

0.0.1 $\langle \Delta H_{SD} \rangle$ Derivation

The equation $H = -q(1 - q)$ defines a constant of motion for the SD part of the game, where q is the fraction of players in the 4th strategy. Using the transition probabilities of the different process we can derive an expression for the expected change in H within the simplex.

Where $i, j, k, N - i - j - k$ are the players playing R, P, S, and the 4th strategy respectively.

$$\begin{aligned}\Delta H &= H(t+1) - H(t), \\ \Delta H &= -x_{t+1}(1 - x_{t+1}) - (-x_t(1 - x_t)) \\ \Delta H &= -x_{t+1}(1 - x_{t+1}) + x_t(1 - x_t) \\ \Delta H &= x_t(1 - x_t) - x_{t+1}(1 - x_{t+1})\end{aligned}$$

$$\langle \Delta H \rangle = \sum_{i,j,k} (H_s - H_{s'}) T^{s \rightarrow s'}, \text{ } s \text{ is a particular state in the simplex.}$$

$$\begin{aligned}\langle \Delta H_{SD} \rangle = \text{scaling? } & \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \left[(N - i - j - k)(1 - N + i + j + k)(T^{R+} + T^{P+} + T^{S+} + T^{+R} + T^{+P} + T^{+S}) \right. \\ & - (N - i - j - k + 1)(-N + i + j + k)T^{R+} \\ & - (N - i - j - k + 1)(-N + i + j + k)T^{P+} \\ & - (N - i - j - k - 1)(2 - N + i + j + k)T^{+R} \\ & - (N - i - j - k - 1)(2 - N + i + j + k)T^{+P} \\ & \left. - (N - i - j - k - 1)(2 - N + i + j + k)T^{+S} \right] \end{aligned} \quad (1)$$

$p = N - i - j - k$, the number of players playing the 4th strategy.

$$\begin{aligned}\langle \Delta H_{SD} \rangle = \text{scaling? } & \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \left[p(1 - p)(T^{R+} + T^{P+} + T^{S+} + T^{+R} + T^{+P} + T^{+S}) \right. \\ & - (p + 1)(-p)(T^{R+} + T^{P+} + T^{S+}) \\ & \left. - (p - 1)(2 - p)(T^{+R} + T^{+P} + T^{+S}) \right] \end{aligned} \quad (2)$$

The continuous limit, where $x = i/N$, $y = j/N$, $z = k/N$, and $q = p/N$ and $q = 1 - x - y - z$ leads to:

$$\begin{aligned}\langle \Delta H_{SD} \rangle = \text{scaling? } & \int_0^1 dx \int_0^1 dy \int_0^1 dz \left[q(1 - q)(T^{R+} + T^{P+} + T^{S+} + T^{+R} + T^{+P} + T^{+S}) \right. \\ & \left. - (q + \frac{1}{N})(1 - q - \frac{1}{N})(T^{R+} + T^{P+} + T^{S+}) - (q - \frac{1}{N})(1 - q + \frac{1}{N})(T^{+R} + T^{+P} + T^{+S}) \right] \end{aligned} \quad (3)$$

This can then be solved numerically and the critical population values can be found where $\langle \Delta H_{SD} \rangle = 0$.

Critical N found matches nicely with the simulated versions. Numerical integration in python code `./augRps.py`, shows change of sign as expected. Matches nicely with the approximated values for the Moran process. The specific expression for Moran process is very long. Computed numerically and solved with `scipy.integrate` (reference `scipy`)