Geometry

MOP 2020

Lecture 5

July 21, 2020

Poll

- How was the test?
- How was the CMC?

Learning Outcome

Today's learning outcomes:

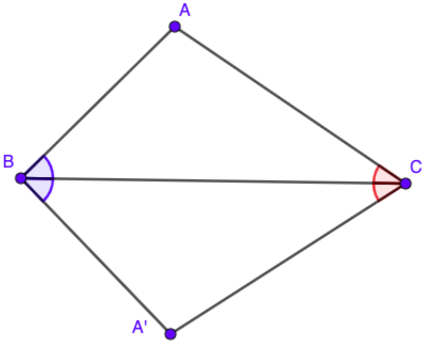
- Isometric Transformations
- Spiral Similarities
- Radical Axis

Last Week

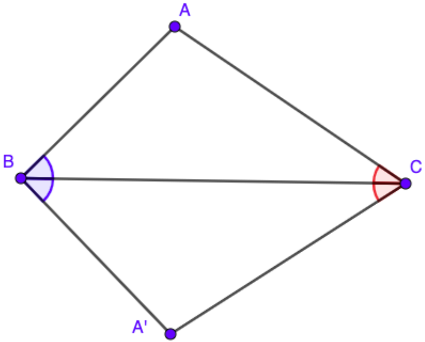
Last week:

- Isometric Transformations: translation, reflection, and rotation
- Homothety.

Isometric Transformation



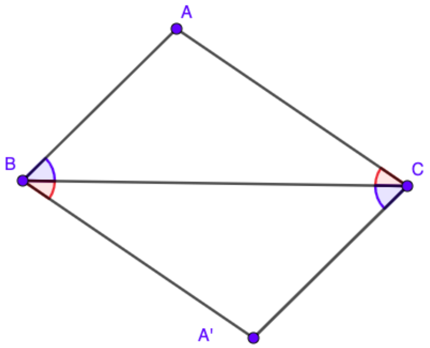
Find an isometric transformation that maps $\triangle ABC$ to $\triangle A'BC$, where these two triangles are congruent.



Find an isometric transformation that maps $\triangle ABC$ to $\triangle A'BC$, where these two triangles are congruent.

Reflection over BC.

Isometric Transformation



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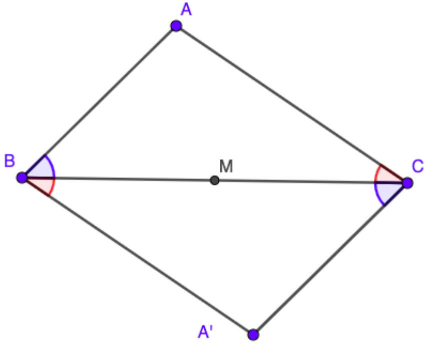
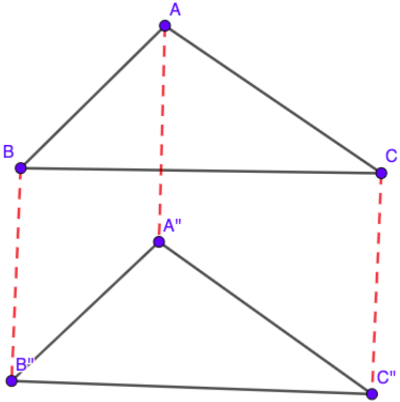
Isometric Transformation



Find an isometric transformation that maps $\triangle ABC$ to $\triangle A'CB$, where these two triangles are congruent.

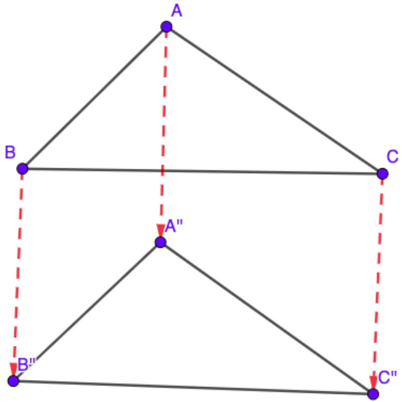
Reflection about M, the midpoint of BC.

Isometric Transformation



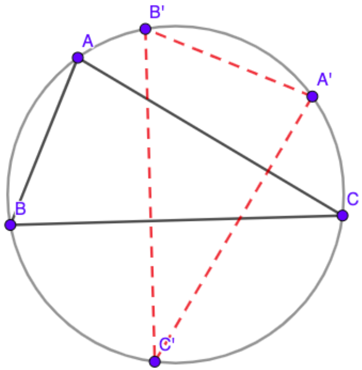
Find an isometric transformation that maps $\triangle ABC$ to $\triangle A''B''C''$, where these two triangles are congruent. (We have 3 parallelograms.)

Isometric Transformation



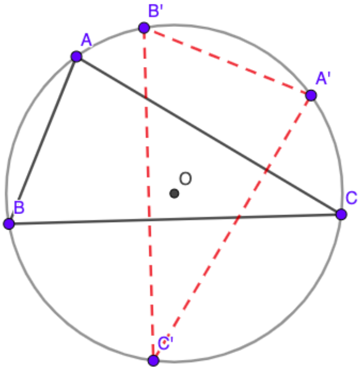
Find an isometric transformation that maps $\triangle ABC$ to $\triangle A''B''C''$, where these two triangles are congruent. (We have 3 parallelograms.)

Translation that takes A to A'' (or B to B'', and C to C'').



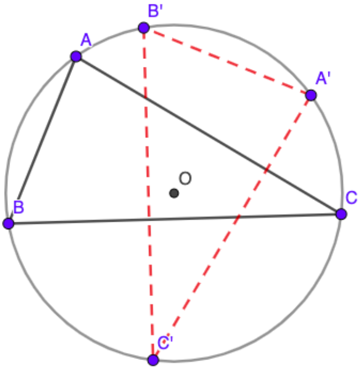
Find an isometric transformation that maps $\triangle ABC$ to $\triangle A'B'C'$, where these two triangles are congruent. (We have all six points lie on a same circle.)

Isometric Transformation



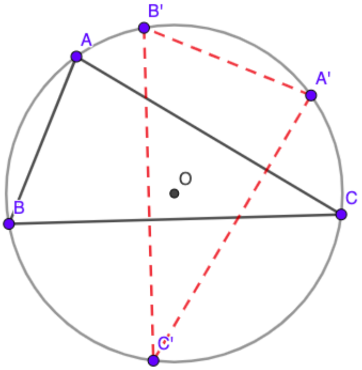
Find an isometric transformation that maps $\triangle ABC$ to $\triangle A'B'C'$, where these two triangles are congruent. (We have all six points lie on a same circle.)

Rotation about O, the circumcenter of $\triangle ABC$.



Rotation about O, the circumcenter of $\triangle ABC$.

What's the rotation angle?

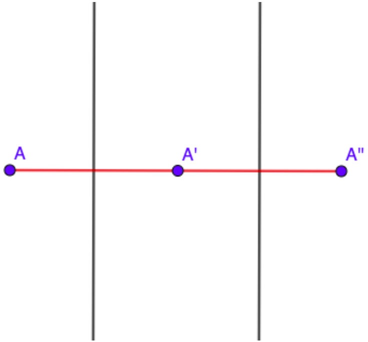


Rotation about O, the circumcenter of $\triangle ABC$.

What's the rotation angle?

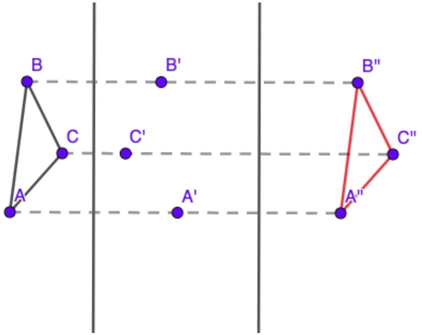
The angle between AB and $A'B'$ (or between AC and $A'C'$, and between BC and $B'C'$.)

Isometric Transformations



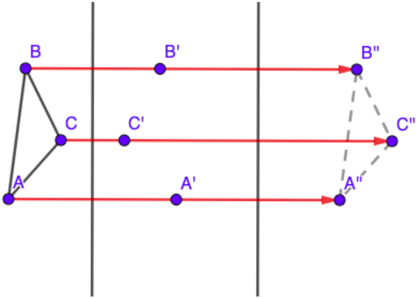
What is the composition of two reflections about parallel lines?

Isometric Transformations



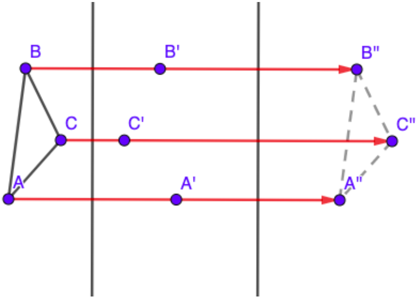
What is the composition of two reflections about parallel lines?

Hint: Figure.



What is the composition of two reflections about parallel lines?

