Composite games evolutionary dynamics in finite populations.

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Abstract

Final year masters project. Replicator dynamics in rock paper scissors (RPS) and composition of RPS and the snowdrift game (SD) in both the infinite and finite population cases. Payoff matrices and replicator derivations, stochastic simulations and drift analysis under different interaction process. Extensive simulation tool for graphing and animation of the different processes and large ensemble averages for general games. Derivation and analytical solution for observation variable H and exact calculation of critical population sizes under different parameters.

1 Standard RPS

The standard (general) rock paper scissors (RPS) payoff matrix is defined as:

$$\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} \tag{1}$$

Let x, y, and z represent the fraction of players using rock, paper, and scissors respectively. Since z can be eliminated, we define:

$$z = 1 - x - y$$

Expected payoffs for each strategy can then be defined as follows:

$$\pi_R = ax + b\left(-x - y + 1\right) + cy \tag{2}$$
$$\pi_P = ay + bx + c\left(-x - y + 1\right) \tag{3}$$
$$\pi_S = a\left(-x - y + 1\right) + by + cx \tag{4}$$

The replicator equations can be defined (ref Evo games and pop dynamics hofbauer) for (x, y) using the standard formula:

$$\dot{x}_i = x_i(\pi_i(x) - \langle \pi(x) \rangle) \tag{5}$$

where the average payoff (or mean fitness) in the population is:

$$\langle \pi(x) \rangle = x\pi_R + y\pi_P + z\pi_S$$

This leads to the following replicator equations for x and y:

$$\begin{aligned} \dot{x} &= xy\left(ax - ay - bx + b\left(-x - y + 1\right) + cy - c\left(-x - y + 1\right)\right) \\ &\quad + x\left(-x - y + 1\right)\left(ax - a\left(-x - y + 1\right) - by + b\left(-x - y + 1\right) - cx + cy\right) \end{aligned} \tag{6}$$

$$\begin{aligned} \dot{y} &= xy\left(-ax + ay + bx - b\left(-x - y + 1\right) - cy + c\left(-x - y + 1\right)\right) \\ &\quad + y\left(-x - y + 1\right)\left(ay - a\left(-x - y + 1\right) + bx - by - cx + c\left(-x - y + 1\right)\right) \end{aligned} \tag{7}$$

Defining $F = \dot{x}, G = \dot{y}$, the following derivatives form the Jacobian used to determine the stability of the system:

$$\begin{aligned}\frac{dF}{dx} &= xy(a-2b+c) + x(2a-b-c)(-x-y+1)\\ &- x\left(ax - a(-x-y+1) - by + b(-x-y+1) - cx + cy\right)\\ &+ y\left(ax - ay - bx + b(-x-y+1) + cy - c(-x-y+1)\right)\\ &+ (-x-y+1)\left(ax - a(-x-y+1) - by + b(-x-y+1) - cx + cy\right)\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{dF}{dy} &= xy(-a-b+2c) + x(a-2b+c)(-x-y+1)\\ &+ x\left(ax - ay - bx + b(-x-y+1) + cy - c(-x-y+1)\right)\\ &- x\left(ax - a(-x-y+1) - by + b(-x-y+1) - cx + cy\right)\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{dG}{dx} &= xy(-a+2b-c) + y(a+b-2c)(-x-y+1)\\ &+ y(-ax + ay + bx - b(-x-y+1) - cy + c(-x-y+1))\\ &- y(ay - a(-x-y+1) + bx - by - cx + c(-x-y+1))\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{dG}{dy} &= xy(a+b-2c) + x\left(-ax + ay + bx - b(-x-y+1) - cy + c(-x-y+1)\right)\\ &+ y(2a-b-c)(-x-y+1)\\ &- y\left(ay - a(-x-y+1) + bx - by - cx + c(-x-y+1)\right)\\ &+ (-x-y+1)\left(ay - a(-x-y+1) + bx - by - cx + c(-x-y+1)\right)\end{aligned} \tag{11}$$

The Jacobian is as follows:

$$J = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix}$$

Eigenvalues at the standard RPS game (a = 0, b = 1, c = -1):

$$\left\{-\frac{1}{2}\sqrt{16x^2+16xy-16x+16y^2-16y+4},\quad \frac{1}{2}\sqrt{16x^2+16xy-16x+16y^2-16y+4}\right\}$$

At the internal fixed point of $(x, y) = (\frac{1}{3}, \frac{1}{3})$, $\lambda_{1,2} = \left\{-\frac{\sqrt{3}}{3}i, \frac{\sqrt{3}}{3}i\right\}$
No real parts of any eigenvalues, and imaginary components exist therefore neutral stability in standard RPS.

2 Augmented RPS

The augmented game can be described by the following payoff matrix.

$$\begin{bmatrix} a & c & b & \gamma \\ b & a & c & \gamma \\ c & b & a & \gamma \\ a+\beta & a+\beta & a+\beta & 0 \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} \langle RPS \rangle & \gamma \\ \beta & 0 \end{bmatrix}$$

Where the strategies are labelled as R,P,S (as before) and L (the loner strategy). From this the Jacobian matrix can be derived with dimensions 3×3 after using the trick to remove one replicator equation. (q = 1 - x - y - z)

2.1 Payoffs

Payoffs for the augmented game:

$$\pi_R = ax + bz + cy + \gamma(-x - y - z + 1) \tag{13}$$
$$\pi_P = ay + bx + cz + \gamma(-x - y - z + 1) \tag{14}$$
$$\pi_S = az + by + cx + \gamma(-x - y - z + 1) \tag{15}$$
$$\pi_L = x(a + \beta) + y(a + \beta) + z(a + \beta) \tag{16}$$

2.2 Replicator equations

$$\begin{aligned}\dot{x} = x(&ax + bz + cy + \gamma(-x - y - z + 1) - x(ax + bz + cy + \gamma(-x - y - z + 1))\\ &- y(ay + bx + cz + \gamma(-x - y - z + 1)) - z(az + by + cx + \gamma(-x - y - z + 1))\\ &- (x(a + \beta) + y(a + \beta) + z(a + \beta))(-x - y - z + 1))\end{aligned} \tag{17}$$

$$\begin{aligned}\dot{y} = y\big(&ay + bx + cz + \gamma(-x - y - z + 1) - x(ax + bz + cy + \gamma(-x - y - z + 1))\\ &- y(ay + bx + cz + \gamma(-x - y - z + 1)) - z(az + by + cx + \gamma(-x - y - z + 1))\\ &- (x(a + \beta) + y(a + \beta) + z(a + \beta))(-x - y - z + 1)\big)\end{aligned} \tag{18}$$

$$\begin{aligned}\dot{z} = z\big(&az + by + cx + \gamma(-x - y - z + 1) - x(ax + bz + cy + \gamma(-x - y - z + 1))\\ &- y(ay + bx + cz + \gamma(-x - y - z + 1)) - z(az + by + cx + \gamma(-x - y - z + 1))\\ &- (x(a + \beta) + y(a + \beta) + z(a + \beta))(-x - y - z + 1)\big)\end{aligned} \tag{19}$$

Reduced the dimensions to just 3 replicator equations in x, y and z as q = 1 - x - y - z.
Jacobian formed as before, $F = \dot{x}, G = \dot{y}, P = \dot{z}$

$$J = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial z} \end{bmatrix}$$

2.3 Stability

Eigenvalues computed using Sympy python library, simplified by hand - floating point precision issue starting to come into play but values are clear (0, 2/9, -2/3 etc).

With a = 0, b = 1, c = -1, $\gamma = 0.2, \beta = 0.1$ we have an internal fixed point at $(2/9, 2/9, 2/9, 1/3)$ (see matrix 43) with eigenvalues (Re, Im):

$$\left\{(0,\ 0.222222222222222\sqrt{3}), (-0.0666666666666667,\ 0), (0,\ -0.222222222222222\sqrt{3})\right\} \tag{20}$$

All the real parts are <= 0, indicating neutral stability (oscillations at fixed distance around the interior fixed point) also visible in the simulations 1.

See figure 1 for the 4 simplex plot of these dynamics for the local update and Moran processes aswell as numerical solution of the replicator system of ODE's derived from the standard replicator equation (local update in the limit $N \to \infty$). With a suitably large population the local and Moran processes are following the deterministic trajectory but with added noise.

