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1. LTI discrete-time system described with the difference equation

$$y(k) - y(k-1) + 0.21y(k-2) = 3x(k) + 2x(k-2)$$

1.1. Find the characteristic polynomial $a(z)$ and the input polynomial $b(z)$. Express $a(z)$ in factored form.

Ans

The characteristic polynomial can be obtained in case of zero input, $x(k) = 0$. The trial solution, $y(k) = z^k$, is substituted into the difference equation expressed as

$$z^k - z^{k-1} + 0.21z^{k-2} = 0.$$

By multiplying the equation with z^{2-k}, the equation becomes

$$z^2 - z + 0.21 = 0.$$

It appears that the polynomial on the left side is the characteristic polynomial given as

$$a(z) = z^2 - z + 0.21.$$

The $a(z)$ can be written in the factored form expressed as

$$a(z) = (z - 0.7)(z - 0.3).$$

The input polynomial can be obtained by considering zero output, $y(k) = 0$. By $x(k) = z^k$ substitution, the equation is expressed as

$$3z^k + 2z^{k-2} = 0.$$

By multiplying the equation with z^{2-k}, the equation becomes

$$3z^2 + 2 = 0.$$

It appears that the polynomial on the left side is the input polynomial given as

$$b(z) = 3z^2 + 2.$$

1.2. Write down the general form of the zero-input response $y_{zi}(k)$.

Ans

The general from of the zero-input response is expressed as

$$y_{zi}(k) = \sum_{i=1}^{N} c_i p_i^k \qquad \text{when } k \geq -N.$$

For the given system, the y_{zi} is

$$y_{zi}(k) = c_1 0.7^k + c_2 0.3^k \qquad \text{when } k \geq -2.$$

1.3. Find the zero-input response when the initial condition is $y(-1) = 1$ and $y(-2) = -1$.

Ans

By applying the initial conditions, the $y_{zi}(k)$ can be expressed as

$$y_{zi}(-1) = 1 = c_1 0.7^{-1} + c_2 0.3^{-1}$$
$$y_{zi}(-2) = -1 = c_1 0.7^{-2} + c_2 0.3^{-2}$$

The system of equations can be written in the matrix form given as

$$\begin{bmatrix} 0.7^{-1} & 0.3^{-1} \\ 0.7^{-2} & 0.3^{-2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0.7^{-1} & 0.3^{-1} \\ 0.7^{-2} & 0.3^{-2} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1.5925 \\ -0.3825 \end{bmatrix}$$

Therefore, the zero-input is response of the given initial conditions is

$$y_{zi}(k) = 1.5925(0.7)^k - 0.3825(0.3)^k.$$

1.4. Write down the general form of the zero-state response when the input is $x(k) = 2(0.5)^{k-1}\mu(k)$.

Ans

The casual exponential input can be expressed as

$$x(k) = \frac{2}{0.5}(0.5)^k\mu(k).$$
$$x(k) = 4(0.5)^k\mu(k)$$

It appears that the amplitude is $A = 4$ and the exponential factor is $p_0 = 0.5$. For the simple root of the characteristic polynomial, the general form of the zero-state response is given as

$$y_{zs}(k) = \sum_{i=0}^{N} d_i p_i^k \mu(k),$$

where $d_i = \frac{A(z-p_i)b(z)}{(z-p_0)a(z)}\Big|_{z=p_i}$, $0 \le i \le N$.

For the given system, $p_1 = 0.7, p_2 = 0.3$. $y_{zs}(k)$ is expressed as

$$y_{zs}(k) = [d_0(0.5)^k + d_1(0.7)^k + d_2(0.3)^k]\mu(k).$$

1.5. Find the zero-state response using the input in Problem 1.4.

Ans

The weighting coefficients (d_i) are as the following

$$d_0 = \frac{4(z-0.5)(3z^2+2)}{(z-0.5)(z-0.7)(z-0.3)}\Bigg|_{z=0.5}$$
$$d_0 = \frac{4(3(0.5)^2+2)}{(0.5-0.7)(0.5-0.3)}$$
$$d_0 = -275$$
$$d_1 = \frac{4(z-0.7)(3z^2+2)}{(z-0.5)(z-0.7)(z-0.3)}\Bigg|_{z=0.7}$$
$$d_1 = \frac{4(3(0.7)^2+2)}{(0.7-0.5)(0.7-0.3)}$$
$$d_1 = 173.5$$
$$d_2 = \frac{4(z-0.3)(3z^2+2)}{(z-0.5)(z-0.7)(z-0.3)}\Bigg|_{z=0.3}$$
$$d_2 = \frac{4(3(0.3)^2+2)}{(0.3-0.5)(0.3-0.7)}$$
$$d_2 = 113.5$$

