Canonical Energy-Momentum Tensor of General Relativity

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Abstract

In this article, we provide the natural derivation of symmetrical, coordinate independent canonical energy-momentum tensor for gravitational field.

1 Introduction

The stress-energy-momentum pseudotensor is traditional method to describe the concept of energy-momentum of gravitational field. The Landau-Lifshitz pseudotensor $t^{\mu\nu}_{LL}$ is derived from the Einstein field equation $\mathbb{G}^{\mu\nu} = \kappa T^{\mu\nu}$ to satisfied the conservation law $(T^{\mu\nu} + t^{\mu\nu}_{LL})_{,\nu} = 0$, where $\mathbb{G}^{\mu\nu}$ is Einstein field tensor, $\kappa = \frac{8\pi G}{c^4}$, and $T^{\mu\nu}$ is the energy-momentum tensor of source.

$$t^{\mu\nu}_{LL} = -\frac{1}{2\kappa}\mathbb{G}^{\mu\nu} + \frac{1}{2\kappa(-g)}[(-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})]_{,\alpha\beta}$$

$t^{\mu\nu}_{LL}$ is symmetrical but depend explicit in the Christoffel symbols (i.e., coordinate dependent and vanish in specific coordinate). The Dirac pseudotensor $t^{\mu\nu}_{D}$ starting from the equivalent action $\mathscr{L}^*$ and derived using standard Noether's theorem derivation. The original Einstein-Hilbert action $\mathscr{L} = R\sqrt{-g} = \mathscr{L}[g^{\mu\nu}, g^{\mu\nu}{}_{,\gamma}, g^{\mu\nu}{}_{,\gamma\eta}]$ depend on the second derivative of metric tensor, where R and g are Ricci scalar and $g = \det(g_{\mu\nu})$, respectively. The Ostrogradsky instability indicate the Lagrangian should not depend on higher order derivative more than 1st order. The equivalent action

$$\begin{aligned}\mathscr{L}^* &= \mathscr{L} - \partial_\mu(\sqrt{-g}g^{\mu\nu}\Gamma^\sigma_{\nu\sigma} - \sqrt{-g}g^{\sigma\nu}\Gamma^\mu_{\nu\sigma}) \\ &= \sqrt{-g}g^{\mu\nu}\left(\Gamma^\tau_{\mu\nu}\Gamma^\sigma_{\tau\sigma} - \Gamma^\tau_{\mu\sigma}\Gamma^\sigma_{\tau\nu}\right) \\ &= \mathscr{L}^*[g^{\mu\nu}, g^{\mu\nu}{}_{,\gamma}]\end{aligned}$$

have same EoM with advantage only depend on the 1st order derivative of metric tensor, which can apply standard Noether's theorem derivation. However, the equivalent action $\mathscr{L}^*$ lost the scalar property. Also, the Dirac pseudotensor $t^{\mu\nu}_{D}$:

$$t^{\mu\nu}_{D} = \frac{1}{2\kappa(-g)}\left[g^{\mu\gamma}\left(g^{\alpha\beta}\sqrt{-g}\right)_{,\gamma}\left(\Gamma^\nu_{\alpha\beta} - \delta^\nu_\beta\Gamma^\sigma_{\alpha\sigma}\right) - g^{\mu\nu}g^{\alpha\beta}\left(\Gamma^\rho_{\alpha\beta}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\alpha\sigma}\Gamma^\sigma_{\beta\rho}\right)\right]$$

lost the symmetric property, and is coordinate dependent and vanishes in specific coordinate as $t^{\mu\nu}_{LL}$.

As we previously derive the symmetrical, gauge invariant canonical energy-momentum tensor for abelian and non-abelian gauge field, in this article we generalize to general relativity. To reserve the scalar property of the Lagrangian and avoid the higher-order derivatives, we will use Palatini variation for derivation the EoM. To further apply Noether's theorem, we will use the vielbeins technique for deriving the conservation law. A more detailed derivation can be found in the supplementary derivation.

2 Derivation of Palatini variation

The Palatini variation treat metric tensor $g^{\omega\sigma}$ and the connection $\Gamma^{\varepsilon}_{\kappa\gamma}$ as independent field. The curvature tensor $R^{\varepsilon}_{\kappa\omega\sigma}$:

$$R^{\varepsilon}_{\kappa\omega\sigma} = \Gamma^{\varepsilon}_{\kappa\sigma,\omega} - \Gamma^{\varepsilon}_{\kappa\omega,\sigma} + \Gamma^{\varepsilon}_{\gamma\omega}\Gamma^{\gamma}_{\kappa\sigma} - \Gamma^{\varepsilon}_{\gamma\sigma}\Gamma^{\gamma}_{\kappa\omega} \tag{1}$$

and the torsion tensor $T^{\alpha}_{\beta\gamma}$:

$$T^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta} = -T^{\alpha}_{\gamma\beta} \tag{2}$$

In the following, we will NOT assume the connection to be torsion-free, i.e., $T^{\alpha}_{\beta\gamma} \neq 0 \leftrightarrow \Gamma^{\alpha}_{\beta\gamma} \neq \Gamma^{\alpha}_{\gamma\beta}$ in general. The Einstein-Hilbert action:

$$\mathscr{L} = \frac{1}{2\kappa}\sqrt{-g}g^{\mu\nu}\delta^{\omega}_{\varepsilon}R^{\varepsilon}_{\kappa\omega\sigma} = \mathscr{L}[g^{\mu\nu}, \Gamma^{\kappa}_{\mu\nu}, \Gamma^{\kappa}_{\mu\nu,\gamma}]$$

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}g^{\kappa\sigma}\delta^{\omega}_{\epsilon}R^{\epsilon}_{\kappa\omega\sigma}$$

Here we note that the Ricci tensor $R_{\kappa\sigma}$ and Ricci scalar R are:

$$R_{\kappa\sigma} = \delta^{\omega}_{\varepsilon}R^{\varepsilon}_{\kappa\omega\sigma}$$

and

$$R = g^{\kappa\sigma}R_{\kappa\sigma}$$

, so $\mathscr{L} = \frac{1}{2\kappa}\sqrt{-g}R$. Since the metric tensor $g^{\mu\nu}$ and the connection $\Gamma^{\alpha}_{\mu\nu}$ are independent field, the Lagrangian only depend on 1st order derivative of the connection. The Palatini variation:

$$\begin{aligned}
\delta S &= \int\left(\frac{\partial\mathscr{L}}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\partial\mathscr{L}}{\partial\Gamma^{\alpha}_{\mu\nu}}\delta\Gamma^{\alpha}_{\mu\nu} + \frac{\partial\mathscr{L}}{\partial\Gamma^{\alpha}_{\mu\nu,\gamma}}\delta\Gamma^{\alpha}_{\mu\nu,\gamma}\right)d^4x \\
&= \int\underbrace{\frac{\partial\mathscr{L}}{\partial g^{\mu\nu}}}_{EoM\#1}\delta g^{\mu\nu}d^4x + \int\underbrace{\left[\frac{\partial\mathscr{L}}{\partial\Gamma^{\alpha}_{\mu\nu}} - \left(\frac{\partial\mathscr{L}}{\partial\Gamma^{\alpha}_{\mu\nu,\gamma}}\right)_{,\gamma}\right]}_{EoM\#2}\delta\Gamma^{\alpha}_{\mu\nu}d^4x + \int\left[\frac{\partial\mathscr{L}}{\partial\Gamma^{\alpha}_{\mu\nu,\gamma}}\delta\Gamma^{\alpha}_{\mu\nu}\right]_{,\gamma}d^4x
\end{aligned} \tag{3}$$

Two equations of motion (EoM) are

$$\frac{\partial\mathscr{L}}{\partial g^{\mu\nu}} = 0$$

$$\frac{\partial\mathscr{L}}{\partial\Gamma^{\alpha}_{\mu\nu}} = \left(\frac{\partial\mathscr{L}}{\partial\Gamma^{\alpha}_{\mu\nu,\gamma}}\right)_{,\gamma}$$

3 EoM#1 - Einstein Field Equation

Since the curvature tensor do not depend on metric tensor, the first EoM:

$$\frac{\partial\mathscr{L}}{\partial g^{\mu\nu}} = \frac{1}{2\kappa}\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 0 \tag{4}$$

Define the Einstein field tensor $G_{\mu\nu}$:

$$\mathbb{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

For vacuum, $\mathbb{G}_{\mu\nu} = 0$ is the Einstein field equation in vacuum.
If the matter presents, $\mathscr{L} = \mathscr{L}_G + \mathscr{L}_M$, where $\mathscr{L}_G = \frac{1}{2\kappa}\sqrt{-g}R$ and $\mathscr{L}_M$ is matters. The variation gives:

$$\frac{\partial(\mathscr{L}_G + \mathscr{L}_M)}{\partial g^{\mu\nu}} = \frac{\partial\mathscr{L}_G}{\partial g^{\mu\nu}} + \frac{\partial\mathscr{L}_M}{\partial g^{\mu\nu}} = \frac{1}{2\kappa}\sqrt{-g}\,\mathbb{G}_{\mu\nu} + \frac{\partial\mathscr{L}_M}{\partial g^{\mu\nu}} = 0$$

$$\rightarrow \mathbb{G}_{\mu\nu} = \kappa\left(\frac{-2}{\sqrt{-g}}\frac{\partial\mathscr{L}_M}{\partial g^{\mu\nu}}\right) \equiv \kappa T_{\mu\nu}$$

4 EoM#2 Relate to The Metric Compatible Condition

To show the second EoM relates to the metric compatible condition, we will use the following relations:

$$\frac{1}{\sqrt{-g}}(\sqrt{-g})_{,\gamma} = \Gamma^{\eta}_{\eta\gamma} \tag{5}$$

$$g^{\phi\psi}{}_{,\gamma} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{\phi\psi}\right)_{,\gamma} - g^{\phi\psi}\Gamma^{\eta}_{\eta\gamma} \tag{6}$$

