Discrete-time Systems in the Frequency Domain

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Introduction

- Discrete-time signals with discrete amplitude are **digital signals**.
- A discrete-time system that processes a digital input signal x(k) to produce a digital output signal y(k) is called a digital signal processor or a **digital filter**.
- "**Z transform** is an essential tool for analyzing discrete-time systems."
- The Z transform maps a discrete-time signal $x(k)$ into a function $X(z)$ of a complex variable z: $X(z) = Z\{x(k)\}$
- Applying the Z transform to a difference equation describing a discrete-time system yields an algebraic equation of the Z transform of the output $Y(z)$.

Schilling and Harris (2012, p. 145)

Z Transform

The Z transform of discrete-time signal $x(k)$ is defined as

$$X(z) = Z\left\{x(k)\right\} \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$

which is a power series in the variable z^{-1}.

"The Z transform of most signals can be expressed in factored form as a ratio of two polynomials:"

$$X(z) = \frac{b_0 (z - z_1)(z - z_2)\cdots(z - z_m)}{(z - p_1)(z - p_2)\cdots(z - p_n)}$$

The roots of numerator polynomials z_i, $i = 1,\ldots,m$, are called the **zeroes** of $X(z)$ while the roots of denominator polynomial p_i, $i = 1,\ldots,n$, are called the **poles** of $X(z)$.

Schilling and Harris (2012, p. 149-150)

Z Transform

The **generalized geometric series**

$$\sum_{k=m}^{\infty} z^k = \frac{z^m}{1-z}, \quad m \geq 0, \; |z| < 1$$

can be used to directly compute the Z transform.

Region of Convergence

Any signal can be decomposed into causal and anti-causal parts:

$$x(k) = x_c(k) + x_a(k)$$

where

$$x_c(k) \overset{\text{def}}{=} x(k)\mu(k)$$

$$x_a(k) \overset{\text{def}}{=} x(k)\mu(-k-1)$$

"The causal part $x_c(k)$ has one region of convergence (ROC) while the anti-causal part $x_a(k)$ has another region of convergence."

"The overall region of convergence Ω_{ROC} includes the intersection of the two regions."

Schilling and Harris (2012, p. 150)

Region of Convergence: Example

Consider a two-sided exponential signal

$$x(k)=\begin{cases} a^k, & k\geq 0 \\ b^k, & k<0 \end{cases} = \underbrace{a^k\mu(k)}_{x_c(k)}+\underbrace{b^k\mu(-k-1)}_{x_a(k)}$$

"Since the Z transform is a linear operation, the transforms for the two parts can be computed separately and then added."

Schilling and Harris (2012, p. 150-151)

ROC of Causal Part

The Z transform of the causal part is

$$X_c(z) = \sum_{k=-\infty}^{\infty} \left(a^k \mu(k) \right) z^{-k}$$

$$= \sum_{k=0}^{\infty} a^k z^{-k}$$

Lower limit becomes 0 due to the step function

$$= \sum_{k=0}^{\infty} \left(a/z \right)^k$$

$$= \frac{\left(a/z \right)^0}{1 - a/z} = \frac{1}{1 - a/z}, \qquad \frac{|a|}{|z|} < 1$$

Using the generalized geometric series

$$= \frac{z}{z - a}, \qquad |z| > |a|$$

"Region of convergence is **outside** the circle of radius $|a|$ centered at the origin."

Schilling and Harris (2012, p. 151)

ROC of Anti-causal Part

The Z transform of the anti-causal part is

$$X_a(z) = \sum_{k=-\infty}^{\infty} \left(b^k \mu(-k-1)\right) z^{-k} = \sum_{k=-\infty}^{\infty} \left(b^k \mu(-(k+1))\right) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} b^k z^{-k}$$

Upper limit becomes -1 due to the step function
What does μ(-(k+1)) look like?

$$= \sum_{i=\infty}^{1} b^{-i} z^{i} = \sum_{i=1}^{\infty} (z/b)^i, \quad i = -k$$

$$= \frac{z/b}{1 - z/b}, \quad \frac{|z|}{|b|} < 1$$

Using the generalized geometric series

$$= \frac{-z}{z-b}, \quad |z| < |b|$$

"Region of convergence is **inside** the circle of radius |*b*| centered at the origin."

Schilling and Harris (2012, p. 151)

ROC Example (cont.)

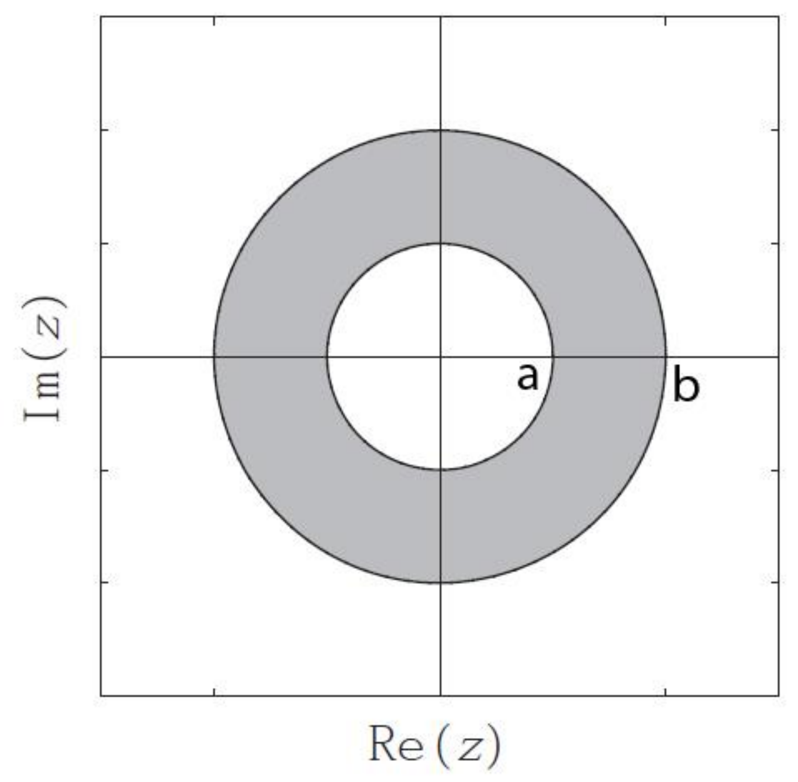
"The complete Z transform is then"

$$X(z) = X_c(z) + X_a(z)$$

$$= \frac{z}{z-a} - \frac{z}{z-b} = \frac{(a-b)z}{(z-a)(z-b)}, \quad |a| < |z| < |b|$$

"The region of convergence of $X(z)$ is an annulus with inner radius $|a|$ and outer radius $|b|$ centered at the origin of the complex plane."

"If $|a| \geq |b|$, then the Z transform does not exist."



Schilling and Harris (2012, p. 151)

ROC: Generalization

Suppose $\{p_1, \ldots, p_n\}$ are the poles of causal part $x_c(k)$ and $\{q_1, \ldots, q_r\}$ are the poles of anti-causal parts $x_a(k)$ that survive zero-pole cancellation in $X_c(z) + X_a(z)$.

"Define the radius of innermost anti-causal pole R_m and the outermost causal pole R_M as follows."

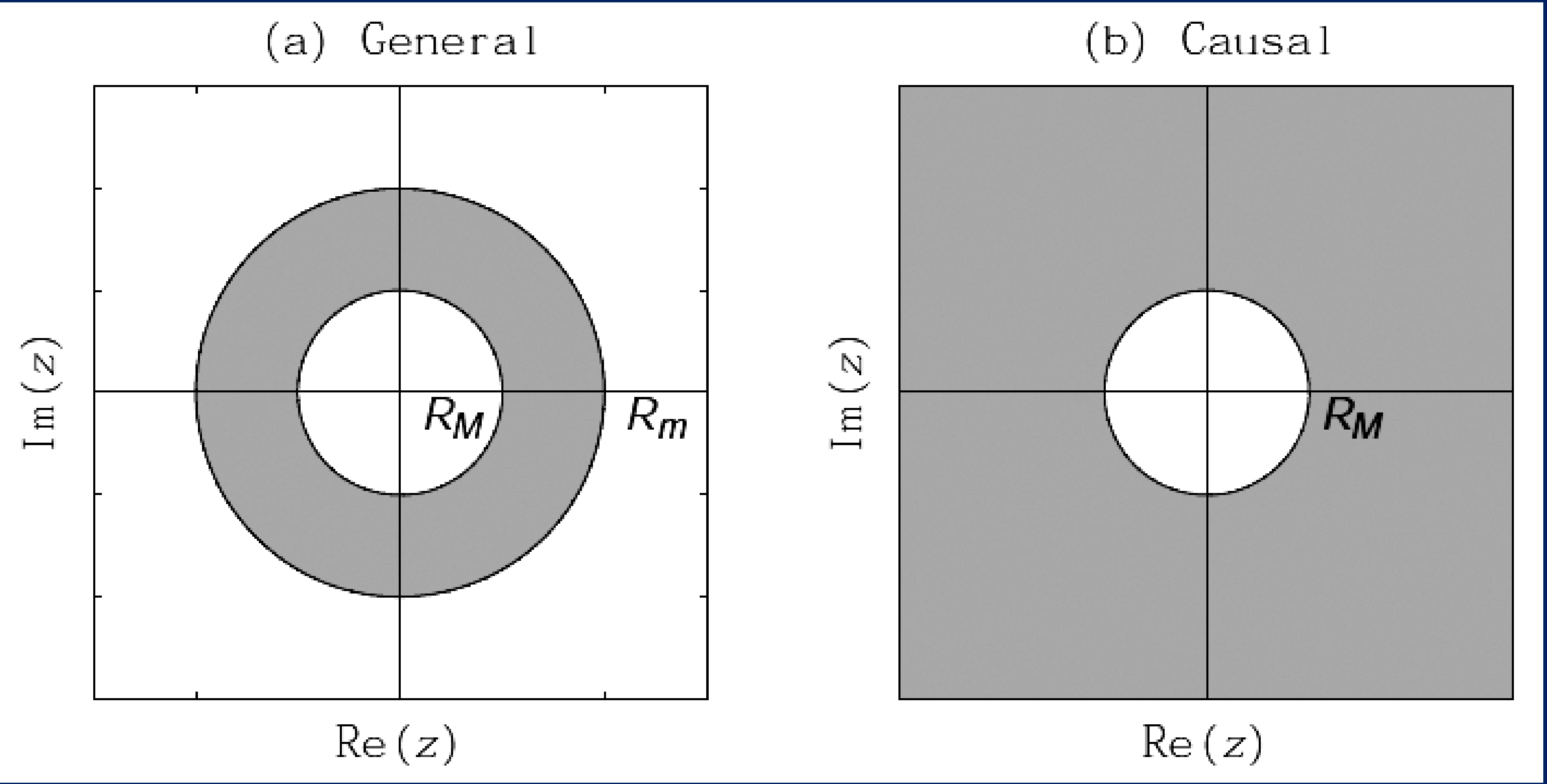
$$R_\mathrm{m} \overset{\text{def}}{=} \min_i |q_i|, \quad R_\mathrm{M} \overset{\text{def}}{=} \max_i |p_i|$$

