Discrete-time Systems in the Time Domain

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August 18, 2020

Discrete-time Systems

- “A **discrete-time system** is an entity that processes a discrete-time input signal $x(k)$ to produce a discrete-time output signal $y(k)$.”
- **Digital signals** are in both amplitude and time.
- A system associated with digital signals is a **digital filter**.
- “A finite-dimensional linear time-invariant (LTI) discrete-time system can be represented in the time domain by a constant-coefficient difference equation”

$$y(k)+\sum_{i=1}^{N} a_i y(k-i)=\sum_{i=0}^{M} b_i x(k-i), \quad \text{i.e., } a_0=1$$

- “Many problems can be modeled as discrete-time systems using difference equations.”

Schilling and Harris (2012, p.70-71)

Example: Radar

- "A radar antenna transmits an EM wave $x(k)$ into space. When a target is illuminated by the radar, some of signal energy is reflected back and returns to the radar receiver."
- "The received signal $y(k)$ can be modeled using the difference equation"

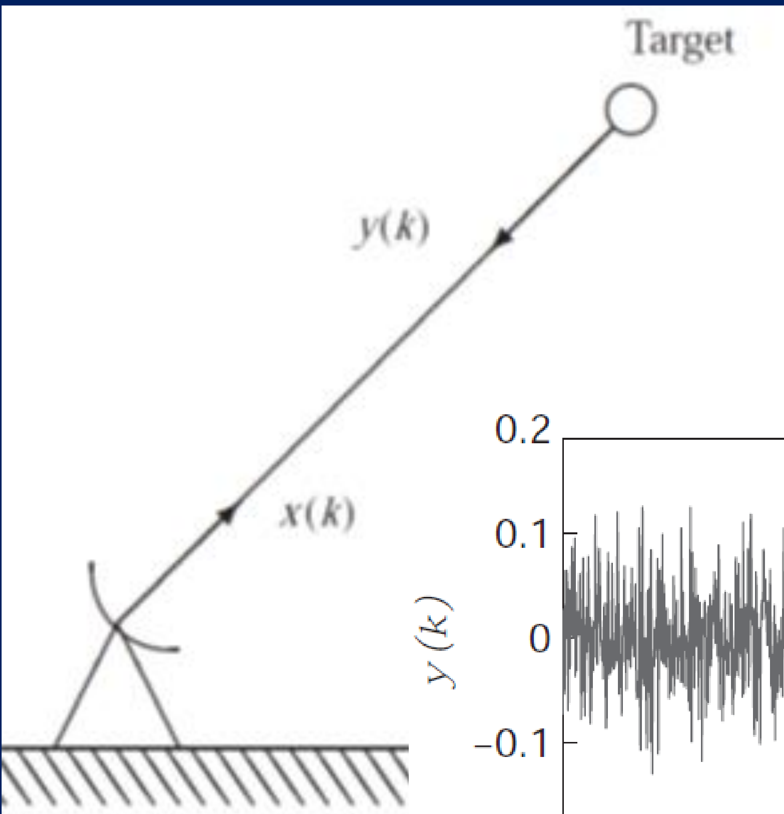
$$y(k) = ax(k-d) + \eta(k)$$

- "The first term represents the echo of the transmitted signal with delay *d* proportional to the time of flight."
- "The second term accounts for random noise picked up and amplified by the receiver."
- "Typically, the echo is very faint, i.e., $0 < a \ll 1$."

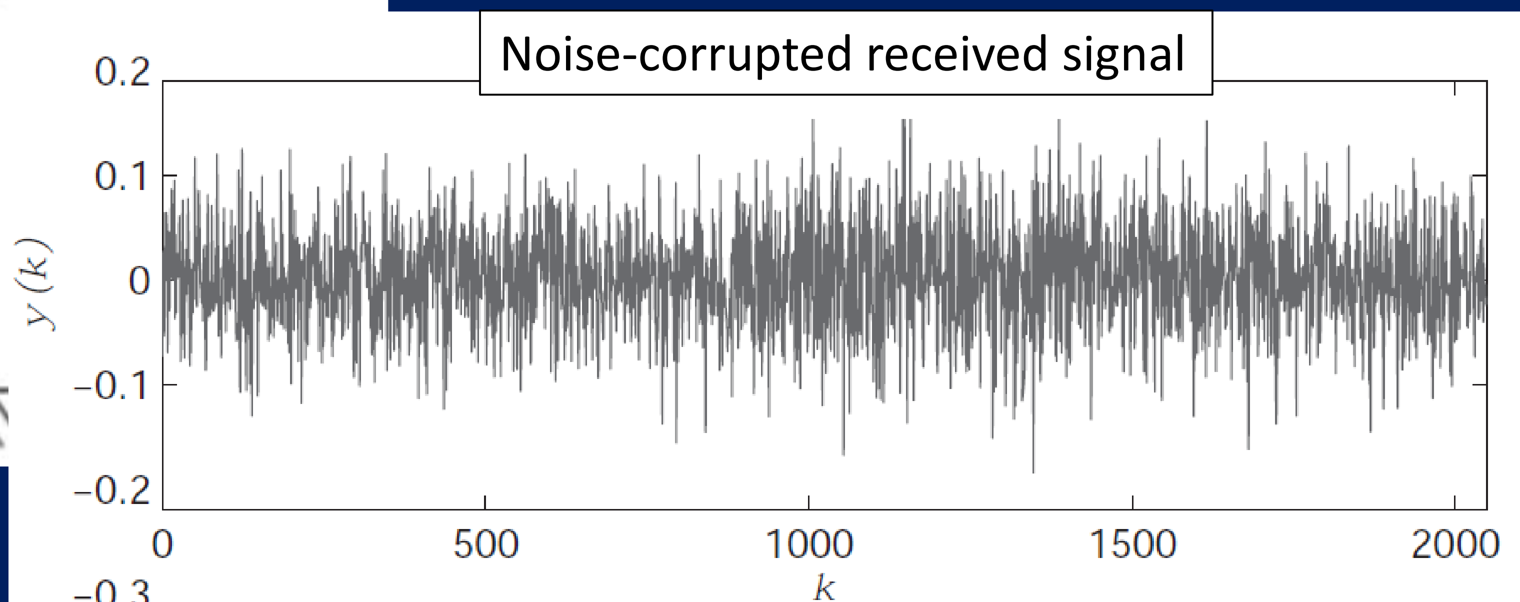
Schilling and Harris (2012, p.70-71)

Example: Radar

“The objective in processing the received signal is to determine whether or not an echo is present.”



“If an echo is detected, then the distance to the target can be obtained from the delay *d*.”

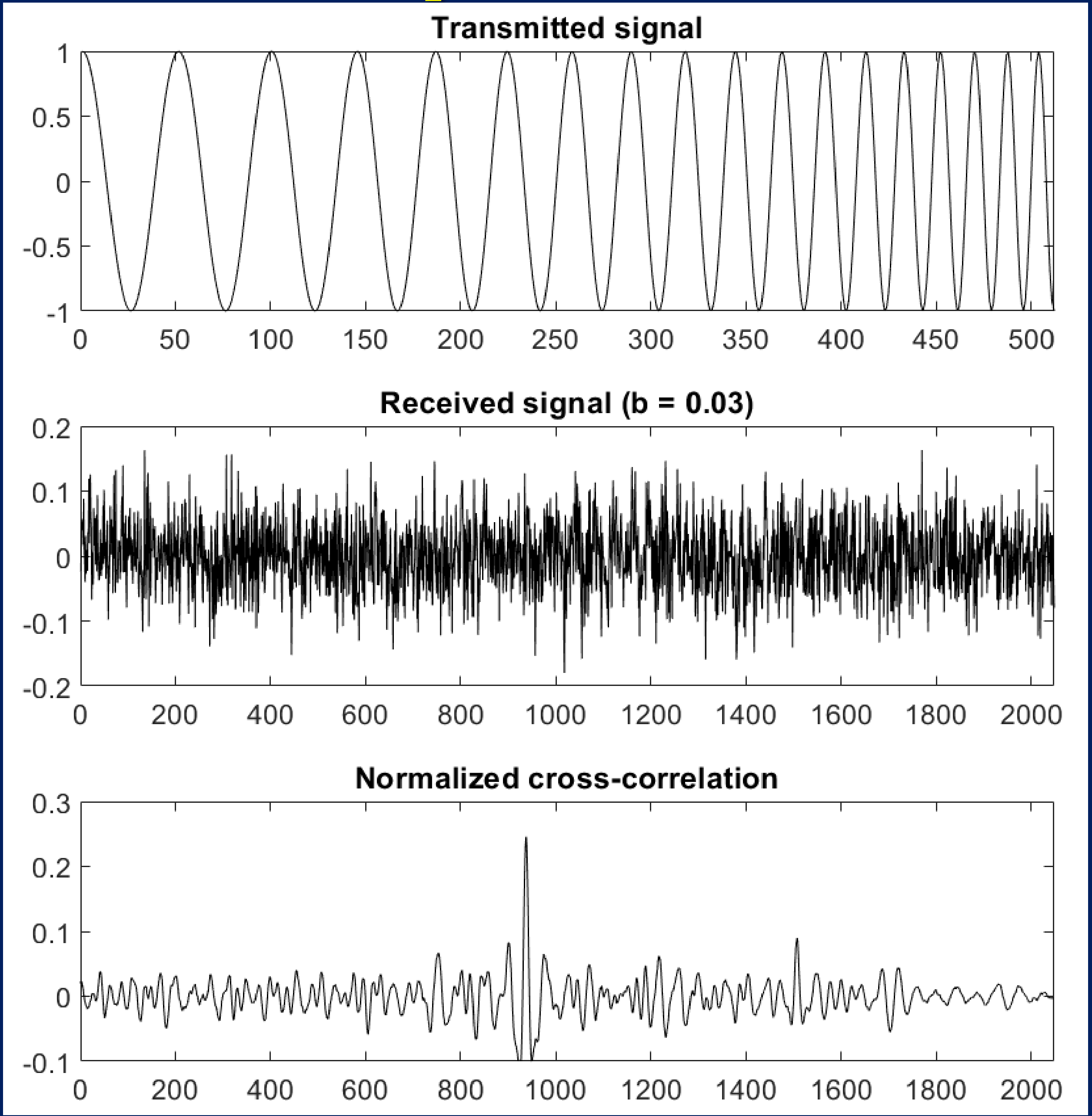


Example: Radar

- "If T is the sampling interval, then the time of flight in seconds is $\tau = dT$."
- The distance to the target is $r = cdT/2$ where c is the speed of light and the factor 2 arises because the time of flight is a two-way travel time.
- "To detect if an echo is present in the received signal, we compute the normalized cross-correlation which measures the degree to which the received signal $y(k)$ is similar to the transmitted signal $x(k)$."