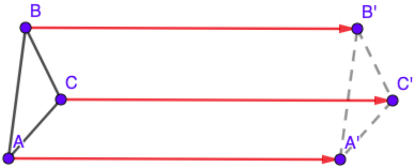
It is a translation.



What is the composition of two reflections about parallel lines?

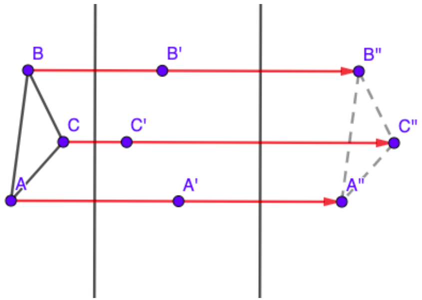
It is a translation.

What is the distance?



Now, show that any translation can be written as the composition of two reflections.

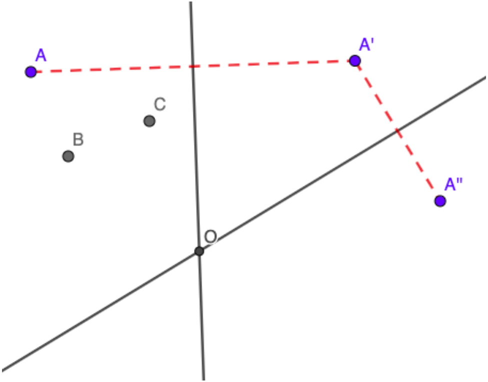
Isometric Transformations



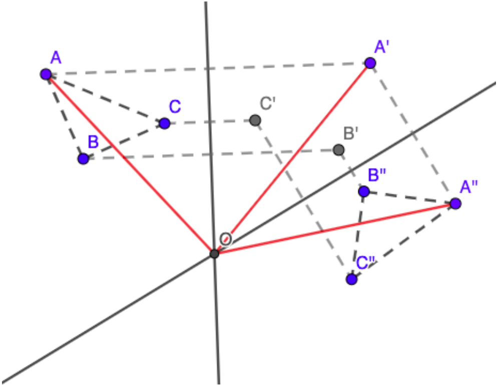
Now, show that any translation can be written as the composition of two reflections.

Take two lines that:

- are perpendicular to the translation vector
- have distance equal to half of the translation distance.

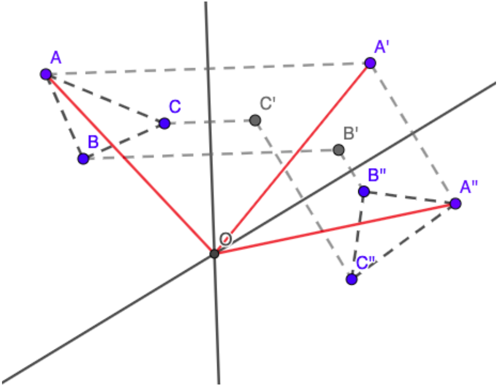


What about the composition of two reflections with axes that are not parallel?



What about the composition of two reflections with axes that are not parallel?

Rotation centered at O, the intersection of two axes.

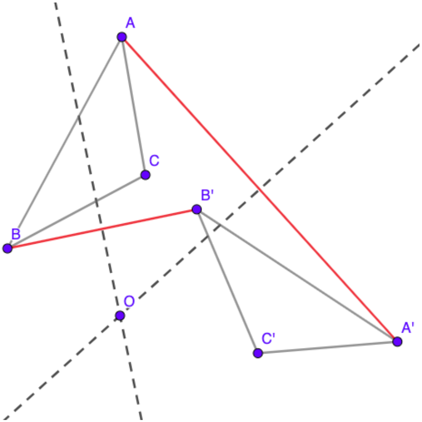


What about the composition of two reflections with axes that are not parallel?

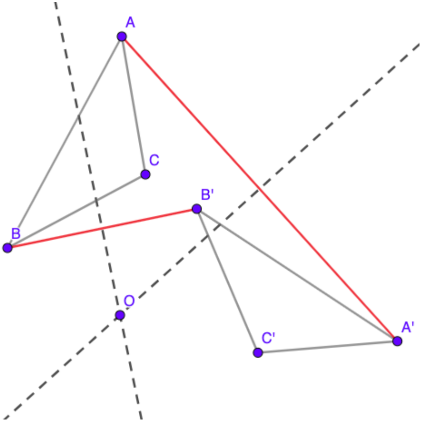
Rotation centered at O, the intersection of two axes.

What's the rotation angle?

Isometric Transformations



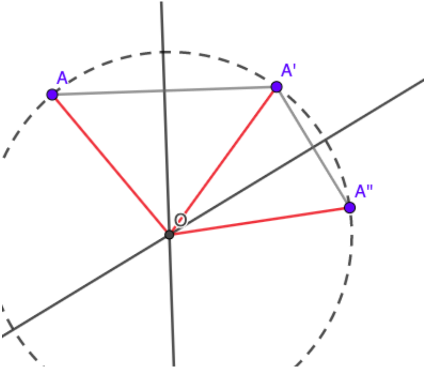
$\triangle ABC$ and $\triangle A'B'C'$ have same orientation, and corresponding sides are not parallel. Then there is a rotation that takes $\triangle ABC$ to $\triangle A'B'C'$.



$\triangle ABC$ and $\triangle A'B'C'$ have same orientation, and corresponding sides are not parallel. Then there is a rotation that takes $\triangle ABC$ to $\triangle A'B'C'$.

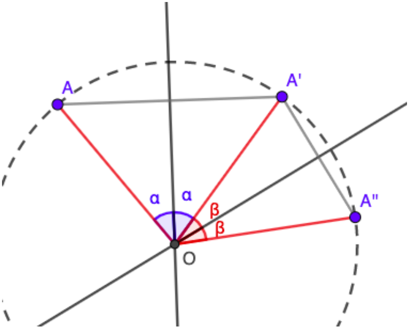
To find the center of the rotation, we can construct perpendicular bisectors of AA' and BB'. Let O be the intersection. Then, O will be the center of rotation, and the rotation angle is $\angle AOA'$.

Isometric Transformations



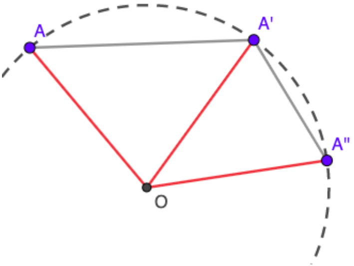
Now, show that any rotation can be written as the composition of two reflections.

Isometric Transformations



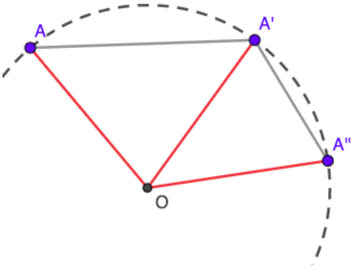
Now, show that any rotation can be written as the composition of two reflections.

Take any two axes that intersect with angle that is half of the rotation angle.



The composition of 2 rotations with same center is a rotation.

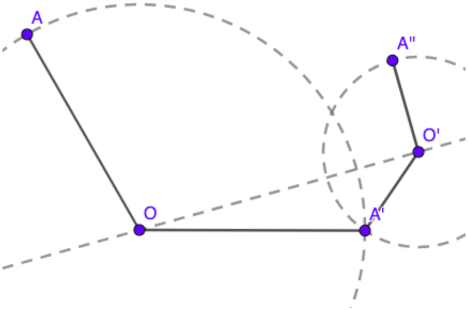
Isometric Transformations



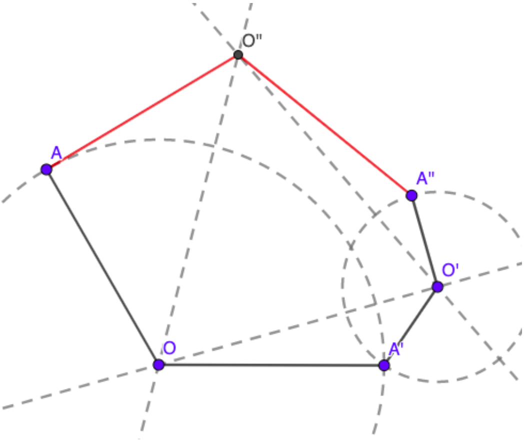
The composition of 2 rotations with same center is a rotation.

Trivial.

Isometric Transformations



The composition of 2 rotations with different centers is a translation or a rotation.

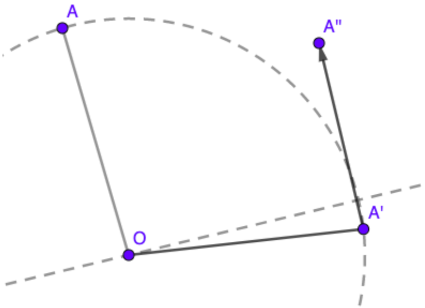


The composition of 2 rotations with different centers is a translation or a rotation.