ROC of causal part is $|z| > R_\mathrm{M}$ while ROC of anti-causal part is $|z| < R_\mathrm{m}$. ROC for general signals is then

$$\Omega_\mathrm{ROC} = \{ z \in \mathbb{C} : R_\mathrm{M} < |z| < R_\mathrm{m} \}$$

Schilling and Harris (2012, p. 152)

ROC: Generalization



Schilling and Harris (2012, p. 152)

ROC: Finite Causal Signals

Let $x(k)$ be a finite causal signal of length m.
The Z transform of $x(k)$ is

$$X(z) = x(0) + x(1)z^{-1} + \cdots + x(m-1)z^{1-m}$$
$$= \frac{x(0)z^{m-1} + x(1)z^{m-2} + \cdots + x(m-1)}{z^{m-1}}$$

"Thus, a finite causal signal of length $m > 1$ has $m - 1$ poles at $z = 0$, which means its ROC is $|z| > 0$."

"If $m = 1$ as in $x(k) = \delta(k)$, then there is no pole and the ROC is the entire complex plane."

Schilling and Harris (2012, p. 152)

ROC: Finite Anti-causal Signals

If $x(k)$ be a finite anti-causal signal of length r.
The Z transform of $x(k)$ is

$$X(z) = x(-r)z^{r} + x(-r+1)z^{r-1} + \cdots + x(-1)$$

"Thus, a finite anti-causal signal has no poles, which means the ROC is the entire complex plane."

Schilling and Harris (2012, p. 152)

ROC: Summary

Signal	Length	Ω_{ROC}
Causal	Infinite	$\|z\| > R_M$
Anti-causal	Infinite	$\|z\| < R_m$
General	Infinite	$R_M < \|z\| < R_m$
Causal	Finite, $m > 1$	$\|z\| > 0$

Schilling and Harris (2012, p. 153)

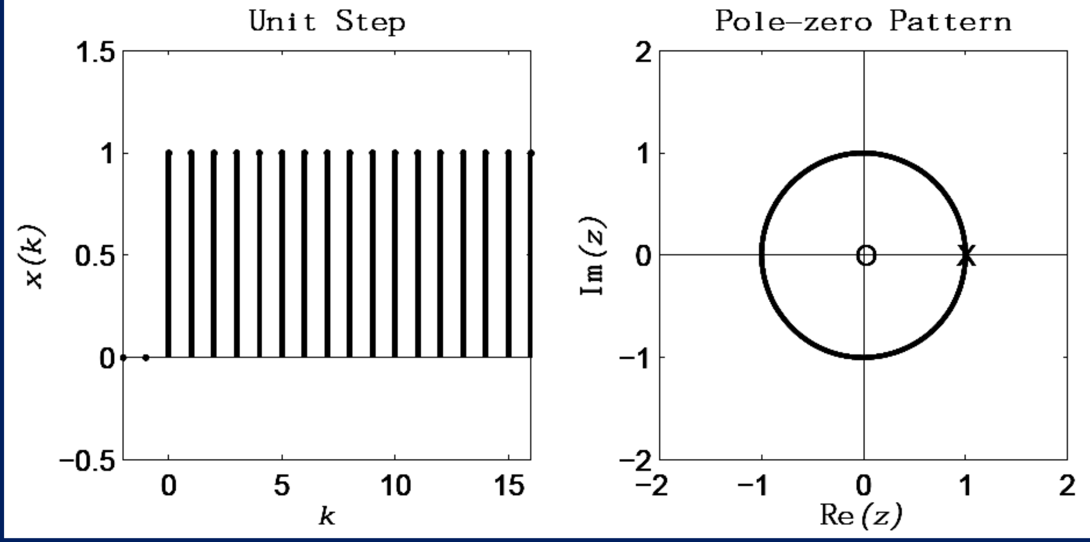
Z Transforms of Common Signals

Unit impulse: $Z\{\delta(k)\} = \sum_{k=-\infty}^{\infty} \delta(k) z^{-k} = z^0 = 1$

Unit step: $Z\{\mu(k)\} = \sum_{k=-\infty}^{\infty} \mu(k) z^{-k} = \sum_{k=0}^{\infty} z^{-k}$

$$= \sum_{k=0}^{\infty} \left(1/z\right)^k = \frac{1}{1-1/z}, \quad \frac{1}{|z|} < 1$$

$$= \frac{z}{z-1}, \quad |z| > 1$$



"The Z transform has a zero at $z = 0$ and a pole at $z = 1$."

ROC: $|z| > 1$

Z Transforms of Common Signals

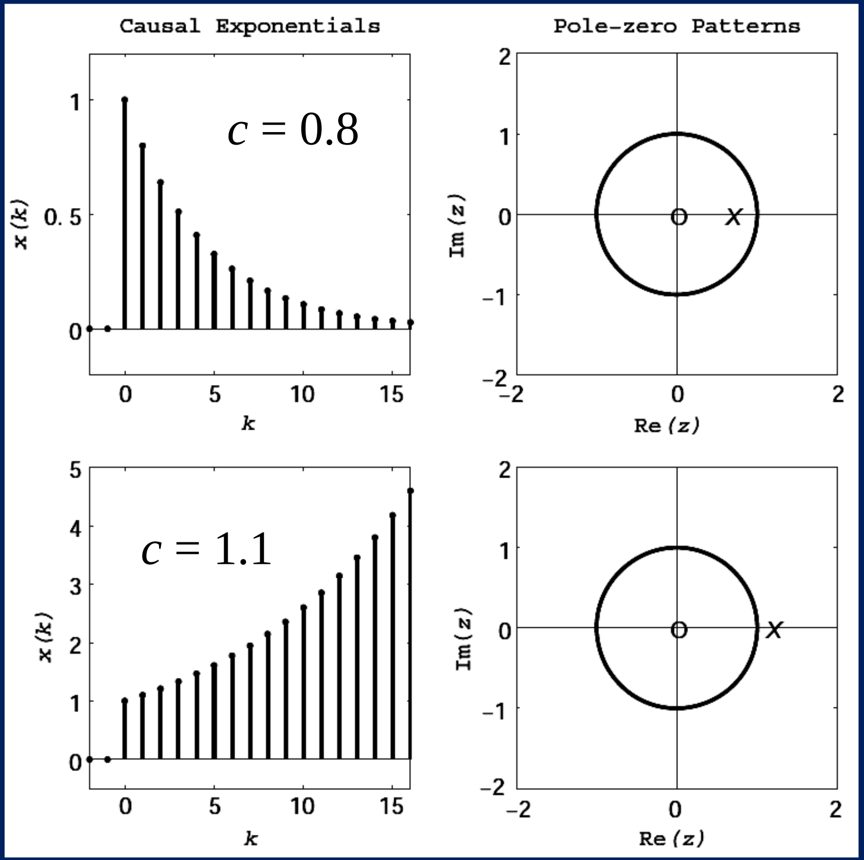
Causal exponential: $x(k) = c^k \mu(k)$

$$Z\left\{c^k \mu(k)\right\} = \sum_{k=0}^{\infty} c^k z^{-k} = \sum_{k=0}^{\infty} \left(\frac{c}{z}\right)^k = \frac{1}{1-c/z}, \qquad \left|\frac{c}{z}\right| < 1$$

$$= \frac{z}{z-c}, \qquad |z| > |c|$$

"The Z transform has a zero at $z = 0$ and a pole at $z = c$."

ROC: $|z| > |c|$



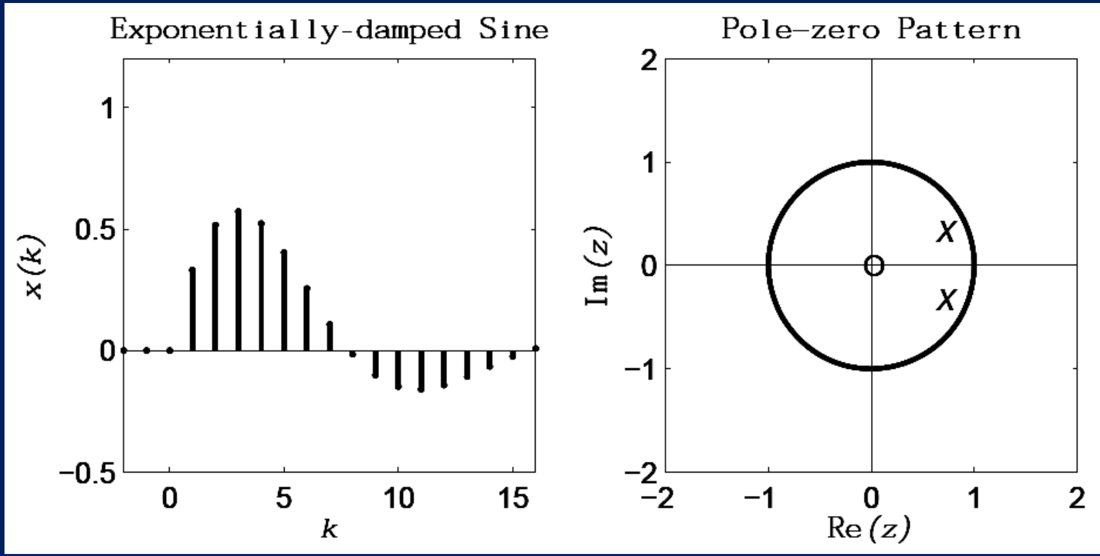
Schilling and Harris (2012, p. 154-155)

