^A in (3.1.19) is indeed the same as that given by Definition 3.4.8. We will show this in Remark 3.4.14 in a more general context. Once we know this, it will in particular follow from Theorem 3.4.12 that the resulting notions of parallel transport coincide as well:

$$\mathcal{P}_\gamma^\nabla \equiv \mathcal{P}_\gamma^{\nabla^{A^\nabla}} = \mathcal{P}_\gamma^{TM,A^\nabla}.$$

3.4.14 Remark. Generalizing Example 3.4.13, let F be a vector bundle that is associated to a $\mathrm{GL}(r,\mathbb{K})$-principal fiber bundle P, $F = P \times_{\mathrm{GL}(r,\mathbb{K})} \mathbb{K}^r$. Let ∇^F be any covariant derivative on F. Then we show that there exists a connection form A on P such that $\nabla^A = \nabla^F$.

To find such an A we adapt the procedure used in Example 3.1.10: we define local matrix valued 1-forms A_s that have the right transformation behaviour, namely (3.1.14). Locally we have $F \cong U \times \mathbb{K}^r$ and $P \cong U \times \mathrm{GL}(r,\mathbb{K})$. Let s be a local section of P, so $s(x)$ is an invertible $r \times r$ matrix $s = (f_1, \dots, f_r)$ with columns $f_i \in \Gamma(F)$. Since the f_i form a basis of $\mathbb{K}^r$ we can write

$$\nabla^F f_i = \sum_{j=1}^r \omega_{ji} \otimes f_j,$$

with $\omega_{ji} \in \Omega^1(U)$. Now set

$$A_s := (\omega_{ij})_{i,j=1}^r = \sum_{i,j=1}^r \omega_{ij} B_{ij} \in \Omega^1(U, \mathfrak{gl}(r,\mathbb{K})).$$

It then follows exactly as in Example 3.1.10 that the A_s indeed obey (3.1.14). So we are left with proving that $\nabla^F = \nabla^A$ for the resulting connection form A on P.

Since this is a local statement we may assume $F \cong U \times \mathbb{K}^r$ and $P \cong U \times \mathrm{GL}(r, \mathbb{K})$. Thus any $p \in P$ is of the form (x, g) and the bundle chart (2.3.2) becomes ψ_U : $[p, v] = [(x, g), v] \mapsto (x, gv)$. Then in Theorem 3.4.9 we have $[s, v] = s \cdot v$, and any element of $\Gamma(F)$ is of this form. We are going to use (3.4.14) to show that for each vector field X we have

$$\nabla^F_X([s, v]) = \nabla^A_X([s, v]).$$

As above we have $s = (f_1, \dots, f_r)$ with $f_i \in \Gamma(F)$. Therefore,

$$\begin{aligned}\nabla^F_X([s, v]) &= \nabla^F_X(s \cdot v) = \nabla^F_X\Big(\sum_i v_i f_i\Big) = \sum_i X(v_i) f_i + \sum_{i,j} v_i \omega_{ji}(X) f_j \\ &= s(x) \cdot Tv(X) + \sum_{i,j} v_i \omega_{ji}(X) f_j.\end{aligned}$$

Because of (3.4.15) it therefore only remains to show that the second summand here equals $s(x) A_s(X) v(x)$. Indeed we have (writing ω_{ij} instead of $\omega_{ij}(X)$ for brevity):

$$\begin{aligned}s(x) \cdot A_s(X) \cdot v(x) &= \begin{pmatrix} f_{11} & \dots & f_{r1} \\ \vdots & & \vdots \\ f_{1r} & \dots & f_{rr} \end{pmatrix} \cdot \begin{pmatrix} \omega_{11} & \dots & \omega_{1r} \\ \vdots & & \vdots \\ \omega_{r1} & \dots & \omega_{rr} \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} \\ &= \begin{pmatrix} \sum_j f_{j1} \omega_{j1} v_1 + & \dots & + \sum_j f_{j1} \omega_{jr} v_r \\ \vdots & & \vdots \\ \sum_j f_{jr} \omega_{j1} v_1 + & \dots & + \sum_j f_{jr} \omega_{jr} v_r \end{pmatrix} = \sum_{i,j} f_j \omega_{ji} v_i,\end{aligned}$$

as claimed. In particular, by Theorem 3.4.12 we have $\mathcal{P}^{\nabla^F}_\gamma = \mathcal{P}^{\nabla^A}_\gamma = \mathcal{P}^{F,A}_\gamma$.

3.4.15 Example. For metric connections we have seen in Example 3.1.12 and Corollary 3.1.13 that $A \mapsto \nabla^A$ is a bijection between the set of connection forms on $\mathrm{O}(M, g)$ and the set of metric covariant derivatives on M, and the analogue of (3.1.19) holds:

$$\nabla^{A^\nabla} = \nabla \quad \text{and} \quad A^{\nabla^A} = A.$$

This applies, in particular, to the Levi-Civita connection ∇^{LC} on a semi-Riemannian manifold (M, g) of signature (k, l), for which we have $TM \cong \mathrm{O}(M, g) \times_{\mathrm{O}(k,l)} \mathbb{R}^{k+l}$. It therefore induces a connection form A^{LC} on $\mathrm{O}(M, g)$ with $\nabla^{\mathrm{LC}} = \nabla^{A^{\mathrm{LC}}}$. Finally, by Theorem 3.4.12 all parallel transports coincide:

$$\mathcal{P}^{\nabla^{\mathrm{LC}}}_\gamma = \mathcal{P}^{\nabla^{A^{\mathrm{LC}}}}_\gamma = \mathcal{P}^{TM, A^{\mathrm{LC}}}_\gamma.$$

3.5 Curvature of a connection

The curvature of a connection on a principal fiber bundle is a 2-form that in a sense measures the non-flatness of the bundle. As we will see, it underlies all other notions of curvature that are used in differential geometry. Throughout this section let (P, π, M, G) be a principal fiber bundle with connection Th and connection form A. Moreover, let $\rho : G \to \mathrm{GL}(V)$ be a representation of G and let $E = P \times_G V$ be the associated vector bundle over M.

3.5.1 Definition. *The* 2*-form*

$$F^A := D_A A \in \Omega^2(P, \mathfrak{g}) \tag{3.5.1}$$

is called the curvature form *of A (resp. of Th).*

It follows directly from the definition of D_A (cf. (3.4.2)) that F^A is horizontal. Moreover, since A is of type Ad, so is F^A (see the proof of Theorem 3.4.4). Given a local section $s : U \to P$, we call the pullback of F^A under s,

$$F^s := s^* F^A = F^A(Ts(\,.\,), Ts(\,.\,)) \in \Omega^2(U, \mathfrak{g}) \tag{3.5.2}$$

the *local curvature form* with respect to s. Similar to (3.1.8) for A^s, we can also derive the transformation behaviour of F^s. Thus let $\tau : U \to P$ be another section and let $\tau = s \cdot g$ for a smooth function $g : U \to G$. Now given $X \in T_x M$, let c be a smooth curve in M with $\dot{c}(0) = X$ and apply Lemma 1.2.3 to $x = s \circ c$, $g = g \circ c$ to obtain

$$T\tau(X) = TR_g(Ts(X)) + (TL_{g^{-1}} Tg(X))^{\sim}.$$

Since the second summand here is vertical while F^A is horizontal and of type Ad, we conclude from this

$$\begin{aligned} F^\tau(X,Y) = F^A(T\tau(X), T\tau(Y)) &= F^A(TR_g(Ts(X)), TR_g(Ts(Y))) \\ &= (R_g^* F^A)(Ts(X), Ts(Y)) = \mathrm{Ad}(g^{-1})(F^A(Ts(X), Ts(Y))) \\ &= \mathrm{Ad}(g^{-1}) \circ F^s(X,Y). \end{aligned}$$

Thus we arrive at the transformation forumula

$$F^\tau = \mathrm{Ad}(g^{-1}) \circ F^s. \tag{3.5.3}$$

In particular, if $G \subseteq \mathrm{GL}(m, \mathbb{K})$ this reads

$$F^\tau = g^{-1} \circ F^s \circ g. \tag{3.5.4}$$

To formulate further properties of F^A we need some new operations on differential forms. Let N be a manifold and $\mathfrak{g}$ a Lie algebra with basis $(a_1, \dots, a_r)$. Then for $\omega \in \Omega^k(N, \mathfrak{g})$, $\tau \in \Omega^l(N, \mathfrak{g})$ we can write

$$\omega = \sum_{i=1}^r \omega^i a_i \qquad \tau = \sum_{i=1}^r \tau^i a_i,$$

with ω^i, τ^i real valued differential forms on N. Then we define the commutator of ω and τ by

$$[\omega, \tau] := \sum_{i,j} (\omega^i \wedge \tau^j)[a_i, a_j]_{\mathfrak{g}} \in \Omega^{k+l}(N, \mathfrak{g}). \tag{3.5.5}$$

It is easily seen that this definition does not depend on the choice of basis in $\mathfrak{g}$.

3.5.2 Lemma. *The map*

$$\begin{aligned} [\,.\,,\,.\,] : \Omega^k(N, \mathfrak{g}) \times \Omega^l(N, \mathfrak{g}) &\to \Omega^{k+l}(N, \mathfrak{g}) \\ (\omega, \tau) &\mapsto [\omega, \tau] \end{aligned}$$

has the following properties:

(i) $[\omega, \tau] = (-1)^{kl+1}[\tau, \omega]$.

(ii) $d[\omega, \tau] = [d\omega, \tau] + (-1)^k [\omega, d\tau]$.

(iii) If $\omega \in \Omega^1(N, \mathfrak{g})$ *and* $X, Y \in \mathfrak{X}(N)$, *then* $[\omega, \omega](X,Y) = 2[\omega(X), \omega(Y)]_{\mathfrak{g}}$.

Proof. (i) and (ii) are seen directly by inserting, so let us prove (iii). We have

$$\begin{aligned} [\omega,\omega](X,Y) = \sum_{i,j} (\omega^i \wedge \omega^j)(X,Y)[a_i, a_j] &= \sum_{i,j} (\omega^i(X)\omega^j(Y) - \omega^i(Y)\omega^j(X))[a_i, a_j] \\ &= \sum_{i,j} \omega^i(X)\omega^j(Y)[a_i, a_j] + \sum_{i,j} \omega^i(X)\omega^j(Y)[a_i, a_j] = 2[\omega(X), \omega(Y)], \end{aligned}$$

where in the second sum we interchanged i and j. □

3.5.3 Theorem. *Let $F^A \in \Omega^2(P, \mathfrak{g})$ be the curvature form of A. Then*

(i) Structure equation: $F^A = dA + \frac{1}{2}[A, A]$.

(ii) Bianchi identity: $D_A F^A = 0$.

(iii) If $\omega \in \Omega^k_{\text{hor}}(P, V)^{(G,\rho)}$, then $D_A D_A \omega = \rho_*(F^A) \wedge \omega$.

Proof. (i) It suffices to check the identity on $X, Y \in T_pP$ that are either horizontal or vertical. If both are horizontal, then $A(X) = A(Y) = 0$ and by (3.4.2) we have

$$F^A(X, Y) = (D_A A)(X, Y) = dA(X, Y),$$

giving the claim (due to Lemma 3.5.2 (iii)). Next, let X be horizontal and Y vertical. Then $F^A(X, Y) = 0$ since F^A is horizontal. We may extend X, Y such that $X = V^*(p)$ is a horizontal lift for some $V \in \mathfrak{X}(M)$ and $Y = \tilde{T}(p)$ is the value of a fundamental vector field induced by $T \in \mathfrak{g}$. Then by Theorem 3.3.2 we have $[V^*, \tilde{T}] = 0$, so (3.4.1) gives

$$dA(X, Y) = dA(V^*, \tilde{T})_p = V^*(\underbrace{A(\tilde{T})}_{\equiv T}) - \tilde{T}(\underbrace{A(V^*)}_{=0}) = 0.$$

By Lemma 3.5.2 (iii) also $[A, A](X, Y) = 0$, so the claim follows also in this case. Finally, let both X and Y be vertical, $X = \tilde{T}(p)$, $Y = \tilde{S}(p)$, $T, S \in \mathfrak{g}$. Then $F^A(X, Y) = 0$ and (3.4.1) and Theorem 1.2.2 give

$$\begin{aligned} dA(X, Y) &= X(A(\tilde{S})) - Y(A(\tilde{T})) - A([\tilde{T}, \tilde{S}]) = -A([T, S]^{\sim}) = -[T, S]_{\mathfrak{g}} \\ &= -[A(\tilde{T}(p)), A(\tilde{S}(p))]_{\mathfrak{g}} = -[A(X), A(Y)]_{\mathfrak{g}} \underset{3.5.2}{=} -\frac{1}{2}[A, A](X, Y). \end{aligned}$$

(ii)Differentiating (i) and using Lemma 3.5.2 (i) and (ii) we get

$$dF^A = ddA + \frac{1}{2}d[A, A] = \frac{1}{2}([dA, A] - [A, dA]) = [dA, A],$$

so Lemma 3.5.2 (iii) and the definition of D_A give

$$D_A F^A = dF^A \circ \text{pr}_h = [dA \circ \text{pr}_h, A \circ \text{pr}_h] = 0,$$

where $\circ\text{pr}_h$ is an abbreviation for applying pr_h to each argument.
(iii) Using that ω and $D_A\omega$ are horizontal and of type ρ and that $\rho_* : \mathfrak{g} \to L(V, V)$ is linear, (3.4.3) implies

$$\begin{aligned} D_A(D_A\omega) &= d(d\omega + \rho_*(A) \wedge \omega) + \rho_*(A) \wedge (d\omega + \rho_*(A) \wedge \omega) \\ &= dd\omega + d(\rho_*(A)) \wedge \omega - \rho_*(A) \wedge d\omega + \rho_*(A) \wedge d\omega + \rho_*(A) \wedge (\rho_*(A) \wedge \omega) \\ &= \rho_*(dA) \wedge \omega + \rho_*(A) \wedge (\rho_*(A) \wedge \omega). \end{aligned}$$

Here, by applying (3.4.4) twice (and using the remark following (3.4.5)) we see that $\rho_*(A) \wedge (\rho_*(A) \wedge \omega) = (\rho_*(A) \wedge \rho_*(A)) \wedge \omega$, with

$$\begin{aligned} (\rho_*(A) \wedge \rho_*(A))(X, Y) &:= \rho_*(A(X)) \circ \rho_*(A(Y)) - \rho_*(A(Y)) \circ \rho_*(A(X)) \\ &= [\rho_*(A(X)), \rho_*(A(Y))]_{\mathfrak{gl}(V)} = \rho_*([A(X), A(Y)]_{\mathfrak{g}}) = \frac{1}{2}\rho_*([A, A](X, Y)), \end{aligned}$$

where we used that ρ_* is a Lie algebra homomorphism, as well as Lemma 3.5.2 (iii) in the last step. Thus, finally,

$$D_A D_A \omega = \rho_*(dA + \frac{1}{2}[A, A]) \wedge \omega = \rho_*(F^A) \wedge \omega.$$

□

Since $F^A \in \Omega^2_{\mathrm{hor}}(P,\mathfrak{g})^{(G,\mathrm{Ad})}$, by Theorem 3.2.3 we can consider it as a 2-form in the adjoint bundle $\mathrm{Ad}(P) = P \times_G \mathfrak{g}$, and we shall notationally not distinguish between these versions:

$$F^A \in \Omega^2(M, \mathrm{Ad}(P)). \tag{3.5.6}$$

In order to formulate the identities from Theorem 3.5.3 from this point of view we introduce a bundle homomorphism

$$\rho_* : \mathrm{Ad}(P) \to \mathrm{End}(E,E)$$

as follows (recall that $E = P \times_G V$): Let $\varphi \in \mathrm{Ad}(P)_x$, $e \in E_x$, and fix some $p \in P_x$. Then $\varphi = [p, X]$ and $e = [p, v]$ for some $X \in \mathfrak{g}$, $v \in V$ and we define

$$\rho_*(\varphi)e := [p, \rho_*(X)v]. \tag{3.5.7}$$

To check that this is well-defined, let also $q \in P_x$, $q = p \cdot g$. Then

$$\begin{aligned} \varphi &= [p, X] = [p \cdot g, \mathrm{Ad}(g^{-1})X] = [q, \mathrm{Ad}(g^{-1})X], \\ e &= [p, v] = [q, \rho(g^{-1})v], \end{aligned}$$

so we have to show that $[q, \rho_*(\mathrm{Ad}(g^{-1})X)(\rho(g^{-1})v)] = [p, \rho_*(X)v]$. We have[3]

$$\rho_* \circ \mathrm{Ad}(g^{-1}) = T_e(\rho \circ \mathrm{conj}_{g^{-1}}) = T_e(h \mapsto \rho(g^{-1})\rho(h)\rho(g)),$$

implying $\rho_*(\mathrm{Ad}(g^{-1})X) = \rho(g^{-1}) \circ \rho_*(X) \circ \rho(g)$ and

$$\rho_*(\mathrm{Ad}(g^{-1})X)(\rho(g^{-1})v) = \rho(g^{-1}) \circ \rho_*(X) \circ \rho(g) \circ \rho(g^{-1})v = \rho(g^{-1})\rho_*(X)v.$$

Thus indeed

$$[q, \rho_*(\mathrm{Ad}(g^{-1})X)(\rho(g^{-1})v)] = [pg, \rho(g^{-1})\rho_*(X)v] = [p, \rho_*(X)v].$$

3.5.4 Lemma. *The map $\rho_* : \mathrm{Ad}(P) \to \mathrm{End}(E,E)$ is a vector bundle homomorphism.*

Proof. See the Appendix. □

Using this map we can now define the wedge product of differential forms with values in $\mathrm{Ad}(P)$ with those that take values in E:

$$\begin{aligned} \wedge : \Omega^k(M, \mathrm{Ad}(P)) \times \Omega^l(M,E) &\to \Omega^{k+l}(M,E) \\ (\sigma, \omega) &\mapsto \sigma \wedge \omega, \end{aligned}$$

where, for $t_1, \dots, t_{k+l} \in T_xM$,

$$\begin{aligned} &(\sigma \wedge \omega)_x(t_1, \dots, t_{k+l}) \\ &:= \frac{1}{k!l!} \sum_{\tau \in \mathcal{S}_{k+l}} \mathrm{sign}(\tau)\rho_*(\sigma_x(t_{\tau(1)}, \dots, t_{\tau(k)}))\omega_x(t_{\tau(k+1)}, \dots, t_{\tau(k+l)}). \end{aligned} \tag{3.5.8}$$

Using this notation, the identification of $\Omega^2_{\mathrm{hor}}(P,\mathfrak{g})^{(G,\mathrm{Ad})}$ with $\Omega^2(M, \mathrm{Ad}(P))$ gives:

3.5.5 Theorem. *Let $F^A \in \Omega^2(M, \mathrm{Ad}(P))$ be the curvature form of A. Then F^A satisfies the Bianchi identity*

$$d_A F^A = 0,$$

and for the differential $d_A : \Omega^k(M,E) \to \Omega^{k+1}(M,E)$ we get

$$d_A d_A \omega = F^A \wedge \omega.$$

[3] Here e appears in two meanings, the one in the lower index is the unit in G.

Proof. Using the notation from Theorem 3.2.3 and de-identifying for the purpose of the present proof, Theorem 3.5.3 actually refers to $\overline{F^A} \in \Omega^2_{\mathrm{hor}}(P,\mathfrak{g})^{(G,\mathrm{Ad})}$, and (ii) there says $D_A\overline{F^A} = 0$. By (3.4.6), therefore,

$$\overline{d_A F^A} = D_A \overline{F^A} = 0,$$

giving the Bianchi identity in the claimed form. Also, Theorem 3.5.3 (iii) reads, for $\bar{\omega} \in \Omega^k_{\mathrm{hor}}(P,V)^{(G,\rho)}$:

$$D_A D_A \bar{\omega} = \rho_*(\overline{F^A}) \wedge \bar{\omega},$$

where ρ_* denotes the post-composition with the derivative of ρ. By (3.4.6),

$$D_A D_A \bar{\omega} = D_A(\overline{d_A\omega}) = \overline{d_A d_A \omega},$$

so we are left with proving that $\rho_*(\overline{F^A}) \wedge \bar{\omega} = \overline{F^A \wedge \omega}$. To verify this, let (cf. the proof of Theorem 3.2.3) $t_i \in T_xM$ and $X_i \in T_pP$ with $T\pi(X_i) = t_i$. Then

$$\begin{aligned}
&[p,\rho_*(\overline{F^A}) \wedge \bar{\omega})_p(X_1,\dots,X_{k+2})] \\
&\underset{(3.4.5)}{=} \Big[p, \frac{1}{2!k!}\sum_{\tau\in\mathcal{S}_{k+2}} \mathrm{sign}(\tau)\rho_*((\overline{F^A})_p(X_{\tau(1)},X_{\tau(2)}))(\bar{\omega}_p(X_{\tau(3)},\dots,X_{\tau(k+2)}))\Big] \\
&\underset{(3.5.7)}{=} \frac{1}{2!k!}\sum_{\tau\in\mathcal{S}_{k+2}} \mathrm{sign}(\tau)\rho_*((F^A)_x(t_{\tau(1)},t_{\tau(2)}))(\omega_x(t_{\tau(3)},\dots,t_{\tau(k+2)})) \\
&\underset{(3.5.8)}{=} (F^A \wedge \omega)_x(t_1,\dots,t_{k+2}) = [p, \overline{F^A \wedge \omega}_p(X_1,\dots,X_{k+2})].
\end{aligned}$$

□

In this sense the curvature form F^A measures the failure of $d_A \circ d_A$ to vanish. The following result shows that F^A determines the vertical component of the commutator of horizontal vector fields.