We first calculate $\frac{\partial \mathscr{L}}{\partial \Gamma^{\alpha}_{\mu\nu}}$ and $\frac{\partial \mathscr{L}}{\partial \Gamma^{\alpha}_{\mu\nu,\gamma}}$ for later use:

$$\frac{\partial \mathscr{L}}{\partial \Gamma^{\alpha}_{\mu\nu}} = \sqrt{-g}\left(g^{\kappa\sigma}\delta^{\nu}_{\alpha}\Gamma^{\mu}_{\kappa\sigma} + g^{\mu\nu}\Gamma^{\omega}_{\alpha\omega} - g^{\kappa\nu}\Gamma^{\mu}_{\kappa\alpha} - g^{\mu\sigma}\Gamma^{\nu}_{\alpha\sigma}\right) \tag{7}$$

$$\frac{\partial \mathscr{L}}{\partial \Gamma^{\alpha}_{\mu\nu,\gamma}} = \sqrt{-g}\left(g^{\mu\nu}\delta^{\gamma}_{\alpha} - g^{\mu\gamma}\delta^{\nu}_{\alpha}\right) \tag{8}$$

$$\left(\frac{\partial \mathscr{L}}{\partial \Gamma^{\alpha}_{\mu\nu,\gamma}}\right)_{,\gamma} = (\sqrt{-g}\,g^{\mu\nu})_{,\alpha} - (\sqrt{-g}\,g^{\mu\gamma})_{,\gamma}\delta^{\nu}_{\alpha} \tag{9}$$

Substituting Eq.(7) and Eq.(9) into EoM(more detail derivation, see supplement Eq.(10) below):

$$\begin{aligned}
\underbrace{(\sqrt{-g}\,g^{\mu\nu})_{,\alpha} - (\sqrt{-g}\,g^{\mu\gamma})_{,\gamma}\delta^{\nu}_{\alpha}}_{\left(\frac{\partial \mathscr{L}}{\partial \Gamma^{\alpha}_{\mu\nu,\gamma}}\right)_{,\gamma}} &= \underbrace{\sqrt{-g}\,g^{\kappa\sigma}\delta^{\nu}_{\alpha}\Gamma^{\mu}_{\kappa\sigma} + \sqrt{-g}\,g^{\mu\nu}\Gamma^{\omega}_{\alpha\omega} - \sqrt{-g}\,g^{\kappa\nu}\Gamma^{\mu}_{\kappa\alpha} - \sqrt{-g}\,g^{\mu\sigma}\Gamma^{\nu}_{\alpha\sigma}}_{\frac{\partial \mathscr{L}}{\partial \Gamma^{\alpha}_{\mu\nu}}} \\
g^{\mu\nu}{}_{,\alpha} + g^{\kappa\nu}\Gamma^{\mu}_{\kappa\alpha} + g^{\mu\kappa}\Gamma^{\nu}_{\kappa\alpha} &= \left[g^{\mu\gamma}{}_{,\gamma} + g^{\kappa\sigma}\Gamma^{\mu}_{\kappa\sigma} + g^{\mu\eta}\Gamma^{\gamma}_{\eta\gamma} + g^{\mu\gamma}\,T^{\eta}_{\eta\gamma}\right]\delta^{\nu}_{\alpha} + g^{\mu\nu}T^{\gamma}_{\alpha\gamma} + g^{\mu\kappa}T^{\nu}_{\kappa\alpha}
\end{aligned} \tag{10}$$

Define the "covariant derivative"

$$g^{\mu\nu}{}_{;\alpha} = g^{\mu\nu}{}_{,\alpha} + g^{\kappa\nu}\Gamma^{\mu}_{\kappa\alpha} + g^{\mu\kappa}\Gamma^{\nu}_{\kappa\alpha}$$

Eq.(10) becomes to

$$g^{\mu\nu}{}_{;\alpha} = \left(g^{\mu\gamma}{}_{;\gamma} + g^{\mu\gamma}\,T^{\eta}_{\eta\gamma}\right)\delta^{\nu}_{\alpha} + g^{\mu\nu}T^{\gamma}_{\alpha\gamma} + g^{\mu\kappa}T^{\nu}_{\kappa\alpha} \tag{11}$$

After some calculation (more detail derivation, see supplement Eq.(12) below), we arrive

$$g^{\mu\nu}{}_{;\alpha} = \frac{1}{3}g^{\mu\gamma}\,T^{\eta}_{\eta\gamma}\,\delta^{\nu}_{\alpha} + g^{\mu\nu}T^{\gamma}_{\alpha\gamma} + g^{\mu\kappa}T^{\nu}_{\kappa\alpha} \tag{12}$$

Eq.(12) indicate that:

Torsion-free then metric compatible	Metric compatible then torsion-free
If torsion-free: $T^{\alpha}_{\beta\gamma} = 0$, then $g^{\mu\nu}{}_{;\alpha} = 0$	If Metric compatible: $g^{\mu\nu}{}_{;\alpha} = 0$, then $T^{\alpha}_{\beta\gamma} = 0$

Table 1: Comparison of Torsion-free and Metric Compatibility Conditions

If torsion-free, then

$$g^{\mu\nu}{}_{;\alpha} = g^{\mu\nu}{}_{,\alpha} + g^{\kappa\nu}\Gamma^{\mu}_{\kappa\alpha} + g^{\mu\kappa}\Gamma^{\nu}_{\kappa\alpha} = 0$$

is the metric compatible condition.

5 Vielbeins Formalism

In previous tutorials, the variation of the Lie-algebra value gauge 1-form (gauge connection) $\mathbf{B}$ on principal bundle is :

$$\Delta\mathbf{B} = \delta\mathbf{B} + \hat{\mathcal{L}}_{\delta x}\mathbf{B} \tag{13}$$

In local coordinate $p \to \{x^\mu\}$, the expressions of $\mathbf{B}$ is

$$\mathbf{B}(p) \to B^a_\mu(x^\gamma)\hat{T}_a$$

, where $\hat{T}_a$ is the generator of Lie algebra. For a given Lie algebra representation:

$$\hat{T}_a \to \left(\hat{T}_a\right)^b_{\ c}$$

We define the notation:

$$\hat{B}^b_{c\mu} \equiv B^a_\mu \left(\hat{T}_a\right)^b_{\ c}$$

. The curvature tensor:

$$\mathbf{G} \equiv d\mathbf{B} + [\mathbf{B} \wedge \mathbf{B}] \to \hat{G}^a_{b\mu\nu} = \hat{B}^a_{b\nu,\mu} - \hat{B}^a_{b\mu,\nu} + \hat{B}^a_{c\mu}\hat{B}^c_{b\nu} - \hat{B}^a_{c\nu}\hat{B}^c_{b\mu} \tag{14}$$

Eq.(13) in local coordinate with Lie algebra representation is:

$$\Delta\hat{B}^b_{c\mu} = \delta\hat{B}^b_{c\mu} + \underbrace{\delta x^\nu\partial_\nu\hat{B}^b_{c\mu} + \hat{B}^b_{c\mu}\partial_\mu\delta x^\nu}_{\hat{\mathcal{L}}_{\delta x}\mathbf{B}} \tag{15}$$

However, the connection on tangent bunble is more subtle. The curvature on tangent bundle is Eq.(1):

$$R^\varepsilon_{\kappa\omega\sigma} = \Gamma^\varepsilon_{\kappa\sigma,\omega} - \Gamma^\varepsilon_{\kappa\omega,\sigma} + \Gamma^\varepsilon_{\gamma\omega}\Gamma^\gamma_{\kappa\sigma} - \Gamma^\varepsilon_{\gamma\sigma}\Gamma^\gamma_{\kappa\omega} \tag{1}$$

The principal curvature Eq.(14) is Lie-algebra value 2-form. In contrast, the tangent bundle curvature takes value in tangent vector. If we directly apply Lie derivative on tangent connection:

$$\left(\hat{\mathcal{L}}_{\delta x}\Gamma\right)^\alpha_{\ \mu\nu} = \delta x^\epsilon\Gamma^\alpha_{\mu\nu,\epsilon} + \Gamma^\alpha_{\mu\epsilon}\delta x^\epsilon_{,\nu} + \underbrace{\delta x^\alpha_{,\mu\nu} - \Gamma^\epsilon_{\mu\nu}\delta x^\alpha_{,\epsilon} + \Gamma^\alpha_{\epsilon\nu}\delta x^\epsilon_{,\mu}}_{due\,to\,tangent\,vector\,value}$$

We can use Vielbeins formalism. In the vielbeins formalism, the tangent connection $\Gamma^\varepsilon_{\kappa\sigma}$ becomes the spin connection $\omega^a_{b\sigma}$ in the frame bundle, which is $\mathfrak{gl}$-value 1-form. From the tangent formalism $\{\partial_\mu\}$ to the vielbeins formalism $\{\hat{e}_a\}$, define the transformation e^μ_a such that(more detail can be found in Supplementary: Vielbeins):

$$\partial_\mu = e^a_\mu\hat{e}_a$$
$$g_{\mu\nu} = e^a_\mu e^b_\nu\eta_{ab}$$
$$e^\mu_a e^a_\nu = \delta^\mu_\nu \;\; and \;\; e^\mu_a e^b_\mu = \delta^a_b$$

The spin conncection:

$$\omega^a_{b\sigma} = e^a_\varepsilon e^\kappa_b\Gamma^\varepsilon_{\kappa\sigma} + e^a_\mu e^\mu_{b,\sigma}$$

The frame bundle curvature $\mathscr{R}^a_{b\omega\sigma}$ is

$$\begin{aligned}\mathscr{R}^a_{b\omega\sigma} &= \omega^a_{b\sigma,\omega} - \omega^a_{b\omega,\sigma} + \omega^a_{c\omega}\omega^c_{b\sigma} - \omega^a_{c\sigma}\omega^c_{b\omega}\\ &= e^a_\varepsilon e^\kappa_b R^\varepsilon_{\kappa\omega\sigma}\end{aligned}$$