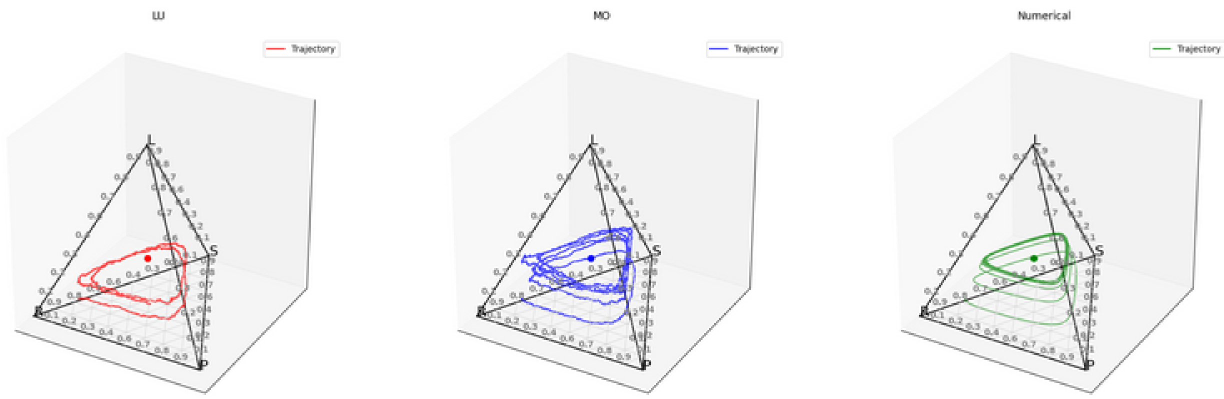


Figure 1: N=1000, iterations=1,000,000

3 Interaction processes simulations

3.1 Moran

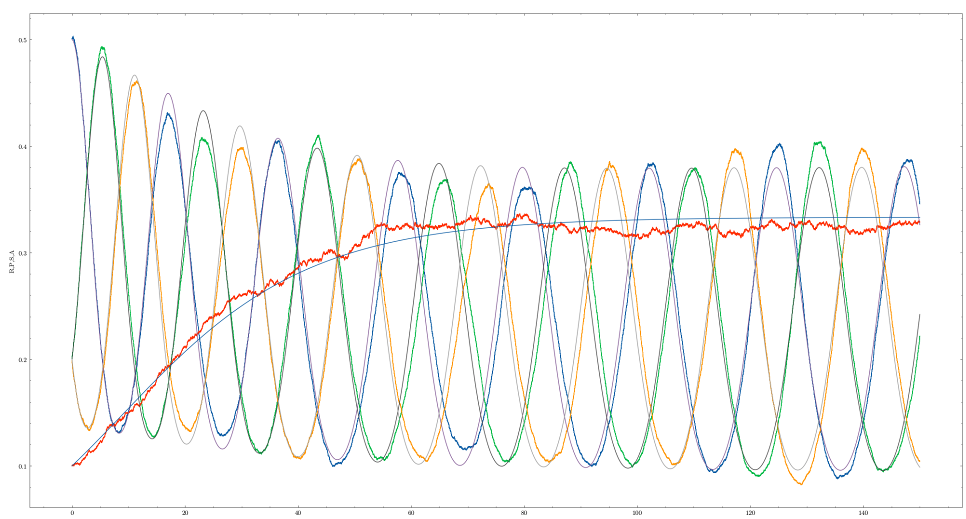


Figure 2: population size 60000 Moran process, along with numerical solution to adjusted dynamics.

3.2 Local update

3.3 Fermi process

3.4 Link to deterministic equations $\lim_{N\to\infty}$

Adjusted deterministic equations for the interaction processes, derived from the fokker planck expansions of the interaction processes defined (master equation).

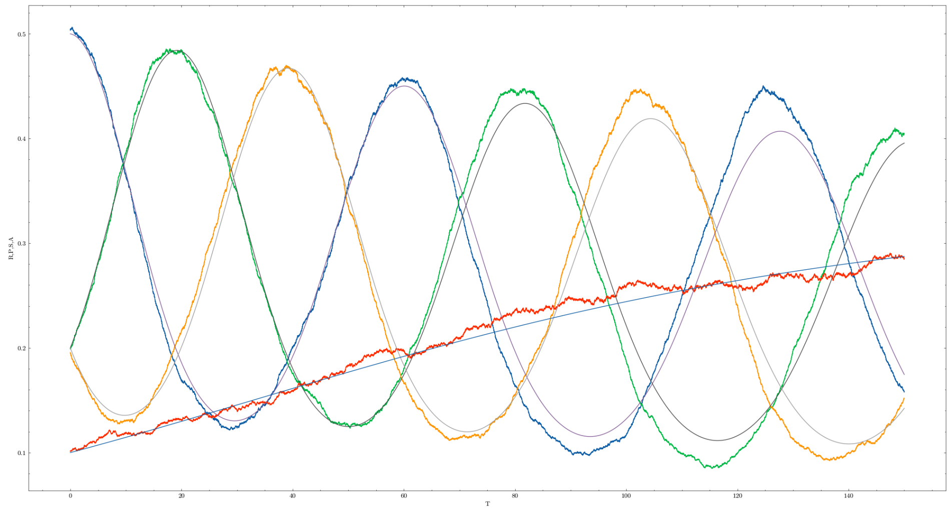


Figure 3: Time series of local update with population 20000, simulation trajectory normalized, compared with the numerical solution of the replicator equations derived from FP equation [3]

3.5 Transition probabilities

Finite population, i rock players, j paper players, k scissors players, and $N-i-j-k$ + players. In the Moran process an individual is chosen proportionally to their fitness in the population, therefore we need the global average fitness $\langle\pi\rangle$, the average payoff is $\langle\pi\rangle = \frac{i}{N}\pi_R + \frac{j}{N}\pi_P + \frac{k}{N}\pi_S + \frac{N-i-j-k}{N}\pi_+$

For the Moran process the transition probabilities are as follows:

$$T^{R\to+} = \frac{1}{2}\frac{1-w+w\pi_+}{1-w+w\langle\pi\rangle}\frac{i}{N}\frac{N-i-j-k}{N} \tag{21}$$

$$T^{+\to R} = \frac{1}{2}\frac{1-w+w\pi_R}{1-w+w\langle\pi\rangle}\frac{i}{N}\frac{N-i-j-k}{N} \tag{22}$$

Also transitions within RPS

$$T^{R\to S} = \frac{1}{2}\frac{1-w+w\pi_S}{1-w+w\langle\pi\rangle}\frac{i}{N}\frac{k}{N} \tag{23}$$

$$T^{S\to R} = \frac{1}{2}\frac{1-w+w\pi_R}{1-w+w\langle\pi\rangle}\frac{i}{N}\frac{k}{N} \tag{24}$$

The rest of the transitions, $T^{R\to P}, T^{P\to R}, T^{S\to P}, T^{P\to S}, T^{P\to+}, T^{+\to P}, T^{S\to+}, T^{+\to S}$ formed from permuations $(R, P, S, +)$ and $(i, j, k, N-i-j-k)$

For the local update process we compare payoffs between only 2 randomly selected individuals a and b.

$$T^{R\to+} = \left(\frac{1}{2} + \frac{w}{2}\frac{\pi_+ - \pi_R}{\Delta\pi_{\max}}\right)\frac{i}{N}\frac{N-i-j-k}{N} \tag{25}$$

Can simply define these transitions as:

$$\phi(b\to a)\frac{N_a N_b}{N^2} \tag{26}$$

Given an appropriate ϕ reproductive function.