Z Transforms of Common Signals

Exponentially damped sine $x(k) = c^k \sin(dk)\mu(k)$
where c and d are real and $c > 0$

$$Z\left\{c^k \sin(dk)\mu(k)\right\} = \frac{c\sin(d)z}{z^2 - 2c\cos(d)z + c^2}, \quad |z| > c$$

"The Z transform has
a zero at $z = 0$ and
a pole at $z = c$."
ROC: $|z| > |c|$



Common Z Transforms

Signal	Z-transform	Poles	Ω_{ROC}
$\delta(k)$	1	none	$z \in C$
$\mu(k)$	$\dfrac{z}{z-1}$	$z = 1$	$\|z\| > 1$
$k\mu(k)$	$\dfrac{z}{(z-1)^2}$	$z = 1$	$\|z\| > 1$
$c^k\mu(k)$	$\dfrac{z}{z-c}$	$z = c$	$\|z\| > \|c\|$
$k(c)^k\mu(k)$	$\dfrac{cz}{(z-c)^2}$	$z = c$	$\|z\| > \|c\|$
$\sin(dk)\mu(k)$	$\dfrac{\sin(d)z}{z^2 - 2\cos(d)z + 1}$	$z = \exp(\pm jd)$	$\|z\| > 1$
$\cos(dk)\mu(k)$	$\dfrac{[z - \cos(d)]z}{z^2 - 2\cos(d)z + 1}$	$z = \exp(\pm jd)$	$\|z\| > 1$
$c^k \sin(dk)\mu(k)$	$\dfrac{c\sin(d)z}{z^2 - 2c\cos(d)z + c^2}$	$z = c\exp(\pm jd)$	$\|z\| > c$
$c^k \cos(dk)\mu(k)$	$\dfrac{[z - c\cos(d)]z}{z^2 - 2c\cos(d)z + c^2}$	$z = c\exp(\pm jd)$	$\|z\| > c$

Schilling and Harris (2012, p. 157)

Z Transform Properties: Linearity

Z-transform operator is linear.

Given two signals $x(k)$ and $y(k)$, and two arbitrary constant a and b, we have

$$Z\left\{ax(k)+by(k)\right\} = \sum_{k=-\infty}^{\infty}\left[ax(k)+by(k)\right]z^{-k}$$

$$= a\sum_{k=-\infty}^{\infty} x(k)z^{-k} + b\sum_{k=-\infty}^{\infty} y(k)z^{-k}$$

$$= aZ\left\{x(k)\right\} + bZ\left\{y(k)\right\}$$

$$= aX(z) + bY(z)$$

Schilling and Harris (2012, p. 157)

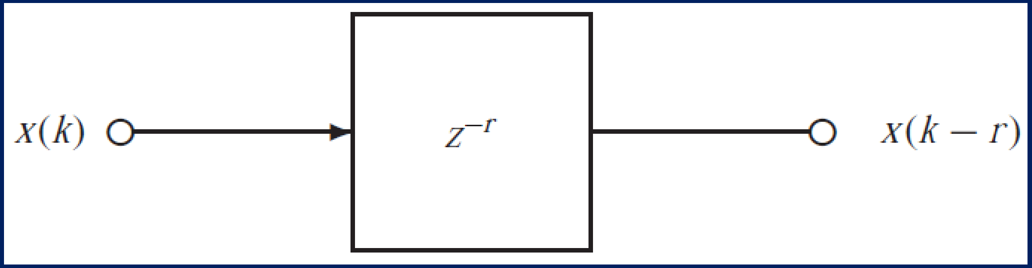
Z Transform Properties: Delay

Let $r \geq 0$ denote the number of samples by which a *causal* discrete-time signal $x(k)$ is delayed.

Using the change of variable $i = k - r$, we have

$$
\begin{aligned}
Z\{x(k-r)\} &= \sum_{k=0}^{\infty} x(k-r) z^{-k} \\
&= \sum_{i=-r}^{\infty} x(i) z^{-(i+r)}, \qquad i = k - r \\
&= z^{-r} \sum_{i=0}^{\infty} x(i) z^{-i} \\
&= z^{-r} X(z)
\end{aligned}
$$

Lower limit becomes 0 since $x(i)$ is causal



Schilling and Harris (2012, p. 158)

Z Transform: Example 1

Consider a signal $x(k) = b\left[\mu(k) - \mu(k-r)\right]$

Applying the linearity and delay properties of the Z transform yields

$$X(z) = b\left(1 - z^{-r}\right) Z\left\{\mu(k)\right\}$$

$$= \frac{b\left(1 - z^{-r}\right) z}{z - 1}$$

$$= \frac{b\left(z^{r} - 1\right)}{z^{r-1}\left(z - 1\right)}, \qquad |z| > 1$$

Schilling and Harris (2012, p. 158)

Z Transform Properties: Z Scale

Let c be constant and consider the signal $c^k x(k)$

$$Z\left\{c^k x(k)\right\} = \sum_{k=-\infty}^{\infty} c^k x(k) z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \left(\frac{z}{c}\right)^{-k}$$

$$= X\left(\frac{z}{c}\right)$$

Schilling and Harris (2012, p. 159)

Z Transform Properties: Time Multiplication

Taking the derivative of the Z transform of a signal yields

$$\frac{dX(z)}{dz} = \frac{d}{dz}\sum_{k=-\infty}^{\infty} x(k)z^{-k}$$

$$= -\sum_{k=-\infty}^{\infty} kx(k)z^{-(k+1)}$$

$$= -z^{-1}\sum_{k=-\infty}^{\infty} kx(k)z^{-k} = -z^{-1}Z\{kx(k)\}$$

$$Z\{kx(k)\} = -z\frac{dX(z)}{dz}$$

Multiplying a signal by the time variable *k* is equivalent to taking the derivative of the Z transform and scaling by –*z*.

Z Transform: Example 2

Consider a unit ramp $x(k) = k\mu(k)$

Using the time multiplication property of the Z transform yields

$$
\begin{aligned}
Z\{k\mu(k)\} &= -z\frac{d}{dz}Z\{\mu(k)\} \\
&= -z\frac{d}{dz}\left(\frac{z}{z-1}\right) \\
&= \frac{z}{(z-1)^2}, \qquad |z| > 1
\end{aligned}
$$

"The time multiplication property can be applied repeatedly to obtain the Z transform of $k^m\mu(k)$ for $m \geq 0$.

Schilling and Harris (2012, p. 160)

Z Transform: Example 3

Consider a causal exponential $x(k) = kc^k \mu(k)$

Using the Z scale property and Z transform of the unit ramp yields

$$Z\left\{kc^k \mu(k)\right\} = Z\left\{k\mu(k)\right\}\Big|_{z=z/c}$$

$$= \frac{z}{(z-1)^2}\Bigg|_{z=z/c}$$

$$= \frac{cz}{(z-c)^2}, \qquad |z| > |c|$$

Z Transform Properties: Convolution

"Suppose $h(k)$ and $x(k)$ are zero extended so that they are defined for all k."

$$Z\left\{h(k)*x(k)\right\} = \sum_{k=-\infty}^{\infty}\left[h(k)*x(k)\right]z^{-k}$$

$$= \sum_{k=-\infty}^{\infty}\left[\sum_{i=-\infty}^{\infty} h(i)x(k-i)\right]z^{-k} = \sum_{i=-\infty}^{\infty} h(i)\sum_{k=-\infty}^{\infty} x(k-i)z^{-k}$$

$$= \sum_{i=-\infty}^{\infty} h(i)\sum_{m=-\infty}^{\infty} x(m)z^{-(m+i)}, \quad m = k-i$$

$$= \sum_{i=-\infty}^{\infty} h(i)z^{-i}\sum_{m=-\infty}^{\infty} x(m)z^{-m} = H(z)X(z)$$

Schilling and Harris (2012, p. 160-161)

Z Transform Properties: Time Reversal

Reversing time by replacing $x(k)$ by $x(-k)$.

$$Z\{x(-k)\} = \sum_{k=-\infty}^{\infty} x(-k) z^{-k}$$

$$= \sum_{i=\infty}^{-\infty} x(i) z^{i}, \quad i = -k$$

$$= \sum_{i=\infty}^{-\infty} x(i) \left(\frac{1}{z}\right)^{-i}$$

$$= X(1/z)$$

Schilling and Harris (2012, p. 161)

Z Transform Properties: Correlation

Cross-correlation of $y(k)$ with $x(k)$ is

$$r_{yx}(k) = \frac{1}{L}\sum_{i=-\infty}^{\infty} y(i)x(i-k) = \frac{1}{L}\sum_{i=-\infty}^{\infty} y(i)x(-(k-i)) = \frac{y(k) * x(-k)}{L}$$

Using the convolution property yields

$$Z\left\{r_{yx}(k)\right\} = \frac{Y(z)Z\left\{x(-k)\right\}}{L}$$

$$= \frac{Y(z)X\left(1/z\right)}{L}$$

Z Transform Properties: Causality

Suppose $x(k)$ is a causal signal. Then, $X(z)=\sum_{k=0}^{\infty} x(k) z^{-k}$

As $z \to \infty$, all terms go to zero except the initial value. This leads to the **initial value theorem**:

$$x(0)=\lim_{z \to \infty} X(z)$$

Consider the function $X_1(z) = (z\text{-}1)X(z)$ whose poles are inside the unit circle. Using the **final value theorem**, we have

$$x(\infty)=\lim_{z \to 1}(z-1) X(z)$$

We can use these formulas to partially check the validity of the computed Z transform.