With this preparation, the Hilbert-Einstein action becomes as follows:

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}g^{\kappa\sigma}\delta^{\omega}_{\varepsilon}R^{\varepsilon}_{\kappa\omega\sigma} = \frac{1}{2\kappa}\int d^4x\,\eta^{ae}e^{\sigma}_{e}e^{\omega}_{c}\mathscr{R}^{c}_{a\omega\sigma} \tag{16}$$

, where $e = \det(e^a_\mu) = \sqrt{-g}$.
The variation is similar to Eq.(3):

$$\begin{aligned}\delta S &= \int\left(\frac{\partial\mathscr{L}}{\partial e^{\mu}_{a}}\delta e^{\mu}_{a} + \frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu}}\delta\omega^{b}_{c\mu} + \frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}}\delta\omega^{b}_{c\mu,\gamma}\right)d^4x \\ &= \int\underbrace{\frac{\partial\mathscr{L}}{\partial e^{\mu}_{a}}}_{EoM\#3}\delta e^{\mu}_{a}d^4x + \int\underbrace{\left[\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu}} - \left(\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}}\right)_{,\gamma}\right]}_{EoM\#4}\delta\omega^{b}_{c\mu}d^4x + \int\left[\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}}\delta\omega^{b}_{c\mu}\right]_{,\gamma}d^4x\end{aligned} \tag{17}$$

Since $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$, the variation $\delta g_{\mu\nu} = 2\,\eta_{ab}e^a_\mu\delta e^b_\nu$, the $EoM\#3$ is similar to $EoM\#1$. $EoM\#4$ is also similar to $EoM\#2$. Now, we apply the Noether variation:

$$\Delta S = \int[EoM]\,d^4x + \int\left[\partial_\gamma\left(\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}}\delta\omega^{b}_{c\mu}\right) + (\mathscr{L}\delta x^{\gamma})_{,\gamma}\right]d^4x$$

The spin connection $\omega^b_{c\mu}$ is $\mathfrak{gl}$-value 1-form, we apply similar variation as Eq.(15):

$$\Delta\omega^{b}_{c\mu} = \delta\omega^{b}_{c\mu} + \underbrace{\delta x^{\varepsilon}\partial_{\varepsilon}\omega^{b}_{c\mu} + \omega^{b}_{c\varepsilon}\partial_{\mu}\delta x^{\varepsilon}}_{\hat{\mathcal{L}}_{\delta x}\omega} \tag{18}$$

We have:

$$\begin{aligned}\Delta S &= \int[EoM]\,d^4x + \int\left[\partial_\gamma\left(\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}}\left(\Delta\omega^{b}_{c\mu} - \delta x^{\varepsilon}\partial_{\varepsilon}\omega^{b}_{c\mu} - \omega^{b}_{c\varepsilon}\partial_{\mu}\delta x^{\varepsilon}\right)\right) + (\mathscr{L}\delta x^{\gamma})_{,\gamma}\right]d^4x \\ &= \int[EoM]\,d^4x + \int\underbrace{\partial_\gamma}_{(*)}\left[\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}}\Delta\omega^{b}_{c\mu} - \left(\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}}\omega^{b}_{c\mu,\varepsilon}\delta x^{\varepsilon} + \underbrace{\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}}\omega^{b}_{c\varepsilon}\delta x^{\varepsilon}_{,\mu}}_{(*)} - \delta^{\gamma}_{\varepsilon}\mathscr{L}\delta x^{\varepsilon}\right)\right]d^4x\end{aligned}$$

Evaluate the $(*)$ term:

$$\underbrace{\partial_\gamma\left[\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}}\omega^{b}_{c\varepsilon}\delta x^{\varepsilon}_{,\mu}\right]}_{(*)} = \underbrace{\partial_\gamma\left[\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}}\omega^{b}_{c\varepsilon}\delta x^{\varepsilon}\right]_{,\mu}}_{(*1)} - \underbrace{\partial_\gamma\left[\left(\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}}\right)_{,\mu}\omega^{b}_{c\varepsilon}\delta x^{\varepsilon}\right]}_{(*2)} - \underbrace{\partial_\gamma\left[\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}}\omega^{b}_{c\varepsilon,\mu}\delta x^{\varepsilon}\right]}_{(*3)}$$

We first calculate $\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}}$ for later use:

$$\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}} = \frac{1}{2\kappa}e\,\eta^{ce}\left(e^{\mu}_{e}e^{\gamma}_{b} - e^{\gamma}_{e}e^{\mu}_{b}\right)\left(= -\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\gamma,\mu}}\right) \tag{19}$$

The $(*1)$ term:

$$\underbrace{\partial_\gamma\left[\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}}\omega^{b}_{c\varepsilon}\delta x^{\varepsilon}\right]_{,\mu}}_{(*1)} = \left[\underbrace{\frac{1}{2\kappa}e\,\eta^{ce}\left(e^{\mu}_{e}e^{\gamma}_{b} - e^{\gamma}_{e}e^{\mu}_{b}\right)}_{(19)}\omega^{b}_{c\varepsilon}\delta x^{\varepsilon}\right]_{,\mu\gamma} = 0$$

The $(*2)$ term rely on $EoM\#4$, using

$$\underbrace{-\partial_\gamma\left[\left(\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\mu,\gamma}}\right)_{,\mu}\omega^{b}_{c\varepsilon}\delta x^{\varepsilon}\right]}_{(*2)}\underbrace{=}_{(19)} -\partial_\gamma\left[\left(-\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\gamma,\mu}}\right)_{,\mu}\omega^{b}_{c\varepsilon}\delta x^{\varepsilon}\right]\underbrace{=}_{EoM\#4}\partial_\gamma\left(\frac{\partial\mathscr{L}}{\partial\omega^{b}_{c\gamma}}\omega^{b}_{c\varepsilon}\delta x^{\varepsilon}\right)$$

and

$$\frac{\partial \mathscr{L}}{\partial \omega^b_{c\gamma}} = \frac{1}{2\kappa} e\, \eta^{ae} \left(e^\gamma_e e^\mu_b - e^\mu_e e^\gamma_b\right) \omega^c_{a\mu} + \frac{1}{2\kappa} e\, \eta^{ce} \left(e^\mu_e e^\gamma_f - e^\gamma_e e^\mu_f\right) \omega^f_{b\mu} \tag{20}$$

We can calulate

$$\frac{\partial \mathscr{L}}{\partial \omega^b_{c\gamma}} \omega^b_{c\varepsilon} = \frac{\partial \mathscr{L}}{\partial \omega^b_{c\mu,\gamma}} \left(\omega^b_{d\varepsilon}\omega^d_{c\mu} - \omega^b_{d\mu}\omega^d_{c\varepsilon}\right) \tag{21}$$

The $(*)$ term than become:

$$\underbrace{\partial_\gamma \left[\frac{\partial \mathscr{L}}{\partial \omega^b_{c\mu,\gamma}} \omega^b_{c\varepsilon} \delta x^\varepsilon_{,\mu}\right]}_{(*)} = \underbrace{0}_{(*1)} - \underbrace{\partial_\gamma \left[\frac{\partial \mathscr{L}}{\partial \omega^b_{c\mu,\gamma}} \omega^b_{c\varepsilon,\mu} \delta x^\varepsilon\right]}_{(*3)} + \underbrace{\partial_\gamma \left[\frac{\partial \mathscr{L}}{\partial \omega^b_{c\mu,\gamma}} \left(\omega^b_{d\varepsilon}\omega^d_{c\mu} - \omega^b_{d\mu}\omega^d_{c\varepsilon}\right) \delta x^\varepsilon\right]}_{(*2)}$$

We have:

$$\begin{aligned}
\Delta S &= \int [EoM]\, d^4x + \int \partial_\gamma \left[\frac{\partial \mathscr{L}}{\partial \omega^b_{c\mu,\gamma}} \Delta\omega^b_{c\mu} - \left(\frac{\partial \mathscr{L}}{\partial \omega^b_{c\mu,\gamma}} \omega^b_{c\mu,\varepsilon} \delta x^\varepsilon - \underbrace{\frac{\partial \mathscr{L}}{\partial \omega^b_{c\mu,\gamma}} \omega^b_{c\varepsilon,\mu} \delta x^\varepsilon}_{(*3)} + \underbrace{\frac{\partial \mathscr{L}}{\partial \omega^b_{c\mu,\gamma}} \left(\omega^b_{d\varepsilon}\omega^d_{c\mu} - \omega^b_{d\mu}\omega^d_{c\varepsilon}\right) \delta x^\varepsilon}_{(*2)} - \delta^\gamma_\varepsilon \mathscr{L} \delta x^\varepsilon\right)\right] d^4x \\
&= \int [EoM]\, d^4x + \int \partial_\gamma \left[\frac{\partial \mathscr{L}}{\partial \omega^b_{c\mu,\gamma}} \Delta\omega^b_{c\mu} - \left(\frac{\partial \mathscr{L}}{\partial \omega^b_{c\mu,\gamma}} \left(\omega^b_{c\mu,\varepsilon} - \omega^b_{c\varepsilon,\mu} + \omega^b_{d\varepsilon}\omega^d_{c\mu} - \omega^b_{d\mu}\omega^d_{c\varepsilon}\right) \delta x^\varepsilon - \delta^\gamma_\varepsilon \mathscr{L} \delta x^\varepsilon\right)\right] d^4x \\
&= \int [EoM]\, d^4x + \int \partial_\gamma \left[\frac{\partial \mathscr{L}}{\partial \omega^b_{c\mu,\gamma}} \Delta\omega^b_{c\mu} - \left(\frac{\partial \mathscr{L}}{\partial \omega^b_{c\mu,\gamma}} \mathscr{R}^b_{c\varepsilon\mu} - \delta^\gamma_\varepsilon \mathscr{L}\right) \delta x^\varepsilon\right] d^4x
\end{aligned}$$