3.5.6 Theorem. *Let X and Y be horizontal vector fields on P and let $F^A \in \Omega^2(P,\mathfrak{g})$ be the curvature form of A. Then*

(i) $F^A(X,Y) = -A([X,Y])$.

(ii) $\mathrm{pr}_v([X,Y]) = -F^A(X,Y)^\sim$.

Proof. Since $A(X) = A(Y) = 0$, it follows from Lemma 3.5.2 (iii) that $[A,A](X,Y) = 0$. Thus Theorem 3.5.3 (i) shows

$$F^A(X,Y) = dA(X,Y) \underset{(3.4.1)}{=} X(A(Y)) - Y(A(X)) - A([X,Y]) = -A([X,Y]).$$

Set $Z := [X,Y]$, then by Theorem 3.1.1 for each $p \in P$ there is a unique $U \in \mathfrak{g}$ such that $Z_p = \tilde{U}_p + \mathrm{pr}_h(Z_p)$. Thus $A(Z_p) = A(\tilde{U}_p) = U$, so $\mathrm{pr}_v(Z_p) = A(Z_p)^\sim_p = -(F^A(X,Y))^\sim_p$. □

This result allows us to characterize the integrability (i.e., by the Frobenius theorem [9, 17.33], the involutivity) of the horizontal and vertical distributions:

3.5.7 Theorem. *Let (P,π,M,G) be a principal fiber bundle with connection form A. Then:*

(i) The vertical bundle TvP is integrable.

(ii) The horizontal bundle Th is integrable if and only if the curvature form vanishes, $F^A = 0$.

Proof. (i) By Theorem 1.2.2, for any two fundamental vector fields $\tilde{T}, \tilde{S}$ we have $[\tilde{T}, \tilde{S}] = [T, S]^{\sim}$. Since these vector fields span TvP, the claim follows.

(ii) Let X, Y be horizontal vector fields. Then by Theorem 3.5.6 (ii) we have $\mathrm{pr}_v([X,Y]) = -(F^A(X,Y))^{\sim}$. Thus $[X,Y]$ is horizontal for each X, Y horizontal (i.e., ThP is involutive) if and only if $F^A(X,Y) = 0$ for all such X, Y. By (3.4.2) and (3.5.1) F^A is horizontal, so this condition is equivalent to the vanishing of F^A. □

By the global version of the Frobenius theorem (cf. [9, 17.25]), $F^A = 0$ if and only if any point of P lies in a maximal connected submanifold H of P (a leaf) transversal to the fibers of the bundle, with tangent bundle $TH = ThP|_H$.

3.5.8 Example. Let $P_0 = M \times G$ be the trivial principal fiber bundle over M, equipped with the canonical flat connection ThP_0, corresponding to the connection form A_0 (cf. Example 3.1.8). Then the maximal integral manifold of ThP_0 through (x, g) is obviously $M \times \{g\} \subseteq M \times G$. Thus $F^{A_0} = 0$, as can also be seen directly: By Theorem 3.5.6, $F^{A_0}(X,Y) = -A_0([X,Y]) = 0$, because $[X,Y]$ is horizontal in this case.

3.5.9 Definition. *A connection Th and the corresponding connection form A on P are called* flat *if the curvature form of A vanishes: $F^A = 0$.*

3.5.10 Theorem. *The following are equivalent:*

(i) The connection form A is flat, i.e., $F^A = 0$.

(ii) The horizontal distribution Th is integrable.

(iii) There exists an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of M such that each subbundle (P_{U_i}, A) is isomorphic to the trivial G-principal bundle over U_i with the canonical flat connection.

(iv) There exists an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of M such that every subbundle P_{U_i} possesses an A-horizontal section.

(v) There exists an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of M such that parallel transport in every P_{U_i} is path-independent.

Proof. (i)⇔(ii) is Theorem 3.5.7 (ii). Point 1.) in the proof of Theorem 3.3.9 shows that (iii)⇔(iv). It also follows from Theorem 3.3.9 that (v)⇒(iii), while (iii)⇒(v) is Example 3.3.8, so altogether (iii)⇔(iv)⇔(v).

(iii)⇒(i): Any $x \in M$ has a neighborhood U and an isomorphism $\phi : P_U \to P_{0U} = U \times G$ of principal fiber bundles with $\phi^* A_0 = A|_U$. Then $F^A|_U = \phi^* F^{A_0} = 0$, so $F^A = 0$.

(ii)⇒(iv): Let $p \in P$ and let $H(p) \subseteq P$ be a maximal connected integral manifold of Th, so $T_q(H(p)) = Th_q P$ for each $q \in H(p)$. Now choose any Riemannian metric on M, let $p \in P$ and let U be a normal neighborhood of $x := \pi(p)$. For any $y \in U$ let $\gamma_y : [0,1] \to U$ be the radial geodesic from x to y. Then we define a section over U by

$$\begin{aligned} s : U &\to P \\ y &\mapsto \gamma_p^*(1), \end{aligned}$$

where γ_p^* is the horizontal lift of γ_y with starting point p. Since γ_y depends smoothly on y it follows as in the proof of Theorem 3.3.9 that s is smooth. However, contrary to the situation there we do not (yet) know that s is independent of the chosen path (rather, each point in its domain is reached by a unique radial geodesic, whose lift is used to define s), so we need a different argument to show that it is horizontal, i.e., that $T_x s(T_x M) = Th_{s(x)}P = T_{s(x)}H(p)$. As $\pi \circ s = \mathrm{id}$ we have $\mathrm{rk}(s) = \dim M = \dim H(p)$, so we only need to prove inclusion. Since $H(p)$ is a leaf of Th, γ_p^* lies in $H(p)$[4], so $s(U) \subseteq H(p)$. As $H(p)$ is an integral manifold of an integrable distribution and $s : U \to P$ is smooth, it is also smooth when viewed as a map $s : U \to H(p)$ (see [9, 17.27]). But then $T_x s(T_x U) \subseteq T_{s(x)}H(p)$ for each $x \in U$, and we are done. □

3.5.11 Theorem. *Let (P, π, M, G) be a principal fiber bundle with connection form A and suppose that M is simply connected. Then $F^A = 0$ if and only (P, A) is isomorphic to the trivial bundle $M \times G$ with the canonical flat connection.*

Proof. One direction is immediate from Theorem 3.5.10 (iii)⇒(i). Conversely, by Theorem 3.3.9 it will suffice to show that $F^A = 0$ implies path independence of parallel transport. So let $x, y \in M$ and let $\gamma, \delta : I = [0,1] \to M$ be two paths from x to y. Since M is simply connected there exists a (piecewise smooth)[5] homotopy $H : I \times I \to M$ between γ and δ that leaves x and y fixed. For $s \in I$, denote by H_s^* the horizontal lift of $H_s := H(\,.\,, s)$. We show that the corresponding endpoints $H_s^*(1)$ all coincide. To this end we subdivide $I \times I$ into sufficiently small rectangles K_i such that each $H(K_i)$ lies in a neighborhood as in Theorem 3.5.10 (v), so that parallel transport there does not depend on the path. It follows that $s \mapsto H_s^*(1)$ is locally constant and continuous on I, hence constant. □

Similar to the definition of the Riemann tensor one can introduce a curvature operator on any vector bundle equipped with a covariant derivative:

3.5.12 Definition. *Let $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ be a covariant derivative on the vector bundle E. Then the 2-form on the endomorphism bundle*

$$R^\nabla \in \Gamma(\Lambda^2(T^*M) \otimes \mathrm{End}(E,E)) = \Omega^2(M, \mathrm{End}(E,E))$$

defined by

$$R^\nabla(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \qquad (X, Y \in \mathfrak{X}(M))$$

is called the curvature endomorphism *of ∇.*

In particular, this applies to the situation from Definition 3.4.8, where we saw how to assign a covariant derivative ∇^A on the vector bundle $E = P \times_G V$ associated to a principal fiber bundle P with connection form A (and due to Theorem 2.4.9 and Remark 3.4.14 any covariant derivative comes about in this way). In this case the curvature endomorphism on E is completely determined by the curvature form on P:

3.5.13 Theorem. *Let $p \in P_x$ and $[p] : V \to E_x$ the corresponding fiber diffeomorphism (see (2.3.7)). Then*

$$R_x^{\nabla^A}(X,Y) = [p] \circ \rho_*(F_p^A(X^*, Y^*)) \circ [p]^{-1},$$

where $X, Y \in T_x M$ and $X^*, Y^* \in T_p P$ are their horizontal lifts.

[4]This follows by using flat charts for ThP, cf. the the proof of [9, Th. 21.12]

[5]See, e.g., [12, Lem. 6.6].

Proof. Given $\varphi \in \Gamma(E)$, let $\bar{\varphi} \in \mathcal{C}^\infty(P,V)^{(G,\rho)}$, $\bar{\varphi}(p) = [p]^{-1} \circ \varphi(\pi(p))$ be the G-equivariant function corresponding to φ by Theorem 2.3.4. Then for $X \in \mathfrak{X}(M)$ with horizontal lift X^*, by (3.4.7) and Definition 3.4.8 we have

$$(\nabla^A_X \varphi)(x) = [p, d\bar{\varphi}_p(X^*)] = [p, X^*(\bar{\varphi})(p)] \qquad (p \in P_x). \tag{3.5.9}$$

Therefore,

$$\begin{aligned}
R^{\nabla^A}_x(X_x, Y_x)\varphi(x) &\underset{2.3.4}{=} R^{\nabla^A}_x(X_x, Y_x)[p, \bar{\varphi}(p)] \\
&\underset{(3.5.9)}{=} \big[p, (X^*Y^*(\bar{\varphi}) - Y^*X^*(\bar{\varphi}) - [X,Y]^*(\bar{\varphi}))(p)\big] \\
&= \big[p, ([X^*,Y^*](\bar{\varphi}) - [X,Y]^*(\bar{\varphi}))(p)\big] \\
&\underset{3.3.2}{=} \big[p, (\mathrm{pr}_v[X^*,Y^*](\bar{\varphi}))(p)\big] \\
&\underset{3.5.6}{=} \big[p, -(F^A_p(X^*_p, Y^*_p))\tilde{\ }(\bar{\varphi})(p)\big].
\end{aligned}$$

Here,

$$\begin{aligned}
(F^A_p(X^*_p, Y^*_p))\tilde{\ }(\bar{\varphi})(p) &\underset{3.1.1}{=} \frac{d}{dt}\Big|_0 \bar{\varphi}\big(p \cdot \exp(tF^A_p(X^*_p, Y^*_p))\big) \\
&\underset{(2.3.9)}{=} \frac{d}{dt}\Big|_0 \big(\rho(\exp(-tF^A_p(X^*_p, Y^*_p)))\bar{\varphi}(p)\big) \\
&= -\rho_*\big(F^A_p(X^*_p, Y^*_p)\big)\bar{\varphi}(p).
\end{aligned}$$

Altogether, since $[p] = v \mapsto [p, v]$ we have shown that

$$R^{\nabla^A}_x(X_x, Y_x)[p](\bar{\varphi}(p)) = [p](-(F^A_p(X^*_p, Y^*_p))\tilde{\ }_p(\bar{\varphi})) = [p](\rho_*\big(F^A_p(X^*_p, Y^*_p)\big)\bar{\varphi}(p)),$$

as claimed. □

3.5.14 Remark. As was shown in Example 3.4.15, for the case of the Levi-Civita connection ∇^{LC} on a semi-Riemannian manifold (M,g) of signature (k,l), we have $TM \cong \mathrm{O}(M,g) \times_{\mathrm{O}(k,l)} \mathbb{R}^{k+l}$, and ∇^{LC} induces a connection form A^{LC} on $\mathrm{O}(M,g)$ with $\nabla^{\mathrm{LC}} = \nabla^{A^{\mathrm{LC}}}$. The previous result then shows how the Riemann curvature tensor on M, which in this case is the curvature endomorphism $R^{\nabla^{\mathrm{LC}}} \in \Gamma(\Lambda^2(T^*M) \otimes \mathrm{End}(TM, TM))$ relates to the curvature form $F^{A^{\mathrm{LC}}} \in \Omega^2(\mathrm{O}(M,g), \mathfrak{o}(k,l))$. In this sense semi-Riemannian geometry can ultimately be traced back to the study of the principal fiber bundle $\mathrm{O}(M,g)$ with connection form A^{LC}.

For $H \in \Omega^k(M, \mathrm{End}(E,E))$ and $\omega \in \Omega^l(M,E)$ we define a wedge product in accordance with (3.4.5):

$$\begin{aligned}
&(H \wedge \omega)(X_1, \ldots, X_{k+l}) \\
&\quad := \frac{1}{k!l!} \sum_{\tau \in \mathcal{S}_{k+l}} \mathrm{sign}(\tau) H(X_{\tau(1)}, \ldots, X_{\tau(k)})\big(\omega(X_{\tau(k+1)}, \ldots, X_{\tau(k+l)})\big).
\end{aligned} \tag{3.5.10}$$

Using this notation we have:

3.5.15 Theorem. *Let (P, π, M, G) be a principal fiber bundle with connection form A and let $E = P \times_G V$. Then for any $\omega \in \Omega^k(M,E)$,*

$$d_A d_A \omega = R^{\nabla^A} \wedge \omega. \tag{3.5.11}$$

Proof. Using the de-identification from the proof of Theorem 3.5.5, we have shown there that

$$\overline{d_A d_A \omega} = \rho_*(\overline{F^A}) \wedge \bar{\omega}.$$

Let $X_1, \dots, X_{k+2} \in \mathfrak{X}(M)$ with horizontal lifts $X_1^*, \dots, X_{k+2}^* \in \mathfrak{X}(P)$. Then (suppressing the arguments, x for X_i, p for X_i^*)

$$\begin{aligned}
&(R^{\nabla^A} \wedge \omega)_x(X_1, \dots, X_{k+2}) \\
&\underset{(3.5.10)}{=} \frac{1}{2!k!} \sum_{\tau \in \mathcal{S}_{k+2}} \operatorname{sign}(\tau) R_x^{\nabla^A}(X_{\tau(1)}, X_{\tau(2)})\big(\omega_x(X_{\tau(3)}, \dots, X_{\tau(k+2)})\big) \\
&\underset{3.5.13}{=} \frac{1}{2!k!} \sum_{\tau \in \mathcal{S}_{k+2}} \operatorname{sign}(\tau)[p] \circ \rho_*(\overline{F^A}_p(X^*_{\tau(1)}, X^*_{\tau(2)})) \circ [p]^{-1}\big(\omega_x(X_{\tau(3)}, \dots, X_{\tau(k+2)})\big) \\
&\underset{(3.2.3)}{=} [p]\Big(\frac{1}{2!k!} \sum_{\tau \in \mathcal{S}_{k+2}} \operatorname{sign}(\tau) \rho_*(\overline{F^A}_p(X^*_{\tau(1)}, X^*_{\tau(2)})) \bar{\omega}_p(X^*_{\tau(3)}, \dots, X^*_{\tau(k+2)})\Big) \\
&\underset{(3.4.5)}{=} [p]\big((\rho_*(\overline{F^A}) \wedge \bar{\omega})_p(X_1^*, \dots, X_{k+2}^*)\big) = [p]\big(\overline{d_A d_A \omega}_p(X_1^*, \dots, X_{k+2}^*)\big) \\
&\underset{(3.2.3)}{=} (d_A d_A \omega)_x(X_1, \dots, X_{k+2}).
\end{aligned}$$

□

3.5.16 Definition. *Let (P, π, M, G) be a principal fiber bundle. A diffeomorphism $f: P \to P$ is called a* gauge transformation *if (f, id_G) is a bundle morphism in the sense of Definition 2.2.3, i.e.,*

(i) $\pi \circ f = \pi$, *and*

(ii) $f(p \cdot g) = f(p) \cdot g$ *for all* $p \in P$, $g \in G$.

By $\mathcal{G}(P)$ we denote the group of gauge transformations on P.

By $\mathcal{C}^\infty(P, G)^G$ we denote the set of G-equivariant smooth maps from P to G:

$$\mathcal{C}^\infty(P, G)^G := \{\sigma \in \mathcal{C}^\infty(P, G) \mid \sigma(pg) = g^{-1}\sigma(p)g\}. \tag{3.5.12}$$

3.5.17 Lemma. *The group of gauge transformations $\mathcal{G}(P)$ can be identified with $\mathcal{C}^\infty(P, G)^G$ via the bijection*

$$\begin{aligned} S : \mathcal{G}(P) \ni f &\mapsto \sigma_f \in \mathcal{C}^\infty(P, G)^G \\ f(p) &=: p \cdot \sigma_f(p). \end{aligned} \tag{3.5.13}$$

Proof. Given $f \in \mathcal{G}(P)$ and $p \in P$, $\pi(f(p)) = \pi(p)$, so there is a unique $\sigma_f(p) \in G$ with $f(p) = p \cdot \sigma_f(p)$. The resulting map σ_f is smooth, since for a bundle chart ϕ as in Definition 2.2.1 with $\phi(p) = (x, g)$ we have

$$\phi(f(p)) = (x, f_2(p)) = (x, g) \cdot (g^{-1} f_2(p)) = \phi(p) \cdot (g^{-1} f_2(p)) = \phi(p \cdot (g^{-1} f_2(p))),$$

so that $\sigma_f(p) = g^{-1} f_2(p) = (\mathrm{pr}_2 \circ \phi(p))^{-1} \cdot f_2(p)$, which is obviously smooth.
$\sigma_f \in \mathcal{C}^\infty(P, G)^G$: $pg\sigma_f(pg) = f(pg) = f(p)g = p\sigma_f(p)g$, so $pg\sigma_f(pg) = p\sigma_f(p)g$, and thereby $\sigma_f(pg) = g^{-1}\sigma_f(p)g$.
S is bijective: Its inverse is obviously given by $\sigma \mapsto f_\sigma := p \mapsto p \cdot \sigma(p)$, so it only remains to show that $f_\sigma \in \mathcal{G}(P)$. It is a diffeomorphism with inverse $f_{\sigma^{-1}}$ (with σ^{-1} the pointwise inverse in G) and one readily verifies Definition 3.5.16 for f_σ. □

3.5.18 Theorem. *Let (P, π, M, G) be a principal fiber bundle with connection form A and let $f \in \mathcal{G}(P)$. Then also f^*A is a connection form on P and we have* *(i)* $f^*A = \mathrm{Ad}(\sigma_f^{-1}) \circ A + \sigma_f^* \mu_G$.

(ii) $f \circ \mathcal{P}_\gamma^{f^*A} = \mathcal{P}_\gamma^A \circ f$.

(iii) $D_{f^*A} = f^* \circ D_A \circ f^{*-1}$.

(iv) $F^{f^*A} = f^*F^A = \mathrm{Ad}(\sigma_f^{-1}) \circ F^A$.

