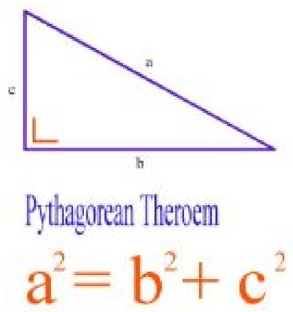
TRIGONOMETRY

The word "Trigonometry" is derived from the Greek word "Tri-Gonia-Metron" where Tri means three, Gonia means angles and Metron mean measure. So, trigonometry is a branch of mathematics which concerned with the measurement of sides, angles and their relation to a triangle.

PYTHAGORAS THEOREM



The relationship between the three sides of a triangle is simply known as Pythagoras Theorem. The relation was given by the popular Mathematician Pythagoras which is called Pythagoras theorem.
In mathematics, the Pythagorean theorem, also known as Pythagoras's theorem, is a relation in Euclidean geometry among the three sides of a right triangle. It states that the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides. According to this theorem "In any right-angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas of squares of perpendicular and base".

By Pythagoras Theorem

Hypotenuse (h^2) = Perpendicular (p^2) + Base (b^2)
or, $h^2 = p^2 + b^2$
From this theory we can derive,
h = $\sqrt{p^2+b^2}$
p = $\sqrt{h^2-b^2}$
b = $\sqrt{h^2-p^2}$

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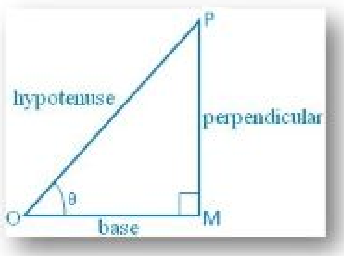
TRIGONOMETRICAL RATIOS



There are six trigonometric ratios which relate the sides of a right triangle to its angles. Trigonometric ratios are generally used to calculate the unknown lengths and angles in a right triangle. The six trigonometric ratios are tabulated below: -

1) Sinθ $\frac{perpendicuar}{hypotenuse}$ $(\frac{p}{h})$
2) Cosθ $\frac{base}{hypotenuse}$ $(\frac{b}{h})$
3) Tanθ $\frac{perpendicuar}{base}$ $(\frac{p}{b})$
4) Cosecθ $\frac{hypotenuse}{perpendicuar}$ $(\frac{h}{p})$
5) Secθ $\frac{hypotenuse}{base}$ $(\frac{h}{b})$
6) Cotθ $\frac{base}{perpendicuar}$ $(\frac{b}{p})$

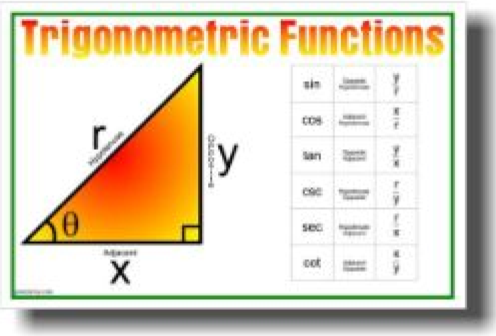
NOTE: - Since, Sinθ ≠ Sin×θ, Cosθ ≠ Cos×θ, etc.

Operation of Trigonometric Ratios

As we know that the trigonometric ratios are numbers like fraction, decimal or whole numbers. We can operate the operations of addition, subtraction, multiplication and division on them that we do in algebra.

Operations	In algebra	In Trigonometry
Addition	$2a + 4a = 6a$ $4x^2 + 7x^2 = 11x^2$	$2\sin\theta + 4\sin\theta = 6\sin\theta$ $4\cos^2\theta + 7\cos^2\theta = 11\cos^2\theta$
Subtraction	$11z - 6z = 5z$ $8x^2 - 5x^2 = 3x^2$	$11\tan\theta - 6\tan\theta = 5\tan\theta$ $8\sec^2\theta - 5\sec^2\theta = 3\sec^2\theta$
Multiplication	$7a \times 4a = 28a^2$ $5y^2 \times 3y^2 = 15y^4$	$7\sin\theta \times 4\sin\theta = 28\sin^2\theta$ $5\tan^2\theta \times 4\tan^2\theta = 20\tan^4\theta$
Division	$\frac{q^5}{q} = q^4$ $10b \div 5b = 2$	$\frac{\cos\theta^5}{\cos\theta} = \cos\theta^4$ $10\sec\theta \div 5\sec\theta = 2$

Trigonometric Expressions



Trigonometric Expressions are the expressions in which variables are written under the signs of trigonometric functions.

