Introduction to Signal Processing and Sampling and Reconstructing Continuous-time Signals

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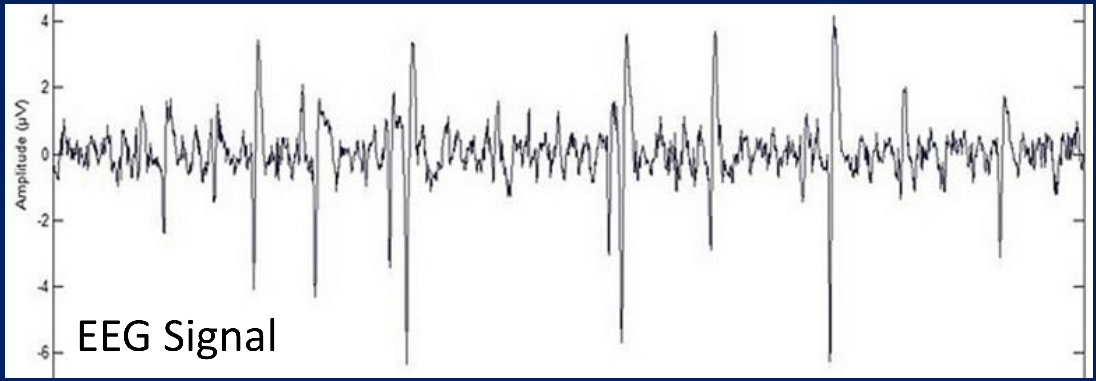
What is Signal?

- Signal is a variable or function that contains information of a physical system.
- Examples:
  - Audio and speech signals
  - Image and video signals
  - Medical signals: EEG, EKG, ultrasound
  - Geophysical signals: earthquakes, tide gauge, LIDAR
  - Climate signals
  - SONAR, RADAR



http://www.intechopen.com/source/html/18890/media/image4.jpeg

Types of Signals

- **Analog signal**
  - **Continuous signal** with infinite resolution
  - Typically referred to **electrical signal** converted from a physical variable by a **transducer**, e.g. microphone converting audio signal into electrical (analog) signal.
- **Discrete signal**: signal values are available at some discrete points in space or time
- **Digital signal**: value of discrete signal is stored with a **finite precision**, e.g., in computer as fixed-point or floating-point numbers.

Sampling

A **discrete-time signal** $x(n)$ can be obtained by taking samples of an **analog signal** $x_a(t)$

$$x(n) \stackrel{\text{def}}{=} x_a(nT), \quad |n| = 0, 1, 2, \ldots$$

where T is the **sampling interval** or time between samples, and sampling frequency or **sampling rate** $f_s \stackrel{\text{def}}{=} 1/T$ (Hz).

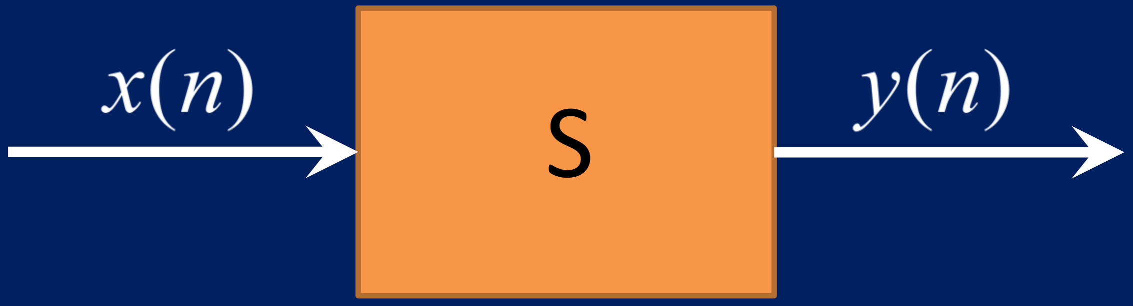
“When finite precision is used to represent the value of $x(n)$, the sequence of quantized values is called a **digital signal**.”

Schilling and Harris (2012; p.3,12)

Digital Signal Processor

In exploration seismology, signals are in digital form.

Any system or algorithm which processes input digital signal $x(n)$ and produces an output digital signal $y(n)$ is a **digital signal processor**.



Schilling and Harris (2012, p.3)

Causal and Acausal Signal

Causal signal: $x_a(t) = 0 \quad \text{for } t < 0$

Acausal signal: $x_a(t) \neq 0 \quad \text{for } \exists t < 0$

Examples of causal signals:

Heaviside step function: $\mu_a(t) \overset{\text{def}}{=} \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$

Dirac delta function:

$$\int_{-\infty}^{t} \delta_a(\tau)\, d\tau = \mu_a(t), \int_{-\infty}^{\infty} \delta_a(t)\, dt = 1; \qquad \delta_a(t) = 0, t \neq 0$$

$$\int_{-\infty}^{\infty} x_a(t)\delta_a(t - t_0)\, dt = x_a(t_0)$$ Sifting property of delta function

Schilling and Harris (2012, p.15-16)

System Classification

input system output

$$x \longrightarrow \boxed{S} \longrightarrow y$$

System is considered as a function or operator:

$$y = S(x)$$

- Continuous system: continuous input and output
- Discrete system: discrete input and output
- Linear system: $S(ax_1 + bx_2) = aS(x_1) + bS(x_2)$
- Time-invariant system: $S(x_a(t-\tau)) = y_a(t-\tau)$
- Bounded signal: $|x_a(t)| \le B_x \quad \text{for } t \in \mathbb{R}, B_x > 0$
- Stable system: every bounded input produces a bounded output (BIBO)

Schilling and Harris (2012, p.16-18)

Magnitude and Phase

- Forward and inverse Fourier transforms:

$$X_a(f) = F\left\{x_a(t)\right\} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x_a(t) e^{-i2\pi ft} dt$$

$$x_a(t) = F^{-1}\left\{X_a(f)\right\} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} X_a(f) e^{i2\pi ft} df$$

- Magnitude spectrum: $A_a(f) = \left|X_a(f)\right|$
- Phase spectrum: $\phi_a(f) = \angle X_a(f)$
- Polar form: $X_a(f) = A_a(f)\exp\left[i\phi_a(f)\right]$
- "For real $x_a(t)$, the magnitude spectrum is even function of *f*, and the phase spectrum is odd function of *f*."

Schilling and Harris (2012, p.18-19)

Filter and Frequency Response

- **Filter** is a system designed to reshape the spectrum of a signal.
- For linear time-invariant (LTI) continuous-time system S with input $x_a(t)$ and output $y_a(t)$, the **frequency response** $H_a(f)$ is defined as

$$H_a(f) \overset{\text{def}}{=} \frac{Y_a(f)}{X_a(f)} = A_a(f)\exp\left[i\phi_a(f)\right]$$

where $A_a(f)$ is magnitude response of S

$\phi_a(f)$ is phase response of S

Schilling and Harris (2012, p.19)

Impulse Response

- **Impulse response** is the system output when the input is the unit impulse (Dirac function)

$$h_a(t) \stackrel{\text{def}}{=} S\left(\delta_a(t)\right)$$

- It can be shown that $F\left\{\delta_a(t)\right\} = 1$
- As a result, when the system input is the unit impulse, the frequency response of the system is