Proof. We begin by verifying the conditions from Definition 3.1.3 to show that f^*A is a connection form. First, since $f \circ R_g = R_g \circ f$ we have

$$R_g^*(f^*A) = f^*(R_g^*A) = f^*(\mathrm{Ad}(g^{-1}) \circ A) = \mathrm{Ad}(g^{-1}) \circ f^*A.$$

Moreover, using Theorem 3.1.1, for $X \in \mathfrak{g}$ we have

$$T_pf(\tilde{X}(p)) = \frac{d}{dt}\Big|_0 (f(p \cdot \exp(tX))) = \frac{d}{dt}\Big|_0 (f(p) \cdot \exp(tX)) = \tilde{X}(f(p)),$$

so that $(f^*A)_p(\tilde{X}(p)) = A_{f(p)}(\tilde{X}(f(p))) = X$.
(i) Let $X \in T_pP$ and pick a smooth curve γ in P with $\gamma(0) = p$ and $\gamma'(0) = X$. Then setting $\sigma := \sigma_f$, by Lemma 1.2.3 we get

$$\begin{aligned} T_pf(X) = \frac{d}{dt}\Big|_0 f(\gamma(t)) &\underset{(3.5.13)}{=} \frac{d}{dt}\Big|_0 (\gamma(t) \cdot \sigma(\gamma(t))) \\ &= TR_{\sigma(p)}(X) + (TL_{\sigma(p)^{-1}} T_p\sigma(X))^{\tilde{}}(f(p)) \end{aligned} \tag{3.5.14}$$

Therefore,

$$\begin{aligned} (f^*A)_p(X) &= A_{f(p)}(TR_{\sigma(p)}(X)) + A_{f(p)}((TL_{\sigma(p)^{-1}} T_p\sigma(X))^{\tilde{}}(f(p))) \\ &= \mathrm{Ad}(\sigma(p)^{-1}) \circ A_p(X) + TL_{\sigma(p)^{-1}} T_p\sigma(X) \\ &= \mathrm{Ad}(\sigma(p)^{-1}) \circ A_p(X) + (\sigma^*\mu_G)_p(X). \end{aligned} \tag{3.5.15}$$

(ii) We have

$$Th^{f^*A}P = \ker f^*A = \ker(A \circ Tf) = Tf^{-1}(\ker A) = Tf^{-1}(Th^AP). \tag{3.5.16}$$

Denote by $\gamma_{f(p)}^A$ the A-horizontal lift of a path γ with initial point $f(p)$, and by $\gamma_p^{f^*A}$ the f^*A-horizontal lift of γ with initial point p. Then by (3.5.16)

$$\gamma_p^{f^*A} = f^{-1} \circ \gamma_{f(p)}^A, \quad \text{hence} \quad \mathcal{P}_\gamma^{f^*A} = f^{-1} \circ \mathcal{P}_\gamma^A \circ f.$$

(iii) We note first that $Tf \circ \mathrm{pr}_h^{f^*A} = \mathrm{pr}_h^A \circ Tf$: To see this, let $X = X_h^A + X_v^A \in Th^AP \oplus Tv^AP = TP$. Then

$$(T_pf)^{-1}(X) = \underbrace{(T_pf)^{-1}(X_h^A)}_{\underset{(3.5.16)}{\in} Th_p^{f^*A}P} + \underbrace{(T_pf)^{-1}(X_v^A)}_{\underset{3.5.16}{\in} Tv_pP},$$

so $\mathrm{pr}_h^{f^*A}((T_pf)^{-1}(X)) = (T_pf)^{-1}(X_h^A) = (T_pf)^{-1} \circ \mathrm{pr}_h^A(X)$. Using this, we calculate

$$\begin{aligned} (D_{f^*A}f^*\omega)(X_1, \ldots, X_k) &\underset{(3.4.2)}{=} d(f^*\omega)(\mathrm{pr}_h^{f^*A}X_1, \ldots, \mathrm{pr}_h^{f^*A}X_k) \\ &= d\omega(Tf \circ \mathrm{pr}_h^{f^*A}X_1, \ldots, Tf \circ \mathrm{pr}_h^{f^*A}X_k) = d\omega(\mathrm{pr}_h^ATfX_1, \ldots, \mathrm{pr}_h^ATfX_k) \\ &= (D_A\omega)(TfX_1, \ldots, TfX_k) = (f^*D_A\omega)(X_1, \ldots, X_k). \end{aligned}$$

(iv) From (iii) we get for the curvature form

$$F^{f^*A} = D_{f^*A} f^*A = f^*D_A(f^*)^{-1}f^*A = f^*D_AA = f^*F^A.$$

Finally, since F^A is horizontal, (3.5.14) gives

$$\begin{aligned}(f^*F^A)_p(X,Y) &= F^A_{f(p)}(TR_{\sigma(p)}X, TR_{\sigma(p)}Y) = (R^*_{\sigma(p)}F^A)_p(X,Y)\\ &= \mathrm{Ad}(\sigma(p)^{-1}) \circ F^A_p(X,Y),\end{aligned}$$

where the last equality holds since F^A is of type Ad. □

3.6 S^1-connections

In applications, in particular in mathematical physics, connections in principal fiber bundles with $S^1 = \mathrm{U}(1)$ as structure group play an important role. Since S^1 is abelian, many of the constructions from previous sections simplify, which warrants a separate treatment. Throughout this section, let (P, π, M, S^1) be a principal fiber bundle over S^1.

Recall from Remark 3.4.6 that with $\rho : G \to \mathrm{GL}(V)$ the trivial representation ($g \mapsto \mathrm{id}_V$ for all $g \in G$), $\rho_* = T_e\rho = 0$, and being of type ρ means being right-invariant, so

$$\Omega^k_{\mathrm{hor}}(P,V)^{(G,\rho)} = \{\bar\omega \in \Omega^k(P,V) \mid \bar\omega \text{ right invariant and horizontal}\}.$$

Moreover, $D_A = d$ by (3.4.3) and

$$\begin{aligned}E = P \times_G V &\cong M \times V\\ [p,v] &\mapsto (\pi(p), v),\end{aligned}$$

so that $\Omega^k(M,E) \cong \Omega^k(M,V)$. Composing this with the isomorphism from Theorem 3.2.3 we obtain

$$\Omega^k_{\mathrm{hor}}(P,V)^{(G,\rho)} \xrightarrow{\Phi} \Omega^k(M,E) \cong \Omega^k(M,V).$$

Explicitly (see (3.2.2)) to $\bar\omega \in \Omega^k_{\mathrm{hor}}(P,V)^{(G,\rho)}$ one assigns $\omega \in \Lambda^k(T^*_xM) \otimes V$ by

$$\omega_x(v_1,\ldots,v_k) := \bar\omega_p(v_1^*,\ldots,v_k^*), \tag{3.6.1}$$

where $\pi(p) = x$, $v_1,\ldots,v_k \in T_xM$ and $v_1^*,\ldots,v_k^* \in T_pP$ with $T\pi(v_j^*) = v_j$. Thus ω is the unique differential form with $\pi^*\omega = \bar\omega$. So this map is the inverse of the map $\omega \mapsto \bar\omega$ (cf. (3.2.3)), $\Omega^k(M,V) \to \Omega^k_{\mathrm{hor}}(P,V)^{(G,\rho)}$ and we denote it by $\alpha \mapsto \hat\alpha$, $\Omega^k_{\mathrm{hor}}(P,V)^{(G,\rho)} \to \Omega^k(M,V)$. In particular for $k = 0$ we get that $\bar f \in \mathcal{C}^\infty(P,V)$ if and only if $f(x) := \bar f(p)$ ($p \in P_x$ arbitrary) is in $\mathcal{C}^\infty(M,V)$.

We are interested in the case where V is the Lie algebra of S^1. Since $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, any smooth curve through $1 \in S^1$ is of the form $\gamma(t) = e^{i\delta(t)}$ with $\delta(0) = 0$. Then $\dot\gamma(0) = i\dot\delta(0) \in i\mathbb{R}$, so $V = i\mathbb{R}$ and the above isomorphisms take the form

$$\begin{aligned}\Omega^k(M,i\mathbb{R}) &\to \Omega^k_{\mathrm{hor}}(P,i\mathbb{R})^{(S^1,\mathrm{id})}\\ \omega &\mapsto \bar\omega\end{aligned}$$

with inverse

$$\begin{aligned}\Omega^k_{\mathrm{hor}}(P,i\mathbb{R})^{(S^1,\mathrm{id})} &\to \Omega^k(M,i\mathbb{R})\\ \alpha &\mapsto \hat\alpha.\end{aligned}$$

Recall also that by (3.4.11) we have

$$d\bar{\omega} = \overline{d\omega}, \quad d\hat{\alpha} = (d\alpha)^\wedge. \tag{3.6.2}$$

Denote by $\mu = \mu_{S^1}$ the Maurer–Cartan form of S^1. Then

$$\mu_z = z^{-1} dz. \tag{3.6.3}$$

Indeed, for $X \in T_z S^1$ and γ a curve in S^1 with $\gamma(0) = z$ and $\dot{\gamma}(0) = X$ we have

$$\mu_z(X) = TL_{z^{-1}} X = \frac{d}{dt}\Big|_0 (L_{z^{-1}}(\gamma(t))) = \frac{d}{dt}\Big|_0 (z^{-1} \cdot \gamma(t)) = z^{-1} X = z^{-1} dz(X).$$

Next we want to determine the group of gauge transformations of P. By (3.5.13) the map $f \mapsto \sigma_f$, $f(p) = p \cdot \sigma_f(p)$ is an isomorphism from $\mathcal{G}(P)$ onto

$$\mathcal{C}^\infty(P, S^1)^{S^1} = \{\sigma \in \mathcal{C}^\infty(P, S^1) \mid \sigma(pz) = z^{-1}\sigma(p)z = \sigma(p)\},$$

where we exploitet the commutativity of S^1. So $\mathcal{G}(P)$ is identified with the set of right invariant smooth functions $P \to S^1$. But by [9, 15.13] (analogously to Lemma 3.4.5) these in turn can be identified with the smooth functions $M \to S^1$, so

$$\mathcal{G}(P) \cong \mathcal{C}^\infty(M, S^1).$$

3.6.1 Lemma. *The set $\mathcal{C}(P)$ of all connection forms on P is an affine space over the vector space $\Omega^1(M, i\mathbb{R})$.*

Proof. By Remark 3.2.4, $\mathcal{C}(P)$ is an affine space over $\Omega^1_{\mathrm{hor}}(P, i\mathbb{R})^{(S^1, \mathrm{Ad})}$, where $\mathrm{Ad}(P) := P \times_G \mathfrak{g}$. In the current situation $\mathrm{Ad}(z) = \mathrm{id}_{i\mathbb{R}}$ for all z because S^1 is abelian. Thus Ad is the trivial representation and we are in the setting discussed at the beginning of this section, giving

$$\Omega^1_{\mathrm{hor}}(P, i\mathbb{R})^{(S^1, \mathrm{Ad})} \cong \Omega^1(M, i\mathbb{R}).$$

□

Explicitly, if $A_1, A_2 \in \mathcal{C}(P)$, then $A_1 - A_2 \in \Omega^1_{\mathrm{hor}}(P, i\mathbb{R})$ and

$$\eta := (A_1 - A_2)^\wedge \in \Omega^1(M, i\mathbb{R}). \tag{3.6.4}$$

Note however that, strictly speaking, one cannot write $\hat{A}_1 = \hat{A}_2 + \eta$ because the A_i themselves are not horizontal, so the hat-map isn't available on them.

If $f \in \mathcal{G}(P)$ with $\sigma_f \in \mathcal{C}^\infty(P, S^1)$ and $A \in \mathcal{C}(P)$ then due to Theorem 3.5.18 and the fact that $\mathrm{Ad}(z) = \mathrm{id}_{i\mathbb{R}}$ we have

$$f^* A = A + \sigma_f^* \mu_{S^1} \underset{(3.6.3)}{=} A + \sigma_f^{-1} d\sigma_f. \tag{3.6.5}$$

By Theorem 3.5.18 (iv),

$$f^* F^A = F^A \qquad \forall f \in \mathcal{G}(P),\ A \in \mathcal{C}(P).$$

Next we show that F^A can be viewed as an element of $\Omega^2(M, i\mathbb{R})$. Since the Lie bracket on $i\mathbb{R}$ vanishes, Theorem 3.5.3 shows that

$$F^A \equiv D_A A = dA. \tag{3.6.6}$$

Here, $F^A \in \Omega^2_{\mathrm{hor}}(P, i\mathbb{R})$, so $(F^A)^\wedge \in \Omega^2(M, i\mathbb{R})$. However $A \notin \Omega^1_{\mathrm{hor}}(P, i\mathbb{R})$, so there is no $\hat{A}$. Nevertheless,

$$d((F^A)^\wedge) \underset{(3.6.2)}{=} (dF^A)^\wedge = (d^2 A)^\wedge = 0.$$

Consequently, $(F^A)^\wedge \in \Omega^2(M, i\mathbb{R})$ is a closed form, and therefore defines a de Rham cohomology class $[i(F^A)^\wedge] \in H^2_{\mathrm{dR}}(M,\mathbb{R})$. Now given $A, \tilde{A} \in \mathcal{C}(P)$, $A - \tilde{A} \in \Omega^1_{\mathrm{hor}}(M, i\mathbb{R})$, $\eta := (\tilde{A} - A)^\wedge \in \Omega^1(M, i\mathbb{R})$, and

$$(F^{\tilde{A}} - F^A)^\wedge \underset{(3.6.6)}{=} (d(\tilde{A} - A))^\wedge \underset{(3.6.2)}{=} d((\tilde{A} - A)^\wedge) = d\eta. \tag{3.6.7}$$

In particular,

$$c_1(P) := \Big[-\frac{1}{2\pi i}(F^A)^\wedge \Big] \in H^2_{\mathrm{dR}}(M,\mathbb{R})$$

is independent of A! $c_1(P)$ is called the *first real Chern class* of P.

3.6.2 Theorem. *Let $\omega \in c_1(P)$. Then there exists some $A_\omega \in \mathcal{C}(P)$ such that $\omega = -\frac{1}{2\pi i}(F^{A_\omega})^\wedge$. Any two such connection forms $A_{1\omega}, A_{2\omega}$ differ from each other by a closed 1-form on M, i.e., $A_{1\omega} = A_{2\omega} + 2\pi i\nu$ for some $\nu \in \Omega^1(M,\mathbb{R})$ with $d\nu = 0$. If M is simply connected then A_ω is unique up to gauge transformations.*

Proof. Fix some $A_0 \in \mathcal{C}(P)$. Then $F^{A_0} \in \Omega^2_{\mathrm{hor}}(P, i\mathbb{R})$, so $(F^{A_0})^\wedge \in \Omega^2(M, i\mathbb{R})$, and

$$\tilde{\omega} := -\frac{1}{2\pi i}(F^{A_0})^\wedge \in c_1(P).$$

Thus there exists some $\eta \in \Omega^1(M,\mathbb{R})$ with $\omega - \tilde{\omega} = d\eta$. Then $\bar{\eta} \in \Omega^1_{\mathrm{hor}}(P,\mathbb{R})$ and $A := A_0 - 2\pi i\bar{\eta} \in \mathcal{C}(P)$ by Lemma 3.6.1. Also, (3.6.7) implies $(F^A - F^{A_0})^\wedge = -2\pi i d\eta$, so that

$$(F^A)^\wedge = (F^{A_0})^\wedge - 2\pi i d\eta = (F^{A_0})^\wedge - 2\pi i(\omega - \tilde{\omega}) = (F^{A_0})^\wedge - 2\pi i\omega - (F^{A_0})^\wedge = -2\pi i\omega.$$

Now suppose that $A_1, A_2 \in \mathcal{C}(P)$ are such that $\omega = -\frac{1}{2\pi i}(F^{A_j})^\wedge$ for $j = 1, 2$. Then in particular $F^{A_1} = F^{A_2}$ and $(A_1 - A_2)^\wedge = 2\pi i\nu$ for some $\nu \in \Omega^1(M,\mathbb{R})$. Moreover, by (3.6.7) $(2\pi i)d\nu = (F^{A_1} - F^{A_2})^\wedge = 0$, i.e., ν is closed.

Finally, if M is simply connected then ν is exact, so there exists some $h \in \mathcal{C}^\infty(M,\mathbb{R})$ with $(A_1 - A_2)^\wedge = 2\pi i dh$. Set $\sigma := e^{2\pi i h} \in \mathcal{C}^\infty(M, S^1)$. Then $d\sigma = 2\pi i\sigma dh$, so $(A_1 - A_2)^\wedge = \sigma^{-1}d\sigma$. We arrive at

$$A_1 - A_2 = \overline{\sigma^{-1}d\sigma} = \sigma^{-1}\overline{d\sigma} = \sigma^{-1}d\bar{\sigma} \equiv \sigma^{-1}d\sigma,$$

because S^1 is abelian and we can identify σ with $\bar{\sigma}$ (cf. the remark preceding Lemma 3.6.1). By (3.6.5) then $A_1 = f^*A_2$ for some $f \in \mathcal{G}(P)$, i.e., A_1 and A_2 are gauge equivalent. $\square$

Chapter 4

Holonomy theory

4.1 Reduction and extension of connections

In this section we study the effect of reducing or extending principal fiber bundles on connections on these bundles.

4.1.1 Theorem. *Let (P, π_P, M, G) be a principal fiber bundle, $\lambda : H \to G$ a Lie group homomorphism and $((Q, \pi_Q, M, H), f)$ a λ-reduction of P. Also, let A be a connection form on Q. Then there is a unique connection form $\tilde{A}$ on P such that, for each $q \in Q$,*

$$T_q f(Th_q^A Q) = Th_{f(q)}^{\tilde{A}} P. \tag{4.1.1}$$

For the corresponding connection and curvature forms we have

$$f^* \tilde{A} = \lambda_* \circ A, \tag{4.1.2}$$

$$f^* F^{\tilde{A}} = \lambda_* \circ F^A. \tag{4.1.3}$$

Proof. Recall from Section 2.5 that we have

$$\begin{tikzcd} Q \times H \arrow[r, "\cdot"] \arrow[d, "f\times\lambda"'] & Q \arrow[d, "f"] \arrow[dr, "\pi_Q"] & \\ P \times G \arrow[r, "\cdot"] & P \arrow[r, "\pi_P"] & M \end{tikzcd}$$

Now let $p \in P_x$, pick any $q \in Q_x$ and let g be the unique element of G with $f(q)g = p$. Then set

$$Th_p P := TR_g Tf(Th_q^A Q) \subseteq T_p P. \tag{4.1.4}$$

We want to show that

$$Th : P \ni p \mapsto Th_p P \subseteq T_p P$$

defines a connection on P. To begin with, $Th_p P$ is well-defined, i.e., independent of q: If $p = f(q_1)g_1$, then $q_1 = qh$ for some $h \in H$. Then

$$f(q)g = p = f(qh)g_1 = f(q)\lambda(h)g_1,$$

and since G acts simply transitively on P_x we have $g = \lambda(h)g_1$. This, together with the fact that $f \circ R_h = R_{\lambda(h)} \circ f$ (see Definition 2.2.3), implies

$$\begin{aligned} TR_{g_1} Tf(Th_{q_1}^A Q) &= TR_{g_1} Tf(TR_h(Th_q^A Q)) = TR_{g_1} TR_{\lambda(h)} Tf(Th_q^A Q) \\ &= TR_g Tf(Th_q^A Q). \end{aligned}$$

Th is right invariant: $f(q)ga = pa$, so

$$TR_a(Th_pP) = TR_aTR_gTf(Th_q^AQ) = TR_{ga}Tf(Th_q^AQ) = Th_{pa}P.$$

Next, Th_pP is complementary to Tv_pP: By Definition 2.2.3, $\pi_P \circ f = \pi_Q$, so $T_{f(q)}\pi_P \circ T_qf = T_q\pi_Q : Th_qQ \to T_xM$, which is an isomorphism. Thus $T_{f(q)}\pi_P : T_qf(Th_qQ) \to T_xM$ is an isomorphism, and in particular $\dim Th_pP = \dim T_qf(Th_qQ) = \dim M$, so $\dim Th_pP + \dim Tv_pP = \dim P$ and it remains to show that $Th_pP \cap \ker(T_p\pi_P) = \{0\}$. So let $v = TR_g \circ T_qf(w)$ for some $w \in Th_qQ$, and

$$0 = T\pi_P(v) = T\pi_P \circ TR_g \circ T_qf(w) = T\pi_P \circ T_qf(w) = T\pi_Q(w).$$

Then $w = 0$, so $v = 0$.

Finally, to see that the distribution ThP is smooth, we have to show that q and g in (4.1.4) can be chosen to depend smoothly on p. Let ϕ_P and ϕ_Q be bundle charts and set $q := \phi_Q^{-1}(\pi(p), e)$ and $g := \varphi_P(f(q))^{-1} \cdot \varphi_P(p)$, both clearly smooth. Then

$$\phi_P(f(q)\cdot g) = (\pi_P(f(q)\cdot g), \varphi_P(f(q)\cdot g)) = (x, \varphi_P(f(q))\varphi_P(f(q))^{-1}\varphi_P(p)) = \phi_P(p).$$

So we have proved that ThP is a connection on P. Uniqueness follows since (4.1.1) completely fixes the horizontal tangent spaces (due to right invariance, which forces (4.1.4) to hold), and ThP satisfies that equation (set $g = e$ in (4.1.4)).

Denote by $\tilde{A}$ the connection form corresponding to ThP via Theorem 3.1.4. To verify (4.1.2), let $X \in Th_q^AQ$. Then

$$\lambda_*(A(X)) = 0 \quad \text{and} \quad (f^*\tilde{A})(X) = \tilde{A}(Tf(X)) = 0$$

because $Tf(X)$ is horizontal by (4.1.1). If, on the other hand, $\tilde{Y}(q) \in Tv_qQ$ ($Y \in \mathfrak{h}$) is an arbitrary vertical vector, then first note that due to $\exp^G \circ \lambda_* = \lambda \circ \exp^H$ ([9, 8.8]),

$$\begin{aligned}(\lambda_*Y)^{\sim}(f(q)) &\underset{3.1.1}{=} \frac{d}{dt}\Big|_0 \big(f(q)\cdot \exp^G(t\lambda_*Y)\big) = \frac{d}{dt}\Big|_0 \big(f(q)\lambda(\exp^H(tY))\big) \\ &= \frac{d}{dt}\Big|_0 \big(f(q\cdot \exp^H(tY))\big) = Tf\Big(\frac{d}{dt}\Big|_0 (q\cdot \exp^H(tY))\Big) \underset{3.1.1}{=} Tf(\tilde{Y}(q)).\end{aligned}$$

Therefore,

$$(f^*\tilde{A})(\tilde{Y}(q)) = \tilde{A}(Tf(\tilde{Y}(q))) = \tilde{A}((\lambda_*(Y))^{\sim}(f(q))) = \lambda_*(Y) = \lambda_*(A(\tilde{Y}(q))).$$

To prove the remaining claim, we use the structure equation from Theorem 3.5.3:

$$\begin{aligned}f^*F^{\tilde{A}} &= f^*d\tilde{A} + \frac{1}{2}f^*[\tilde{A},\tilde{A}] \underset{(3.5.5)}{=} d(f^*\tilde{A}) + \frac{1}{2}[f^*\tilde{A}, f^*\tilde{A}] \\ &\underset{(4.1.2)}{=} d(\lambda_* \circ A) + \frac{1}{2}[\lambda_* \circ A, \lambda_* \circ A] = \lambda_* \circ dA + \frac{1}{2}\lambda_* \circ [A,A] = \lambda_* \circ F^A,\end{aligned}$$

where we used the fact that λ_* is a Lie algebra homomorphism. □

4.1.2 Remark. The Theorem shows that (4.1.2) follows from (4.1.1). Conversely, if there exists a connection form $\tilde{A}$ on P such that (4.1.2) holds, then also (4.1.1) is satisfied, so these conditions on A and $\tilde{A}$ are indeed equivalent. To see this, let $X \in Th_q^AQ$, then $\tilde{A}(T_qf(X)) = (f^*\tilde{A})(X) = \lambda_*(A(X)) = 0$, so $T_qf(X) \in Th_{f(q)}^{\tilde{A}}P$. On the other hand, given $Y \in Th_{f(q)}^{\tilde{A}}P$, there is a unique A-horizontal lift $X \in Th_q^AQ$ of $T_{f(q)}\pi_P(Y) \in T_xM$ (with $x = \pi_P(f(q)) = \pi_Q(q)$). Then by what we just showed $T_qf(X)$ is $\tilde{A}$-horizontal and satisfies $T_{f(q)}\pi_P(T_qf(X)) = T\pi_Q(X) = T_{f(q)}\pi_P(Y)$, implying $Y = T_qf(X) \in T_qf(Th_q^AQ)$ by uniqueness of horizontal lifts, showing (4.1.1).

