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Assignment 4

Instruction: Put your answers, source codes, and/or figures in your report file and save it as a PDF file. Then, submit your PDF file via the online submission system in the class website by 11.59 PM of September 24, 2020.

1. Consider a discrete-time system described by the difference equation

$$y(k) = y(k-1) - 0.24y(k-2) + 2x(k-1) - 1.6x(k-2)$$

1.1 Find the transfer function $H(z)$.

Ans

We take the Z-transform to the difference equation. We obtain that

$$Y(z) = Y(z)z^{-1} - 0.24Y(z)z^{-2} + 2X(z)z^{-1} - 1.6X(z)z^{-2}$$
$$[1 - z^{-1} + 0.24z^{-2}]Y(z) = [2z^{-1} - 1.6z^{-2}]X(z)$$
$$\frac{Y(z)}{X(z)} = \frac{2z^{-1} - 1.6z^{-2}}{1 - z^{-1} + 0.24z^{-2}}$$
$$H(z) = \frac{2z^{-1} - 1.6z^{-2}}{1 - z^{-1} + 0.24z^{-2}}.$$

1.2 Write down the form of the natural mode terms of this system.

Ans

The natural modes are generated by the poles of $H(z)$.

$$H(z) = \frac{2z^{-1} - 1.6z^{-2}}{1 - z^{-1} + 0.24z^{-2}}$$
$$H(z) = \frac{2z - 1.6}{z^2 - z + 0.24}$$
$$H(z) = \frac{2(z - 0.8)}{(z - 0.6)(z - 0.4)}$$

So, the natural modes of the system are $z = 0.6$ and $z = 0.4$.

1.3 Find the zero-state response to the step input $x(k) = 10\mu(k)$.

Ans

Firstly, we find the Z-transform of $x(k)$

$$X(z) = \frac{10z}{z - 1}$$

The zero-state response can be evaluated by

$$Y(z) = H(z)X(z)$$
$$Y(z) = \frac{2(z - 0.8)}{(z - 0.6)(z - 0.4)} \times \frac{10z}{z - 1}$$
$$Y(z) = \frac{20z(z - 0.8)}{(z - 0.6)(z - 0.4)(z - 1)}$$

To find $y(k)$, we have to take the inverse of Z-transform. In this case, we should the partial fraction technique.

$$Y(z) = 20\left(\frac{3/2}{z - 0.6} + \frac{-4/3}{z - 0.4} + \frac{5/6}{z - 1}\right)$$
$$Z^{-1}\{Y(z)\} = 20\left(\frac{3}{2}(0.6)^k - \frac{4}{3}(0.4)^k + 5/6\right)\mu(k)$$
$$y(k) = 20\left(\frac{3}{2}(0.6)^k - \frac{4}{3}(0.4)^k + \frac{5}{6}\right)\mu(k).$$

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1.4 Find the zero-state response to the causal exponential input $x(k)=0.8^k\mu(k).$ Does a forced mode term appear in $y(k)$? If not, why not?

Ans

$$X(z)=\frac{z}{z-0.8}$$
$$Y(z)=H(z)X(z)$$
$$Y(z)=\frac{2(z-0.8)}{(z-0.6)(z-0.4)}\times\frac{z}{z-0.8}$$
$$Y(z)=\frac{2z}{(z-0.6)(z-0.4)}$$
$$Y(z)=2\left(\frac{3}{z-0.6}-\frac{2}{z-0.4}\right)$$

The zero-state response is

$$y(k)=2[3(0.6)^k-2(0.4)^k]\mu(k).$$

The forced mode does not appear in $y(k)$ because it is suppressed by the zero of the transfer functions.

1.5 Find the zero-state response to the causal exponential input $x(k)=0.4^k\mu(k).$ Is this an example of harmonic forcing? Why or why not?

