Geometry

MOP 2020

Lecture 1

June 23, 2020

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Poll

- How was the pre-camp test?
- How was the homework?

Questions

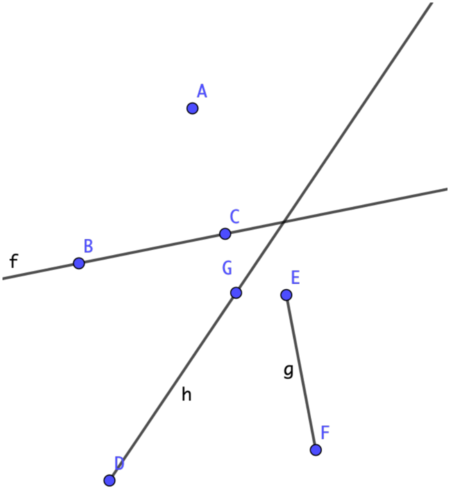
- How would you like this course to be?

Learning Outcome

- Basics in Euclidean Geometry
- Proofs

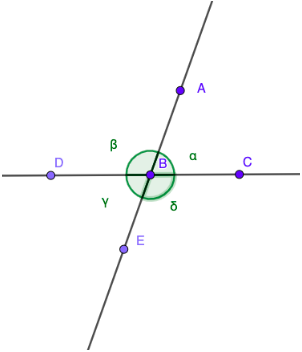
Euclidean Postulates

1. A straight line may be drawn from any point to any other point
2. A finite straight line may be extended continuously in a straight line
3. A circle may be described with any center and any radius
4. All right angles are equal to one another.
5. If a straight line meets two other straight lines so as to make the two interior angles on one side of it together less than two right angles, the two other straight lines, if extended indefinitely, will meet on that side on which the angles are less than two right angles.



- Point
- Line
- Ray
- Segment (line segment)

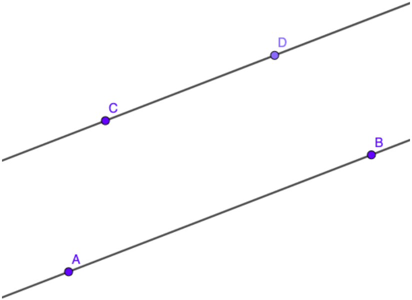
Angle



Notation: e.g. α, $\angle ABC$

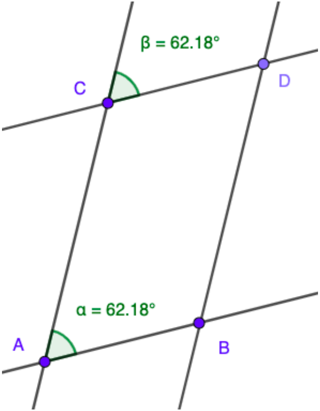
- Supplementary angles: ...
- Vertical angles: ...
- Show vertical angles are equal.

Parallel Lines



Notation: e.g. $AB \parallel CD$, $l \parallel m$

Parallel Lines

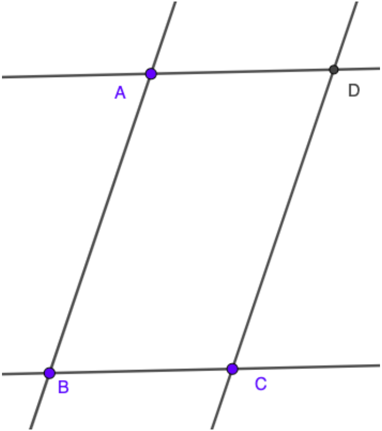


Notation: e.g. $AB \parallel CD$, $l \parallel m$

Corresponding angles are equal, i.e. $\alpha = \beta$.

- Show $\angle BAC + \angle ACD = 180°$.
- Show $\angle BAC = \angle BDC$ if $AC \parallel BD$.