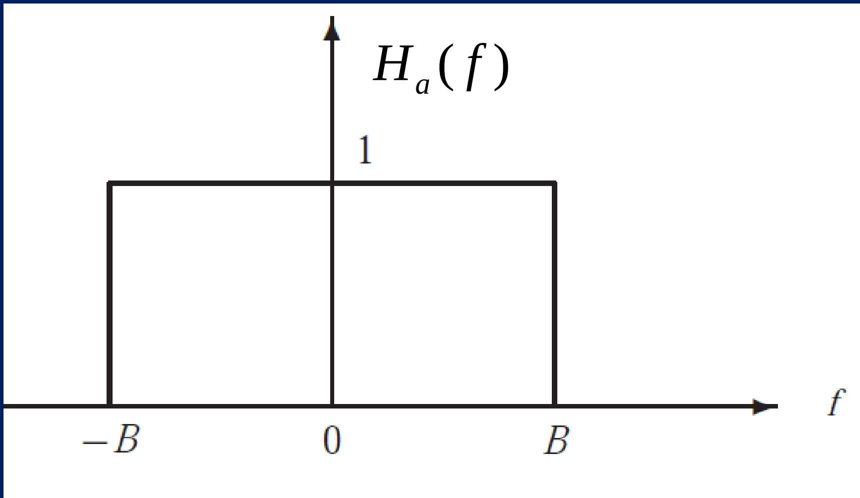
$$H_a(f) \stackrel{\text{def}}{=} \frac{Y_a(f)}{X_a(f)} = Y_a(f) = F\left\{h_a(t)\right\}$$

Schilling and Harris (2012, p.20)

Example of Continuous-Time System

Ideal low-pass filter with cut-off frequency B has frequency response

$$H_a(f) = \rho_B(f) \overset{\text{def}}{=} \begin{cases} 1, |f| \le B \\ 0, |f| > B \end{cases}$$



Recall that $Y_a(f) = H_a(f) X_a(f)$

So, the frequency component of $x_a(t)$ in the range [- B, B] passes through the filter without distortion.

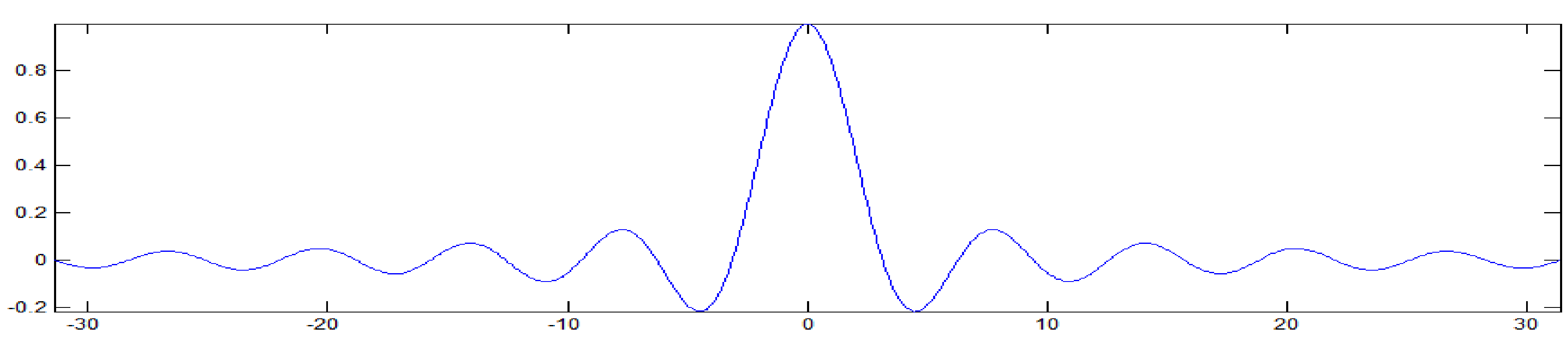
Schilling and Harris (2012, p.20)

Impulse Response of Ideal Low-pass Filter

Using the inverse Fourier transform on the frequency response, we obtain the impulse response of this filter as

$$h_a(t) = 2B \ \mathrm{sinc}\left(2\pi B t\right)$$

where $\mathrm{sinc}(x) \stackrel{\text{def}}{=} \sin(x)/x$



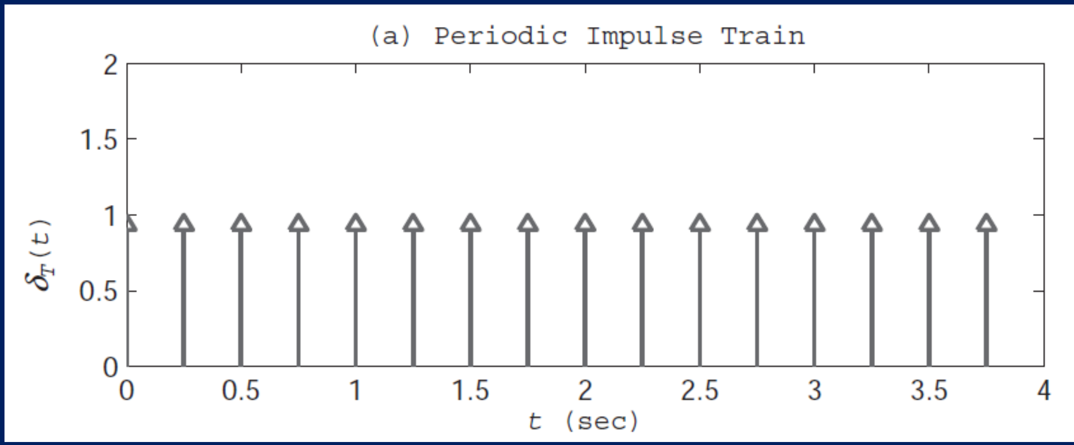
Sinc function is an acausal output of the filter when the input is causal.
"So, this filter cannot be realized with physical hardware."

Schilling and Harris (2012, p.20-21)

Sampling of Continuous Signals

Periodic impulse train with period T is defined as

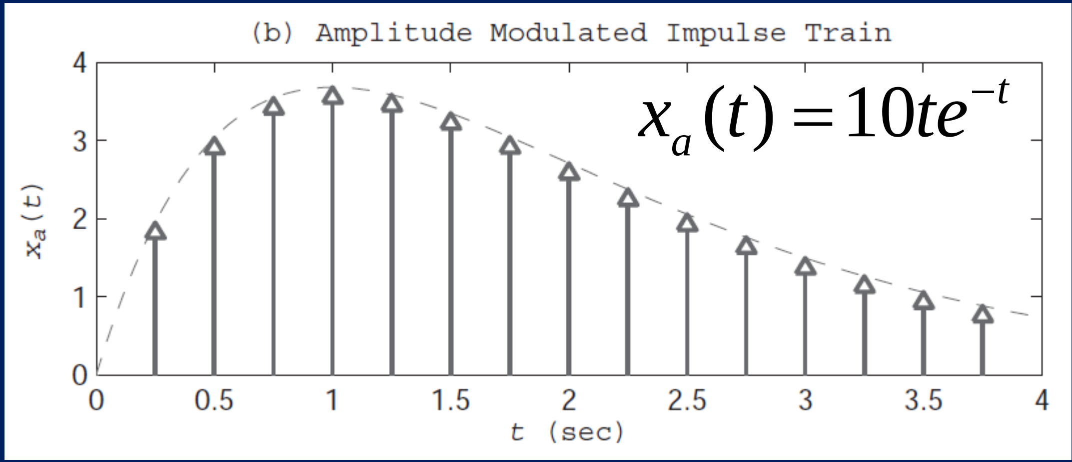
$$\delta_T(t) \overset{\text{def}}{=} \sum_{n=-\infty}^{\infty} \delta_a(t - nT)$$



The sampled version of signal $x_a(t)$ denoted by $\hat{x}_a(t)$ is defined as

$$\hat{x}_a(t) \overset{\text{def}}{=} x_a(t)\delta_T(t)$$

Amplitude modulation



Schilling and Harris (2012, p.21-23)

Sampling as Modulation

We then have

$$\hat{x}_a(t) \stackrel{\text{def}}{=} x_a(t)\delta_T(t)$$

$$= \sum_{n=-\infty}^{\infty} x_a(t)\delta_a(t-nT)$$

Using the property

$$= \sum_{n=-\infty}^{\infty} x_a(nT)\delta_a(t-nT); \qquad \delta_a(t-t_0)=0, t\neq t_0$$

$$= \sum_{n=-\infty}^{\infty} x(n)\delta_a(t-nT); \qquad x(n)=x_a(nT)$$

$x(n)$ is a discrete-time signal.