4.1.3 Definition. *The connection form $\tilde{A} \in \mathcal{C}(P)$ from Theorem 4.1.1 is called the λ-extension of $A \in \mathcal{C}(Q)$. Conversely, given $\tilde{A} \in \mathcal{C}(P)$, a connection form $A \in \mathcal{C}(Q)$ that satisfies* (4.1.1) *(or, equivalently,* (4.1.2)*) is called a λ-reduction of $\tilde{A}$. If $H \subseteq G$ is a Lie subgroup and $Q \subseteq P$ is an H-reduction of P, then A is simply called a reduction of $\tilde{A}$ to Q and $\tilde{A}$ is called reducible to Q if such an $A \in \mathcal{C}(Q)$ exists.*

While a λ-extension always exists (due to Theorem 4.1.1), the same is not necessarily true for λ-reductions. For Lie subgroups, the next Theorem will give a simple characterization. In its proof we shall make use of:

4.1.4 Lemma. *Let $j : M^m \hookrightarrow N^n$ be an immersion, let $X \in \mathfrak{X}(M)$ and let $p \in M$. Then there exists a neighborhood U of p in M and some $\hat{X} \in \mathfrak{X}(N)$ such that $\hat{X} \circ j|_U = Tj \circ X|_U$.*

Proof. See Appendix. □

4.1.5 Theorem. *Let $H \subseteq G$ be a Lie subgroup, $Q \subseteq P$ an H-reduction of the principal fiber bundle P and $\tilde{A}$ a connection form on P. The following are equivalent:*

(i) $\tilde{A}$ is reducible to Q.

(ii) On TQ, $\tilde{A}$ only takes values in $\mathfrak{h}$.

(iii) $Th_q^{\tilde{A}}P \subseteq T_qQ$ for all $q \in Q$.

Proof. As before (but contrary to what we did in [9]) we will notationally suppress the tangent map of the inclusion $\iota : Q \hookrightarrow P$ and view T_qQ directly as a subspace of T_qP, and similarly for $H \subseteq G$, where for $\lambda = H \hookrightarrow G$ we identify $T_e\lambda = \lambda_*$ with $\mathfrak{h} \hookrightarrow \mathfrak{g}$. Then (4.1.1) takes the form $Th_q^AQ = Th_q^{\tilde{A}}P$, and (4.1.2) reads $\tilde{A}|_{TQ} = A$.

(i)⇒(ii): By Definition 4.1.3 there exists a connection form A on Q with $\tilde{A}|_{TQ} = A$, and A takes values in $\mathfrak{h}$ by definition.

(ii)⇒(i): Recall from Remark 2.5.3 that in terms of suitable bundle charts the inclusion map $Q \hookrightarrow P$ can locally be written as $\mathrm{id}_U \times \iota_{H \hookrightarrow G}$. This shows that Q is an immersive submanifold of P, and in particular $Q \hookrightarrow P$ is smooth. Now set $A := \tilde{A}|_{TQ}$. To see that A is smooth, let X be a smooth local vector field on Q and let $\hat{X}$ be a smooth local extension to P as in Lemma 4.1.4. Then $A(X) = q \mapsto \tilde{A}_q(\hat{X}(q)) = \tilde{A}(\hat{X}) \circ \mathrm{incl}_{Q \hookrightarrow P}$ is smooth. By assumption, $A \in \Omega^1(Q, \mathfrak{h})$. That it is a connection form is seen as follows: For $h \in H \subseteq G$ and $X \in T_qQ$,

$$(R_h^* A)_q(X) = \tilde{A}_{qh}(TR_hX) = \mathrm{Ad}(h^{-1}) \circ \tilde{A}(X) = \mathrm{Ad}(h^{-1}) \circ A(X).$$

Moreover, for $X \in \mathfrak{h}$ we can calculate the corresponding vertical vector field $\tilde{X}^Q$ in q as

$$\tilde{X}^Q(q) = \frac{d}{dt}\Big|_0 (q \cdot \exp^H(tX)) = \frac{d}{dt}\Big|_0 (q \cdot \exp^G(tX)) = \tilde{X}(q),$$

so $A(\tilde{X}^Q)|_q = \tilde{A}(\tilde{X})|_q = X$.

The connection form A satisfies (4.1.2) by its definition and what we said at the beginning of the proof.

(i)⇒(iii): Let A be a connection form on Q satisfying (4.1.2), i.e., $\tilde{A}|_{TQ} = A$, then by Remark 4.1.2 it also satisfies (4.1.1), which in the current setup reads $Th_q^{\tilde{A}}P = Th_q^AQ \subseteq T_qQ$ for all $q \in Q$.

(iii)⇒(i): Set $Th_qQ := Th_q^{\tilde{A}}P$ for all $q \in Q$. This defines a smooth distribution on Q, for if $X_1, \dots, X_k \in \mathfrak{X}_{\mathrm{loc}}(P)$ is a local basis for $Th^{\tilde{A}}P$, then the X_i are tangential to

Q by (iii), hence by [9, 17.15], their restrictions to Q form a smooth local basis in $\mathfrak{X}_{\mathrm{loc}}(Q)$ for ThQ. Also, that ThQ is right invariant (under H) and complementary to TvQ is immediate from the corresponding properties of ThP (and the fact that $q \cdot h \in Q$ for $q \in Q$, $h \in H$). The connection form A on Q corresponding to ThQ by Theorem 3.1.4 satisfies (4.1.1) by definition. □

4.2 Holonomy group and holonomy bundle of a connection

Our aim here is to find the smallest possible reduction of a given principal fiber bundle to a Lie subgroup of G. In general this subgroup will not be closed (hence will only be an immersive submanifold, cf. [9, 21.9]). In this section, let (P, π, M, G) be a principal fiber bundle over a connected manifold M and let A be a connection form on P. For a path $\gamma : [0,1] \to M$ denote by $\mathcal{P}^A_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$ the parallel transport along γ. For $x \in M$ let

$$\begin{aligned}\Omega(x) &:= \{\gamma \mid \gamma \text{ is a path in } M \text{ that is closed in } x\}\\ \Omega_0(x) &:= \{\gamma \mid \gamma \text{ is a path in } M \text{ that is closed in } x \text{ and null-homotopic}\}.\end{aligned}$$

Let $\gamma : [0,1] \to M$ be closed in x and fix $u \in P_x$. Since G acts simply transitively on the fibers there is a unique element $\mathrm{hol}_u(\gamma) \in G$ with

$$\mathcal{P}^A_\gamma(u) = u \cdot \mathrm{hol}_u(\gamma). \tag{4.2.1}$$

$\mathrm{hol}_u(\gamma)$ is called the holonomy of γ with respect to u.

4.2.1 Lemma. *Let $\gamma, \delta \in \Omega(x)$, $u \in P_x$, $a \in G$ and $\mu : [0,1] \to M$ a path in M with initial point x. Then:*

(i) $\mathrm{hol}_u(\gamma * \delta) = \mathrm{hol}_u(\gamma) \cdot \mathrm{hol}_u(\delta)$.

(ii) $\mathrm{hol}_{ua}(\gamma) = a^{-1} \cdot \mathrm{hol}_u(\gamma) \cdot a$.

(iii) $\mathrm{hol}_{\mathcal{P}^A_\mu(u)}(\mu * \gamma * \mu^-) = \mathrm{hol}_u(\gamma)$.

Proof. All these properties follow from Theorem 3.3.7 and simple transitivity of the action on fibers:

$$\begin{aligned}u \cdot \mathrm{hol}_u(\gamma * \delta) &= \mathcal{P}^A_{\gamma*\delta}(u) = \mathcal{P}^A_\gamma(\mathcal{P}^A_\delta(u)) = \mathcal{P}^A_\gamma(u \cdot \mathrm{hol}_u(\delta)) = \mathcal{P}^A_\gamma(u) \cdot \mathrm{hol}_u(\delta)\\ &= u \cdot \mathrm{hol}_u(\gamma) \cdot \mathrm{hol}_u(\delta) \Rightarrow \text{(i)}.\\ (u \cdot a) \cdot \mathrm{hol}_{u\cdot a}(\gamma) &= \mathcal{P}^A_\gamma(u \cdot a) = \mathcal{P}^A_\gamma(u) \cdot a = u \cdot \mathrm{hol}_u(\gamma) \cdot a \Rightarrow \text{(ii)}\\ \mathcal{P}^A_\mu(u) \cdot \mathrm{hol}_{\mathcal{P}^A_\mu(u)}(\mu * \gamma * \mu^-) &= \mathcal{P}^A_{\mu*\gamma*\mu^-}(\mathcal{P}^A_\mu(u)) = \mathcal{P}^A_{\mu*\gamma}(u)\\ &= \mathcal{P}^A_\mu(\mathcal{P}^A_\gamma(u)) = \mathcal{P}^A_\mu(u \cdot \mathrm{hol}_u(\gamma)) = \mathcal{P}^A_\mu(u) \cdot \mathrm{hol}_u(\gamma) \Rightarrow \text{(iii)}.\end{aligned}$$

□

4.2.2 Definition. *For $u \in P_x$, the group*

$$\mathrm{Hol}_u(A) := \{\mathrm{hol}_u(\gamma) \mid \gamma \in \Omega(x)\} \subseteq G$$

is called the holonomy group *of A with respect to $u \in P$. The group*

$$\mathrm{Hol}^0_u(A) := \{\mathrm{hol}_u(\gamma) \mid \gamma \in \Omega_0(x)\} \subseteq G$$

is called the reduced holonomy group *of A with respect to $u \in P$.*

Both sets are indeed subgroups of G by Lemma 4.2.1. For $\gamma \in \Omega(x)$ and $\gamma_0 \in \Omega_0(x)$ we have $\gamma^- * \gamma_0 * \gamma \in \Omega_0(x)$, hence

$$\mathrm{hol}_u(\gamma)^{-1} \cdot \mathrm{hol}_u(\gamma_0) \cdot \mathrm{hol}_u(\gamma) = \mathrm{hol}_u(\gamma^- * \gamma_0 * \gamma) \in \mathrm{Hol}_u^0(A),$$

implying that $\mathrm{Hol}_u^0(A)$ is a normal subgroup of $\mathrm{Hol}_u(A)$. Moreover, Lemma 4.2.1 (ii) shows that the holonomy groups corresponding to different reference points in the same fiber are conjugate to one another:

$$\mathrm{Hol}_{ua}(A) = a^{-1} \cdot \mathrm{Hol}_u(A) \cdot a \qquad (u \in P_x,\ a \in G).$$

Let $g \in \mathrm{Hol}_u(A)$, $u \cdot g = \mathcal{P}_\gamma^A(u)$ and let $\alpha := \mu * \gamma * \mu^-$, μ a path from x to y. Then $\alpha \in \Omega(y)$ and Theorem 3.3.7 gives

$$\mathcal{P}_\alpha^A(\mathcal{P}_\mu^A(u)) = \mathcal{P}_{\mu*\gamma}^A(u) = \mathcal{P}_\mu^A(\mathcal{P}_\gamma^A(u)) = \mathcal{P}_\mu^A(u \cdot g) = \mathcal{P}_\mu^A(u) \cdot g,$$

so $g \in \mathrm{Hol}_{\mathcal{P}_\mu^A(u)}(A)$. Applying the same reasoning to μ^- we also get $\mathrm{Hol}_{\mathcal{P}_\mu^A(u)}(A) \subseteq \mathrm{Hol}_u(A)$, so altogether

$$\mathrm{Hol}_u(A) = \mathrm{Hol}_{\mathcal{P}_\mu^A(u)}(A). \tag{4.2.2}$$

The next result shows that $\mathrm{Hol}_u(A)$ is indeed even a Lie subgroup of G:

4.2.3 Theorem. *The holonomy group* $\mathrm{Hol}_u(A)$ *is either discrete or is a Lie subgroup of G. The reduced holonomy group* $\mathrm{Hol}_u^0(A)$ *is the connected component of the unit element of* $\mathrm{Hol}_u(A)$. *In particular, if M is simply connected then* $\mathrm{Hol}_u(A)$ *is connected.*

Proof. We begin by showing that if $\mathrm{Hol}_u^0(A) \neq \{e\}$ then it is a connected Lie subgroup of G. To do so, according to [9, 21.12] it suffices to show that any element $g \in \mathrm{Hol}_u^0(A)$ can be connected to $e \in G$ by a (piecewise smooth) path that lies entirely in $\mathrm{Hol}_u^0(A)$. Let $\gamma \in \Omega_0(x)$ be a null-homotopic path that is closed in $x = \pi(u)$ and for which $\mathcal{P}_\gamma^A(u) = ug$. Let $H : [0,1] \times [0,1] \to M$ be a homotopy with $H_s = H(\,.\,, s) \in \Omega_0(x)$ between the constant path H_0 and $H_1 = \gamma$. Since γ is piecewise smooth, we can also choose H piecewise smooth.[1] Let H_s^* be the horizontal lift of H_s with initial point u. By the proof of Theorem 3.3.5, H_s^* is the solution of an ODE whose right hand side is piecewise smooth, so H_s^* is itself piecewise smooth as well. Let g_s be the unique element of G with

$$\mathcal{P}_{H_s}^A(u) = H_s^*(1) = ug_s.$$

Then $[0,1] \ni s \mapsto g_s \in G$ is a piecewise smooth curve in G whose image lies entirely in $\mathrm{Hol}_u^0(A)$ and which connects e to g, proving our claim.

Next, consider the map

$$\begin{aligned} \rho : \pi_1(M, x) &\to \mathrm{Hol}_u(A)/\mathrm{Hol}_u^0(A) \\ [\gamma] &\mapsto \mathrm{hol}_u(\gamma) \bmod \mathrm{Hol}_u^0(A), \end{aligned} \tag{4.2.3}$$

with γ a piecewise smooth representative of the homotopy class.[1] To see that ρ is well-defined, let γ_1, γ_2 be paths with $[\gamma_1] = [\gamma_2]$. Then $\tau := \gamma_2^- * \gamma_1$ is null homotopic and $\mathrm{hol}_u(\gamma_1) = \mathrm{hol}_u(\gamma_2) \cdot \mathrm{hol}_u(\tau)$. Thus $\mathrm{hol}_u(\gamma_1)$ and $\mathrm{hol}_u(\gamma_2)$ lie in the same equivalence class of $\mathrm{Hol}_u(A)/\mathrm{Hol}_u^0(A)$. In addition, ρ is a group homomorphism:

$$\rho([\gamma] \cdot [\delta]) = \rho([\gamma * \delta]) = [\mathrm{hol}_u(\gamma * \delta)] = [\mathrm{hol}_u(\gamma) \cdot \mathrm{hol}_u(\delta)] = \rho([\gamma]) \cdot \rho([\delta]).$$

The fundamental group of a smooth manifold is at most countable ([11, Th. 8.11]). Since ρ is surjective, also $\mathrm{Hol}_u(A)/\mathrm{Hol}_u^0(A)$ is at most countable. In particular,

[1] See, e.g., [12, Lem. 6.6].

$\mathrm{Hol}_u(A)$ is the union of at most countably many disjoint orbits of the form $g_n \cdot \mathrm{Hol}^0_u(A)$, with $g_n \in \mathrm{Hol}_u(A)$. By declaring $x \mapsto g_n \cdot x$ to be a diffeomorphism we may introduce a smooth structure on each such orbit so that $\mathrm{Hol}_u(A)$ becomes an immersive submanifold of G, and since $\mathrm{Hol}^0_u(A)$ is an integral manifold of an integrable distribution (by [9, 19.3]), so is $\mathrm{Hol}_u(A)$. The group multiplication μ on $\mathrm{Hol}_u(A)$ can be decomposed according to

$$\begin{aligned} \mathrm{Hol}_u(A) \times \mathrm{Hol}_u(A) &\to \mathrm{Hol}^0_u(A) \times \mathrm{Hol}^0_u(A) \to \mathrm{Hol}_u(A) \\ (g_n x, g_m y) &\mapsto (x, y) \to g_n x g_m y \end{aligned}$$

and is therefore smooth as a map into G. But then by [9, 17.26], also $\mu : \mathrm{Hol}_u(A) \times \mathrm{Hol}_u(A) \to \mathrm{Hol}_u(A)$ is smooth, hence $\mathrm{Hol}_u(A)$ is a Lie subgroup of G. By construction, $\mathrm{Hol}^0_u(A)$ is the connected component of e. Finally, if M is simply connected then any closed path is null homotopic, so $\mathrm{Hol}_u(A) = \mathrm{Hol}^0_u(A)$ and thereby $\mathrm{Hol}_u(A)$ is connected. □

The following fundamental result shows that any connection on a principal fiber bundle can be reduced to its holonomy group.

4.2.4 Theorem. *Let (P, π, M, G) be a principal fiber bundle with connection A over a connected manifold M and fix $u \in P$. Let $\mathrm{Hol}_u(A)$ be non-discrete and set*

$$P^A(u) := \{p \in P \mid \exists A\text{-horizontal path from } u \text{ to } p\}.$$

Then:

(i) $(P^A(u), \pi|_{P^A(u)}, M, \mathrm{Hol}_u(A))$ is a principal fiber bundle.

(ii) (P, π, M, G) and A reduce to $(P^A(u), \pi|_{P^A(u)}, M, \mathrm{Hol}_u(A))$.

Proof. We verify the three conditions from Theorem 2.5.4 to show that $P^A(u) \subseteq P$ carries the structure of a $\mathrm{Hol}_u(A)$-reduction of P. Let $x = \pi(u)$.

1.) Let $p \in P^A(u)$ and $h \in \mathrm{Hol}_u(A)$. This means that there exists a horizontal path δ from u to p and a $\mu \in \Omega(x)$ with $\mathcal{P}^A_\mu(u) = uh$ (i.e., the horizontal lift μ^*_u of μ through u connects u to $u \cdot h$). From Theorem 3.3.7 (iii) we gather that $R_h \circ \delta$ is a horizontal path from uh to ph. Thus $(R_h \circ \delta) * \mu^*_u$ is horizontal and connects u to ph, implying that $R_h(P^A(u)) \subseteq P^A(u)$. The same argument with h^{-1} in place of h shows that indeed we have $R_h(P^A(u)) = P^A(u)$.

2.) Let $p, \tilde{p} \in P^A(u) \cap P_y$ and $p = \tilde{p}g$. Pick horizontal paths δ from u to p and $\tilde{\delta}$ from u to $\tilde{p}$, and set $\mu := \pi(\tilde{\delta}^-) * \pi(\delta) \in \Omega(x)$. Again by Theorem 3.3.7, the horizontal lift of μ through u is given by $\mu^*_u = (\tilde{\delta}^- \cdot g) * \delta$, which has endpoint ug. Thus $g \in \mathrm{Hol}_u(A)$.

3.) Let $y \in M$. Then since M is connected there exists a path δ in M from x to y. Then the endpoint $v = \delta^*_u(1) \in P_y$ of the horizontal lift of δ through u lies in $P^A(u)$. Fix any Riemannian metric on M and let U_y be a normal neighborhood of y in M. For $z \in U_y$ denote by γ_{yz} the radial geodesic from y to z in U_y. We define a section $s : U_y \to P$ by

$$s(z) := \mathcal{P}^A_{\gamma_{yz}}(v) = \mathcal{P}^A_{\gamma_{yz}} \circ \mathcal{P}^A_\delta(u).$$

Then s is smooth since γ_{yz} depends smoothly on z and horizontal lifts are solutions of ODEs (proof of Theorem 3.3.7). The second form of s shows that $s(U_y) \subseteq P^A(u)$.

Thus Theorem 2.5.4 implies that $P^A(u)$ has the structure of a $\mathrm{Hol}_u(A)$-principal fiber bundle that is a reduction of P and it only remains to show that also the connection form A reduces to $P^A(u)$. According to Theorem 4.1.5 it suffices to show that $Th^A_q P \subseteq T_q P^A(u)$ for all $q \in P^A(u)$. So let $q \in P^A(u)$ and let $X \in Th^A_q P$. Let σ be a smooth curve in P with $\dot{\sigma}(0) = X$ and set $\gamma := \pi \circ \sigma$. Then by (3.1.3),

$\dot{\gamma}_q^*(0) = X$. Since q can be connected to u by a horizontal path, the image of γ_q^* lies in $P^A(u)$. If we we can show that γ_q^* is smooth as a map into $P^A(u)$ then we can conclude that $X \in T_q P^A(u)$ and are done. Since $P^A(u)$ is only an immersive submanifold of P this requires an argument: By the proof of Theorem 3.3.5 we have $\gamma_q^*(t) = \delta(t) \cdot g(t)$ and since the only requirements on δ are that $\delta(0) = q$ and that it projects to γ, we can choose it as a smooth curve into $P^A(u)$: In fact, with s as in 3.) around $\pi(q)$, let $\hat{\delta}(t) := s(\gamma(t))$ and pick $h \in H$ such that $\hat{\delta}(\gamma(0)) \cdot h = q$. Then $\delta(t) := \hat{\delta}(t) \cdot h$ is smooth by Theorem 2.5.4 and has the required properties. Also, $g : I \to G$ is smooth. Since $\delta(t) \in P^A(u)$ and $\gamma_q^*(t) = \delta(t) \cdot g(t) \in P^A(u)$, point 2.) above implies that $g(t) \in \mathrm{Hol}_u(A)$ for all t. Also, as $\mathrm{Hol}_u(A)$ is a Lie subgroup, [9, 19.3,17.27] show that $g : I \to \mathrm{Hol}_u(A)$ is smooth, hence so is $t \mapsto \gamma_q^*(t) = \delta(t) \cdot g(t)$ as a map into $P^A(u)$.