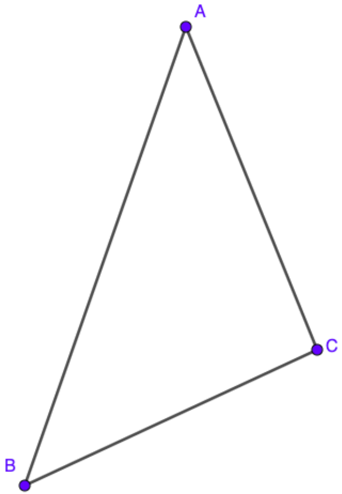
Parallelogram



The following statements are equivalent:

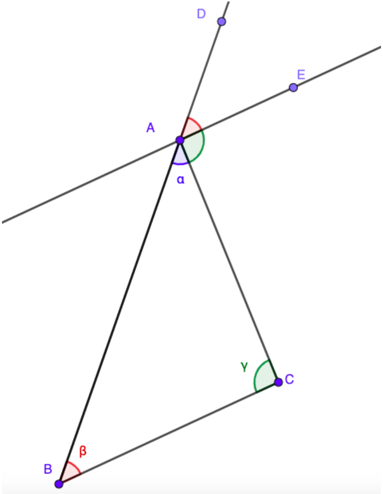
- $ABCD$ is a parallelogram
- $AB \parallel CD$ and $AB = CD$

Triangle



Notation: $\triangle ABC$

- Show that the interior angles of $\triangle ABC$ sum to 180°.

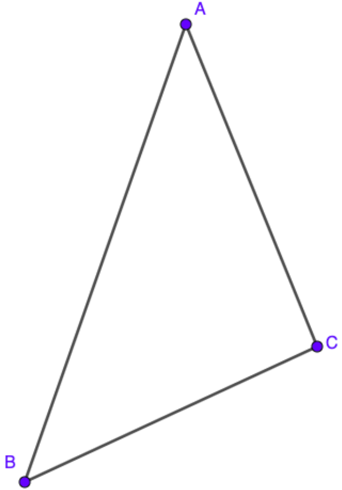


Show that the interior angles of $\triangle ABC$ sum to $180°$.

Proof.
D is on AB and $AE \parallel BC$.

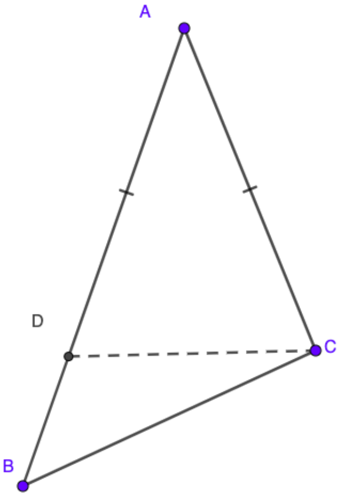
- $\angle ABC = \angle DAE$
- $\angle ACB = \angle EAC$
- $\angle DAB = \ldots$

Triangle



Notation: $\triangle ABC$

- Show that if $\angle ABC < \angle ACB$, then $AC < AB$.



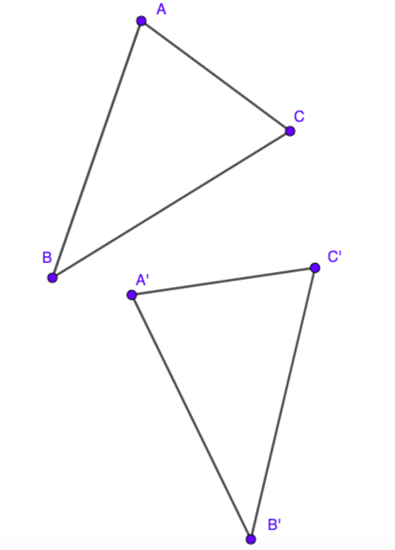
Show that if $\angle ABC < \angle ACB$, then $AC < AB$.

Proof.
D on AB such that $AD = AC$.

$\angle DCA = \frac{\beta+\gamma}{2} < \gamma = \angle BCA$.

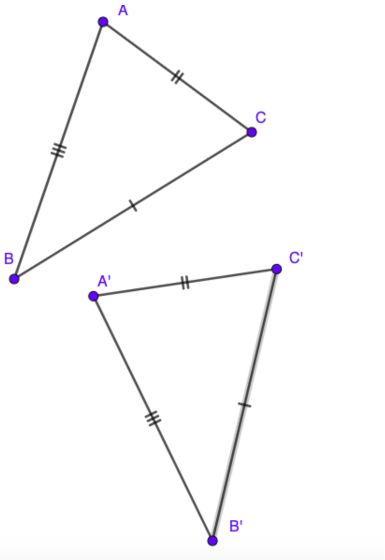
So, D lies on the side AB.