Schilling and Harris (2012, p.22)

Laplace Transform

Note that $\hat{x}_a(t)$ is still a continuous-time signal.
If $\hat{x}_a(t)$ is causal, we can apply the Laplace transform to the signal.

$$X_a(s) = L\{x_a(t)\} \stackrel{\text{def}}{=} \int_0^\infty x_a(t) \exp(-st)\, dt$$

For causal signals, the Fourier transform is the Laplace transform with $s = i2\pi f$
Therefore, the spectrum of a causal signal can be obtained from its Laplace transform, i.e.,

$$X_a(f) = X_a(s)\Big|_{s=i2\pi f}$$

Schilling and Harris (2012, p.22-23)

Spectrum of Sampled Signal

Taking the Laplace transform of $\hat{x}_a(t)$ and then using $s = i2\pi f$, we then obtain the spectrum of $\hat{x}_a(t)$, the sampled version of $x_a(t)$, as

$$\hat{X}_a(f) = f_s \sum_{n=-\infty}^{\infty} X_a(f - nf_s)$$

Aliasing formula

where $X_a(f) = F\{x_a(t)\}$

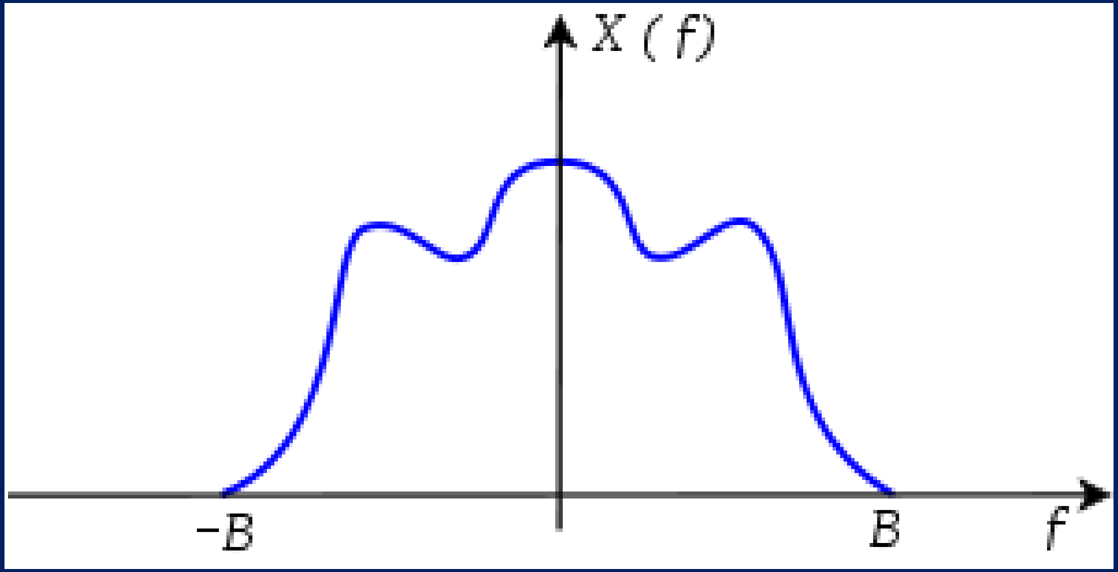
$$\hat{X}_a(f) = F\{\hat{x}_a(t)\}$$

Schilling and Harris (2012, p.23)

Band-Limited Signal

"A continuous-time signal $x_a(t)$ is band-limited to bandwidth B if and only if its magnitude spectrum satisfies $|X_a(f)| = 0 \quad \text{for} \quad |f| > B$."

Schilling and Harris (2012, p.23)



https://upload.wikimedia.org/wikipedia/commons/thumb/f/f7/Bandlimited.svg/300px-Bandlimited.svg.png

Aliasing Formula

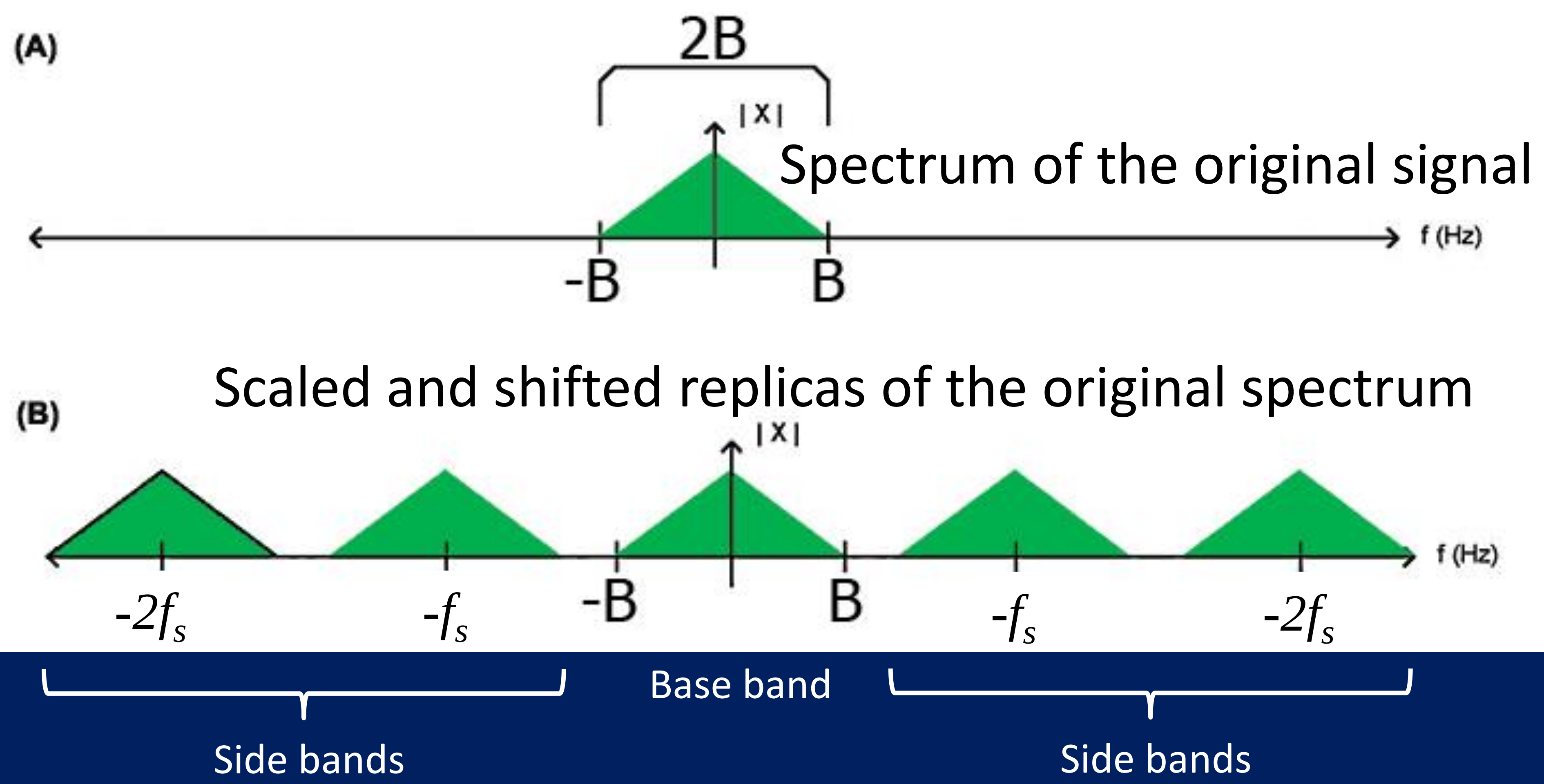
$$\hat{X}_a(f) = f_s \sum_{n=-\infty}^{\infty} X_a(f - nf_s)$$

"The spectrum of the sampled version of a signal is a sum of scaled and shifted spectra of the original signal."