Schilling and Harris (2012, p.72-73)

Example: Radar



Classification of Discrete-time Signals

- **Finite signal**: signal is nonzero for a finite number of samples $x(k)=0, k \notin [N_1, N_2]$ where $N_1 \le N_2$. Otherwise, it is called *infinite signal*.
- **Causal signal**: $x(k)=0,\ k<0$. Otherwise, it is called *noncausal signal*.
- **Periodic signal**: $x(k+N)=x(k)$ where N is the period.
- **Bounded signal**: $|x(k)| \le B_x,\ B_x > 0$. Otherwise, it is an *unbounded signal*.

Norm of Signals

- Signals can be thought of as vectors.
- The length of vectors can be measured by
  - l_1 norm $\|x\|_1 \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} |x(k)|$
  - l_2 norm $\|x\|_2 \stackrel{\text{def}}{=} \left[\sum_{k=-\infty}^{\infty} |x(k)|^2 \right]^{1/2}$
- **Absolutely summable signal**: $\|x\|_1 < \infty$
- **Square summable signal**: $\|x\|_2 < \infty$

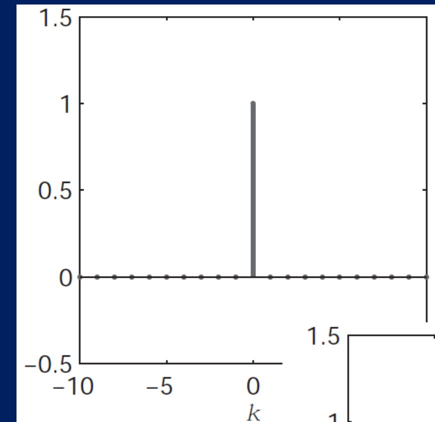
Schilling and Harris (2012, p.76-77)

Energy and Power Signals

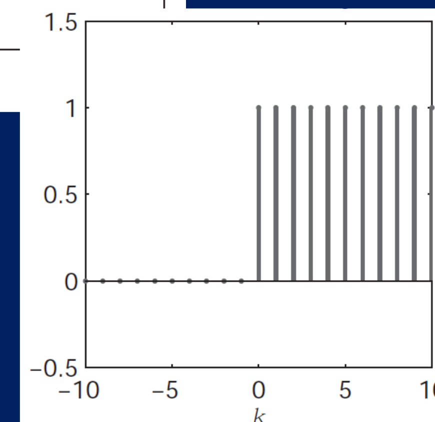
- **Energy** of signal is defined as $E_x \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} |x(k)|^2$
- Since $E_x = \|x\|_2^2$, the energy is finite if $x(k)$ is square summable.
- **Energy signal**: a signal with finite energy $E_x < \infty$
- **Instantaneous power**: $p(x) = |x(k)|^2$
- **Average power**: $P_x \stackrel{\text{def}}{=} \lim_{N \to \infty} \frac{1}{2N+1} \sum_{k=-N}^{N} |x(k)|^2$
- **Power signal**: a finite signal with nonzero average power

Common Signals

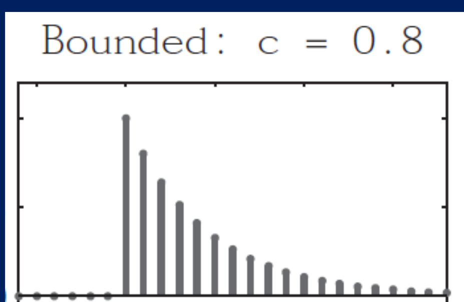
- **Unit impulse:** $$\delta(k) \stackrel{\text{def}}{=} \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

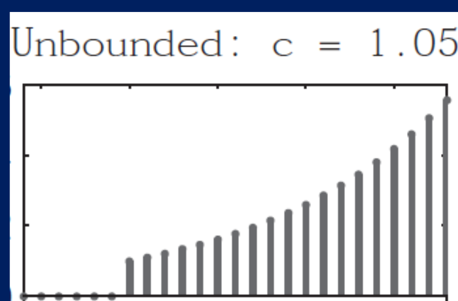


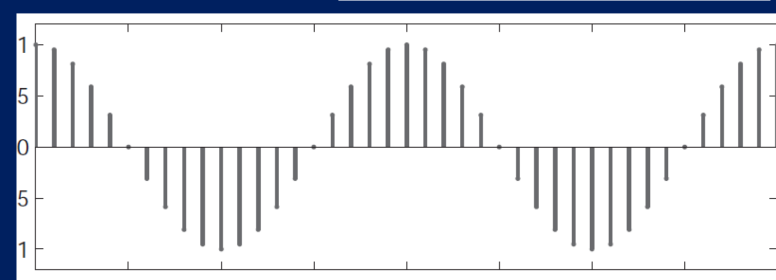
- **Unite step:** $$\mu(k) \stackrel{\text{def}}{=} \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$



- **Causal exponential:** $x(k) = c^n \mu(k)$







- **Periodic signal:** $x(k) = \cos(2\pi f k T)$

Schilling and Harris (2012, p.79-80)

Discrete-time Systems

- **Linear system**: S is a linear system if

$$x(k) = ax_1(k) + bx_2(k) \;\Rightarrow\; y(k) = ay_1(k) + by_2(k)$$

  Otherwise, the system S is nonlinear.
- **Time-invariant system**: S is a time-invariant system if the output produced by the shifted input $x(k-m)$ is $y(k-m)$. Otherwise, the system S is time-varying.
- **Causal system**: "For a physical system operating in *real time*, the output at the present time k cannot depend on the future input $x(m),\ m > k$, because the input has not yet occurred."
- "A discrete-time system is causal if for each time k"

$$x_1(m) = x_2(m),\ \text{for } m \le k \Rightarrow y_1(k) = y_2(k)$$

Schilling and Harris (2012, p.82-83)

Discrete-time Systems

- "When signal processing is not performed in real time, noncausal systems can be used to process the data *offline* in batch mode, where future samples of the input are available."
- **Stable system**: S is a bounded-input bounded-output (BIBO) stable if every bounded input $x(k)$ produces a bounded output $y(k)$:

$$|x(k)| \le B_x \Rightarrow |y(k)| \le B_y$$

- **Passive system**: energy does not increase: $E_y \le E_x$
- **Lossless system**: energy stays the same: $E_y = E_x$
- "Lossless *physical* systems contain energy storage elements (spring, mass, capacitor, inductor) without energy dissipative elements (resistor, damper)

Schilling and Harris (2012, p.85-86)

Difference Equations

- "The output of a causal LTI system S at time k can be expressed as"

$$y(k)=\sum_{i=0}^{M} b_i x(k-i)-\sum_{i=1}^{N} a_i y(k-i)$$

- "When causal inputs are used, the output or response of a discrete-time system depends on both the input $x(k)$ and the initial condition represented by a vector $y_0 \in \mathbb{R}^N$ of past outputs,"

$$y_0 \overset{\text{def}}{=} \left[y(-1), y(-2), \cdots, y(-N)\right]^T$$

- "For a linear system, the contributions to the output from initial condition y_0 and input $x(k)$ can be considered separately."

Schilling and Harris (2012, p.86)

Zero-input Response

- Zero-input response of a discrete-time system S is the solution of the system

$$y(k) + \sum_{i=1}^{N} a_i y(k-i) = 0, \quad y_0 \neq 0$$

when the input is $x(k) = 0$ and is denoted as $y_{zi}(k)$.

- To solve the system for a zero-input response, let's use the trial solution of the form $y(k) = z^k$ where z is a complex scalar to be determined.
- Substituting the trial solution into the system and multiplying both sides by z^{N-k} yields the **characteristic polynomial**,

$$a(z) = z^N + a_1 z^{N-1} + \cdots + a_N = 0$$

Schilling and Harris (2012, p.87)

Zero-input Response: Simple Mode

- The factored form of the characteristic polynomial is
$$a(z) = (z - p_1)(z - p_2)\cdots(z - p_N) = 0$$
where p_i are the roots. So, z must be equal to the roots.
- If the characteristic polynomial has N distinct roots, also called simple roots, the general solution is the linear combination
$$y_{zi}(k) = \sum_{i=1}^{N} c_i p_i^k, \quad k \geq -N$$
where $c_i p_i^k$ are called the **simple natural modes** of the system.
- Applying the initial conditions yields a linear system for determining the value of the coefficients c_i.

Schilling and Harris (2012, p.87)

Example

Consider the 2D discrete-time system

$$y(k)-0.6y(k-1)+0.05y(k-2)=2x(k)+x(k-1)$$

with initial condition $y(-1) = 3$ and $y(-2) = 2$.

