Reinventing the wheel: designing wheels for staircases

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1 Introduction

It is a well known fact that circular wheels are well suited to roll over flat surfaces. Explaining why this is so is another matter, and there is certainly a lot to find out if one starts investigating the physics of motion in detail. There is, however, one convincing argument that is simple to understand and doesn't require any physics: if the wheel spins around an axle, a vehicle mounted on top of this axle will always be at the same height from the road. Thus, if the road is (approximately) flat, the vehicle will follow a straight line, free of bumps and depressions. In a word, the journey could be described as smooth.

Unfortunately, staircases are not flat and climbing or descending them on circular wheels is anything but a smooth experience. This leads us to wonder whether it would be possible to design wheels specialized to roll over staircases while providing a comfortable ride. These wheels could potentially be mounted on wheelchairs to help disabled people, or used in small transportation vehicles in places where lifts or ramps are unavailable.

In this document we propose to study the shape of wheels for staircases. Perhaps surprisingly, we will find out that there is, for each staircase, an infinite number of wheels matching our criteria for comfort and smoothness, and we will develop an algorithm to design them all. Our work follows from that of Hall and Wagon [1], who focused on the more general problem of designing wheels for roads of arbitrary shape. They haven't, however, investigated in detail the particular case of roads shaped like staircases.

The rest of this document is organised as follows. In section 2 we express the problem of pairing roads and wheels mathematically and derive a model to solve it. We then apply this model, first in section 3 for the simpler case of straight roads and then for staircases in section 4. This leads us to section 5, where we systematize all our findings and describe an algorithm to build wheels for staircases, step by step. Then, in section 6 we present a simulator that implements this algorithm and is available online. Finally, we conclude with some remarks regarding the feasibility of using these wheels in the real world.

2 Pairing roads and wheels

The question of designing wheels for arbitrary roads was posed by Hall and Wagon in [1], where they proved the equivalence to a certain initial value problem. We begin by describing the mathematical model they developed before applying it to the particular case of staircases in subsequent sections.

Let us define a road as a two-dimensional curve given in parametric coordinates

$$(x, y) = (x(t), y(t)), \quad t \geq 0 \tag{1}$$

We can think of t as the time since the motion started and $(x(t), y(t))$ as the point of contact between the road and the wheel at time t. Similarly, a wheel is a curve in polar form

$$(r, \theta) = (r(\theta(t)), \theta(t)), \quad t \geq 0 \tag{2}$$

$(r(\theta(t)), \theta(t))$ represents the point on the border of the wheel that will be in contact with the road after rolling for a time t. Without loss of generality, we may assume the wheel starts centred at the origin and the initial point of contact with the road occurs right beneath it, that is,

$$x(0) = 0, \quad y(0) = -y_0 < 0, \quad \theta(0) = -\frac{\pi}{2} \tag{3}$$

In order to be smooth, we require the motion to obbey the following constraints:

No-bouncing means the centre of the wheel must travel along a linear path, without bumps or breaks. To simplify calculations, we will presume this path is just the horizontal axis. Consequently, the road is always below $y = 0$ and the distance between the centre of the wheel and the point of contact with the road equals the height of the road. That is,

$$r(\theta(t)) = -y(t) \tag{4}$$

No-sliding forces the wheel to roll over the road without sliding. This means that the arc length between two points on the wheel that touch the road at two different times must be the same as the corresponding arc length on the road. Mathematically, this is expressed as:

$$\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{-\frac{\pi}{2}}^{\theta(t)} \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{5}$$

It turns out that these two conditions simplify into a differential equation that determines the wheel, as we shall see now.