Altogether, $(P^A(u), \pi|_{P^A(u)}, M, \mathrm{Hol}_u(A))$ is a principal fiber bundle that is a reduction of (P, A). □

4.2.5 Definition. *The principal fiber bundle $(P^A(u), \pi|_{P^A(u)}, M, \mathrm{Hol}_u(A))$ is called the* holonomy bundle *of A with respect to u.*

4.2.6 Remark. For $u, v \in P$ it follows immediately from the definitions that $P^A(u) = P^A(v)$ if and only if u and v can be joined by a horizontal path. Since being joined by such a path gives an equivalence relation, either $P^A(u) = P^A(v)$ or $P^A(u) \cap P^A(v) = \varnothing$, so P is decomposed into the disjoint union of its holonomy subbundles.

4.2.7 Definition. *Let (P, π, M, G) be a principal fiber bundle over a connected manifold M. A connection form $A \in \mathcal{C}(P)$ is called* irreducible *if (P, A) cannot be reduced to a proper Lie subgroup of G.*

The holonomy bundle $P^A(u)$ is the "smallest" possible reduction of P in the following sense:

4.2.8 Theorem. *Let (P, A) be a principal fiber bundle with connection and let $(Q, \hat{A})$ be a reduction of (P, A) to a Lie subgroup H of G, where $Q \subseteq P$ is an immersive submanifold of P. Then*

(i) $P^A(u) \subseteq Q$ for each $u \in Q$.

(ii) $\hat{A}|_{TP^A(u)} = A|_{TP^A(u)}$, i.e., $\hat{A}$ reduces to the connection induced (by Theorem 4.2.4) on $P^A(u)$ by A.

Proof. (i) Let $p \in P^A(u)$ and let γ^* be an A-horizontal path $[0,1] \to P$ with $\gamma^*(0) = u$ and $\gamma^*(1) = p$, so that γ^* is the A-horizontal lift of $\gamma := \pi \circ \gamma^* : [0,1] \to M$. By the proof of Theorem 4.1.5 we have $Th_q^{\hat{A}} Q = Th_q^A P$ for all $q \in Q$. Let $\gamma^{\hat{A}}$ be the $\hat{A}$-horizontal lift of γ in Q through u. Then $\gamma^{\hat{A}}$ is also A-horizontal and by the uniqueness of horizontal lifts it follows that $\gamma^{\hat{A}} = \gamma^*$, which thereby lies entirely in Q. In particular, $p = \gamma^*(1) \in Q$. Note that we have thereby shown that for each $u \in Q$, the A-horizontal paths in P emanating from u are precisely the $\hat{A}$-horizontal paths in Q emanating from u, so $P^A(u) = Q^{\hat{A}}(u)$. For the latter set we know by Theorem 4.2.4 (and Theorem 2.5.4) that it is an immersive submanifold of Q.

(ii) Since both f and λ from Theorem 4.1.1 are the inclusion maps, (4.1.2) says that $\hat{A} = A|_{TQ}$. As noted in (i), $P^A(u)$ is an immersive submanifold of Q, so $T_q P^A(u) \subseteq T_q Q$, i.e., $T_q P^A(u) \cap T_q Q = T_q P^A(u)$ for all $q \in Q$. Since (by Theorem 4.2.4) $A|_{TP^A(u)}$ is the reduced connection on $P^A(u)$, we obtain

$$\hat{A}|_{TP^A(u)} = (A|_{TQ})|_{TP^A(u)} = A|_{TQ \cap TP^A(u)} = A|_{TP^A(u)}.$$

$\square$

4.2.9 Remark. By Theorem 4.2.4, therefore, $A \in \mathcal{C}(P)$ is irreducible if and only if $P = P^A(u)$ and $G = \mathrm{Hol}_u(A)$ for all $u \in P$.

The following Theorem gives a criterion for irreducibility and also a measure of the size of the holonomy group. In its proof we will make use of a direct consequence of the Frobenius theorem:

4.2.10 Lemma. *Let E be an integrable distribution on a manifold M and let $x \in M$. Then the leaf of E through x, i.e., the maximal connected integral manifold of E through x is given by the set of all $a \in M$ that can be connected to x by a (piecewise smooth) path $\gamma : I \to M$ with $\dot{\gamma}(t) \in E_{\gamma(t)}$ for all $t \in I$.*

Proof. See the Appendix. $\square$

4.2.11 Theorem. *(Holonomy Theorem of Ambrose and Singer) Let (P, π, M, G) be a principal fiber bundle over a connected manifold M, with connection form A and curvature form $F^A = D_A A$. Then for the Lie algebra $\mathfrak{hol}_u(A)$ of the holonomy group $\mathrm{Hol}_u(A)$ we have*

$$\mathfrak{hol}_u(A) = \mathrm{span}\{F^A_p(X, Y) \mid p \in P^A(u),\ X, Y \in Th^A_p P\} \subseteq \mathfrak{g}. \tag{4.2.4}$$

If G is connected and M is simply connected then A is irreducible if and only if $P = P^A(u)$ and

$$\mathfrak{g} = \mathrm{span}\{F^A_p(X, Y) \mid p \in P^A(u),\ X, Y \in Th^A_p P\}.$$

for all $u \in P$.

Proof. To prove (4.2.4), without loss of generality we may assume that $P = P^A(u)$ and $G = \mathrm{Hol}_u(A)$, since otherwise by Theorem 4.2.4 we may first reduce (P, A) to $P^A(u)$ (without changing the right hand side of (4.2.4), cf. (4.1.1) and (4.1.3)). Let

$$\mathfrak{m} := \mathrm{span}\{F^A_p(X, Y) \mid p \in P,\ X, Y \in Th^A_p P\}.$$

Then we need to prove that $\mathfrak{g} = \mathfrak{m}$. We first show that $\mathfrak{m}$ is an ideal in $\mathfrak{g}$. Let $F^A_p(X, Y) \in \mathfrak{m}$ and $W \in \mathfrak{g}$. Let $g := \exp(tW)$ and $p \in P$, then $T_p R_g(X) \in Th^A_{pg} P$, so for any $t \in \mathbb{R}$,

$$(R^*_{\exp(tW)} F^A)_p(X, Y) = F^A_{pg}(T_p R_g(X), T_p R_g(Y)) \in \mathfrak{m},$$

and hence the same is true for the derivative. Using that F^A is of type Ad, we have

$$\begin{aligned}\mathfrak{m} \ni \frac{d}{dt}\Big|_0 &\big(F^A_{p\exp(tW)}(TR_{\exp(tW)}X, TR_{\exp(tW)}Y)\big) \\ = \frac{d}{dt}\Big|_0 &\big(\mathrm{Ad}(\exp(-tW))(F^A_p(X, Y))\big) = -\mathrm{ad}(W)(F^A_p(X, Y)) = [F^A_p(X, Y), W].\end{aligned}$$

Consequently, $\mathfrak{m}$ is indeed an ideal in $\mathfrak{g}$.
Next we claim that the smooth distribution

$$E : P \ni p \to E_p := Th_p P \oplus \{\tilde{W}(p) \mid W \in \mathfrak{m}\} \subseteq T_p P$$

is involutive. If X is a horizontal vector field and $W \in \mathfrak{m}$, then by Theorem 3.3.2 $[X, \tilde{W}]$ is horizontal, hence lies in E. If $V, W \in \mathfrak{m}$, then $[V, W] \in \mathfrak{m}$ since it is an ideal, and so by Theorem 1.2.2 $[\tilde{V},\tilde{W}] = [V,W]^{\sim} \in E$. Finally, if X and Y are horizontal vector fields then by Theorem 3.5.6 the vertical component of $[X,Y]$ is

$$[X,Y]^v = -F^A(X,Y)^{\sim},$$

and $F^A(X,Y) \in \mathfrak{m}$ by definition. By the Frobenius theorem there exists a maximal connected integral manifold $Q \subseteq P$ of E through $u \in P$ (the leaf of E through u). By Lemma 4.2.10, $q \in P$ lies in Q if and only if there exists a path $\gamma : [0,1] \to P$ from u to q with $\dot{\gamma}(t) \in E_{\gamma(t)}$ for each $t \in [0,1]$. By definition of E, $ThP \subseteq E$, and by that of $P^A(u)$ we therefore get $P = P^A(u) \subseteq Q$. Thus indeed $P = Q$ and we conclude that $TP = TQ = E$. Consequently,

$$\dim Q = \dim E_p = \dim(Th_pP) + \dim \mathfrak{m} = \dim M + \dim \mathfrak{m},$$

so

$$\dim \mathfrak{g} = \dim P - \dim M = \dim Q - \dim M = \dim \mathfrak{m},$$

implying $\mathfrak{g} = \mathfrak{m}$.

Finally, suppose that G is connected and M is simply connected. By Remark 4.2.9, A is irreducible if and only if $P = P^A(u)$ and $G = \mathrm{Hol}_u(A)$ for all $u \in P$. $\mathrm{Hol}_u(A)$ is connected by Theorem 4.2.3 and connected subgroups of a Lie group coincide if and only if their Lie algebras coincide ([9, 19.5]). Thus

$$G = \mathrm{Hol}_u(A) \Leftrightarrow \mathfrak{g} = \mathfrak{hol}_u(A) = \mathfrak{m}.$$

$\square$

Given a vector bundle E with a covariant derivative ∇ one can also define a corresponding holonomy group, as follows: Let $x \in M$ and $\gamma \in \Omega(x)$. Then (cf. the remark preceding Theorem 3.4.12) $\mathcal{P}_\gamma^\nabla \in \mathrm{GL}(E_x)$ is a linear isomorphism of the fiber E_x.

4.2.12 Definition. *The group*

$$\mathrm{Hol}_x(\nabla) := \{\mathcal{P}_\gamma^\nabla \mid \gamma \in \Omega(x)\} \subseteq \mathrm{GL}(E_x)$$

is called the holonomy group *of ∇ with respect to x. The group*

$$\mathrm{Hol}_x^0(\nabla) := \{\mathcal{P}_\gamma^\nabla \mid \gamma \in \Omega_0(x)\} \subseteq \mathrm{GL}(E_x)$$

is called the reduced holonomy group *of ∇ with respect to x.*

A straightforward modification of the proof of Theorem 4.2.3 shows that $\mathrm{Hol}_x(\nabla)$ is either discrete or is a Lie subgroup of $\mathrm{GL}(E_x)$.

In Section 3.4 we studied the interrelation between connection forms on principal fiber bundles and covariant derivatives on associated vector bundles. The relation of the corresponding holonomy groups is clarified in the next Theorem. In its proof we shall make use of the following result on Lie group homomorphisms:

4.2.13 Lemma. *Let $\varphi : G \to H$ be a surjective Lie group homomorphism. Then also $\varphi_* : \mathfrak{g} \to \mathfrak{h}$ is surjective.*

Proof. See the Appendix. $\square$

4.2.14 Theorem. *Let (P,π,M,G) be a principal fiber bundle, $\rho : G \to \mathrm{GL}(V)$ a representation of G and $E := P \times_G V$ the associated vector bundle. Let A be a* *connection form on P and ∇^A the induced covariant derivative on E. Finally, for $u \in P_x$ let $[u] : V \to E_x$ be the fiber isomorphism induced by u (cf. (2.3.7)). Then*

$$\mathrm{Hol}_x(\nabla^A) = [u] \circ \rho(\mathrm{Hol}_u(A)) \circ [u]^{-1}. \tag{4.2.5}$$

In particular, $\mathrm{Hol}_u(A)$ and $\mathrm{Hol}_x(\nabla^A)$ are isomorphic if ρ is injective.
Let R^{∇^A} be the curvature endomorphism of ∇^A and $\mathcal{P}_\gamma := \mathcal{P}_\gamma^{\nabla^A}$ the parallel transport defined by ∇^A in E. Then the Lie algebra of $\mathrm{Hol}_x(\nabla^A)$ is given by

$$\mathfrak{hol}_x(\nabla^A) = \mathrm{span}\left\{\mathcal{P}_\gamma^{-1} \circ R_y^{\nabla^A}(v,w) \circ \mathcal{P}_\gamma \mid v,w \in T_yM,\ \gamma \text{ path from } x \text{ to } y\right\}.$$

Proof. Let δ be a path in M that is closed in x. Then by Theorem 3.4.12 and (3.3.2) with $e = [u,z] \in E_x$ we have

$$\mathcal{P}_\delta^{\nabla^A}(e) = [\mathcal{P}_\delta^A(u), z] = [u \cdot \mathrm{hol}_u(\delta), z] = [u, \rho(\mathrm{hol}_u(\delta))z] = [u] \circ \rho(\mathrm{hol}_u(\delta)) \circ [u]^{-1}(e),$$

giving the first claim.
Let γ^* be an A-horizontal path from u to $p \in P_y$, so $p \in P^A(u)_y$ and let $\gamma := \pi \circ \gamma^*$. Then for any $v \in V$,

$$\mathcal{P}_\gamma^{\nabla^A} \circ [u](v) = \mathcal{P}_\gamma^{\nabla^A}([u,v]) = [\mathcal{P}_\gamma^A(u), v] = [p,v] = [p](v).$$

Let $X, Y \in T_yM$ with A-horizontal lifts X^*, Y^*. Then by Theorem 3.5.13,

$$\begin{aligned}\rho_*(F_p^A(X^*,Y^*)) &= [p]^{-1} \circ R_y^{\nabla^A}(X,Y) \circ [p] \\ &= [u]^{-1} \circ \left((\mathcal{P}_\gamma^{\nabla^A})^{-1} \circ R_y^{\nabla^A}(X,Y) \circ \mathcal{P}_\gamma^{\nabla^A}\right) \circ [u].\end{aligned} \tag{4.2.6}$$

By (4.2.5), $\mathrm{Hol}_x(\nabla^A)$ is the image of $\mathrm{Hol}_u(A)$ under the Lie group homomorphism $\Psi : G \to \mathrm{GL}(E_x)$, $g \mapsto [u] \circ \rho(g) \circ [u]^{-1}$. Hence Lemma 4.2.13 shows that

$$\begin{aligned}\mathfrak{hol}_x(\nabla^A) &= \Psi_*(\mathfrak{hol}_u(A)) = [u] \circ \rho_*(\mathfrak{hol}_u(A)) \circ [u]^{-1} \\ &\underset{(4.2.4)}{=} [u] \circ \mathrm{span}\{\rho_*(F_p^A(X,Y)) \mid p \in P^A(u),\ X,Y \in Th_p^AP\} \circ [u]^{-1}.\end{aligned}$$

Noting that $\{F_p^A(X,Y) \mid X,Y \in Th_p^AP\} = \{F_p^A(X^*,Y^*) \mid X,Y \in T_yM\}$, the second claim follows from (4.2.6). □

4.3 Holonomy groups and parallel sections

Let (E,π,M) be a real or complex vector bundle over a connected manifold M with covariant derivative ∇^E and denote by

$$\mathrm{Par}(E,\nabla^E) := \{\varphi \in \Gamma(E) \mid \nabla^E\varphi = 0\}$$

the set of parallel sections of E.
A subbundle $F \subseteq E$ is called ∇^E-invariant if

$$\nabla_X^E\Gamma(F) \subseteq \Gamma(F) \qquad \forall X \in \mathfrak{X}(M).$$

In this case ∇^E induces a covariant derivative $\nabla^F := \nabla^E|_{\Gamma(F)}$ on F.

4.3.1 Theorem. *Let M be simply connected, $F \subseteq E$ a ∇^E-invariant subbundle of rank $r > 0$, ∇^F the induced covariant derivative on F and suppose that $R^{\nabla^F} = 0$. Then $\dim \mathrm{Par}(E,\nabla^E) \geq r$.*

Proof. Fix $x \in M$ and a basis $(v_1, \dots, v_r)$ of F_x. We generate sections φ_i in F by parallel transporting v_i $(1 \le i \le r)$:

$$\varphi_i : M \ni y \mapsto \mathcal{P}^{\nabla^F}_{\gamma_{xy}}(v_i) \in F_y,$$

where γ_{xy} is a path in M from x to y. To see that φ_i is well-defined we have to show that it does not depend on the choice of path γ_{xy}. We know from Remark 2.4.4 that F is associated to a $\mathrm{GL}(r, \mathbb{K})$-frame bundle P, i.e., $F = P \times_{\mathrm{GL}(r,\mathbb{K})} \mathbb{K}^r$. Also, by Remark 3.4.14 there is a connection form A on P such that $\nabla^A = \nabla^F$. Then Theorem 3.5.13 (with $\rho = \mathrm{id}$ the natural representation of $\mathrm{GL}(r, \mathbb{K})$ on $\mathbb{K}^r$) together with our assumption on R give

$$0 = R^{\nabla^F}(X, Y)\varphi = [p, F^A_p(X^*, Y^*)v], \qquad \text{for } \varphi = [p, v] = [p](v),$$

where X^*, Y^* are horizontal lifts of X and Y, respectively. Since $[p]$ is bijective, $F^A(X^*, Y^*)(v) = 0$ for all v, hence $F^A(X^*, Y^*) = 0$ for all X^*, Y^*, i.e., $F^A = 0$ since it is a horizontal form. As M is assumed to be simply connected, parallel transport $\mathcal{P}^A$ in P then is path-independent due to (the proof of) Theorem 3.5.11. Again by Remark 3.4.14, for any $v \in F$,

$$\mathcal{P}^{\nabla^F}_\gamma \circ [p](v) = \mathcal{P}^{F,A}_\gamma \circ [p](v) = \mathcal{P}^{F,A}_\gamma([p, v]) \underset{(3.3.2)}{=} [\mathcal{P}^A_\gamma(p), v] = [\mathcal{P}^A_\gamma(p)](v),$$

implying that $\mathcal{P}^{\nabla^F}$ is path-independent, and so φ_i is indeed well-defined. It is smooth by the smooth dependence of solutions of ODEs on the initial conditions. Since parallel transport is a linear isomorphism (cf. (3.4.17)), the φ_i are linearly independent.

To conclude the proof it therefore suffices to show that the sections φ_i are parallel. Using Theorem 3.4.11 we calculate:

$$\begin{aligned}
\big(\nabla^F_X \varphi_i\big)(y) &= \frac{d}{dt}\Big|_0 \big(\mathcal{P}^{\nabla^F}_{\gamma(t),y}(\varphi_i(\gamma(t)))\big) \\
&= \frac{d}{dt}\Big|_0 \big(\mathcal{P}^{\nabla^F}_{\gamma(t),y} \mathcal{P}^{\nabla^F}_{x,\gamma(t)}(v_i)\big) = \frac{d}{dt}\Big|_0 \big(\mathcal{P}^{\nabla^F}_{x,y}(v_i)\big) = 0.
\end{aligned}$$

□

This result shows that one strategy for finding parallel sections in E consists in detecting flat subbundles of E. Another possibility makes use of the holonomy group: Let (P, π, M, G) be a principal fiber bundle with M connected and connection form A, let $\rho : G \to \mathrm{GL}(V)$ be a representation of G, $E = P \times_G V$ the associated vector bundle and ∇^E the covariant derivative induced on E via Definition 3.4.8. In this situation we have:

4.3.2 Theorem. *(The holonomy principle) There exists a bijective correspondence between the space of parallel sections of E and the set of holonomy invariant vectors in V:*

$$\mathrm{Par}(E, \nabla^E) \overset{1:1}{\longleftrightarrow} \{v \in V \mid \rho(\mathrm{Hol}_u(A))v = v\}.$$

If M is simply connected, then in addition we have

$$\{v \in V \mid \rho(\mathrm{Hol}_u(A))v = v\} = \{v \in V \mid \rho_*(\mathfrak{hol}_u(A))v = 0\}.$$

Proof. Let $x = \pi(u)$ and abbreviate $\mathrm{Hol}_u(A)$ by H. Given $v \in V$ such that $\rho(H)v = v$, define $\varphi_v \in \Gamma(E)$ by

$$\varphi_v : M \ni y \mapsto [\mathcal{P}^A_\gamma(u), v] \in E_y,$$

where γ is a path in M from x to y. To show that φ_v is well-defined we have to establish independence from the choice of γ. So let μ be another path from x to y. Then $\mu^- * \gamma \in \Omega(x)$ and by Theorem 3.3.7

$$\mathcal{P}^A_{\mu^- * \gamma}(u) = \mathcal{P}^A_{\mu^-}(\mathcal{P}^A_\gamma(u)) = u \cdot h$$

for some $h \in H$. Thus $\mathcal{P}^A_\gamma(u) = \mathcal{P}^A_\mu(uh)$ and thereby

$$[\mathcal{P}^A_\gamma(u), v] = [\mathcal{P}^A_\mu(u)h, v] = [\mathcal{P}^A_\mu(u), \rho(h)v] = [\mathcal{P}^A_\mu(u), v].$$

Now

$$\varphi_v(y) = [\mathcal{P}^A_\gamma(u), v] \underset{(3.3.2)}{=} \mathcal{P}^{E,A}_\gamma([u,v]) \underset{3.4.12}{=} \mathcal{P}^{\nabla^A}_\gamma([u,v]),$$

so exactly as in the proof of Theorem 4.3.1 it follows that φ_v is smooth and parallel.

