Master equation for Rock

$$
\begin{aligned}
P^{\tau+1}(i) - P^{\tau}(i) = {} & P^{\tau}(i-1)T^{PR}(i-1) - P^{\tau}(i)T^{PR}(i) + P^{\tau}(i-1)T^{SR}(i-1) - P^{\tau}(i)T^{SR}(i) \\
& + P^{\tau}(i-1)T^{+R}(i-1) - P^{\tau}(i)T^{+R}(i) + P^{\tau}(i+1)T^{RP}(i+1) - P^{\tau}(i)T^{RP}(i) \\
& + P^{\tau}(i+1)T^{RS} - P^{\tau}(i)T^{RS}(i) + P^{\tau}(i+1)T^{R+} - P^{\tau}(i)T^{R+}(i)
\end{aligned} \tag{1}
$$

Where i is the number of rock players.

$$
\begin{aligned}
T(x \pm \frac{1}{N}) = T(x) \pm \frac{1}{N}T'(x) + \frac{1}{2N^2}T''(x)\dots \\
P(x \pm \frac{1}{N}) = P(x) \pm \frac{1}{N}P'(x) + \frac{1}{2N^2}P''(x)\dots
\end{aligned}
$$

$$
\begin{aligned}
P^{\tau+1}(i) - P^{\tau}(i) = {} & (P - \frac{1}{N}P' + \frac{1}{2N^2}P'')(T^{PR} - \frac{1}{N}T^{PR'} + \frac{1}{2N^2}T^{PR''}) - PT^{PR} \\
& + (P - \frac{1}{N}P' + \frac{1}{2N^2}P'')(T^{SR} - \frac{1}{N}T^{SR'} + \frac{1}{2N^2}T^{SR''}) - PT^{SR} \\
& + (P - \frac{1}{N}P' + \frac{1}{2N^2}P'')(T^{+R} - \frac{1}{N}T^{+R'} + \frac{1}{2N^2}T^{+R''}) - PT^{+R} \\
& + (P + \frac{1}{N}P' + \frac{1}{2N^2}P'')(T^{RP} + \frac{1}{N}T^{RP'} + \frac{1}{2N^2}T^{RP''}) - PT^{RP} \\
& + (P + \frac{1}{N}P' + \frac{1}{2N^2}P'')(T^{RS} + \frac{1}{N}T^{RS'} + \frac{1}{2N^2}T^{RS''}) - PT^{RS} \\
& + (P + \frac{1}{N}P' + \frac{1}{2N^2}P'')(T^{R+} + \frac{1}{N}T^{R+'} + \frac{1}{2N^2}T^{R+''}) - PT^{R+}
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
P^{\tau+1}(i) - P^{\tau}(i) = {} & -\frac{1}{N}\Big[P\big(T^{PR'} + T^{SR'} + T^{+R'} - T^{RP'} - T^{RS'} - T^{R+'}\big) \\
& \qquad + P'\big(T^{PR} + T^{SR} + T^{+R} - T^{RP} - T^{RS} - T^{R+}\big)\Big] \\
& + \frac{1}{2N^2}\Big[P\big(T^{PR''} + T^{SR''} + T^{+R''} + T^{RP''} + T^{RS''} + T^{R+''}\big) \\
& \qquad + 2P'\big(T^{PR'} + T^{SR'} + T^{+R'} + T^{RP'} + T^{RS'} + T^{R+'}\big) \\
& \qquad + P''\big(T^{PR} + T^{SR} + T^{+R} + T^{RP} + T^{RS} + T^{R+}\big)\Big] \\
= {} & \frac{1}{N}\Big[-\big(P\big(T^{PR} + T^{SR} + T^{+R} - T^{RP} - T^{RS} - T^{R+}\big)\big)' \\
& + \frac{1}{2}\big(P(\frac{T^{PR} + T^{SR} + T^{+R} + T^{RP} + T^{RS} + T^{R+}}{N})\big)''\Big]
\end{aligned} \tag{3}
$$

We then have for rock (i) $a(i) = T^{PR} + T^{SR} + T^{+R} - T^{RP} - T^{RS} - T^{R+}$ and $b^2(i) = \frac{T^{PR}+T^{SR}+T^{+R}+T^{RP}+T^{RS}+T^{R+}}{N}$.