3.5.1 Master equation

Master equation for Rock

$$\begin{aligned}P^{\tau+1}(i)-P^{\tau}(i)&=P^{\tau}(i-1)T^{PR}(i-1)-P^{\tau}(i)T^{PR}(i)+P^{\tau}(i-1)T^{SR}(i-1)-P^{\tau}(i)T^{SR}(i)\\&+P^{\tau}(i-1)T^{+R}(i-1)-P^{\tau}(i)T^{+R}(i)+P^{\tau}(i+1)T^{RP}(i+1)-P^{\tau}(i)T^{RP}(i)\\&+P^{\tau}(i+1)T^{RS}-P^{\tau}(i)T^{RS}(i)+P^{\tau}(i+1)T^{R+}-P^{\tau}(i)T^{R+}(i)\end{aligned}\tag{27}$$

Where i is the number of rock players.

$$\begin{aligned}T(x\pm\frac{1}{N})&=T(x)\pm\frac{1}{N}T'(x)+\frac{1}{2N^2}T''(x)\dots\\P(x\pm\frac{1}{N})&=P(x)\pm\frac{1}{N}P'(x)+\frac{1}{2N^2}P''(x)\dots\end{aligned}$$

$$\begin{aligned}P^{\tau+1}(i)-P^{\tau}(i)&=(P-\frac{1}{N}P'+\frac{1}{2N^2}P'')(T^{PR}-\frac{1}{N}T^{PR'}+\frac{1}{2N^2}T^{PR''})-PT^{PR}\\&+(P-\frac{1}{N}P'+\frac{1}{2N^2}P'')(T^{SR}-\frac{1}{N}T^{SR'}+\frac{1}{2N^2}T^{SR''})-PT^{SR}\\&+(P-\frac{1}{N}P'+\frac{1}{2N^2}P'')(T^{+R}-\frac{1}{N}T^{+R'}+\frac{1}{2N^2}T^{+R''})-PT^{+R}\\&+(P+\frac{1}{N}P'+\frac{1}{2N^2}P'')(T^{RP}+\frac{1}{N}T^{RP'}+\frac{1}{2N^2}T^{RP''})-PT^{RP}\\&+(P+\frac{1}{N}P'+\frac{1}{2N^2}P'')(T^{RS}+\frac{1}{N}T^{RS'}+\frac{1}{2N^2}T^{RS''})-PT^{RS}\\&+(P+\frac{1}{N}P'+\frac{1}{2N^2}P'')(T^{R+}+\frac{1}{N}T^{R+'}+\frac{1}{2N^2}T^{R+''})-PT^{R+}\end{aligned}\tag{28}$$

$$\begin{aligned}P^{\tau+1}(i)-P^{\tau}(i)&=-\frac{1}{N}\Big[P\big(T^{PR'}+T^{SR'}+T^{+R'}-T^{RP'}-T^{RS'}-T^{R+'}\big)\\&\qquad+P'\big(T^{PR}+T^{SR}+T^{+R}-T^{RP}-T^{RS}-T^{R+}\big)\Big]\\&+\frac{1}{2N^2}\Big[P\big(T^{PR''}+T^{SR''}+T^{+R''}+T^{RP''}+T^{RS''}+T^{R+''}\big)\\&\qquad+2P'\big(T^{PR'}+T^{SR'}+T^{+R'}+T^{RP'}+T^{RS'}+T^{R+'}\big)\\&\qquad+P''\big(T^{PR}+T^{SR}+T^{+R}+T^{RP}+T^{RS}+T^{R+}\big)\Big]\\&=\frac{1}{N}\Big[-\big(P\big(T^{PR}+T^{SR}+T^{+R}-T^{RP}-T^{RS}-T^{R+}\big)\big)'\\&+\frac{1}{2}\big(P\big(\frac{T^{PR}+T^{SR}+T^{+R}+T^{RP}+T^{RS}+T^{R+}}{N}\big)\big)''\Big]\end{aligned}\tag{29}$$

We then have for rock (i) $a(i)=T^{PR}+T^{SR}+T^{+R}-T^{RP}-T^{RS}-T^{R+}$ and $b^2(i)=\frac{T^{PR}+T^{SR}+T^{+R}+T^{RP}+T^{RS}+T^{R+}}{N}$.

For paper and scissors we get the same but with the corresponding ingoing and outgoing transition probabilities.

$$\begin{aligned}P^{\tau+1}(j)-P^{\tau}(j)&=\frac{1}{N}\Big[-\big(P\big(T^{RP}+T^{SP}+T^{+P}-T^{PR}-T^{PS}-T^{P+}\big)\big)'\\&+\frac{1}{2}\big(P\big(\frac{T^{RP}+T^{SP}+T^{+P}+T^{PR}+T^{PS}+T^{P+}}{N}\big)\big)''\Big]\end{aligned}\tag{30}$$

$a(j)=T^{RP}+T^{SP}+T^{+P}-T^{PR}-T^{PS}-T^{P+}$ and $b^2(j)=\frac{T^{RP}+T^{SP}+T^{+P}+T^{PR}+T^{PS}+T^{P+}}{N}$.

$$P^{\tau+1}(k) - P^{\tau}(k) = \frac{1}{N}\Big[-\big(P\big(T^{RS} + T^{PS} + T^{+S} - T^{SR} - T^{SP} - T^{S+}\big)\big)' \\ + \frac{1}{2}\big(P\big(\frac{T^{RS} + T^{PS} + T^{+S} + T^{SR} + T^{SP} + T^{S+}}{N}\big)\big)''\Big] \tag{31}$$

$a(k) = T^{RS} + T^{PS} + T^{+S} - T^{SR} - T^{SP} - T^{S+}$ and $b^2(k) = \frac{T^{RS}+T^{PS}+T^{+S}+T^{SR}+T^{SP}+T^{S+}}{N}$.

From these we can form the fokker planck equations for each strategies population ($x \in \{i, j, k\}$):

$$\frac{d}{dt}P^{\tau}(x) = \frac{1}{N}\Big[-\frac{d}{dx}\big(a(x)P^{\tau}(x)\big) + \frac{1}{2}\frac{d^2}{dx^2}\big(b^2(x)P^{\tau}(x)\big)\Big] \tag{32}$$

Langevin equation with noise term $b(x)\xi$:

$$\dot{x} = a(x) + b(x)\xi \tag{33}$$

$a(x)$ is our deterministic replicator equation as $N \to \infty$ noise dissapears ($b^2(x)$ is divided by N).

For the Moran process, the infinite population dynamics reduce to a scaled version of the ordinary replicator equation by the factor $\frac{1}{\Gamma+\langle\pi(x)\rangle}$

$$a(x) = \frac{1}{\Gamma + \langle\pi(x)\rangle} x_i(\pi_i(x) - \langle\pi(x)\rangle), \Gamma = \frac{1-w}{w} \tag{34}$$

$$\begin{aligned}\langle\pi(x)\rangle = {} & x\left(ax + bz + cy + \gamma\left(-x - y - z + 1\right)\right) \\ & + y\left(ay + bx + cz + \gamma\left(-x - y - z + 1\right)\right) \\ & + z\left(az + by + cx + \gamma\left(-x - y - z + 1\right)\right) \\ & + \left(x\left(a + \beta\right) + y\left(a + \beta\right) + z\left(a + \beta\right)\right)\left(-x - y - z + 1\right)\end{aligned}$$

The RHS of the equation corresponding to the un-scaled replicator dynamics is the following.

$$\frac{\begin{aligned}\Big[x\big(&2awxy + 2awxz + 2awyz - awy - awz\\ &- bwxy - bwxz - bwyz + bwz\\ &+ \beta wx^2 + 2\beta wxy + 2\beta wxz - \beta wx\\ &+ \beta wy^2 + 2\beta wyz - \beta wy\\ &+ \beta wz^2 - \beta wz\\ &- cwxy - cwxz - cwyz + cwy\\ &+ \gamma wx^2 + 2\gamma wxy + 2\gamma wxz - 2\gamma wx\\ &+ \gamma wy^2 + 2\gamma wyz - 2\gamma wy\\ &+ \gamma wz^2 - 2\gamma wz + \gamma w\big)\Big]\end{aligned}}{\begin{aligned}\Big[&- 2awxy - 2awxz + awx - 2awyz + awy + awz\\ &+ bwxy + bwxz + bwyz\\ &- \beta wx^2 - 2\beta wxy - 2\beta wxz + \beta wx\\ &- \beta wy^2 - 2\beta wyz + \beta wy\\ &- \beta wz^2 + \beta wz\\ &+ cwxy + cwxz + cwyz\\ &- \gamma wx^2 - 2\gamma wxy - 2\gamma wxz + \gamma wx\\ &- \gamma wy^2 - 2\gamma wyz + \gamma wy\\ &- \gamma wz^2 + \gamma wz - w + 1\Big]\end{aligned}} \tag{35}$$

Next to do - look at $b^2(x)$ and see how it is different between the Moran and local update, focus on whether or not N is present as it can tell us about drift reversal. If N is not present then the drift does not depend on pop size and therfore no reversal. If there is an N then drift is possible.