Conversely, let $\varphi \in \Gamma(E)$ be parallel. By Theorem 4.2.4, (P, A) reduces to the holonomy bundle $Q := P^A(u)$, and Theorem 2.5.8 shows that

$$E = P \times_G V \cong Q \times_H V.$$

Using Theorem 2.3.4 we may therefore write φ in the form

$$\varphi(\pi(p)) = [p, \bar\varphi(p)]_{P\times_G V} = [q, \bar\psi(q)]_{Q\times_H V} = \varphi(\pi_Q(q))$$

for functions $\bar\varphi \in \mathcal{C}^\infty(P, V)^G$ and $\bar\psi \in \mathcal{C}^\infty(P^A(u), V)^H$. The proof of Theorem 2.3.4 shows that, since for $q \in Q$ we have $\pi(q) = \pi_Q(q)$,

$$[q, \bar\varphi(q)]_{P\times_G V} = \varphi(\pi(q)) = \varphi(\pi_Q(q)) = [q, \bar\psi(q)]_{Q\times_H V} \underset{2.5.8}{=} [q, \bar\psi(q)]_{P\times_G V},$$

and since $[q]$ is bijective this implies that $\bar\varphi(q) = \bar\psi(q)$.

By Definition 3.4.8 and (3.4.7),

$$\big(\nabla^E_X \varphi\big)(\pi(p)) = [p, d\bar\varphi(X^*(p))] \underset{(3.4.1)}{=} [p, X^*(\bar\varphi)(p)],$$

with X^* the horizontal lift of X. From this relation we read off that φ is parallel if and only if $X^*(\bar\varphi) = 0$ for each horizontal vector X^*, i.e., if and only if $\bar\varphi$ is constant along every horizontal path. Thus there exists a vector $v \in V$ with $\bar\psi = \bar\varphi|_{P^A(u)} \equiv v$. By the invariance property of $\bar\psi \in \mathcal{C}^\infty(P^A(u), V)^H$, for each $h \in H$ we have

$$v \equiv \bar\psi(qh) = \rho(h)^{-1}\bar\psi(q) = \rho(h^{-1})v,$$

so $\rho(H)v = v$. Moreover, taking for γ the constant path $\gamma \equiv x$,

$$\varphi_v(x) = \varphi_v(\pi(u)) = [\mathcal{P}^A_\gamma(u), v] = [u, v] = \varphi(x),$$

and since both sections are parallel we obtain that $\varphi = \varphi_v$, concluding the proof of the first claim.

If $\rho(H)v = v$, then for any $X \in \mathfrak{hol}_u(A)$ we get by differentiating $\rho(\exp(tX))v = v$ at $t = 0$ that $\rho_*(X)v = 0$. Conversely, if $\rho_*(X)v = 0$ for each $X \in \mathfrak{hol}_u(A)$, then by [9, 8.8]

$$\rho(\exp(X))v = \exp(\rho_*(X))v = e^{\rho_*(X)}v = v.$$

If M is simply connected then $\mathrm{Hol}_u(A)$ is connected by Theorem 4.2.3. It is therefore generated by $\exp(\mathfrak{hol}_u(A))$, and so in this case

$$\{v \in V \mid \rho(\mathrm{Hol}_u(A))v = v\} = \{v \in V \mid \rho_*(\mathfrak{hol}_u(A))v = 0\}.$$

□

4.4 Holonomy groups of semi-Riemannian manifolds

In this final section we want to briefly look at the important special case of holonomy theory for semi-Riemannian manifolds with their natural Levi-Civita connection, referring to [1, Ch. 5] for much more information. Thus let (M,g) be a connected manifold with semi-Riemannian metric g and denote by $\nabla^g \equiv \nabla^{\mathrm{LC}}$ the Levi-Civita covariant derivative on M induced by g (cf. Example 3.1.12).

For any path $\gamma : [a,b] \to M$ from x to y and $v \in T_x M$ denote by X_v the vector field along γ that is generated by parallel transport of v along γ, i.e., (cf. (3.4.16))

$$\frac{\nabla^g X_v}{dt} = 0, \qquad X_v(a) = v.$$

Since the Levi-Civita connection is metric, the parallel transport defined via X_v (cf. (3.4.17))

$$\begin{aligned}\mathcal{P}^g_\gamma : T_x M &\to T_y M\\ v &\mapsto X_v(b)\end{aligned}$$

is a linear isometry between the spaces $(T_x M, g_x)$ and $(T_y M, g_y)$. In particular, if γ is a closed path then $\mathcal{P}^g_\gamma$ is an orthogonal map on $(T_x M, g_x)$.

The *holonomy group* of (M,g) with respect to $x \in M$ is the holonomy group of the Levi-Civita connection ∇^g (see Definition 4.2.12):

$$\mathrm{Hol}_x(M,g) := \{\mathcal{P}^g_\gamma : T_x M \to T_x M \mid \gamma \in \Omega(x)\} \subseteq \mathrm{O}(T_x M, g_x).$$

The reduced holonomy group of (M,g) with respect to x is

$$\mathrm{Hol}^0_x(M,g) := \{\mathcal{P}^g_\gamma : T_x M \to T_x M \mid \gamma \in \Omega_0(x)\} \subseteq \mathrm{Hol}_x(M,g).$$

Holonomy groups in different points are conjugated (cf. the discussion following Definition 4.2.2):

$$\mathrm{Hol}_y(M,g) = \mathcal{P}^g_\sigma \circ \mathrm{Hol}_x(M,g) \circ \mathcal{P}^g_{\sigma^-}, \tag{4.4.1}$$

with σ any path from x to y.

If two semi-Riemannian manifolds are isometric then obviously their holonomy groups are isomorphic, but the converse is not true in general. This raises two fundamental questions:

1. Which groups can occur as holonomy groups of semi-Riemannian manifolds?
2. Which geometric properties of (M,g) are characterized by its holonomy group?

For Riemannian manifolds the answers to these questions are known, whereas in the general semi-Riemannian setting the situation is not yet completely understood.

If g has signature (p,q) then by (2.5.3) we have

$$TM \cong \mathrm{O}(M,g) \times_{\mathrm{O}(p,q)} \mathbb{R}^{p+q}. \tag{4.4.2}$$

Also, we know from Example 3.4.15 that ∇^g is associated to a connection form A^g on $\mathrm{O}(M,g)$, such that $\nabla^g = \nabla^{A_g}$. From Theorem 4.2.3 and the remark following Definition 4.2.12 we therefore immediately obtain:

4.4.1 Theorem. *The holonomy group* $\mathrm{Hol}_x(M,g)$ *is either discrete or is a Lie subgroup of the orthogonal group* $O(T_x M, g_x)$*. The reduced holonomy group* $\mathrm{Hol}^0_x(M,g)$ *is the connected component of the unit element of* $\mathrm{Hol}_x(M,g)$*. If* M *is simply connected then* $\mathrm{Hol}_x(M,g)$ *is connected.*

4.4.2 Example. *The holonomy group of* $\mathbb{R}^{p,q}$

Let $\mathbb{R}^{p,q}$ be the pseudo-Euclidean space $\mathbb{R}^n$, $n = p + q$ with scalar product

$$\langle x, y\rangle_{p,q} := -x_1 y_1 - \cdots - x_p y_p + x_{p+1} y_{p+1} + \cdots + x_{p+q} y_{p+q}.$$

The Levi-Civita connection on $\mathbb{R}^{p,q}$ then is given by the directional derivative of vector fields:

$$\nabla^g_X Y = X(Y) \qquad (X, Y \in \mathfrak{X}(\mathbb{R}^{p,q})).$$

In particular, if $Z \in \mathfrak{X}_\gamma$ is a vector field along a path γ in $\mathbb{R}^n$, then

$$\frac{\nabla^g Z}{dt}(t) = Z'(t).$$

Thus Z is parallel along γ if and only if $t \mapsto Z(t)$ is constant. Therefore

$$\mathrm{Hol}_x(\mathbb{R}^{p,q}) = \{\mathrm{id}_{T_x\mathbb{R}^n}\}.$$

Note that, conversely, if the holonomy groups of a semi-Riemannian manifold are trivial, then parallel transport is path independent, which implies that the curvature tensor vanishes, i.e., that M is flat (cf. [10, Rem. 3.1.8]).

Next we want to formulate the holonomy theorem of Ambrose and Singer (Theorems 4.2.11, 4.2.14) in the present context. Denote by R^g the Riemann curvature tensor of (M, g). Since $\nabla^g = \nabla^{A_g}$, $R^g = R^{\nabla^g}$ (cf. Definition 3.5.12). By the symmetry properties of R^g (cf. [10, 3.1.2]), the endomorphism $R^g_x(v, w) : T_xM \to T_xM$ is skew-symmetric for all $x \in M$ and $v, w \in T_xM$, i.e., it is an element of the Lie algebra $\mathfrak{so}(T_xM, g_x)$ of $\mathrm{O}(T_xM, g_x)$.[2] Let γ be a path in M from x to y and let $v, w \in T_xM$. By $(\gamma^* R^g)_x(v, w)$ we denote the endomorphism

$$(\gamma^* R^g)_x(v, w) := \mathcal{P}^g_{\gamma^-} \circ R^g_y(\mathcal{P}^g_\gamma(v), \mathcal{P}^g_\gamma(w)) \circ \mathcal{P}^g_\gamma \in \mathfrak{so}(T_xM, g_x).$$

Since $\mathcal{P}^g_\gamma : T_xM \to T_yM$ is a bijection,

$$\{\mathcal{P}^g_{\gamma^-} \circ R^g_y(v, w) \circ \mathcal{P}^g_\gamma \mid v, w \in T_yM\} = \{\mathcal{P}^g_{\gamma^-} \circ R^g_y(\mathcal{P}^g_\gamma(v), \mathcal{P}^g_\gamma(w)) \circ \mathcal{P}^g_\gamma \mid v, w \in T_xM\}.$$

Theorem 4.2.14 therefore takes the form:

4.4.3 Theorem. *(Holonomy theorem of Ambrose and Singer) The Lie algebra of the holonomy group of* (M, g) *is given by*

$$\mathfrak{hol}_x(M, g) = \mathrm{span}\{(\gamma^* R^g)_x(v, w) \mid v, w \in T_xM,\ \gamma \textit{ path starting in } x\}.$$

Setting $\gamma \equiv x$ it follows, in particular, that all curvature operators $R_x(v, w)$ lie in the Lie subalgebra $\mathfrak{hol}_x(M, g)$ of $\mathfrak{so}(T_xM, g_x)$. Hence if the reduced holonomy group is smaller than $\mathrm{SO}(T_xM, g_x)$ (the connected component of I in $\mathrm{O}(T_xM, g_x)$) then Theorem 4.4.3 imposes additional curvature restrictions. On the other hand, if there exists even one point $x_0 \in M$ in which the curvature operators $R^g_{x_0}(v, w)$ generate the entire Lie algebra of skew symmetric endomorphisms, then the reduced holonomy group (of *every* point in M, cf. (4.4.1)) is the maximal possible group $\mathrm{SO}(T_xM, g_x)$.

4.4.4 Example. If (M, g) is a simply connected manifold that is isometric to the flat space $\mathbb{R}^n$ in the neighborhood of a point y, but isometric to the upper half of the sphere S^n in the neighborhood of another point x, then also $\mathrm{Hol}_y(M, g) \cong \mathrm{SO}(n)$.

[2] $\mathrm{SO}(T_xM, g_x)$ is the connected component of the identity in $\mathrm{O}(T_xM, g_x)$, so both Lie groups have the same Lie algebra, namely the space of skew-symmetric linear endomorphisms $\mathfrak{so}(T_xM, g_x)$.

Thus the holonomy group of M is a global object, and in general it does not suffice to analyze parallel transport only locally.

Finally, we want to examine the form the holonomy principle (Theorem 4.3.2) takes here. On any tensor bundle $\mathcal{T}$ the Levi-Civita covariant derivative induces a covariant derivative (tensor derivative, cf. [10, Sec. 1.3])

$$\nabla^g : \Gamma(\mathcal{T}) \to \Gamma(T^*M \otimes \mathcal{T})$$

E.g., for the homomorphism bundle $\mathcal{T} = \mathrm{Hom}(TM, TM)$ we have

$$(\nabla^g_X F)(Y) := \nabla^g_X(F(Y)) - F(\nabla^g_X Y),$$

and for (r, s)-tensors the following product rule holds:

$$\nabla^g_X(T \otimes S) = \nabla^g_X T \otimes S + T \otimes \nabla^g_X S.$$

Any tensor bundle is associated to a principal fiber bundle with linear structure group (cf. Example 2.4.6), and the above defines a covariant derivative on it. So we find ourselves precisely in the situation of Theorem 4.3.2, allowing us to conclude:

4.4.5 Theorem. *(Holonomy Principle) Let (M, g) be a connected semi-Riemannian manifold, $\mathcal{T}$ a tensor bundle over M and $x \in M$. Then:*

(i) Let $T \in \Gamma(\mathcal{T})$ with $\nabla^g T = 0$. Then $\mathrm{Hol}_x(M, g)T(x) = T(x)$, where the action of the holonomy group is canonically extended to the tensors $\mathcal{T}_x$.

(ii) Let $T_x \in \mathcal{T}_x$ with $\mathrm{Hol}_x(M, g)T_x = T_x$. Then there exists a unique tensor field $T \in \Gamma(\mathcal{T})$ with $\nabla^g T = 0$ and $T(x) = T_x$. This tensor field is obtained by parallel transport of T_x, i.e., $T(y) := \mathcal{P}^g_\gamma(T_x)$, where $y \in M$ and γ is any path from x to y.

This result very nicely displays the geometric meaning of the holonomy group: If the holonomy group $\mathrm{Hol}_x(M, g)$ lies in the invariance group of a tensor T_x at x then there exists an additional *global* structure on M, namely the tensor field resulting from T_x via parallel transport. The following special cases illustrate this principle:

4.4.6 Examples.

(i) For $\mathcal{T} = TM$, the holonomy principle says that (M, g) possesses a non-trivial global parallel vector field if and only if there exists a vector $0 \neq v \in T_xM$ with $\mathrm{Hol}_x(M, g)v = v$.

(ii) Parallel transport along curves in oriented manifolds is orientation preserving. Thus the holonomy group of an oriented semi-Riemannian manifold lies in $\mathrm{SO}(T_xM, g_x)$. Another way to express this is to note that the volume form on an orientable semi-Riemannian manifold is parallel. Conversely, let $\mathcal{T} = \Lambda^n T^*M$ and let $(dV_g)_x \in \Lambda^n T^*_xM$ be the volume form of (T_xM, g_x) (for some fixed orientation in T_xM). Then the invariance group of $(dV_g)_x$ is $\mathrm{SO}(T_xM, g_x)$ and by the holonomy principle it can be extended by parallel transport to a volume form on M if the holonomy group is contained in $\mathrm{SO}(T_xM, g_x)$. Altogether, a semi-Riemannian manifold (M, g) is orientable if and only if its holonomy group is contained in $\mathrm{SO}(T_xM, g_x)$.

(iii) Let $\mathcal{T} = \mathrm{Hom}(TM, TM)$ and $J_x : T_xM \to T_xM$ be a linear orthogonal map with $J_x^2 = -\mathrm{id}_{T_xM}$. Then the invariance group of J_x is precisely the unitary group $\mathrm{U}(T_xM, g_x, J_x) \subseteq \mathrm{SO}(T_xM, g_x)$:

$$\mathrm{U}(T_xM, g_x, J_x) = \{A \in \mathrm{SO}(T_xM, g_x) \mid AJ_x = J_xA\}.$$

The holonomy principle in this case says that the holonomy group $\mathrm{Hol}_x(M,g)$ lies in the unitary group $\mathrm{U}(T_xM, g_x, J_x)$ if and only if (M,g) is a *Kaehler manifold*, i.e., if and only if there exists a parallel almost complex structure J on M. By this we mean a homomorphism $J : TM \to TM$ with

$$g(JX, JY) = g(X, Y), \quad J^2 = -\mathrm{id}_{TM} \text{ and } \nabla^g J = 0.$$

Chapter 5

The Yang–Mills equation

5.1 The Maxwell equations as Yang–Mills equation

The Maxwell equations of electrodynamics provide a complete description of the generation of electric and magnetic fields as well as their interaction. In particular, for time-dependent fields they predict the occurence of electromagnetic waves. Classically, the electric and magnetic field are described by vector fields E and H on some open region $U \subseteq \mathbb{R}^3$:

$$E : U \times \mathbb{R} \to \mathbb{R}^3, \qquad H : U \times \mathbb{R} \to \mathbb{R}^3.$$

The charge density is described by a time-dependent function ρ, and the current density by a time-dependent vector J:

$$\rho : U \times \mathbb{R} \to \mathbb{R}, \qquad J : U \times \mathbb{R} \to \mathbb{R}^3.$$

Then the Maxwell equations read

$$\mathrm{rot}(E) = -\frac{1}{c}\frac{\partial H}{\partial t}, \qquad \mathrm{div}(H) = 0, \tag{5.1.1}$$

$$\mathrm{rot}(H) = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}J, \qquad \mathrm{div}(E) = 4\pi\rho, \tag{5.1.2}$$

with c the speed of light. Equivalently, the electromagentic field (E, H) can be described by a 2-form F on the subset $U \times \mathbb{R}$ of Minkowski space $\mathbb{R}^{1,3}$, with Lorentzian metric

$$g = dx^2 + dy^2 + dz^2 - c^2 dt^2.$$

Let $E = (E_x, E_y, E_z)$, $H = (H_x, H_y, H_z)$, and $J = (J_x, J_y, J_z)$, then we set

$$F := (E_x dx + E_y dy + E_z dz) \wedge cdt + H_x dy \wedge dz + H_y dz \wedge dx + H_z dx \wedge dy,$$

and

$$J_\rho := \frac{1}{c}J_x dx + \frac{1}{c}J_y dy + \frac{1}{c}J_z dz - \rho c dt.$$

To continue, we define ad-hoc the Hodge-star operator on Minkowski space, but postpone the general definition on semi-Riemannian manifolds to the next section. Let

$$* : \Omega^k(\mathbb{R}^{1,3}) \to \Omega^{4-k}(\mathbb{R}^{1,3})$$

where, for $k = 2$:

$$
\begin{aligned}
*(cdt \wedge dx) &:= dy \wedge dz, & *(dx \wedge dy) &:= -cdt \wedge dz,\\
*(cdt \wedge dy) &:= -dx \wedge dz, & *(dx \wedge dz) &:= cdt \wedge dy,\\
*(cdt \wedge dz) &:= dx \wedge dy, & *(dy \wedge dz) &:= -cdt \wedge dx,
\end{aligned}
$$

and for $k = 3$

$$
\begin{aligned}
*(cdt \wedge dx \wedge dy) &= dz, & *(cdt \wedge dx \wedge dz) &= -dy,\\
*(cdt \wedge dy \wedge dz) &= dx, & *(dx \wedge dy \wedge dz) &= cdt.
\end{aligned}
$$

Using this, we define the *codifferential*

$$
\begin{gathered}
\delta : \Omega^k(\mathbb{R}^{3,1}) \to \Omega^{k-1}(\mathbb{R}^{3,1})\\
\delta := *d* .
\end{gathered}
$$

With these operations we have:

5.1.1 Theorem. *The Maxwell equations are equivalent to*

$$dF = 0 \qquad \textit{and} \qquad \delta F = 4\pi J_\rho. \tag{5.1.3}$$

Proof. This follows by a direct calculation, see the Appendix. □

Consider now $M := U \times \mathbb{R} \subseteq \mathbb{R}^{3,1}$ and let P_0 be the trivial S^1-principal bundle $M \times S^1$ over M. Recall from Section 3.6 that $\mathrm{Lie}(S^1) = i\mathbb{R}$ and that $\mathcal{C}(P_0)$ is an affine space over $\Omega^1(M, i\mathbb{R})$. Furthermore, $F^A = dA$ and $d_A = d$ (see (3.6.6) and (3.4.12)), and by (3.6.5), $A, \tilde{A} \in \mathcal{C}(P_0)$ are gauge-equivalent if and only if there exists some $\sigma \in \mathcal{C}^\infty(M, S^1)$ such that

$$\tilde{A} = A + \sigma^{-1} d\sigma. \tag{5.1.4}$$

Let us additionally suppose that U is contractible and consider the Maxwell equations on M, i.e.,

$$dF = 0 \qquad \text{and} \qquad \delta F = 4\pi J_\rho. \tag{5.1.5}$$

By the Poincaré Lemma (e.g., [12, Th. 3.15]) there exists some $A \in \Omega^1(M, i\mathbb{R})$ with $dA = iF$. We call A the *potential* of the electromagentic field. Let $\bar{A}$ be the corresponding element of $\Omega^1_{\mathrm{hor}}(P_0, i\mathbb{R})$ (see (3.6.1)). Then $\bar{A}$ itself is not a connection form because it is horizontal, hence violates (ii) from Definition 3.1.3. Nevertheless, since $\mathcal{C}(P_0)$ is an affine space over $\Omega^1(M, i\mathbb{R})$, if we pick any $A_0 \in \mathcal{C}(P_0)$, then

$$\tilde{A} := A_0 + \bar{A} \in \mathcal{C}(P_0).$$

We choose A_0 to be the canonical flat connection form on P_0. Then by Theorem 3.5.11 we have $F^{A_0} = 0$, so $F^{\tilde{A}} = F^{\bar{A}}$. Moreover,

$$(F^{\bar{A}})^\wedge = (d\bar{A})^\wedge \underset{(3.6.2)}{=} d((\bar{A})^\wedge) = dA = iF. \tag{5.1.6}$$

The electromagentic potential is not uniquely determined: By the Poincaré Lemma any two potentials A, A' differ by some df:

$$A' = A + df \qquad (f \in \mathcal{C}^\infty(M, i\mathbb{R})).$$

Now set $\sigma := e^f$, then $\sigma \in \mathcal{C}^\infty(M, S^1)$ and $d\sigma = \sigma df$, i.e., $df = \sigma^{-1} d\sigma$. By (5.1.4) it follows that A and A' are gauge equivalent.