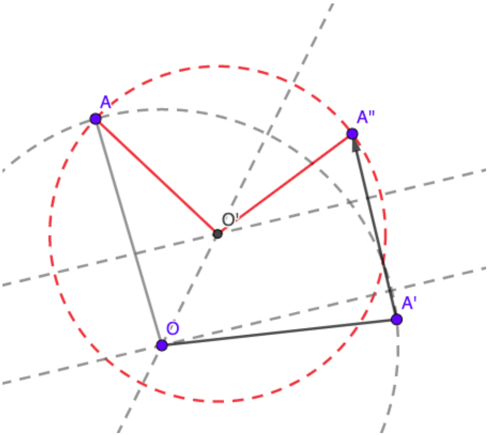
Change the rotations into 2 reflections:

- 1. l and OO'
- 2. OO' and m

Now this composition becomes reflection along l and then m.



The composition of a rotation and a translation is a rotation.

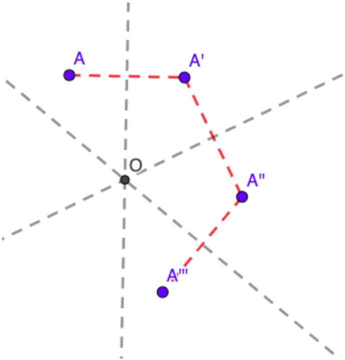


The composition of a rotation and a translation is a rotation.
Change each transformation into 2 reflections:

- 1. l and m, where $O \in m$ and $m \perp A'A''$
- 2. m and k

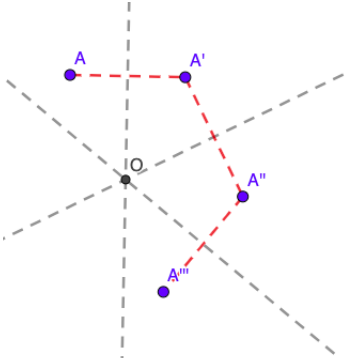
Now this composition becomes reflection along l and then k. And their composition is a rotation.

Isometric Transformations



The composition of three reflections with concurrent axes is a reflection.

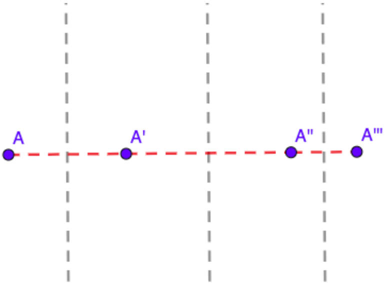
Isometric Transformations



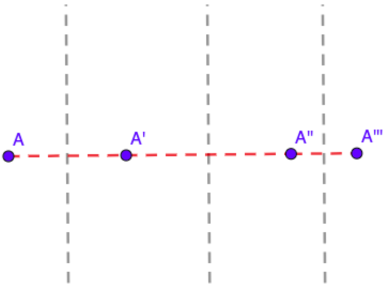
The composition of three reflections with concurrent axes is a reflection.

Change the first two reflections to a rotation. Change that rotation to two reflections, where the latter is the third original reflection. Now, we only have one reflection.

Isometric Transformations



The composition of three reflections with parallel axes is a reflection.



The composition of three reflections with parallel axes is a reflection.

Change the first two reflections to a translation. Change the translation to two reflection, where the latter is the third original reflection. Now, we have left with only one reflection.

Isometric Transformations

Suppose we have three reflections with axes, l, m, and k, where l and m are not parallel. We can make these transformations into three reflections with axes l', m', and k', where $l' \parallel m'$ and $m' \perp k'$ (this is called a glide reflection.)

Isometric Transformations

$$(l, m, k) \to (l', m', k'), l' \parallel m', m' \perp k'.$$

$$(l, m, k) \rightarrow (l', m', k'), l' \parallel m', m' \perp k'.$$

Change $(l, m) \rightarrow$ a rotation, r.
So, $(l, m, k) \rightarrow (r, k)$

Isometric Transformations

$$(r, k) \to (l', m', k'), l' \parallel m', m' \perp k'.$$

Change $r \to (l'', m'')$, where $m'' \perp k$.
So, $(r, k) \to (l'', m'', k)$ where $m'' \perp k$.

$$(l'', m'', k), m'' \perp k \to (l', m', k'), l' \parallel m', m' \perp k'.$$

Change $(m'', k) \to$ a rotation, r'.
So, $(l'', m'', k) \to (l'', r')$.

Isometric Transformations

$$(l'', r') \to (l', m', k'), l' \parallel m', m' \perp k'.$$

Change $r' \to (m', k')$, where $m' \parallel l''$.
So, $(l'', r') \to (l'', m', k')$, where $l'' \parallel m'$.
And we know that $m' \perp k'$.

Isometric Transformations

$$(l'', m', k'), l'' \parallel m', m' \perp k' \to (l', m', k'), l' \parallel m', m' \perp k'.$$

Take l' as l''. Now, we are done.

Isometric Transformations

Suppose we have three reflections with axes, l, m, and k, where l and m are parallel. We can can these transformations into three reflections with axes l', m', and k', which results a glide reflections, i.e. $l' \parallel m'$ and $m' \perp k'$.

Isometric Transformations

$$(l, m, k) \to (l', m', k'), l' \parallel m', m' \perp k'.$$

$$(l, m, k) \to (l', m', k'), l' \parallel m', m' \perp k'.$$

Change $(m, k) \to$ a rotation r.
So, $(l, m, k) \to (l, r)$.

$$(l, r) \to (l', m', k'), l' \parallel m', m' \perp k'.$$

Change $r \to (m'', k'')$ where m'' is not parallel to l.
So, $(l, r) \to (l, m'', k'')$, which is the first case we have considered.

Main Theorem: Suppose we have a isometric transformation that is a composition of translation, rotation, and reflection. Then, this isometric transformation can be reduced to no more than three reflections.

Isometric Transformations

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E.g. The composition can be:

- $Re_1 Re_2 T_1 Ro_1 Ro_1 T_2 T_3$
- $T_1 T_2 T_3 T_4$
- etc.

Isometric Transformations

Main Theorem: Suppose we have a isometric transformation that is a composition of translation, rotation, and reflection. Then, this isometric transformation can be reduced to no more than three reflections.

E.g. The composition can be:

- $Re_1 Re_2 T_1 Ro_1 Ro_1 T_2 T_3$
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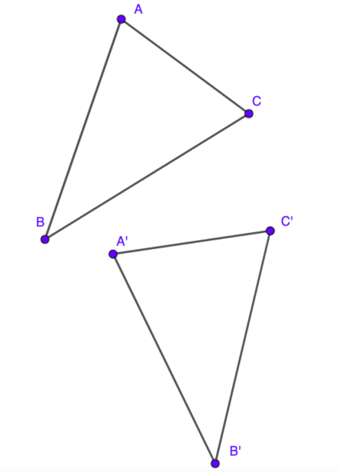
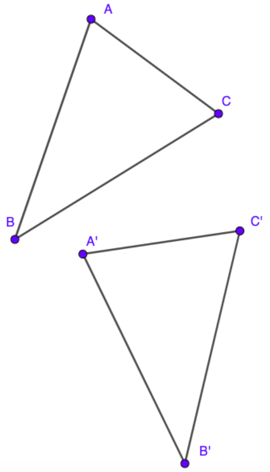
- etc.

Proof.

Use previous results, and then a little combinatorics problem? :)



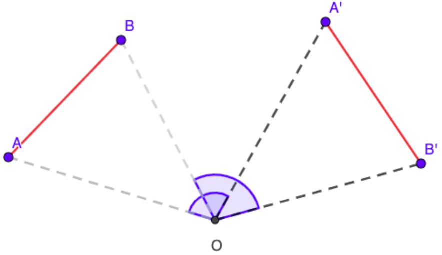
Suppose we have two congruent triangles. Show that we can find a composition that is no more than three reflections that maps $\triangle ABC \to \triangle A'B'C'$



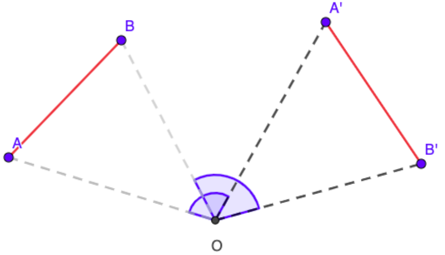
Suppose we have two congruent triangles. Show that we can find a composition that is no more than three reflections that maps $\triangle ABC \to \triangle A'B'C'$

Hint: Take midpoint of AA', BB', and CC'. They will tell you what kind of transformations we need.

Spiral Similarity

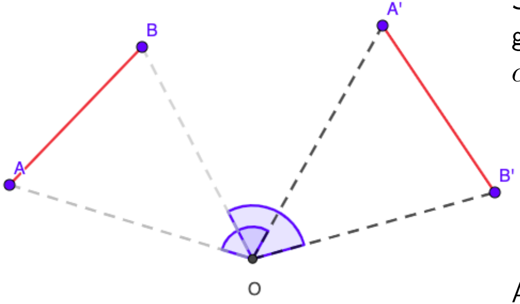


Scale and *rotate* with given center O and angle α and scale factor k.