We can write the $y_{zs}(k)$ as

$$y_{zs}(k) = [-275(0.5)^k + 173.5(0.7)^k + 113.5(0.3)^k]\mu(k).$$

1.6. Find the complete response using the initial condition in Problem 1.3 and the input in Problem 1.4.

Ans

The complete response system is the sum of $y_{zi}(k)$ and $y_{zs}(k)$.

$$y(k) = 1.5925(0.7)^k - 0.3825(0.3)^k + [d_0(0.5)^k + d_1(0.7)^k + d_2(0.3)^k]\mu(k)$$

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2. Compute and plot the impulse response of the LTI discrete-time system described the difference equation

$$y(k) + 0.25y(k-2) = x(k-1).$$

Ans

The characteristic polynomial and the input polynomial of the system is expressed as the following equations.

$$a(z) = z^2 + 0.25$$

$$b(z) = z$$

It appears that $p_1 = 0.5i, p_2 = -0.5i$. For the LTI system, the general impulse response is given as

$$h(k) = c_0\delta(k) + \sum_{i=1}^{N} c_i(p_i)^k\mu(k),$$

where $c_i = \frac{(z-p_i)b(z)}{za(z)}\Big|_{z=p_i}$, $0 \le i \le N$ and $p_0 = 0$.

For the given system, the impulse response is expressed as

$$h(k) = c_0\delta(k) + [c_1(0.5i)^k + c_2(-0.5i)^k]\mu(k).$$

The coefficients are evaluated by

$$c_0 = \frac{(z-0)z}{z(z+0.5i)(z-0.5i)}\Bigg|_{z=0}$$

$$c_0 = 0$$

$$c_1 = \frac{(z-0.5i)z}{z(z+0.5i)(z-0.5i)}\Bigg|_{z=0.5i}$$

$$c_1 = \frac{1}{(0.5i+0.5i)}$$

$$c_1 = -i$$

$$c_2 = \frac{(z+0.5i)z}{z(z+0.5i)(z-0.5i)}\bigg|_{z=-0.5i}$$

$$c_2 = \frac{1}{(-0.5i-0.5i)}$$

$$c_2 = i$$

The impulse response is expressed as

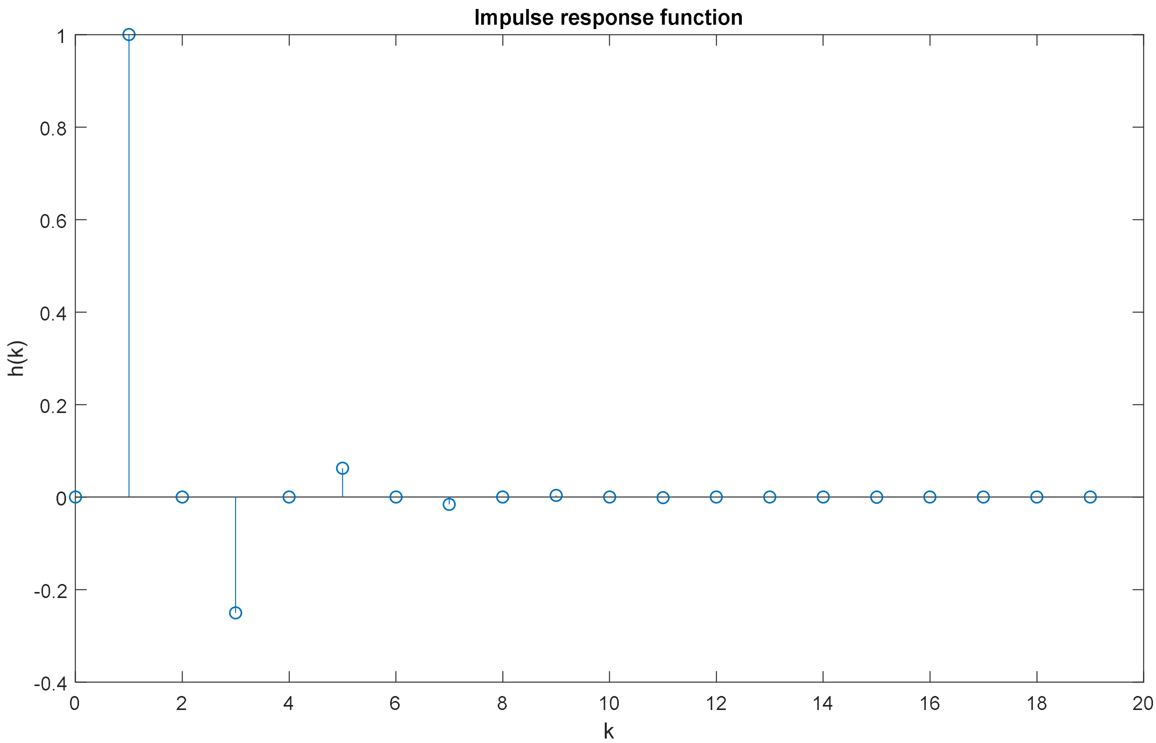
$$h(k) = [-i(0.5i)^k + i(-0.5i)^k]\mu(k)$$