Z Transform: Example 4

Consider the signal $x(k) = b\left[\mu(k) - \mu(k-M)\right]$

whose Z transform is $X(z) = \dfrac{b\left(z^M - 1\right)}{z^{M-1}(z-1)}$

Using the initial value theorem, the initial value of $x(k)$ is

$$x(0) = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{b\left(z^M - 1\right)}{z^{M-1}(z-1)} = b$$

Since all poles of $(z\text{-}1)X(z)$ are inside the unit circle, the final value of $x(k)$ is

$$x(\infty) = \lim_{z\to 1}(z-1)X(z) = \lim_{z\to 1} \frac{b\left(z^M - 1\right)}{z^{M-1}} = 0$$

Schilling and Harris (2012, p. 163)

Summary of Z Transform Properties

Property	Description
Linearity	$Z\{ax(k) + by(k)\} = aX(z) + bY(z)$
Delay	$Z\{x(k-r)\} = z^{-r}X(z)$
Time multiplication	$Z\{kx(k)\} = -z\dfrac{dX(z)}{dz}$
Time reversal	$Z\{x(-k)\} = X(1/z)$
Z-scale	$Z\{a^k x(k)\} = X(z/a)$
Complex conjugate	$Z\{x^*(k)\} = X^*(z^*)$
Convolution	$Z\{h(k) \star x(k)\} = H(z)X(z)$
Correlation	$Z\{r_{yx}(k)\} = \dfrac{Y(z)X(1/z)}{L}$
Initial value	$x(0) = \lim\limits_{z\to\infty} X(z)$
Final value	$x(\infty) = \lim\limits_{z\to 1}(z-1)X(z)$, stable

Schilling and Harris (2012, p. 163)

Inverse Z Transform

Once we apply the Z transform to a difference equation and obtain $X(z)$, the actual solution $x(k)$ can then be obtained using the inverse Z transform:

$$x(k) = Z^{-1}\left\{X(z)\right\}$$

"Z transforms associated with LTI systems take the form of a ratio of two polynomials in z, that is, they are **rational polynomials** in z:"

$$X(z) = \frac{b(z)}{a(z)} = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n}, \qquad a_0 = 1$$

Schilling and Harris (2012, p. 164)

Inverse Z Transform

"If $x(k)$ is a causal signal, then $X(z)$ will be a **proper rational polynomial**: when $m \leq n$, that is, the number of zeroes of $X(z)$ will be less than or equal to the number of poles." If $m > n$, then using long division we can write

$$X(z) = Q(z) + \frac{R(z)}{a(z)},$$

where the quotient polynomial $Q(z)$ will be of degree $m - n$ and can be expressed as $Q(z) = \sum_{i=0}^{m-n} q_i z^i$

and $R(z)$ is a remainder polynomial.

Schilling and Harris (2012, p. 164)

Inverse Z Transform

"When $m = n$, the quotient polynomial is the constant b_0."

Compared to the definition $X(z) \equiv \sum_{k=-\infty}^{\infty} x(k) z^{-k}$

the inverse Z-transform of $Q(z)$ is

$$q(k) = \sum_{i=0}^{m-n} q_i \delta(k + i)$$

"Thus, $Q(z)$ represents anti-causal part of signal, plus the constant term. The remaining part of $X(z)$ representing the signal with $k > 0$ can be obtained using the following techniques which assumes that $X(z)$ represents a causal signal, i.e., $m \le n$."

Schilling and Harris (2012, p. 164)

Synthetic Division Method

Rewriting the rational polynomial as

$$X(z)=\frac{b_0 z^m + b_1 z^{m-1}+\cdots+b_m}{z^n + a_1 z^{n-1}+\cdots+a_n}=\frac{z^{-r}\left(b_0+b_1 z^{-1}+\cdots+b_m z^{-m}\right)}{1+a_1 z^{-1}+\cdots+a_n z^{-n}}$$

where $r = n - m$ and $m \le n$.

"Synthetic division method performs long division of the numerator polynomial by the denominator polynomial to produce an infinitely long quotient polynomial"

$$X(z)=z^{-r}\left[q_0+q_1 z^{-1}+q_2 z^{-2}+\cdots\right]=z^{-r}Z\{q(k)\}$$

So $x(k)$ is $q(k)$ delayed by r samples, that is,

$$x(k)=q(k-r)\mu(k-r)$$

The unit step function ensures that $x(k)$ is causal.

Schilling and Harris (2012, p. 164-165)

Synthetic Division: Example

Consider Z transform $X(z)=\dfrac{z+1}{z^2-2z+3}=\dfrac{z^{-1}(1+z^{-1})}{1-2z^{-1}+3z^{-2}}$

There are n = 2 poles and m = 1 zero. Here, $r = n - m = 1$.

Performing long division yields

$$
\begin{array}{r|l}
 & 1+3z^{-1}+3z^{-2}-3z^{-3}-15z^{-4}+\cdots \\
\hline
1-2z^{-1}+3z^{-2} & 1+z^{-1} \\
 & \underline{1-2z^{-1}+3z^{-2}} \\
 & 3z^{-1}-3z^{-2} \\
 & \underline{3z^{-1}-6z^{-2}+9z^{-3}} \\
 & 3z^{-2}-9z^{-3} \\
 & \underline{3z^{-1}-6z^{-2}+9z^{-3}} \\
 & -3z^{-3}-9z^{-3} \\
 & \underline{-3z^{-3}+6z^{-4}-9z^{-4}} \\
 & -15z^{-4}+9z^{-5}
\end{array}
$$

Thus, we have

$$q(k)=[1,3,3,-3,-15,\ldots]$$

and

$$x(k)=[0,1,3,3,-3,-15,\ldots]$$

since r = 1.

Impulse Response Method

"The inverse Z transform of $X(z)$ can be obtained by interpreting $x(k)$ as the impulse response of a discrete-time system with characteristic polynomial $a(z)$ and input polynomial $b(z)$."

Using the MATLAB function filter, one can obtain the impulse response from $a(z)$ and $b(z)$ as follows.

```
% given vectors a and b
N = 100;
delta = [1; zeros(N-1,1)];
x = filter(b,a,delta);
```

Schilling and Harris (2012, p. 165-166)

Partial Fractions

- Partial fractions can provide a closed-form expression for $x(k)$ as a function of k.
- Using a table of Z-transform pairs, we can find the inverse Z transform of $X(z)$.
- When $X(z)$ is not in the table, partial fractions can be used to rewrite $X(z)$ as a sum of terms in the table.

Simple Poles: When n poles of $X(z)$ are nonzero and simple, $X(z)$ can be written in factored form as

$$X(z) = \frac{b(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

Schilling and Harris (2012, p. 166-167)

Partial Fractions: Simple Poles

Express $X(z)/z$ using partial fractions

$$\frac{X(z)}{z} = \sum_{i=0}^{n} \frac{R_i}{z - p_i}$$

where R_i is called the residue of $X(z)/z$ at pole p_i.

- Note that $X(z)/z$ has n + 1 simple poles with $p_0 = 0$.
- "To find the residues, put the $n + 1$ partial fraction terms over a common denominator and then equate numerators, and then equate the coefficients of like power of z."
- "This results in $n + 1$ equations in the $n + 1$ unknown residues."

Schilling and Harris (2012, p. 166)

Partial Fractions: Simple Poles

Another approach is to multiply both sides of

$$\frac{X(z)}{z} = \sum_{i=0}^{n} \frac{R_i}{z - p_i}$$

by $z - p_k$ and then evaluate the result at $z = p_k$ and obtain

$$\left.\frac{(z - p_k)X(z)}{z}\right|_{z=p_k} = \sum_{i=0}^{n} \left.\frac{(z - p_k)R_i}{z - p_i}\right|_{z=p_k} = R_k, \quad 0 \le k \le n$$

"The term residue comes from the fact that R_i is what remains of $X(z)/z$ at $z = p_i$, after the pole at $z = p_i$ has been removed."

Schilling and Harris (2012, p. 166)

Partial Fractions: Simple Poles

"Once the partial fraction representation is obtained,

multiply both sides of $\dfrac{X(z)}{z} = \sum_{i=0}^{n} \dfrac{R_i}{z - p_i}$ by z to obtain"

$$X(z) = R_0 + \sum_{i=1}^{n} \frac{R_i z}{z - p_i}$$

"Using the linearity property and the Z transform table, we obtain"

$$x(k) = R_0 \delta(k) + \left[R_1 p_1^k + R_2 p_2^k + \cdots + R_n p_n^k \right] \mu(k)$$

Schilling and Harris (2012, p. 166-167)

Partial Fractions: Example 1

Simple Pole Case: Consider the Z transform

$$\frac{X(z)}{z} = \frac{10\left(z^2+4\right)}{z\left(z^2-2z-3\right)} = \frac{10\left(z^2+4\right)}{z\left(z+1\right)\left(z-3\right)}$$

Using $R_k = \left.\frac{\left(z-p_k\right)X(z)}{z}\right|_{z=p_k}$, we obtain

$$R_0 = \left.\frac{10\left(z^2+4\right)}{\left(z+1\right)\left(z-3\right)}\right|_{z=p_0} = \frac{40}{-3}, R_1 = \left.\frac{10\left(z^2+4\right)}{z\left(z-3\right)}\right|_{z=p_1} = \frac{50}{4}, R_2 = \left.\frac{10\left(z^2+4\right)}{z\left(z+1\right)}\right|_{z=p_2} = \frac{130}{12}$$

Closed-form expression of inverse Z transform of $X(z)$ is

$$x(k) = \frac{-40}{3}\delta(k) + \left[\frac{25(-1)^k}{2} + \frac{65(3)^k}{6}\right]\mu(k)$$

Schilling and Harris (2012, p. 167)

Partial Fractions: Multiple Poles

"Suppose a nonzero pole at $z = p_1$ is repeated n times."
"Then the partial fraction expansion is"

$$\frac{X(z)}{z} = \frac{R_0}{z - p_0} + \frac{c_1}{z - p_1} + \frac{c_2}{\left(z - p_1\right)^2} + \cdots + \frac{c_n}{\left(z - p_1\right)^n}$$

"To find the coefficients c_i, one can put the terms on RHS over a common denominator and equate numerators."

"Equating the coefficients of like powers of z results in $n + 1$ equations in the $n + 1$ variables $\left\{R_0, c_1, \ldots, c_n\right\}$."