Scale and *rotate* with given center O and angle α and scale factor k:

- $OA' = kOA$
- $\angle AOA' = \alpha$
- $OB' = kOB$
- $\angle BOB' = \alpha$



Scale and *rotate* with given center O and angle α and scale factor k:

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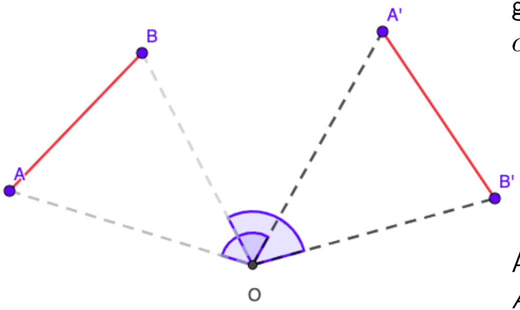
As a consequence, $A'B' = kAB$.



Scale and *rotate* with given center O and angle α and scale factor k:

- $OA' = kOA$
- $\angle AOA' = \alpha$
- $OB' = kOB$
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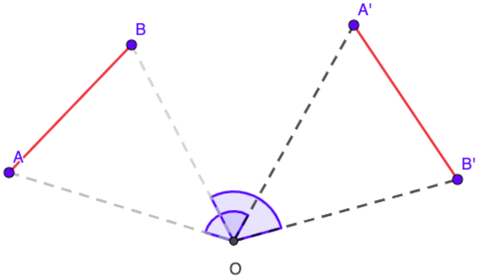
As a consequence, $A'B' = kAB$.
Also, $OA \cdot OB' = OB \cdot OA'$.

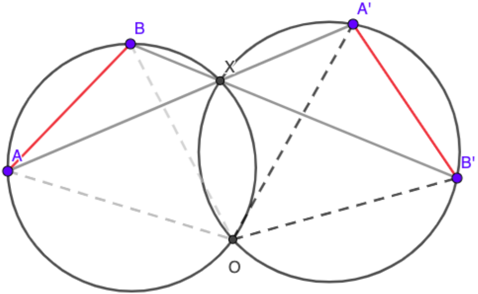
Spiral Similarity



Suppose we don't know the spiral similarity center but only know A and B goes to A' and B'.

How do we find the center?

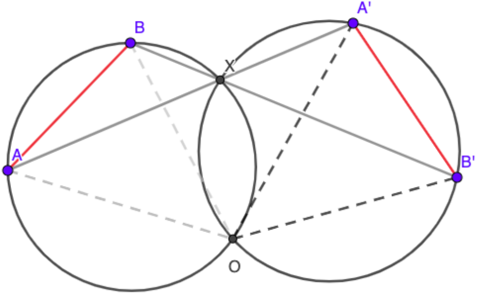
Spiral Similarity



How do we find the center?

$$X = AA' \cap BB'$$
$$O = \omega(ABX) \cap \omega(A'B'X').$$

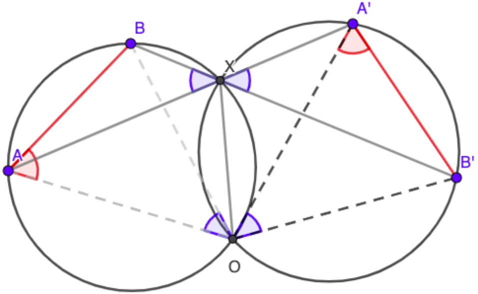
Spiral Similarity



$X = AA' \cap BB'$
$O = \omega(ABX) \cap \omega(A'B'X')$.

Show that O is the center of the spiral similarity.

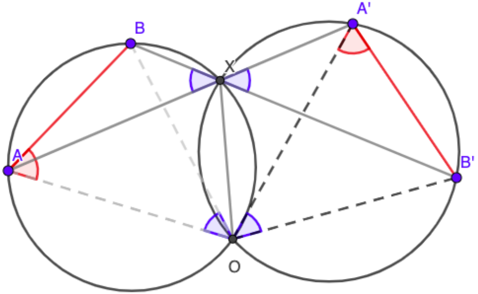
Spiral Similarity



Show that O is the center of the spiral similarity.

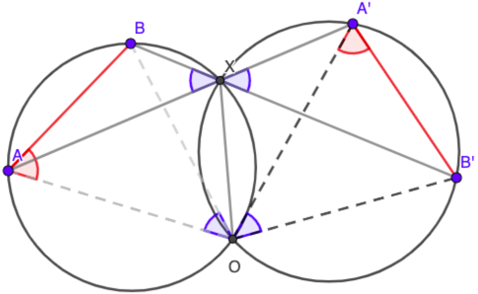
In other words, we just need to show $\triangle AOB \sim \triangle A'OB'$.

Spiral Similarity



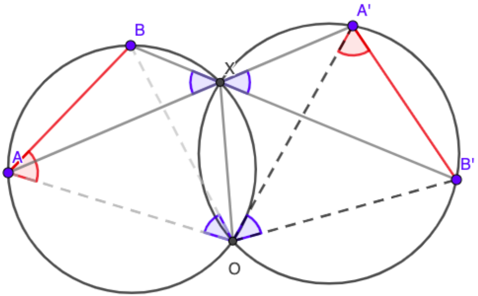
Show that O is the center of the spiral similarity.

In other words, we just need to show $\triangle AOB \sim \triangle A'OB'$. This is true by AA.



Suppose a spiral similarity centered at O takes AB to $A'B'$. Then, show that there is a spiral similarity centered at O takes AA' to BB'.

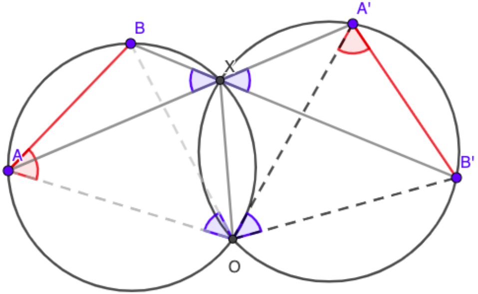
Spiral Similarity



Suppose a spiral similarity centered at O takes AB to $A'B'$. Then, show that there is a spiral similarity centered at O takes AA' to BB'.

We just need to show $\triangle OAA' \sim \triangle OBB'$.

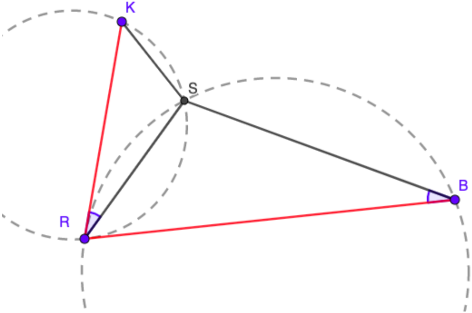
Spiral Similarity



Suppose a spiral similarity centered at O takes AB to $A'B'$. Then, show that there is a spiral similarity centered at O takes AA' to BB'.

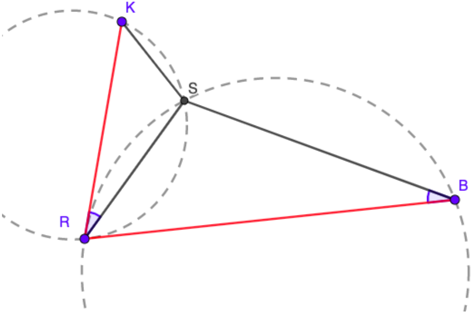
We just need to show $\triangle OAA' \sim \triangle OBB'$:

- $\angle AOA' = \angle BOB'$
- $\frac{OA}{OB} = \frac{kOA}{kOB} = \frac{OA'}{OB'}$



Consider spiral similarity centered at S that takes KR to RB. What can we tell about this situation?

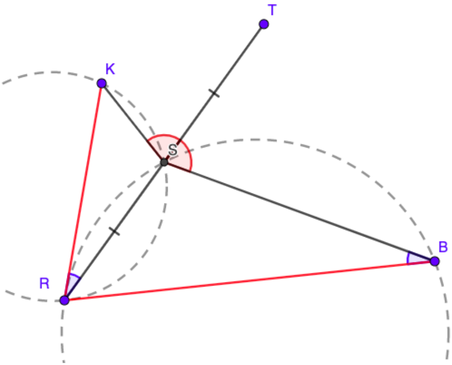
Spiral Similarity



Consider spiral similarity centered at S that takes KR to RB. What can we tell about this situation?

We can deduce that KR is tangent to $\omega(SRB)$. Similarly, RB is tangent to $\omega(SKR)$.

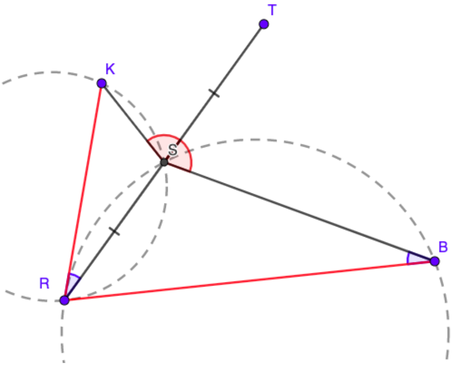
Spiral Similarity



Consider spiral similarity centered at S that takes KR to RB. What can we tell about this situation? RB is tangent to $\omega(SKR)$.

