Inequalities associated with certain arithmetic functions

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Abstract

In this project, we study and collect some inequalities associated with certain arithmetic functions in number theory. For example, if φ is the Euler-phi function, then $\varphi(n) \geq \sqrt{n}$ for every positive integer n with $n \neq 2$ and $n \neq 6$.

Preliminary

Divisor function: $d(n)$, the number of positive divisors of n

$$d(p^k) = k+1$$

Sum-of-divisors function: $\sigma(n)$, the sum of all positive divisors of n

$$\sigma(p^k) = 1 + p + p^2 + \cdots + p^k = \frac{p^{k+1}-1}{p-1}$$

Euler-phi function: $\varphi(n)$, the number of positive integers $m \leq n$ that are relatively prime to n

$$\varphi(p^k) = p^k - p^{k-1} = p^k\left(1-\frac{1}{p}\right)$$

Dedekind-psi function:

$$\psi(p^k) = p^k + p^{k-1} = p^k\left(1+\frac{1}{p}\right)$$

φ^* and σ^* function: $\varphi^*(p^a) = p^a - 1$
$\sigma^*(p^a) = p^a + 1$

Result

Theorem 1 [6] If φ is the Euler-phi function, then $\varphi(n) \geq \sqrt{n}$ for every positive integer n with $n \neq 2$ and $n \neq 6$.

Theorem 2 [2] If $n \geq 3$, then $\sigma(n) < n\sqrt{n}$.

Theorem 3 [4] For all $n \geq 1$,

$$\varphi(n)\psi(n)\sigma(n) \geq \varphi^*(n)(\sigma^*(n))^2 \geq (n-1)(n+1)^2$$

Theorem 4 [3] For any $n \geq 2$,

$$\varphi\left(n\left\lfloor\frac{\psi(n)}{n}\right\rfloor\right) < n,$$

where $\lfloor x \rfloor$ denotes the integer part of x.

Theorem 5 [1] Let a and n be an integer such that $a > 6$ and $n > 2$. Then $a^{\varphi(n)} > an$.

Result

Theorem 6 [5] Let n be a positive integer. Then $\varphi(n)d(n) \geq n$.

Theorem 7 [3]

(i) There are infinitely many n such that
$$\psi(\varphi(n)) < \varphi(\psi(n)) < n$$
(ii) There are infinitely many m such that
$$\varphi(\psi(m)) < \psi(\varphi(m)) < m$$
(iii) There are infinitely many h such that
$$\varphi(\psi(h)) < h < \psi(\varphi(h))$$
(iv) There are infinitely many k such that
$$\varphi(\psi(k)) = \frac{1}{2}\psi(\varphi(k))$$

Theorem 8 [3] Let $1 < n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime factorization of n and suppose that the odd part of n is squarefull, i.e. $\alpha_i \geq 2$ for all i with $p_i \geq 3$. Then $\varphi(\psi(n)) = \psi(\varphi(n))$ if and only if for every prime p dividing $(p_1-1)(p_2-1)\cdots(p_r-1)$, one has $p \in \{p_1, p_2, \ldots, p_r\}$ and for every prime p dividing $(p_1+1)(p_2+1)\cdots(p_r+1)$, one has $p \in \{p_1, p_2, \ldots, p_r\}$

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