Schilling and Harris (2012, p. 167)

Partial Fractions: Example 2

Multiple Pole Case: Consider the Z transform

$$\frac{X(z)}{z}=\frac{2(z+3)}{z(z^2-4z+4)}=\frac{2(z+3)}{z(z-2)^2}$$

The partial fraction expansion is

$$\frac{X(z)}{z}=\frac{R_0}{z}+\frac{c_1}{z-2}+\frac{c_2}{(z-2)^2}$$

"The residue of the pole at $p_0 = 0$ is"

$$R_0=\left.\frac{2(z+3)}{(z-2)^2}\right|_{z=0}=\frac{6}{4}=1.5$$

Example 2 (continued)

"Substituting R_0 = 1.5 and putting the terms of the partial fraction expansion over a common denominator yields"

$$\frac{X(z)}{z} = \frac{1.5(z-2)^2 + c_1 z(z-2) + c_2 z}{z(z-2)^2}$$

$$= \frac{(1.5+c_1)z^2 + (c_2 - 2c_1 - 6)z + 6}{z(z-2)^2} = \frac{2z+6}{z(z-2)^2}$$

"Equating coefficients of like powers of z yields"

$$1.5 + c_1 = 0, \qquad c_2 - 2c_1 + 6 = 2 \quad \Rightarrow \quad c_1 = -1.5, c_2 = -7$$

We then obtain $X(z) = 1.5 - \dfrac{1.5z}{z-2} - \dfrac{(3.5)2z}{(z-2)^2}$

Schilling and Harris (2012, p. 168)

Example 2 (continued)

Using the Z-transform table (Slide 18), we then obtain

Signal	Z-transform
$\delta(k)$	1
$c^k \mu(k)$	$\dfrac{z}{z-c}$
$k(c)^k \mu(k)$	$\dfrac{cz}{(z-c)^2}$

$$x(k) = 1.5\delta(k) - \left[1.5(2)^k + 3.5k(2)^k\right]\mu(k)$$

$$= 1.5\delta(k) - (1.5 + 3.5k)2^k \mu(k)$$

Schilling and Harris (2012, p. 168)

Partial Fractions: Complex Poles

- "The poles of $X(z)$ can be real or complex."
- "If the signal $x(k)$ is real, then complex poles appear in conjugate pairs."
- Suppose $X(z)$ has poles at $z = c\exp\left(\pm jd\right)$ and

$$X(z) = \frac{b_0 z + b_1}{\left[z - c\exp(jd)\right]\left[z - c\exp(-jd)\right]}$$

- Recall the Z transform table.

Signal	Z-transform
$c^k \sin(dk)\mu(k)$	$\dfrac{c\sin(d)z}{z^2 - 2c\cos(d)z + c^2}$
$c^k \cos(dk)\mu(k)$	$\dfrac{[z - c\cos(d)]z}{z^2 - 2c\cos(d)z + c^2}$

We want to turn the nominator into a linear combination of $c\sin(d)$ and $(z - c\cos(d))$ so that we can use the table.

Schilling and Harris (2012, p. 169)

Partial Fractions: Complex Poles

This can be done by rewriting $X(z)$ as

$$X(z) = \frac{z^{-1}\left(b_0 z^2 + b_1 z\right)}{\left[z - c\exp(jd)\right]\left[z - c\exp(-jd)\right]}$$

Then, we seek f_1 and f_2 such that

$$\begin{aligned} b_0 z^2 + b_1 z &= f_1\left[c\sin(d)z\right] + f_2\left[z - c\cos(d)\right]z \\ &= f_2 z^2 + c\left[f_1\sin(d) - f_2\cos(d)\right]z \end{aligned}$$

Equating coefficients of like powers of z yields

$$f_1 = \left[b_1 + cb_0\cos(d)\right]\big/\sin(d), \quad f_2 = b_0$$

With the delay z^{-1}, we then obtain

$$x(k) = c^{k-1}\left(f_1\sin\left[d(k-1)\right] + f_2\cos\left[d(k-1)\right]\right)\mu(k-1)$$

Schilling and Harris (2012, p. 169)

Partial Fractions: Example 3

Complex Pole Case: Consider the Z transform

$$X(z) = \frac{3z+5}{z^2 - 4z + 13}$$

with complex conjugate poles $p_{1,2} = 2 \pm j3 = c\exp(\pm jd)$
where $c = \sqrt{4+9} = 3.61, \quad d = \tan^{-1}(3/2) = 0.983$

We then have $f_1 = \dfrac{5 + 3.61(3)\cos(0.983)}{\sin(0.983)} = 13.2, \quad f_2 = 3$

The inverse Z transform is

$$x(k) = c^{k-1}\left(f_1 \sin\left[d(k-1)\right] + f_2 \cos\left[d(k-1)\right]\right)\mu(k-1)$$

$$= 3.61^{k-1}\left(13.2\sin\left[0.983(k-1)\right] + 3\cos\left[0.983(k-1)\right]\right)\mu(k-1)$$

Schilling and Harris (2012, p. 170)

Residue Method

- "The partial fraction method is effective for simple poles, but it can become cumbersome for multiple poles and furthermore it requires the Z transform table."
- The residue method based on the theory of functions of complex variables is more effective for multiple poles and does not require a table.
- **Cauchy integral formula**:

$$f(a) = \frac{1}{j2\pi} \oint_C \frac{f(z)}{z-a} dz$$

where $f(z)$ is an analytic function and C is a circle oriented counterclockwise forming a boundary of $D = \{z{:}|z-z_0|\leq r\}$

Schilling and Harris (2012, p. 170)

Residue Method

It can be shown that

$$Z^{-1}\{X(z)\} = \frac{1}{j2\pi}\oint_C X(z)z^{k-1}dz$$

where C is a counterclockwise contour in the region of convergence of $X(z)$ that encloses all of the poles.

Using the Cauchy residue theorem, we can compute $x(k)$ by

$$x(k) = \sum_{i=1}^{q} \mathrm{Res}(p_i, k)$$

where p_i are the q distinct poles of $X(z)z^{k-1}$ and $\mathrm{Res}(p_i,k)$ is the residue of $X(z)z^{k-1}$ at the pole $z = p_i$.

Schilling and Harris (2012, p. 170-171)

Residue Method

To compute the residue, $X(z)$ is rewritten as

$$X(z) = \frac{b(z)}{(z-p_1)^{m_1}(z-p_2)^{m_2}\cdots(z-p_q)^{m_q}}$$

"Here, p_i is a pole of multiplicity m_i for $1 \le i \le q$."

"If p_i is a **simple pole** ($m_i = 1$), then the residue is what remains of $X(z)z^{k-1}$ at the pole after the pole has been removed:" $\operatorname{Res}(p_i,k) = (z-p_i)X(z)z^{k-1}\Big|_{z=p_i} \quad \text{if } m_i = 1$

If p_i is a **multiple pole** of multiplicity $m_i > 1$, then the residue is

$$\operatorname{Res}(p_i,k) = \frac{1}{(m_i-1)!}\frac{d^{m_i-1}}{dz^{m_i-1}}\left\{(z-p_i)^{m_i}X(z)z^{k-1}\right\}\Big|_{z=p_i}, \quad m_i > 1$$

Residue Method

- "The residue method requires the same amount of computational effort as the partial fraction method for the case of simple poles, but it requires less effort for multiple poles and does not requires a table."
- "The residue method has a drawback when some value of k cause $X(z)z^{k-1}$ to have a pole at $z = 0$. This problem can be solved by separating $x(k)$ into two cases, $k = 0$ and $k > 0$."
- "The initial value theorem can be used to computer $x(0)$ as"

$$x(0) = \lim_{z \to \infty} X(z) = b_0 \delta(n-m), \quad m \le n$$

Schilling and Harris (2012, p. 171)

Residue Method: Algorithm

1. Factor the denominator polynomial of $X(z)$ as in

$$X(z) = b(z) \big/ (z-p_1)^{m_1} (z-p_2)^{m_2} \cdots (z-p_q)^{m_q}$$

2. Set $x(0) = b_0 \delta(n-m)$
3. For i =1 to q do

$$\operatorname{Res}(p_i, k) = \begin{cases} (z-p_i) X(z) z^{k-1} \Big|_{z=p_i}, & m_i = 1 \\ \dfrac{1}{(m_i-1)!} \dfrac{d^{m_i-1}}{dz^{m_i-1}} \left\{ (z-p_i)^{m_i} X(z) z^{k-1} \right\} \Big|_{z=p_i}, & m_i > 1 \end{cases}$$

4. Set

$$x(k) = x(0)\delta(k) + \left[\sum_{i=1}^{q} \operatorname{Res}(p_i, k) \right] \mu(k-1)$$

Schilling and Harris (2012, p. 172)

Residue Method: Simple Poles Example

Consider a Z transform $X(z) = \dfrac{z^2}{(z-a)(z-b)}$

The initial value of $x(k)$ is $x(0) = 1$. The two residues are

$$\mathrm{Res}(a,k) = \left.\frac{z^{k+1}}{(z-b)}\right|_{z=a} = \frac{a^{k+1}}{a-b}, \quad \mathrm{Res}(b,k) = \left.\frac{z^{k+1}}{(z-a)}\right|_{z=a} = \frac{b^{k+1}}{b-a}$$

Thus,

$$x(k) = x(0)\delta(k) + \left[\mathrm{Res}(a,k) + \mathrm{Res}(b,k)\right]\mu(k-1)$$

$$= \delta(k) + \left(\frac{a^{k+1} - b^{k+1}}{a-b}\right)\mu(k-1) = \left(\frac{a^{k+1} - b^{k+1}}{a-b}\right)\mu(k)$$

Schilling and Harris (2012, p. 172)

Residue Method: Mixed Poles Example

Consider a Z transform $X(z) = \dfrac{1}{(z-a)^2(z-b)}$

The initial value of $x(k)$ is $x(0) = 0$.