For paper and scissors we get the same but with the corresponding ingoing and outgoing transition probabilities.

$$
\begin{aligned}
P^{\tau+1}(j) - P^{\tau}(j) = {} & \frac{1}{N}\Big[-\big(P\big(T^{RP} + T^{SP} + T^{+P} - T^{PR} - T^{PS} - T^{P+}\big)\big)' \\
& + \frac{1}{2}\big(P(\frac{T^{RP} + T^{SP} + T^{+P} + T^{PR} + T^{PS} + T^{P+}}{N})\big)''\Big]
\end{aligned} \tag{4}
$$

$a(j) = T^{RP} + T^{SP} + T^{+P} - T^{PR} - T^{PS} - T^{P+}$ and $b^2(j) = \frac{T^{RP}+T^{SP}+T^{+P}+T^{PR}+T^{PS}+T^{P+}}{N}$.

$$P^{\tau+1}(k) - P^{\tau}(k) = \frac{1}{N}\Big[-\big(P\big(T^{RS} + T^{PS} + T^{+S} - T^{SR} - T^{SP} - T^{S+}\big)\big)' + \frac{1}{2}\big(P\big(\frac{T^{RS} + T^{PS} + T^{+S} + T^{SR} + T^{SP} + T^{S+}}{N}\big)\big)''\Big] \tag{5}$$

$a(k) = T^{RS} + T^{PS} + T^{+S} - T^{SR} - T^{SP} - T^{S+}$ and $b^2(k) = \frac{T^{RS}+T^{PS}+T^{+S}+T^{SR}+T^{SP}+T^{S+}}{N}$.

From these we can form the fokker planck equations for each strategies population ($x \in \{i, j, k\}$):

$$\frac{d}{dt}P^{\tau}(x) = \frac{1}{N}\Big[-\frac{d}{dx}\big(a(x)P^{\tau}(x)\big) + \frac{1}{2}\frac{d^2}{dx^2}\big(b^2(x)P^{\tau}(x)\big)\Big] \tag{6}$$

Langevin equation with noise term $b(x)\xi$:

$$\dot{x} = a(x) + b(x)\xi \tag{7}$$

$a(x)$ is our deterministic replicator equation as $N \to \infty$ noise dissapears ($b^2(x)$ is divided by N).

For the Moran process, the infinite population dynamics reduce to a scaled version of the ordinary replicator equation by the factor $\frac{1}{\Gamma + \langle\pi(x)\rangle}$

$$a(x) = \frac{1}{\Gamma + \langle\pi(x)\rangle} x_i(\pi_i(x) - \langle\pi(x)\rangle), \Gamma = \frac{1-w}{w} \tag{8}$$

$$\begin{aligned}\langle\pi(x)\rangle = {} & x\,(ax + bz + cy + \gamma\,(-x - y - z + 1)) \\ & + y\,(ay + bx + cz + \gamma\,(-x - y - z + 1)) \\ & + z\,(az + by + cx + \gamma\,(-x - y - z + 1)) \\ & + (x\,(a + \beta) + y\,(a + \beta) + z\,(a + \beta))\,(-x - y - z + 1)\end{aligned}$$

The RHS of the equation corresponding to the un-scaled replicator dynamics is the following.

$$\frac{\begin{aligned}\Big[x\big(&2awxy + 2awxz + 2awyz - awy - awz\\ &- bwxy - bwxz - bwyz + bwz\\ &+ \beta wx^2 + 2\beta wxy + 2\beta wxz - \beta wx\\ &+ \beta wy^2 + 2\beta wyz - \beta wy\\ &+ \beta wz^2 - \beta wz\\ &- cwxy - cwxz - cwyz + cwy\\ &+ \gamma wx^2 + 2\gamma wxy + 2\gamma wxz - 2\gamma wx\\ &+ \gamma wy^2 + 2\gamma wyz - 2\gamma wy\\ &+ \gamma wz^2 - 2\gamma wz + \gamma w\big)\Big]\end{aligned}}{\begin{aligned}\Big[&- 2awxy - 2awxz + awx - 2awyz + awy + awz\\ &+ bwxy + bwxz + bwyz\\ &- \beta wx^2 - 2\beta wxy - 2\beta wxz + \beta wx\\ &- \beta wy^2 - 2\beta wyz + \beta wy\\ &- \beta wz^2 + \beta wz\\ &+ cwxy + cwxz + cwyz\\ &- \gamma wx^2 - 2\gamma wxy - 2\gamma wxz + \gamma wx\\ &- \gamma wy^2 - 2\gamma wyz + \gamma wy\\ &- \gamma wz^2 + \gamma wz - w + 1\Big]\end{aligned}} \tag{9}$$

Next to do - look at $b^2(x)$ and see how it is different between the Moran and local update, focus on whether or not N is present as it can tell us about drift reversal. If N is not present then the drift does not depend on pop size and therfore no reversal. If there is an N then drift is possible.