A replicated version of the spectrum of original signal is scaled by f_s, the sampling frequency , and shifted by nf_s where n is integer.

Schilling and Harris (2012, p.23)

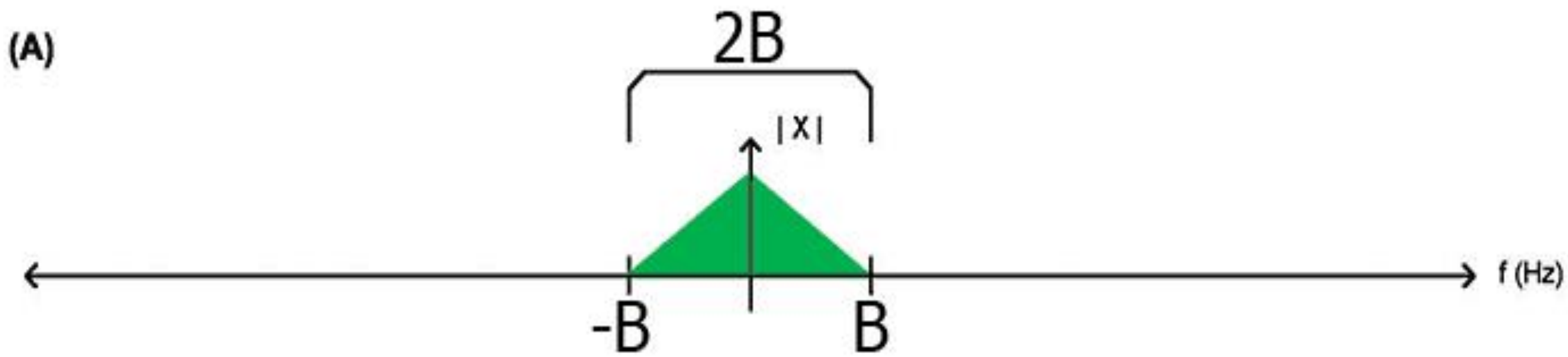
Aliasing Formula



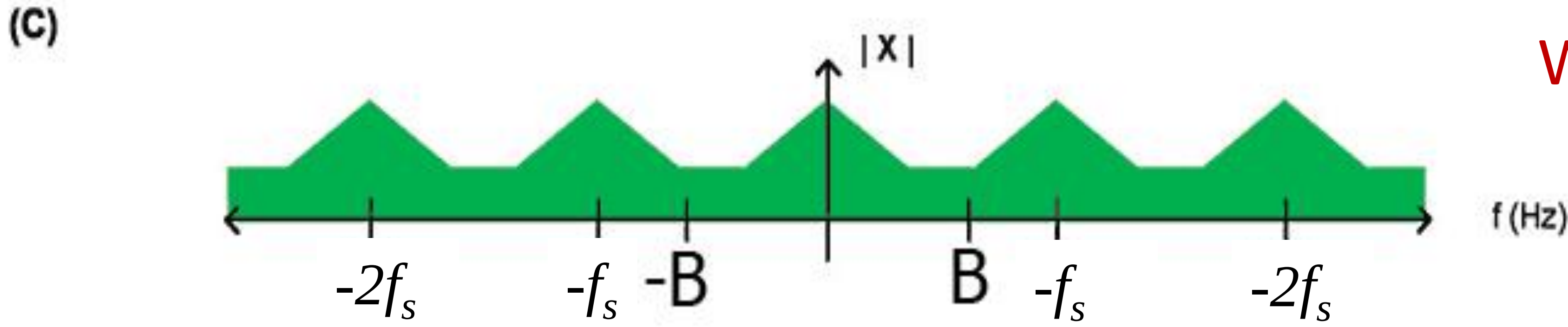
http://archives.sensorsmag.com/articles/0103/38/fig5_big.gif

Aliasing

The overlap of the replicas of the original spectrum (aliasing) occurs when $f_s \leq 2B$.



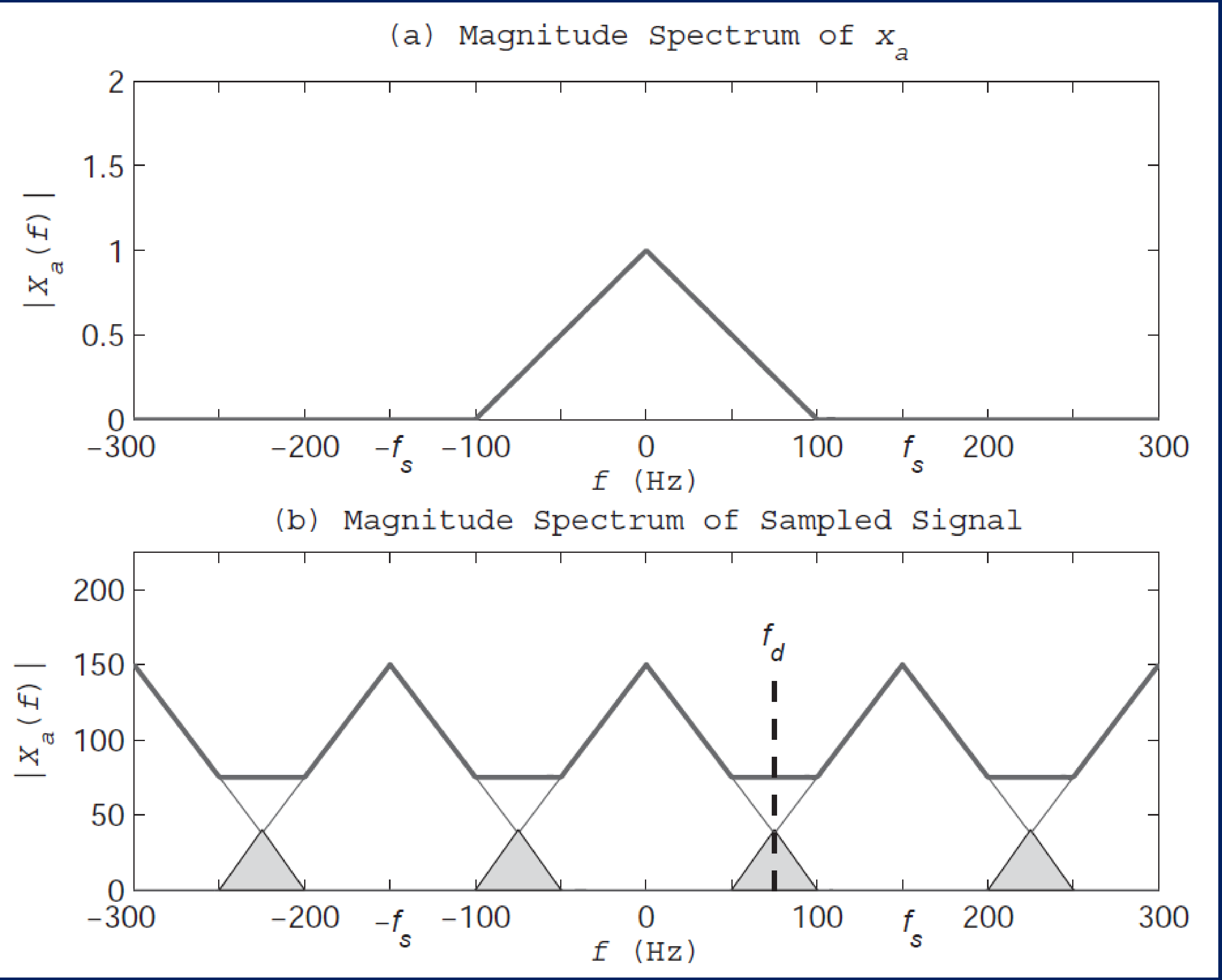
The original signal can no longer be exactly reconstructed!