3.6 Constant of motion ΔH definition (drift)

3.6.1 $\langle \Delta H_{SD} \rangle$ Derivation

The equation $H = -q(1-q)$ defines a constant of motion for the SD part of the game, where q is the fraction of players in the 4th strategy. Using the transition probabilities of the different process we can derive an expression for the expected change in H within the simplex.

Where $i, j, k, N-i-j-k$ are the players playing R, P, S, and the 4th strategy respectively.

$\Delta H = H(t+1) - H(t)$,
$\Delta H = -x_{t+1}(1-x_{t+1}) - (-x_t(1-x_t))$
$\Delta H = -x_{t+1}(1-x_{t+1}) + x_t(1-x_t)$
$\Delta H = x_t(1-x_t) - x_{t+1}(1-x_{t+1})$

$\langle \Delta H \rangle = \sum_{i,j,k} \Big(H_s - H_{s'}\Big) T^{s\to s'}$, s is a particular state in the simplex.

$$
\begin{aligned}
\langle \Delta H_{SD} \rangle = \text{scaling ?} \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} \Big[&(N-i-j-k)(1-N+i+j+k)(T^{R+}+T^{P+}+T^{S+}+T^{+R}+T^{+P}+T^{+S}) \\
&-(N-i-j-k+1)(-N+i+j+k)T^{R+} \\
&-(N-i-j-k+1)(-N+i+j+k)T^{P+} \\
&-(N-i-j-k-1)(2-N+i+j+k)T^{+R} \\
&-(N-i-j-k-1)(2-N+i+j+k)T^{+P} \\
&-(N-i-j-k-1)(2-N+i+j+k)T^{+S}\Big]
\end{aligned} \tag{36}
$$

The terms with transitions within the RPS simplex can be ignored as q would not change between these states, therefore the term $H_s - H_{s'} = 0$

$p = N - i - j - k$, the number of players playing the 4th strategy.

$$
\begin{aligned}
\langle \Delta H_{SD} \rangle = \text{scaling ?} \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} \Big[&p(1-p)(T^{R+}+T^{P+}+T^{S+}+T^{+R}+T^{+P}+T^{+S}) \\
&-(p+1)(-p)(T^{R+}+T^{P+}+T^{S+}) \\
&-(p-1)(2-p)(T^{+R}+T^{+P}+T^{+S})\Big]
\end{aligned} \tag{37}
$$

The continuous limit, where $x = i/N$, $y = j/N$, $z = k/N$, and $q = p/N$ and $q = 1 - x - y - z$ leads to:

$$
\begin{aligned}
\langle \Delta H_{SD} \rangle = \text{scaling?} \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \Big[&q(1-q)(T^{R+}+T^{P+}+T^{S+}+T^{+R}+T^{+P}+T^{+S}) \\
-(q+\frac{1}{N})(1-q-\frac{1}{N})(T^{R+}+T^{P+}+T^{S+}) &- (q-\frac{1}{N})(1-q+\frac{1}{N})(T^{+R}+T^{+P}+T^{+S})\Big]
\end{aligned} \tag{38}
$$

This can then be solved numerically and the critical population values can be found where $\langle \Delta H_{SD} \rangle = 0$.

Moran process equation (needs to be finalised with correct scale factor):

$$
\begin{aligned}
\langle \Delta H_{SD} \rangle_{MO} = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \Bigg[\\
\Big(&(N^2(x+y+z)(x+y+z-1)\big(x(w(ax+bz+cy-\gamma(x+y+z-1))-w+1)+x(w(a+\beta)(x+y+z)-w+1) \\
&+y(w(ay+bx+cz-\gamma(x+y+z-1))-w+1)+y(w(a+\beta)(x+y+z)-w+1) \\
&+z(w(az+by+cx-\gamma(x+y+z-1))-w+1)+z(w(a+\beta)(x+y+z)-w+1)\big) \\
&-(N(x+y+z)-1)(N(x+y+z-1)-1)(x+y+z)(w(a+\beta)(x+y+z)-w+1) \\
&-(N(x+y+z)+1)(N(x+y+z-1)+1)\big(x(w(ax+bz+cy-\gamma(x+y+z-1))-w+1) \\
&+y(w(ay+bx+cz-\gamma(x+y+z-1))-w+1)+z(w(az+by+cx-\gamma(x+y+z-1))-w+1)\big)\big)(x+y+z-1)\Big) \\
\Big/ \Big(&N^4\big(w\big(x(ax+bz+cy-\gamma(x+y+z-1))+y(ay+bx+cz-\gamma(x+y+z-1)) \\
&+z(az+by+cx-\gamma(x+y+z-1))-(a+\beta)(x+y+z)(x+y+z-1)\big)-w+1\big)\Big)\Bigg]
\end{aligned} \tag{39}
$$

With $w = 0$, this reduces to:

$$
\langle \Delta H_{SD} \rangle_{MO} = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \frac{-2\left((x+y+z-1)(x+y+z)\right)}{N^4} \tag{40}
$$

Critical N found matches nicely with the simulated versions. Numerical integration in python code ./augRps.py, shows change of sign as expected. Matches nicely with the approximated values for the Moran process. The specific expression for Moran process is very long. Computed numerically and solved with scipy.integrate (reference scipy) Maybe can plot the simulated critical population sizes against the analyical on the same graph.

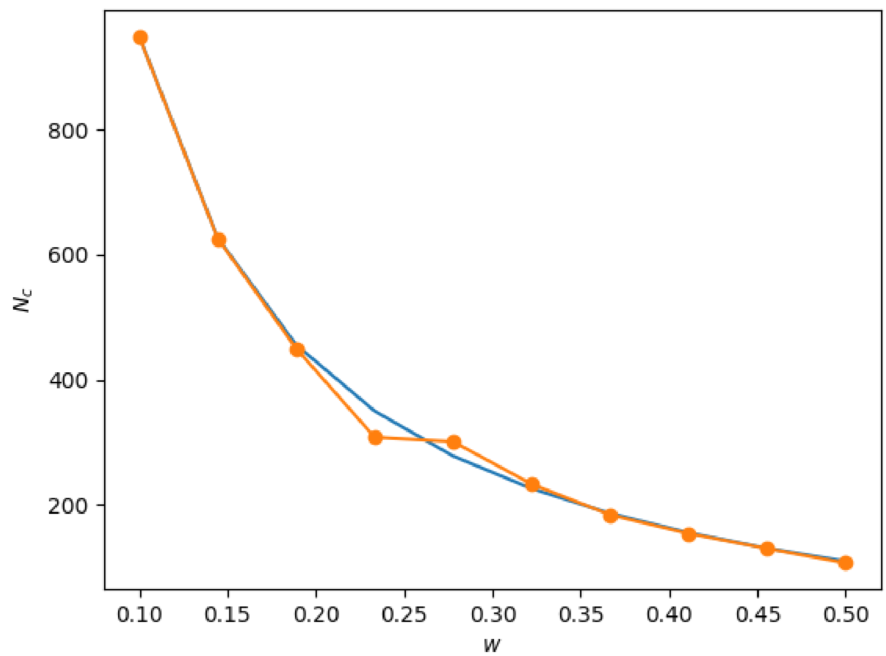


Figure 4: Comparison of simulated critical population sizes against the derived expected change in H for Moran process. Small deviations due to stochasticity in the simulations. $\gamma = 0.2, \beta = 0.1$ standard RPS.