Differentiating the no-sliding equation (5) with respect to t and squaring we get

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2\right)\left(\frac{d\theta}{dt}\right)^2 \tag{6}$$

But the no-bouncing condition (4) tells us that

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{dy}{dt} \quad \Leftrightarrow \quad \frac{dr}{d\theta} = -\frac{dy}{dt}\frac{dt}{d\theta} \tag{7}$$

Substituting into (6) and noting that we want $\theta'(t)$ to be positive, we get

$$\frac{d\theta}{dt} = -\frac{1}{y(t)}\frac{dx}{dt} \tag{8}$$

The solution to this differential equation subject to the initial condition $\theta(0) = -\pi/2$ determines $\theta(t)$. If $\theta(t)$ is invertible, we can obtain $t(\theta)$ and use the no-bouncing condition (4) again to determine the radius of the wheel at any angle θ:

$$r(\theta) = -y(t(\theta)) \tag{9}$$

Theoretically, if we can solve the initial value problem stated above for a particular road, we get a wheel that fits it. In practice it is not so simple, because the resulting wheel might not be closed (that is, $r(\theta)$ is not periodic) or it might penetrate the road during the motion.

Remark 2.1. *If the road is given by a cartesian equation $y = f(x)$, we can parametrize it by $x(t) = t$, $y(t) = f(t)$ and the initial value problem reduces to*

$$\frac{d\theta}{dt} = -\frac{1}{y(t)}, \quad \theta(0) = -\pi/2 \tag{10}$$

All roads in the remaining of this text will be of this kind.

3 Wheels for straight roads

As a simple application of the model described above, let's confirm our intuitive guess that circular wheels are perfect for flat roads.

The trivial example. If the road is $y(x) = -y_0$ where $y_0 > 0$ is constant, then the wheel is a circle of radius y_0.

Proof. It is a direct consequence of the no-bouncing condition (4).

Let's confirm that this wheel does not slide. The distance travelled on the road after time t is

$$\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^t dt = t$$

The initial value problem (10) gives $\theta(t) = \frac{1}{y_0}t - \frac{\pi}{2}$. Inverting we get $t(\theta) = (\theta + \pi/2)\, y_0$. But this is just the arc length of a circle of radius y_0 between angles $\theta(0) = -\pi/2$ and $\theta(t) = \theta$, i.e., the distance covered by the wheel's border during the same amount of time t. Thus, this wheel obeys the no-sliding condition. $\square$

The wheel becomes more complicated when the road has a slope. This is because we force the centre of the wheel to move along the horizontal axis, so its radius grows to infinity.

Proposition 3.1. *If the road is given by $y(x) = -mx - y_0$, where $m > 0$ and $y_0 > 0$ are both constant, the corresponding wheel is given by $r(\theta) = y_0 e^{m\left(\theta + \frac{\pi}{2}\right)}$*

Proof. Solving the initial value problem (10) we get

$$\theta(t) = \frac{\ln\left(\frac{mx}{y_0} + 1\right)}{m} - \frac{\pi}{2} \tag{11}$$

Inverting to get $t(\theta)$ gives

$$t(\theta) = \frac{y_0}{m}\left(e^{m\left(\theta + \frac{\pi}{2}\right)} - 1\right) \tag{12}$$

Finally, substituting the above into the no-bouncing equation (4) results in

$$r(\theta) = y_0 e^{m\left(\theta + \frac{\pi}{2}\right)} \tag{13}$$

which is the desired equation. $\square$

Remark 3.1. *This wheel in Proposition 3.1 is not closed, because the radius grows exponentially.*

The wheel above is a Bernoulli or logarithmic spiral, whose general equation in polar coordinates is $r(\theta) = ae^{b\theta}$, where a and b are positive constants (the case $b = 0$ degenerates into a circle). Bernoulli spirals are equiangular, meaning that, at all points, the angle between the radial and tangent lines remains constant (see Figure 1). This remarkable property is so important for the remainder of our work that it merits its own proposition and proof.