Ans

$$X(z)=\frac{z}{z-0.4}$$
$$Y(z)=\frac{2(z-0.8)}{(z-0.6)(z-0.4)}\times\frac{z}{z-0.4}$$
$$Y(z)=\frac{2z(z-0.8)}{(z-0.6)(z-0.4)^2}$$
$$Y(z)=2\frac{z^2-0.8z}{(z-0.6)(z-0.4)^2}$$
$$y(k)=[Res(0.6,k)+Res(0.4,k)]\mu(k-1)$$

$$y(k)=\left[\frac{2(0.6)(0.6-0.8)(0.6)^{k-1}}{(0.6-0.4)}+2\frac{d}{dz}\left(\frac{(z^2-0.8z)z^{k-1}}{z-0.6}\right)\bigg|_{z=0.4}\right]\mu(k-1)$$
$$y(k)=\left[-\frac{6}{5}(0.6)^{k-1}+2\frac{(z-0.6)\left((k+1)z^k-0.8kz^{k-1}\right)-(z^{k+1}-0.8z^k)}{(z-0.6)^2}\Bigg|_{z=0.4}\right]\mu(k-1)$$
$$y(k)=\left[-\frac{6}{5}(0.6)^{k-1}+2\frac{(-0.2)((k+1)(0.4)^k-0.8k(0.4)^{k-1})-(0.4)^{k+1}+0.8(0.4)^k)}{(0.2)^2}\right]\mu(k-1)$$
$$y(k)=[-2(0.6)^k+10(k+1)(0.4)^k]\mu(k-1)$$

It appears that $y(k)$ is stable because the 0.4^k is going to 0 faster than $k+1$. The $y(k)$ is stable. I think that this is not the example of the harmonic forcing.

$$\lim_{k\to\infty}\frac{k+1}{0.4^{-k}}=\lim_{k\to\infty}\frac{k+1}{e^{-k\log(0.4)}}=\lim_{k\to\infty}\frac{1}{-\log(0.4)\,e^{-k\log 0.4}}=0$$

2. Consider a discrete-time system described by the following transfer function.

$$H(z)=\frac{3(z-0.4)}{z+0.8}$$

2.1 Suppose the zero-state response to an input $x(k)$ is $y(k)=\mu(k).$ Find $X(z).$

Ans

$$Y(z)=\frac{z}{z-1}$$

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$$X(z) = \frac{Y(z)}{H(z)}$$
$$X(z = \frac{z}{z-1} \times \frac{z+0.8}{3(z-0.4)}$$
$$X(z) = \frac{z(z+0.8)}{3(z-1)(z-0.4)}$$

2.2 Find $x(k)$.

Ans

$$X(z) = \frac{1}{3}\left[\frac{3}{z-1} - \frac{\frac{4}{5}}{z-0.4}\right]$$
$$x(k) = \frac{1}{3}\left[3 - \frac{4}{5}(0.4)^k\right]\mu(k)$$
$$x(k) = \left[1 - \frac{4}{15}(0.4)^k\right]\mu(k).$$

3. Consider the following discrete-time signal $x = [1,2,1,0]^T$.

3.1 Find $X(i) = \mathrm{DFT}\{x(k)\}$.

Ans

Firstly, we construct the transformation matrix W

$$W_{ij} = W_4^{ij}$$

where, $i = 0,1,2,3$ $k = 0,1,2,3$ and $W_4 = e^{-\frac{2\pi i}{4}} = -i$.

$$W = [1,1,1,1; 1,-i,-1,i; 1,-1,1,-1; 1,i,-1,-i]$$
$$X = Wx$$
$$X = [1,1,1,1; 1,-i,-1,i; 1,-1,1,-1; 1,i,-1,-i][1; 2; 1; 0]$$
$$X = [4; -2i; 0; 2i].$$

So, $X(0) = 4, X(1) = -2i, X(2) = 0, x(3) = 2i$.

3.2 Compute the magnitude spectrum $A_x(i)$.

Ans

Magnitude spectrum is $A(i) = |X(i)|$

So, $A(0) = 4, A(1) = 2, A(2) = 0, A(3) = 2$.

3.3 Compute the phase spectrum $\phi_x(i)$.

Ans

The phase spectrum is that $\phi(0) = 0, \phi(1) = -\frac{\pi}{2}, \phi(2) = 0, \phi(3) = \frac{\pi}{2}$.

3.4 Compute the power density spectrum $S_x(i)$.

Ans

$$S_x(i) = \frac{|X(i)|^2}{N}$$

So, $S_x(0) = 4, S_x(1) = 1, S_x(2) = 0, S_x(3) = 1$.

