

Master equation for Rock

$$\begin{aligned}
P^{\tau+1}(i) - P^\tau(i) &= P^\tau(i-1)T^{PR}(i-1) - P^\tau(i)T^{PR}(i) + P^\tau(i-1)T^{SR}(i-1) - P^\tau(i)T^{SR}(i) \\
&\quad + P^\tau(i-1)T^{+R}(i-1) - P^\tau(i)T^{+R}(i) + P^\tau(i+1)T^{RP}(i+1) - P^\tau(i)T^{RP}(i) \\
&\quad + P^\tau(i+1)T^{RS} - P^\tau(i)T^{RS}(i) + P^\tau(i+1)T^{R+} - P^\tau(i)T^{R+}(i)
\end{aligned} \tag{1}$$

Where  $i$  is the number of rock players.

$$\begin{aligned}
T(x \pm \frac{1}{N}) &= T(x) \pm \frac{1}{N}T'(x) + \frac{1}{2N^2}T''(x) \dots \\
P(x \pm \frac{1}{N}) &= P(x) \pm \frac{1}{N}P'(x) + \frac{1}{2N^2}P''(x) \dots \\
P^{\tau+1}(i) - P^\tau(i) &= (P - \frac{1}{N}P' + \frac{1}{2N^2}P'')(T^{PR} - \frac{1}{N}T^{PR'} + \frac{1}{2N^2}T^{PR''}) - PT^{PR} \\
&\quad + (P - \frac{1}{N}P' + \frac{1}{2N^2}P'')(T^{SR} - \frac{1}{N}T^{SR'} + \frac{1}{2N^2}T^{SR''}) - PT^{SR} \\
&\quad + (P - \frac{1}{N}P' + \frac{1}{2N^2}P'')(T^{+R} - \frac{1}{N}T^{+R'} + \frac{1}{2N^2}T^{+R''}) - PT^{+R} \\
&\quad + (P + \frac{1}{N}P' + \frac{1}{2N^2}P'')(T^{RP} + \frac{1}{N}T^{RP'} + \frac{1}{2N^2}T^{RP''}) - PT^{RP} \\
&\quad + (P + \frac{1}{N}P' + \frac{1}{2N^2}P'')(T^{RS} + \frac{1}{N}T^{RS'} + \frac{1}{2N^2}T^{RS''}) - PT^{RS} \\
&\quad + (P + \frac{1}{N}P' + \frac{1}{2N^2}P'')(T^{R+} + \frac{1}{N}T^{R+'} + \frac{1}{2N^2}T^{R+''}) - PT^{R+} \\
P^{\tau+1}(i) - P^\tau(i) &= -\frac{1}{N} \left[ P(T^{PR'} + T^{SR'} + T^{+R'} - T^{RP'} - T^{RS'} - T^{R+'}) \right. \\
&\quad \left. + P'(T^{PR} + T^{SR} + T^{+R} - T^{RP} - T^{RS} - T^{R+}) \right] \\
&\quad + \frac{1}{2N^2} \left[ P(T^{PR''} + T^{SR''} + T^{+R''} + T^{RP''} + T^{RS''} + T^{R+''}) \right. \\
&\quad \left. + 2P'(T^{PR'} + T^{SR'} + T^{+R'} + T^{RP'} + T^{RS'} + T^{R+'}) \right. \\
&\quad \left. + P''(T^{PR} + T^{SR} + T^{+R} + T^{RP} + T^{RS} + T^{R+}) \right] \\
&= \frac{1}{N} \left[ - (P(T^{PR} + T^{SR} + T^{+R} - T^{RP} - T^{RS} - T^{R+}))' \right. \\
&\quad \left. + \frac{1}{2} (P(\frac{T^{PR} + T^{SR} + T^{+R} + T^{RP} + T^{RS} + T^{R+}}{N}))'' \right]
\end{aligned} \tag{2}$$

We then have for rock (i)  $a(i) = T^{PR} + T^{SR} + T^{+R} - T^{RP} - T^{RS} - T^{R+}$  and  $b^2(i) = \frac{T^{PR} + T^{SR} + T^{+R} + T^{RP} + T^{RS} + T^{R+}}{N}$ .