We simplify the impulse response as the following

$$h(k) = i^{k+1}[-(0.5)^k + (-0.5)^k]\mu(k)$$

$$h(k) = \begin{cases} 0 & ,k \text{ is even} \\ (-1)^{\frac{k-1}{2}} \times 2(0.5)^k\mu(k) & ,k \text{ is odd} \end{cases}$$

The impulse response of the given system is shown in the below figure.



* In MATLAB, h(k) is computed by

```
h = 2*(-1).^((k-1)/2).*(0.5).^k.*(k>=0).*(mod(k,2)==1);
```

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3. Echo detection: Let the input signal $x(k)$ be the multifrequency chirp described by

$$f(k) = \frac{kf_s}{2(M-1)}, \qquad M = 512, f_s = 1 \text{ MHz}$$

$$x(k) = \sin 2\pi f(k)kT\,, \quad T = 1/f_s$$

where k = 1, 2, ..., M. Suppose the received signal consists of L = 2048 samples and is given by

$$y(k) = \alpha x_z(k-d) + \eta(k), \qquad 0 \le k \le L$$

where $x_z(k)$ the transmitted signal which is zero-extended such that its length becomes L samples, and $\eta(k)$ is the atmospheric noise.

3.1. Generate and plot the M-point input signal using the above formulas.

Ans

The frequency function, $f(k)$, is defined as a function in the MATLAB code.

```
function f = cal_f(k)
    f_s = 1E6;
    M = 512;
    f = k*f_s/2/(M-1);
end
```

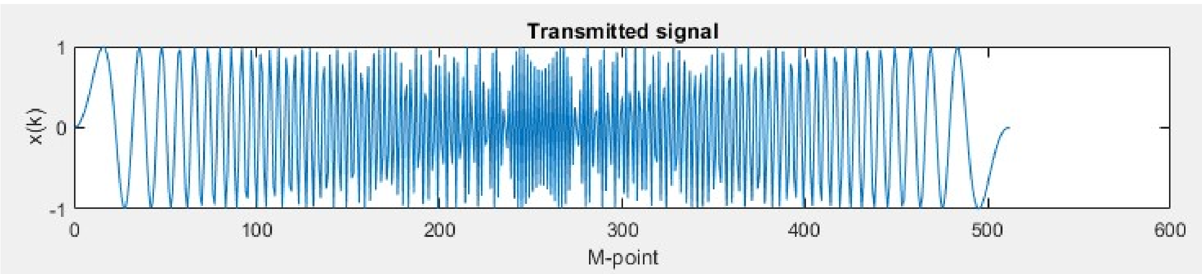
The input signal, $x(k)$, is also defined as a function in the MATLAB code. The logical operations are used as a step function, $\mu(k)$, and the limitation for M-point ($k \le 512$).

```
function x = input(k)
    f_s = 1E6;
    x = sin(2*pi.*cal_f(k).*k/f_s).*(k>=0).*(k<=512);
end
```

We generate the input signal by the following code.

```
%3.1
M = 512;
k_M = 1:M;
x = input(k_M);
subplot(5,1,1)
plot(k_M,x);
title('Transmitted signal');
xlabel('M-point');
ylabel('x(k)');
```