Congruent Triangles



Notation: $\triangle ABC \cong \triangle A'B'C'$

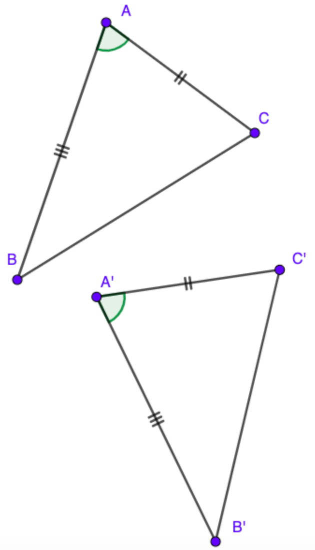
Congruent Triangles



Notation: $\triangle ABC \cong \triangle A'B'C'$

Side-Side-Side *(SSS)*:

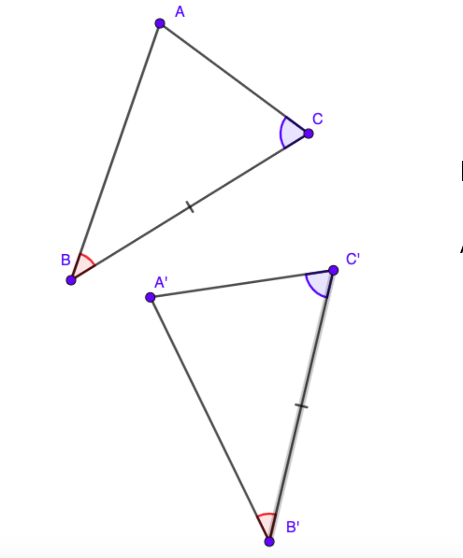
- $BA = B'A'$
- $AC = A'C'$
- $CB = C'B'$



Notation: $\triangle ABC \cong \triangle A'B'C'$

Side-Angle-Side *(SAS)*:

- $BA = B'A'$
- $\angle BAC = \angle B'A'C'$
- $AC = A'C'$

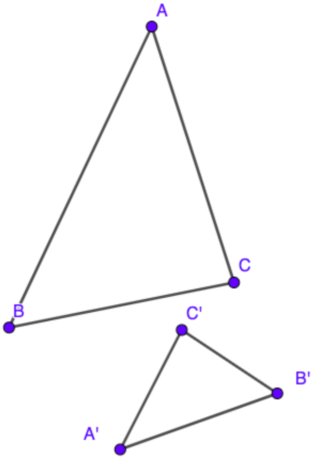


Notation: $\triangle ABC \cong \triangle A'B'C'$

Angle-Side-Angle *(ASA)*:

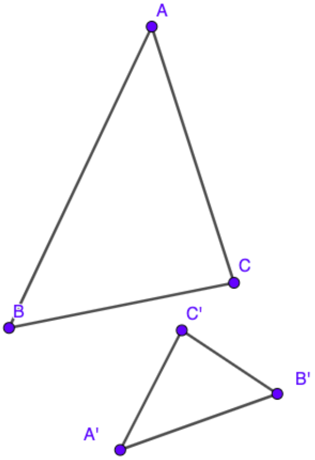
- $\angle ABC = \angle A'B'C'$
- $BC = B'C'$
- $\angle BCA = \angle B'C'A'$

Similar Triangles



Notation: $\triangle ABC \sim \triangle A'B'C'$

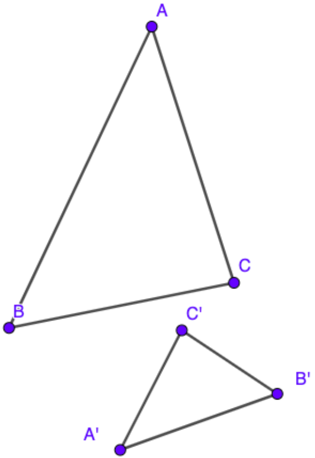
Similar Triangles



Notation: $\triangle ABC \sim \triangle A'B'C'$

Side-Side-Side *(SSS)*:

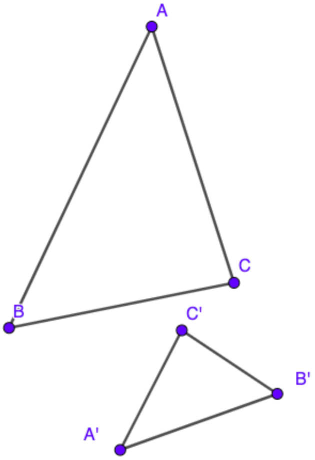
$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$



Notation: $\triangle ABC \sim \triangle A'B'C'$

Angle-Angle *(AA)*:

The triangles ABC and $A'B'C$ share two common angles.