Proposition 3.2. *Bernoulli spirals are equiangular.*

Proof. The parametric equation of the spiral,

$$\begin{cases} x(\theta) = r(\theta)\cos\theta \\ y(\theta) = r(\theta)\sin\theta \end{cases} \tag{14}$$

allows us to find the derivative at $(x(\theta), y(\theta))$:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta} = \frac{b\sin\theta + \cos\theta}{b\cos\theta - \sin\theta} \tag{15}$$

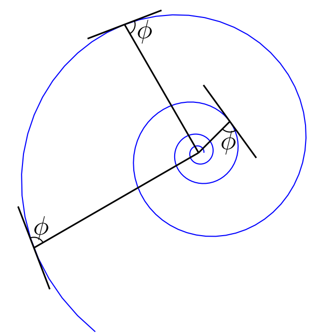


Figure 1: Bernoulli spirals are equiangular

where the last equality stems from the fact that $r'(\theta) = br(\theta)$. Hence, the tangent vector at every point $(x(\theta), y(\theta))$ is

$$\vec{T} = \left(1, \frac{b\sin\theta + \cos\theta}{b\cos\theta - \sin\theta}\right) \tag{16}$$

while the radial unitary vector is just $\hat{r} = (\cos\theta, \sin\theta)$.

Now, $\tan\alpha$, where α is the angle between the radial and tangent vectors is just

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\left(\hat{r} \times \vec{T}\right)_z / \|\vec{T}\|}{\left(\hat{r} \cdot \vec{T}\right) / \|\vec{T}\|} = \frac{1}{b} \tag{17}$$

$\left(\hat{r} \times \vec{T}\right)_z$ denotes the third component of the cross product between $\hat{r}$ and $\vec{T}$ (extended into $\mathbb{R}^3$ by setting $\hat{r}_z = \vec{T}_z = 0$).

Finally, we see that $\tan\alpha$ is constant and thus α must be constant as well.

$\square$

We shall now study what happens when the road is shaped like a staircase.

4 Finding a wheel for a staircase

4.1 The first step

To simplify, we will suppose the staircase is laid down in such a way that the handrail is parallel to the horizontal axis, so that the staircase looks more like a triangle wave. In this way the (hypothetical) wheel will roll horizontally and we can apply the model developed in section 2 to find it. Furthermore, let's consider first the case of a staircase composed of a single step with tread width T and riser height R, as in Figure 2.

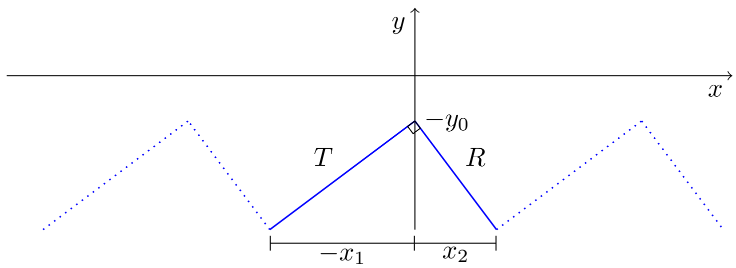


Figure 2: One step of the staircase

The equation of such a road is

$$y(x) = \begin{cases} y_1(x) = mx - y_0, & x \in [x_1, 0] \\ y_2(x) = -\frac{1}{m}x - y_0, & x \in [0, x_2] \end{cases} \tag{18}$$

where $x_1 < 0$, $x_2 > 0$ and both m and y_0 are positive constants.

It is not difficult to check that both line segments are perpendicular to each other and that $y(x)$ is continuous at $x = 0$. Moreover, a simple application of the Pythagorean theorem shows that $m = R/T$, the slope of the staircase.

To find a wheel for this road, we will apply Proposition 3.1 to both branches of $y(x)$. This gives the polar curve

$$r(\theta) = \begin{cases} r_1(\theta) = y_0 e^{-m\left(\theta + \frac{\pi}{2}\right)}, & \theta \in \left[-\frac{\pi}{2} - \theta_1, -\frac{\pi}{2}\right] \\ r_2(\theta) = y_0 e^{\frac{1}{m}\left(\theta + \frac{\pi}{2}\right)}, & \theta \in \left[-\frac{\pi}{2}, -\frac{\pi}{2} + \theta_2\right] \end{cases} \tag{19}$$

for some unknown θ_1 and θ_2.

