

MIT Linear Algebra Lecture 1

Wenqing Cao

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2 equations, 2 unknowns:

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$$

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

where $\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$ is the coefficient matrix.

Row picture:

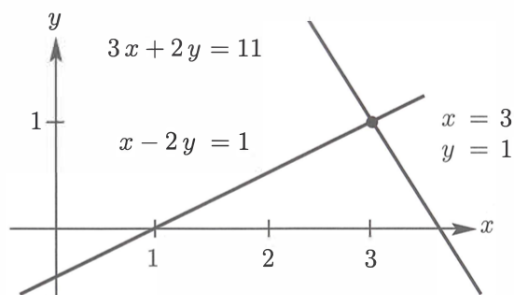


Figure 2.1: *Row picture*: The point $(3, 1)$ where the lines meet solves both equations.



*** Column picture:

Now look at the columns of the matrix:

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ one component \rightarrow 2 dimension vector
 $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ one component

The left diagram shows a 2D coordinate system with a grid. Two vectors originate from the origin (0,0). The first vector, labeled "column 1", points to the point (3, 9) and is labeled with the column vector $\begin{bmatrix} 3 \\ 9 \end{bmatrix} = 3 \text{ (column 1)}$. The second vector, labeled "column 2", points to the point (-2, 2) and is labeled with the column vector $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

The right diagram shows the same coordinate system. It includes the same two vectors. A third vector, labeled b in blue, originates from the origin and points to the point (7, 11). This vector b is the result of the linear combination $3(\text{column 1}) + 1(\text{column 2}) = b$. Dashed lines show the construction of vector b by adding three times the first vector and one times the second vector.

A: Will give the whole plane.

Now consider 3 equations, 3 unknowns:

$$\begin{cases} x + 2y + 3z = 6 \\ 2x + 5y + 2z = 4 \\ 6x + 3y + z = 2 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} \quad (Ax = b, \text{ will talk about it in later lecture})$$

where $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}$ is the coefficient matrix.

Ax : multiply a matrix by a vector.

Row picture (complex):

Each row in a 3×3 matrix gives us a plane in 3 dimensions.

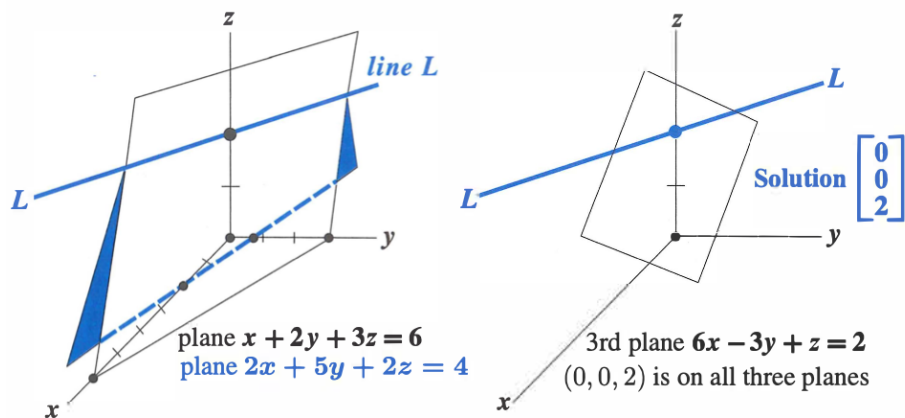


Figure 2.3: Row picture: Two planes meet at a line L . Three planes meet at a point.

Column picture:

$$\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} \text{ one component} \\ \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} \text{ one component} \rightarrow 3 \text{ dimension vector} \\ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ one component}$$

$$x \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} \text{ (linear combination of the columns, those vectors have 3 components)}$$

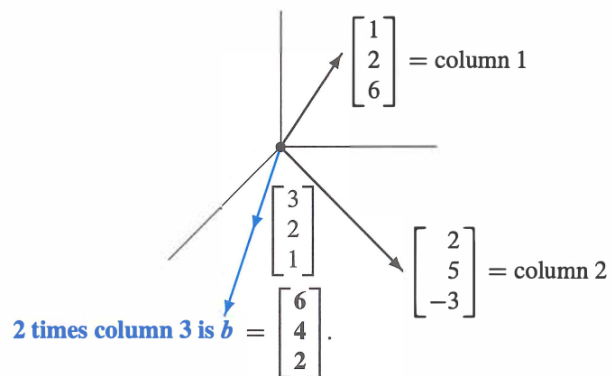


Figure 2.4: Column picture: Combine the columns with weights $(x, y, z) = (0, 0, 2)$.

Summary:

Consider changing the value b , then the column picture will not change, but the row picture will change to 3 different plane.

Next chapter will talk about elimination, which is the systematic way when solving the problem.

Now think about all b 's.

A: For the previous matrix A , the answer is yes. However, there could be other matrix that the answer is no. Consider 3 columns all lie in the same plane, then their combination will also lie in that same plane. Then it could only solve b in the same plane, but most b would be out of the plane and unreachable. **Singular case, the matrix would be not invertible. There would not be a solution for every b .**

A: if there is, elimination will give the way to find it.

The matrix form of the equation: $Ax = b$, where A is the matrix, x is the vector. **A times x is a linear combination of the columns of A**

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 5 \cdot 2 \\ 1 \cdot 1 + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix} \quad \text{row at a time (dot product)}$$

Introduction to Linear Algebra 5th by Gilbert Strang