Why?

http://archives.sensorsmag.com/articles/0103/38/fig5_big.gif

Aliasing

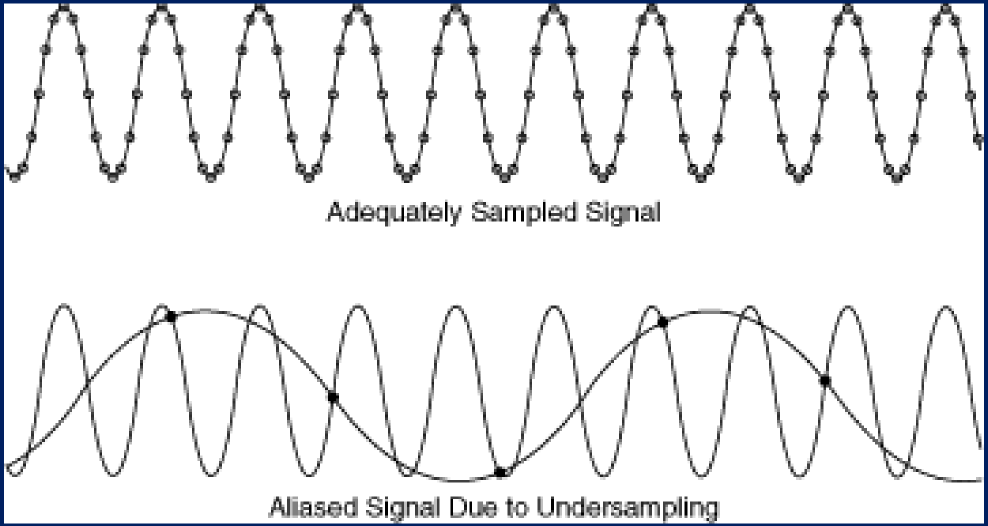


Schilling and Harris (2012, p.24)

Shannon Sampling Theorem

"Suppose a continuous-time signal $x_a(t)$ is band-limited to B Hz. Let $\hat{x}_a(t)$ denote the sampled version of $x_a(t)$ using impulse sampling with a sampling frequency f_s. Then the samples contain all the information necessary to recover the original signal $x_a(t)$ if $f_s > 2B$."

Schilling and Harris (2012, p.25)



http://zone.ni.com/images/reference/en-XX/help/370051T-01/aliasing_effects.gif

Nyquist Frequency

Nyquist frequency is defined as

$$f_n \stackrel{\text{def}}{=} \frac{f_s}{2}$$

If $x_a(t)$ has any frequency component outside of f_n then in $\hat{x}_a(t)$ these frequencies get reflected about f_n and folded back into the range $[-f_n, f_n]$ This is call **aliasing**.

Schilling and Harris (2012, p.26)

Under- and Over-Sampling

Undersampling: $f_s < 2B$

- Aliasing occurs.
- Original signal **cannot** be reconstructed.

Oversampling: $f_s > 2B$

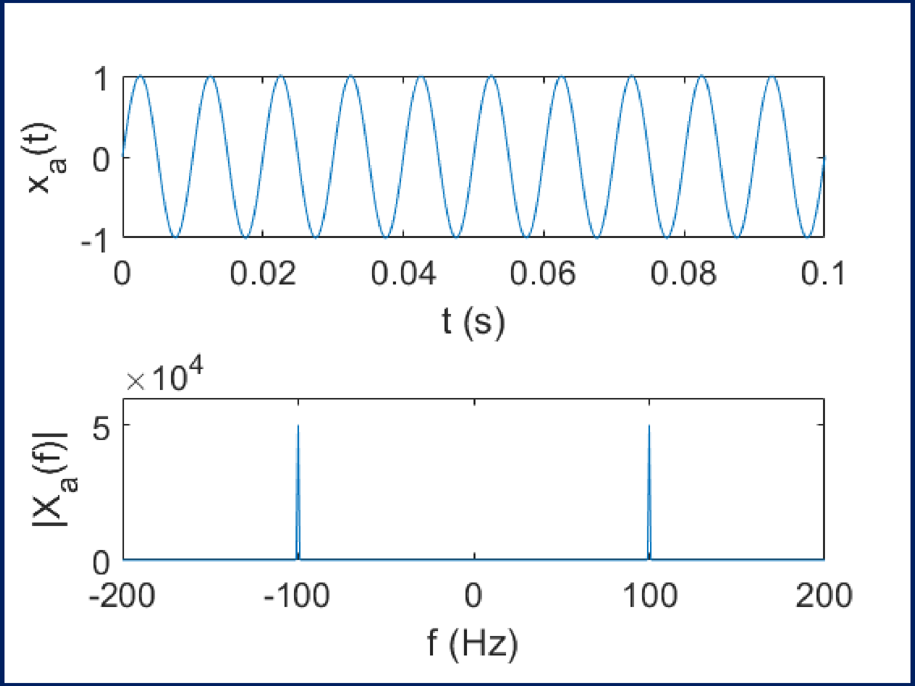
- No aliasing.
- Original signal can be reconstructed.

Example: Aliasing

Consider the signal $x_a(t) = \sin(2\pi B t), \quad B = 100$

The spectrum of $x_a(t)$ is

$$X_a(t) = \frac{i}{2}\left[\delta(f+100) - \delta(f-100)\right]$$



Adapted from Example 1.5 of Schilling and Harris (2012, p.25)

Example: Aliasing

To avoid aliasing, we need $f_s > 200$ Hz.

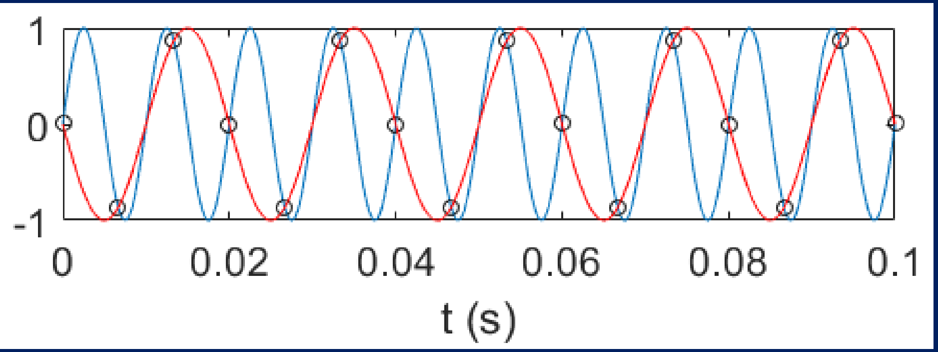
Let's sample $x_a(t)$ at the rate $f_s = 150$ Hz.