The canonical energy-momentum tensor:

$$\begin{aligned}
t^\gamma_\varepsilon &= \frac{\partial \mathscr{L}}{\partial \omega^b_{c\mu,\gamma}} \mathscr{R}^b_{c\varepsilon\mu} - \delta^\gamma_\varepsilon \mathscr{L} \\
&= \frac{1}{2\kappa} e\, \eta^{ce} \left(e^\gamma_e e^\mu_b - e^\mu_e e^\gamma_b\right) \mathscr{R}^b_{c\varepsilon\mu} - \delta^\gamma_\varepsilon \mathscr{L} \\
&= \frac{1}{2\kappa} \left(g^{\beta\mu} R^\gamma_{\beta\varepsilon\mu} + g^{\beta\gamma} R_{\beta\varepsilon} - \delta^\gamma_\varepsilon R\right) \sqrt{-g}
\end{aligned} \tag{22}$$

The canonical energy-momentum 2-form:

$$t_{\alpha\varepsilon} = g_{\alpha\gamma} t^\gamma_\varepsilon = \frac{1}{2\kappa} \left(g_{\alpha\gamma} g^{\beta\mu} R^\gamma_{\beta\varepsilon\mu} + R_{\alpha\varepsilon} - g_{\alpha\varepsilon} R\right) \sqrt{-g}$$

If metric compatible, we have:

$$t_{\alpha\varepsilon} = \frac{1}{\kappa} \mathbb{G}_{\alpha\varepsilon} \sqrt{-g} \tag{23}$$

This canonical energy-momentum 2-form has following properties:
1. symmetric
2. coordinate independent
3. vanish when vacuum, which does not depend on the choice of the connection.

References

1. Canonical Energy-Momentum Tensor of Abelian Fields, arXiv:2503.15031.

Supplementary Derivation

Eq.(4)

$$\begin{aligned}\frac{\partial \mathscr{L}}{\partial g^{\mu\nu}} &= \frac{1}{2\kappa}\frac{\partial}{\partial g^{\mu\nu}}\left(\sqrt{-g}g^{\kappa\sigma}\delta^{\omega}_{\epsilon}R^{\epsilon}_{\kappa\omega\sigma}\right) = \frac{1}{2\kappa}\sqrt{-g}\left(\delta^{\omega}_{\epsilon}\delta^{\kappa}_{\mu}\delta^{\sigma}_{\nu}R^{\epsilon}_{\kappa\omega\sigma} - \frac{1}{2}g_{\mu\nu}g^{\kappa\sigma}\delta^{\omega}_{\epsilon}R^{\epsilon}_{\kappa\omega\sigma}\right)\\ &= \frac{1}{2\kappa}\sqrt{-g}\left(\delta^{\omega}_{\epsilon}\delta^{\kappa}_{\mu}\delta^{\sigma}_{\nu}R^{\epsilon}_{\kappa\omega\sigma} - \frac{1}{2}g_{\mu\nu}g^{\kappa\sigma}R_{\kappa\sigma}\right)\\ &= \frac{1}{2\kappa}\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\\ &= \frac{1}{2\kappa}\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\end{aligned}$$

Eq.(5)

$$\begin{aligned}(\sqrt{-g})_{,\gamma} &= -\frac{1}{2}\sqrt{-g}\,g_{\phi\psi}g^{\phi\psi}{}_{,\gamma}\\ &= -\frac{1}{2}\sqrt{-g}\,g_{\phi\psi}(-g^{\eta\psi}\Gamma^{\phi}_{\eta\gamma} - g^{\phi\eta}\Gamma^{\psi}_{\eta\gamma})\\ &= \frac{1}{2}\sqrt{-g}(g_{\phi\psi}g^{\eta\psi}\Gamma^{\phi}_{\eta\gamma} + g_{\phi\psi}g^{\phi\eta}\Gamma^{\psi}_{\eta\gamma})\\ &= \frac{1}{2}\sqrt{-g}(\delta^{\eta}_{\phi}\Gamma^{\phi}_{\eta\gamma} + \delta^{\eta}_{\psi}\Gamma^{\psi}_{\eta\gamma})\\ &= \frac{1}{2}\sqrt{-g}(\Gamma^{\eta}_{\eta\gamma} + \Gamma^{\eta}_{\eta\gamma})\\ &= \sqrt{-g}\Gamma^{\eta}_{\eta\gamma}\end{aligned}$$

Eq.(6)

$$g^{\phi\psi}{}_{,\gamma} = \frac{\sqrt{-g}}{\sqrt{-g}}g^{\phi\psi}{}_{,\gamma} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{\phi\psi}\right)_{,\gamma} - \underbrace{\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\right)_{,\gamma}}_{(5)=\Gamma^{\eta}_{\eta\gamma}}g^{\phi\psi} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{\phi\psi}\right)_{,\gamma} - g^{\phi\psi}\Gamma^{\eta}_{\eta\gamma}$$

Eq.(7)
Evaluate $\frac{\partial \mathscr{L}}{\partial \Gamma^{\alpha}_{\mu\nu}}$:

$$\begin{aligned}\frac{\partial \mathscr{L}}{\partial \Gamma^{\alpha}_{\mu\nu}} &= \frac{\partial}{\partial \Gamma^{\alpha}_{\mu\nu}}\left(\frac{1}{2\kappa}\sqrt{-g}\,g^{\kappa\sigma}\delta^{\omega}_{\epsilon}R^{\epsilon}_{\kappa\omega\sigma}\right)\\ &= \frac{1}{2\kappa}\sqrt{-g}\,g^{\kappa\sigma}\delta^{\omega}_{\epsilon}\frac{\partial}{\partial \Gamma^{\alpha}_{\mu\nu}}\left(\Gamma^{\epsilon}_{\kappa\sigma,\omega} - \Gamma^{\epsilon}_{\kappa\omega,\sigma} + \Gamma^{\epsilon}_{\eta\omega}\Gamma^{\eta}_{\kappa\sigma} - \Gamma^{\epsilon}_{\eta\sigma}\Gamma^{\eta}_{\kappa\omega}\right)\\ &= \frac{1}{2\kappa}\sqrt{-g}\,g^{\kappa\sigma}\delta^{\omega}_{\epsilon}\left(\delta^{\eta}_{\alpha}\delta^{\epsilon}_{\mu}\delta^{\omega}_{\nu}\Gamma^{\eta}_{\kappa\sigma} + \Gamma^{\epsilon}_{\eta\omega}\delta^{\eta}_{\alpha}\delta^{\mu}_{\kappa}\delta^{\nu}_{\sigma} - \delta^{\eta}_{\alpha}\delta^{\epsilon}_{\mu}\delta^{\sigma}_{\nu}\Gamma^{\eta}_{\kappa\omega} - \Gamma^{\epsilon}_{\eta\sigma}\delta^{\eta}_{\alpha}\delta^{\mu}_{\kappa}\delta^{\nu}_{\omega}\right)\\ &= \frac{1}{2\kappa}\left(\sqrt{-g}\,g^{\kappa\sigma}\delta^{\nu}_{\alpha}\Gamma^{\mu}_{\kappa\sigma} + \sqrt{-g}\,g^{\mu\nu}\Gamma^{\omega}_{\alpha\omega} - \sqrt{-g}\,g^{\kappa\nu}\Gamma^{\mu}_{\kappa\alpha} - \sqrt{-g}\,g^{\mu\sigma}\Gamma^{\nu}_{\alpha\sigma}\right)\end{aligned}$$

Eq.(8)

Evaluate $\frac{\partial \mathscr{L}}{\partial \Gamma^\alpha_{\mu\nu,\gamma}}$:

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial \Gamma^\alpha_{\mu\nu,\gamma}} &= \frac{\partial}{\partial \Gamma^\alpha_{\mu\nu,\gamma}}\left(\frac{1}{2\kappa}\sqrt{-g}\,g^{\kappa\sigma}\delta^\omega_\epsilon R^\epsilon_{\kappa\omega\sigma}\right) \\
&= \frac{1}{2\kappa}\sqrt{-g}\,g^{\kappa\sigma}\delta^\omega_\epsilon \frac{\partial}{\partial \Gamma^\alpha_{\mu\nu,\gamma}}\left(\Gamma^\epsilon_{\kappa\sigma,\omega} - \Gamma^\epsilon_{\kappa\omega,\sigma} + \Gamma^\epsilon_{\eta\omega}\Gamma^\eta_{\kappa\sigma} - \Gamma^\epsilon_{\eta\sigma}\Gamma^\eta_{\kappa\omega}\right) \\
&= \frac{1}{2\kappa}\sqrt{-g}\,g^{\kappa\sigma}\delta^\omega_\epsilon\left(\delta^\epsilon_\alpha\delta^\mu_\kappa\delta^\nu_\sigma\delta^\gamma_\omega - \delta^\epsilon_\alpha\delta^\mu_\kappa\delta^\nu_\omega\delta^\gamma_\sigma\right) \\
&= \frac{1}{2\kappa}\sqrt{-g}\left(g^{\mu\nu}\delta^\gamma_\alpha - g^{\mu\gamma}\delta^\nu_\alpha\right)
\end{aligned}$$

Eq.(9)
Evaluate $\left(\frac{\partial \mathscr{L}}{\partial \Gamma^\alpha_{\mu\nu,\gamma}}\right)_{,\gamma}$:

$$\begin{aligned}
\left(\frac{\partial \mathscr{L}}{\partial \Gamma^\alpha_{\mu\nu,\gamma}}\right)_{,\gamma} &= \frac{1}{2\kappa}\left[\sqrt{-g}\left(g^{\mu\nu}\delta^\gamma_\alpha - g^{\mu\gamma}\delta^\nu_\alpha\right)\right]_{,\gamma} \\
&= \frac{1}{2\kappa}(\sqrt{-g}\,g^{\mu\nu})_{,\gamma}\delta^\gamma_\alpha - (\sqrt{-g}\,g^{\mu\gamma})_{,\gamma}\delta^\nu_\alpha \\
&= \frac{1}{2\kappa}(\sqrt{-g}\,g^{\mu\nu})_{,\alpha} - \frac{1}{2\kappa}(\sqrt{-g}\,g^{\mu\gamma})_{,\gamma}\delta^\nu_\alpha
\end{aligned}$$