We see that this curve is made up of two pieces of Bernoulli spirals glued together. Because the spiral r_1 is tangent to y_1 at $x = 0$ and r_2 is tangent to y_2 at the same location, we conclude that both spirals intersect at an angle of $\pi/2$ and the curve fits the road at the edge of the step (see Figure 3). Hence, $r(\theta)$ is the wheel we were looking for.

To find the angles θ_1 and θ_2 we can apply the no-sliding constraint (5), which reduces to

$$\begin{aligned} T &= \int_{-\frac{\pi}{2}-\theta_1}^{-\frac{\pi}{2}} r_1(\theta)\sqrt{1+m^2}\, d\theta \\ R &= \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+\theta_2} r_2(\theta)\sqrt{1+\frac{1}{m^2}}\, d\theta \end{aligned} \tag{20}$$

Solving for θ_1 and θ_2 one gets

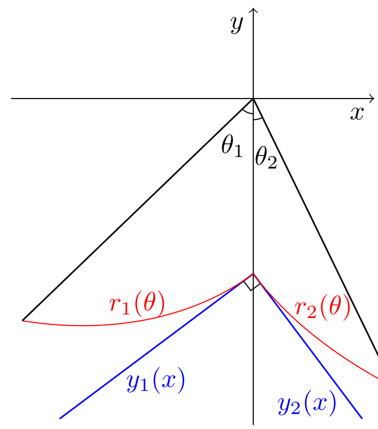


Figure 3: The wheel for the first step

$$
\begin{aligned}
\theta_1 &= \frac{1}{m} \ln \left(\frac{R}{y_0\sqrt{1+m^2}} + 1 \right) \\
\theta_2 &= m \ln \left(\frac{R}{y_0\sqrt{1+m^2}} + 1 \right)
\end{aligned}
\tag{21}
$$

4.2 Completing the wheel

Our strategy to build a wheel for a staircase with multiple steps will be to stitch together multiple single-step wheels constructed in the previous section, so that $r(\theta)$ becomes a periodic function of period $\theta_1 + \theta_2$. For reasons that will become apparent soon, we will call these "single-step wheels" by *petals* from now on.

Is this wheel able to roll over the staircase? To show that it is, we must prove some of its properties.

First, the wheel must be continuous at the point where two petals intersect. That is, the radius at the rightmost point of a petal must be equal to the radius at the leftmost point of the next one.

Proposition 4.1. *The wheel is continuous at the point where two petals meet.*

Proof. Simple calculations show that

$$
r_1\left(-\frac{\pi}{2} - \theta_1\right) = r_2\left(-\frac{\pi}{2} + \theta_2\right) = \frac{R}{\sqrt{1+m^2}} + y_0 \tag{22}
$$

□

Second, because two consecutive steps meet at an angle of $\pi/2$, petals must intersect at the same angle, so that the wheel fits the road. Coincidentally, this is the case.

Proposition 4.2. *Petals intersect at an angle of $\pi/2$.*

Proof. For this proof, we will refer to Figure 4. All the angles labelled α in this figure are congruent, due to the equiangular property of Bernoulli spirals proved in Proposition 3.2, and the same holds for the angles labelled β. But we already showed above that the two spirals that make up one petal intersect each other perpendicularly. Thus, $\alpha + \beta = \pi/2$, which concludes the proof. $\square$

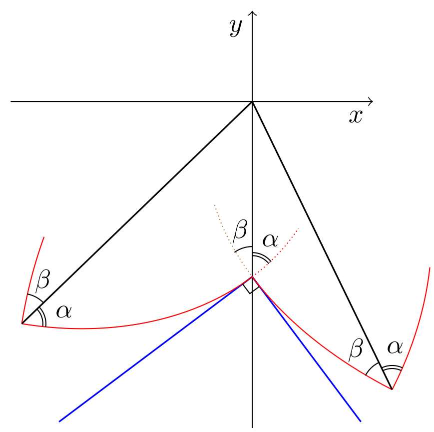


Figure 4: Two petals meet at an angle of $\pi/2$

Finally, we must show the wheel is closed, so that it returns to the initial position after one complete revolution. This basically amounts to saying that we want $r(\theta)$ to be periodic of period 2π. But we already know that $r(\theta)$ is periodic of period $\theta_1+\theta_2$, so the road will be closed if 2π is a multiple of $\theta_1+\theta_2$. As the next proposition shows, we can make this happen by varying the height of the wheel.

