

0.0.1 $\langle \Delta H_{SD} \rangle$ Derivation

The equation $H = -q(1 - q)$ defines a constant of motion for the SD part of the game, where q is the fraction of players in the 4th strategy. Using the transition probabilities of the different process we can derive an expression for the expected change in H within the simplex.

Where $i, j, k, N - i - j - k$ are the players playing R, P, S, and the 4th strategy respectively.

$$\begin{aligned}\Delta H &= H(t+1) - H(t), \\ \Delta H &= -x_{t+1}(1 - x_{t+1}) - (-x_t(1 - x_t)) \\ \Delta H &= -x_{t+1}(1 - x_{t+1}) + x_t(1 - x_t) \\ \Delta H &= x_t(1 - x_t) - x_{t+1}(1 - x_{t+1})\end{aligned}$$

$$\langle \Delta H \rangle = \sum_{i,j,k} (H_s - H_{s'}) T^{s \rightarrow s'}, \text{ } s \text{ is a particular state in the simplex.}$$

$$\begin{aligned}\langle \Delta H_{SD} \rangle &= \frac{6}{N^5} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \left[(N - i - j - k)(1 - N + i + j + k)(T^{R+} + T^{P+} + T^{S+} + T^{+R} + T^{+P} + T^{+S}) \right. \\ &\quad - (N - i - j - k + 1)(-N + i + j + k)T^{R+} \\ &\quad - (N - i - j - k + 1)(-N + i + j + k)T^{P+} \\ &\quad - (N - i - j - k - 1)(2 - N + i + j + k)T^{+R} \\ &\quad - (N - i - j - k - 1)(2 - N + i + j + k)T^{+P} \\ &\quad \left. - (N - i - j - k - 1)(2 - N + i + j + k)T^{+S} \right] \end{aligned} \quad (1)$$

The terms with transitions within the RPS simplex can be ignored as q would not change between these states, therefore the term $H_s - H_{s'} = 0$

$p = N - i - j - k$, the number of players playing the 4th strategy.

$$\begin{aligned}\langle \Delta H_{SD} \rangle &= \frac{6}{N^5} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \left[p(1 - p)(T^{R+} + T^{P+} + T^{S+} + T^{+R} + T^{+P} + T^{+S}) \right. \\ &\quad - (p + 1)(-p)(T^{R+} + T^{P+} + T^{S+}) \\ &\quad \left. - (p - 1)(2 - p)(T^{+R} + T^{+P} + T^{+S}) \right] \end{aligned} \quad (2)$$

$$\langle \Delta H_{SD} \rangle = \frac{6}{N^5} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \left[T^{R+} (p(1 - p) + p(p + 1)) \right] \quad (3)$$

The continuous limit, where $x = i/N$, $y = j/N$, $z = k/N$, and $q = p/N$ and $q = 1 - x - y - z$ leads to:

$$\begin{aligned}\langle \Delta H_{SD} \rangle &= \text{scaling?} \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \left[q(1 - q)(T^{R+} + T^{P+} + T^{S+} + T^{+R} + T^{+P} + T^{+S}) \right. \\ &\quad \left. - (q + \frac{1}{N})(1 - q - \frac{1}{N})(T^{R+} + T^{P+} + T^{S+}) - (q - \frac{1}{N})(1 - q + \frac{1}{N})(T^{+R} + T^{+P} + T^{+S}) \right] \end{aligned} \quad (4)$$

This can then be solved numerically and the critical population values can be found where $\langle \Delta H_{SD} \rangle = 0$.

Moran process equation (needs to be finalised with correct scale factor):

$$\begin{aligned}
\langle \Delta H_{SD} \rangle_{MO} = & \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \left[\right. \\
& (N^2(x+y+z)(x+y+z-1)(x(w(ax+bz+cy-\gamma(x+y+z-1))-w+1) + x(w(a+\beta)(x+y+z)-w+1) \\
& + y(w(ay+bx+cz-\gamma(x+y+z-1))-w+1) + y(w(a+\beta)(x+y+z)-w+1) \\
& + z(w(az+by+cx-\gamma(x+y+z-1))-w+1) + z(w(a+\beta)(x+y+z)-w+1)) \\
& - (N(x+y+z)-1)(N(x+y+z-1)-1)(x+y+z)(w(a+\beta)(x+y+z)-w+1) \\
& - (N(x+y+z)+1)(N(x+y+z-1)+1)(x(w(ax+bz+cy-\gamma(x+y+z-1))-w+1) \\
& + y(w(ay+bx+cz-\gamma(x+y+z-1))-w+1) + z(w(az+by+cx-\gamma(x+y+z-1))-w+1)))(x+y+z-1) \\
& \left. \frac{N^4(w(x(ax+bz+cy-\gamma(x+y+z-1))+y(ay+bx+cz-\gamma(x+y+z-1)) + z(az+by+cx-\gamma(x+y+z-1)) - (a+\beta)(x+y+z)(x+y+z-1)) - w+1)}{N^4} \right]
\end{aligned} \tag{5}$$