Eq.(10)

$$\begin{aligned}
\left(\frac{\partial \mathscr{L}}{\partial \Gamma^\alpha_{\mu\nu,\gamma}}\right)_{,\gamma} &= \frac{\partial \mathscr{L}}{\partial \Gamma^\alpha_{\mu\nu}} \\
\underbrace{(\sqrt{-g}\,g^{\mu\nu})_{,\alpha}}_{(A)} - \underbrace{(\sqrt{-g}\,g^{\mu\gamma})_{,\gamma}\delta^\nu_\alpha}_{(B)} &= \underbrace{\sqrt{-g}\,g^{\kappa\sigma}\delta^\nu_\alpha\Gamma^\mu_{\kappa\sigma}}_{(C)} + \underbrace{\sqrt{-g}\,g^{\mu\nu}\Gamma^\omega_{\alpha\omega}}_{(D)} - \underbrace{\sqrt{-g}\,g^{\kappa\nu}\Gamma^\mu_{\kappa\alpha}}_{(E)} - \underbrace{\sqrt{-g}\,g^{\mu\sigma}\Gamma^\nu_{\alpha\sigma}}_{(F)} \\
\underbrace{(\sqrt{-g}\,g^{\mu\nu})_{,\alpha}}_{(A)} + \underbrace{\sqrt{-g}\,g^{\kappa\nu}\Gamma^\mu_{\kappa\alpha}}_{(E)} &= \underbrace{(\sqrt{-g}\,g^{\mu\gamma})_{,\gamma}\delta^\nu_\alpha}_{(B)} + \underbrace{\sqrt{-g}\,g^{\kappa\sigma}\delta^\nu_\alpha\Gamma^\mu_{\kappa\sigma}}_{(C)} + \underbrace{\sqrt{-g}\,g^{\mu\nu}\Gamma^\omega_{\alpha\omega}}_{(D)} - \underbrace{\sqrt{-g}\,g^{\mu\sigma}\Gamma^\nu_{\alpha\sigma}}_{(F)} \\
\frac{1}{\sqrt{-g}}(\sqrt{-g}\,g^{\mu\nu})_{,\alpha} + g^{\kappa\nu}\Gamma^\mu_{\kappa\alpha} &= \underbrace{\left[\frac{1}{\sqrt{-g}}(\sqrt{-g}\,g^{\mu\gamma})_{,\gamma} + g^{\kappa\sigma}\Gamma^\mu_{\kappa\sigma}\right]}_{(B)+(C)}\delta^\nu_\alpha + g^{\mu\nu}\Gamma^\gamma_{\alpha\gamma} - g^{\mu\kappa}\Gamma^\nu_{\alpha\kappa}
\end{aligned}$$

$$\underbrace{\frac{1}{\sqrt{-g}}(\sqrt{-g}\,g^{\mu\nu})_{,\alpha} - g^{\mu\nu}\Gamma^\gamma_{\gamma\alpha}}_{(6)=g^{\mu\nu}{}_{,\alpha}} + g^{\kappa\nu}\Gamma^\mu_{\kappa\alpha} + g^{\mu\kappa}\Gamma^\nu_{\kappa\alpha} = \underbrace{\left[\frac{1}{\sqrt{-g}}(\sqrt{-g}\,g^{\mu\gamma})_{,\gamma} + g^{\eta\gamma}\Gamma^\mu_{\eta\gamma}\right]}_{(\#)}\delta^\nu_\alpha + \underbrace{g^{\mu\nu}\Gamma^\gamma_{\alpha\gamma} - g^{\mu\nu}\Gamma^\gamma_{\gamma\alpha}}_{g^{\mu\nu}T^\gamma_{\alpha\gamma}} + \underbrace{g^{\mu\kappa}\Gamma^\nu_{\kappa\alpha} - g^{\mu\kappa}\Gamma^\nu_{\alpha\kappa}}_{g^{\mu\kappa}T^\nu_{\kappa\alpha}} \tag{24}$$

Calculate (#) term:

$$\begin{aligned}
\underbrace{\left[\frac{1}{\sqrt{-g}}(\sqrt{-g}\,g^{\mu\gamma})_{,\gamma} + g^{\eta\gamma}\Gamma^\mu_{\eta\gamma}\right]}_{(\#)} &= \underbrace{\frac{1}{\sqrt{-g}}(\sqrt{-g}\,g^{\mu\gamma})_{,\gamma} - g^{\mu\gamma}\Gamma^\eta_{\eta\gamma}}_{(6)=g^{\mu\gamma}{}_{,\gamma}} + g^{\eta\gamma}\Gamma^\mu_{\eta\gamma} + g^{\mu\eta}\Gamma^\gamma_{\eta\gamma} + \underbrace{g^{\mu\gamma}\Gamma^\eta_{\eta\gamma} - g^{\mu\eta}\Gamma^\gamma_{\eta\gamma}}_{g^{\mu\gamma}T^\eta_{\eta\gamma}} \\
&= g^{\mu\gamma}{}_{,\gamma} + g^{\eta\gamma}\Gamma^\mu_{\eta\gamma} + g^{\mu\eta}\Gamma^\gamma_{\eta\gamma} + g^{\mu\gamma}T^\eta_{\eta\gamma}
\end{aligned}$$

Eq.(24) become

$$g^{\mu\nu}{}_{,\alpha} + g^{\kappa\nu}\Gamma^\mu_{\kappa\alpha} + g^{\mu\kappa}\Gamma^\nu_{\kappa\alpha} = \underbrace{\left[g^{\mu\gamma}{}_{,\gamma} + g^{\kappa\sigma}\Gamma^\mu_{\kappa\sigma} + g^{\mu\eta}\Gamma^\gamma_{\eta\gamma} + g^{\mu\gamma}T^\eta_{\eta\gamma}\right]}_{(\#)}\delta^\nu_\alpha + g^{\mu\nu}T^\gamma_{\alpha\gamma} + g^{\mu\kappa}T^\nu_{\kappa\alpha} \tag{25}$$

Eq.(12)
Starting from Eq.(11)

$$\begin{aligned}
g^{\mu\nu}{}_{;\alpha} &= \left[g^{\mu\gamma}{}_{;\gamma} + g^{\mu\gamma}\, T^{\eta}_{\eta\gamma}\right]\delta^{\nu}_{\alpha} + g^{\mu\nu}T^{\gamma}_{\alpha\gamma} + g^{\mu\kappa}T^{\nu}_{\kappa\alpha} \\
g^{\mu\nu}{}_{;\alpha} - \delta^{\nu}_{\alpha}g^{\mu\gamma}{}_{;\gamma} &= \delta^{\nu}_{\alpha}g^{\mu\gamma}\, T^{\eta}_{\eta\gamma} + g^{\mu\nu}T^{\gamma}_{\alpha\gamma} + g^{\mu\kappa}T^{\nu}_{\kappa\alpha} \\
\delta^{\alpha}_{\nu}g^{\mu\nu}{}_{;\alpha} - \delta^{\alpha}_{\nu}\delta^{\nu}_{\alpha}g^{\mu\gamma}{}_{;\gamma} &= \delta^{\alpha}_{\nu}\delta^{\nu}_{\alpha}g^{\mu\gamma}\, T^{\eta}_{\eta\gamma} + \delta^{\alpha}_{\nu}g^{\mu\nu}T^{\gamma}_{\alpha\gamma} + \delta^{\alpha}_{\nu}g^{\mu\kappa}T^{\nu}_{\kappa\alpha} \\
g^{\mu\nu}{}_{;\nu} - 4\, g^{\mu\gamma}{}_{;\gamma} &= 4\, g^{\mu\gamma}\, T^{\eta}_{\eta\gamma} + g^{\mu\nu}T^{\gamma}_{\nu\gamma} + g^{\mu\kappa}T^{\nu}_{\kappa\nu} \\
-3g^{\mu\nu}{}_{;\nu} &= 2\, g^{\mu\gamma}\, T^{\eta}_{\eta\gamma} \\
g^{\mu\nu}{}_{;\nu} &= -\frac{2}{3}\, g^{\mu\gamma}\, T^{\eta}_{\eta\gamma}
\end{aligned}$$

Substituting back into Eq.(11)

$$\begin{aligned}
g^{\mu\nu}{}_{;\alpha} &= \left(-\frac{2}{3}\, g^{\mu\gamma}\, T^{\eta}_{\eta\gamma} + g^{\mu\gamma}\, T^{\eta}_{\eta\gamma}\right)\delta^{\nu}_{\alpha} + g^{\mu\nu}T^{\gamma}_{\alpha\gamma} + g^{\mu\kappa}T^{\nu}_{\kappa\alpha} \\
&= \frac{1}{3}\, g^{\mu\gamma}\, T^{\eta}_{\eta\gamma}\, \delta^{\nu}_{\alpha} + g^{\mu\nu}T^{\gamma}_{\alpha\gamma} + g^{\mu\kappa}T^{\nu}_{\kappa\alpha}
\end{aligned}$$

If trosion-free, i.e., $T^{\alpha}_{\beta\gamma} = 0$, hence:

$$g^{\mu\nu}{}_{;\alpha} = 0$$

is metric compatible.
On the otherhand, if metric compatible, i.e., $g^{\mu\nu}{}_{;\alpha} = 0$, hence:

$$\begin{aligned}
0 &= \frac{1}{3}\, g^{\mu\gamma}\, T^{\eta}_{\eta\gamma}\, \delta^{\nu}_{\alpha} + g^{\mu\nu}T^{\gamma}_{\alpha\gamma} + g^{\mu\kappa}T^{\nu}_{\kappa\alpha} \qquad (26)\\
0 &= \delta^{\alpha}_{\nu}\frac{1}{3}\, g^{\mu\gamma}\, T^{\eta}_{\eta\gamma}\, \delta^{\nu}_{\alpha} + \delta^{\alpha}_{\nu}g^{\mu\nu}T^{\gamma}_{\alpha\gamma} + \delta^{\alpha}_{\nu}g^{\mu\kappa}T^{\nu}_{\kappa\alpha} \\
0 &= \frac{4}{3}\, g^{\mu\gamma}\, T^{\eta}_{\eta\gamma} + g^{\mu\nu}T^{\gamma}_{\nu\gamma} + g^{\mu\kappa}T^{\nu}_{\kappa\nu} \\
0 &= \frac{4}{3}\, g^{\mu\gamma}\, T^{\eta}_{\eta\gamma} - 2g^{\mu\nu}T^{\gamma}_{\gamma\nu} \\
0 &= T^{\nu}_{\kappa\nu}
\end{aligned}$$

Put back into Eq.(26):

$$\begin{aligned}
0 &= 0 + 0 + g^{\mu\kappa}T^{\nu}_{\kappa\alpha} \\
0 &= T^{\nu}_{\kappa\alpha}
\end{aligned}$$

imply torsion-free.