Proposition 4.3. *Suppose we create a wheel as described with N petals. The wheel is closed if and only if*

$$y_0 = \frac{R}{\sqrt{1+m^2}\left(e^{\frac{2\pi m}{N(1+m^2)}}-1\right)} \tag{23}$$

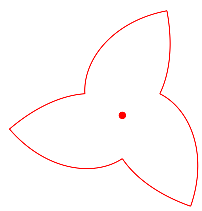
Proof. Suppose the wheel is closed. Then, using the formulas (21) for θ_1 and θ_2 we get

$$\begin{aligned} \theta_1+\theta_2 = \tfrac{2\pi}{N} \quad &\Leftrightarrow \quad \tfrac{m^2+1}{m}\ln\left(\tfrac{R}{y_0\sqrt{1+m^2}}+1\right) = \tfrac{2\pi}{N} \\ &\Leftrightarrow \quad y_0 = \tfrac{R}{\sqrt{1+m^2}\left(e^{\frac{2\pi m}{N(1+m^2)}}-1\right)} \end{aligned} \tag{24}$$

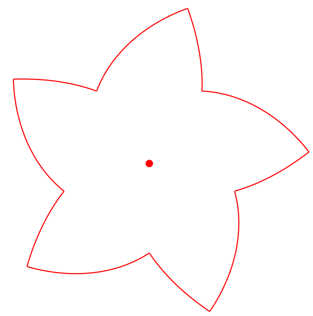
The converse follows in the opposite direction. $\square$

The proposition above allows us to conclude that, by varying the height of the (imaginary) handrail where the centre of the wheel passes, we can build a closed wheel with any number of petals. We have just proved the following theorem:

Theorem 4.1. *For every staircase, there is an infinite number of wheels, each with a different number of petals.*



(a) A wheel with 3 petals



(b) A wheel with 5 petals

Figure 5: Two wheels of different sizes for the same staircase

5 An algorithm to build a wheel

Suppose we are given a staircase with tread width T and riser height R. We can build a wheel for it made up of N petals with the following procedure:

Step 1 Find m, which is just the slope of the staircase given by R/T.

Step 2 Compute y_0 according to (23):

$$y_0 = \frac{R}{\sqrt{1+m^2}\left(e^{\frac{2\pi m}{N(1+m^2)}} - 1\right)} \tag{25}$$

Step 3 Determine the angles θ_1 and θ_2 given by (21), i.e.,

$$\begin{aligned} \theta_1 &= \frac{1}{m}\ln\left(\frac{R}{y_0\sqrt{1+m^2}} + 1\right) \\ \theta_2 &= m\ln\left(\frac{R}{y_0\sqrt{1+m^2}} + 1\right) \end{aligned} \tag{26}$$

Step 4 Build the first petal, that rolls over the first step, using the polar equation (19), that is,

$$r(\theta) = \begin{cases} r_1(\theta) = y_0 e^{-m\left(\theta+\frac{\pi}{2}\right)}, & \theta \in \left[-\frac{\pi}{2} - \theta_1, -\frac{\pi}{2}\right] \\ r_2(\theta) = y_0 e^{\frac{1}{m}\left(\theta+\frac{\pi}{2}\right)}, & \theta \in \left[-\frac{\pi}{2}, -\frac{\pi}{2} + \theta_2\right] \end{cases} \tag{27}$$

Step 5 Complete the wheel by doing $N-1$ copies of the first petal, rotating them around the origin by an angle of $2\pi k/N$, $k = 1, 2, \cdots N-1$.

