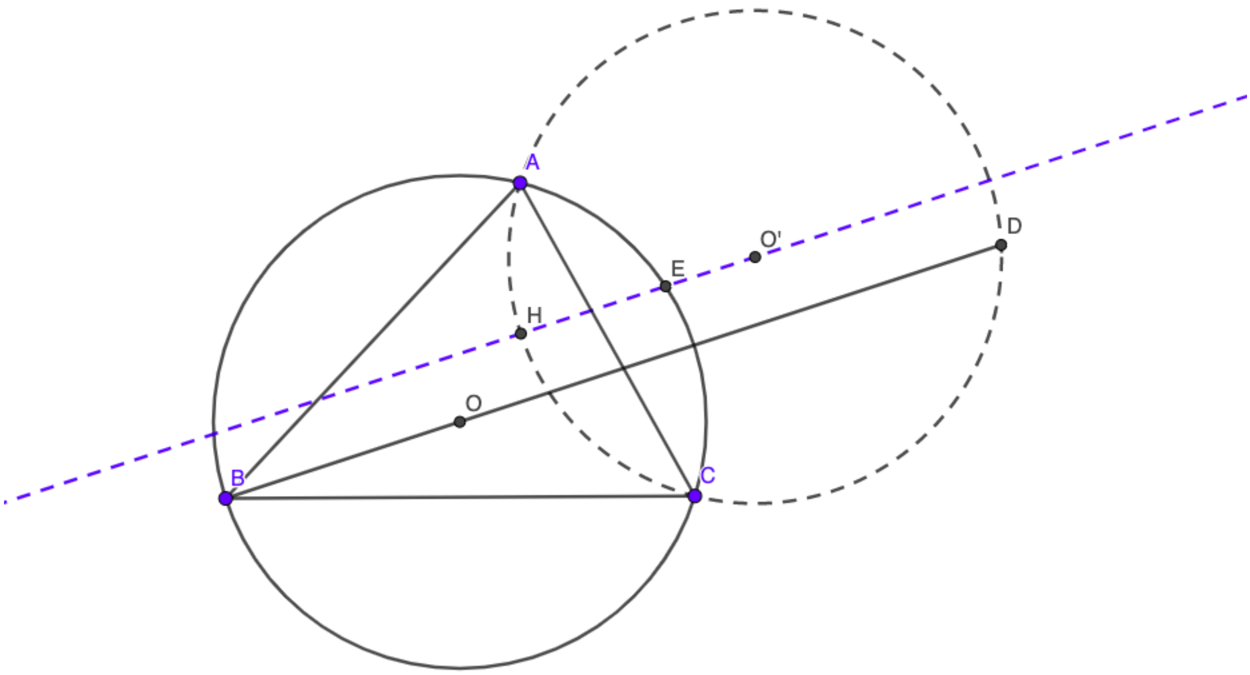
Ghana Mathematical Olympiad
Team Selection Program
Test 1

26th June, 2020

(4.) Let $\triangle ABC$ be a triangle with circumcenter at O, i.e. $\omega(ABC)$ is centered at O, orthocenter at H, and $AB < BC$. Consider a point D on OB such that O is between D and B and $\angle ADC = \angle ABC$. A ray starting at H that is parallel to BO, which intersect AC, meets $\omega(ABC)$ at E. Show that $BH = DE$.

Solution.



Consider the quadrilateral $AHCD$:

$$\begin{aligned}\angle AHC + \angle ADC &= (180^\circ - \angle ACH - \angle HAC) + \angle ABC \\ &= (180^\circ - (90^\circ - \alpha) - (90^\circ - \gamma)) + \beta \\ &= (180^\circ - \beta) + \beta \\ &= 180^\circ.\end{aligned}$$

Therefore, $AHCD$ is cyclic, meaning its vertices lie on a same circle.

Now, let O' be a reflection of O over AC. Then,

$$\begin{aligned}
&O'A = O'B \\
&\angle AO'B = \angle AOB = 2\beta = 2\angle ADC.
\end{aligned}$$

So, O' is the circimcenter of $\omega(ADC)$.

By Law of Sine, the radius of the circumcircles $\omega(ABC)$ and $\omega(ADC)$ are same:

$$2R = \frac{AC}{\sin \angle ABC} = \frac{AC}{\sin \angle ADC}.$$

The quadrilateral $BHO'O$ is a parallelogram because:

$$\begin{aligned}
&HO' = OB \\
&BH \perp AC \text{ and } OO' \perp AC \Longrightarrow BH \parallel OO'.
\end{aligned}$$

So, $O' \in HE$.
Also, the quadrilateral $OO'ED$ is isosceles trapezoid since:

$$\begin{aligned}
&O'E \parallel OD \\
&R = O'D = OE.
\end{aligned}$$

Hence, $BH = OO' = DE$.