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Signal and image processing

Homework 3

1. Consider the following signal

$$x(k) = \begin{cases} 10, & 0 \le k < 4 \\ -2, & 4 \le k < \infty \end{cases}$$

1.1 Write $x(k)$ as a difference of two step signals.

Ans

$$x(k) = 10\mu(k) - 12\mu(k-4)$$

1.2 Use the time shift property to find $X(z)$. Express your final answer as a ratio of two polynomials in z.

Ans

From the table of Z transform, $Z\{\mu(k)\} = z/(z-1)$.

$$\begin{aligned} X(z) &= \left(10 - \frac{12}{z^4}\right)\frac{z}{z-1} \\ &= \frac{(10z^4 - 12)z}{z^4(z-1)} \\ &= \frac{10z^4 - 12}{z^3(z-1)} \\ &= \frac{10 - 12z^{-4}}{1 - z^{-1}} \end{aligned}$$

1.3 Find the region of convergence of $X(z)$.

Ans

The region of convergence is $|z| > 1$.

2. Consider the following finite anti-causal signal where $x(-1) = 4$.

$$x = [\ldots, 0, 0, 3, -7, 2, 9, 4]$$

2.1 Find the Z transform $X(z)$, and express it as a ratio of two polynomials in z.

Ans

The given anti-casual signal is expressed as

$$x(k) = 4\delta(k+1) + 9\delta(k+2) + 2\delta(k+3) - 7\delta(k+4) + 3\delta(k+5).$$

The general finite Z transform of the finite anti-casual signal is expressed as

$$X(z) = x(-r)z^r + x(-r+1)z^{r-1} + \cdots + x(-1)z.$$

For the given signal, the expression of $X(z)$ is following as

$$X(z) = \frac{4z + 9z^2 + 2z^3 - 7z^4 + 3z^5}{1}.$$

2.2 What is the region of convergence of $X(z)$?

Ans

The region convergence is the entire of the complex plane because $X(z)$ has no pole.

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3. Consider the following causal signal

$$x(k) = 2(0.8)^{k-1}\mu(k).$$

3.1 Find the Z transform $X(z)$, and express it as a ratio of two polynomials in z.

Ans

From the Z transform table, $Z\{c^k\mu(k)\} = \frac{z}{z-c}$. The Z transform of the given signal is expressed as

$$\begin{aligned} X(z) &= \frac{2}{0.8}\frac{z}{z-0.8} \\ &= \frac{5}{2}\left[\frac{z}{z-0.8}\right]. \end{aligned}$$

3.2 What is the region of convergence of $X(z)$?

Ans

We arrange $X(z)$ expressed as

$$X(z) = \frac{5}{2}\left[\frac{1}{1-0.8/z}\right]$$

By the convergence of the geometric series, the region of convergence is $|z| > 0.8$.

4. Consider the following Z transform

$$X(z) = \frac{z^4 + 2z^3 + 3z^2 + 2z + 1}{z^4}, |z| > 0.$$

4.1 Rewrite $X(z)$in terms of negative powers of z.

Ans

$$\begin{aligned} X(z) &= 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{2}{z^3} + \frac{1}{z^4} \\ &= 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4} \end{aligned}$$

4.2 Find $x(k)$

Ans

$$x(k) = \delta(k) + 2\delta(k-1) + 3\delta(k-2) + 2\delta(k-3) + \delta(k-4)$$

4.3 Verify that $x(k)$ is consistent with the initial value theorem.

Ans

For the initial value theorem,

$$x(0) = \lim_{z\to\infty} X(z)$$

We firstly evaluate $x(0)$.

$$\begin{aligned} x(0) &= \delta(0) + 2\delta(0-1) + 3\delta(0-2) + 2\delta(0-3) + \delta(0-4) \\ &= 1 \end{aligned}$$

Then, we evaluate $\lim_{z\to\infty} X(z)$.

$$\begin{aligned} \lim_{z\to\infty} X(z) &= \lim_{z\to\infty}(1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}) \\ &= 1 \end{aligned}$$

Therefore,

$$x(0) = \lim_{z\to\infty} X(z).$$

The initial value theorem is verified.