■

It is clear from the previous discussion that one could, theoretically, build a wheelchair with two sets of wheels, one larger at the back and one smaller up front, that is able climb or descend a staircase while still providing a comfortable ride to its passenger.

6 Proof of concept

We have implemented the algorithm developed in the previous section into an interactive animation made available at [3]. It works in any modern web browser, including tablets and smartphones, without the need for any additional software.

This animation allows the user to enter the parameters for the simulation, including the tread width T, the riser height R and the number of petals N on the wheel. As soon as the user changes any attribute, the wheel is rebuilt and it starts to descend the staircase. The user can also choose to view a second wheel, in front of the first, and select its size independently from the first.

To build the wheel we follow exactly the procedure outlined above. The only complication is that, in order to animate the wheel, it was necessary to calculate $\theta(t)$. However, retracing the steps in Proposition 3.1, it is not difficult to verify that, for the first step, $\theta(t)$ is given by

$$\theta(t) = \begin{cases} \theta_1(t) = -\frac{\ln\left(1-\frac{m}{y_0}t\right)}{m} - \frac{\pi}{2}, & t \in [x_1, 0] \\ \\ \theta_2(t) = m \ln\left(\frac{1}{my_0}t + 1\right) - \frac{\pi}{2}, & t \in [0, x_2] \end{cases} \tag{28}$$

7 Conclusion

In this document we have shown that it is possible to design wheels properly suited to climb or descend stairs in such a way that any vehicles attached to them will follow a linear path, free of bumpiness, just like circular wheels do on flat surfaces. Moreover, we have demonstrated that, for each stair, there are many different wheels that can roll smoothly over it, all of them assembled from segments of Bernoulli spirals. Finally, we have introduced an algorithm to design these wheels, based on the dimensions of the staircase and the desired size of the wheel, and implemented this algorithm into a web application available online.

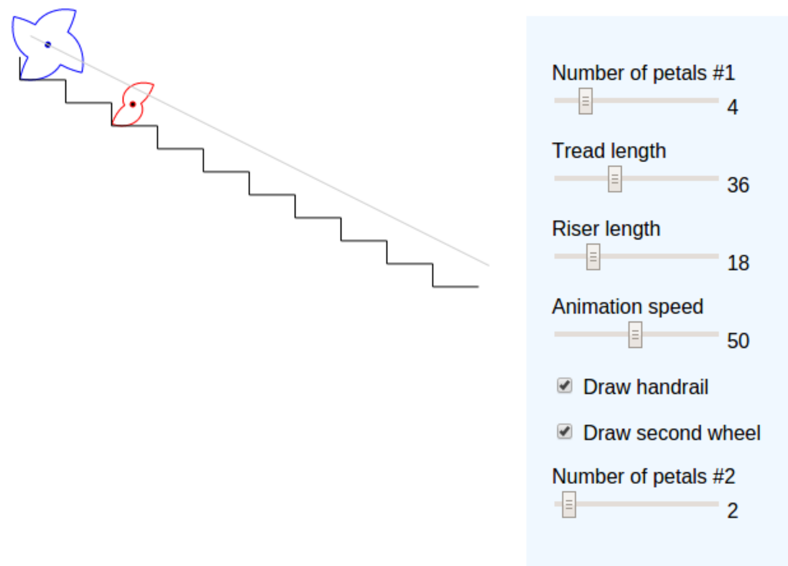


Figure 6: A screenshot of the working prototype showing two different wheels rolling over a staircase

Unfortunately, as we have seen, there is no general purpose wheel that works for all staircases. However, these wheels could still be useful in some particular scenarios, such as public buildings or airplane stairs, where the stairs have well known dimensions and wheels could be built specifically for them. Further work needs to be done in order to assess their applicability.

References

[1] Leon Hall and Stan Wagon, *Roads and Wheels*, Mathematics Magazine, Vol. 65, No. 5 (Dec. 1992), 283–301.

[2] F. Silva Leite et al., *Wheels for staircases*, Unpublished.

[3] Proof of concept, `http://pureza.github.io/app/` (Jun. 2015).