With $w = 0$, this reduces to:

$$\langle \Delta H_{SD} \rangle_{MO} = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \frac{-2((x+y+z-1)(x+y+z))}{N^4} \tag{6}$$

Critical N found matches nicely with the simulated versions. Numerical integration in python code `./augRps.py`, shows change of sign as expected. Matches nicely with the approximated values for the Moran process. The specific expression for Moran process is very long. Computed numerically and solved with `scipy.integrate` (reference `scipy`) Maybe can plot the simulated critical population sizes against the analytical on the same graph.

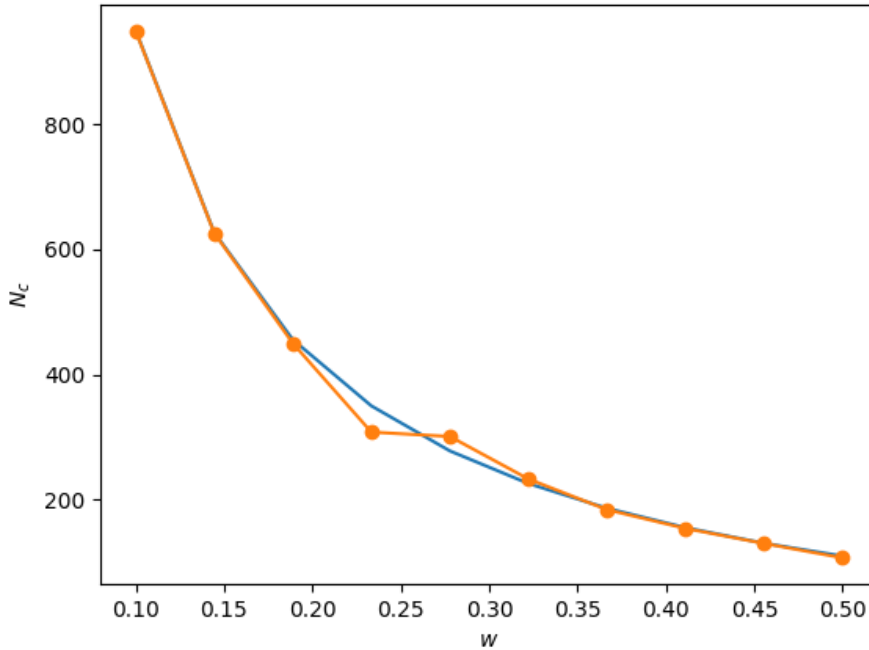


Figure 1: Comparison of simulated critical population sizes against the derived expected change in H for Moran process. Small deviations due to stochasticity in the simulations. $\gamma = 0.2, \beta = 0.1$ standard RPS.

0.0.2 $\langle \Delta H_{RPS} \rangle$ Derivation

$\langle \Delta H \rangle$ within the RPS plane $H = -xyz$.

$$\begin{aligned} \langle \Delta H_{RPS} \rangle = \text{scaling?} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N & \left[ijk(T^{RP} + T^{RS} + T^{R+} + T^{PR} + T^{PS} + T^{P+} \right. \\ & + T^{SR} + T^{SP} + T^{S+} + T^{+R} + T^{+P} + T^{+S}) \\ & - k(i-1)(j+1)T^{RP} - (i-1)j(k+1)T^{RS} - jk(i-1)T^{R+} \\ & - k(i+1)(j-1)T^{PR} - i(j-1)(k+1)T^{PS} - ik(j-1)T^{P+} \\ & - (i+1)j(k-1)T^{SR} - i(j+1)(k-1)T^{SP} - ij(k-1)T^{S+} \\ & \left. - jk(i+1)T^{+R} - ik(j+1)T^{+P} - ij(k+1)T^{+S} \right] \end{aligned} \quad (7)$$