4.4 Verify that $x(k)$ is consistent with the final value theorem.

Ans

For the final value theorem,

$$x(\infty) = \lim_{z\to 1}(z-1)X(z)$$

We firstly evaluate $x(\infty)$.

$$\begin{aligned} x(\infty) &= \delta(\infty) + 2\delta(\infty - 1) + 3\delta(\infty - 2) + 2\delta(\infty - 3) + \delta(\infty - 4) \\ &= 0 \end{aligned}$$

Then, we evaluate $\lim_{z\to 1}(z-1)X(z)$.

$$\begin{aligned} \lim_{z\to 1}(z-1)X(z) &= \lim_{z\to 1}\left[(z-1)\frac{z^4 + 2z^3 + 3z^2 + 2z + 1}{z^4}\right] \\ &= \lim_{z\to 1}(z-1) \times \lim_{z\to 1}\left(\frac{z^4 + 2z^3 + 3z^2 + 2z + 1}{z^4}\right) \\ &= 0 \end{aligned}$$

Therefore,

$$x(\infty) = \lim_{z\to 1}(z-1)X(z)$$

The final value theorem is verified.

5. Consider the following Z transform

$$X(z) = \frac{2z}{z^2 - 1}, |z| > 1$$

5.1 Find $x(k)$ for 0 £ k £ 5 using the synthetic division method.

Ans

We arrange the given signal,

$$\begin{aligned} X(z) &= \frac{2z(1)}{z^2(1 - z^{-2})}. \\ &= \frac{2z^{-1}}{1 - z^{-2}} \end{aligned}$$

For the synthetic division, we need to evaluate

$$\frac{1}{1 - z^{-2}} = 1 + z^{-2} + z^{-4} + O(z^{-6})$$

For $0 \le k \le 5$, we neglect $O(z^{-5})$ term. The $X(z)$ is expressed as

$$\begin{aligned} X(z) &= 2z^{-1}(1 + z^{-2} + z^{-4}). \\ &= 2(z^{-1} + z^{-3} + z^{-5}) \end{aligned}$$

By the synthetic division method, x(k) is

$$x(k) = 2[\delta(k-1) + \delta(k-3) + \delta(k-5)].$$

5.2 Find $x(k)$ using the partial fraction method.

Ans

We arrange the given signal,

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$$
\begin{aligned}
X(z) &= \frac{2z}{z^2-1} \\
&= \frac{2z}{(z-1)(z+1)} \\
&= 2\left(\frac{1}{2(z-1)} + \frac{1}{2(z+1)}\right) \\
&= \frac{1}{z-1} + \frac{1}{z+1}
\end{aligned}
$$

We know that

$$Z\{\mu(k)\} = \frac{z}{z-1}.$$

By the delayed property of the Z transform, we obtain

$$Z\{\mu(k-r)\} = z^{-r} \times \frac{z}{z-1}.$$

Therefore,

$$Z^{-1}\left\{\frac{1}{z-1}\right\} = \mu(k-1).$$

By the Z scale property of the Z transform, we know that

$$Z\{c^k\mu(k)\} = \frac{z}{z-c}.$$

Then,

$$Z\{(-1)^k\mu(k)\} = \frac{z}{z+1}.$$

By the delayed property of the Z transform, we obtain

$$Z\{(-1)^{k-r}\mu(k-r)\} = z^{-r} \times \frac{z}{z+1}.$$

Therefore,

$$Z\{(-1)^{k-1}\mu(k-1)\} = \frac{1}{z+1}$$

$$Z^{-1}\left\{\frac{1}{z+1}\right\} = (-1)^{k-1}\mu(k-1).$$

Finally, we obtain $x(k)$ expressed as

$$
\begin{aligned}
x(k) &= \mu(k-1) + (-1)^{k-1}\mu(k-1) \\
&= [1 + (-1)^{k-1}]\mu(k-1) \\
&= [1 + (-1)^{k-1}]\mu(k).
\end{aligned}
$$

5.3 Find $x(k)$ using the residue method

Ans

From the given signal,

$$X(z) = \frac{2z}{(z+1)(z-1)}$$

By the residue method, $x(k)$ is expressed as

$$x(k) = \mu(k-1)\sum_{i=1}^{q} \mathrm{Res}(p_i, k)$$

For the given signal, there are two simple poles at $z = \pm 1$.