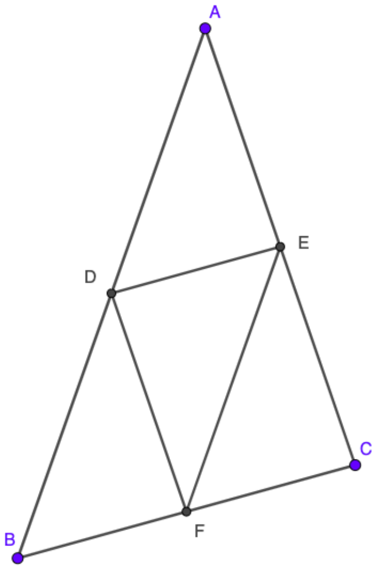
Notation: $\triangle ABC \sim \triangle A'B'C'$

Side-Angle-Side *(SAS)*

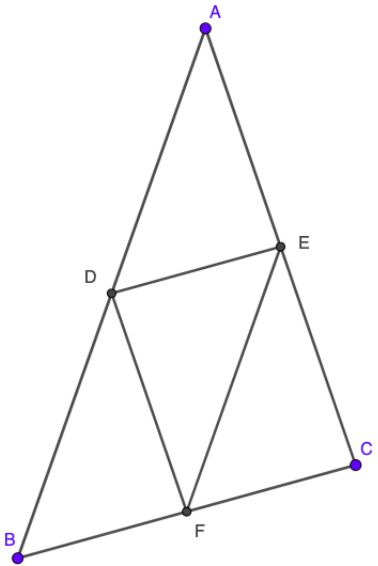
$\frac{AB}{A'B'} = \frac{AC}{A'C'}$ and

$\angle BAC = \angle B'A'C'$

Medial Triangle



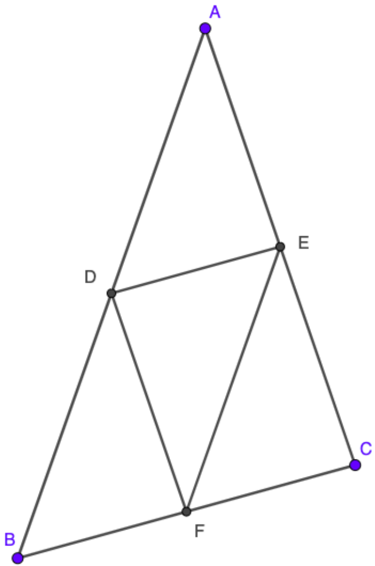
Show that $\triangle ABC \sim \triangle FED$, where F, E, and D are midpoints, e.g. $AD = BD$.



Show that $\triangle ABC \sim \triangle FED$, where F, E, and D are midpoints.

Hint: Use SAS to show $\triangle ADE \sim \triangle ABC$.

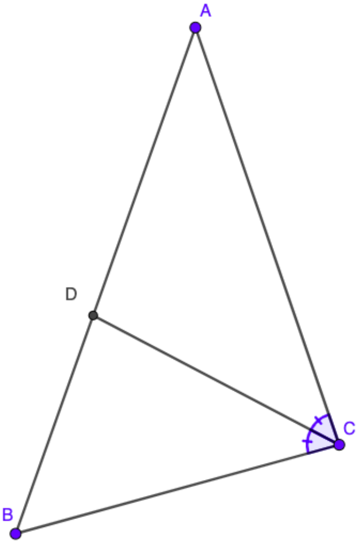
Medial Triangle



Show that $\triangle ABC \sim \triangle FED$, where F, E, and D are midpoints.

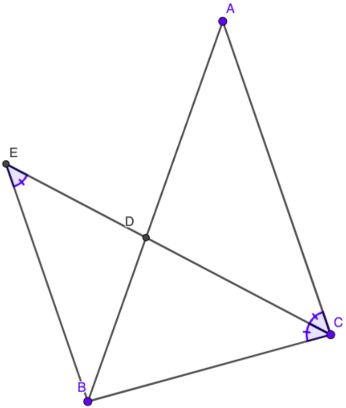
Proof.
$\triangle ADE \sim \triangle ABC$.
$ADFE$ is parallelogram.