Eq.(16)

$$\begin{aligned}
R &= g^{\kappa\sigma}\delta^{\omega}_{\epsilon}R^{\varepsilon}_{\kappa\omega\sigma} \\
&= \left(\eta^{de}e^{\kappa}_{d}e^{\sigma}_{e}\right)\delta^{\omega}_{\varepsilon}\left(e^{\varepsilon}_{c}e^{a}_{\kappa}\mathscr{R}^{c}_{a\omega\sigma}\right) \\
&= \eta^{ae}e^{\sigma}_{e}e^{\omega}_{c}\mathscr{R}^{c}_{a\omega\sigma}
\end{aligned}$$

Eq.(19)

$$
\begin{aligned}
\frac{\partial \mathscr{L}}{\partial \omega^{b}_{c\mu,\gamma}} &= \frac{\partial}{\partial \omega^{b}_{c\mu,\gamma}} \left(\frac{1}{2\kappa} e\, \eta^{ae} e^{\sigma}_{e} e^{\omega}_{f} \mathscr{R}^{f}_{a\omega\sigma} \right) \\
&= \frac{1}{2\kappa} e\, \eta^{ae} e^{\sigma}_{e} e^{\omega}_{f} \frac{\partial}{\partial \omega^{b}_{c\mu,\gamma}} \left(\omega^{f}_{a\sigma,\omega} - \omega^{f}_{a\omega,\sigma} + \omega^{f}_{d\omega}\omega^{d}_{a\sigma} - \omega^{f}_{d\sigma}\omega^{d}_{a\omega} \right) \\
&= \frac{1}{2\kappa} e\, \eta^{ae} e^{\sigma}_{e} e^{\omega}_{f} \left(\delta^{f}_{b}\delta^{c}_{a}\delta^{\mu}_{\sigma}\delta^{\gamma}_{\omega} - \delta^{f}_{b}\delta^{c}_{a}\delta^{\mu}_{\omega}\delta^{\gamma}_{\sigma} \right) \\
&= \frac{1}{2\kappa} e\, \eta^{ae} \delta^{c}_{a} \left(e^{\sigma}_{e} e^{\omega}_{f} \delta^{f}_{b}\delta^{\mu}_{\sigma}\delta^{\gamma}_{\omega} - e^{\sigma}_{e} e^{\omega}_{f} \delta^{f}_{b}\delta^{\mu}_{\omega}\delta^{\gamma}_{\sigma} \right) \\
&= \frac{1}{2\kappa} e\, \eta^{ce} \left(e^{\mu}_{e} e^{\gamma}_{b} - e^{\gamma}_{e} e^{\mu}_{b} \right)
\end{aligned}
$$

Similarly,

$$
\frac{\partial \mathscr{L}}{\partial \omega^{b}_{c\gamma,\mu}} = \frac{1}{2\kappa} e\, \eta^{ce} \left(e^{\gamma}_{e} e^{\mu}_{b} - e^{\mu}_{e} e^{\gamma}_{b} \right) = -\frac{\partial \mathscr{L}}{\partial \omega^{b}_{c\mu,\gamma}}
$$

Eq.(20)

$$
\begin{aligned}
\frac{\partial \mathscr{L}}{\partial \omega^{b}_{c\gamma}} &= \frac{\partial}{\partial \omega^{b}_{c\gamma}} \left(\frac{1}{2\kappa} e\, \eta^{ae} e^{\sigma}_{e} e^{\omega}_{f} \mathscr{R}^{f}_{a\omega\sigma} \right) \\
&= \frac{1}{2\kappa} e\, \eta^{ae} e^{\sigma}_{e} e^{\omega}_{f} \frac{\partial}{\partial \omega^{b}_{c\gamma}} \left(\omega^{f}_{a\sigma,\omega} - \omega^{f}_{a\omega,\sigma} + \omega^{f}_{d\omega}\omega^{d}_{a\sigma} - \omega^{f}_{d\sigma}\omega^{d}_{a\omega} \right) \\
&= \frac{1}{2\kappa} e\, \eta^{ae} e^{\sigma}_{e} e^{\omega}_{f} \left(\delta^{f}_{b}\delta^{c}_{d}\delta^{\gamma}_{\omega}\omega^{d}_{a\sigma} + \omega^{f}_{d\omega}\delta^{d}_{b}\delta^{c}_{a}\delta^{\gamma}_{\sigma} - \delta^{f}_{b}\delta^{c}_{d}\delta^{\gamma}_{\sigma}\omega^{d}_{a\omega} - \omega^{f}_{d\sigma}\delta^{d}_{b}\delta^{c}_{a}\delta^{\gamma}_{\omega} \right) \\
&= \frac{1}{2\kappa} e\, \eta^{ae} e^{\sigma}_{e} e^{\omega}_{f} \left(\delta^{f}_{b}\delta^{\gamma}_{\omega}\omega^{c}_{a\sigma} + \omega^{f}_{b\omega}\delta^{c}_{a}\delta^{\gamma}_{\sigma} - \delta^{f}_{b}\delta^{\gamma}_{\sigma}\omega^{c}_{a\omega} - \omega^{f}_{b\sigma}\delta^{c}_{a}\delta^{\gamma}_{\omega} \right) \\
&= \frac{1}{2\kappa} e \left(\eta^{ae} e^{\sigma}_{e} e^{\omega}_{f} \delta^{f}_{b}\delta^{\gamma}_{\omega}\omega^{c}_{a\sigma} + \eta^{ae} e^{\sigma}_{e} e^{\omega}_{f} \omega^{f}_{b\omega}\delta^{c}_{a}\delta^{\gamma}_{\sigma} - \eta^{ae} e^{\sigma}_{e} e^{\omega}_{f} \delta^{f}_{b}\delta^{\gamma}_{\sigma}\omega^{c}_{a\omega} - \eta^{ae} e^{\sigma}_{e} e^{\omega}_{f} \omega^{f}_{b\sigma}\delta^{c}_{a}\delta^{\gamma}_{\omega} \right) \\
&= \frac{1}{2\kappa} e \left(\eta^{ae} e^{\sigma}_{e} e^{\gamma}_{b} \omega^{c}_{a\sigma} + \eta^{ce} e^{\gamma}_{e} e^{\omega}_{f} \omega^{f}_{b\omega} - \eta^{ae} e^{\gamma}_{e} e^{\omega}_{b} \omega^{c}_{a\omega} - \eta^{ce} e^{\sigma}_{e} e^{\gamma}_{f} \omega^{f}_{b\sigma} \right) \\
&= \frac{1}{2\kappa} e\, \eta^{ae} e^{\mu}_{e} e^{\gamma}_{b} \omega^{c}_{a\mu} + \frac{1}{2\kappa} e\, \eta^{ce} e^{\gamma}_{e} e^{\mu}_{f} \omega^{f}_{b\mu} - \frac{1}{2\kappa} e\, \eta^{ae} e^{\gamma}_{e} e^{\mu}_{b} \omega^{c}_{a\mu} - \frac{1}{2\kappa} e\, \eta^{ce} e^{\mu}_{e} e^{\gamma}_{f} \omega^{f}_{b\mu} \\
&= \frac{1}{2\kappa} e\, \eta^{ae} \left(e^{\mu}_{e} e^{\gamma}_{b} - e^{\gamma}_{e} e^{\mu}_{b} \right) \omega^{c}_{a\mu} - \frac{1}{2\kappa} e\, \eta^{ce} \left(e^{\mu}_{e} e^{\gamma}_{f} - e^{\gamma}_{e} e^{\mu}_{f} \right) \omega^{f}_{b\mu}
\end{aligned}
$$