3.6.2 $\langle \Delta H_{RPS} \rangle$ Derivation

$\langle \Delta H \rangle$ within the RPS plane $H = -xyz$.

$$\begin{aligned}\langle \Delta H_{RPS} \rangle = \text{scaling ?} \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} \Big[& ijk(T^{RP} + T^{RS} + T^{R+} + T^{PR} + T^{PS} + T^{P+} \\ & + T^{SR} + T^{SP} + T^{S+} + T^{+R} + T^{+P} + T^{+S}) \\ & - k(i-1)(j+1)T^{RP} - (i-1)j(k+1)T^{RS} - jk(i-1)T^{R+} \\ & - k(i+1)(j-1)T^{PR} - i(j-1)(k+1)T^{PS} - ik(j-1)T^{P+} \\ & - (i+1)j(k-1)T^{SR} - i(j+1)(k-1)T^{SP} - ij(k-1)T^{S+} \\ & - jk(i+1)T^{+R} - ik(j+1)T^{+P} - ij(k+1)T^{+S} \Big]\end{aligned} \tag{41}$$

$$
\begin{aligned}
\langle \Delta H_{RPS} \rangle = \text{scaling?} \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \Big[& xyz(T^{RP} + T^{RS} + T^{R+} + T^{PR} + T^{PS} + T^{P+} \\
& + T^{SR} + T^{SP} + T^{S+} + T^{+R} + T^{+P} + T^{+S}) \\
& - z(x - \frac{1}{N})(y + \frac{1}{N})T^{RP} - (x - \frac{1}{N})y(z + \frac{1}{N})T^{RS} - yz(x - \frac{1}{N})T^{R+} \\
& - z(x + \frac{1}{N})(y - \frac{1}{N})T^{PR} - x(y - \frac{1}{N})(z + \frac{1}{N})T^{PS} - xz(y - \frac{1}{N})T^{P+} \\
& - (x + \frac{1}{N})y(z - \frac{1}{N})T^{SR} - x(y + \frac{1}{N})(z - \frac{1}{N})T^{SP} - xy(z - \frac{1}{N})T^{S+} \\
& - yz(x + \frac{1}{N})T^{+R} - xz(y + \frac{1}{N})T^{+P} - xy(z + \frac{1}{N})T^{+S} \Big]
\end{aligned}
\tag{42}
$$

Rough figures of rps and SD delta H values.

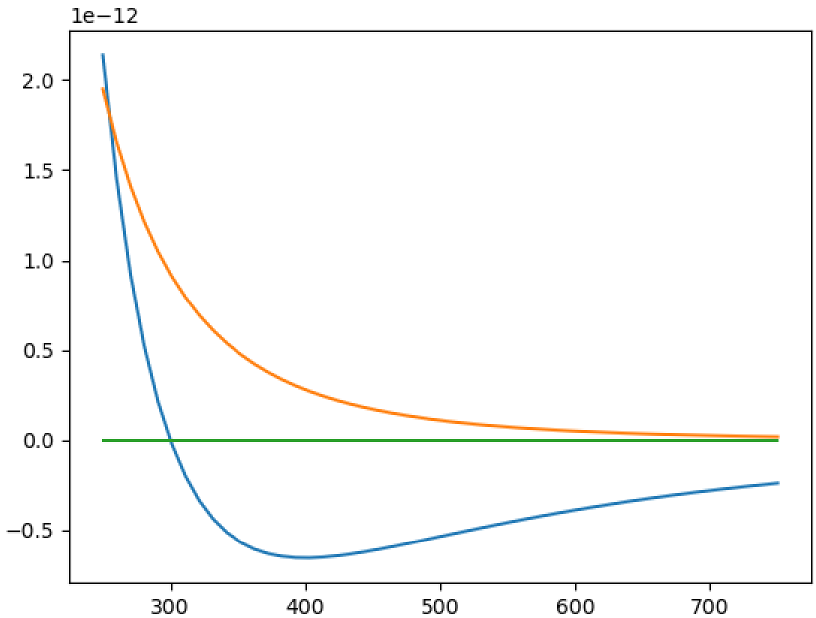


Figure 5: Blue - SD

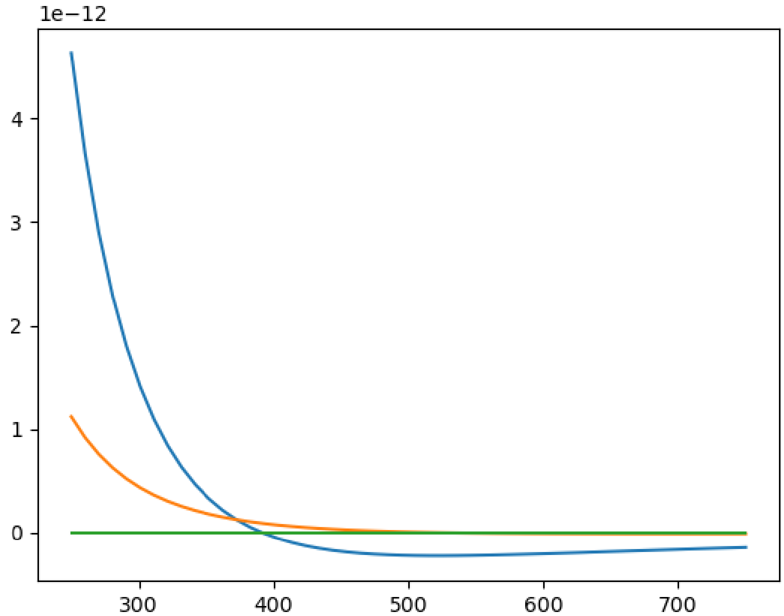


Figure 6: Double reversal case. Blue - SD

3.6.3 $\langle \Delta H_4 \rangle$ Derivation

All 4 strategies, $H = -xyz(1 - x - y - z)$ As above.

4 Drift

$$\begin{bmatrix} 0 & -s & 1 & 0.2 \\ 1 & 0 & -s & 0.2 \\ -s & 1 & 0 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0 \end{bmatrix} \tag{43}$$

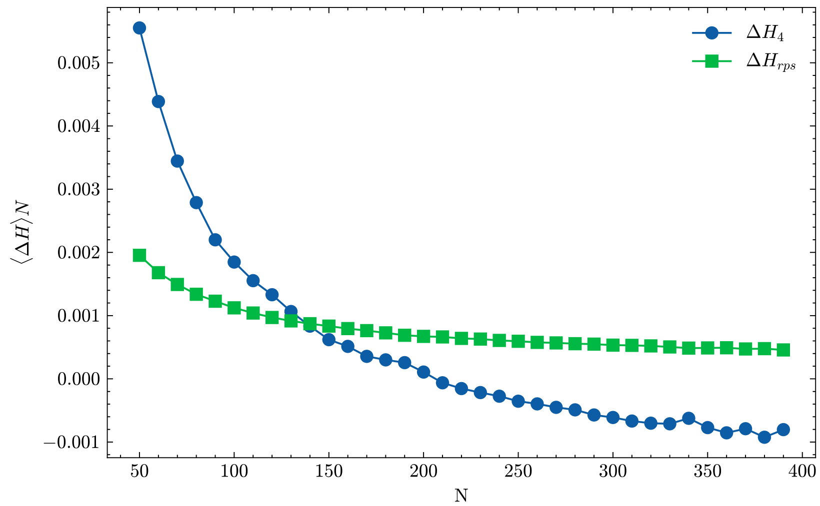


Figure 7: Drift reversal in the above game 43, in the 4th strategy $\langle \Delta H \rangle$ changes sign. Averages over 10,000,000 realisations, fixed $w = 0.45$, ΔH_{rps} approaches 0 but does not reverse, $s = 1$

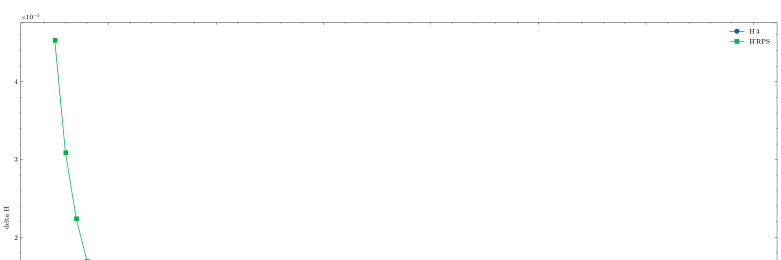


Figure 8: Drift reversal in the standard RPS game - 0 playing 4th strategy, $s = 0.8$, deterministic case the internal fixed point is stable ($\langle \Delta H_{rps} \rangle > 0$) but again it changes sign at a small population size, average over 10,000,000 realisations. Change of sign at $N \approx 470$

$$\begin{bmatrix} 0 & -s & 1 & 0.5 \\ 1 & 0 & -s & 0.5 \\ -s & 1 & 0 & 0.5 \\ 0+\beta & 0+\beta & 0+\beta & 0 \end{bmatrix} \tag{44}$$