3.5 Verify the Parseval's identity in this case.

Ans

We need to show that

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$$\sum_{k=0}^{N-1} |x(k)|^2 = \frac{1}{N} \sum_{i=0}^{N-1} |X(i)|^2$$
$$1^2 + 2^2 + 1^2 + 0^2 = \frac{1}{4}(4^2 + |-2i|^2 + 0^2 + |2i|^2)$$
$$6 = \frac{1}{4}(16 + 4 + 0 + 4)$$
$$6 = 6$$

The Parseval's identity is verified.

4. Let $x_a(t)$ be a periodic pulse train of period $T_0 = 1$. Suppose the pulse amplitude is $a = 10$ and the pulse duration is $\tau = T_0/5$ as shown in Figure 1. This signal can be represented by the Fourier series

$$x_a(t) = \frac{d_0}{2} + \sum_{i=1}^{\infty} d_i \cos\left(\frac{2\pi it}{T_0} + \theta_i\right)$$

Write a program that uses the DFT to compute the value of the coefficients d_0 and (d_i, θ_i) for $1 \le i < 16$. Plot d_i and θ_i using the function `stem`.

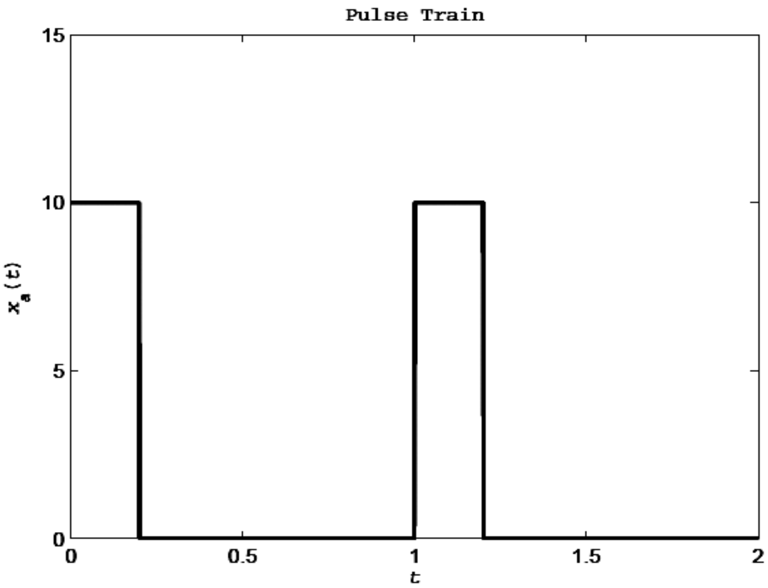

Figure 1. Periodic pulse train.

Ans

The code to compute d_0 and θ_i is shown as the following figures.

hw4_4.m | hw4_5.m | hw4_6.m | spectrogram_water_sound.m

```matlab
%hw 4.4
clear;
%create analog pulse
T0 = 1; T_pulse = T0/5; F0 = 1/T0;
t_f = 5*T0;
t = 0:1e-4:t_f;
x = 10*(mod(t,T0)<=T_pulse).*(t>=0);
%create digital pulse
N_T = 16;
fs = F0*(N_T);
t_i = 0:1/fs:t_f;
N =length(t_i);
x_i = 10*(mod(t_i,T0)<=T_pulse).*(t_i>=0);
%calculate DFT
W_N = exp(-2*pi*1i/N_T);
W = zeros(N,N);
for indx_i = 1:N
    for indx_k = 1:N
        W(indx_i,indx_k)= W_N^((indx_i-1)*(indx_k-1));
    end
end
X = W*x_i';

%Magnetude spectrum (M)
M = abs(X);
d = 2*M(1:N_T/2)/N;
subplot(3,1,2); stem(d); xlabel('i'); ylabel('d_i');
%phase spectrum (phi)
phi = angle(X(1:N_T/2));
subplot(3,1,3); stem(phi/pi); xlabel('i'); ylabel('\phi_i (unit of \pi)');
%reconstruct signal
x_r = 0.5*d(1)*ones(1,length(t));
for indx_i = 2:N_T/2
    x_r = x_r + d(indx_i)*cos(2*pi*(indx_i-1)*F0*t+phi(indx_i));
end
%plot signal
subplot(3,1,1);
hold on;
plot(t,x);
stem(t_i,x_i);
plot(t,x_r);
hold off;
title('Pulse signal and the fourier coefficients')
legend('x_a','x_i','reconstrcted x_a');
ylabel('x(t)');
xlabel('time');
```

The results are shown in the below figure.