The characteristic polynomial of the system is

$$a(z)=z^2-0.6z+0.05=\left(z-0.5\right)\left(z-0.1\right)$$

and has simple roots $p_1=0.5, p_2=0.1$

The zero-input response is then of the form

$$y_{zi}(k)=c_1\left(0.5\right)^k+c_2\left(0.1\right)^k$$

Applying the initial conditions $y_{zi}(-1)=3, y_{zi}(-2)=2$ yields the linear system

$$\begin{bmatrix}2 & 10\\ 4 & 100\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}3\\ 2\end{bmatrix}\rightarrow c_1=1.75, c_2=-0.05$$

Schilling and Harris (2012, p.88)

Zero-input Response: Multiple Mode

When root p occurs r times, p is referred to as a root of multiplicity r, and generates a **multiple natural mode** of the form $\left(c_1 + c_2 k + \cdots + c_r k^{r-1}\right) p^k$

Example: System $y(k) + y(k-1) + 0.25y(k-2) = 3x(k)$ with initial condition $y(-1) = -1$ and $y(-2) = 6$ has the corresponding characteristic polynomial as

$$a(z) = z^2 + z + 0.25 = \left(z + 0.5\right)^2$$

whose root is $p_1 = -0.5$ of multiplicity 2.

The zero-input response is then of the form

$$y_{zi}(k) = \left(c_1 + c_2 k\right)\left(-0.5\right)^k$$

Applying the initial conditions yields the linear system

$$\begin{bmatrix} -2 & 2 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} \rightarrow c_1 = -0.5, c_2 = -1$$

Zero-input Response: Complex Mode

When roots are complex, they always occur in conjugate pairs since the coefficients of $a(z)$ are real. Complex conjugate roots $p_{1,2} = r\exp\left(\pm j\theta\right)$ form the **complex mode** of the form $r^k\left[c_1\cos(k\theta)+c_2\sin(k\theta)\right]$

Hint: use Euler identity $\exp(\pm j\theta)=\cos\theta \pm j\sin\theta$

Example: System $y(k)+0.49y(k-2)=3x(k)$ with initial condition $y(-1) = 4$ and $y(-2) = 2$ has the corresponding characteristic polynomial as

$$a(z)=z^2+1=\left(z-j0.7\right)\left(z+j0.7\right)$$

The zero-input response is then of the form

$$y_{zi}(k)=\left(0.7\right)^k\left[c_1\cos(\pi k/2)+c_2\sin(\pi k/2)\right], \quad \theta=\pi/2$$

Applying the initial conditions yields $c_1=0.98, c_2=-2.8$

Schilling and Harris (2012, p.90)

Zero-state Response

Zero-state response of a LTI discrete-time system is the output corresponding to an arbitrary input $x(k)$ when the initial condition vector is zero, i.e.,

$$y(k)+\sum_{i=1}^{N} a_i y(k-i)=\sum_{i=0}^{M} b_i x(k-i), \quad y_0=0$$

Consider a special case of **causal exponential input** of the form

$$x(k)=Ap_0^k\mu(k)$$

where A is amplitude and p_0 exponential factor.

If $M \le N,$ then the input polynomial associated with the coefficients of the input is

$$b(z)=b_0 z^N+b_1 z^{N-1}+\cdots+b_M b^{N-M}$$

Schilling and Harris (2012, p.90)

Zero-state Response

Suppose that the input is a causal exponential and that the characteristic polynomial $a(z)$ has N+1 distinct roots. The zero-state response has a form similar to the zero-input response:

$$y_{zs}(k) = \sum_{i=1}^{N} d_i p_i^k \mu(k)$$

"The weighting coefficient $d \in \mathbb{R}^{N+1}$ can be computed from the polynomials $a(z)$ and $b(z)$ as

$$d_i = \left. \frac{A(z - p_i) b(z)}{(z - p_0) a(z)} \right|_{z=p_i}, \quad 0 \le i \le N$$

where p_i are the roots of the denominator."

Schilling and Harris (2012, p.90-91)

Example

Consider the system

$$y(k) - 0.6y(k-1) + 0.05y(k-2) = 2x(k) + x(k-1)$$

with the input $x(k) = 0.8^{k+1}\mu(k)$, i.e., $A = 0.8$, $p_0 = 0.8$

"The characteristic polynomial of this system is"

$$a(z) = z^2 - 0.6z + 0.05 = \left(z - 0.5\right)\left(z - 0.1\right)$$

"The input polynomial of this system is"

$$b(z) = 2z^2 + z = 2z\left(z + 0.5\right)$$

We then have $$\frac{Ab(z)}{(z - p_0)a(z)} = \frac{1.6z(z+0.5)}{(z-0.8)(z-0.5)(z-0.1)}$$

Schilling and Harris (2012, p.91)

Example (continued)

The weighting coefficients are

$$d_0 = \frac{1.6(0.8)(1.3)}{0.3(0.7)} = 7.92, \quad p_0 = 0.8$$

$$d_1 = \frac{1.6(0.5)(1.0)}{-0.3(0.4)} = -6.67, \quad p_1 = 0.5$$

$$d_2 = \frac{1.6(0.1)(0.6)}{-0.7(-0.4)} = 0.343, \quad p_2 = 0.1$$

The zero-state response is then

$$y_{zs}(k) = \left[7.92(0.8)^k - 6.67(0.5)^k + 0.343(0.1)^k\right]\mu(k)$$

The complete response is $y(k) = y_{zi}(k) + y_{zs}(k)$

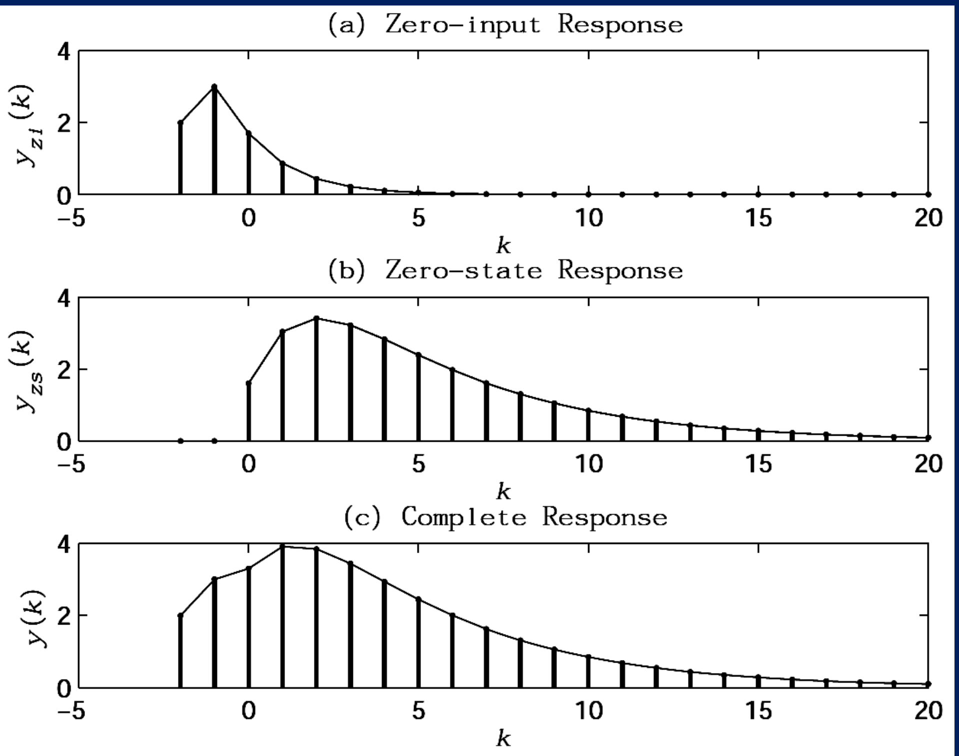
Example (continued)

Suppose the initial condition is $y_0 = [3, 2]$, that is $y(-1) = 3, y(-2) = 2$. The zero-input response is

$$y_{zi}(k) = 1.75(0.5)^k - 0.05(0.1)^k$$

So, the complete response of the system is

$$y(k) = 1.75(.5)^k - .05(.1)^k + \left[7.92(.8)^k - 6.67(.5)^k + .343(.1)^k\right]\mu(k)$$