The residue of multiple pole at $z = a$ is

$$\mathrm{Res}(a,k) = \frac{d}{dz}\left\{\frac{z^{k-1}}{z-b}\right\}\Bigg|_{z=a} = \frac{\left[(a-b)(k-1)-a\right]a^{k-2}}{(a-b)^2}$$

The residue of simple pole at $z = b$ is

$$\mathrm{Res}(b,k) = \frac{z^{k-1}}{(z-a)^2}\Bigg|_{z=b} = \frac{b^{k-1}}{(b-a)^2}$$

Schilling and Harris (2012, p. 172-173)

Residue Method: Mixed Poles Example

Thus,

$$x(k) = x(0)\delta(k) + \left[\mathrm{Res}(a,k) + \mathrm{Res}(b,k)\right]\mu(k-1)$$

$$= \left[\frac{\left[(a-b)(k-1)-a\right]a^{k-2}}{(a-b)^2} + \frac{b^{k-1}}{(b-a)^2}\right]\mu(k-1)$$

$$= \left[\frac{\left[(a-b)(k-1)-a\right]a^{k-2} - b^{k-1}}{(a-b)^2}\right]\mu(k-1)$$

Schilling and Harris (2012, p. 173)

Residue Method: MATLAB

"MATLAB has a built-in function called residue that can be used to compute the residue terms for partial fraction expansions."

```
[r,p,k] = residue(b,a)
[b,a] = residue(r,p,k)
```

where

`b` = vector of length m+1 containing numerator coefficients ($m \leq n$)

`a` = vector of length n+1 containing denominator coefficients

`r` = vector of length n containing the residues R_i in

$$x(k) = R_0\delta(k) + \left[R_1 p_1^k + R_2 p_2^k + \cdots + R_n p_n^k \right] \mu(k)$$

`p` = vector of length n containing the poles

`q` = residue R_0

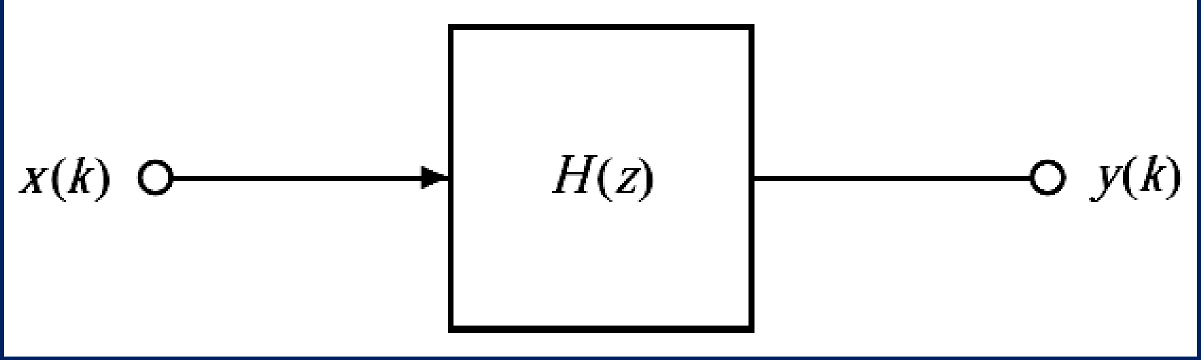
Schilling and Harris (2012, p. 173)

Transfer Functions

- In addition to difference equations, LTI systems can also be represented by transfer functions H(z).
- The transfer function is defined as the ratio of the Z transform of the zero-state response and the Z transform of the input:

$$H(z) \stackrel{\text{def}}{=} \frac{Y_{zs}(z)}{X(z)}, \qquad \text{i.e., initial condition is zero}$$

$$Y_{zs}(z) = H(z)X(z)$$



Schilling and Harris (2012, p. 174)

Transfer Function

Taking the Z-transform of the difference equation

$$y(k)+\sum_{i=1}^{n} a_i y(k-i)=\sum_{i=0}^{m} b_i x(k-i)$$

with zero initial condition yields

$$Y(z)+\sum_{i=1}^{n} a_i z^{-i} Y(z)=\sum_{i=0}^{m} b_i z^{-i} X(z)$$

$$\left(1+\sum_{i=1}^{n} a_i z^{-i}\right) Y(z)=\left(\sum_{i=1}^{n} b_i z^{-i}\right) X(z)$$

Thus,

$$H(z)=\frac{Y(z)}{X(z)}=\frac{b_0+b_1 z^{-1}+\cdots+b_m z^{-m}}{1+a_1 z^{-1}+\cdots+a_n z^{-n}}$$

Schilling and Harris (2012, p. 175)

Transfer Function

Multiplying the nominator and denominator of the transfer function by z^n converts it to positive powers of z:

$$H(z) = \frac{z^{n-m}\left(b_0 z^m + b_1 z^{m-1} + \cdots + b_m\right)}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

When $m \le n$, $H(z)$ has $n - m$ zeros at $z = 0$.
When $m > n$, $H(z)$ has $m - n$ poles at $z = 0$.

The zero-state response can be computed from the transfer function using

$$y_{zs}(k) = Z^{-1}\left\{H(z)X(z)\right\}$$

Schilling and Harris (2012, p. 175-176)

Transfer Function: Example 1

Consider the system

$$y(k)-1.2y(k-1)+0.32y(k-2)=10x(k-1)+6x(k-2)$$

The transfer function of this system is

$$H(z)=\frac{b_0+b_1z^{-1}+\cdots+b_mz^{-m}}{1+a_1z^{-1}+\cdots+a_nz^{-n}}$$

$$=\frac{10z^{-1}+6z^{-2}}{1-1.2z^{-1}+0.32z^{-2}}$$

$$=\frac{10z+6}{z^2-1.2z+0.32}$$

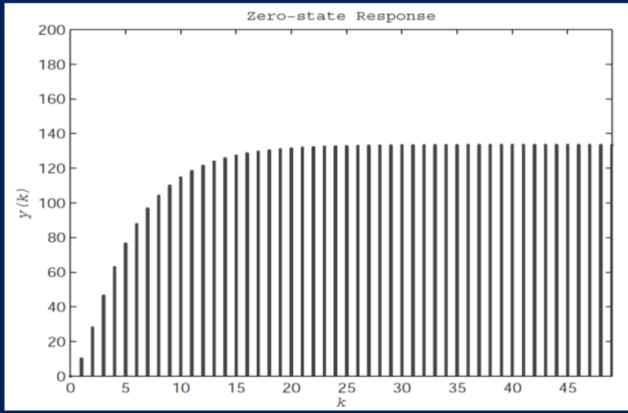
Schilling and Harris (2012, p. 175)

Transfer Function: Example 1

The zero-state response of this system can be computed by $y_{zs}(k) = Z^{-1}\{H(z)X(z)\}$. If the input is $x(k) = \mu(k)$, then

$$Y(z) = H(z)X(z) = \left(\frac{10z+6}{z^2-1.2z+0.32}\right)\frac{z}{z-1}$$

$$= \frac{(10z+6)z}{(z-0.8)(z-0.4)(z-1)}$$



The initial value is $y(0) = b_0\delta(m\text{-}n) = 0$. Using the residue method yields

$$\text{Res}(0.8,k) = -175(0.8)^k, \text{Res}(0.4,k) = 41.7(0.4)^k, \text{Res}(1,k) = 133.3$$

and

$$y_{zs}(k) = \left(133.3 - 175\times 0.8^k + 41.7\times 0.4^k\right)\mu(k)$$

Schilling and Harris (2012, p. 176-177)

TF: Poles, Zeros, Modes

Recall the transfer function

$$H(z) = \frac{b_0 z^{n-m}(z - z_1)(z - z_2)\cdots(z - z_m)}{(z - p_1)(z - p_2)\cdots(z - p_n)}$$

- "The roots of the numerator polynomial are called the **zeroes** of the discrete-time system while the roots of the denominator polynomial are called the **poles**."
- Recall that $Y(z) = H(z)X(z)$. The output $y(k)$ can be decomposed into two types of modes:

$$y(k) = \text{natural modes} + \text{forced modes}$$

where natural modes are generated by poles of $H(z)$ and forced modes are generated by poles of $X(z)$.

Schilling and Harris (2012, p. 176-177)

TF: Poles, Zeros, Modes

- For a **simple pole** at $z = p$, the corresponding mode is of the form
$$cp^k \mu(k)$$
- For a **multiple pole** of multiplicity r, the corresponding mode is of the form
$$\left(c_0 + c_1 k + \cdots + c_{r-1} k^{r-1}\right) p^k \mu\left(k\right), \quad r \geq 1$$
- If $H(z)$ has poles at p_i for $1 \leq i \leq n$, and $X(z)$ has poles at q_i for $1 \leq i \leq r$, and if all the poles are simple, then
$$y(k) = \sum_{i=1}^{n} \underbrace{c_i p_i^k \mu(k)}_{\text{Natural mode}} + \sum_{i=1}^{r} \underbrace{d_i q_i^k \mu(k)}_{\text{Forced mode}}$$

where c_i and d_i are coefficients to be determined.

Schilling and Harris (2012, p. 178)

Pole-Zero Cancellation

- "Since $Y(z) = H(z)X(z)$, pole-zero cancellation can happen when some zeros of $H(z)$ are the same as poles of X(z)."
- Zeros of $H(z)$ can suppress forced modes of $X(z)$.
- Zeros of $X(z)$ can suppress natural modes of $H(z)$.

Example: Consider the transfer function and input signal

$$H(z)=\frac{10(z+0.6)}{(z-0.8)(z-0.4)}, x(k)=10(-0.6)^{k}\mu(k)-4(-0.6)^{k-1}\mu(k-1)$$

$$X(z)=10\left(\frac{z}{z+0.6}\right)-4z^{-1}\left(\frac{z}{z+0.6}\right)=\frac{10(z-0.4)}{(z+0.6)}$$

Thus, $y(k)=Z^{-1}\{H(z)X(z)\}=Z^{-1}\left\{\frac{100}{z-0.8}\right\}=100(0.8)^{k-1}\mu(k-1)$

Schilling and Harris (2012, p. 178-179)

Stable Modes

A multiple mode can be written as $c(k)\,p^k\mu(k)$ where $c(k)$ is polynomial of degree r-1 with r being multiplicity of pole.

$$r > 1:\ \ |c(k)| \to \infty \text{ as } k \to \infty$$

$$|p| < 1:\ \ \left|p^k\right| \to 0 \text{ as } k \to \infty$$

$$\lim_{k\to\infty}\left|c(k)\,p^k\right| = \lim_{k\to\infty}\frac{|c(k)|}{|p|^{-k}} = \lim_{k\to\infty}\frac{|c(k)|}{\exp\left[-k\log\left(|p|\right)\right]}$$

To evaluate this limit, we need the L'Hospital's rule.