The plot of the transmitted signal is shown as



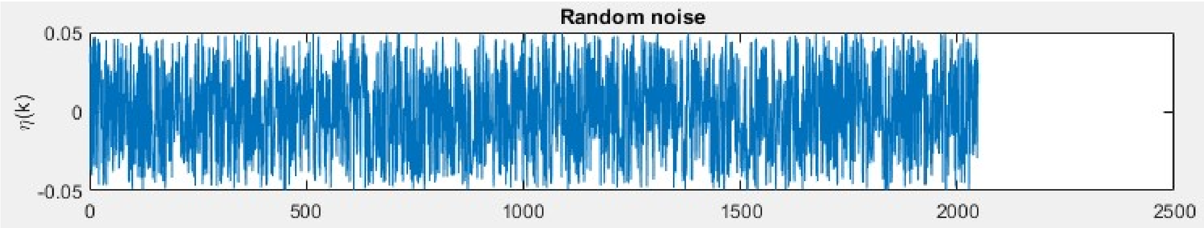
3.2. Generate the L-point noise $\eta(k)$using the MATLAB command rand such that its value is in the range [-0.05,0.05].

Ans

The noise in range [-0.05,0.05] is generated by the following code.

```
%3.2
L = 2048;
k_L = 1:L;
noise = rand(1,L)/10-0.05;
subplot(5,1,2)
plot(k_L,noise);
title('Random noise')
xlabel('L-point');
ylabel('\eta(k)');
```

We plot the noise function shown as



3.3. Generate and plot the received signal $y(k)$ with attenuation factor $a = 0.01$, and delay $d = 500$.

Ans

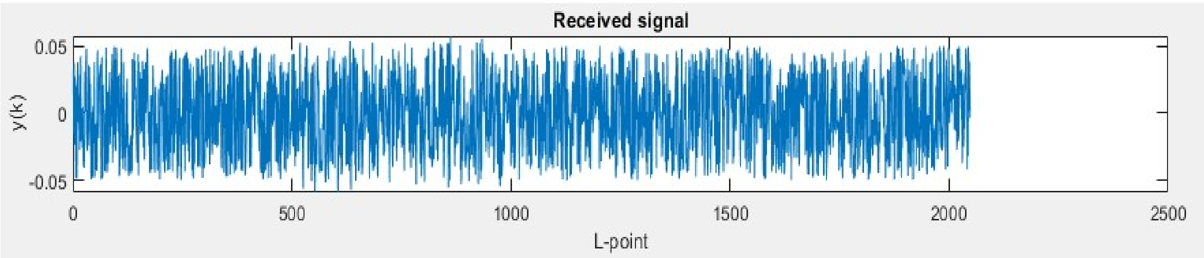
We define the output function as the following code.

```
function y = output(k,noise)
    alpha = 0.01;
    d = 500;
    y = alpha*input(k-d)+noise;
end
```

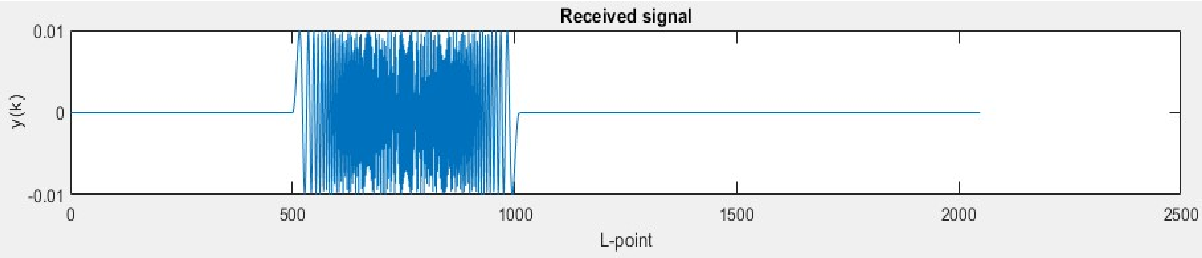
The received signal is generated by the given code.

```
%3.3
y = output(k_L,noise);
subplot(5,1,3)
plot(k_L,y);
title('Received signal')
xlabel('L-point');
ylabel('y(k)');
```

The plot of the output signal is shown as



To confirm the delayed signal, we remove the noise shown in the below plot.