We are now going to show that the Maxwell equations (5.1.3) can be viewed as a differential equation for the curvature form of an S^1-connection (namely $\tilde{A}$ from above) in a trivial S^1-bundle over M. We have $\tilde{A} = A_0 + \bar{A}$, and

$$0 = idF = d((F^{\bar{A}})^\wedge) = d((F^{\bar{A}})^\wedge + (\underbrace{F^{A_0}}_{=0})^\wedge) = d((F^{\tilde{A}})^\wedge) \underset{(3.6.2)}{=} (dF^{\tilde{A}})^\wedge$$

$$\underset{(3.4.12)}{=} (d_{\tilde{A}} F^{\tilde{A}})^\wedge \Leftrightarrow d_{\tilde{A}} F^{\tilde{A}} = 0.$$

Furthermore,

$$4\pi i J_\rho = \delta(iF) = *d*(F^{\bar{A}})^\wedge = *d*(F^{\tilde{A}})^\wedge \underset{(3.6.2)}{=} (*d*F^{\tilde{A}})^\wedge$$

$$\underset{(3.4.12)}{=} (*d_{\tilde{A}} * F^{\tilde{A}})^\wedge \Leftrightarrow *d_{\tilde{A}} * F^{\tilde{A}} = 4\pi i \overline{J_\rho}.$$

Altogether, we obtain

$$d_{\tilde{A}} F^{\tilde{A}} = 0 \tag{5.1.7}$$

$$*d_{\tilde{A}} * F^{\tilde{A}} = 4\pi i \overline{J_\rho} \tag{5.1.8}$$

Note that (5.1.7) is just the Bianchi identity from Theorem 3.5.5. The entire dynamical content of the Maxwell equations is therefore encoded in the second equation (5.1.8). In vacuum (i.e., for $J = 0$ and $\rho = 0$), (5.1.8) reduces to the so-called *Yang–Mills equation*:

$$\delta_{\tilde{A}} F^{\tilde{A}} := *d_{\tilde{A}} * F^{\tilde{A}} = 0. \tag{5.1.9}$$

In theoretical physics it has turned out that also interactions other than the electromagentic one can be described in a similar way, replacing S^1 by other (non-abelian) Lie groups.

5.2 The Yang–Mills equation as an Euler–Lagrange equation

To formulate the Yang–Mills equation in the general case we first need to define the Hodge-star operator for a general n-dimensional oriented semi-Riemannian manifold (M, g), with g of signature $(p, n-p)$. Let dV_g be the volume form of g on M. We note first that g induces a bundle metric on $\Lambda^k T^*M$ as follows: for $\omega, \eta \in \Lambda^k T_x^*M$ set

$$\langle \omega, \eta \rangle_x := \sum_{i_1 < \cdots < i_k} \varepsilon_{i_1} \cdots \varepsilon_{i_k} \omega(s_{i_1}, \ldots, s_{i_k}) \cdot \eta(s_{i_1}, \ldots, s_{i_k}), \tag{5.2.1}$$

where $(s_1, \ldots, s_n)$ is a positively oriented orthonormal basis of $(T_x M, g_x)$ and $\varepsilon_i = g_x(s_i, s_i) = \pm 1$ is the causal type of s_i. If $(\sigma^1, \ldots, \sigma^n)$ is the dual basis of $(s_1, \ldots, s_n)$, then

$$\sigma^{i_1} \wedge \cdots \wedge \sigma^{i_k}, \qquad 1 \le i_1 < \cdots < i_k \le n$$

is an orthonormal basis of $\Lambda^k T_x^* M$ of causal type (see (5.2.1) and [7, 4.3.9])

$$\langle \sigma^{i_1} \wedge \cdots \wedge \sigma^{i_k}, \sigma^{i_1} \wedge \cdots \wedge \sigma^{i_k} \rangle = \varepsilon_{i_1} \cdot \cdots \cdot \varepsilon_{i_k}. \tag{5.2.2}$$

An alternative, manifestly basis-independent way of introducing the scalar product (5.2.1) on $\Lambda^k T_x^* M$ is to view the scalar product on $T_x^* M$ itself as a linear bijection $\tilde{g} : T_x^* M \to T_x M$ and then apply the universal property of $\Lambda^k T_x^* M$ to the map

$$T_x^* M \times \cdots \times T_x^* M \to \Lambda^k T_x M$$
$$(\alpha_1, \ldots, \alpha_k) \mapsto \tilde{g}(\alpha^1) \wedge \cdots \wedge \tilde{g}(\alpha^k).$$

Indeed, from this it follows that for $\alpha_1, \omega_1, \dots, \alpha_k, \omega_k$ one-forms on T_xM we have

$$\begin{aligned}\langle \alpha_1 \wedge \dots \wedge \alpha_k, \omega_1 \wedge \dots \wedge \omega_k \rangle &= (\tilde{g}(\alpha^1) \wedge \dots \wedge \tilde{g}(\alpha^k))(\omega_1 \wedge \dots \wedge \omega_k) \\ &= \det(\langle \alpha_i, \omega_j \rangle) = \sum_{\tau \in S_k} \operatorname{sgn}\tau \langle \alpha_1, \omega_{\tau(1)} \rangle \dots \langle \alpha_k, \omega_{\tau(k)} \rangle,\end{aligned}$$

which again confirms (5.2.2).
For $\omega, \eta \in \Omega^k(M)$ we obtain a smooth function on M,

$$\langle \omega, \eta \rangle : x \mapsto \langle \omega, \eta \rangle(x) := \langle \omega(x), \eta(x) \rangle_x \in \mathbb{R}.$$

5.2.1 Definition. *The Hodge-star operator on (M, g) is the linear operator*

$$\begin{aligned} * : \Omega^k(M) &\to \Omega^{n-k}(M) \\ \omega &\mapsto *\omega, \end{aligned}$$

where

$$\omega \wedge \sigma = \langle *\omega, \sigma \rangle dV_g \qquad \forall \sigma \in \Omega^{n-k}(M). \tag{5.2.3}$$

5.2.2 Theorem. *The Hodge-star operator on (M, g) has the following properties:*

(i) Existence: Let $(s_1, \dots, s_n)$ be a local positively oriented basis in TM and let $(\sigma^1, \dots, \sigma^n)$ be the dual basis in T^*M. Then $*$ is given by

$$*(\sigma^{i_1} \wedge \dots \wedge \sigma^{i_k}) = \varepsilon_{j_1} \cdots \varepsilon_{j_{n-k}} \cdot \operatorname{sgn}(IJ) \cdot \sigma^{j_1} \wedge \dots \wedge \sigma^{j_{n-k}}, \tag{5.2.4}$$

where $(IJ) = (i_1 \dots i_k\, j_1 \dots j_{n-k})$ is a permutation of $(1 \dots n)$ and $\operatorname{sgn}(IJ)$ *is its sign.*

(ii)

$$* \circ *|_{\Omega^k(M)} = (-1)^{k(n-k)+p} \operatorname{id}_{\Omega^k(M)}. \tag{5.2.5}$$

(iii) The $*$-operator is isometric or skew-isometric:

$$\langle *\omega, *\tilde{\omega} \rangle = (-1)^p \langle \omega, \tilde{\omega} \rangle \qquad (\omega, \tilde{\omega} \in \Omega^k(M)). \tag{5.2.6}$$

(iv) Let $\omega, \tilde{\omega} \in \Omega^k(M)$ and $\sigma \in \Omega^{n-k}(M)$, then

$$\omega \wedge *\tilde{\omega} = \tilde{\omega} \wedge *\omega = (-1)^p \langle \omega, \tilde{\omega} \rangle dV_g \tag{5.2.7}$$

$$\omega \wedge \sigma = (-1)^{k(n-k)} \langle \omega, *\sigma \rangle dV_g. \tag{5.2.8}$$

Proof. (i) Since $dV_g = \sigma^1 \wedge \dots \wedge \sigma^n$,

$$\sigma^{i_1} \wedge \dots \wedge \sigma^{i_k} \wedge \sigma^{j_1} \wedge \dots \wedge \sigma^{j_{n-k}} = \operatorname{sgn}(IJ) dV_g. \tag{5.2.9}$$

Therefore, defining $*$ by (5.2.4), due to (5.2.1) we indeed obtain the required property (5.2.3), because

$$\langle *(\sigma^{i_1} \wedge \dots \wedge \sigma^{i_k}), \sigma^{j_1} \wedge \dots \wedge \sigma^{j_{n-k}} \rangle = \operatorname{sgn}(IJ)$$

whenever (IJ) is a permutation of $(1 \dots n)$. Otherwise, both sides of (5.2.3) vanish since $*(\sigma^{i_1} \wedge \dots \wedge \sigma^{i_k})$ is perpendicular to the other base vectors of $\Lambda^{n-k}T^*M$.
(ii) It suffices to verify the claim on vectors from the given base. We have

$$\begin{aligned} * * (\sigma^{i_1} \wedge \dots \wedge \sigma^{i_k}) &\underset{(5.2.4)}{=} \varepsilon_{j_1} \cdots \varepsilon_{j_{n-k}} \cdot \operatorname{sgn}(IJ) \cdot *(\sigma^{j_1} \wedge \dots \wedge \sigma^{j_{n-k}}) \\ &= (-1)^p \operatorname{sgn}(JI) \operatorname{sgn}(IJ) \sigma^{i_1} \wedge \dots \wedge \sigma^{i_k} \\ &= (-1)^p (-1)^{k(n-k)} \sigma^{i_1} \wedge \dots \wedge \sigma^{i_k}. \end{aligned}$$

(iii)

$$\langle *\omega, *\tilde{\omega}\rangle dV_g \underset{(5.2.3)}{=} \omega \wedge *\tilde{\omega} = (-1)^{k(n-k)} *\tilde{\omega} \wedge \omega \underset{(5.2.3)}{=} (-1)^{k(n-k)} \langle **\tilde{\omega}, \omega\rangle dV_g$$
$$\underset{(5.2.5)}{=} (-1)^p \langle \tilde{\omega}, \omega\rangle dV_g.$$

(iv) By (5.2.3), $\omega \wedge *\tilde{\omega} = \langle *\omega, *\tilde{\omega}\rangle dV_g$, and $\tilde{\omega} \wedge *\omega = \langle *\tilde{\omega}, *\omega\rangle dV_g$, which together with(5.2.6) gives (5.2.7). Finally (5.2.8) follows from

$$\omega \wedge \sigma = (-1)^{k(n-k)} \sigma \wedge \omega \underset{(5.2.3)}{=} (-1)^{k(n-k)} \langle *\sigma, \omega\rangle dV_g.$$

□

Let (P, π, M, G) be a principal fiber bundle over the oriented semi-Riemannian manifold (M, g), let $\rho : G \to \mathrm{GL}(V)$ be a representation of G, and let $E = P \times_G V$ be the corresponding associated vector bundle. Then the Hodge-star operator can be extended to a linear map

$$* : \Omega^k(M, E) \to \Omega^{n-k}(M, E)$$

as follows: Fix a basis $(e_1, \dots, e_r)$ in E_x. Then any $\omega \in \Lambda^k T_x^* M \otimes E_x$ can be written as $\omega = \sum_{j=1}^r \omega_j \otimes e_j$, and we set

$$*\omega := \sum_{j=1}^r *\omega_j \otimes e_j. \tag{5.2.10}$$

5.2.3 Definition. *Let A be a connection form on P and let $d_A : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ be the differential induced by A (see* (3.4.6)*). The* codifferential *$\delta_A : \Omega^{k+1}(M, E) \to \Omega^k(M, E)$ is defined by*

$$\delta_A := (-1)^{nk+p+1} * d_A * .$$

On V we fix a G-invariant (not necessarily positive definite) scalar product $\langle\, .\, , .\, \rangle_V$ By Theorem 2.4.10, to $\langle\, .\, , .\, \rangle_V$ there corresponds a bundle metric on E via

$$\langle e, \hat{e}\rangle_{E_x} := \langle v, \hat{v}\rangle_V \qquad (e, \hat{e} \in E_x),$$

where $e = [p, v]$ and $\hat{e} = [p, \hat{v}]$ for some $p \in P_x$. Then the metric g on M and the bundle metric $\langle\, .\, , .\, \rangle_E$ induce a bundle metric on $\Lambda^k(T^*M) \otimes E$ of E-valued k-forms on M: For $\omega, \eta \in \Lambda^k(T_x^*M) \otimes E$ we set

$$\langle \omega, \eta\rangle_x := \sum_{i_1 < \dots < i_k} \varepsilon_{i_1} \cdot \dots \cdot \varepsilon_{i_k} \langle \omega(s_{i_1}, \dots, s_{i_k}), \eta(s_{i_1}, \dots, s_{i_k})\rangle_{E_x}, \tag{5.2.11}$$

where as before $(s_1, \dots, s_n)$ is a positively oriented orthonormal basis of $(T_x M, g_x)$ and $\varepsilon_i = g_x(s_i, s_i) = \pm 1$ is the causal type of s_i. Then for $\omega, \eta \in \Omega^k(M, E)$ we obtain a smooth scalar function

$$\langle \omega, \eta\rangle : x \mapsto \langle \omega, \eta\rangle(x) := \langle \omega(x), \eta(x)\rangle_x \in \mathbb{R}.$$

Based on this we introduce an L^2-scalar product on the space $\Omega_0^k(M, E)$ of compactly supported E-valued k-forms on M:

$$\langle \omega, \eta\rangle_{L^2} := \int_M \langle \omega, \eta\rangle \, dV_g \qquad (\omega, \eta \in \Omega_0^k(M, E)). \tag{5.2.12}$$

If (M, g) is Riemannian and $\langle\, .\, , .\, \rangle_V$ is positive definite, then so is $\langle\, .\, , .\, \rangle_{L^2}$.

An essential fact for our further considerations is that δ_A is adjoint to d_A with respect to $\langle\, .\, , .\, \rangle_{L^2}$:

5.2.4 Theorem. *Let $\omega \in \Omega_0^k(M,E)$, $\eta \in \Omega_0^{k+1}(M,E)$. Then*

$$\langle d_A\omega, \eta\rangle_{L^2} = \langle \omega, \delta_A\eta\rangle_{L^2}.$$

Proof. Since both d_A and δ_A are linear and using a partition of unity, it suffices to work on a trivializing open set U and to verify the claim on forms $\omega = \sigma\otimes e$ with $\sigma \in \Omega_0^k(M)$, $e \in \Gamma(E)$, and $\eta = \mu\otimes f$ with $\mu \in \Omega_0^{k+1}(M)$ and $f \in \Gamma(E)$. Indeed we may choose e and f from a local orthonormal basis in E, so that

$$\langle e, f\rangle = \text{const} \tag{5.2.13}$$

on U. We have

$$\begin{aligned} d_A\omega = d_A(\sigma\otimes e) &\underset{(3.4.14)}{=} d\sigma\otimes e + (-1)^k\sigma\wedge\nabla^A e \\ &= d\sigma\otimes e + (-1)^k\sum_{i=1}^n(\sigma\wedge\sigma^i)\otimes\nabla^A_{s_i}e, \end{aligned} \tag{5.2.14}$$

where $(s_1,\dots,s_n)$ is a local orthonormal frame on U and $(\sigma^1,\dots,\sigma^n)$ is the corresponding dual frame. Then

$$(-1)^{nk+p+1}\delta_A\eta = *d_A*\eta \underset{(3.4.14)}{=} *\Big((d*\mu)\otimes f + (-1)^{n-k-1}\sum_i(*\mu\wedge\sigma^i)\otimes\nabla^A_{s_i}f\Big), \tag{5.2.15}$$

and therefore, using (5.2.11) and (5.2.1),

$$\begin{aligned} \langle d_A\omega,\eta\rangle dV_g &\underset{(5.2.14)}{=} \langle d\sigma,\mu\rangle\langle e,f\rangle dV_g + (-1)^k\sum_i\langle\sigma\wedge\sigma^i,\mu\rangle\langle\nabla^A_{s_i}e,f\rangle dV_g \\ &\underset{(5.2.7)}{=} (-1)^p(d\sigma\wedge*\mu)\langle e,f\rangle + (-1)^{p+k}\sum_i(\sigma\wedge\sigma^i\wedge*\mu)\langle\nabla^A_{s_i}e,f\rangle \\ &\underset{3.4.10}{=} (-1)^p\big(d(\sigma\wedge*\mu) - (-1)^k\sigma\wedge d(*\mu)\big)\langle e,f\rangle \\ &\qquad + (-1)^{p+n-1}\sum_i\sigma\wedge*\mu\wedge\sigma^i\big(\underbrace{s_i(\langle e,f\rangle)}_{=0\text{ by }(5.2.13)} - \langle e,\nabla^A_{s_i}f\rangle\big) \\ &\underset{(5.2.8)}{=} (-1)^p d(\sigma\wedge*\mu\langle e,f\rangle) + (-1)^{k+p+k(n-k)+1}\langle\sigma,*d*\mu\rangle\langle e,f\rangle dV_g \\ &\qquad + (-1)^{p+n+k(n-k)}\sum_i\langle\sigma,*(*\mu\wedge\sigma^i)\rangle\langle e,\nabla^A_{s_i}f\rangle dV_g \\ &\underset{(5.2.10)}{=} (-1)^p d(\sigma\wedge*\mu\langle e,f\rangle) \\ &\qquad\quad + (-1)^{p+nk+1}\langle\omega,*\big((d*\mu)\otimes f + (-1)^{n-k-1}(*\mu)\wedge\nabla^A f\big)\rangle dV_g \\ &\underset{(5.2.15)}{=} (-1)^p d(\sigma\wedge*\mu\langle e,f\rangle) + \langle\omega,\delta_A\eta\rangle dV_g. \end{aligned}$$

Finally, since σ has compact support we get by Stokes' theorem

$$\int_M\langle d_A\omega,\eta\rangle\,dV_g = \int_M\langle\omega,\delta_A\eta\rangle\,dV_g.$$

□

Consider now the adjoint bundle $\mathrm{Ad}(P) = P\times_{(G,\mathrm{Ad})}\mathfrak{g}$ from Remark 3.2.4. By (3.5.6), the curvature form F^A of a connection form A on P can be viewed as a 2-form on M taking values in $\mathrm{Ad}(P)$. Analogous to the case of S^1-bundles we therefore define:

5.2.5 Definition. *A connection form A (as well as the corresponding connection) on P is called a Yang–Mills connection if its curvature form $F^A \in \Omega^2(M,\mathrm{Ad}(P))$ satisfies the Yang–Mills equation*

$$\delta_A F^A = 0.$$

We want to show that the Yang–Mills equation is the Euler–Lagrange equation of a Lagrangian functional on the space $\mathcal{C}(P)$ of connection forms on P. To do so, consider a principal fiber bundle P over a compact oriented semi-Riemannian manifold (M,g) and fix an Ad-invariant scalar product $\langle\,.\,,\,.\,\rangle_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$ of G:

$$\langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y\rangle_{\mathfrak{g}} = \langle X, Y\rangle_{\mathfrak{g}} \qquad \forall X,Y \in \mathfrak{g},\ g \in G. \tag{5.2.16}$$

We equip the space $\Omega^2(M,\mathrm{Ad}(P))$ with the scalar product induced by g and $\langle\,.\,,\,.\,\rangle_{\mathfrak{g}}$, as described after Definition 5.2.3. A suitable Lagrange functional is then given as follows:

5.2.6 Definition. *The functional* $L : \mathcal{C}(P) \to \mathbb{R}$,

$$L(A) := \int_M \langle F^A, F^A\rangle \, dV_g$$

is called the Yang–Mills functional *(corresponding to* $\langle\,.\,,\,.\,\rangle_{\mathfrak{g}}$*).*

5.2.7 Theorem. *The Yang–Mills functional is invariant under the action of the group of gauge transformations* $\mathcal{G}(P)$ *on the space* $\mathcal{C}(P)$*, i.e.,* $L(A) = L(f^*A)$ *for each* $A \in \mathcal{C}(P)$ *and each* $f \in \mathcal{G}(P)$*.*

Proof. By Theorem 3.5.18 we have

$$F^{f^*A} = f^*F^A = \mathrm{Ad}(\sigma_f^{-1}) \circ F^A, \tag{5.2.17}$$

where $\sigma_f \in \mathcal{C}^\infty(P,G)^G$ satisfies

$$f(p) = p \cdot \sigma_f(p) \qquad \forall p \in P,$$

see (3.5.13). As was stated before Definition 5.2.5, we consider F^A here as an element of $\Omega^2(M,\mathrm{Ad}(P))$. By (3.2.2), the relation between $\overline{F^A} \in \Omega^2_{\mathrm{hor}}(P,\mathfrak{g})^{(G,\mathrm{Ad})}$ and F^A is

$$F^A_x(t_1,t_2) = [p, \overline{F^A_p}(X_1,X_2)],$$

where $t_i \in T_xM$, $\pi(p) = x$, and $T\pi(X_i) = t_i$. In particular, if $s : M \supseteq U \to P$ is a local section, then $T\pi \circ Ts = \mathrm{id}$, so we may pick $X_i = Ts(t_i)$. Using this, since $\langle\,.\,,\,.\,\rangle_{\mathfrak{g}}$ is Ad-invariant, by (5.2.11) we get for $(s_1,\dots,s_n)$ a local orthonormal frame in TM:

$$\begin{aligned}
\langle F^{f^*A}, F^{f^*A}\rangle_x &= \sum_{i<j} \varepsilon_i\varepsilon_j \langle \overline{F^{f^*A}_{s(x)}}(Ts(s_i),Ts(s_j)), \overline{F^{f^*A}_{s(x)}}(Ts(s_i),Ts(s_j))\rangle_{\mathfrak{g}} \\
&\underset{(5.2.16),(5.2.17)}{=} \sum_{i<j} \varepsilon_i\varepsilon_j \langle \overline{F^{A}_{s(x)}}(Ts(s_i),Ts(s_j)), \overline{F^{A}_{s(x)}}(Ts(s_i),Ts(s_j))\rangle_{\mathfrak{g}} \\
&= \langle F^A, F^A\rangle_x.
\end{aligned}$$

Thus $L(f^*A) = L(A)$, as claimed. □

To determine the first variation of the Yang–Mills functional, recall from Remark 3.2.4 that the space $\mathcal{C}(P)$ of connection forms on P is an affine space over the vector space $\Omega^1(M,\mathrm{Ad}(P))$. In this sense we may consider $T_A\mathcal{C}(P) := \Omega^1(M,\mathrm{Ad}(P))$ as the tangent space of $\mathcal{C}(P)$ at A. Then the differential of L in the point A in direction $\omega \in T_A\mathcal{C}(P)$ is defined by

$$dL_A(\omega) := \frac{d}{dt}\Big|_0 \big(L(A + t\omega)\big).$$

5.2.8 Definition. *A connection form* $A \in \mathcal{C}(P)$ *is called a* critical point *of* L *if* $dL_A = 0$.

To characterize critical points of L we need one more auxiliary result:

5.2.9 Lemma. *Let (P, π, M, G) be a principal fiber bundle, A a connection form on P and $\sigma \in \Omega^1(M, \mathrm{Ad}(P))$. Then $A + \sigma \in \mathcal{C}(P)$ by Remark 3.2.4, and if $\rho : G \to \mathrm{GL}(V)$ is a representation of G with $E = P \times_\rho G$ the associated vector bundle, then*

(i) $d_{A+\sigma}\omega = d_A\omega + \rho_*(\sigma) \wedge \omega$ *for all* $\omega \in \Omega^p(M, E)$.