Simplification and factorisation

We can simplify or factorise the trigonometric expressions as in algebra. For examples,

1. $(a+b)(a-b) = a^2 - b^2$
   $(\text{Sin}A + \text{Cos}A)(\text{Sin}A - \text{Cos}A) = \text{Sin}^2A - \text{Cos}^2A$
2. $(a+b)^2 = a^2 + 2ab + b^2$
   $(\text{Sin}A + \text{Cos}A)^2 = \text{Sin}^2A + 2\text{Sin}A.\text{Cos}A + \text{Cos}^2A$
3. $a^3+b^3 = (a+b)(a^2-ab+b^2)$
   $\text{Sin}^3A + \text{Cos}^3A = (\text{Sin}A + \text{Cos}A)(\text{Sin}^2A - 2\text{Sin}A.\text{Cos}A + \text{Cos}^2A)$
4. $a^3-b^3 = (a+b)(a^2+ab+b^2)$
   $\text{Sin}^3A - \text{Cos}^3A = (\text{Sin}A - \text{Cos}A)(\text{Sin}^2A + \text{Sin}A.\text{Cos}A + \text{Cos}^2A)$

Relation between the Trigonometric Ratios of an Angle

Simply there are four fundamental relations of trigonometric ratios. They are,

1. Reciprocal Relation
2. Quotient Relation
3. Pythagoras Relation
4. Derived Relation

Reciprocal Relation

Reciprocal Identities

$\sin u = \frac{1}{\csc u}$ $\cos u = \frac{1}{\sec u}$ $\tan u = \frac{1}{\cot u}$

$\csc u = \frac{1}{\sin u}$ $\sec u = \frac{1}{\cos u}$ $\cot u = \frac{1}{\tan u}$

Pythagorean Identities

$\sin^2 u + \cos^2 u = 1$ $1 + \tan^2 u = \sec^2 u$ $1 + \cot^2 u = \csc^2 u$

Quotient Identities

$\tan u = \frac{\sin u}{\cos u}$ $\cot u = \frac{\cos u}{\sin u}$

Reciprocal relations of trigonometric ratios are explained to represent the relationship between the three pairs of trigonometric ratios as well as their reciprocals. The reciprocals relations are given below: -

Sinθ = $\frac{p}{h}$ and Cosecθ = $\frac{h}{p}$
Then, sinθ × cosecθ = $\frac{p}{h} \times \frac{h}{p}$ = 1

∴ Sinθ × cosecθ = 1

sinθ = $\frac{1}{cosec\theta}$, cosecθ = $\frac{1}{sin\theta}$
Also, cosθ = $\frac{b}{h}$ and secθ = $\frac{h}{b}$
Then, cosθ × secθ = $\frac{b}{h} \times \frac{h}{b}$ = 1

∴ cosθ × secθ = 1

cosθ = $\frac{1}{sec\theta}$, secθ $\frac{1}{cos\theta}$
And, tanθ = $\frac{p}{b}$, cotθ = $\frac{b}{p}$
Then tanθ × cotθ = 1

∴ tanθ × cotθ = 1

tanθ = $\frac{1}{cot\theta}$, cotθ = $\frac{1}{tan\theta}$

Quotient Relation

We have, sinθ = $\frac{p}{h}$, cosθ = $\frac{b}{h}$
Then, $\frac{sin\theta}{cos\theta} = \frac{p}{h} \times \frac{h}{b} = \frac{p}{b}$
We have tanθ = $\frac{p}{b}$
∴ tanθ = $\frac{sin\theta}{cos\theta}$
Similarly,
cotθ = $\frac{cos\theta}{sin\theta}$

