MIT Linear Algebra Lecture 2

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The method of finding the solution will not be determinants in this lecture (will talk about it later). Instead, using **elimination**(the way every software package solves equation).

$$\text{Elimination} \begin{cases} \text{normally succeed (if coefficient matrix A is a good matrix)} \rightarrow \text{get answer} \\ \text{fail} \rightarrow \text{how could?} \end{cases}$$

Q: How elimination decides whether the matrix is a good one or has problem (talk later in the lecture.)? Also note that, to complete the answer there's an obvious step of **back substitution**.

We'll see elimination expressed in matrix language: (key idea in whole course: matrix operation(e.g. matrix multiplication))

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$$\begin{cases} x + 2y + z = 2 \\ 3x + 8y + z = 12 \\ \quad\quad 4y + z = 2 \end{cases}$$

$$\text{Ax} = \text{b}$$

First step: Forward Elimination

$$\underset{\text{A}}{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}} \xrightarrow{pos(2,1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

1 is the first pivot, 2 is the second pivot. **Pivot can't be 0**.
For pos(2,1) it eliminate x of equation 2.

Second step: Still Forward Elimination

$$\underset{A}{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}} \xrightarrow{pos(2,1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{pos(3,2)} \underset{\text{U(Upper Triangular)}}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}}$$

The whole purpose of elimination is to get from A to U.

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Fail for elimination:

1. The very first number was 0 Figure: exchange rows.(if there is a non-0 below this troublesome 0)

2. Example:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{pos(2,1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

0 couldn't be pivot. Figure: exchange rows(if there is a non-0 below this troublesome 0).

3.Example:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{bmatrix} \xrightarrow{pos(2,1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Don't have third pivot. **Matrix would not been invertible**.

1.Temporary Failure: can do row exchange.
2. Complete Failure: when get a 0 in pivot position and there's nothing below that we can use.

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Back substituion:
Recall that

$$\underset{A}{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}} \xrightarrow{pos(2,1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{pos(3,2)} \underset{\text{U(Upper Triangular)}}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}}$$

Now it's better to bring the right-hand side in and as an extra column:

$$\underset{\text{A}\qquad\qquad\text{b}}{\begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 3 & 8 & 1 & | & 12 \\ 0 & 4 & 1 & | & 2 \end{bmatrix}} \xrightarrow{pos(2,1)} \begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 0 & 2 & -2 & | & 6 \\ 0 & 4 & 1 & | & 2 \end{bmatrix} \xrightarrow{pos(3,2)} \underset{\text{U(Upper Triangular)}\qquad\text{c}}{\begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 0 & 2 & -2 & | & 6 \\ 0 & 0 & 5 & | & -10 \end{bmatrix}}$$

Argumented matrix(Ab)

Intuition: when working with equations, will do the same thing to both sides.
U is what happend to A.
c is what happened to b.
Then the elimination clean.

For back substituion:

$$\underset{\text{U}\qquad\qquad\text{c}}{\begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 0 & 2 & -2 & | & 6 \\ 0 & 0 & 5 & | & -10 \end{bmatrix}}$$

Solve this equation based on U and c:

$$\left\{ \begin{aligned} x + 2y + z &= 2 \\ 2y - 2z &= 6 \\ 5z &= -10 \end{aligned} \right. \Rightarrow \left\{ z = -2 \right. \Rightarrow \left\{ \begin{aligned} y &= 1 \\ z &= -2 \end{aligned} \right. \Rightarrow \left\{ \begin{aligned} x &= 2 \\ y &= 1 \\ z &= -2 \end{aligned} \right.$$

Back substitution: solve the equation in reverse order because the system is triangular.

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$\star\star\star$ Matrix Operations:
The result of multiplying a matrix by vector is a combination of the columns of the matrix.
Multiply a matrix by a right-hand side:

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = 3\times\text{col } 1 + 4\times\text{col } 2 + 5\times\text{col } 3$$

Matrix $\times$ column = column

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = 1\times\text{row } 1 + 2\times\text{row } 2 + 7\times\text{row } 3$$

Linear combination of the rows

When multiply on the right, is doing the column operation.
When multiply on the left, is doing the row operation.
When do matrix multiplication, pay attention on what it's doing with the whole vector.

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For now, will express the whole elimination process in matrix language.
Elimination Matrices:

Previous Method:

$$\underset{A\qquad\quad b}{\begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 3 & 8 & 1 & | & 12 \\ 0 & 4 & 1 & | & 2 \end{bmatrix}} \xrightarrow{pos(2,1)} \begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 0 & 2 & -2 & | & 6 \\ 0 & 4 & 1 & | & 2 \end{bmatrix} \xrightarrow{pos(3,2)} \underset{\text{U(Upper Triangular)}\qquad c}{\begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 0 & 2 & -2 & | & 6 \\ 0 & 0 & 5 & | & -10 \end{bmatrix}}$$

New Method:

Want to eliminate $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$

Step 1: Find Matrices(subtract 3 $\times$ row 1 from row 2)

$$\text{Recall}\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = 1\times\text{row } 1 + 2\times\text{row } 2 + 7\times\text{row } 3$$

$$\text{Then } \underset{E_{21}\text{ (Elementary matrix or Elimination matrix)}}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

Step 2: Find Matrices(subtract 2 $\times$ row 2 from row 3)

$$\text{Recall}\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = 1\times\text{row } 1 + 2\times\text{row } 2 + 7\times\text{row } 3$$

$$\text{Then } \underset{E_{32}\text{ (Elementary matrix or Elimination matrix)}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

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Put these steps together (into a matrix that does it all):

$E_{32}(E_{21}A) = U$

$(E_{32}E_{21})A = U$

⋆⋆⋆ **Important fact about matrix multiplication: Can't mess around with the order of the matrices, but can change the order that do the multiplication.**

Note that $AB \neq BA$. Associative law: can move the parenthese.

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Permutation Matrix P (another type of elementary matrix)

Excahnge the rows of a matrix:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

P(exchange the rows of the identity matrix)

Take 0 of row 1 and take 1 of row2, get the first row $\begin{bmatrix} b & a \end{bmatrix}$.

When multiply on the left, is doing the row operation.

Excahnge the columns of a matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

P(exchange the columns of the identity matrix)

Take 0 of col 1 and take 1 of col 2, get the first col $\begin{bmatrix} b \\ d \end{bmatrix}$.

When multiply on the right, is doing the column operation.

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The better way to think is not how to get $A \to U$, instead think **how to get U back to A**. So now coming **reversing(inverse)** steps.

Inverse:

$$\underset{\text{E}}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \text{ This step: subtract } 3 \times \text{row 1 from row 2}$$

$\star\star\star$**Want to find the matrix undoes elimination. The matrix which multiplies E to give the identity.**

$$\underset{E^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

(inverse matrix)
This step: subtract $3 \times$ row 1 from row 2 $\rightarrow$ Inverse step: Add $3 \times$ row 1 to row 2 (what substraced away, add it back).

Reference:
Introduction to Linear Algebra 5th by Gilbert Strang