(ii) $F^{A+\sigma} = F^A + d_A\sigma + \frac{1}{2}[\sigma, \sigma]$.

Proof. By Theorem 3.2.3 and (3.4.6) it suffices to prove these claims for the corresponding horizontal forms on P and the absolute differential D_A. Thus let $\bar{\sigma} \in \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{(\mathrm{Ad},G)}$ be the 1-form corresponding to σ. Then by Theorem 3.4.4, for any $\bar{\omega} \in \Omega^p_{\mathrm{hor}}(P, \mathfrak{g})^{(\mathrm{Ad},G)}$ we have

$$D_{A+\bar{\sigma}}\bar{\omega} = d\bar{\omega} + \rho_*(A + \bar{\sigma}) \wedge \bar{\omega} = d\bar{\omega} + \rho_*(A) \wedge \bar{\omega} + \rho_*(\bar{\sigma}) \wedge \bar{\omega} = D_A\bar{\omega} + \rho_*(\bar{\sigma}) \wedge \bar{\omega},$$

which gives (i).
For (ii), by (3.4.3) we have

$$D_A\bar{\sigma} = d\bar{\sigma} + \mathrm{Ad}_*(A) \wedge \bar{\sigma}.$$

Here, by (3.4.4) and since $\mathrm{Ad}_* = \mathrm{ad}$ we have

$$\begin{aligned}(\mathrm{Ad}_*(A) \wedge \bar{\sigma})(t_0, t_1) &= \mathrm{ad}(A(t_0))(\bar{\sigma}(t_1)) - \mathrm{ad}(A(t_1))(\bar{\sigma}(t_0)) \\ &= [A(t_0), \bar{\sigma}(t_1)] - [A(t_1), \bar{\sigma}(t_0)].\end{aligned}$$

Using (3.5.5) and writing $A = \sum_i A^i v_i$, $\bar{\sigma} = \sum_i \bar{\sigma}^i v_i$ in terms of a basis (v_i) of $\mathfrak{g}$, we calculate

$$\begin{aligned}[A(t_0), \bar{\sigma}(t_1)] - [A(t_1), \bar{\sigma}(t_0)] &= \sum_{i,j} (A^i(t_0)\bar{\sigma}^j(t_1) - A^i(t_1)\bar{\sigma}^j(t_0))[v_i, v_j] \\ &= \Big(\sum_{i,j} A^i \wedge \bar{\sigma}^j [v_i, v_j]\Big)(t_0, t_1) = [A, \bar{\sigma}](t_0, t_1),\end{aligned}$$

so that

$$D_A\bar{\sigma} = d\bar{\sigma} + [A, \bar{\sigma}]. \tag{5.2.18}$$

By Lemma 3.5.2 (i), $[A, \bar{\sigma}] = [\bar{\sigma}, A]$, so

$$\begin{aligned}F^{A+\bar{\sigma}} &\underset{3.5.3(i)}{=} d(A + \bar{\sigma}) + \frac{1}{2}[A + \bar{\sigma}, A + \bar{\sigma}] \\ &= dA + d\bar{\sigma} + \frac{1}{2}\big([A, A] + [\bar{\sigma}, \bar{\sigma}] + [A, \bar{\sigma}] + [\bar{\sigma}, A]\big) \\ &= dA + \frac{1}{2}[A, A] + d\bar{\sigma} + [A, \bar{\sigma}] + \frac{1}{2}[\bar{\sigma}, \bar{\sigma}] \\ &\underset{(5.2.18)}{=} F^A + D_A\bar{\sigma} + \frac{1}{2}[\bar{\sigma}, \bar{\sigma}].\end{aligned}$$

□

Using this, we can finally prove:

5.2.10 Theorem. *A connection form $A \in \mathcal{C}(P)$ is a critical point of the Yang–Mills functional L if and only if it satisfies the Yang–Mills equation $\delta_A F^A = 0$.*

Proof. Let $A \in \mathcal{C}(P)$ and $\omega \in \Omega^1(M, \mathrm{Ad}(P))$. By Lemma 5.2.9,

$$F^{A+t\omega} = F^A + t d_A \omega + \frac{1}{2} t^2 [\omega, \omega].$$

Therefore,

$$\begin{aligned} dL_A(\omega) &= \frac{d}{dt}\Big|_0 \big(L(A + t\omega)\big) = \frac{d}{dt}\Big|_0 \langle F^{A+t\omega}, F^{A+t\omega} \rangle_{L^2} \\ &= \langle F^A, d_A\omega \rangle_{L^2} + \langle d_A \omega, F^A \rangle_{L^2} \\ &= 2\langle d_A\omega, F^A \rangle_{L^2} \underset{5.2.4}{=} 2\langle \omega, \delta_A F^A \rangle_{L^2}. \end{aligned}$$

Since $\langle\, . \,, \,. \,\rangle_{L^2}$ is non-degenerate, this implies that $dL_A = 0$ if and only if $\delta_A F^A = 0$. □

We proved this result under the assumption that M is compact. This restriction can, however, be lifted by considering connection forms for whose curvature form F^A the integral $\int_M \langle F^A, F^A \rangle \, dV_g$ converges, a property that is gauge invariant by Theorem 5.2.7. Then in the proof of Theorem 5.2.10 it suffices to consider variations in the direction of ω, where ω is compactly supported.

Appendices

Appendix A

Proofs of auxilliary results

Proof of Lemma 1.1.10 It suffices to take for W the domain of a chart χ in N. Let p be any point in $\psi^{-1}(W)$ and choose charts $(U, \varphi = (x^1, \ldots, x^m))$ around p in M and $(V, \eta = (y^1, \ldots, y^n))$ around $\psi(p)$ such that $\psi(U) \subseteq V \subseteq W$ and such that $\eta \circ \psi \circ \varphi^{-1} = (x^1, \ldots, x^m) \to (x^1, \ldots, x^m, 0, \ldots, 0)$. Then $\eta(\psi(U)) \subseteq \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$, so it has Lebesgue measure 0. It follows that also the image $\chi(\psi(U))$ of this set under the smooth map $\chi \circ \eta^{-1}$ has Lebesgue measure 0.

As p varies in $\psi^{-1}(W)$, the domains U cover the set $\psi^{-1}(W)$. As M is second countable we may extract a countable subcover $\{U_k \mid k \in \mathbb{N}\}$ from this collection. Then the sets $\psi(U_k)$ cover $W \cap \psi(M)$, entailing

$$\chi(W \cap \psi(M)) = \bigcup_{k \in \mathbb{N}} \chi(\psi(U_k)).$$

But then $\chi(W \cap \psi(M))$ has Lebesgue measure 0 and so it cannot be all of $\chi(W)$. It follows that $\psi(M)$ cannot contain all of W. □

Proof of Lemma 3.3.4

We prove the result for $g(t)$, the other case being analogous. Also, it suffices to treat the case where v is continuous because higher regularity then is immediate from the equation. Finally, we may assume that v is defined on all of $\mathbb{R}$. Consider the following vector field on $G \times \mathbb{R}$:

$$Z(g, s) := \Big(TL_g(v(s)), \frac{\partial}{\partial s}(s)\Big) \in T_{(g,s)}(G \times \mathbb{R}).$$

Then the flow line of Z through $(e, 0)$ is of the form $\mathrm{Fl}_t^Z(e, 0) = (g(t), t)$, where g is a solution of our ODE with $g(0) = e$ by construction of Z. Thus it only remains to show that g exists on the entire interval $[0, 1]$. Let $(e, s) \in G \times \mathbb{R}$, then since $\{e\} \times [0, 1]$ is compact there exists some $\delta > 0$ such that the integral curves $\mathrm{Fl}_t^Z(e, s)$ exist for all $s \in [0, 1]$ and all $|t| < \delta$ (cf. [7, 2.3.3]). Fix a partition $0 = t_0 < t_1 < \cdots < t_r = 1$ with $|t_i - t_{i-1}| < \delta$ for all i. On the first interval $[0, t_1]$ we already have a solution g of our initial value problem. The integral curve $\mathrm{Fl}_t^Z(e, t_1)$ for $t \in [0, t_2 - t_1]$ is of the form $\mathrm{Fl}_t^Z(e, t_1) = (b(t), t + t_1)$, where $\dot{b}(t) = TL_{b(t)}(v(t + t_1))$ and $b(0) = e$. We now extend the curve g continuously to the interval $[t_1, t_2]$ by setting

$$g(t) := g(t_1) \cdot b(t - t_1), \quad t \in [t_1, t_2].$$

Then on this interval

$$\dot{g}(t) = TL_{g(t_1)} \dot{b}(t - t_1) = TL_{g(t_1)} TLb(t - t_1)(v(t)) = TL_{g(t)}(v(t)),$$

so we obtain an extension of the solution to $[0, t_2]$. Iterating this procedure we conclude the proof. Uniqueness follows from general ODE theory. $\square$

Proof of Lemma 3.5.4 To prove smoothness we use local bundle charts according to (2.3.2):

$$\begin{aligned} \psi_U : \mathrm{Ad}(P)_U \to U \times \mathfrak{g}, \quad & [p, X] \mapsto (\pi(p), \mathrm{Ad}(\varphi_U(p))(X)) \\ \tilde{\psi}_U : E_U \to U \times V, \quad & [p, v] \mapsto (\pi(p), \rho(\varphi_U(p))(v)) \end{aligned}$$

Then by Remark 2.4.3 a bundle chart of $\mathrm{Hom}(E, E)$ is

$$\begin{aligned} \chi_U : \mathrm{Hom}(E, E)_U &\to U \times \mathrm{Hom}(V, V) \\ L_x &\mapsto (x, \tilde{\psi}_{U,x} \circ L_x \circ \tilde{\psi}_{U,x}^{-1}), \end{aligned}$$

so we need to show smoothness of

$$\begin{aligned} M \times \mathfrak{g} \ni (x, X) \mapsto \chi_U \circ \rho_* \circ \psi_U^{-1}(x, X) &\underset{(2.3.3)}{=} \chi_U(\rho_*([\phi_U^{-1}(x, e), X])) \\ &= \chi_U([p, v]_E \mapsto [p, \rho_*(X)v]) =: \chi_U(L_x), \end{aligned}$$

where $p := \phi_U^{-1}(x, e)$. For this it suffices to show that

$$((x, X), v) \mapsto (x, \tilde{\psi}_{U,x}(L_x(\tilde{\psi}_{U,x}^{-1}(v)) \in \mathcal{C}^\infty.$$

Indeed,

$$\begin{aligned} \tilde{\psi}_{U,x}(L_x(\tilde{\psi}_{U,x}^{-1}(v)) = \tilde{\psi}_{U,x}(L_x([p, v])) = \tilde{\psi}_{U,x}([p, \rho_*(X)v]) &= (x, \rho(\varphi_U(p))(\rho_*(X)v)) \\ &= (x, \rho(\varphi_U(\phi_U^{-1}(x, e)))(\rho_*(X)v)), \end{aligned}$$

which, finally, is obviously smooth.

Fiber-linearity of ρ_* is immediate from that of $\rho_* : \mathfrak{g} \to L(V, V)$. $\square$

Proof of Lemma 4.1.4

By the rank theorem ([7, 3.3.3]) we can choose charts $\psi = (y^1, \dots, y^n)$ around $j(p)$ and $\varphi = (U, x^1, \dots, x^m)$ around p in M such that $\psi \circ j \circ \varphi^{-1} = x \mapsto (x, 0)$. In this local representation, $\partial_{x^i} = \partial_{y^i}$ for $1 \le i \le m$, so $X|_U$ is of the form $\sum_{i=1}^m X^i \partial_{y_i}$, where $X^i \in \mathcal{C}^\infty(U)$. This reduces the problem to extending the coefficient functions locally to smooth functions on N. So let $f \in \mathcal{C}^\infty(M)$. By the above we have $\mathrm{pr} \circ \psi \circ j = \varphi$ on U. Thus setting $\hat{f} := f \circ \varphi^{-1} \circ \mathrm{pr} \circ \psi$ gives a smooth local function with $\hat{f} \circ j = f$. $\square$

Proof of Lemma 4.2.10

Call $A(p)$ the set described in the Lemma and L_p the leaf of E through p.

$\subseteq$: Let $q \in A(p)$ and $\gamma : I \to M$ from p to q with $\dot{\gamma}(t) \in E_{\gamma(t)}$ for all $t \in I$. Covering $\gamma(I)$ by flat charts for E, for each $t_0 \in I$ there exists an open interval around t_0 such that $\gamma(J)$ is contained in such a cubic chart $(U, \varphi = (x^1, \dots, x^n))$. Since

$$\dot{\gamma}(t) \in E_{\gamma(t)} = \mathrm{span}\{\partial_{x^i} \mid 1 \le i \le k\},$$

it follows that $\dot{\gamma}^i(t) = 0$ for all $t \in J$ and $k + 1 \le i \le n$, so γ lies entirely in one slice $U_a = \varphi^{-1}(\mathbb{R}^k \times \{a\})$, which is itself a connected integral manifold ([9, 17.33]) and therefore is contained in one leaf. If $\gamma(t_0)$ lies in two such charts then the corresponding leaves intersect, hence coincide. Thus $\gamma(I) \subseteq L_p$.

$\supseteq$: Let $q \in L_p$. Since L_p is a connected manifold, there exists a smooth curve $\sigma : I \to L_p$ connecting p to q. Let $j : L_p \hookrightarrow M$ be the inclusion and set $\gamma : j \circ \tilde{\sigma}$. Then

$$\dot{\gamma}(t) = Tj(\dot{\sigma}) \in T_{\gamma(t)} j(T_{\gamma(t)} L_p) = E_{\gamma(t)}$$

for each t, so $q \in A(p)$. □

Proof of Lemma 4.2.13

By [9, Sec. 22], φ factors over $G/\mathrm{ker}\varphi$ as follows:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & H \\ G/\mathrm{ker}\varphi \arrow[ur, "\hat{\varphi}"'] & \end{tikzcd}$$

Here π is a surjective submersion, so $\hat{\varphi}$ is smooth, and it is bijective. Since $\hat{\varphi}_*$ has the same image as φ_* we may replace φ by $\hat{\varphi}$, i.e., we may without loss of generality assume that φ is bijective. By [9, 8.8], $\varphi \circ \exp^G = \exp^H \circ \varphi_*$. Hence if U is a neighborhood of $0 \in \mathfrak{g}$ on which $\exp^G$ is injective, then $\exp^H \circ \varphi_*$ and thereby φ_* itself is injective on U as well, so $\ker(\varphi_*) \cap U = \{0\}$. Since $\ker(\varphi_*)$ is a linear subspace of $\mathfrak{g}$ it must be trivial, showing that φ_* is injective. For general $g \in G$, $\varphi = L_{\varphi(g)} \circ \varphi \circ L_{g^{-1}}$, so also $T_g\varphi$ is injective, i.e., φ is an immersion. Lemma 1.1.10 then shows that $\dim H = \dim G$, so that φ is a local diffeomorphism, and since it is bijective even a global diffeomorphism. This means it is a Lie group isomorphism, and the claim follows. □

Proof of Theorem 5.1.1

We have

$$\begin{aligned} dF =& c\frac{\partial E_x}{\partial y} dy \wedge dx \wedge dt + c\frac{\partial E_x}{\partial z} dz \wedge dx \wedge dt + c\frac{\partial E_y}{\partial x} dx \wedge dy \wedge dt \\ &+ c\frac{\partial E_y}{\partial z} dz \wedge dy \wedge dt + c\frac{\partial E_z}{\partial x} dx \wedge dz \wedge dt + c\frac{\partial E_z}{\partial y} dy \wedge dz \wedge dt \\ &+ \frac{\partial H_x}{\partial x} dx \wedge dy \wedge dz + \frac{\partial H_x}{\partial t} dt \wedge dy \wedge dz + \frac{\partial H_y}{\partial y} dy \wedge dz \wedge dx \\ &+ \frac{\partial H_y}{\partial t} dt \wedge dz \wedge dx + \frac{\partial H_z}{\partial z} dz \wedge dx \wedge dy + \frac{\partial H_z}{\partial t} dt \wedge dx \wedge dy. \end{aligned}$$

Consequently,

$$dF = 0 \Longleftrightarrow \mathrm{rot}(E) = -\frac{1}{c}\frac{\partial H}{\partial t} \quad \text{and } \mathrm{div}(H) = 0.$$

Furthermore,

$$\begin{aligned} *d*F =& -\frac{\partial E_x}{\partial x} cdt - \frac{1}{c}\frac{\partial E_x}{\partial t} dx - \frac{\partial E_y}{\partial y} cdt - \frac{1}{c}\frac{\partial E_y}{\partial t} dy - \frac{\partial E_z}{\partial z} cdt - \frac{1}{c}\frac{\partial E_z}{\partial t} dz \\ &- \frac{\partial H_x}{\partial y} dz + \frac{\partial H_x}{\partial z} dy + \frac{\partial H_y}{\partial x} dz - \frac{\partial H_y}{\partial z} dx - \frac{\partial H_z}{\partial x} dy + \frac{\partial H_z}{\partial y} dx, \end{aligned}$$

which implies

$$\delta F = 4\pi J_\rho \Longleftrightarrow \mathrm{rot}(H) = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c} J \quad \text{and } \mathrm{div}(E) = 4\pi\rho.$$

□

Bibliography

[1] Baum, H., Eichfeldtheorie, 2nd Ed., Springer, 2014

[2] Brickel, F., Clark, R.S., Differentiable Manifolds. An Introduction. Van Nostrand, 1970.

[3] Dieudonné, J., Foundations of Modern Analysis, Vol. 3.

[4] Hamilton, M., Mathematical Gauge Theory, Springer 2017.

[5] Hilgert, J., Neeb, K.H., Lie-Gruppen und Lie-Algebren, Vieweg 1991.

[6] Kobayashi, S., Nomizu, K., Foundations of Differential Geometry, Vol. 1.

[7] Kunzinger, M., Analysis on Manifolds, Lecture notes, `http://www.mat.univie.ac.at/~mike/teaching/ss22/amf.pdf`

[8] Kunzinger, M., Lie Transformation Groups, Lecture Notes, `arxiv.org/pdf/1506.07131v1.pdf`

[9] Kunzinger, M., Lie Groups, Lecture Notes, `http://www.mat.univie.ac.at/~mike/teaching/ws1920/lg.pdf`.

[10] Kunzinger, M., Steinbauer, R., Riemannian Geometry, Lecture Notes, `http://www.mat.univie.ac.at/~mike/teaching/ws2223/rg.pdf`

[11] Lee, J.M., Introduction to Topological Manifolds. Springer, 2000.

[12] Madsen, I., Tornehave, J., From Calculus to Cohomology, Cambridge Univ. Press, 1997.

[13] O'Neill, B., Semi-Riemannian Geometry with Applications to General Relativity, Academic Press 1983

Index