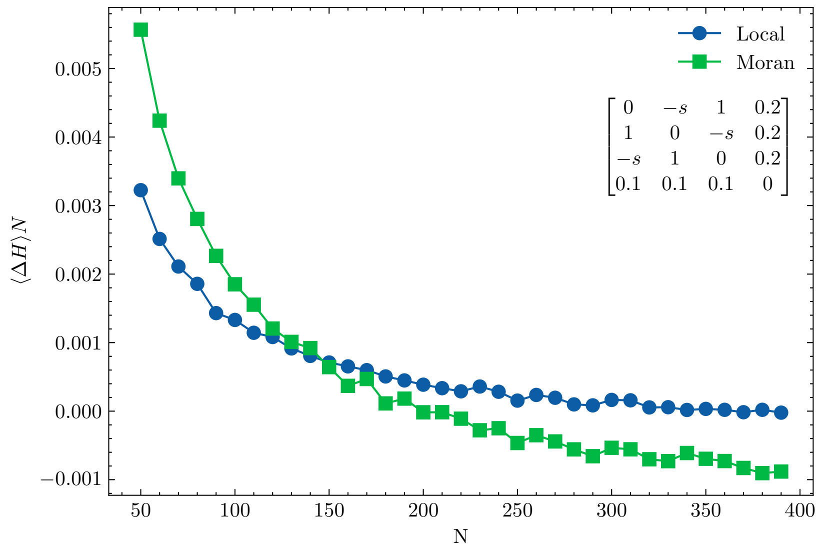


Figure 9: Moran and local update drift for ΔH_4 - 2×10^7 realizations , needs more average but drift reversal by N = 400, with w = 0.45

$$\begin{bmatrix} 0 & -s & 1 & \gamma \\ 1 & 0 & -s & \gamma \\ -s & 1 & 0 & \gamma \\ 0+\beta & 0+\beta & 0+\beta & 0 \end{bmatrix} \tag{45}$$

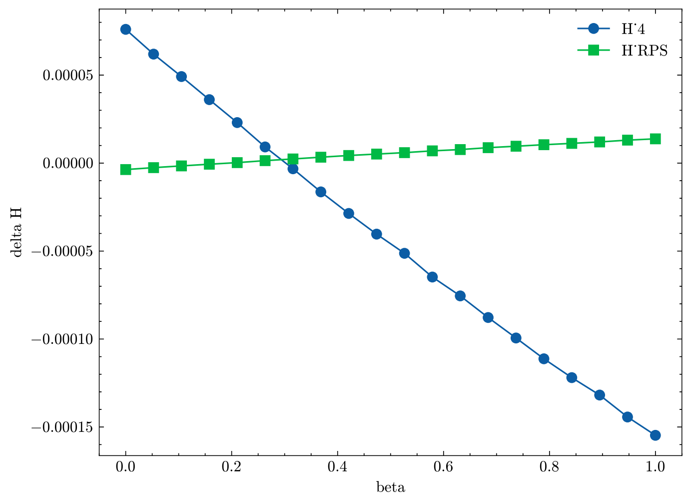


Figure 10: $\langle \Delta H_4 \rangle$ and $\langle \Delta H_{rps} \rangle$ for range of β values (0 to 1) in above matrix, fixed value of $\gamma = 0.5, w = 0.45, N = 200$

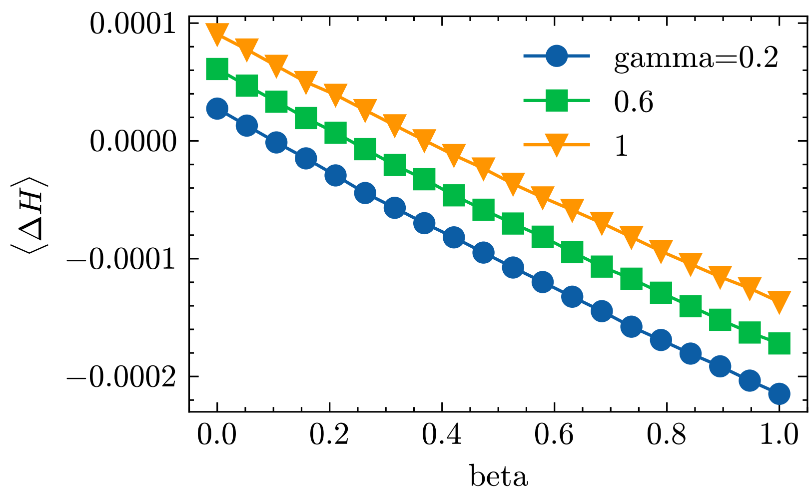


Figure 11: $\langle \Delta H_4 \rangle$ for different β, for 3 different γ

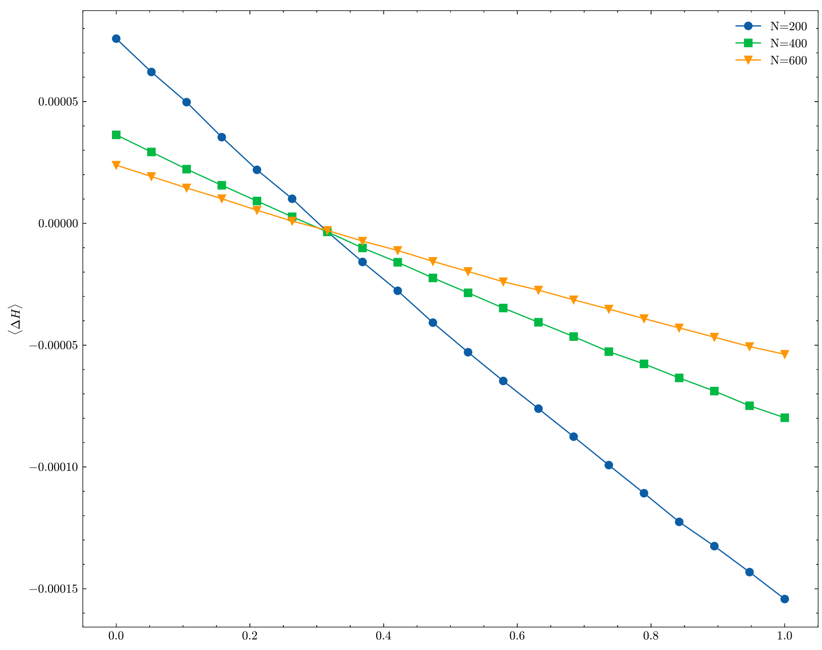


Figure 12: $\langle \Delta H_4 \rangle$ for different β at different pop sizes.

4.1 Critical population size N_c

The critical population size where drift occurs for a varying selection pressure w.
Implemented using the previous simulation code at suitably large simulation repetitions, binary search around the change of sign for the ΔH_4 value and repeated for range of w's.

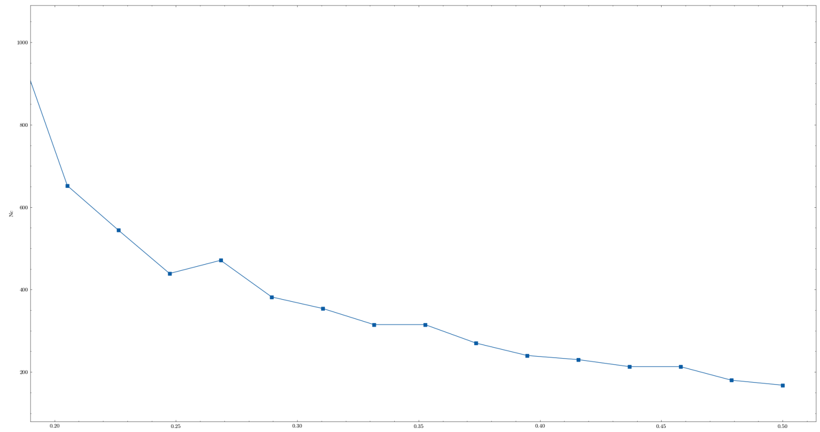


Figure 13: Critical N against W, need to average this with more realizations.