Now, construct T such that $RS = ST$ and $T \in RS$. In other words, T is the reflection of R about S.

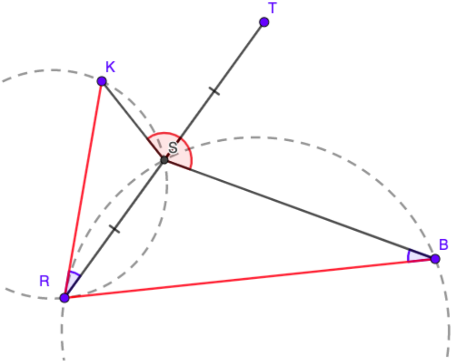
Spiral Similarity



S takes KR to RB.
RB is tangent to $\omega(SKR)$.

T is the reflection of R about S. Can we say S takes KT to TB?

Spiral Similarity

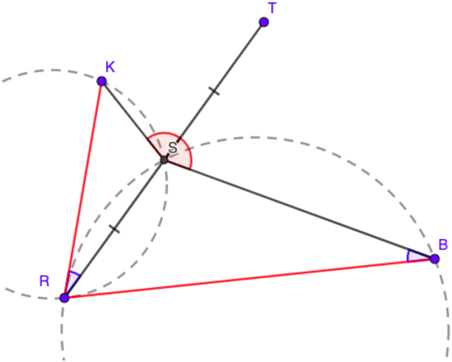


T is the reflection of R about S. Can we say S takes KT to TB?

Yes:

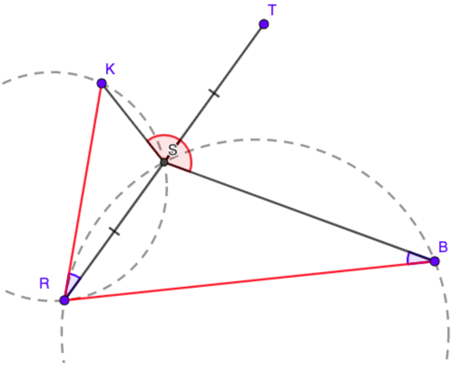
- $\frac{KS}{ST} = \frac{KS}{SR} = \frac{RS}{SB} = \frac{TS}{SB}$
- $\angle KST = \angle BST$

Spiral Similarity



So, S takes KT to TB.

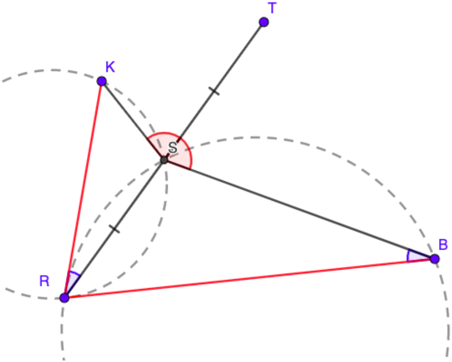
Spiral Similarity



So, S takes KT to TB.

Now, what do we know about in this scenario?

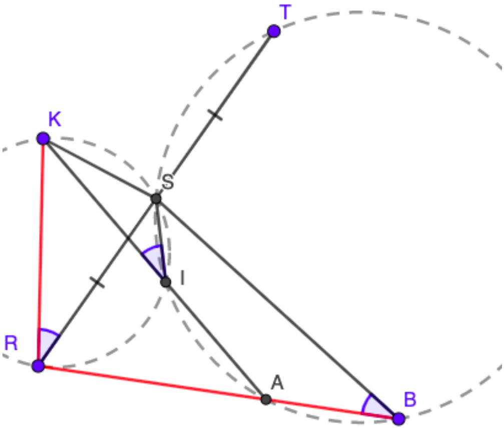
Spiral Similarity



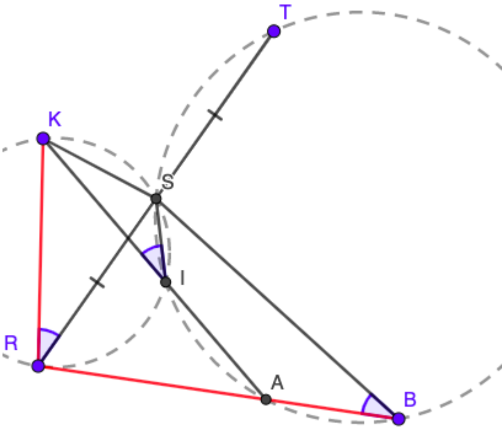
So, S takes KT to TB.

Now, what do we know about in this scenario?

We know that KT is tangent to $\omega(STB)$.



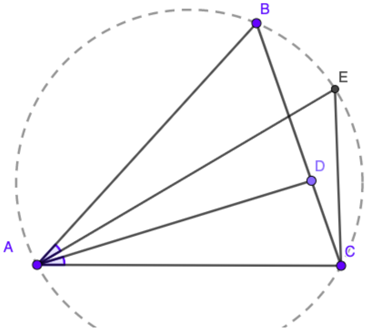
Let R and S be distinct points on circle Ω, and let t denote the tangent line to Ω at R. Point T is the reflection of R with respect to S. A point I is chosen on the smaller arc RS of Ω so that the circumcircle Γ of triangle IST intersect t at two different points. Denote by A the common point of Γ and t that is closest to R. Line AI meets Ω again at K. Show that KT is tangent to Γ.



Show that KT is tangent to Γ.

We just need to see S takes KR to RB. Then, we are done 😄

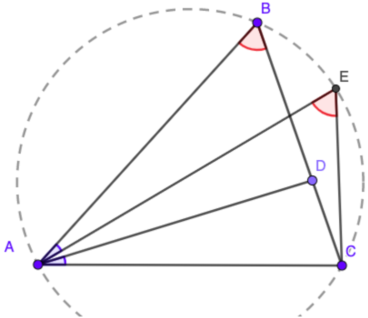
Spiral Similarity



$\triangle ABC$, $AB > BC$,
$D \in BC$, $E \in \omega(ABC)$,
$\angle BAE = \angle DAC$.

Show that a spiral similarity centered at A takes BD to EC.

Spiral Similarity

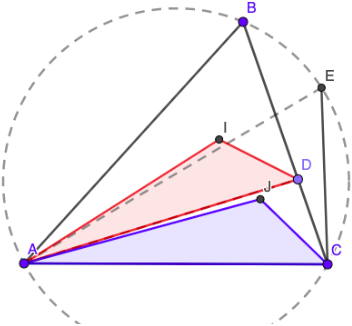


Show that a spiral similarity centered at A takes BD to EC.

By AA:

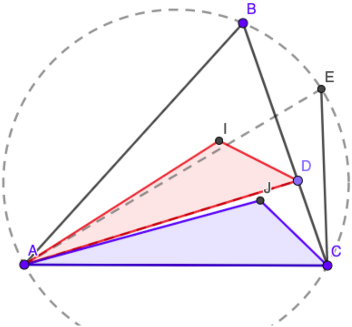
- $\angle BAD = \angle EAC$
- $\angle ABD = \angle AEC$

Spiral Similarity



$\triangle ABD \sim \triangle AEC$
I - incenter of $\triangle ABD$
J - incenter of $\triangle AEC$.

Show that a spiral similarity centered at A takes DI to CJ.

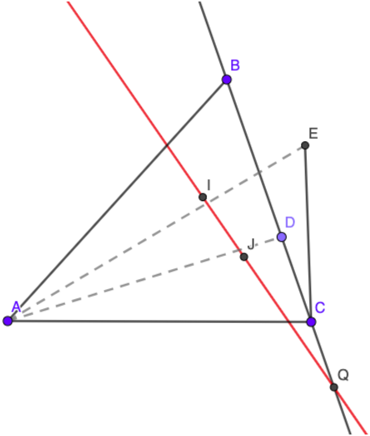


Show that a spiral similarity centered at A takes ID to JC.

Since $\triangle ABD \sim \triangle AEC$, we have $\triangle AID \sim \triangle BJC$ by AA:

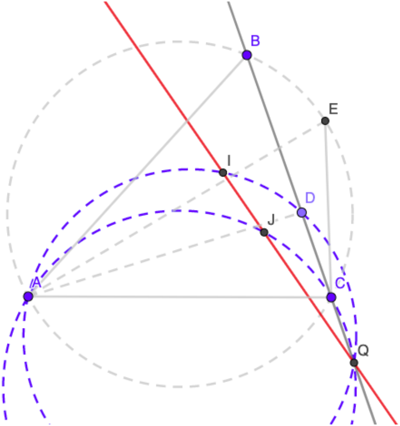
- $\angle AID = 90^\circ \beta/2 = \angle AJC$
- $\angle IAD = \angle BAD/2 = \angle EAC/2 = \angle JAC$

Spiral Similarity



A spiral similarity centered at A takes ID to JC.
$Q = IJ \cap BC$.