Angle Bisector Theorem



Given that CD is the angle bisector, show that

$$\frac{BC}{BD} = \frac{AC}{AD}.$$

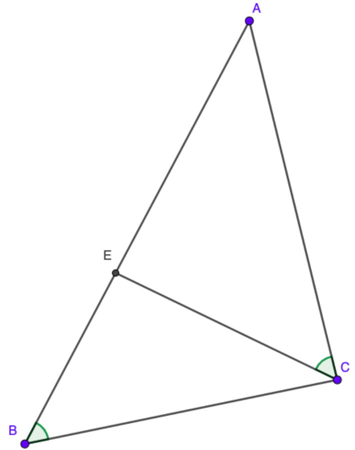


Given that CD is the angle bisector, show that

$$\frac{BC}{BD} = \frac{AC}{AD}.$$

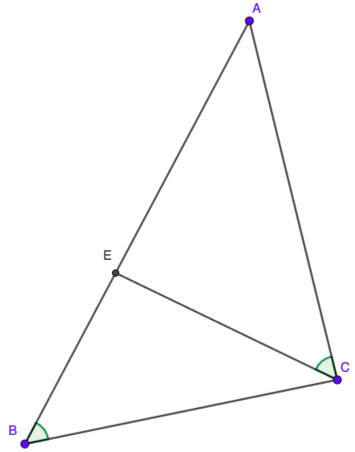
Proof.
Consider E such that $BE \parallel AC$ and $E \in CD$.
$\triangle BDE \sim \triangle ADC$.

Using Similar Triangles



Show that if $\angle ABC < \angle ACB$, then $AC < AB$.

Using Similar Triangles



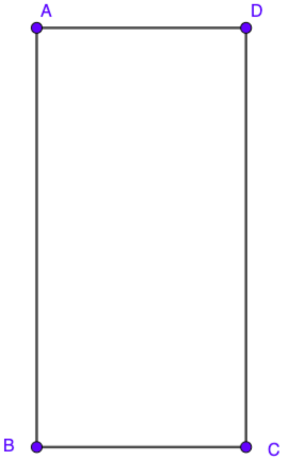
Show that if $\angle ABC < \angle ACB$, then $AC < AB$.

Proof. Consider E on AB s.t. $\angle ACE = \angle ABC$.

By AA, $\triangle ABC \sim \triangle ACE$.

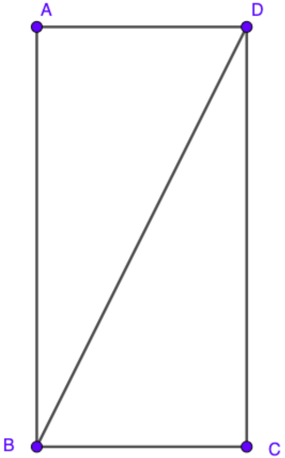
$AC^2 = AE \cdot AB$.

Since $AE < AB$, $AC < AB$.



Area of a rectangle:
$S_{ABCD} = AB \cdot BC$

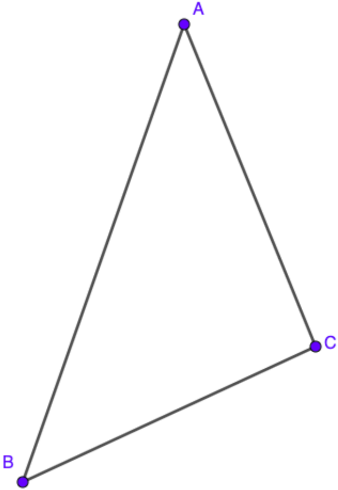
Show that $S_{BCD} = \frac{1}{2} BC \cdot CD$



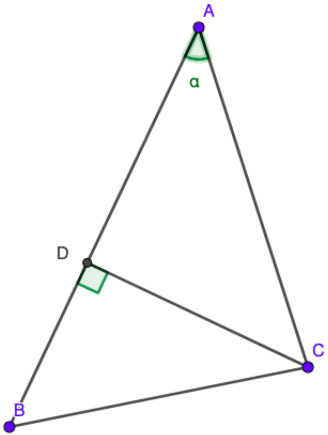
Show that $S_{BCD} = \frac{1}{2} BC \cdot CD$

Proof.
Triangles ADC and BCD are congruent by SSS.
Their areas sum to S_{ABCD}.