In this case, $T = 0.0067$ s and the samples are

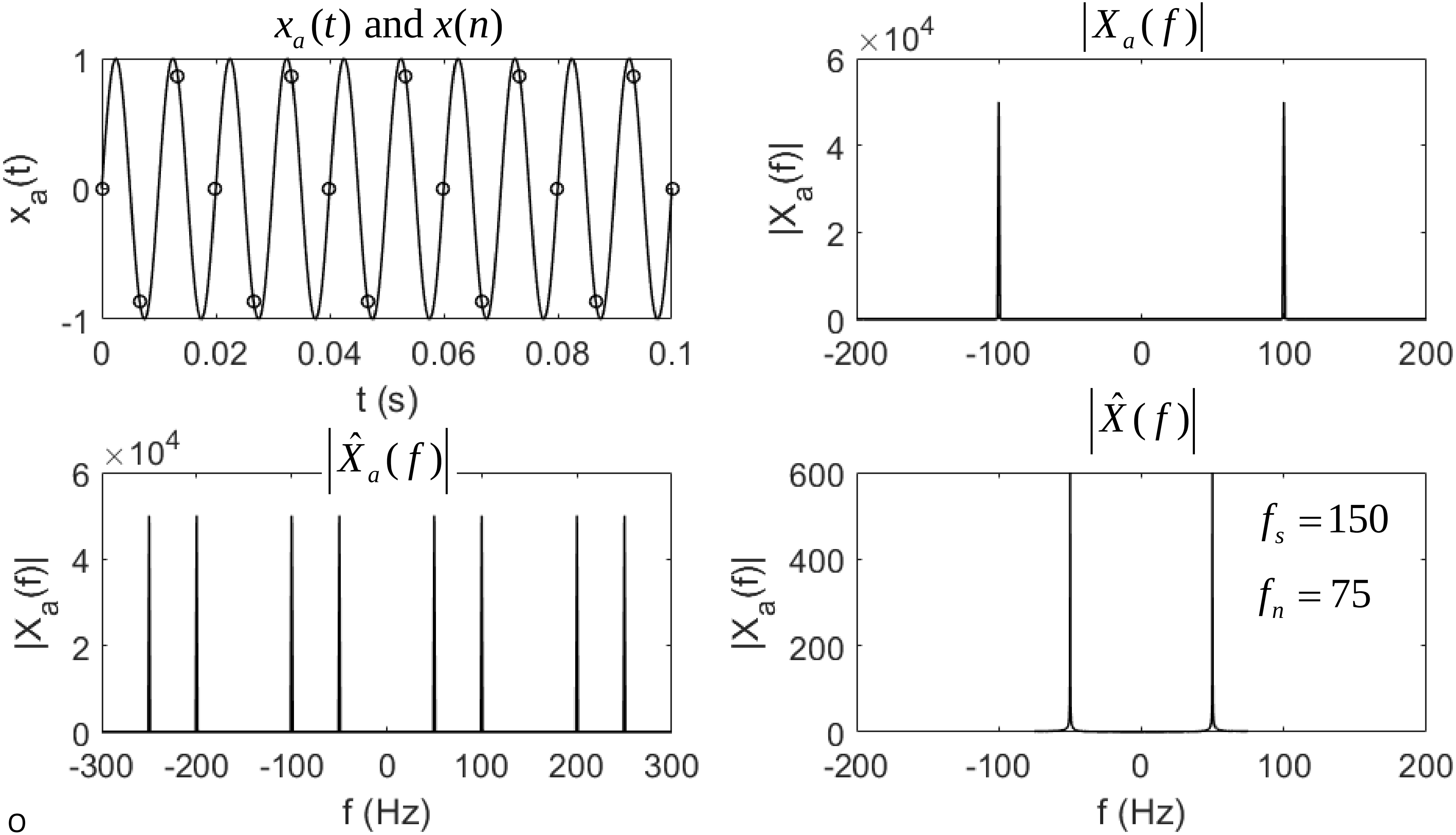
$$
\begin{aligned}
x(n) = x_a(nT) &= \sin(200\pi nT) = \sin(1.\dot{3}\pi n) \\
&= \sin(2\pi n - 0.\dot{6}\pi n) \\
&= \sin(2\pi n)\cos(0.\dot{6}\pi n) - \cos(2\pi n)\sin(0.\dot{6}\pi n) \\
&= -\sin(0.\dot{6}\pi n) = -\sin(100\pi nT)
\end{aligned}
$$

Samples of $x_a(t) = \sin(200\pi t)$ are identical to those of

$x_b(t) = \sin(100\pi t)$



Example: Aliasing



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Reconstruction of Continuous Signal

- Signal reconstruction is to recover the spectrum $X_a(f)$ from the spectrum $\hat{X}_a(f)$.
- This can be done by passing $\hat{x}_a(t)$ through an ideal lowpass reconstruction filter $H_{\text{ideal}}(f)$ that **removes the side bands** and **rescales the base band**.

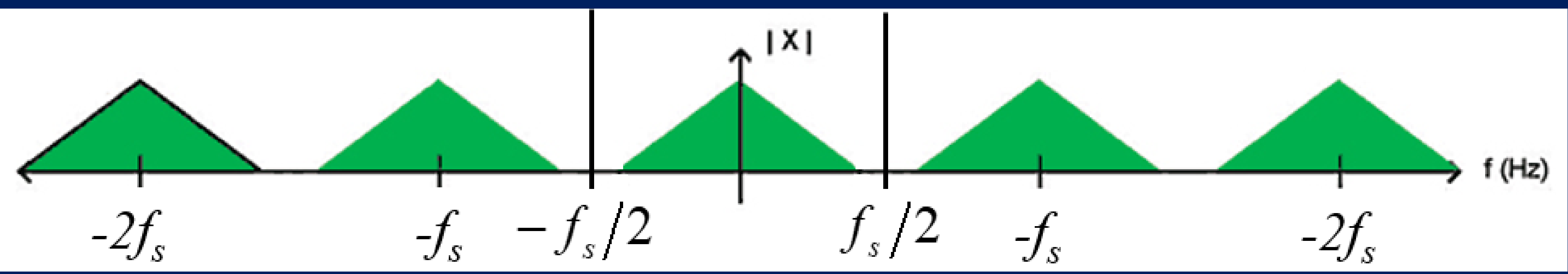
$$H_{\text{ideal}}(f) \equiv \begin{cases} T, & |f| \le f_n \\ 0, & |f| > f_n \end{cases}$$

$$X_a(f) = H_{\text{ideal}}(f)\hat{X}_a(f)$$

Schilling and Harris (2012, p.27)

Reconstruction of Continuous Signal

$$H_{\text{ideal}}(f) \equiv \begin{cases} T, & |f| \le f_n \\ 0, & |f| > f_n \end{cases}$$



Remove side bands

Remove side bands

Reconstruction of Continuous Signal

The impulse response of the ideal reconstruction filter is

$$\begin{aligned} h_{\text{ideal}}(t) &= F^{-1}\left\{H_{\text{ideal}}(f)\right\} \\ &= 2Tf_n \operatorname{sinc}(2\pi f_n t) \\ &= \operatorname{sinc}(\pi f_s t) \end{aligned}$$

The original continuous signal can be perfectly reconstructed by

$$x_a(t) = F^{-1}\left\{H_{\text{ideal}}(f)\hat{X}_a(t)\right\}$$

Schilling and Harris (2012, p.28)

Convolution Theorem

Fourier transform of the convolution of $a(t)$ and $b(t)$, denoted as $a(t)*b(t)$, is the pointwise product of their Fourier transforms:

$$F\{a(t)*b(t)\} = F\{a(t)\}F\{b(t)\} = A(f)B(f)$$

where $$a(t)*b(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} a(t-\tau)b(\tau)d\tau$$

It can be shown that convolution is commutative, that is $$a(t)*b(t) = b(t)*a(t)$$

https://en.wikipedia.org/wiki/Convolution_theorem

Shannon Interpolation Formula

$$x_a(t) = F^{-1}\left\{H_{\text{ideal}}(f)\hat{X}_a(f)\right\} = \int_{-\infty}^{\infty} h_{\text{ideal}}(t-\tau)\hat{x}_a(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} h_{\text{ideal}}(t-\tau)\left[\sum_{n=-\infty}^{\infty} x(n)\delta_a(\tau-nT)\right]d\tau$$

$$= \sum_{n=-\infty}^{\infty} x(n)\left[\int_{-\infty}^{\infty} h_{\text{ideal}}(t-\tau)\delta_a(\tau-nT)d\tau\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n)h_{\text{ideal}}(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x(n)\operatorname{sinc}\left[\pi f_s(t-nT)\right]$$

Schilling and Harris (2012, p.28)

Shannon Interpolation Formula

"Suppose a continuous-time signal $x_a(t)$ is bandlimited to B Hz. Let $x(n) = x_a(nT)$ be the n-th sample of $x_a(t)$ using a sampling frequency of $f_s = 1/T$. If $f_s > 2B$, then $x_a(t)$ can be reconstructed from $x(n)$ as follows."