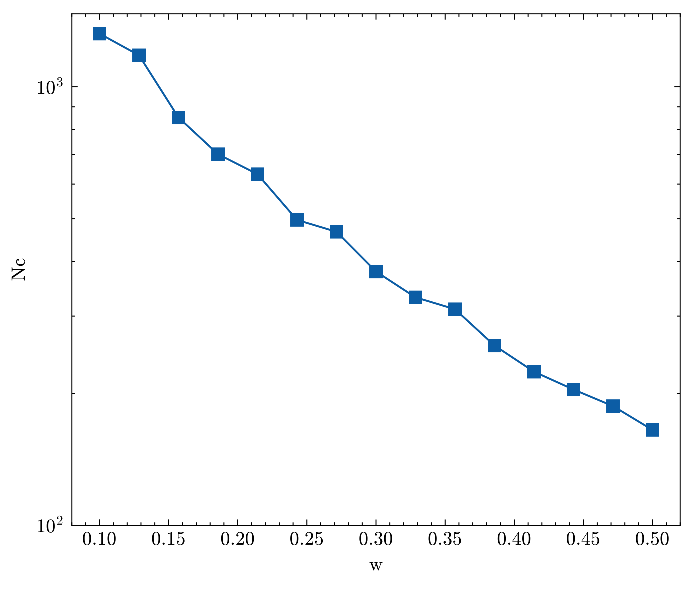


Figure 14: 5×10^7 realizations, critical N for drift reversal in 43

x axis label incorrect below

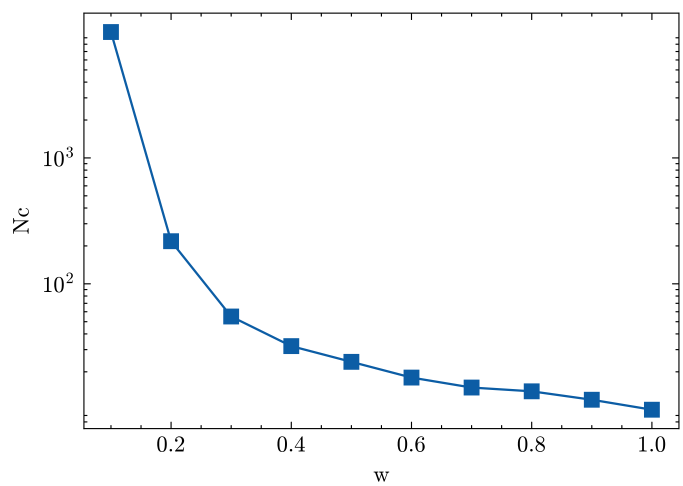


Figure 15: Critical population size N_c for range of β from 0 to 1, fixed $w = 0.45$, fixed $\gamma = 0.5$

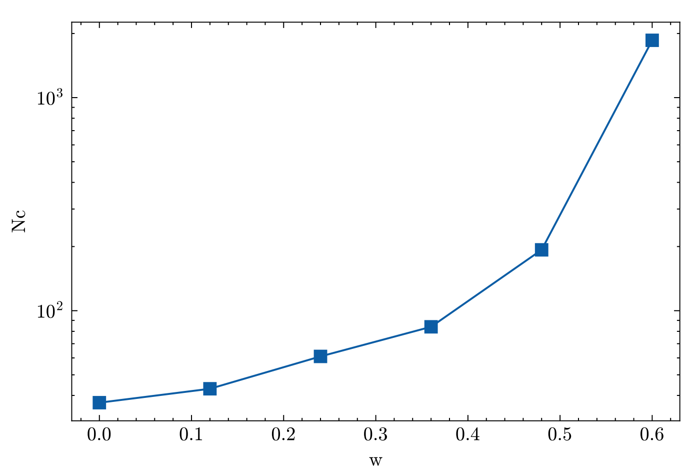
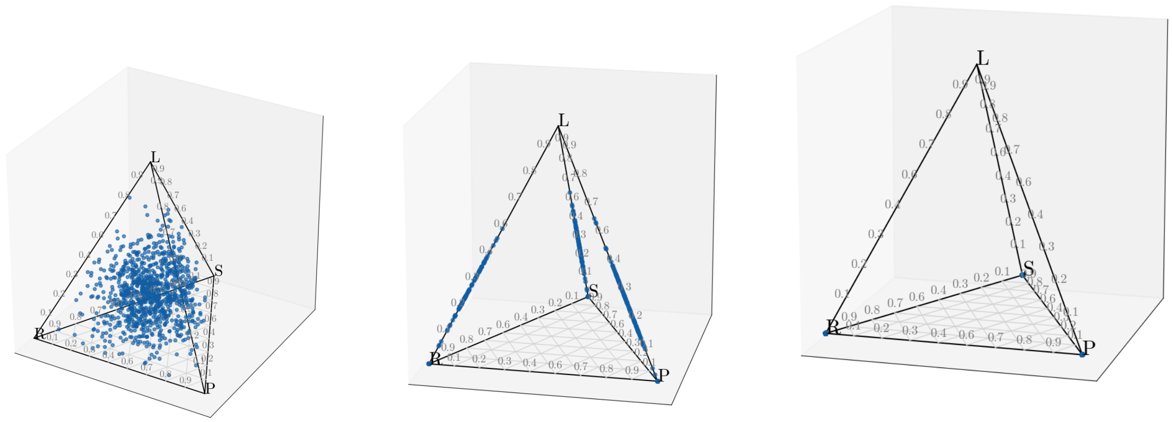


Figure 16: Critical population size N_c for range of γ from 0 to 1, fixed $w = 0.45$, fixed $\beta = 0.2$



(a) Initial

(b) N = 200, rps drift outwards

(c) N = 100, rps drift out and strategy 4 drift down

Figure 17: Initial random starting points, at low population we seee RPS drift occuring first, and the game is still at SD equilibrium, but at lower strategy e.g $N < 150$ as in the drift graph further up we seee reversal of both strategies to the bottom 3 corners, when $c > b$, or to all 4 corners when $b = c$ ($b = \beta, c = \gamma$ in the matrices in the document), similar results for both $s = 1, s = 0.8$ in the small population cases and both time RPS drifst out before SD and SD drifts at low population - only difference in large population case

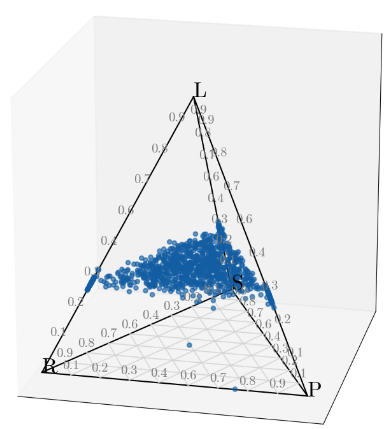


Figure 18: Large pop - fixes in orbit about center, s = 1 - very large population for neutrally stable RPS

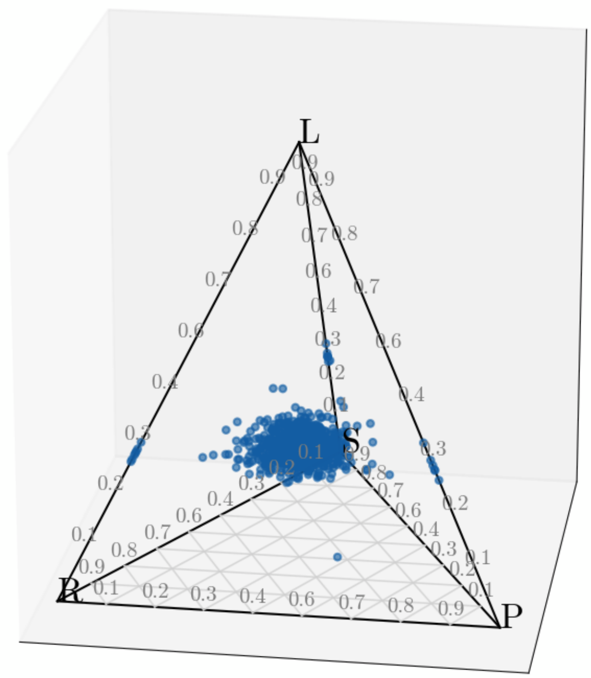


Figure 19: large pop, s = 0.8, converges to central fixed point. - large population for asymptotically stable RPS - $s < 1$