Eq.(21)

$$
\begin{aligned}
\frac{\partial \mathscr{L}}{\partial \omega^{b}_{c\gamma}} \omega^{b}_{c\varepsilon} &= \left[\frac{1}{2\kappa} e\, \eta^{ae} \left(e^{\mu}_{e} e^{\gamma}_{b} - e^{\gamma}_{e} e^{\mu}_{b} \right) \omega^{c}_{a\mu} - \frac{1}{2\kappa} e\, \eta^{ce} \left(e^{\mu}_{e} e^{\gamma}_{f} - e^{\gamma}_{e} e^{\mu}_{f} \right) \omega^{f}_{b\mu} \right] \omega^{b}_{c\varepsilon} \\
&= \frac{1}{2\kappa} e\, \eta^{ae} \left(e^{\mu}_{e} e^{\gamma}_{b} - e^{\gamma}_{e} e^{\mu}_{b} \right) \omega^{b}_{c\varepsilon}\omega^{c}_{a\mu} - \frac{1}{2\kappa} e\, \eta^{ce} \left(e^{\mu}_{e} e^{\gamma}_{f} - e^{\gamma}_{e} e^{\mu}_{f} \right) \omega^{f}_{b\mu}\omega^{b}_{c\varepsilon} \\
&= \frac{1}{2\kappa} e\, \eta^{ce} \left(e^{\mu}_{e} e^{\gamma}_{b} - e^{\gamma}_{e} e^{\mu}_{b} \right) \omega^{b}_{d\varepsilon}\omega^{d}_{c\mu} - \frac{1}{2\kappa} e\, \eta^{ce} \left(e^{\mu}_{e} e^{\gamma}_{b} - e^{\gamma}_{e} e^{\mu}_{b} \right) \omega^{b}_{d\mu}\omega^{d}_{c\varepsilon} \\
&= \frac{1}{2\kappa} e\, \eta^{ce} \left(e^{\mu}_{e} e^{\gamma}_{b} - e^{\gamma}_{e} e^{\mu}_{b} \right) \left(\omega^{b}_{d\varepsilon}\omega^{d}_{c\mu} - \omega^{b}_{d\mu}\omega^{d}_{c\varepsilon} \right) \\
&\underbrace{=}_{(19)} \frac{\partial \mathscr{L}}{\partial \omega^{b}_{c\mu,\gamma}} \left(\omega^{b}_{d\varepsilon}\omega^{d}_{c\mu} - \omega^{b}_{d\mu}\omega^{d}_{c\varepsilon} \right)
\end{aligned}
$$

Eq.(22)

$$
\begin{aligned}
t^{\gamma}_{\varepsilon} &= \frac{\partial \mathscr{L}}{\partial \omega^{b}_{c\mu,\gamma}} \mathscr{R}^{b}_{c\varepsilon\mu} - \delta^{\gamma}_{\varepsilon} \mathscr{L} \\
&= \frac{1}{2\kappa} e\, \eta^{ce} \left(e^{\mu}_{e} e^{\gamma}_{b} - e^{\gamma}_{e} e^{\mu}_{b}\right) \mathscr{R}^{b}_{c\varepsilon\mu} - \delta^{\gamma}_{\varepsilon} \mathscr{L} \\
&= \frac{1}{2\kappa} \sqrt{-g}\, \eta^{ce} \left(e^{\mu}_{e} e^{\gamma}_{b} - e^{\gamma}_{e} e^{\mu}_{b}\right) e^{b}_{\alpha} e^{\beta}_{c} R^{\alpha}_{\beta\varepsilon\mu} - \delta^{\gamma}_{\varepsilon} \mathscr{L} \\
&= \frac{1}{2\kappa} \sqrt{-g}\, \eta^{ce} \left(e^{\mu}_{e} e^{\gamma}_{b} e^{b}_{\alpha} e^{\beta}_{c} - e^{\gamma}_{e} e^{\mu}_{b} e^{b}_{\alpha} e^{\beta}_{c}\right) R^{\alpha}_{\beta\varepsilon\mu} - \delta^{\gamma}_{\varepsilon} \mathscr{L} \\
&= \frac{1}{2\kappa} \sqrt{-g}\, \eta^{ce} \left(e^{\mu}_{e} \delta^{\gamma}_{\alpha} e^{\beta}_{c} - e^{\gamma}_{e} \delta^{\mu}_{\alpha} e^{\beta}_{c}\right) R^{\alpha}_{\beta\varepsilon\mu} - \delta^{\gamma}_{\varepsilon} \mathscr{L} \\
&= \frac{1}{2\kappa} \sqrt{-g}\, \left(\eta^{ce} e^{\mu}_{e} e^{\beta}_{c} \delta^{\gamma}_{\alpha} R^{\alpha}_{\beta\varepsilon\mu} - \eta^{ce} e^{\gamma}_{e} e^{\beta}_{c} \delta^{\mu}_{\alpha} R^{\alpha}_{\beta\varepsilon\mu}\right) - \delta^{\gamma}_{\varepsilon} \mathscr{L} \\
&= \frac{1}{2\kappa} \sqrt{-g}\, \left(g^{\beta\mu} R^{\gamma}_{\beta\varepsilon\mu} {\color{red}+} g^{\beta\gamma} \delta^{\mu}_{\alpha} R^{\alpha}_{\beta{\color{red}\mu\varepsilon}}\right) - \delta^{\gamma}_{\varepsilon} \frac{1}{2\kappa} \sqrt{-g}\, R \\
&= \left(g^{\beta\mu} R^{\gamma}_{\beta\varepsilon\mu} + g^{\beta\gamma} R_{\beta\varepsilon} - \delta^{\gamma}_{\varepsilon} R\right) \frac{1}{2\kappa} \sqrt{-g}
\end{aligned}
$$

Eq.(23)
If metric compatible, the following is valid:

$$
g_{\alpha\gamma} R^{\gamma}_{\beta\varepsilon\mu} = R_{\alpha\beta\varepsilon\mu} = -R_{\beta\alpha\varepsilon\mu} = -R_{\alpha\beta\mu\varepsilon} = R_{\varepsilon\mu\alpha\beta}
$$

$$
\begin{aligned}
t_{\alpha\varepsilon} &= \left(g_{\alpha\gamma} g^{\beta\mu} R^{\gamma}_{\beta\varepsilon\mu} + R_{\alpha\varepsilon} - g_{\alpha\varepsilon} R\right) \frac{1}{2\kappa} \sqrt{-g} \\
&= \left(g^{\beta\mu} R_{\alpha\beta\varepsilon\mu} + R_{\alpha\varepsilon} - g_{\alpha\varepsilon} R\right) \frac{1}{2\kappa} \sqrt{-g} \\
&= \left(g^{\beta\mu} R_{{\color{red}\beta\alpha}{\color{blue}\mu\varepsilon}} + R_{\alpha\varepsilon} - g_{\alpha\varepsilon} R\right) \frac{1}{2\kappa} \sqrt{-g} \\
&= \left(R^{\mu}_{\alpha\mu\varepsilon} + R_{\alpha\varepsilon} - g_{\alpha\varepsilon} R\right) \frac{1}{2\kappa} \sqrt{-g} \\
&= \left(R_{\alpha\varepsilon} + R_{\alpha\varepsilon} - g_{\alpha\varepsilon} R\right) \frac{1}{2\kappa} \sqrt{-g} \\
&= \left(2R_{\alpha\varepsilon} - g_{\alpha\varepsilon} R\right) \frac{1}{2\kappa} \sqrt{-g} \\
&= \frac{1}{\kappa} \mathbb{G}_{\alpha\varepsilon} \sqrt{-g}
\end{aligned}
$$

Supplementary: Vielbeins

The vielbein formalism chooses a new basis frame $\{\hat{e}_a\}$ rather than a natural basis $\{\partial_\alpha\}$,

$$\partial_\mu = e^a_\mu \hat{e}_a \tag{27}$$

, where e^a_μ is transformation called frame field (or vierbein field). Clearly, each point of frame field is an element in $GL(n,\mathbb{R})$ group. We choose the frame field statisfies:

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \tag{28}$$

The inverse transformation e^μ_a:

$$e^\mu_a e^a_\nu = \delta^\mu_\nu \ \ and \ \ e^\mu_a e^b_\mu = \delta^a_b$$

Define $e = det(e^a_\mu)$, we have:

$$g = det(g_{\mu\nu}) \underbrace{=}_{(28)} det(e^a_\mu e^b_\nu \eta_{ab}) = det(e^a_\mu) det(e^b_\nu) det(\eta_{ab}) = \frac{1}{2\kappa} e \, \frac{1}{2\kappa} e \, (-1)$$

$$\rightarrow e = \sqrt{-g}$$

Recall the definition of tangent connection$\Gamma^\beta_{\alpha\mu}$:

$$\nabla_\mu \partial_\alpha = \Gamma^\beta_{\alpha\mu} \partial_\beta$$

We define the spin connection $\omega^b_{a\mu}$ in similar way:

$$\nabla_\mu \hat{e}_a = \omega^b_{a\mu} \hat{e}_b$$

The relation between $\Gamma^\beta_{\alpha\mu}$ and $\omega^b_{a\mu}$:

$$\begin{aligned}
\nabla_\mu \partial_\alpha &\underbrace{=}_{(27)} \nabla_\mu \left(e^a_\alpha \hat{e}_a\right) = e^a_{\alpha,\mu} \hat{e}_a + e^a_\alpha \nabla_\mu \hat{e}_a = e^a_{\alpha,\mu} \hat{e}_a + e^a_\alpha \omega^b_{a\mu} \hat{e}_b \\
&= \Gamma^\beta_{\alpha\mu} \partial_\beta = \Gamma^\beta_{\alpha\mu} e^b_\beta \hat{e}_b \\
&\rightarrow \left(e^b_{\alpha,\mu} + e^a_\alpha \omega^b_{a\mu} - \Gamma^\beta_{\alpha\mu} e^b_\beta\right) \hat{e}_b = 0 \\
&\rightarrow e^b_{\alpha,\mu} + e^a_\alpha \omega^b_{a\mu} - \Gamma^\beta_{\alpha\mu} e^b_\beta = 0
\end{aligned}$$

We can derive 3 useful formula:

$$e^b_{\alpha,\mu} = e^b_\beta \Gamma^\beta_{\alpha\mu} - e^a_\alpha \omega^b_{a\mu} \tag{29}$$

$$\Gamma^\beta_{\alpha\mu} = e^\beta_b e^a_\alpha \omega^b_{a\mu} + e^\beta_b e^b_{\alpha,\mu} \tag{30}$$

$$\omega^b_{a\mu} = e^\alpha_a e^b_\beta \Gamma^\beta_{\alpha\mu} - e^\alpha_a e^b_{\alpha,\mu} \tag{31}$$