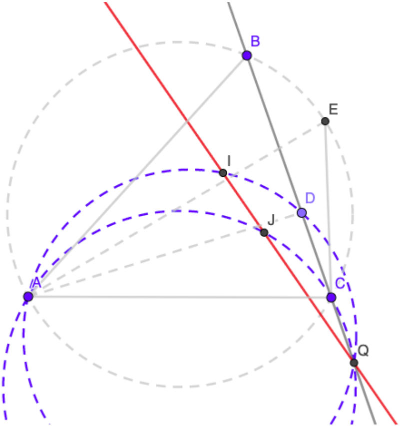
Where are the two cyclic quadrilaterals we have?



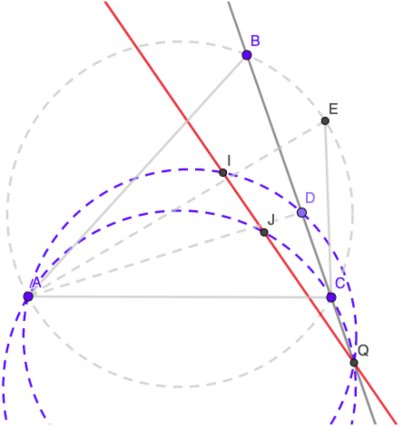
A spiral similarity centered at A takes ID to JC.
$Q = IJ \cap BC$.

Where are the two cyclic quadrilaterals we have?

The quadrilaterals $AIDQ$ and $AJCQ$ are cyclic.

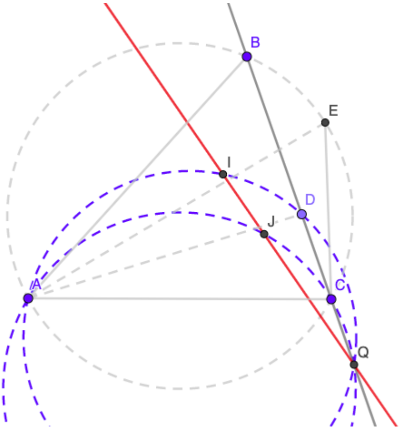


Let ABC be a triangle such that $AB > BC$ and let D be a variable point on the line segment BC. Let E be a point on the circumcircle of triangle ABC, lying on the opposite side of BC from A such that $\angle BAE = \angle DAC$. Let I be the incenter of triangle ABD and let J be the incenter of triangle ACE. Prove that the line IJ passes through a fixed point, that is independent of D.



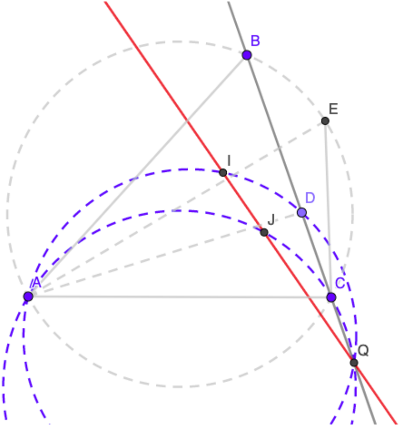
Prove that the line IJ passes through a fixed point, that is independent of D.

We claim that the independent point that the problem is referring is indeed Q.



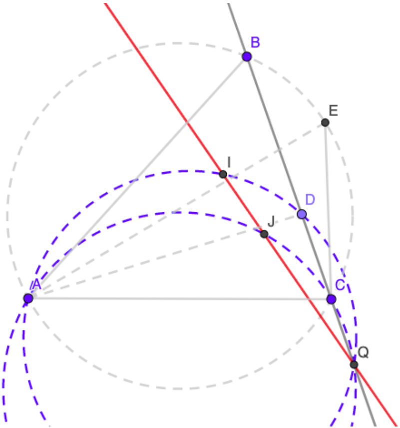
We claim that the independent point that the problem is referring is indeed Q. In other words, no matter how we move D along BC, Q will be in same position.

Moving to P3, CMC 2020



In other words, no matter how we move D along BC, Q will be in same position.

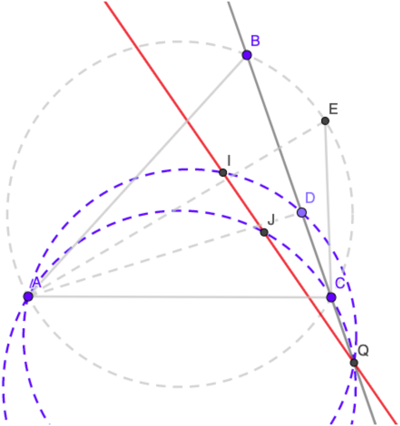
Why?



In other words, no matter how we move D along BC, Q will be in same position.

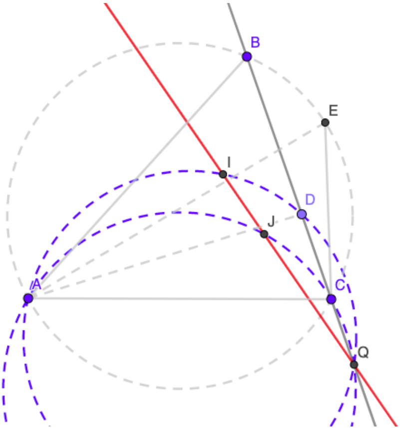
Why?

Take a point, Q', on BC such that $\angle AQ'C = 90^\circ - \beta/2$. We know that Q' is independent of D .



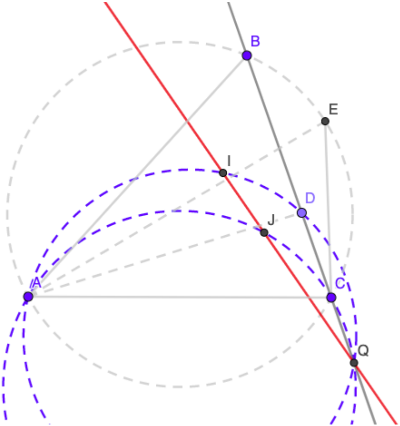
Take a point, Q', on BC such that $\angle AQ'C = 90^\circ - \beta/2$. We know that Q' is independent of D .

Now, consider $\angle AQC$.



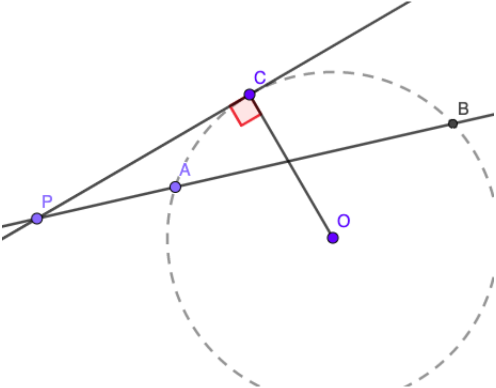
Take a point, Q', on BC such that $\angle AQ'C = 90^\circ - \beta/2$. We know that Q' is independent of D .

$\angle AQC = 180^\circ - \angle AJC = 180^\circ - (90^\circ + \beta/2) = 90^\circ - \beta/2$.



Take a point, Q', on BC such that $\angle AQ'C = 90^\circ - \beta/2$. We know that Q' is independent of D . Hence, Q is Q', meaning Q is independent of D.

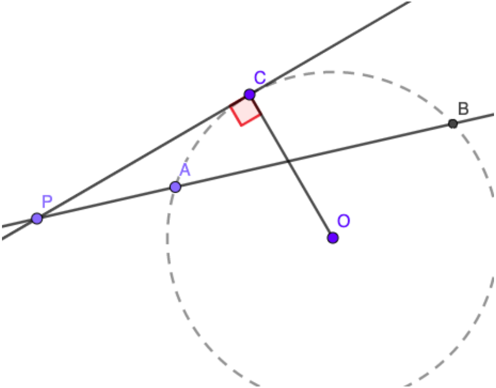
Back to Power of Point



We define power of a point P with respect to circle ω as:

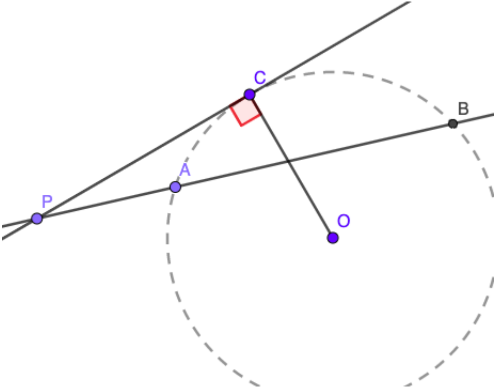
$$Pow(P, \omega) = PA \cdot PB.$$

Back to Power of Point



This is also equal to

$$Pow(P, \omega) = PC^2.$$

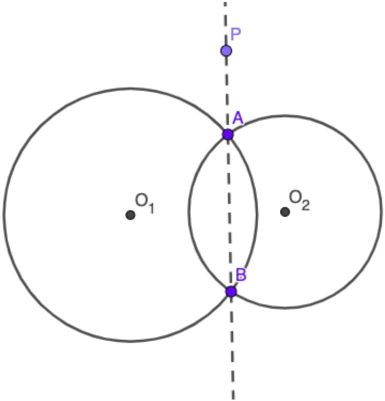


And

$$Pow(P, \omega) = PO^2 - r^2,$$

where r is the radius of ω.