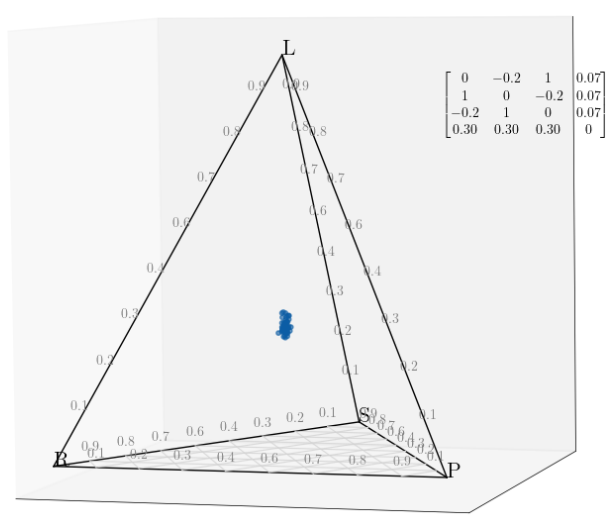
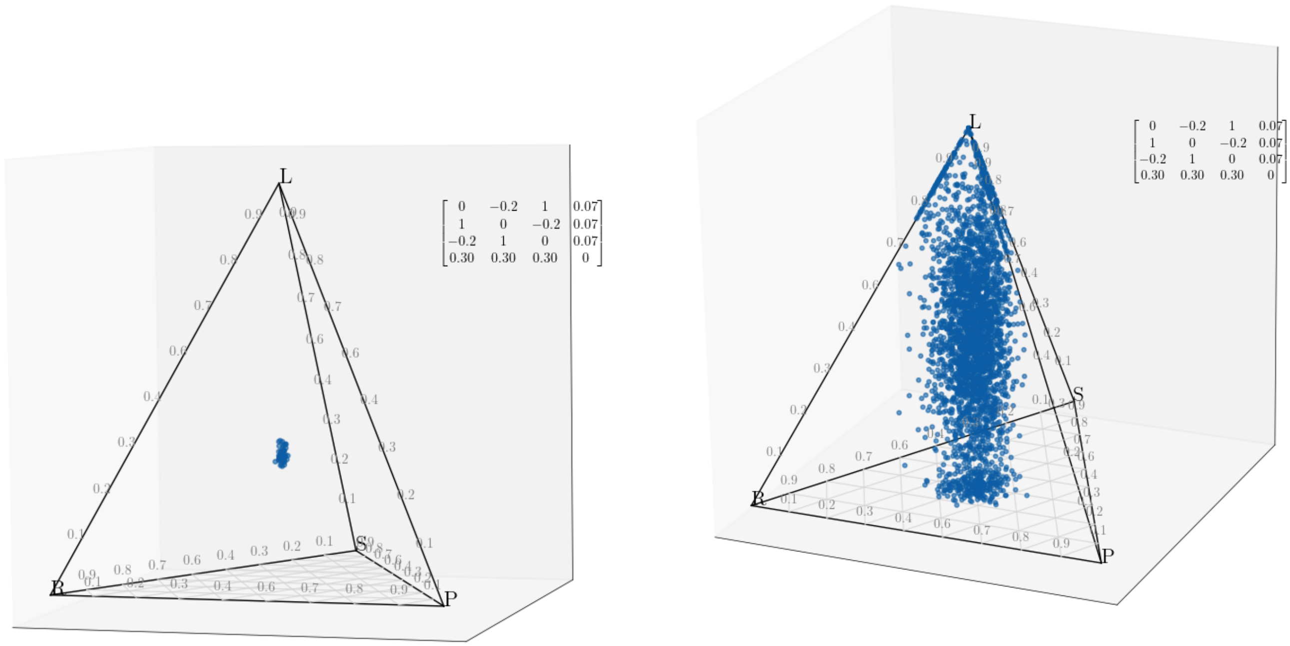


Figure 20: large pop, matrix in diagram



(a) $N = 50000$ (b) $N = 850$

Figure 21: $w = 0.45$, Strong attracting RPS with matrix in figure, reversal seen in SD but not in RPS, large population size tends to the deterministic case. - generate a $\Delta H_4, \Delta H_{RPS}$ plot for the above to show the drift reversal in SD game and not in RPS.

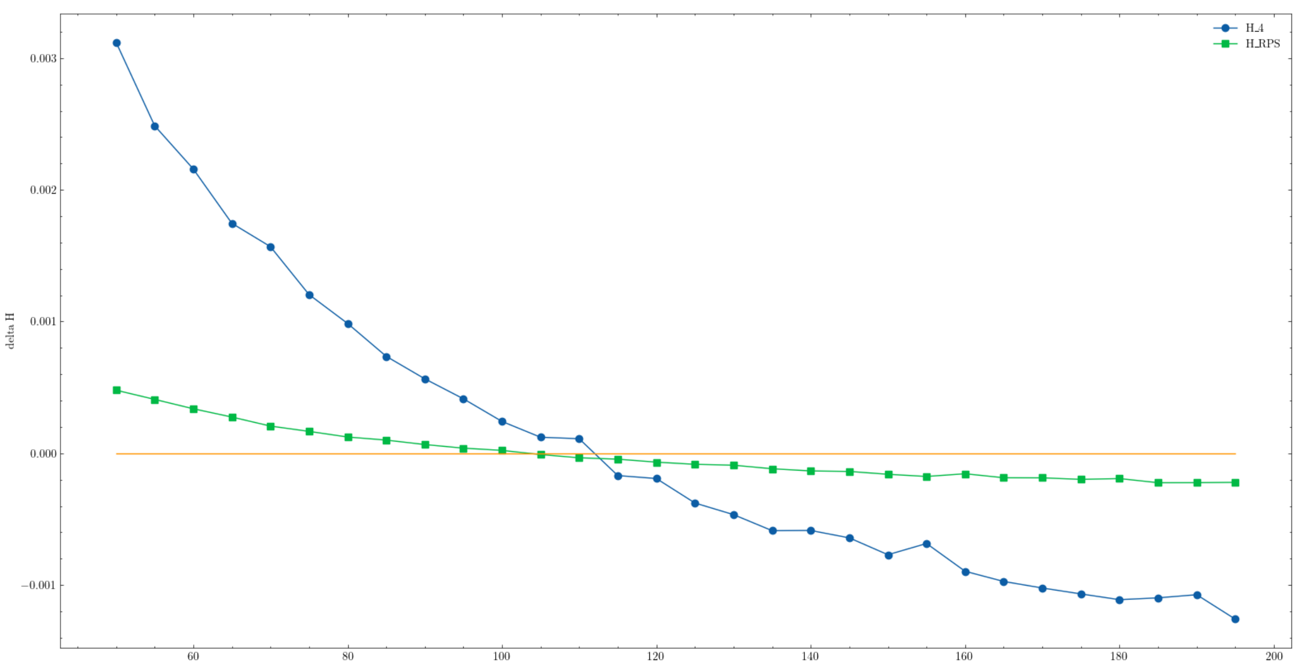


Figure 22: Same game used for above cloud - RPS crossing the x axis at a lower N_c - explains SD drift first

5 Refs

Scienceplots library for simple styling ref. [1]
Numba jit python compiler - used for large optimisations pre compiling intensive functions to fast machine code. [2]
Huge performance increase using numba in no-python mode compiling all the critical simulation methods into machine code and inlining them where appropriate to reduce function call overhead. Reached feasible simulation speed.
Drift reversal paper [3]

References

[1] John D. Garrett. garrettj403/SciencePlots. September 2021.

[2] Siu Kwan Lam, Antoine Pitrou, and Stanley Seibert. Numba: A llvm-based python jit compiler. In *Proceedings of the Second Workshop on the LLVM Compiler Infrastructure in HPC*, pages 1–6, 2015.

[3] Arne Traulsen, Jens Christian Claussen, and Christoph Hauert. Coevolutionary dynamics: From finite to infinite populations. *Phys. Rev. Lett.*, 95:238701, Dec 2005.