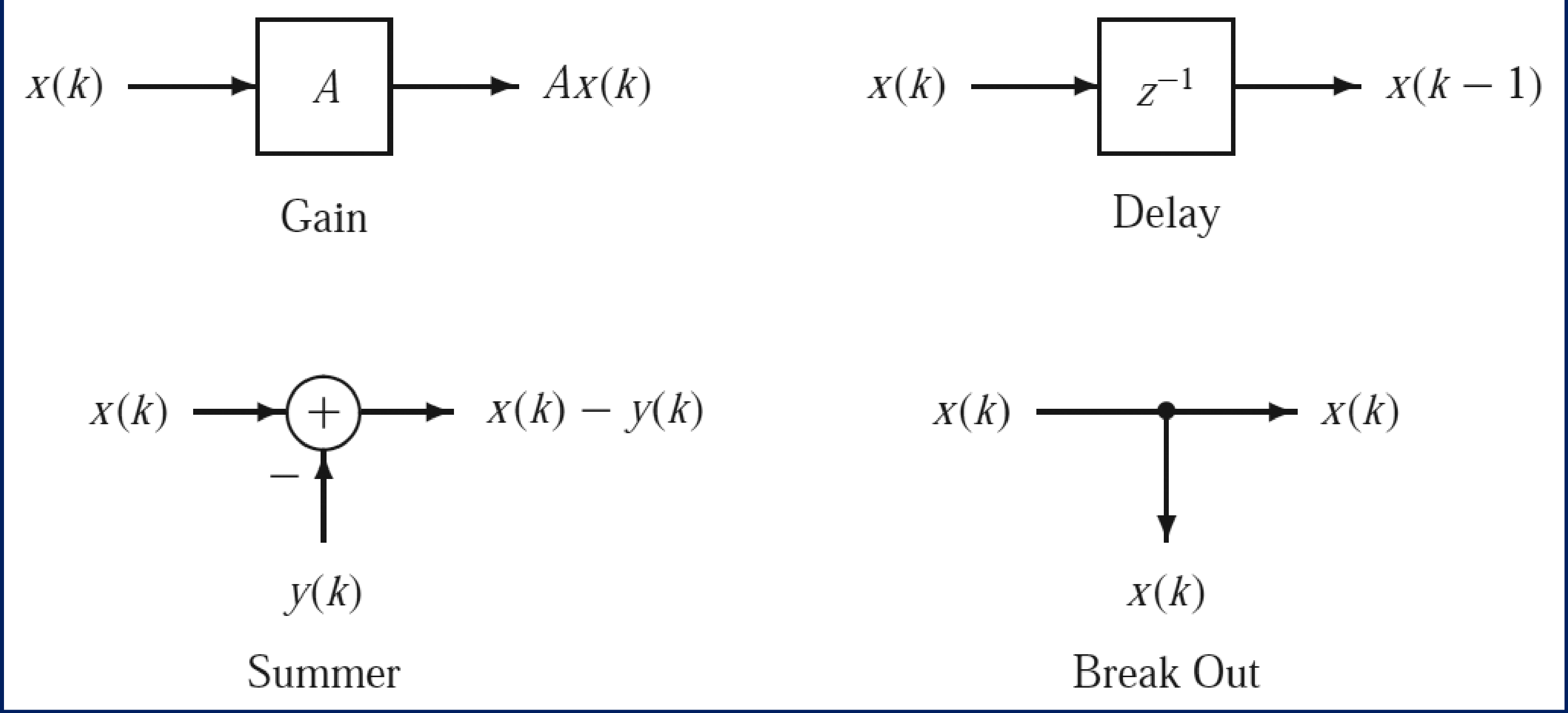


"Note that for $n >> 1$, the complete response is dominated by the zero-state response because the zero-input response quickly dies out."

Schilling and Harris (2012, p.92)

Block Diagram

- "Discrete-time systems can be displayed graphically in the form of block diagrams."
- "A block diagram is a set of blocks that represent processing units interconnected by directed line segments that represent signals."



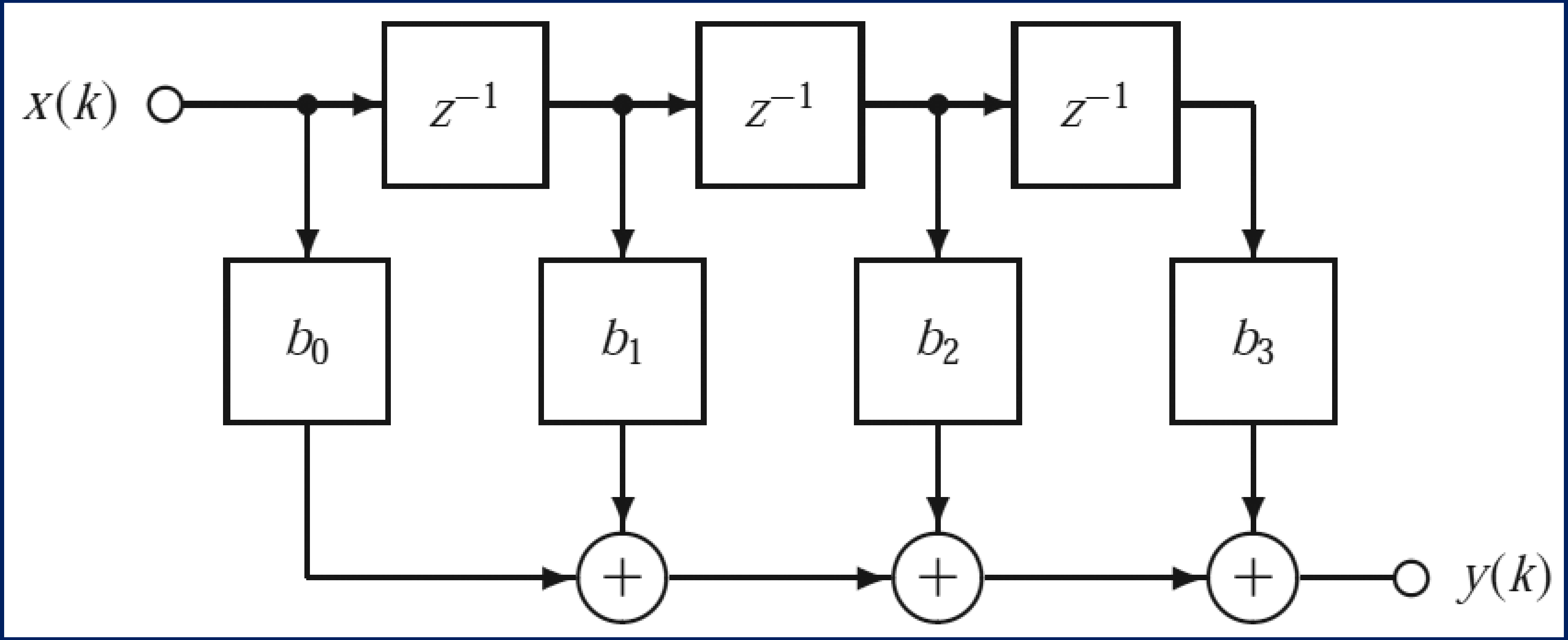
Schilling and Harris (2012, p.94-95)

Block Diagram: Example 1

Consider a **moving average filter** whose output is a weighted sum of the past inputs:

$$y(k) = \sum_{i=0}^{M} b_i x(k-i)$$

When $M = 3$, the block diagram of moving average filter is



Schilling and Harris (2012, p.95)

Block Diagram: Example 2

A LTI system $$y(k) = \sum_{i=0}^{M} b_i x(k-i) - \sum_{i=1}^{N} a_i y(k-i)$$

can be rewritten as

$$y(k) = b_0 x(k) + \sum_{i=1}^{D} \left[b_i x(k-i) - a_i y(k-i) \right]$$

by zero padding the coefficient vectors *a* and *b* so that $a, b \in \mathbb{R}^{D+1}$ where $D = \max\{N, M\}$ is the system dimension. When D = 2, using intermediate signals, the output can be defined recursively by

$$u_2(k) = b_2 x(k) - a_2 y(k)$$

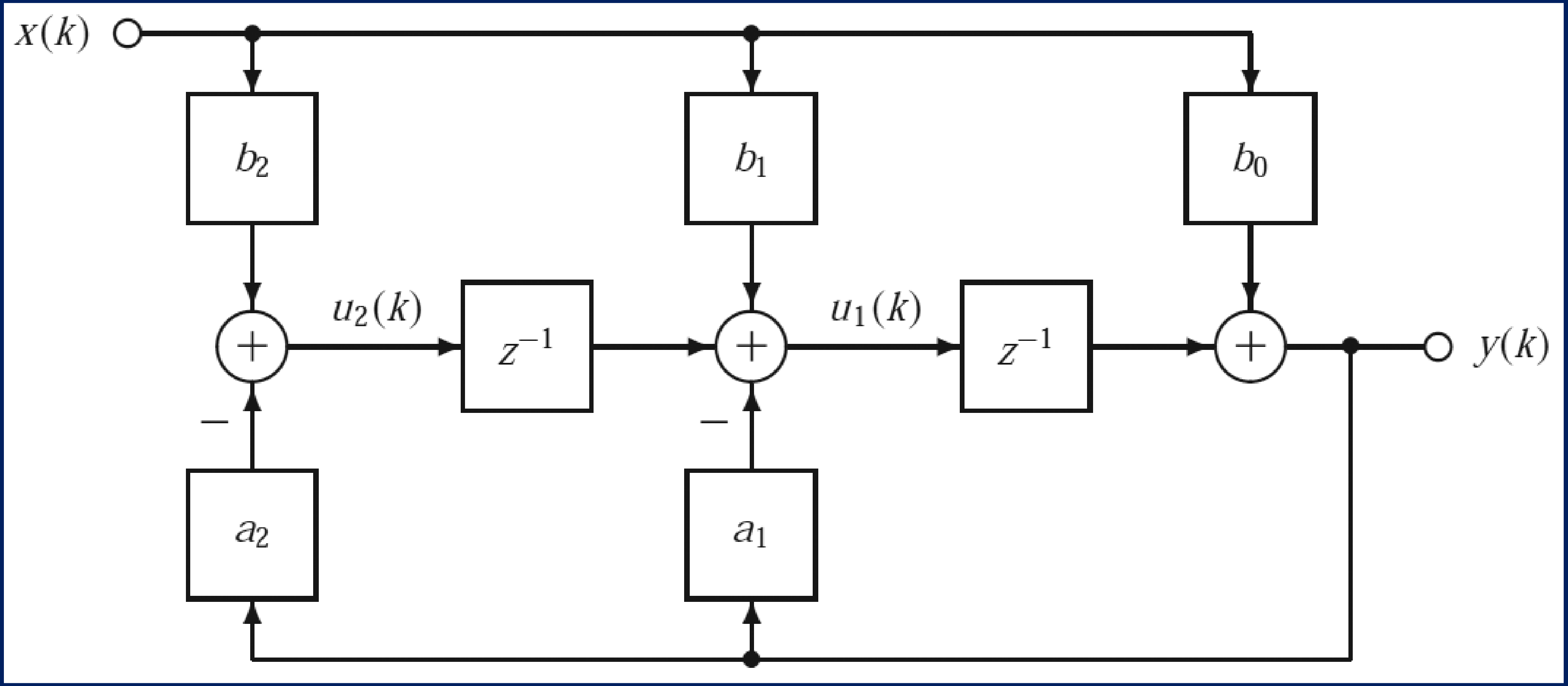
$$u_1(k) = b_1 x(k) - a_1 y(k) + u_2(k-1)$$

$$y(k) = b_0 x(k) + u_1(k-1)$$

Schilling and Harris (2012, p.96)