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \operatorname{sinc}\left[\pi f_s (t - nT)\right]$$

The sinc function is used here as a basis function for interpolation.

Schilling and Harris (2012, p.28)

Transfer Function

"Let $x_a(t)$ be a causal nonzero input to a continuous-time linear system, and let $y_a(t)$ be the corresponding output. The **transfer function** of the system is defined as"

$$H_a(s) \stackrel{\text{def}}{=} \frac{Y_a(s)}{X_a(s)}$$

Since $L\left\{\delta_a(t)\right\} = 1,$ the transfer function is the Laplace transform of the impulse response

$$H_a(s) = L\left\{h_a(t)\right\}$$

Schilling and Harris (2012, p.29)

Example: Time Shift

Consider the system $y_a(t) = x_a(t-\tau)$.

$$Y_a(s) = L\{x_a(t-\tau)\} = \int_0^\infty x_a(t-\tau)\exp(-st)dt$$

$$= \int_0^\infty x_a(t')\exp[-s(t'+\tau)]dt', \quad t' = t-\tau$$

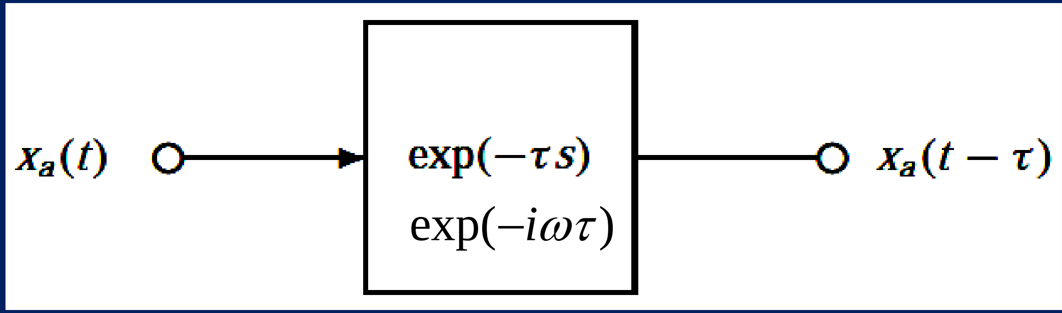
$$= \exp(-s\tau)\int_0^\infty x_a(t')\exp(-st')dt'$$

$$= \exp(-s\tau)X_a(s)$$

So, $H_a(s) = \exp(-\tau s)$

$$H_a(f) = \exp(-i2\pi f\tau)$$

$$H_a(\omega) = \exp(-i\omega\tau)$$



Schilling and Harris (2012, p.29-30)

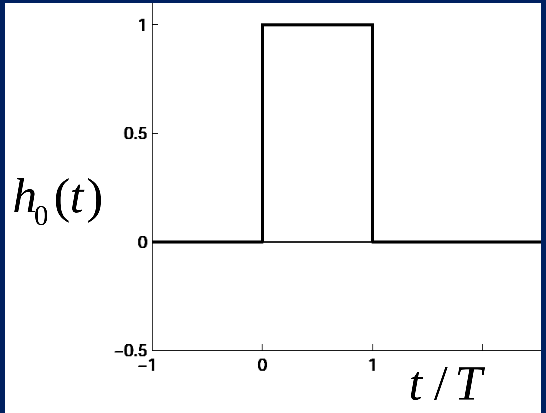
Zero-Order Hold

- Exact reconstruction of $x_a(t)$ using the Shannon interpolation formula required an ideal filter which cannot be realized by a physical system.
- The reconstruction can be approximated by a practical filter such as a zero-order hold filter

$$y_a(t) = \int_0^t \left[x_a(\tau) - x_a(\tau - T)\right] d\tau$$

- The impulse response of a zero-order hold is

$$h_0(t) = \int_0^t \left[\delta_a(\tau) - \delta_a(\tau - T)\right] d\tau$$
$$= \mu_a(t) - \mu_a(t - T)$$

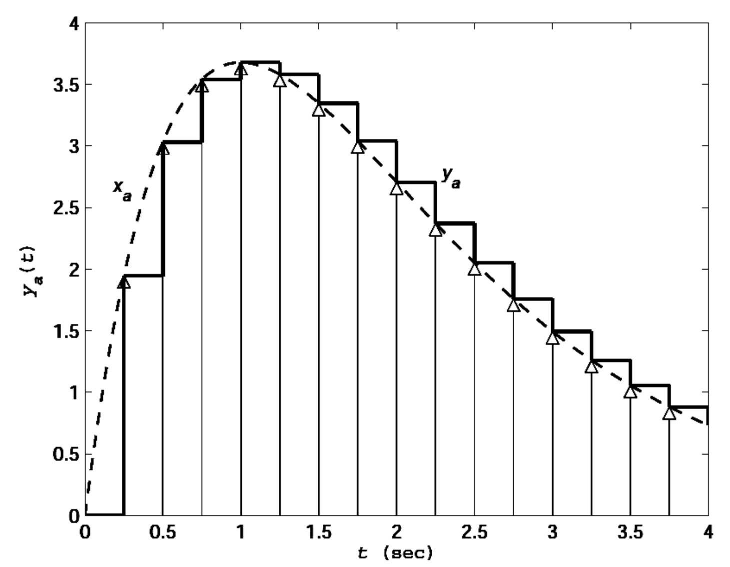


Schilling and Harris (2012, p.30)

Zero-Order Hold

- The zero-order hold is linear and time-invariant.
- The response to an impulse of strength $x(n)$ at time $t = nT$ will be a pulse of height $x(n)$ and width T starting at $t = nT$.
- When the input is $\hat{x}_a(t)$, the output will be a piecewise-constant approximation to $x_a(t)$.
- The transfer function of zero-order hold is

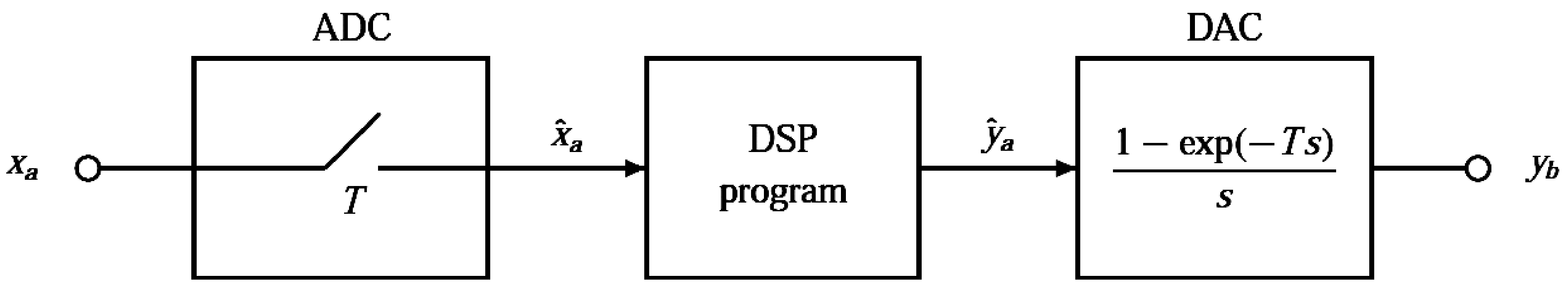
$$H_0(s) = \left[1 - \exp(-Ts)\right] / s$$



Schilling and Harris (2012, p.31)

Digital Signal Processing System

Zero-order hold can be used as a digital-to-analog converter (DAC) while an impulse sampler can be used as an analog-to-digital converter (ADC).



Switch opens and closes every T seconds.

