1. Let ABC be an equilateral triangle and P be a point inside the triangle. If the distances from P to the three sides are 3, 5, and 6, then find the perimeter of the triangle ABC.

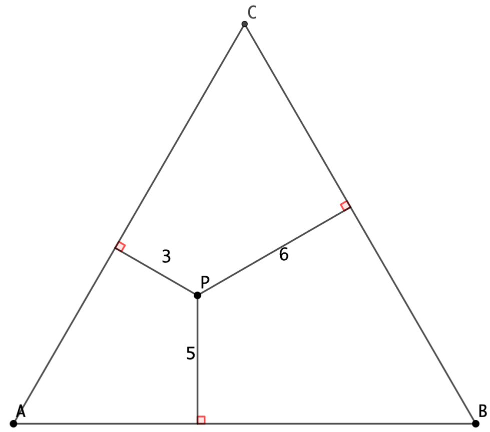

Figure 1:

Solution:
Let x be the length of the side of the equilateral triangle ABC. We see that the area of ABC,

$$S_{ABC} = \frac{1}{2}x^2 \sin 60^\circ = \frac{x^2\sqrt{3}}{4}.$$

On the other hand, the area of ABC is the sum of the areas of APB, BPC, and CPA, i.e.

$$\begin{aligned} S_{ABC} &= S_{APB} + S_{BPC} + S_{CPA} \\ &= \frac{3 \cdot x}{2} + \frac{6 \cdot x}{2} + \frac{5 \cdot x}{2} \\ &= 7x \end{aligned}$$

Therefore,

$$\frac{x^2\sqrt{3}}{4} = 7x \implies x = \frac{28}{\sqrt{3}}.$$

So, the perimeter of the triangle is

$$P_{ABC} = 3 \cdot x = 28\sqrt{3}.$$

2. Let $ABCD$ be a cyclic quadrilateral, i.e. the points A, B, C, and D lie on a same circle, and P be the intersection of diagonals AC and BD. Given that $AB = 20$, $BC = 2\sqrt{22}$, $CD = 5$, $DA = 6\sqrt{22}$, and $DP = 6$, find the lengths of AP and CP.

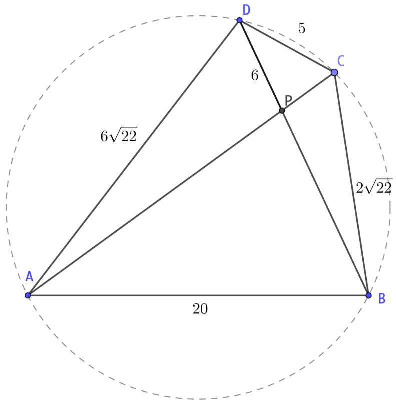

Figure 2:

<u>**Solution**</u>:
By the property of cyclic quadrilateral, we know that

$$\angle PAB = \angle CAB = \angle CDB = \angle PDC.$$

Similarly, we see that

$$\angle PBA = \angle DBA = \angle DCA = \angle DCP.$$

Therefore, the trianles PAB and PDC are similar. So

$$\frac{AB}{DC} = \frac{PA}{PD} \implies PA = 24.$$

Now, we also observe that the triangles PDA and PCB are similar. Hence,

$$\frac{DA}{CB} = \frac{PD}{PC} \implies PC = 2.$$

3. Let ABC be a right triangle with $\angle ACB = 90°$ and P, Q, and R be the midpoints of the sides AB, BC, and CA, respectively. Two equilateral triangles AMR and BQN are constructed outside of triangle ABC. Find $\angle PMN$.

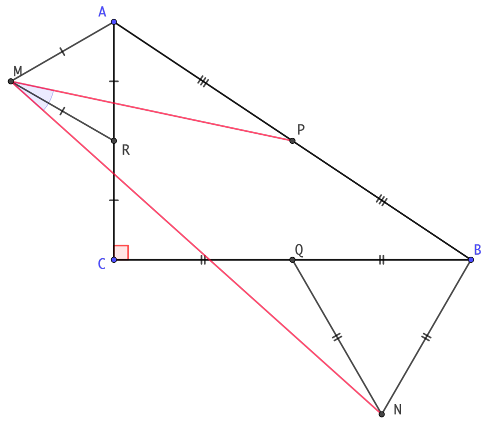

Figure 3:

Solution:
The triangles MRP and PQN are equivalent since

- $MR = AR = \frac{1}{2}AC = PQ \implies MR = PQ$.
- $\angle MRP = \angle MRA + \angle ARP = 150° = \angle PQB + \angle BQN = \angle PQN \implies \angle MRP = PQN$.
- $RP = \frac{1}{2}CB = QB = QN \implies RP = QN$.

Therefore, $MP = PN$ and $\angle PMR = \angle NPQ$. Also,

$$\begin{aligned}
\angle MPN &= \angle MPR + \angle RPQ + \angle QPN \\
&= \angle MPR + 90° + \angle PMR \\
&= 90° + (\angle MPR + \angle PMR) \\
&= 90° + (180° - \angle MRP) \\
&= 90° + (180° - 150°) \\
&= 120°.
\end{aligned}$$

Since we have found that $MP = PN$, the triangle MPN is isosceles with $MPN = 120°$. So,

$$\begin{aligned}
\angle PMN &= \frac{1}{2}(180° - \angle MPN) \\
&= \frac{1}{2}(180° - 120°) \\
&= 30°.
\end{aligned}$$