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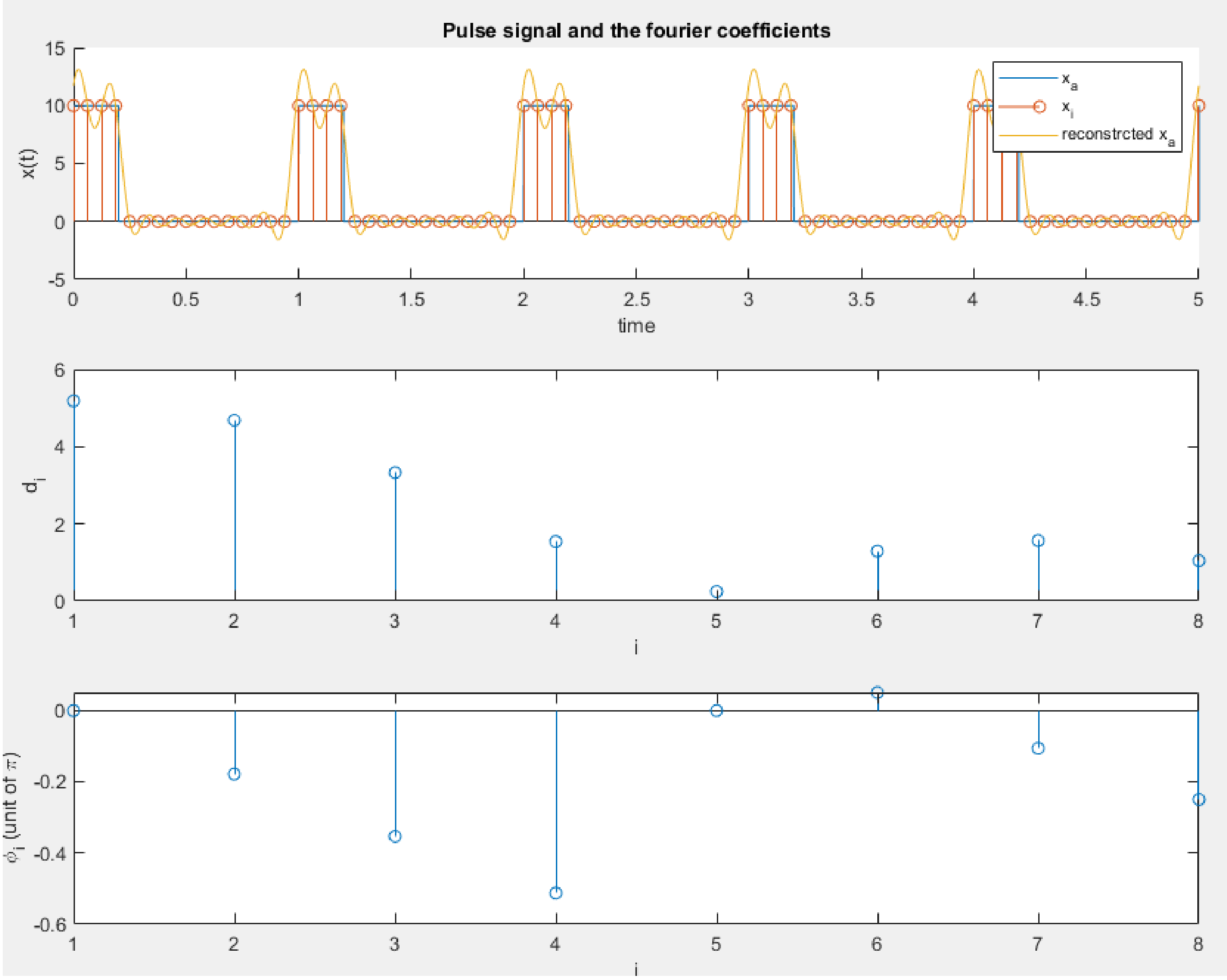

5. Consider the following noisy periodic signal with a sampling frequency of f_s = 1600 Hz and N = 1024.

$$x(k) = \sin^2(400\pi kT)\cos^2(300\pi kT) + v(k), \quad 0 \le k < N$$

Here $v(k)$ is zero-mean Gaussian white noise with a standard deviation of $\sigma = 1/\sqrt{2}$. Write a program that performs the following tasks.