Area



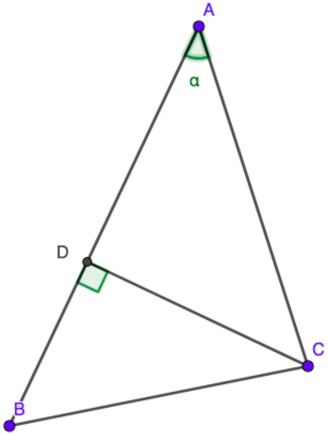
Show that $S_{ABC} = \frac{1}{2} AB \cdot h_C$



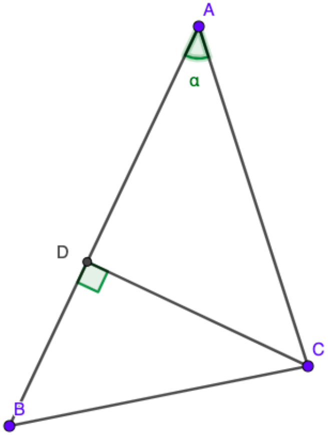
Show that $S_{ABC} = \frac{1}{2} AB \cdot h_C$

Proof.
Drop an altitude from C with foot D. Consider the two right triangles: CDB and CDA.

Area

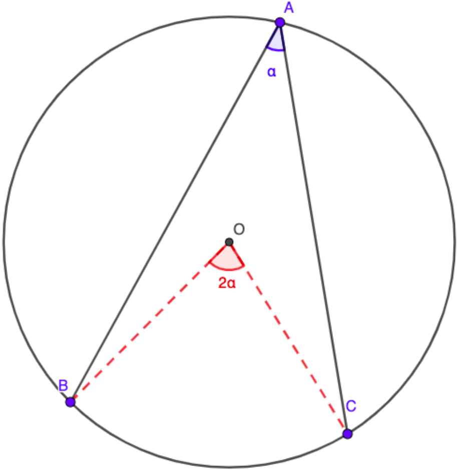


Show that
$S_{ABC} = \frac{1}{2} AB \cdot AC \cdot \sin \alpha$



Show that
$S_{ABC} = \frac{1}{2} AB \cdot AC \cdot \sin\alpha$

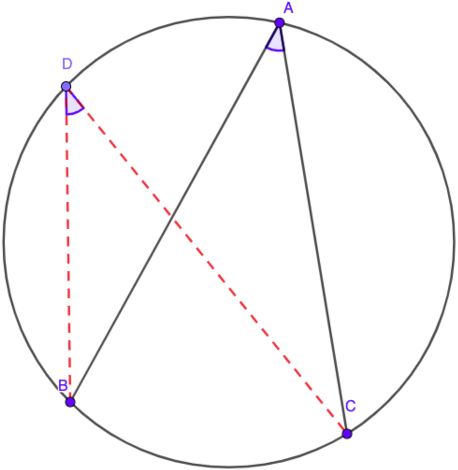
Proof.
$AC \cdot \sin\alpha = h_C$



Notation: $\omega(ABC)$

Property:

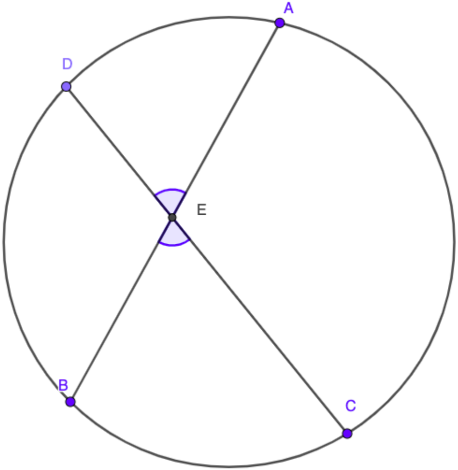
$$\angle BAC = \frac{1}{2}\angle BOC = \frac{1}{2}\overset{\frown}{BC}.$$



