MIT Linear Algebra Lecture 1

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2 equations, 2 unknowns:

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$$

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

where $\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$ is the coefficient matrix.

Row picture:

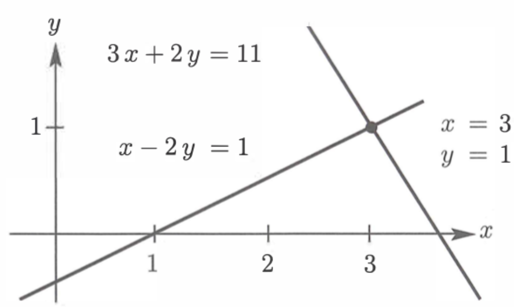

Figure 2.1: *Row picture*: The point $(3, 1)$ where the lines meet solves both equations.

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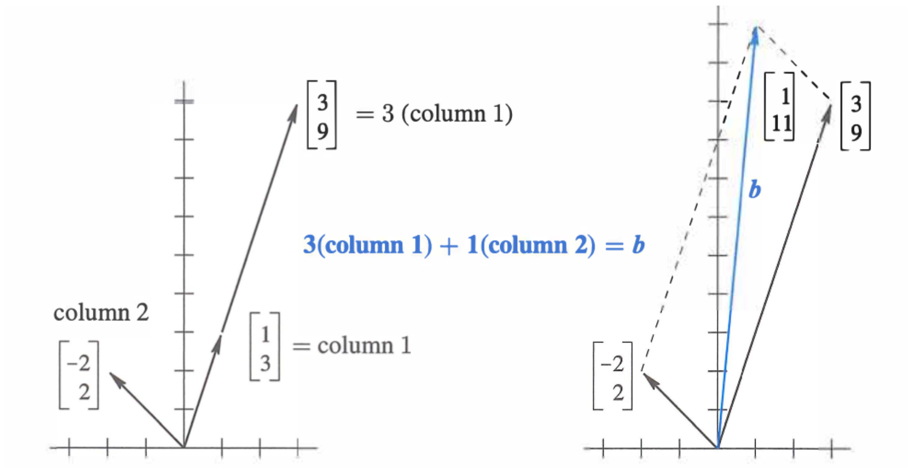
$\star\star\star$ Column picture:
Now look at the columns of the matrix:

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{array}{l} \text{one component} \\ \text{one component} \end{array} \rightarrow 2 \text{ dimension vector}$$

$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$ (**linear combination of the columns**, those vectors have 2 components)



Q: What do all the linear combinations give?
A: Will give the whole plane.

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Now consider 3 equations, 3 unknowns:

$$\begin{cases} x + 2y + 3z = 6 \\ 2x + 5y + 2z = 4 \\ 6x + 3y + z = 2 \end{cases}$$

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$ (Ax = b, will talk about it in later lecture)

where $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}$ is the coefficient matrix.

Ax: multiply a matrix by a vector.

Row picture (complex):
Each row in a 3×3 matrix gives us a plane in 3 dimensions.

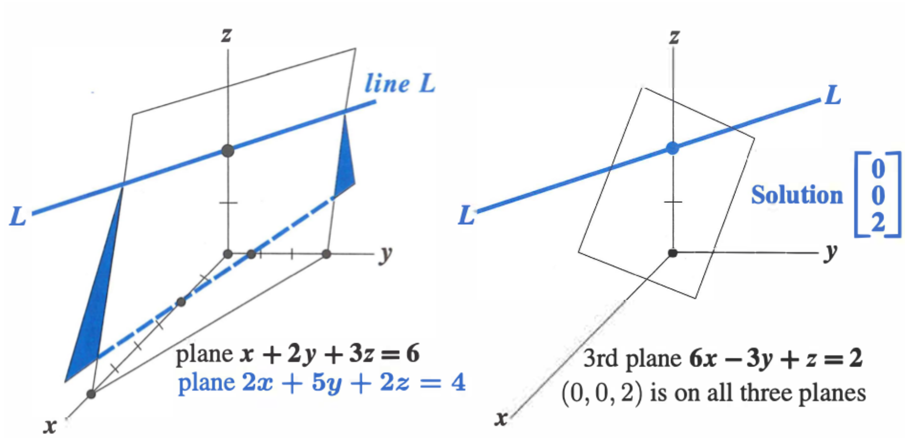

Figure 2.3: *Row picture*: Two planes meet at a line L. Three planes meet at a point.

Column picture:

$\begin{bmatrix}1\\2\\6\end{bmatrix}$ one component, one component $\rightarrow$ 3 dimension vector, one component

$x\begin{bmatrix}1\\2\\6\end{bmatrix} + y\begin{bmatrix}2\\5\\-3\end{bmatrix} + z\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$ (linear combination of the columns, those vectors have 3 components)

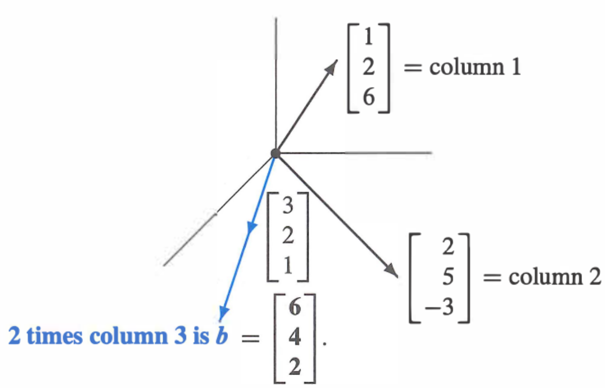

Figure 2.4: *Column picture: Combine the columns with weights* $(x, y, z) = (0, 0, 2)$.

Summary:

It is difficult to visualize this system in the row picture. However, in some cases, it may also be challenging to identify such point in the column picture.

Consider changing the value b, then the column picture will not change, but the row picture will change to 3 different plane.

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Next chapter will talk about elimination, which is the systematic way when solving the problem.

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Now think about all b's.

$\star\star\star$ Q: Can I solve $Ax = b$ for every right hand side b?

A: For the previous matrix A, the answer is yes. However, there could be other matrix that the answer is no. Consider 3 columns all lie in the same plane, them their combination will also lie in that same plane. Then it could only solve b in the same plane, but most b would be out of the plane and unreachable.**Singular case, the matrix would be not invertible. There would not be a solution for every b.**

Q: Do the linear combinations of the columns fill three dimensional space?

Q: If there is a solution?

A: If there is, elimination will give the way to find it.

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$\star\star\star$

The matrix form of the equation: $Ax = b$, where A is the matrix, x is the vector. **A times x is a linear combination of the columns of A**

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix} \quad \textbf{column at a time}$$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 5\cdot 2 \\ 1\cdot 1 + 3\cdot 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix} \quad \textbf{row at a time (dot product)}$$

Reference:

Introduction to Linear Algebra 5th by Gilbert Strang