$x(k) = [\mathrm{Res}(1, k) + \mathrm{Res}(-1, k)]\mu(k-1)$

$$= \left[(Z-1) \times \frac{2z}{(z+1)(z-1)} z^{k-1} \bigg|_{z=1} + (z+1) \times \frac{2z}{(z+1)(z-1)} z^{k-1} \bigg|_{z=-1} \right] \mu(k-1)$$
$$= [1 + (-1)^{k-1}]\mu(k-1)$$
$$= [1 + (-1)^{k-1}]\mu(k).$$

6. Consider the following Z transform. Find $x(k)$ using the time shift property and the residue method.

$$X(z) = \frac{100}{z^2(z-0.5)^3}, |z| > 0.5$$

Ans

We have to find $Z^{-1}\left\{\frac{1}{(z-0.5)^3}\right\}$. There is a multipole $(m = 3)$ at $z = 0.5$.

$$Z^{-1}\left\{\frac{1}{(z-0.5)^3}\right\} = \mu(k-1)\mathrm{Res}(0.5, k)$$
$$= \mu(k-1) \times \frac{1}{(3-1)!} \frac{d^{3-1}}{dz^{3-1}} \left\{ (z-0.5)^3 \frac{1}{(z-0.5)^3} z^{k-1} \right\} \bigg|_{z=0.5}$$
$$= \mu(k-1) \times \frac{1}{2} \frac{d^2 z^{k-1}}{dz^2} \bigg|_{z=0.5}$$
$$a(k) \equiv \frac{1}{2}(k-1)(k-2)(0.5)^{k-3}\mu(k-1).$$

By the delayed property and linearity,

$$Z\{100a(k-2)\} = \frac{100}{z^2(z-0.5)^3}$$

Finally, we obtain $x(k)$,

$$x(k) = 50(k-3)(k-4)(0.5)^{k-5}\mu(k-3).$$

7. Echoes are delayed signals that can be generated by the difference of the form

$$v(k) = y(k) + \alpha y(k-d), \qquad |\alpha| < 1$$

where $y(k)$ is the original signal and v(k) is the resulting signal with a single echo.

7.1 Load and listen to a sound snippet using the following commands.

```
load handel; sound(y,Fs);
```

Ans

After listening to the sound, I think that it's the Hallelujah chorus. The sound signal, $y(k)$, is plotted in Fig 1.

7.2 Let a $= 0.9$ and $d = 5000$. Use the MATLAB function filter to generate the echoed sound. To use the function filter, set the coefficient vectors $a = 1$ and b = [1,zeros(1,d),alpha]. Note that variables d and alpha must be initialized before setting a and b. Then, call the following commands to generate and listen to the echoed sound.

```
v = filter(b,a,y); sound(v,Fs);
```

Ans

After listening to the echo sound, I found that the sound is not clear and there is an echo. Fig 1 shows that the shape of $v(k)$ is different from $y(k)$ due to the echo.

7.3 The echo can be removed by using the difference equation, $w(k) + aw(k - d) = v(k)$. Call the following commands to remove the echo from $v(k)$ and listen to the resulting signal $w(k)$

w = filter(a,b,x); sound(w,Fs);

Ans

To remove the filter (echo), the filter function is used where the arguments between a and b are switched. The sound is reconstructed to the original sound. Fig 1 shows that the shape of $w(k)$ is the same as $y(k)$.