Block Diagram: Example 2

When $D = 2$, the block diagram for the system is



$$u_2(k) = b_2 x(k) - a_2 y(k)$$

$$u_1(k) = b_1 x(k) - a_1 y(k) + u_2(k-1)$$

$$y(k) = b_0 x(k) + u_1(k-1)$$

Schilling and Harris (2012, p.96)

Block Diagram: Example 2

For an arbitrary value of D, the output can then be defined recursively as

$$\begin{aligned}
u_D(k) &= b_D x(k) - a_D y(k) \\
u_{D-1}(k) &= b_{D-1} x(k) - a_{D-1} y(k) + u_D(k-1) \\
&\vdots \\
u_1(k) &= b_1 x(k) - a_1 y(k) + u_2(k-1) \\
y(k) &= b_0 x(k) + u_1(k-1)
\end{aligned}$$

Schilling and Harris (2012, p.96)

Impulse Response

- "The **impulse response** of a LTI system is the zero-state response $h(k)$ produced by the unit impulse input."

$$x(k) = \delta(k) \Rightarrow h(k) = y_{zs}(k)$$

- The impulse response can be used to compute the zero-state response of any input.

Consider a system $$y(k) = \sum_{i=0}^{M} b_i x(k-i)$$

The impulse response of this system is

$$h(k) = \sum_{i=0}^{M} b_i \delta(k-i)$$

Schilling and Harris (2012, p.96-97)

Impulse Response

Recall the property of unit impulse

$$\delta(k-i)=\begin{cases}1, & k=i\\ 0, & k\neq i\end{cases}$$

Consequently, the impulse response is

$$h(k)=\begin{cases}b_k, & 0\le k\le M\\ 0, & M<k\le\infty\end{cases}$$

"A linear system whose impulse response contains a finite number of nonzero samples is called a **finite impulse response (FIR)** system. Otherwise, the system is an **infinite impulse response (IIR)** system."

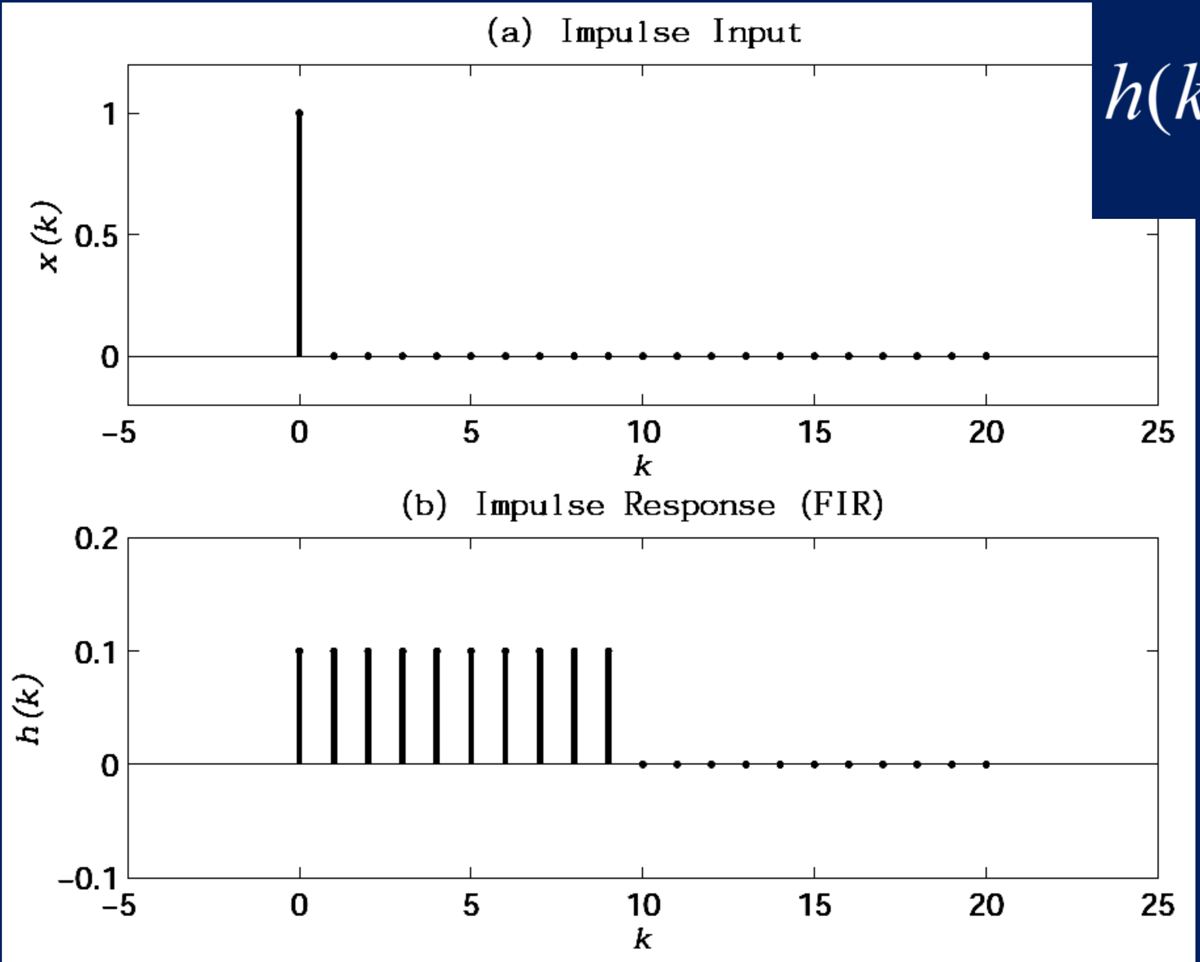
Schilling and Harris (2012, p.97)

FIR System: Example

Consider a moving average filter $$y(k) = \frac{1}{M}\sum_{i=0}^{M-1} x(k-i)$$

The impulse response of this filter is

$$h(k) = \begin{cases} 1/M, & 0 \le k \le M \\ 0, & M < k \le \infty \end{cases}$$



$M = 10$

Schilling and Harris (2012, p.97-98)

IIR System

Consider a LTI system

$$y(k) = \sum_{i=0}^{M} b_i x(k-i) - \sum_{i=1}^{N} a_i y(k-i), \quad N \geq 1$$

A systematic technique for solving this general system is the Z transform which will be introduced later.

"Suppose that $M \leq N$ and the roots of the characteristic polynomial $a(z)$ are simple and nonzero."

The impulse response in this case is

$$h(k) = c_0 \delta(k) + \sum_{i=1}^{N} c_i (p_i)^k \mu(k)$$

where p_i are the roots of $a(z)$.

IIR System

The coefficient vector $c \in \mathbb{R}^{N+1}$ is computed by

$$c_i = \left. \frac{(z - p_i)b(z)}{za(z)} \right|_{z=p_i}, \quad 0 \le i \le N$$

where $b(z)$ is the input polynomial and $p_0 = 0$.

"If $c_i \neq 0$ for some $i > 0$, then the duration of the impulse response $h(k)$ is infinite, in which case S is an IIR system."

Schilling and Harris (2012, p.99)

IIR System: Example

Consider the system S governed by difference equation

$$y(k)-0.2y(k-1)-0.8y(k-2)=2x(k)-3x(k-1)+4x(k-2)$$

The characteristic polynomial of S is

$$a(z)=z^2-0.2z-0.8=\left(z-1\right)\left(z+0.8\right)$$

So, S has simple nonzero roots at $p=[1,-0.8]$.