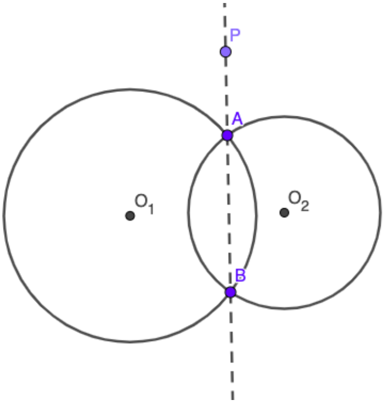
Radical Axis



$Pow(P, \omega) = PO^2 - r^2$.

Suppose, we have two circles, ω_1 and ω_2.

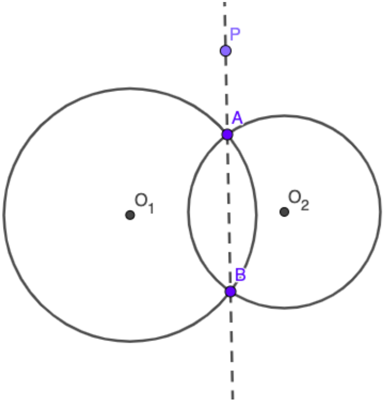
Radical Axis



Suppose, we have two circles, ω_1 and ω_2.
We want to consider a set of points whose powers to these circles are equal:

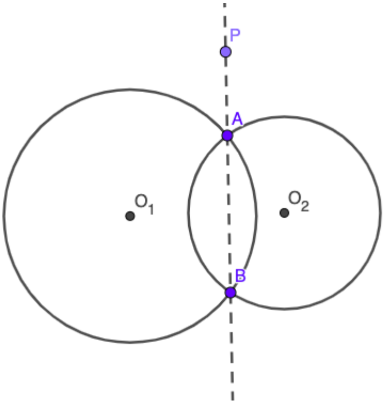
$$\{P | Pow(P, \omega_1) = Pow(P, \omega_2)\}.$$

Radical Axis



We claim that this set forms a line.

$$\{P | Pow(P, \omega_1) = Pow(P, \omega_2)\}.$$

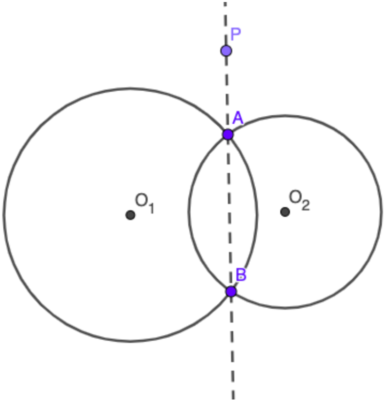


$$P = (x, y)$$
$$O_1 = (a, b)$$
$$O_2 = (c, d).$$

We have,

$$(x-a)^2 + (y-b)^2 - r_1^2 = (x-c)^2 + (y-d)^2 - r_2^2.$$

Radical Axis



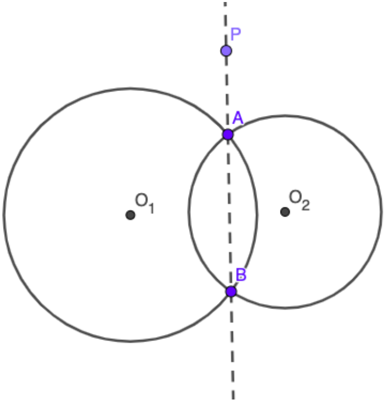
$P = (x, y)$
$O_1 = (a, b)$
$O_2 = (c, d)$.

After cancelling x^2 and y^2, we have:

$$mx + ny + p = 0,$$

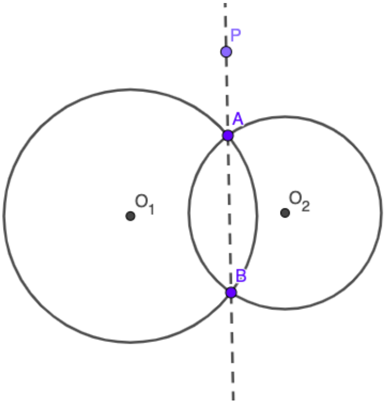
which is the equation of line.

Radical Axis



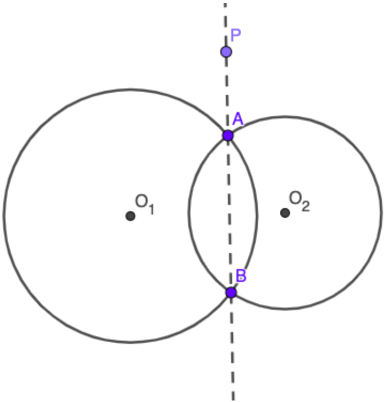
Now, this line is called the radical axis of the circles ω_1 and ω_2.

Radical Axis

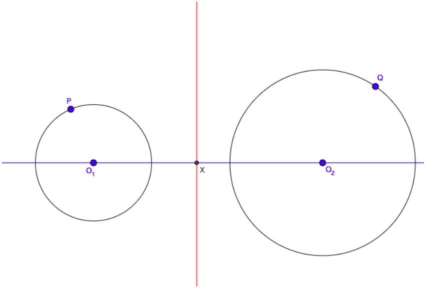


If ω_1 and ω_2 intersects, then the common chord is the radical axis.

Why?

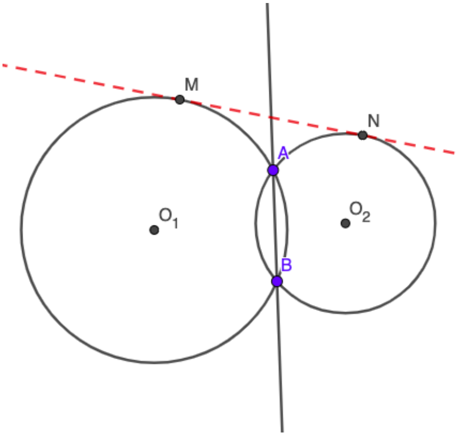


If ω_1 and ω_2 intersects, then the common chord is the radical axis since the powers of the intersections points are zero.



ω_1 and ω_2 does not intersect.

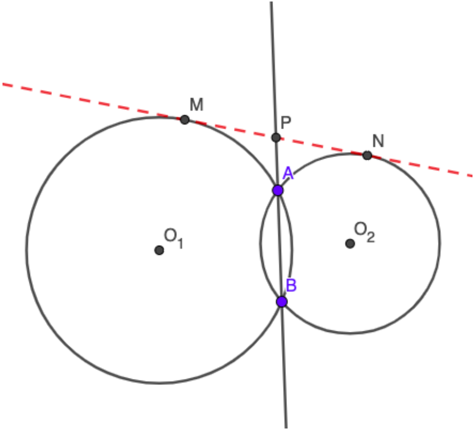
Radical Axis



ω_1 and ω_2 intersects at AB. MN is the common tangent.

Show that AB bisects the segment MN.

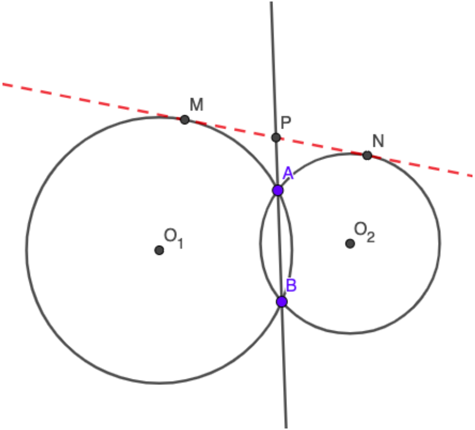
Radical Axis



Show that AB bisects the segment MN.

Let P be the intersection of AB and MN.
We know that AB is the radical axis.
So, $Pow(P, \omega_1) = Pow(P, \omega_2)$.

Radical Axis

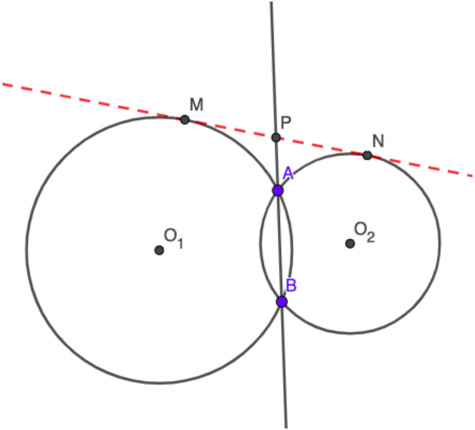


Show that AB bisects the segment MN.

Let P be the intersection of AB and MN.
We know that AB is the radical axis.
So, $Pow(P, \omega_1) = Pow(P, \omega_2)$.