For paper and scissors we get the same but with the corresponding ingoing and outgoing transition probabilities.

$$\begin{aligned}
P^{\tau+1}(j) - P^\tau(j) &= \frac{1}{N} \left[ - (P(T^{RP} + T^{SP} + T^{+P} - T^{PR} - T^{PS} - T^{P+}))' \right. \\
&\quad \left. + \frac{1}{2} (P(\frac{T^{RP} + T^{SP} + T^{+P} + T^{PR} + T^{PS} + T^{P+}}{N}))'' \right]
\end{aligned} \tag{4}$$

$a(j) = T^{RP} + T^{SP} + T^{+P} - T^{PR} - T^{PS} - T^{P+}$  and  $b^2(j) = \frac{T^{RP} + T^{SP} + T^{+P} + T^{PR} + T^{PS} + T^{P+}}{N}$ .

$$\begin{aligned}
P^{\tau+1}(k) - P^\tau(k) &= \frac{1}{N} \left[ - (P(T^{RS} + T^{PS} + T^{+S} - T^{SR} - T^{SP} - T^{S+}))' \right. \\
&\quad \left. + \frac{1}{2} (P(\frac{T^{RS} + T^{PS} + T^{+S} + T^{SR} + T^{SP} + T^{S+}}{N}))'' \right] \\
a(k) &= T^{RS} + T^{PS} + T^{+S} - T^{SR} - T^{SP} - T^{S+} \text{ and } b^2(k) = \frac{T^{RS} + T^{PS} + T^{+S} + T^{SR} + T^{SP} + T^{S+}}{N}.
\end{aligned} \tag{5}$$

From these we can form the fokker planck equations for each strategies population ( $x \in \{i, j, k\}$ ):

$$\frac{d}{dt} P^\tau(x) = \frac{1}{N} \left[ - \frac{d}{dx} (a(x) P^\tau(x)) + \frac{1}{2} \frac{d^2}{dx^2} (b^2(x) P^\tau(x)) \right] \tag{6}$$

Langevin equation with noise term  $b(x)\xi$ :

$$\dot{x} = a(x) + b(x)\xi \tag{7}$$

$a(x)$  is our deterministic replicator equation as  $N \rightarrow \infty$  noise dissapears ( $b^2(x)$  is divided by N).

For the Moran process, the infinite population dynamics reduce to a scaled version of the ordinary replicator equation by the factor  $\frac{1}{\Gamma + \langle \pi(x) \rangle}$

$$a(x) = \frac{1}{\Gamma + \langle \pi(x) \rangle} x_i (\pi_i(x) - \langle \pi(x) \rangle), \Gamma = \frac{1-w}{w} \tag{8}$$

$$\begin{aligned}
\langle \pi(x) \rangle &= x(ax + bz + cy + \gamma(-x - y - z + 1)) \\
&\quad + y(ay + bx + cz + \gamma(-x - y - z + 1)) \\
&\quad + z(az + by + cx + \gamma(-x - y - z + 1)) \\
&\quad + (x(a + \beta) + y(a + \beta) + z(a + \beta))(-x - y - z + 1)
\end{aligned}$$

The RHS of the equation corresponding to the un-scaled replicator dynamics is the following.

$$\begin{aligned}
& \left[ x(2awxy + 2awxz + 2awyz - awy - awz \right. \\
& \quad - bwx y - bwx z - bwy z + bwz \\
& \quad + \beta wx^2 + 2\beta wxy + 2\beta wxz - \beta wx \\
& \quad + \beta wy^2 + 2\beta wyz - \beta wy \\
& \quad + \beta wz^2 - \beta wz \\
& \quad - cwx y - cwx z - cwy z + cw y \\
& \quad + \gamma wx^2 + 2\gamma wxy + 2\gamma wxz - 2\gamma wx \\
& \quad + \gamma wy^2 + 2\gamma wyz - 2\gamma wy \\
& \quad \left. + \gamma wz^2 - 2\gamma wz + \gamma w) \right] \\
& \overline{\left[ -2awxy - 2awxz + awx - 2awyz + awy + awz \right.} \\
& \quad + bwx y + bwx z + bwy z \\
& \quad - \beta wx^2 - 2\beta wxy - 2\beta wxz + \beta wx \\
& \quad - \beta wy^2 - 2\beta wyz + \beta wy \\
& \quad - \beta wz^2 + \beta wz \\
& \quad + cwx y + cwx z + cwy z \\
& \quad - \gamma wx^2 - 2\gamma wxy - 2\gamma wxz + \gamma wx \\
& \quad - \gamma wy^2 - 2\gamma wyz + \gamma wy \\
& \quad \left. - \gamma wz^2 + \gamma wz - w + 1 \right] }
\end{aligned} \tag{9}$$

Next to do - look at  $b^2(x)$  and see how it is different between the Moran and local update, focus on whether or not N is present as it can tell us about drift reversal. If N is not present then the drift does not depend on pop size and therefore no reversal. If there is an N then drift is possible.