Schilling and Harris (2012, p. 179)

Stable Modes

- The (r-1)th derivative of $c(k)$ is the constant $(r\text{-}1)!c_{r\text{-}1}$.
- The (r-1)th derivative of the exponential is

$$\left[-\log\left(\left|p\right|\right)\right]^{r-1} \exp\left[-k\log\left(\left|p\right|\right)\right]$$

- Since $\log(|p|) < 0$, the limit is

$$\lim_{k\to\infty} c(k)\,p^{k} = 0 \;\Leftrightarrow\; \left|p\right| < 1$$

- "That is the exponential factor p^k goes to zero faster than the polynomial factor $c(k)$ goes to infinity if and only if $|p| < 1$."
- "Thus, a **stable mode** is associated with a pole that lies inside the unit circle."

DC Gain

- "When all poles lie inside unit circle, all natural modes decays to zero as time increases."
- Let determine the steady-state response to step input $x(k) = c\mu(k)$ (also called DC input).
- Using $Y(z) = H(z)X(z)$ and the final value theorem

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(z-1)Y(z) = \lim_{z\to 1}(z-1)H(z)Z\left[c\mu(k)\right]$$

$$= \lim_{z\to 1}(z-1)H(z)\left(\frac{cz}{z-1}\right) = H(1)c$$

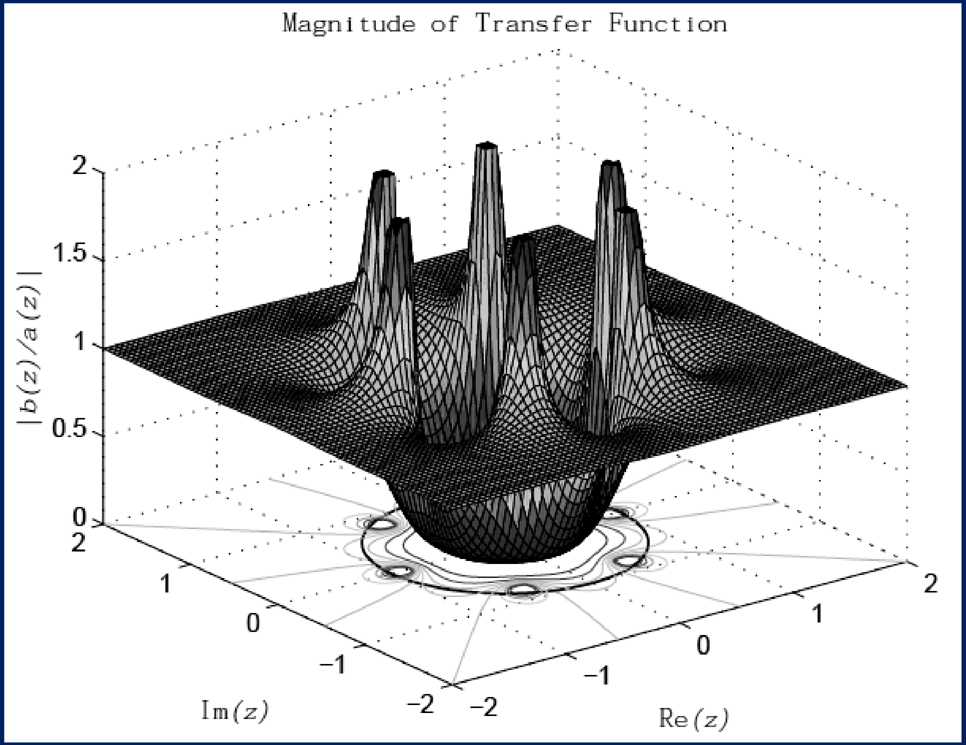
- DC gain is the amount by which the DC input is scaled as it passes through the system to produce a steady-state output:

$$\text{DC gain} = H(1)$$

Schilling and Harris (2012, p. 180)

DC Gain: Example

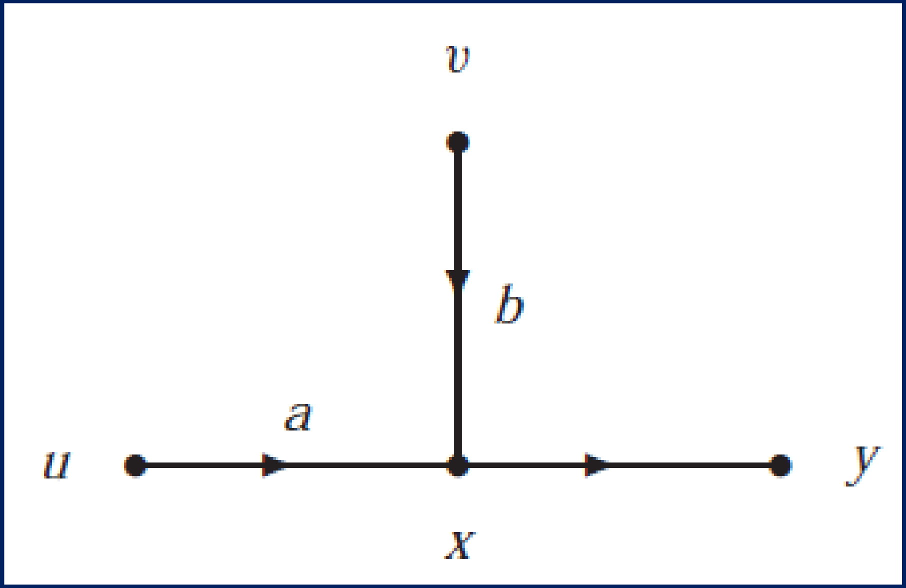
- "Consider the transfer function $H(z) = z^n \big/ \left(z^n - r^n\right)$ of a comb filter where $0 < r < 1$."
- $H(z)$ has n zeroes at the origin and n poles equally spread out on a circle of radius r centered at the origin.
- In case $n = 6$, $r = 0.9$, DC gain $= H(1) = 1 \big/ \left(1 - r^n\right) = 2.134$



Schilling and Harris (2012, p. 180)

Signal Flow Graph

"Signal flow graph is a collection of **nodes** (dots) interconnected by **arcs** (directed line segments)."



$$x = au + bv$$

$$y = x$$

Unlabeled arc has gain = 1.

Schilling and Harris (2012, p. 181)

Signal Flow Graph

Rewrite transfer function as

$$H(z)=\frac{b_0+b_1z^{-1}+\cdots+b_mz^{-m}}{1+a_1z^{-1}+\cdots+a_nz^{-n}}$$

$$=\left(\frac{b_0+b_1z^{-1}+\cdots+b_mz^{-m}}{1}\right)\left(\frac{1}{1+a_1z^{-1}+\cdots+a_nz^{-n}}\right)$$

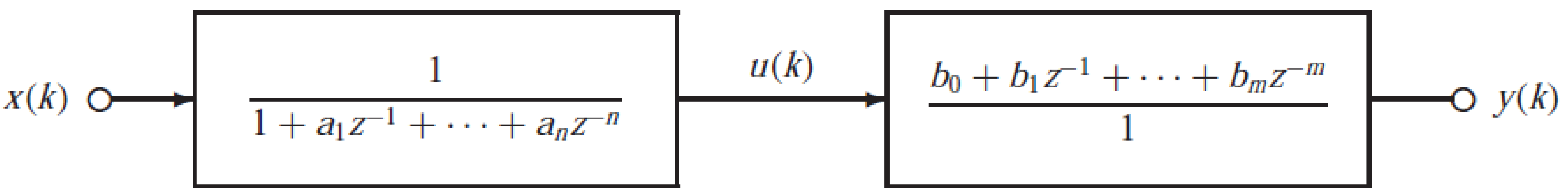
$$=H_b(z)H_a(z)$$

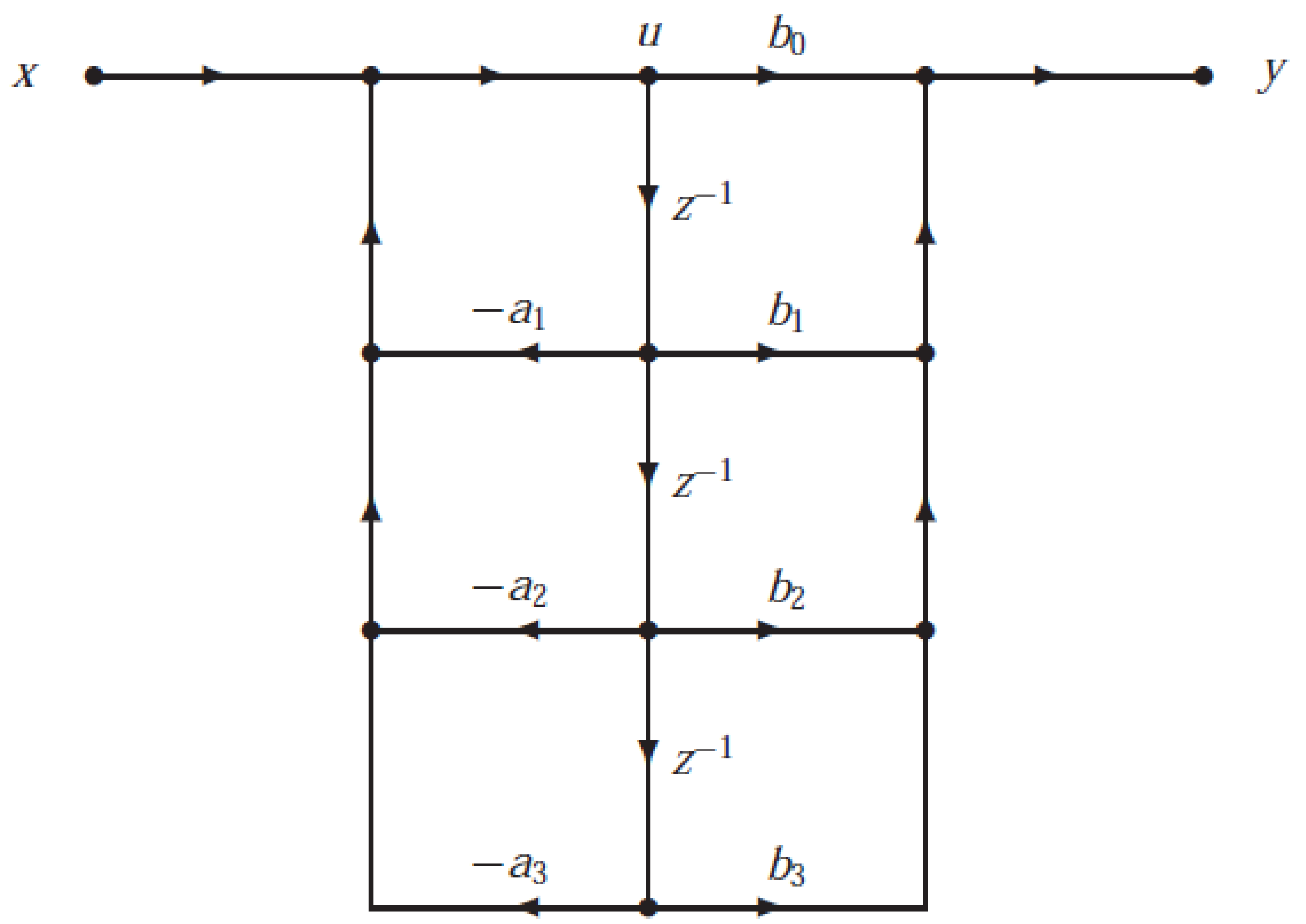
Let $U(z)=H_a(z)X(z)$. We then have

$$u(k)=x(k)-\sum_{i=1}^{n}a_iu(k-i),\ y(k)=\sum_{i=1}^{n}b_iu(k-i)$$

Schilling and Harris (2012, p. 182)