5.1 Compute and plot the power density spectrum $S_x(f)$ for $0 \le f \le f_s/2$.

Ans

The code to compute the power density spectrum is shown as the following figures.

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Editor - C:\Users\hameh\OneDrive\Signal Processing\Signal-processing\Fourier_transform\hw4_5.m

hw4_4.m | hw4_5.m | hw4_6.m | spectrogram_water_sound.m

```
%hw 4.5
clear;
%create spectrum
fs = 1600; N = 1024;
k = 0:(N-1);
t = k/fs;
x = sin(400*pi*t).^2.*cos(300*pi*t).^2;
noise = normrnd(0,1/sqrt(2),[1,N]);
x = x+noise;
plot(t,x);
%calculate fourier transfrom
X = fft(x);
f = [0:N/2 -N/2+1:-1]/(N/fs);
indx = find(f>0);
%calculate power spectrum
S = X.*conj(X)/N;
%calculate average power
P_x = sum(S)/N;
P_v = sum(noise.*noise)/N;
disp(['Average power of x(k) = ',num2str(P_x)]);
disp(['Average power of x(k) = ',num2str(P_v)]);
%plot result
subplot(2,1,1);
plot(t,x);
xlabel('x(t)');
ylabel('t(s)');
title('Signal x(t)');
subplot(2,1,2);
plot(f(indx),S(indx));
xlabel('f(Hz)');
ylabel('S(f)');
title('Power density spectrum');
```

The signal and the power spectrum are shown in the below figure. It appears that there are four outstanding peaks at 100 Hz, 300 Hz, 400 Hz, and 700 Hz.

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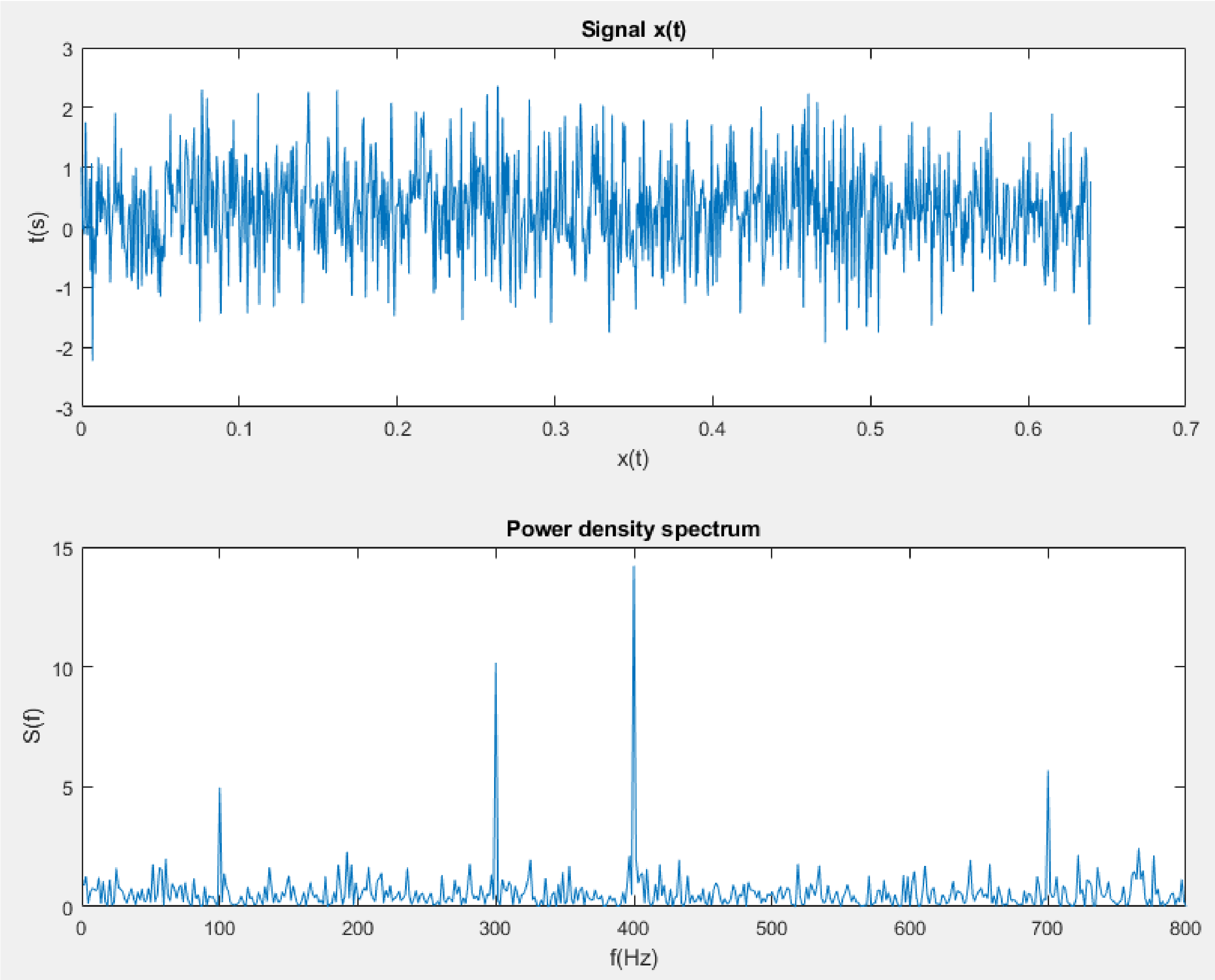