But what is $Pow(P, \omega_1)$?

Radical Axis

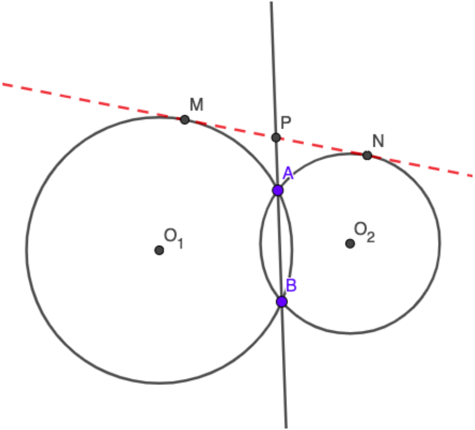


Show that AB bisects the segment MN.

Let P be the intersection of AB and MN.
We know that AB is the radical axis.
So, $Pow(P, \omega_1) = Pow(P, \omega_2)$.

$Pow(P, \omega_1) = PM^2$ and $Pow(P, \omega_2) = PN^2$.

Radical Axis

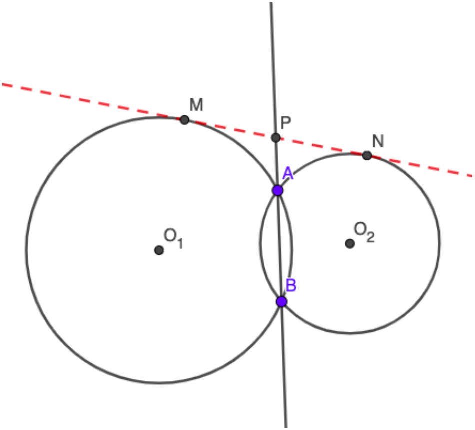


Show that AB bisects the segment MN.

$Pow(P, \omega_1) = PM^2$ and $Pow(P, \omega_2) = PN^2$.

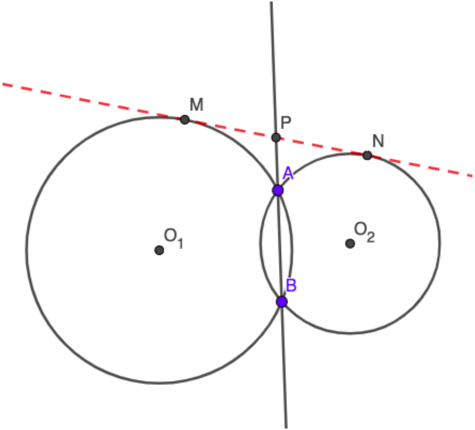
So, $PM = PN$.

Radical Axis



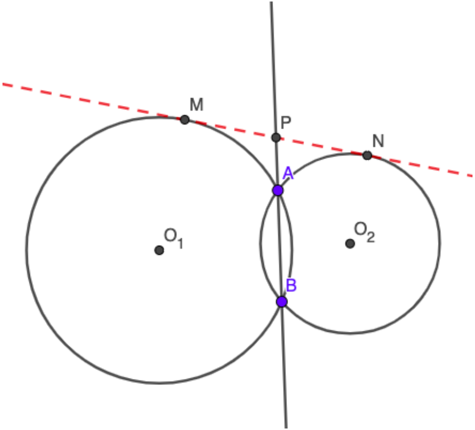
Suppose we have three circles, ω_1, ω_2, and ω_3. Show that radical axes of these circles are concurrent.

Radical Axis



Let r_{12} and r_{13} be the radical axes of (ω_1, ω_2) and (ω_1, ω_3).
$X = r_{12} \cap r_{13}$.

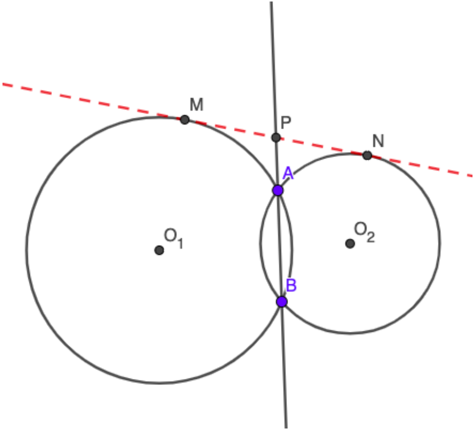
Radical Axis



Let r_{12} and r_{13} be the radical axes of (ω_1, ω_2) and (ω_1, ω_3).
$X = r_{12} \cap r_{13}$.

We need to show that $X \in r_{23}$.

Radical Axis

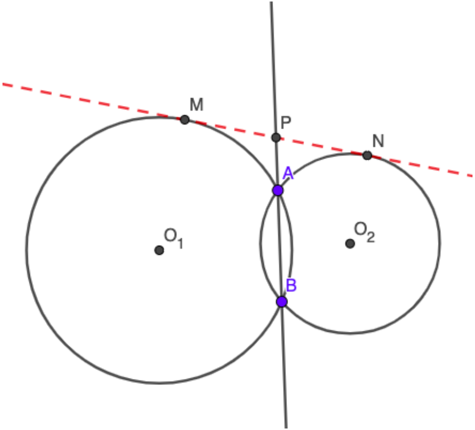


We need to show that $X \in r_{23}$.

Since $X \in r_{12}$,

$Pow(X, \omega_1) = Pow(X, \omega_2)$.

Radical Axis



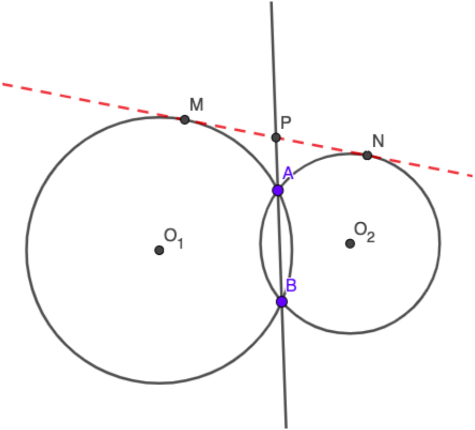
We need to show that $X \in r_{23}$.

Since $X \in r_{12}$,

$$Pow(X, \omega_1) = Pow(X, \omega_2).$$

Since $X \in r_{13}$,

$$Pow(X, \omega_1) = Pow(X, \omega_3).$$



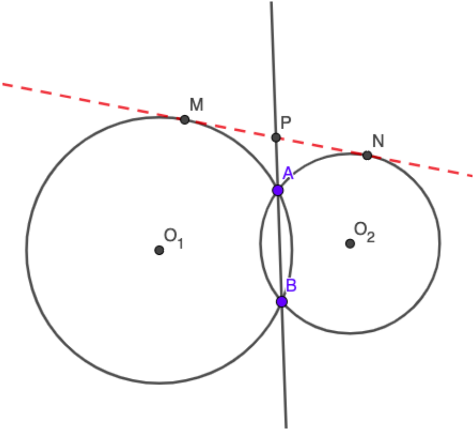
We need to show that $X \in r_{23}$.

So,

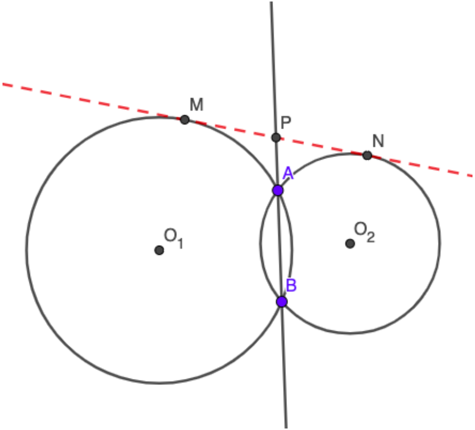
$$Pow(X, \omega_1) = Pow(X, \omega_3),$$

which means $X \in r_{23}$.

Radical Axis

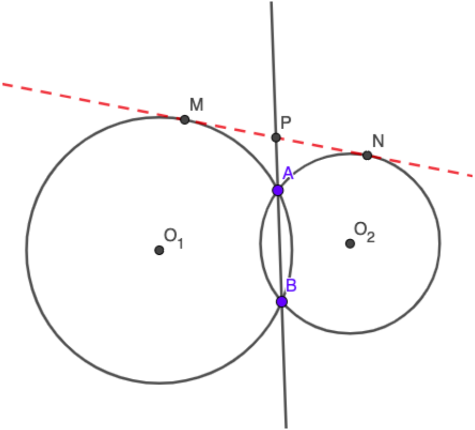


X is called the radical center of the circles ω_1, ω_2, and ω_3.



Question:
What if O_1, O_2, and O_3 are collinear?

Radical Axis



Question:
What if O_1, O_2, and O_3 are collinear?

That means their radical axes are parallel to each other, hence there is no radical center.

Questions

Today's learning outcomes were:

- Isometric Transformations
- Spiral Similarities
- Radical Axis

Thanks for joining us for the past 5 weeks!

Good luck on the TST!