Pythagoras Relation

From the right angled triangle ABC,

$CA^2 = AB^2 + BC^2$

or, $h^2 = p^2 + b^2$

Dividing both sides

or, $\frac{h^2}{h^2} = \frac{p^2+b^2}{h^2}$

or, $1 = \frac{p^2}{h^2} + \frac{b^2}{h^2}$

or, $\frac{p^2}{h^2} + \frac{b^2}{h^2} = 1$

or, $(\frac{p^2}{h^2}) + (\frac{b^2}{h^2}) = 1$

or, $(\sin\theta)^2 + (\cos\theta)^2 = 1$

$\therefore \sin^2\theta = 1 - \cos^2\theta$

$\cos^2\theta = 1 - \sin^2\theta$

Also,

$\sin\theta = \sqrt{1 - cos^2\theta}$

or, $\cos\theta = \sqrt{1 - sin^2\theta}$

Again,

$p^2 + b^2 = h^2$

or, $h^2 - p^2 = b^2$

Dividing both sides by b^2

or, $\frac{h^2}{b2} - \frac{p^2}{b^2} = \frac{b^2}{b^2}$

or, $(\frac{h}{b})^2 + (\frac{p}{b})^2 = 1$

or, $(\sec\theta)^2 - (\tan\theta)^2 = 1$

or, $\sec^2\theta - \tan^2\theta = 1$

or, $\sec^2\theta = 1 + \tan^2\theta$

or, $\tan^2\theta = \sec^2\theta - 1$

Also,

$\sec\theta = \sqrt{1 + tan^2\theta}$

or, $\tan\theta = \sqrt{sec^2\theta - 1}$

Again,

$p^2 + b^2 = h^2$

or, $h^2 - b^2 = p^2$

Dividing both sides by p^2

$\frac{h^2}{p2} - \frac{b^2}{p^2} = \frac{p^2}{p^2}$

or, $(\text{cosec}\,\theta)^2 - (\cot\theta)^2 = 1$

or, $\text{cosec}^2\theta - \cot^2\theta = 1$

or, $\text{cosec}^2\theta = 1 + \cot^2\theta$

or, $\cot^2\theta = \text{cosec}^2\theta - 1$

Also, $\cot\theta = \sqrt{cosec^2\theta - 1}$

or, $\text{cosec}\,\theta = \sqrt{1 + cot^2\theta}$

Methods of proving Trigonometric Identities

To prove trigonometric identities, we may follow any one of the following: -

- Begin from the left-hand side (L.H.S.) and deduct it to the right-hand side (R.H.S.), if L.H.S. is more complex.
- Begin from R.H.S. and deduct it to L.H.S., if L.H.S. is complex.
- Reduce both the L.H.S. and the R.H.S. to the same expression if both the expression are complex.
- By transposition or cross multiplication, change the identity into the appropriate form. Then show that the new L.H.S. = the new R.H.S.

Conversion of a Trigonometric Ratios

With the use of two methods, we can express the trigonometric ratios in terms of other ratios of the same angle. The two methods are as followings: -

- Using trigonometric relations i.e Trigonometric formulae.
- Using Pythagoras theorem.

NOTE: - We can also find remaining ratios if the value of any trigonometric ratio of an angle is given.

Trigonometric Ratios of some angles

As trigonometry have different angles, for those different angles trigonometric ratios also have different values. In the trigonometry the angles 0°, 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°, 210°, 225°, 240°, 270°, 300°, 315°, 330°, 360° are taken as standard angles.
We can find the trigonometric ratios of 0° and 90°. The following table shows the values of trigonometric ratios of the angles 0°, 30°, 45°, 60° and 90°.

Angles→↓	0°	30°	45°	60°	90°
sinθ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosθ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tanθ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cosecθ	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
secθ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
cotθ	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Procedures to find the trigonometric ratios

The standard angles i.e 0°, 30°, 45°, 60°, 90° in prompt way is given below: -

Process 1: -
Put the number from 0 to 4 as follows: -

0°	30°	45°	60°	90°
0	1	2	3	4

Process 2: -

Divide each number by 4

0°	30°	45°	60°	90°
$\frac{0}{4}$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$

Process 3: -
Take square root of all the numbers

0°	30°	45°	60°	90°
$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{4}{4}}$

Process 4: -
The values which are obtained at first are the values of sine of the standard angles.

Angles	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

Process 5: -
Then reversing the order, the value of cosine of theb standard angles are obtained.