Then, the impulse response is

$$h(k)=c_0\delta(k)+\left[c_1+c_2(-0.5)^k\right]\mu(k)$$

The input polynomial is $b(z)=2z^2-3z+4$

The coefficients are $c_0=4/[(-1)(0.8)]=-5$

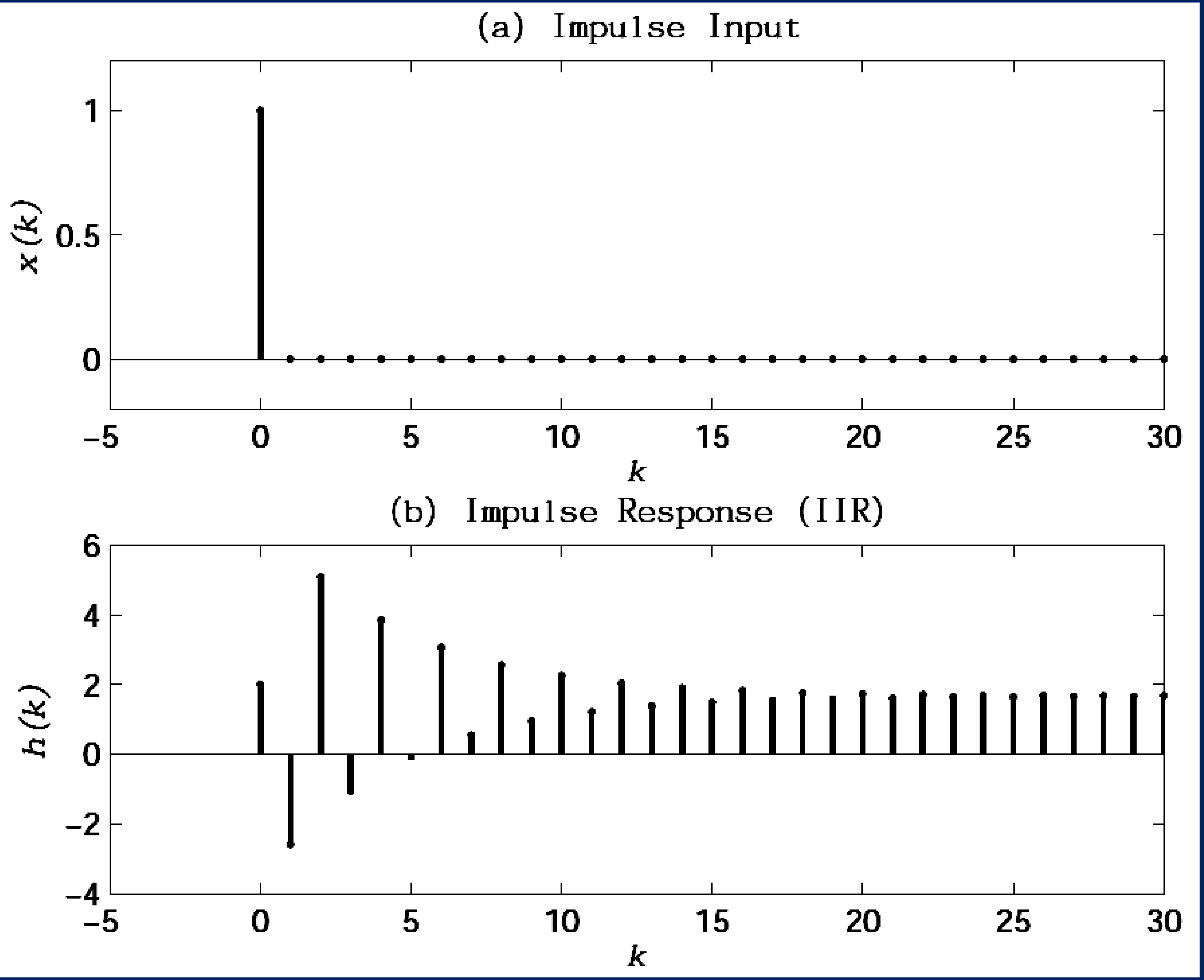
$$c_1=\left[2-3+4\right]/1.8=1.67$$

$$c_2=\left[2(0.64)-3(-0.8)+4\right]/\left[(-0.8)(-1.8)\right]=5.33$$

Schilling and Harris (2012, p.99)

IIR System: Example

The impulse response of *S* has infinite duration. So, *S* is an IIR system.



Schilling and Harris (2012, p.100)

Linear Convolution

Recall that $x(k) = \delta(k) \Rightarrow y(k) = h(k)$ (zero-state output)

For LTI system, $x(k) = \alpha\delta(k-i) \Rightarrow y(k) = \alpha h(k-i)$

"A causal signal $x(k)$ can be written as a weighted sum of unit impulses as"

$$x(k) = \sum_{i=0}^{k} x(i)\delta(k-i)$$

For LTI system, the zero-state output to the causal signal is

$$y(k) = \sum_{i=0}^{k} x(i)h(k-i)$$

The operation on the right-hand side is called **linear convolution** of signal $x(k)$ with signal $h(k)$:

$$x(k) * h(k) \stackrel{\text{def}}{=} \sum_{i=0}^{k} x(i)h(k-i)$$

Schilling and Harris (2012, p.100-101)

Linear Convolution

Convolution is commutative:

$$x(k)*h(k)=\sum_{i=0}^{k}x(i)h(k-i)$$

$$=\sum_{m=k}^{0}x(k-m)h(m), \quad m=k-i$$

$$=\sum_{m=0}^{k}h(m)x(k-m)$$

$$=h(k)*x(k)$$

For FIR system *S*, the impulse response is equal to the input coefficient. So, $y(k)=\sum_{i=0}^{k}b(i)x(k-i)$

Schilling and Harris (2012, p.100-101)

which is the original difference equation

Properties of Linear Convolution

Name	Property		
Commutative	$f \star g$	$=$	$g \star f$
Associative	$f \star (g \star h)$	$=$	$(f \star g) \star h$
Distributive	$f \star (g + h)$	$=$	$f \star g + f \star h$

Schilling and Harris (2012, p.102)

Linear Convolution: Example

Consider the system

$$y(k) - 0.2y(k-1) - 0.8y(k-2) = 2x(k) - 3x(k-1) + 4x(k-2)$$

Suppose the input is $x(k) = 10\sin(0.1\pi k)\mu(k)$

and the initial condition is zero. The output of the system is then

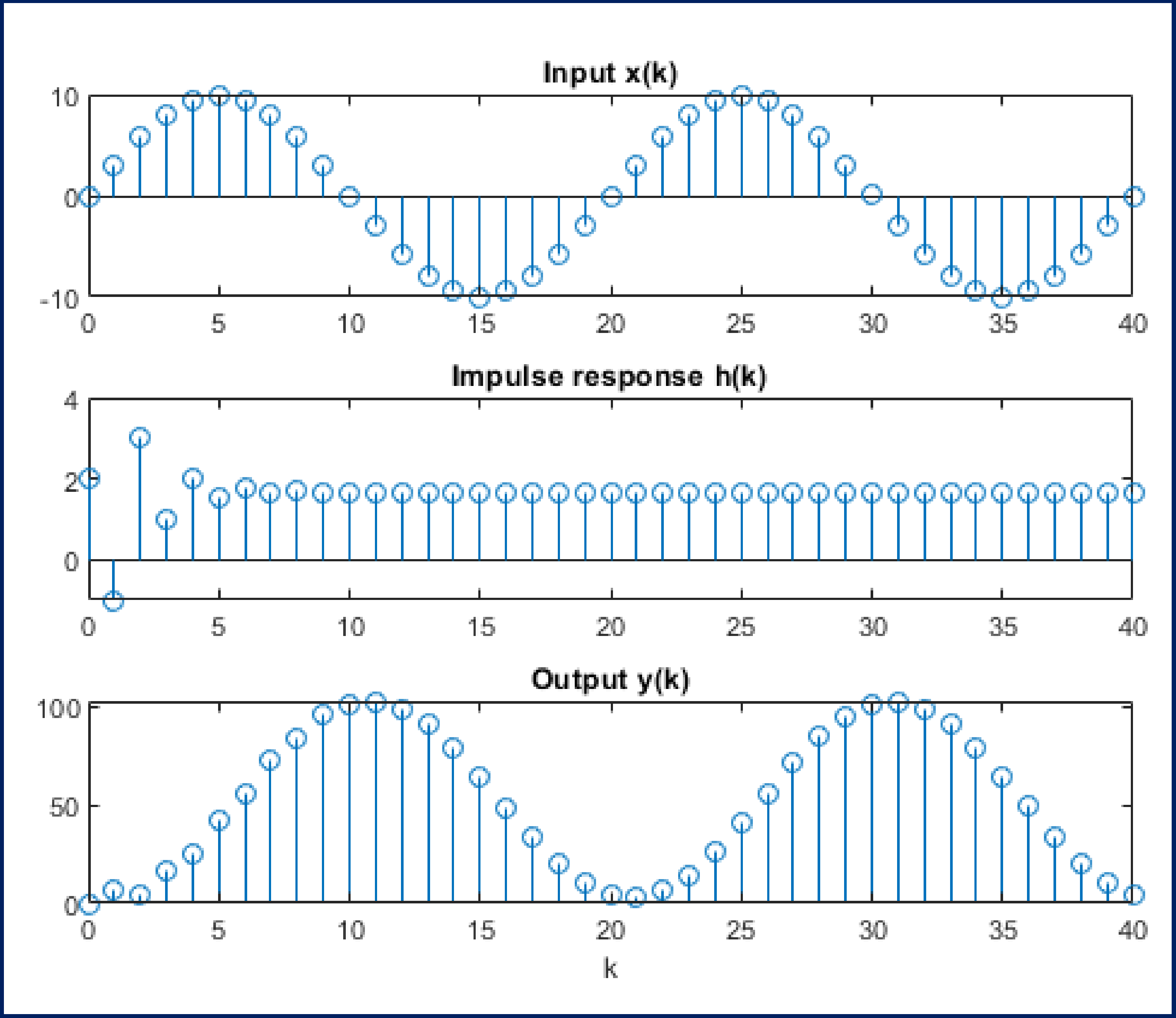
$$y(k) = \sum_{i=0}^{k} h(i)x(k-i)$$

$$= \sum_{i=0}^{k} \left\{-5\delta(i) + [1.67 - 5.33(-0.5)^i]\right\}\left\{10\sin[0.1\pi(k-i)]\right\}$$

MATLAB:
```
k = 0:40; x = 10*sin(0.1*pi*k);
y = filter([2,-3,4],[1,-0.2,-0.8],x);
```

Schilling and Harris (2012, p.102)

Linear Convolution: Example



Linear Convolution

If $h(k)$ is nonzero for $0 \le k < L$ and $x(k)$ is nonzero for $0 \le k < M$, then the linear convolution can be expressed as

$$y(k) = \sum_{i=0}^{L-1} h(i)x(k-i), \quad 0 \le k < L + M - 1$$

- "The upper limit has been changed from k to $L - 1$ because $h(i) = 0$ for $i \ge L$."
- "Linear convolution of an L-point signal with an M-point signal is a signal of length $L + M - 1$."

Schilling and Harris (2012, p.103)

Linear Convolution

Linear convolution of L-point signal $h(k)$ and M-point signal $x(k)$ can be represented as matrix multiplication.