Notation: $\omega(ABCD)$

Show that

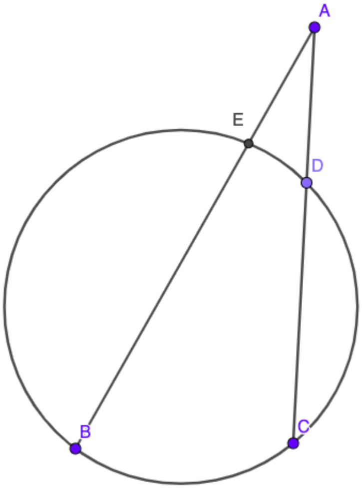
$$\angle BAC = \angle BDC.$$



Notation: $\omega(ABCD)$

Show that

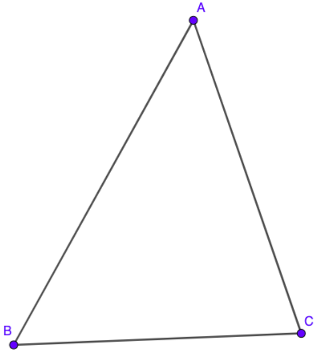
$$\angle BEC = \frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{BC}).$$



Notation: $\omega(ABCD)$

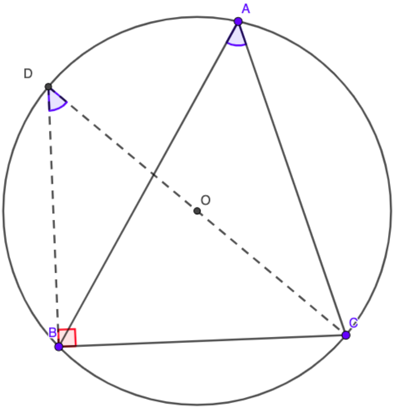
Show that

$$\angle BAC = \frac{1}{2}(\overset{\frown}{\mathrm{BC}} - \overset{\frown}{\mathrm{DE}}).$$



Show that

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R.$$



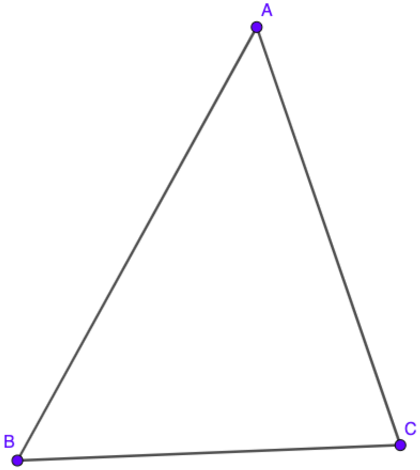
Show that

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R.$$

Proof.
Take $D = \omega(ABC) \cap CO$.

$$\sin\alpha = \frac{a}{CD} = \frac{a}{2R}.$$

Law of Cosines



$$a^2 = b^2 + c^2 - 2bc\cos\alpha.$$

Two-column Proofs

Statements	**Reasons**
Statement 1	Reason 1
Statement 2	Reason 2
Statement 3	Reason 3
Statement 4	Reason 4
Statement 5	Reason 5
...	...

When first writing geometric proofs, two-column proofs helps us to organize our solution in the format of:

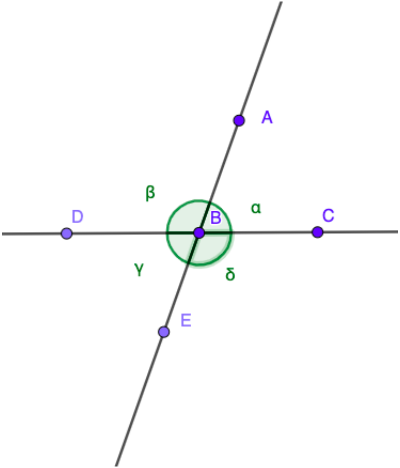
Statement *i* is true by **Reason *i***.

Example.

Vertical angles are equal, i.e.

$$\alpha = \gamma.$$

Statements	Reasons
$\beta = 180^\circ - \alpha$	α and β are supp.
$\beta = 180^\circ - \gamma$	γ and β are supp.
$\alpha = \gamma$	From above



Questions

End

Today's learning outcomes were:

- Reviewing basics in Euclidean Geometry
- Getting comfortable with writing geometric proofs

End

Thanks for attending!