# MIT Linear Algebra Lecture 3

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⋆⋆ Rules for Matrix Multiplication:

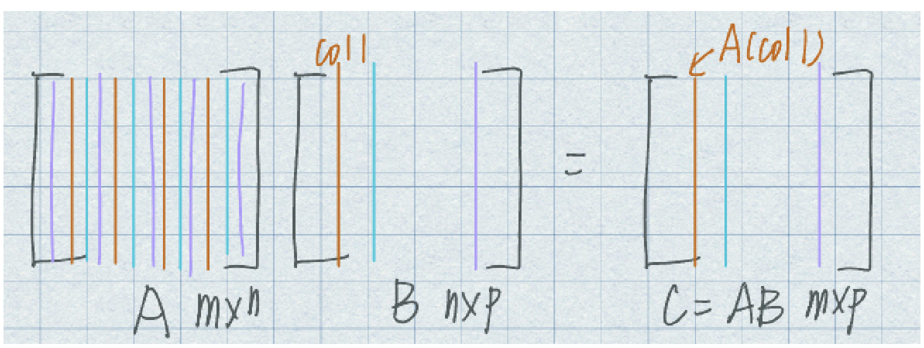
**1. Regular way for row times a column(Dot Product)**

$$\underset{A:\, m\times n}{\begin{bmatrix} & & \\ & & \\ a_{31} & a_{32} & a_{33} \\ & & \end{bmatrix}} \underset{B:\, n\times p}{\begin{bmatrix} & & b_{14} & \\ & & b_{24} & \\ & & b_{34} & \end{bmatrix}} = \underset{C=AB:\, m\times p}{\begin{bmatrix} & & & \\ & & & \\ & & & C_{34} \end{bmatrix}}$$

$C_{34} =$ (row 3 of $A$)$\cdot$ (col 4 of $B$)
$= a_{31} \cdot b_{14} + a_{32} \cdot b_{24} + a_{33} \cdot b_{34}$
$= \sum_{k=1}^{n} a_{3k} b_{k4}$

**2. Looking at whole columns(Multiply a matrix by a column)**

A column at a time:



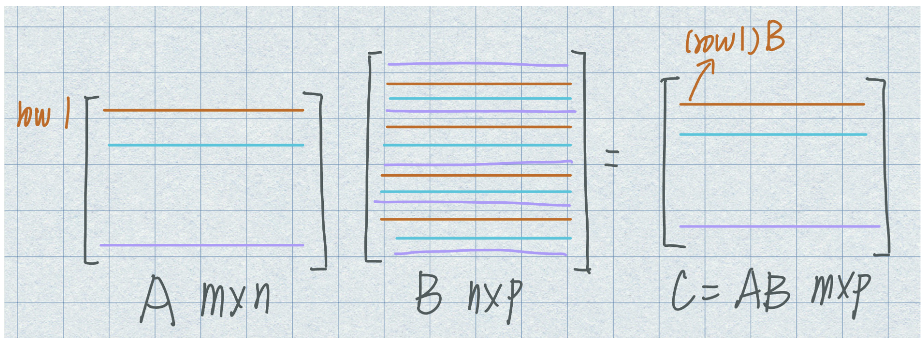

Matrix times 1st column in B is 1st column in C.
Matrix times 2nd column in B is 2nd column in C.
**These columns of C are combinations of columns of A.**
Recall in lecture 2:
Multiply a matrix by a right-hand side:

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = 3 \times \text{col } 1 + 4 \times \text{col } 2 + 5 \times \text{col } 3$$

**Matrix $\times$ column = column**

**2. Looking at whole rows(Multiply a matrix by a row)**

A row at a time:




1st row in A times matrix B is 1st row in C.
2nd row in A times matrix B is 2nd row in C.
**These rows of C are combinations of rows of B.**
Recall in lecture 2:

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = 1 \times \text{row } 1 + 2 \times \text{row } 2 + 7 \times \text{row } 3$$

**Linear combination of the rows**

Knowledge for lecture 2:
**When multiply on the right, is doing the column operation.**
**When multiply on the left, is doing the row operation.**
**When do matrix multiplication, pay attention on what it's doing with the whole vector.**

**4. Column times a row**

Column of A $\times$ row of B
$m \times 1$ $\quad$ $1 \times P$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix} \text{ (A special matrix)}$$

The rows of this special matrix are multiples of $\begin{bmatrix}1 & 6\end{bmatrix}$. ALL those rows lie on the line through(1,6). If draw picture of all these row vectors. they're all the same direction. **Row space, which is all the combinations of the rows is just a line through the vector $\begin{bmatrix}1 & 6\end{bmatrix}$ for this matrix**. (row space come in future lecture)