Signal Flow Graph





Schilling and Harris (2012, p. 182)

ARMA Models

Auto-regressive (AR) model (first subsystem):

$$H_{AR}(z) = \frac{b_0}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

Moving-average (MA) model (second subsystem):

$$H_{MA}(z) = b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}$$

Auto-regressive moving-average (ARMA) model:

$$H_{ARMA}(z) = H_{AR}(z) H_{MA}(z)$$

Schilling and Harris (2012, p. 183)

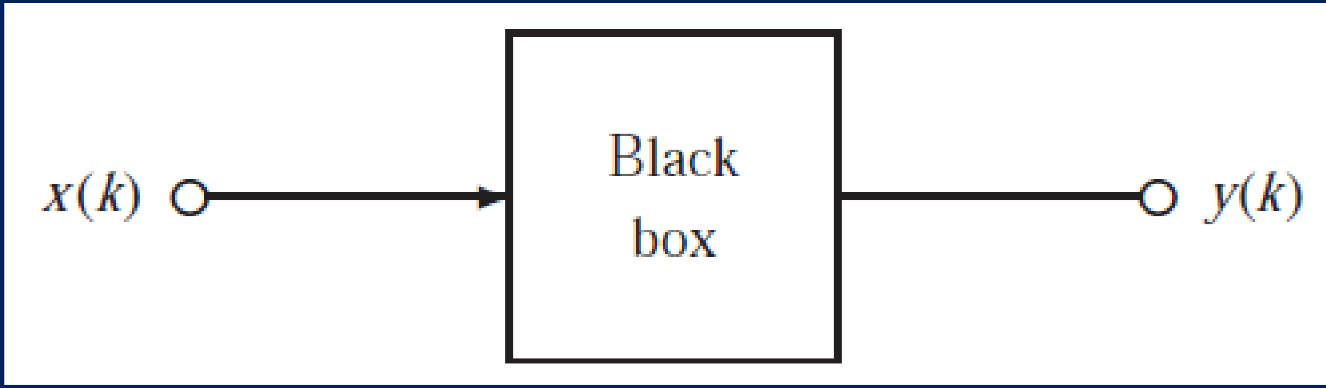
Frequency Response

For stable linear system S with sampling interval T the frequency response is defined as

$$H(f) \equiv H(z)\Big|_{z=\exp(i2\pi fT)}, \quad 0 \le |f| \le \frac{f_s}{2}$$

Schilling and Harris (2012, p. 191)

System Identification



Consider AR model

$$H_{AR}(z) = \frac{1}{a_0 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

Associated difference equation

$$\sum_{i=0}^{n} a_i y(k-i) = x(k)$$

Schilling and Harris (2012, p. 198)

System Identification

Given real input and output

$$D = \{[x(k), y(k)] \in \mathbb{R}^2 \mid 0 \le k < N\}$$

The AR model becomes

$$\sum_{i=0}^{n} a_i y(k-i) = x(k), \quad 0 \le k < N$$

which can be written as a linear system of equations with LS solution

where $\quad \mathbf{Y}\mathbf{a} = \mathbf{x} \qquad \mathbf{a} = \left(\mathbf{Y}^T\mathbf{Y}\right)^{-1}\mathbf{Y}^T\mathbf{x}$

$$\mathbf{Y} = \begin{bmatrix} y(0) & y(-1) & \cdots & y(-n) \\ y(1) & y(0) & \cdots & y(1-n) \\ \vdots & \vdots & \ddots & \vdots \\ y(N-1) & y(N-2) & \cdots & y(N-1-n) \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Schilling and Harris (2012, p. 199-200)

Speech Synthesis

- **Phoneme**: fundamental unit of speech
- "**Unvoiced phonemes** associated with turbulence in vocal tract include fricatives (s, sh, f) and terminal sounds (p, k, t)."
- "**Voiced phonemes** are associated with periodic excitation of vocal chords including vowels, nasal sounds, transient terminal sounds (b, d, g)."

Schilling and Harris (2012, p. 142)

Speech Synthesis

"**Voiced phonemes** can be modeled as response of digital filter to periodic impulse train with period M:

$$x(n) = \sum_{i=0}^{\infty} \delta(n - iM)$$

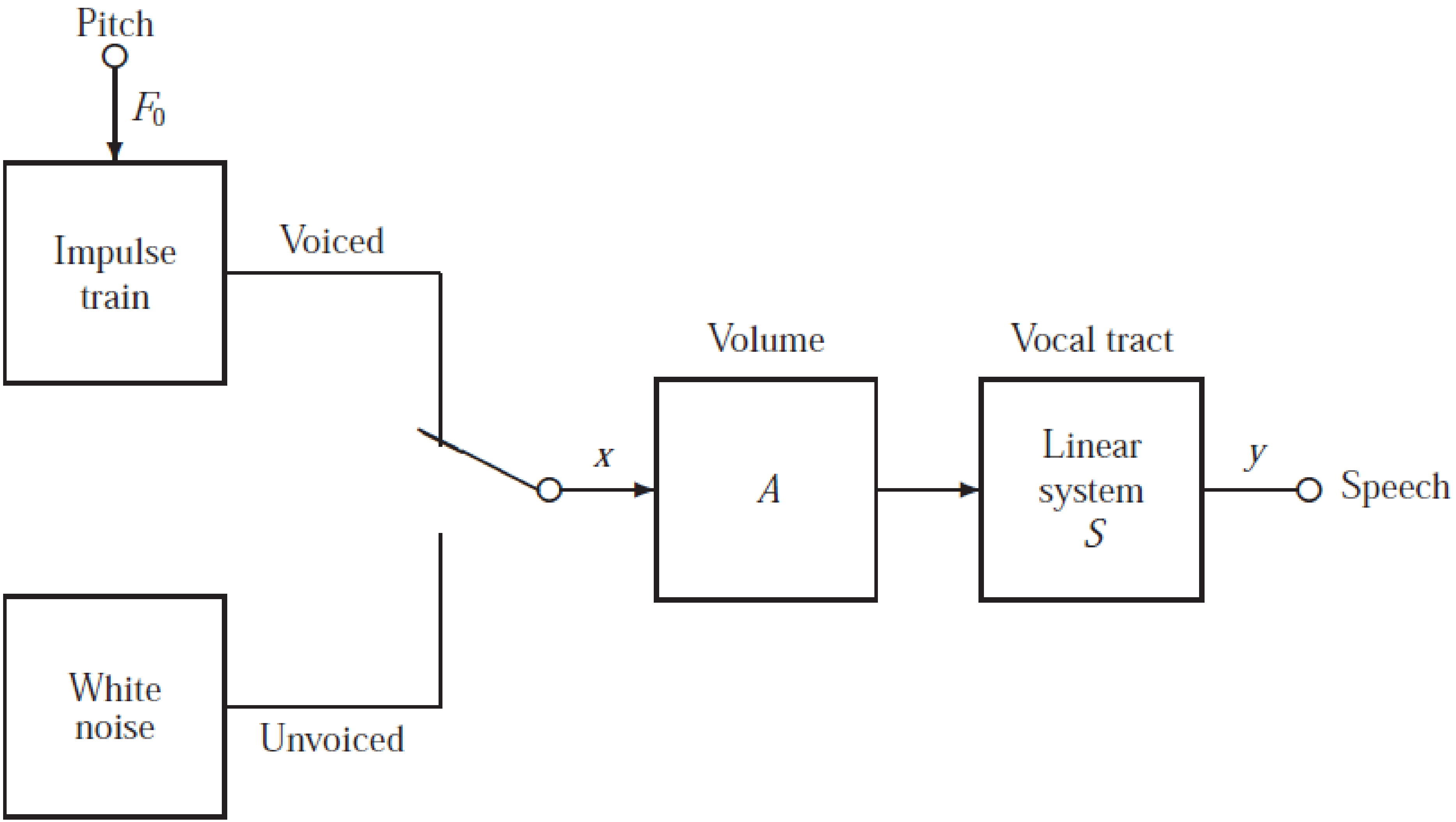
If T is sampling interval, period = MT.

Pitch of fundamental freq. = F_0 = 1/MT

Human pitch range = [50, 400] Hz

Exercise: Determine the pitch of vowel sounds such as a, e, i, o, u

Digital Speech Synthesis



Speech Synthesis

"Auto-regressive system is effective for modeling vocal tract:"

$$y(n) = x(n) - \sum_{i=1}^{n} a_i y(n-i)$$

Given input $x(n)$ and output $y(n)$, finding $a \in \mathbb{R}^n$ is a **system identification** problem.

Schilling and Harris (2012, p. 149)

Reference

- Schilling, R. J. and S. L. Harris, 2012, Fundamentals of Digital Signal Processing using MATLAB, Second Edition, Cengage Learning.