Since $e^\alpha_a e^b_\alpha = \delta^b_a$, then $-e^\alpha_a e^b_{\alpha,\mu} = e^\alpha_{a,\mu} e^b_\alpha$. The Eq.(31) can be rewritten:

$$\begin{aligned}
\omega^b_{a\mu} &= e^\alpha_a e^b_\beta \Gamma^\beta_{\alpha\mu} + e^\alpha_{a,\mu} e^b_\alpha \\
\rightarrow e^\gamma_b \omega^b_{a\mu} &= e^\gamma_b e^\alpha_a e^b_\beta \Gamma^\beta_{\alpha\mu} + e^\alpha_{a,\mu} e^b_\alpha e^\gamma_b \\
\rightarrow e^\gamma_b \omega^b_{a\mu} &= e^\alpha_a \Gamma^\gamma_{\alpha\mu} + e^\gamma_{a,\mu} \\
\rightarrow e^\gamma_{a,\mu} &= e^\gamma_b \omega^b_{a\mu} - e^\alpha_a \Gamma^\gamma_{\alpha\mu}
\end{aligned} \tag{32}$$

Since frame field takes value in GL group, the spin connection $\omega^b_{a\mu}$ is $\mathfrak{gl}$-value 1-form. Next, we will derive the relation between the tangent curvature $R^\kappa_{\lambda\omega\sigma}$ and the spin curvature $\mathscr{R}^a_{b\omega\sigma}$. The definition of these curvature are:

$$R^{\kappa}_{\lambda\omega\sigma} = \Gamma^{\kappa}_{\lambda\sigma,\omega} - \Gamma^{\kappa}_{\lambda\omega,\sigma} + \Gamma^{\kappa}_{\nu\omega}\Gamma^{\nu}_{\lambda\sigma} - \Gamma^{\kappa}_{\nu\sigma}\Gamma^{\nu}_{\lambda\omega}$$
$$\mathscr{R}^{a}_{b\omega\sigma} = \omega^{a}_{b\sigma,\omega} - \omega^{a}_{b\omega,\sigma} + \omega^{a}_{c\omega}\omega^{c}_{b\sigma} - \omega^{a}_{c\sigma}\omega^{c}_{b\omega}$$

First, we take partial derivative on Eq.(30):

$$\begin{aligned}
\Gamma^{\kappa}_{\lambda\sigma,\omega} &= \left(e^{\kappa}_{a}e^{b}_{\lambda}\omega^{a}_{b\sigma} + e^{\kappa}_{b}e^{b}_{\lambda,\sigma}\right)_{,\omega} \\
&= e^{\kappa}_{a,\omega}e^{b}_{\lambda}\omega^{a}_{b\sigma} + e^{\kappa}_{a}e^{b}_{\lambda,\omega}\omega^{a}_{b\sigma} + e^{\kappa}_{a}e^{b}_{\lambda}\omega^{a}_{b\sigma,\omega} + e^{\kappa}_{b,\omega}e^{b}_{\lambda,\sigma} + e^{\kappa}_{b}e^{b}_{\lambda,\sigma\omega} \\
&= \underbrace{\left(e^{\kappa}_{c}\omega^{c}_{a\omega} - e^{\eta}_{a}\Gamma^{\kappa}_{\eta\omega}\right)}_{(32)} e^{b}_{\lambda}\omega^{a}_{b\sigma} + e^{\kappa}_{a}\underbrace{\left(e^{b}_{\eta}\Gamma^{\eta}_{\lambda\omega} - e^{c}_{\lambda}\omega^{b}_{c\omega}\right)}_{(29)}\omega^{a}_{b\sigma} + e^{\kappa}_{a}e^{b}_{\lambda}\omega^{a}_{b\sigma,\omega} + \underbrace{\left(e^{\kappa}_{c}\omega^{c}_{b\omega} - e^{\eta}_{b}\Gamma^{\kappa}_{\eta\omega}\right)}_{(32)}\underbrace{\left(e^{b}_{\gamma}\Gamma^{\gamma}_{\lambda\sigma} - e^{d}_{\lambda}\omega^{b}_{d\sigma}\right)}_{(29)} + e^{\kappa}_{b}e^{b}_{\lambda,\sigma\omega}
\end{aligned}$$

Expand and rearrange:

$$\begin{aligned}
\Gamma^{\kappa}_{\lambda\sigma,\omega} &= \left(e^{\kappa}_{c}\omega^{c}_{a\omega}e^{b}_{\lambda}\omega^{a}_{b\sigma} - e^{\eta}_{a}\Gamma^{\kappa}_{\eta\omega}e^{b}_{\lambda}\omega^{a}_{b\sigma}\right) + \left(e^{\kappa}_{a}e^{b}_{\eta}\Gamma^{\eta}_{\lambda\omega}\omega^{a}_{b\sigma} - e^{\kappa}_{a}e^{c}_{\lambda}\omega^{b}_{c\omega}\omega^{a}_{b\sigma}\right) + e^{\kappa}_{a}e^{b}_{\lambda}\omega^{a}_{b\sigma,\omega} \\
&\quad + \left(e^{\kappa}_{c}\omega^{c}_{b\omega}e^{b}_{\gamma}\Gamma^{\gamma}_{\lambda\sigma} - e^{\kappa}_{c}\omega^{c}_{b\omega}e^{d}_{\lambda}\omega^{b}_{d\sigma} - e^{\eta}_{b}\Gamma^{\kappa}_{\eta\omega}e^{b}_{\gamma}\Gamma^{\gamma}_{\lambda\sigma} + e^{\eta}_{b}\Gamma^{\kappa}_{\eta\omega}e^{d}_{\lambda}\omega^{b}_{d\sigma}\right) + e^{\kappa}_{b}e^{b}_{\lambda,\sigma\omega} \\
&= \left(\underbrace{e^{\kappa}_{c}e^{b}_{\lambda}\omega^{c}_{a\omega}\omega^{a}_{b\sigma}}_{(*a)} - \underbrace{e^{\eta}_{a}e^{b}_{\lambda}\Gamma^{\kappa}_{\eta\omega}\omega^{a}_{b\sigma}}_{(*b)}\right) + \left(e^{\kappa}_{a}e^{b}_{\eta}\Gamma^{\eta}_{\lambda\omega}\omega^{a}_{b\sigma} - e^{\kappa}_{a}e^{b}_{\lambda}\omega^{a}_{c\sigma}\omega^{c}_{b\omega}\right) + e^{\kappa}_{a}e^{b}_{\lambda}\omega^{a}_{b\sigma,\omega} \\
&\quad + \left(e^{\kappa}_{c}e^{b}_{\gamma}\Gamma^{\gamma}_{\lambda\sigma}\omega^{c}_{b\omega} - \underbrace{e^{\kappa}_{c}e^{b}_{\lambda}\omega^{c}_{a\omega}\omega^{a}_{b\sigma}}_{(*a)} \underbrace{-\Gamma^{\kappa}_{\eta\omega}\Gamma^{\eta}_{\lambda\sigma}}_{Mov\,\frac{1}{2\kappa}e\,to\,left} + \underbrace{e^{\eta}_{a}e^{b}_{\lambda}\Gamma^{\kappa}_{\eta\omega}\omega^{a}_{b\sigma}}_{(*b)}\right) + e^{\kappa}_{b}e^{b}_{\lambda,\sigma\omega}
\end{aligned}$$

Remove $(*a)$ and $(*b)$, we have:

$$\Gamma^{\kappa}_{\lambda\sigma,\omega} + \Gamma^{\kappa}_{\eta\omega}\Gamma^{\eta}_{\lambda\sigma} = \left(\underbrace{e^{\kappa}_{a}e^{b}_{\eta}\Gamma^{\eta}_{\lambda\omega}\omega^{a}_{b\sigma}}_{(\#a)} - e^{\kappa}_{a}e^{b}_{\lambda}\omega^{a}_{c\sigma}\omega^{c}_{b\omega}\right) + e^{\kappa}_{a}e^{b}_{\lambda}\omega^{a}_{b\sigma,\omega} + \left(\underbrace{e^{\kappa}_{c}e^{b}_{\gamma}\Gamma^{\gamma}_{\lambda\sigma}\omega^{c}_{b\omega}}_{(\#b)}\right) + \underbrace{e^{\kappa}_{b}e^{b}_{\lambda,\sigma\omega}}_{(\#c)} \tag{33}$$

Similarly, swap $\sigma \Leftrightarrow \omega$:

$$\Gamma^{\kappa}_{\lambda\omega,\sigma} + \Gamma^{\kappa}_{\eta\sigma}\Gamma^{\eta}_{\lambda\omega} = \left(\underbrace{e^{\kappa}_{a}e^{b}_{\gamma}\Gamma^{\gamma}_{\lambda\sigma}\omega^{a}_{b\omega}}_{(\#b)} - e^{\kappa}_{a}e^{b}_{\lambda}\omega^{a}_{c\omega}\omega^{c}_{b\sigma}\right) + e^{\kappa}_{a}e^{b}_{\lambda}\omega^{a}_{b\omega,\sigma} + \left(\underbrace{e^{\kappa}_{c}e^{b}_{\eta}\Gamma^{\eta}_{\lambda\omega}\omega^{c}_{b\sigma}}_{(\#a)}\right) + \underbrace{e^{\kappa}_{b}e^{b}_{\lambda,\omega\sigma}}_{(\#c)} \tag{34}$$

Eq.(33) $-$ Eq.(34):

$$\Gamma^{\kappa}_{\lambda\sigma,\omega} - \Gamma^{\kappa}_{\lambda\omega,\sigma} + \Gamma^{\kappa}_{\eta\omega}\Gamma^{\eta}_{\lambda\sigma} - \Gamma^{\kappa}_{\eta\sigma}\Gamma^{\eta}_{\lambda\omega} = e^{\kappa}_{a}e^{b}_{\lambda}\left(\omega^{a}_{b\sigma,\omega} - \omega^{a}_{b\omega,\sigma} + \omega^{a}_{c\omega}\omega^{c}_{b\sigma} - \omega^{a}_{c\sigma}\omega^{c}_{b\omega}\right)$$
$$R^{\kappa}_{\lambda\omega\sigma} = e^{\kappa}_{a}e^{b}_{\lambda}\mathscr{R}^{a}_{b\omega\sigma}$$