The columns of this special matrix are multiple of $\begin{bmatrix}2\\3\\4\end{bmatrix}$. **Column space, which is all the combinations of the columns is just a line through the vector $\begin{bmatrix}2\\3\\4\end{bmatrix}$ for this matrix**. (column space come in future lecture)

Therefore **the 4th way AB = sumof (cols of A) $\times$ (rows of B)**

$$\begin{bmatrix}2 & 7\\3 & 8\\4 & 9\end{bmatrix}\begin{bmatrix}1 & 6\\0 & 0\end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix}\begin{bmatrix}1 & 6\end{bmatrix} + \begin{bmatrix}7\\8\\9\end{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix} = \begin{bmatrix}2 & 12\\3 & 18\\4 & 24\end{bmatrix}$$

**I. cut the matrix into blocks and do the multiplication by blocks**

$$\begin{bmatrix}A_1 & | & A_2\\ -- & -- & --\\ A_3 & | & A_4\end{bmatrix}\begin{bmatrix}B_1 & | & B_2\\ -- & -- & --\\ B_3 & | & B_4\end{bmatrix} = \begin{bmatrix}A_1B_1 + A_2B_3 & | & \\ -- & -- & --\\ & | & \end{bmatrix}$$

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$\star\star\star$ **Inverses(square matrix)**
If $A^{-1}$ exists:
$A^{-1}A = I = AA^{-1}$ ( invertible, non singular matrix)
**For square matrices, a left inverse is also a right inverse.**
**For rectangular matrices, a left inverse isn't a right inverse, because the shapes won't allow it.**

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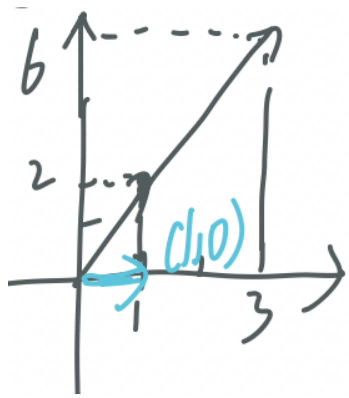
$\star\star\star$ **Singular Case(No inverse)**

A = $\begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}$ has no inverse

Reason:
**1.** $det(A) = 1\times 6 - 2\times 3 = 0$
**2.** Suppose $A\times$ some matrix gave the identity.
The columns of products are combinations of columns of A.

But the column of identity matrix like $\begin{bmatrix}1\\0\end{bmatrix}$ is not a combination of these columns $\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}3\\6\end{bmatrix}$, because these two columns both lie on the same line. Thus every combination is going to be on that line and can't get $\begin{bmatrix}1\\0\end{bmatrix}$.




⋆⋆⋆ **3. A square matrix won't have an inverse if you can find a vector $x \neq 0$ with $Ax = 0$.**

⋆⋆⋆ **Since if $A$ has an inverse, then** $Ax = 0 \longleftrightarrow AA^{-1}x = A^{-1}0 \longleftrightarrow x = 0$.

However, in this case, when $Ax = 0$, it exists $x \neq 0$:

Ax $= \begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix}$ -1$\begin{bmatrix}3\\6\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

⋆⋆⋆ **The matrix can't have an inverse if some combination of the column gives nothing.**

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Matrix has inverse

$$\underset{A}{\begin{bmatrix}1 & 3\\2 & 7\end{bmatrix}}\ \underset{A^{-1}}{\begin{bmatrix}a & c\\b & d\end{bmatrix}} = \underset{I}{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}$$

$A\times$ column $j$ of $A^{-1}$ = column $j$ of I (2 equations)

**Guass-Jordan solve 2 equations at once:**

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{array}{c} \begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 2 & 7 & | & 0 & 1 \end{bmatrix} \\ \text{A} \qquad\qquad \text{I} \end{array}$$

Do elimination steps to make A into I

The inverse will show up

Elimination:

$$\begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 2 & 7 & | & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 0 & 1 & | & -2 & 1 \end{bmatrix}$$

AI(Augmented Matrix) U(Guass quit there, but Jordan will keep going)

$$\xrightarrow{\text{Elimination upwards}} \begin{array}{c} \begin{bmatrix} 1 & 0 & | & 7 & -3 \\ 0 & 1 & | & -2 & 1 \end{bmatrix} \\ I \qquad\qquad A^{-1} \end{array}$$

$\star\star\star$ **Q: Why did we get $A^{-1}$ here?**

**A: The statement of Guass Jordan Elimination:**

$EA = I$ **tells us** $E = A^{-1}$

$E[AI] = [IA^{-1}]$

**Summary(That's how we find the inverse):**

**We can look at it as elimination as solving $N$ equations at the same time and tacking $N$ columns, solving these equations we can get the $N$ columns of $A$ inverse.**

Reference:

Introduction to Linear Algebra 5th by Gilbert Strang