Schilling and Harris (2012, p.31-32)

Anti-aliasing Filter

- When a signal that is not bandlimited is sampled, aliasing will occur. To avoid aliasing, a lowpass filter must be applied to the signal.
- An anti-aliasing filter is a lowpass filter that removes all frequency components outside range $\left[-f_c, f_c\right], f_c < f_n = f_s/2$ where f_c is called the cut-off frequency.
- The ideal lowpass filter is the optimal choice for an anti-aliasing filter. Butterworth filter is a practical filter that has been widely used as an anti-aliasing filter.

Schilling and Harris (2012, p.33)

Butterworth Filter

- "A lowpass Butterworth filter of order n has the magnitude response as follows."

$$\left|H_a(f)\right| = \frac{1}{\sqrt{1+\left(f/f_c\right)^{2n}}}, \quad n \geq 1$$

- At the cut-off frequency f_c,

$$\left|H_a(f_c)\right| = 1/\sqrt{2} \quad \text{and} \quad 20\log_{10}\left\{\left|H_a(f_c)\right|\right\} \approx -3 \text{ dB}$$

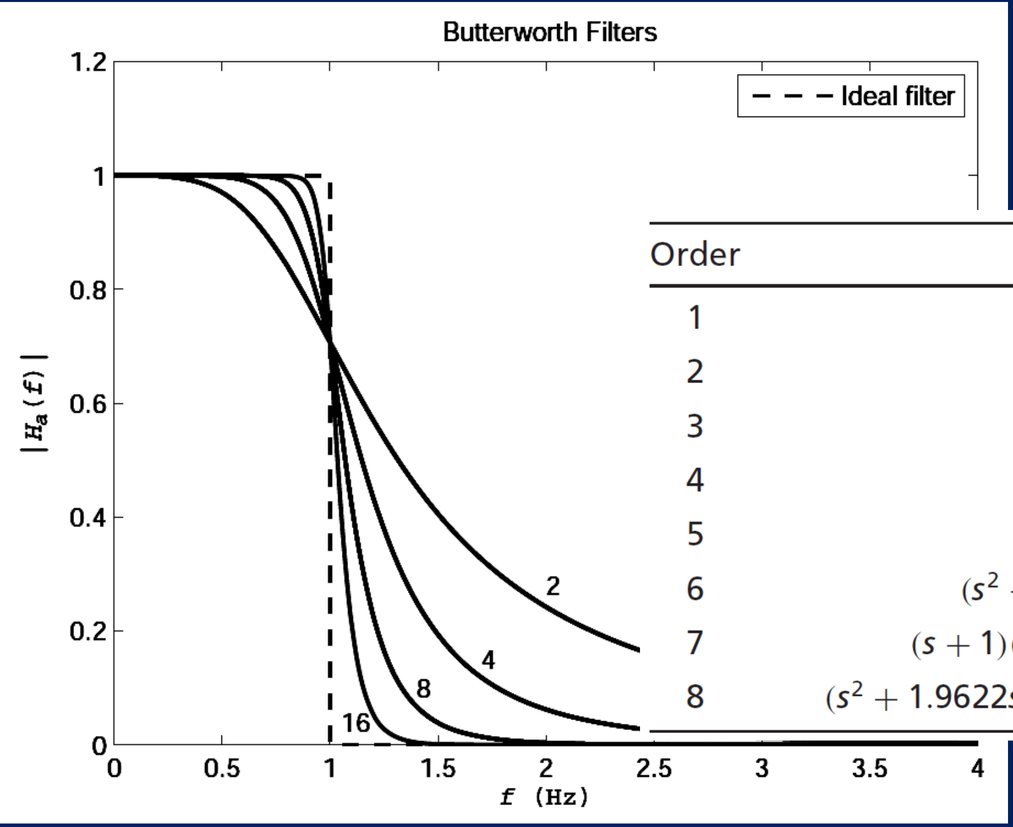
so f_c is called the 3 dB cutoff frequency of the filter.

Schilling and Harris (2012, p.33)

Butterworth Filter

The transfer function of a lowpass Butterworth filter of order n is

$$H_n(s) = \frac{\omega_c^n}{s^n + \omega_c a_1 s^{n-1} + \omega_c^2 a_2 s^{n-2} + \cdots \omega_c^n}, \quad \omega_c = 2\pi f_c$$



"As the order n increases, the magnitude response approaches the ideal lowpass characteristic."

Order	$a(s)$
1	$(s+1)$
2	$(s+1.5142s+1)$
3	$(s+1)(s^2+s+1)$
4	$(s^2+1.8478s+1)(s^2+0.7654s+1)$
5	$(s+1)(s^2+1.5180s+1)(s^2+0.6180s+1)$
6	$(s^2+1.9318s+1)(s^2+1.5142s+1)(s^2+0.5176s+1)$
7	$(s+1)(s^2+1.8022s+1)(s^2+1.2456s+1)(s^2+0.4450s+1)$
8	$(s^2+1.9622s+1)(s^2+1.5630s+1)(s^2+1.1110s+1)(s^2+0.3986s+1)$

$$\omega_c = 1$$

Schilling and Harris (2012, p.33-34)

First-Order Butterworth Filter

- The transfer function of the first-order Butterworth filter is $H_1(s) = \omega_c / (s + \omega_c)$
- The circuit realization of the first-order Butterworth filter is shown below.
- This circuit requires

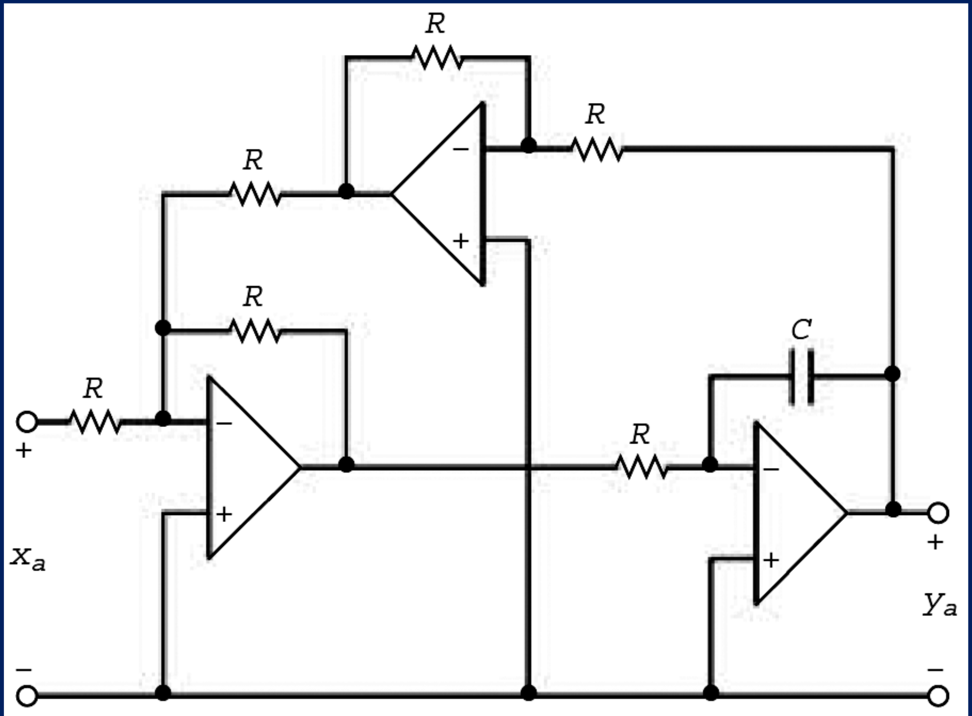
3 operational amplifiers

6 resistors of resistance R

capacitor of capacitance C

and

$$\omega_c = \frac{1}{RC}$$



Schilling and Harris (2012, p.34-35)

References

- Schilling, R. J. and S. L. Harris, 2012, Fundamentals of Digital Signal Processing using MATLAB, Second Edition, Cengage Learning.