Ghana Mathematical Olympiad

July 25, 2020

Problem 1. Problem 1
(a) (4 pts) Given $\triangle ABC$, let I be its incenter and ω be its circumcircle. The ray $\overrightarrow{AI}$ intersects ω again at P. Prove that $PB = PC = PI$.
(b) (3 pts) Given $\triangle ABC$, let H be its orthocenter, O be its circumcenter, and O' be the circumcenter of its medial triangle. Prove that O' is the midpoint of $\overline{HO}$. (Hint: recall from lecture the Euler line: H, the centroid G, and O lie on a line in that order with $HG = 2 \cdot GO$.)

Problem 2. Problem 2 Find, with proof, all non-negative integer pairs (a, b) such that $3^a + 7^b$ is a perfect square.

Problem 3. Problem 3 Let $\mathbb{R}^+$ denote the set of positive real numbers. Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that

$$f(x)f(y) < f(xy) < f(x)f(y) + 2020$$

for all $x, y \in \mathbb{R}^+$.

(a) (2 pts) Prove that there is a $c \in \mathbb{R}^+$ such that $f(x) < c$ for all $x \in \mathbb{R}^+$.
(b) (5 pts) Prove that $f(x) < 1$ for all $x \in \mathbb{R}^+$.

Problem 4. Problem 4
(a) (2 pts) In how many different ways can the six empty circles in the first diagram on the left be filled in with the numbers 2 through 7 such that each number is used once, and each number is either greater than both its neighbors, or less than both its neighbors? Please provide proof.

(b) (5 pts) In how many different ways can the seven empty circles in the second diagram on the right be filled in with the numbers 2 through 8 such that each number is used once, and each number is either greater than both its neighbors, or less than both its neighbors? Please provide proof.