$$\begin{aligned} \langle \Delta H_{RPS} \rangle = \text{scaling?} \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz & \left[xyz(T^{RP} + T^{RS} + T^{R+} + T^{PR} + T^{PS} + T^{P+} \right. \\ & + T^{SR} + T^{SP} + T^{S+} + T^{+R} + T^{+P} + T^{+S}) \\ & - z(x - \frac{1}{N})(y + \frac{1}{N})T^{RP} - (x - \frac{1}{N})y(z + \frac{1}{N})T^{RS} - yz(x - \frac{1}{N})T^{R+} \\ & - z(x + \frac{1}{N})(y - \frac{1}{N})T^{PR} - x(y - \frac{1}{N})(z + \frac{1}{N})T^{PS} - xz(y - \frac{1}{N})T^{P+} \\ & - (x + \frac{1}{N})y(z - \frac{1}{N})T^{SR} - x(y + \frac{1}{N})(z - \frac{1}{N})T^{SP} - xy(z - \frac{1}{N})T^{S+} \\ & \left. - yz(x + \frac{1}{N})T^{+R} - xz(y + \frac{1}{N})T^{+P} - xy(z + \frac{1}{N})T^{+S} \right] \end{aligned} \quad (8)$$

Rough figures of rps and SD delta H values.

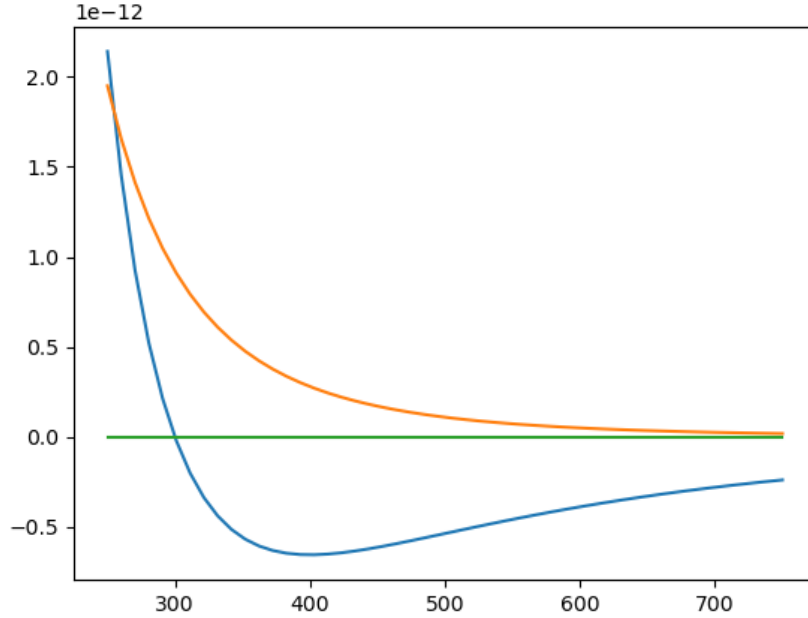


Figure 2: Blue - SD

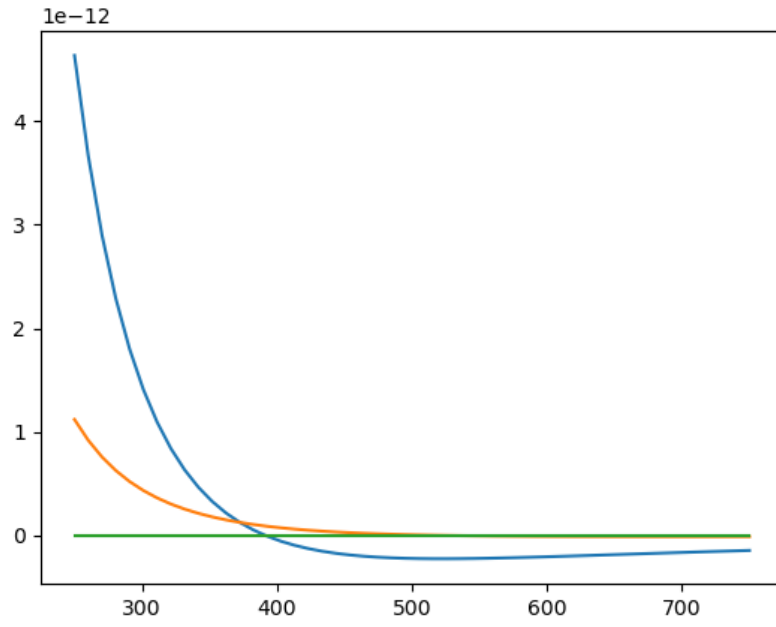


Figure 3: Double reversal case. Blue - SD

0.0.3 $\langle \Delta H_4 \rangle$ Derivation

All 4 strategies, $H = -xyz(1 - x - y - z)$ As above.