Angles	0°	30°	45°	60°	90°
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

Process: -
After dividing the value of sine by the value of cosine, the values of tangent can be obtained: -

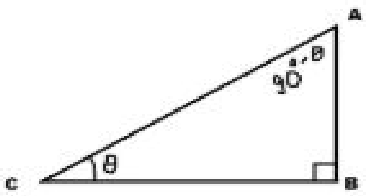
Angle	0°	30°	45°	60°	90°
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

NOTE: - The reciprocal of sine, cosine and tangent are the values of cosecant, secant and cotangent respectively.

Complementary Angles

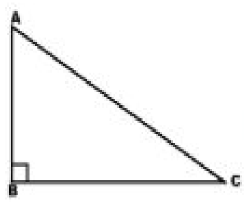
Two angles are called complementary when those angles are added up to 90° or the sum of two angles is 90°.
For example,
60° and 30° are complementary angles.
5° and 85° are complementary angles.
30° and 60°, 50° and 40°, (90° - θ) and θ are complementary angles.

Trigonometric Ratios of Complementary Angles



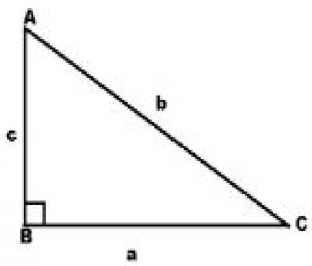
Here ABC is a right angle triangle right angled at B.
Let, $\angle ABC = \theta$, then $\angle CAB = 90° - \theta$
Now taking θ as an angle of reference,
AB = Perpendicular (p)
BC = Base (b)
CA = Hyptenuse (h)
$\sin\theta = \frac{p}{h} = \frac{AB}{CA}$ ....(i)
$\cos\theta = \frac{b}{h} = \frac{BC}{CA}$ ....(ii)
$\tan\theta = \frac{p}{b} = \frac{AB}{BC}$ ....(iii)
$\cot\theta = \frac{b}{p} = \frac{BC}{AB}$ ....(iv)
Taking $(90° - \theta)$ as an angle of reference,
BC = Perpendicular (p)
AB = Base (b)
CA = Hypotenuse (h)
Now,
$\sin\theta(90° - \theta) = \frac{p}{h} = \frac{BC}{CA} = \cos\theta$
$\cos\theta(90° - \theta) = \frac{b}{h} = \frac{AB}{CA} = \sin\theta$
$\tan\theta(90° - \theta) = \frac{p}{b} = \frac{BC}{AB} = \cot\theta$
On the other hand,
$\mathrm{cosec}(90° - \theta) = \sec\theta$
$\sec(90° - \theta) = \mathrm{cosec}\theta$
$\cot(90° - \theta) = \tan\theta$

SOLUTION OF RIGHT ANGLED TRIANGLE



The triangle which consists three sides and three angles with six elements is known as right angled triangle. In right angle triangle, one angle is 90°. If three elements are given, one of which must be the side and remaining others elements can be calculated which is known as a solution of right angle triangle.

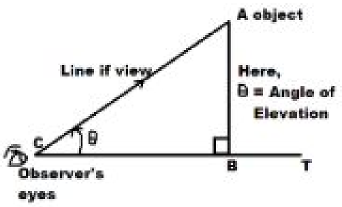
Let,
In a given figure, right angled triangle ABC, right angled at B, ∠A, ∠B and ∠C represent the angles and a, b and c represent their opposite side. Then ΔABC, using a Pythagoras theorem,
$AC^2 = AB^2 + BC^2$
i.e, $b^2 = c^2 + a^2$
and $\angle A + \angle C = 90°$



HEIGHT AND DISTANCE

Angle of Elevation

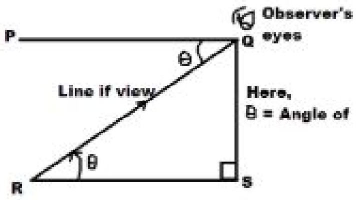
The angle which is above the horizontal line and the condition from which the angle is formed by the line of sight with the horizontal line is called angle of elevation.
In the given figure, the object is at a higher level than the observer's eye. So, ∠ACB = θ is the angle of elevation.



Angle of Elevation

Angle of Depression

The angle which is below the horizontal line and the condition from which the angle is formed with the horizontal line is called angle of depression.
Let, In the given figure, the object is at a lower level than the observer's eye. Therefore ∠PQR = θ is the angle of depression.



Angle of depression