Example: $L = 2$, $M = 3$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \underbrace{\begin{bmatrix} x(0) & 0 \\ x(1) & x(0) \\ x(2) & x(1) \\ 0 & x(2) \end{bmatrix}}_{\text{linear convolution matrix}} \begin{bmatrix} h(0) \\ h(1) \end{bmatrix}$$

Circular Convolution

- Circular convolution is an operator whose result has the same length as the two operands.
- "The periodic extension of an N-point signal $x(k)$ is a signal $x_p(k)$ defined for all integers k as follows."

$$x_p(k) \stackrel{\text{def}}{=} x(k \text{ modulo } N)$$

where the modulo operator returns the remainder of the division between two numbers. Example: 5 modulo 2 = 1
MATLAB: `mod(5,2)`

- "$x_p(k)$ extends $x(k)$ periodically in both positive and negative directions."
- Example: $x_p(N) = x(0)$, $x_p(-1) = x(N-1)$

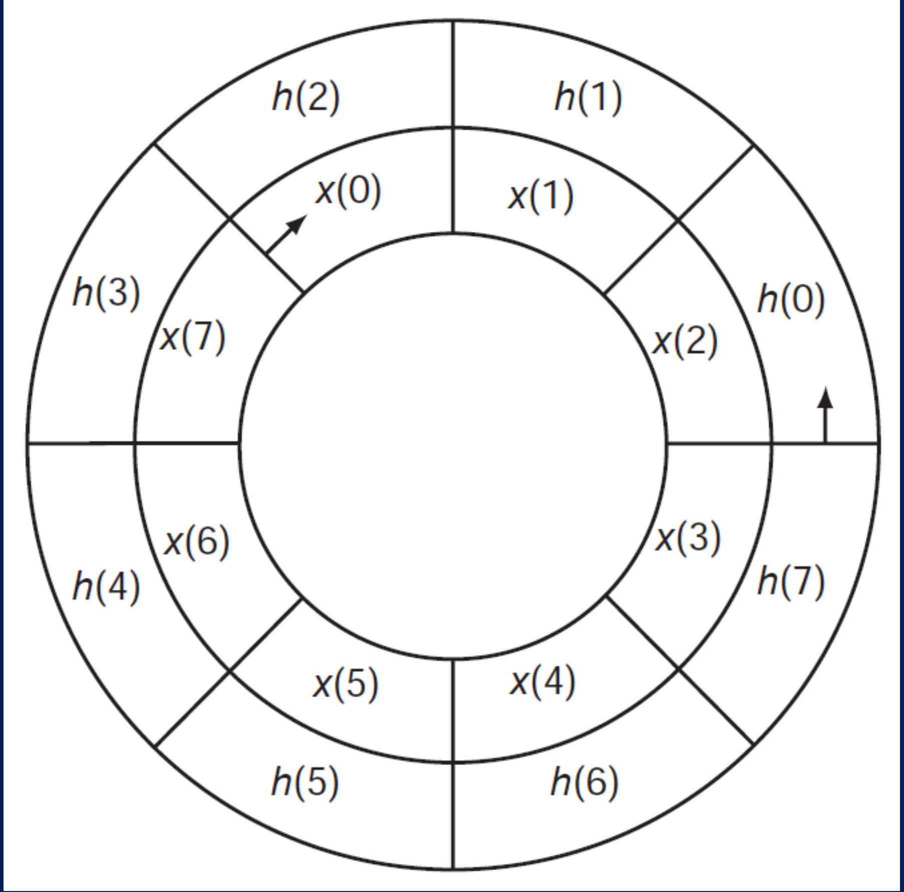
Schilling and Harris (2012, p.103)

Circular Convolution

Circular convolution of N-point signals $h(k)$ and $x(k)$ is defined as

$$h(k) \circ x(k) \stackrel{\text{def}}{=} \sum_{i=0}^{N-1} h(i) x_p(k-i), \quad 0 \le k < N$$

Example: $k = 2$, $N = 8$



Schilling and Harris (2012, p.104)

Circular Convolution

Circular convolution of N-point signals $h(k)$ and $x(k)$ can be represented as matrix multiplication.

Example: $N = 4$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \underbrace{\begin{bmatrix} x(0) & x(3) & x(2) & x(1) \\ x(1) & x(0) & x(3) & x(2) \\ x(2) & x(1) & x(0) & x(3) \\ x(3) & x(2) & x(1) & x(0) \end{bmatrix}}_{\text{Circular convolution matrix}} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \end{bmatrix}$$

Schilling and Harris (2012, p.104-105)

Zero Padding

- Circular convolution can be implemented using the fast Fourier transform (FFT). However, circular convolution produce a different response than linear convolution (zero-state response of a linear discrete-time system).
- Zero padding can be used to modify the *L*-point signal *h*(*k*) and *M*-point signal *h*(*k*) so that circular convolution provides the same result as linear convolution.

$$h_z = [h(0), h(1), \ldots, h(L-1), \overbrace{0, \ldots, 0}^{M\text{-}1}]^T$$

$$x_z = [x(0), x(1), \ldots, x(M-1), \underbrace{0, \ldots, 0}_{L\text{-}1}]^T$$

Schilling and Harris (2012, p.105-106)

Zero Padding

"Thus, h_z and x_z are zero-padded vectors of length $N = L + M - 1$."

$$h_z(k) \circ x_z(k) = \sum_{i=0}^{N-1} h_z(i) x_{zp}(k-i), \qquad 0 \le k < N$$

$x_{zp}(k)$ is periodic extension of $x_z(k)$.

$$= \sum_{i=0}^{L-1} h_z(i) x_{zp}(k-i)$$

$h_z(k)$ has only $L - 1$ nonzero elements

$$= \sum_{i=0}^{L-1} h_z(i) x_z(k-i)$$

$x_z(k)$ has only $L - 1$ zeros padded to the end of it. So, $x_{zp}(k\text{-}i) = x_z(k\text{-}i)$

$$= h_z * x_z \quad = h * x$$

This can be easily verified.

So, $h_z(k) \circ x_z(k) = h(k) * x(k), \qquad 0 \le k < N$

Schilling and Harris (2012, p.105-106)

Exercise

Let $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, h = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Verify that $h * x = h_z \circ x_z$.

Deconvolution

- When the impulse response $h(k)$ and the output $y(k)$ of the system are known, the input $x(k)$ is to be obtained.
- This process is referred to as deconvolution.
- Suppose $h(k)$ and $x(k)$ are both causal and $h(0) \neq 0$.
- Evaluating $y(k) = \sum_{i=0}^{k} x(i)h(k-i)$ at $k = 0$ yields

$$y(0) = x(0)h(0) \Rightarrow x(0) = y(0)/h(0)$$

- Once $x(0)$ is known, the remaining samples of $x(k)$ can be obtained recursively.
- At $k = 1$, $y(1) = x(0)h(1) + x(1)h(0)$. Solving for $x(1)$ yields

$$x(1) = \frac{y(1) - x(0)h(1)}{h(0)}$$

Schilling and Harris (2012, p.108)

Deconvolution

- Repeating the process for $2 \le k < N$, we obtain

$$x(k) = \frac{1}{h(0)}\left[y(k) - \sum_{i=0}^{k-1} x(i)h(k-i) \right], \quad k \ge 1$$

- "Deconvolution also includes finding the impulse response $h(k)$, given the input $x(k)$ and the output $y(k)$." This is a special case of **system identification**.

Exercise: Given the impulse response and the output of a system as $h = [2, -1, 6]^T$, $y = [10, 1, 19, 22, -24]^T$. Find the input $x(k)$.

Polynomial Arithmetic

"Suppose $A(z)$ and $B(z)$ are polynomials of degree L and M, respectively."

$$A(z) = a_0 z^L + a_1 z^{L-1} + \cdots + a_L$$

$$B(z) = b_0 z^M + b_1 z^{M-1} + \cdots + b_M$$

Let $C(z) = A(z)B(z)$ be the product polynomial of degree $N = L + M$, that is $C(z) = c_0 z^{L+M} + c_1 z^{L+M-1} + \cdots + c_{L+M}$

The coefficients of $C(z)$ can be obtained using linear convolution between the coefficients of $A(z)$ and $B(z)$ as

$$c(k) = a(k) * b(k), \quad 0 \le k < L + M + 2$$

where $a(k)$, $b(k)$, $c(k)$ are the coefficients of polynomials $A(z)$, $B(z)$, $C(z)$.

Schilling and Harris (2012, p.109)

Polynomial Arithmetic: Example

Consider the polynomials

$$A(z) = 2z^2 - z + 6, \qquad B(z) = 5z^2 + 3z - 4$$

whose coefficient vectors are $a = [2, -1, 6]^T, b = [5, 3, -4]^T$

Then, $c(k) = a(k) * b(k) = [10, 1, 19, 22, -24]^T$

As a result, $C(z) = A(z)B(z) = 10z^4 + z^3 + 19z^2 + 22z - 24$

MATLAB:

```
a = [2, -1, 6];
b = [5, 3, -4];
c = conv(a,b);
```

Schilling and Harris (2012, p.109-110)

Linear Cross-Correlation

Linear cross-correlation of L-point signal $y(k)$ with M-point signal $x(k)$ where $M \le L$ is denoted as $r_{yx}(k)$ and defined as

$$r_{yx}(k) = y(k) \otimes x(k) \stackrel{\text{def}}{=} \frac{1}{L} \sum_{i=0}^{L-1} y(i) x(i-k), \quad 0 \le k < L$$

- If $x(k)$ is causal, the lower limit can be set to $i = k$.
- "Linear cross-correlation is sometimes defined without the scaling factor $1/L$."
- "The scaling factor is used here so that it is consistent with the statistical definition of cross-correlation between two random signals."