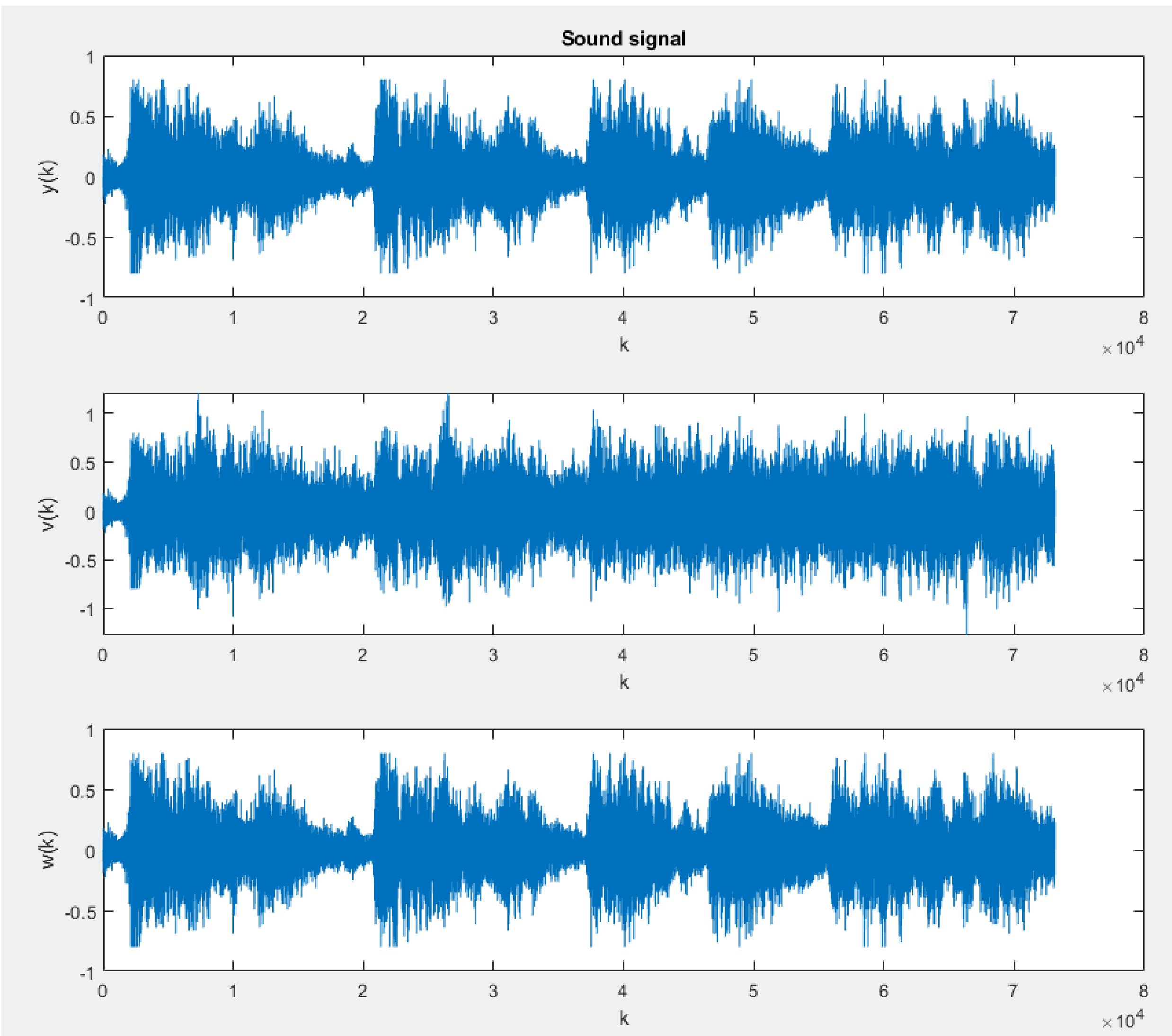

Figure 1 Generated sound: top) original sound, middle) echo sound, bottom) removed echo sound

The code to generate sound in 7.1, 7.2 and 7.3 is shown in Fig 2.

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hw3_7.m

```
%7.1
load handel; sound(y,Fs);
%7.2
alpha = 0.9; d = 5000;
a = 1; b = [1,zeros(1,d),alpha];
v = filter(b,a,y); sound(v,Fs);
%7.3
w = filter(a,b,v); sound(w,Fs);

subplot(3,1,1)
plot(y);
ylabel('y(k)');
xlabel('k');
title('Sound signal');
subplot(3,1,2)
plot(v);
ylabel('v(k)');
xlabel('k');
subplot(3,1,3)
plot(w);
ylabel('w(k)');
xlabel('k');
```

Figure 2 MATLAB code for generating sound signal