3.4. Perform linear cross-correlation of $y(k)$ with the input signal $x(k)$ and plot the result. Also determine the delay d from the graph.

Ans

We define the linear cross-correlation as a function in the MATLAB code as the following

```
function z = linear_corr(x,y)
    L = length(y);
    M = length(x);
    D = zeros(L,L);
    for i=1:L
        for j=1:M
            k = i+j-1;
            if k<=L
                D(i,k) = x(j);
            end
        end
    end
    D = D/L;
    z = D*y';
end
```

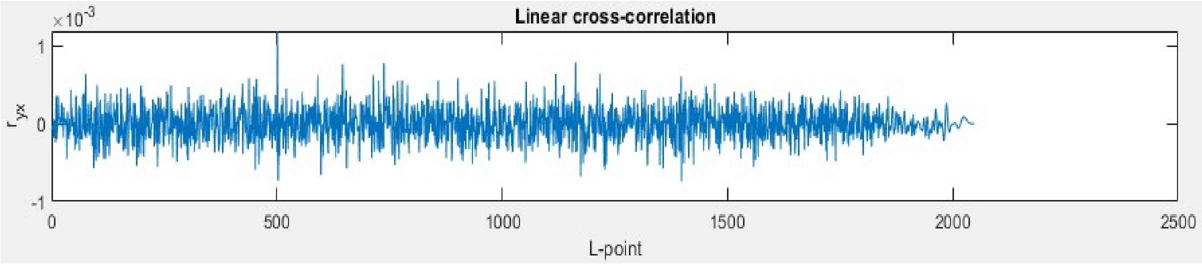
In the function, the cross-correlation matrix, $D(x)$, is constructed. The linear cross-correlation (r_{yx}) is evaluated by

$$r_{yx} = D(x)y(k)$$

The linear cross-correlation of the system is calculated by

```
%3.4
ryx = linear_corr(x,y);
subplot(5,1,4)
plot(k_L,ryx);
title('Linear cross-correlation')
xlabel('L-point');
ylabel('r_{yx}');
```

The result is shown in the following figure.



From the above graph, the delayed value (d_{obs}) can be determined as the peak which is 501.

$$d_{\text{obs}} = 501$$

3.5. Perform normalized linear cross-correlation of $y(k)$ with the input signal $x(k)$ and plot the result.

Ans

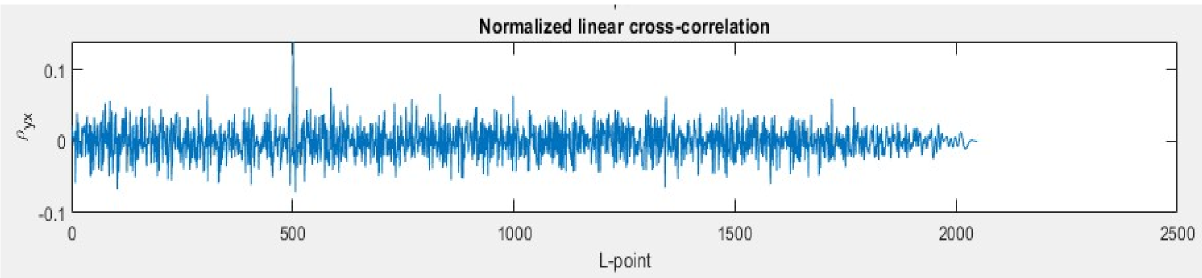
We define the normalized linear cross-correlation (NLCC) as a function in the MATLAB code.

```
function z_norm = norm_licorr(x,y)
    L = length(y);
    M = length(x);
    rxx = linear_corr(x,x);
    ryy = linear_corr(y,y);
    ryx = linear_corr(x,y);
    z_norm = ryx/sqrt((M/L)*rxx(1)*ryy(1));
end
```

LNCC is evaluated by the code.

```
%3.5
ryx_norm = norm_licorr(x,y);
subplot(5,1,5)
plot(k_L,ryx_norm);
title('Normalized linear cross-correlation')
xlabel('L-point');
ylabel('\rho_{yx}');
```

The result of NLCC is shown in the following figure.



It appears that the peak of ρ_{yx} is about 0.1. The correlation between transmitted signal and received signal is not strong. The reason is that the amplitude of the received signal is 0.01 of the transmitted signal.