Schilling and Harris (2012, p.110-111)

Linear Cross-Correlation

Linear cross-correlation of L-point signal $y(k)$ with M-point signal $x(k)$ can be represented as matrix multiplication.

Example: $L = 3$, $M = 2$

$$\begin{bmatrix} r_{yx}(0) \\ r_{yx}(1) \\ r_{yx}(2) \end{bmatrix} = \underbrace{\frac{1}{3}\begin{bmatrix} x(0) & x(1) & 0 \\ 0 & x(0) & x(1) \\ 0 & 0 & x(0) \end{bmatrix}}_{\text{linear cross-correlation matrix}} \begin{bmatrix} y(0) \\ y(1) \\ y(2) \end{bmatrix}$$

Thus, linear cross-correlation of $y(k)$ with $x(k)$ can be expressed as $r_{yx} = D(x)y$ where $D(x)$ is correlation matrix.

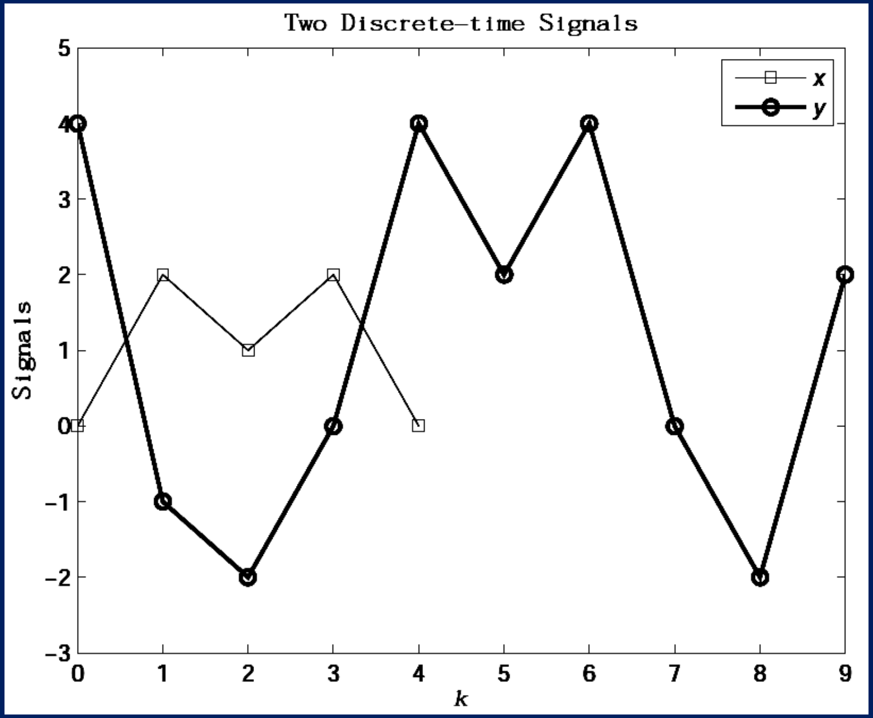
Schilling and Harris (2012, p.110-111)

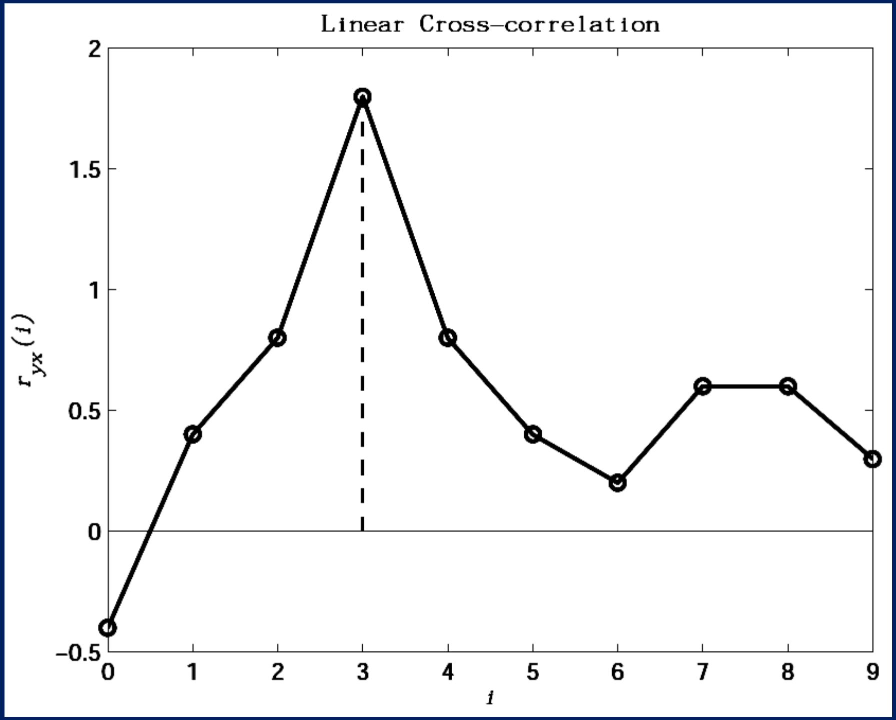
Exercise

"Linear cross-correlation can be used to measure the degree to which the shape of one signal is similar to the shape of the other signal."

Given $x = [0, 2, 1, 2, 0]^T$, $y = [4, -1, -2, 0, 4, 2, 4, 0, -2, 2]^T$,

Compute the linear cross-correlation of $y(k)$ with $x(k)$.





Schilling and Harris (2012, p.111-113)

Normalized Linear Cross-Correlation

Proakis and Manolakis (1992) showed that the square of cross-correlation is bounded

$$r_{yx}^2(k) \le \left(\frac{M}{L}\right) r_{xx}(0) r_{yy}(0), \qquad 0 \le k < L$$

The normalized linear cross-correlation is then defined as

$$\rho_{yx}(k) \overset{\text{def}}{=} \frac{r_{yx}(k)}{\sqrt{(M/L) r_{xx}(0) r_{yy}(0)}}, \qquad 0 \le k < L$$

As a result,

$$-1 \le \rho_{yx}(k) \le 1, \qquad 0 \le k < L$$

When a correlation peak approaches ±1, there is a strong correlation between $y(k)$ and $x(k)$.

Schilling and Harris (2012, p.113-114)

Circular Cross-Correlation

"Let $y(k)$ and $x(k)$ be N-point signals, and let $x_p(k)$ be the periodic extension of $x(k)$. The circular cross-correlation of $y(k)$ with $x(k)$ is denoted as $c_{yx}(k)$ and defined as"

$$c_{yx}(k) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=0}^{N-1} y(i) x_p(i-k), \qquad 0 \le k < N$$

Normalized circular cross-correlation is then defined as

$$\sigma_{yx}(k) \stackrel{\text{def}}{=} \frac{c_{yx}(k)}{\sqrt{c_{xx}(0) c_{yy}(0)}}$$

Schilling and Harris (2012, p.114-115)

Properties of Cross-Correlation

Symmetry: $c_{xy}(k) = c_{yx}(-k), \quad 0 \le k < N$

Relationship with convolution:

$$c_{yx}(k) = \frac{y(k) \circ x(-k)}{N}, \quad 0 \le k < N$$

Relationship between linear and circular correlations:

$y(k)$ = L-point signal, $x(k)$ = M-point signal, $M \le L$.

$$r_{yx}(k) = \left(\frac{N}{L}\right) c_{y_z x_z}(k), \quad 0 \le k < L$$

where $y_z(k)$ and $x_z(k)$ are zero-padded versions of $y(k)$ and $x(k)$, respectively, such that both $y_z(k)$ and $x_z(k)$ are of length $L + M + p$ with $p \ge -1$.

Schilling and Harris (2012, p.115-116)

Lab: Echo Detection

- Let $x(k)$ be transmitted signal and $y(k)$ be received signal.
- Suppose the sampling frequency is f_s = 1 MHz and the number of transmitted samples is M = 512.
- Let $x(k)$ be a multi-frequency chirp, a sinusoidal signal whose frequency varies with time:

$$f(k) = kf_s \big/ \left[2(M-1)\right]$$

$$y(k) = \sin\left[2\pi f(k) kT\right], \quad T = 1\big/ f_s$$

- "The received signal $y(k)$ includes a scaled and delayed version of the transmitted signal plus measurement noise."

Schilling and Harris (2012, p.127)

Lab: Echo Detection

- Suppose received signal consists of L = 2048 samples.
- "If $x_z(k)$ denotes the transmitted signal, zero-extended to L points, then the received signal can be expressed as"

$$y(k) = a x_z(k-d) + \eta(k), \qquad 0 \le k < L$$

- The first term is the echo of the transmitted signal with attenuation factor $a \ll 1$ and delayed by d samples.
- The second term is atmospheric noise. Suppose the noise is uniformly distributed over the interval $[-\sigma, \sigma]$.
- If T is the sampling interval, then the time of flight in seconds is $\tau = dT$. Distance to the target is $r = cdT/2$ where c is the speed of light and the factor 2 arises because the time of flight is a two-way travel time.

Schilling and Harris (2012, p.127)

Lab: Echo Detection

- Generate the input signal $x(k)$, atmospheric noise $\eta(k)$, and received signal $y(k)$ with your own choice of attenuation factor a, delay d, and noise bound σ.
- Perform linear cross-correlation and normalized linear cross-correlation of $y(k)$ with $x(k)$ to determine the delay d. Also plot the correlation results.

References

- Schilling, R. J. and S. L. Harris, 2012, Fundamentals of Digital Signal Processing using MATLAB, Second Edition, Cengage Learning.
- Schilling, R. J. and S. L. Harris, 2012, Introduction to Digital Signal Processing using MATLAB, Second Edition, Cengage Learning.