

The equation  $H = -q(1 - q)$  defines a constant of motion for the SD part of the game, where  $q$  is the fraction of players in the 4th strategy. Using the transition probabilities of the different process we can derive an expression for the expected change in  $H$  within the simplex.

Where  $i, j, k, N - i - j - k$  are the players playing R, P, S, and the 4th strategy respectively.

$$\Delta H = H(t+1) - H(t),$$

$$\Delta H = -x_{t+1}(1 - x_{t+1}) - (-x_t(1 - x_t))$$

$$\Delta H = -x_{t+1}(1 - x_{t+1}) + x_t(1 - x_t)$$

$$\Delta H = x_t(1 - x_t) - x_{t+1}(1 - x_{t+1})$$

Rough equation:

$$\langle \Delta H \rangle = \sum_{i,j,k} \left( \Delta H_s - \Delta H_{s'} \right) T^{s \rightarrow s'}, \text{ } s \text{ is a particular state in the simplex.}$$

$$\begin{aligned} \langle \Delta H \rangle = \text{scaling ? } & \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \left[ (N - i - j - k)(1 - N + i + j + k)(T^{R+} + T^{P+} + T^{S+} + T^{+R} + T^{+P} + T^{+S}) \right. \\ & - (N - i - j - k + 1)(-N + i + j + k)T^{R+} \\ & - (N - i - j - k + 1)(-N + i + j + k)T^{P+} \\ & - (N - i - j - k - 1)(2 - N + i + j + k)T^{+R} \\ & - (N - i - j - k - 1)(2 - N + i + j + k)T^{+P} \\ & \left. - (N - i - j - k - 1)(2 - N + i + j + k)T^{+S} \right] \end{aligned} \quad (1)$$

$$q = 1 - x - y - z, p = N - i - j - k$$

$$\begin{aligned} \langle \Delta H \rangle = \text{scaling ? } & \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \left[ p(1 - p)(T^{R+} + T^{P+} + T^{S+} + T^{+R} + T^{+P} + T^{+S}) \right. \\ & - (p + 1)(-p)(T^{R+} + T^{P+} + T^{S+}) \\ & \left. - (p - 1)(2 - p)(T^{+R} + T^{+P} + T^{+S}) \right] \end{aligned} \quad (2)$$

$$\begin{aligned} \langle \Delta H \rangle = \text{scaling? } & \int_0^1 dx \int_0^1 dy \int_0^1 dz \left[ q(1 - q)(T^{R+} + T^{P+} + T^{S+} + T^{+R} + T^{+P} + T^{+S}) \right. \\ & \left. - \left( q + \frac{1}{N} \right) \left( 1 - q - \frac{1}{N} \right) (T^{R+} + T^{P+} + T^{S+}) - \left( q - \frac{1}{N} \right) \left( 1 - q + \frac{1}{N} \right) (T^{+R} + T^{+P} + T^{+S}) \right] \end{aligned} \quad (3)$$

Numerical integration in python code `./augRps.py`, shows change of sign as expected. Matches nicely with the approximated values for the Moran process. The specific expression for Moran process is very long. Computed numerically and solved with `scipy.integrate` (reference `scipy`)