5.2 Compute and print the average power of $x(k)$ and the average power of $v(k)$.

Ans

We print the average power of $x(k)$ and $v(k)$ in the following figure.

```
Command Window
Average power of v(k) = 0.52419
>> hw4_5
Average power of x(k) = 0.63901
Average power of v(k) = 0.51034
>> hw4_5
Average power of x(k) = 0.60583
Average power of v(k) = 0.46561
>> hw4_5
Average power of x(k) = 0.67305
Average power of v(k) = 0.52441
>> hw4_5
Average power of x(k) = 0.66297
Average power of v(k) = 0.5175
fx >>
```

It appears that the average power of the noise is closed to the given signal. However, we can distinguish between the noise and the signal clearly by observing the power density spectrum.

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6. Compute a spectrogram of the Happy Birthday signal (`hbd.mp3`) using a Hanning window of length L = 4000. Hint: Modify the example code `spectrogram_water_sound.m` and adjust the color scale appropriately so that we can clearly see the pattern in the computed spectrogram.

   Ans

   The code to compute the spectrogram is shown in the following figure.

hw4_4.m hw4_5.m hw4_6.m spectrogram_water_sound.m

```
%hw 4.6
clear;
%load audio source
[sound,fs] = audioread('hbd.mp3');
N = length(sound);
L = 4000; M = ceil(N/L);
y = (sound(:,1)+sound(:,2))/2;
y = [y;zeros(M*L-N,1)];
t = zeros(L,M);
S = zeros(L,M);
%extracting subsignals and compute the spectrum
for ind = 1:2*M-1
    %determine the range of the subsignals
    k_sub = (ind-1)*L/2+1:(ind-1)*L/2+L; k_sub = k_sub';
    %extracting the subsignals and multoply with the Hanning window
    %function
    y_sub = y(k_sub).*(0.5-0.5*cos(2*pi*k_sub/L));
    t(:,ind) = (k_sub-1)/fs;
    %compute the FFT
    Y_sub = fft(y_sub);
    S(:,ind) = abs(Y_sub);
end
f = [0:L/2 -L/2+1:-1]'/(L/fs);
idx = find(f>=0 & f<=5200);
imagesc(f(idx),t(:),S(idx,:)',[0 100]);
xlabel('f(Hz)'); ylabel('t(s)');
set(gca,'ColorScale','log')
colorbar;
title('Spectrogram of the HBD song');
```

We use the logscale for the colorbar to see the signals clearly. The spectrogram is shown in the below figure.

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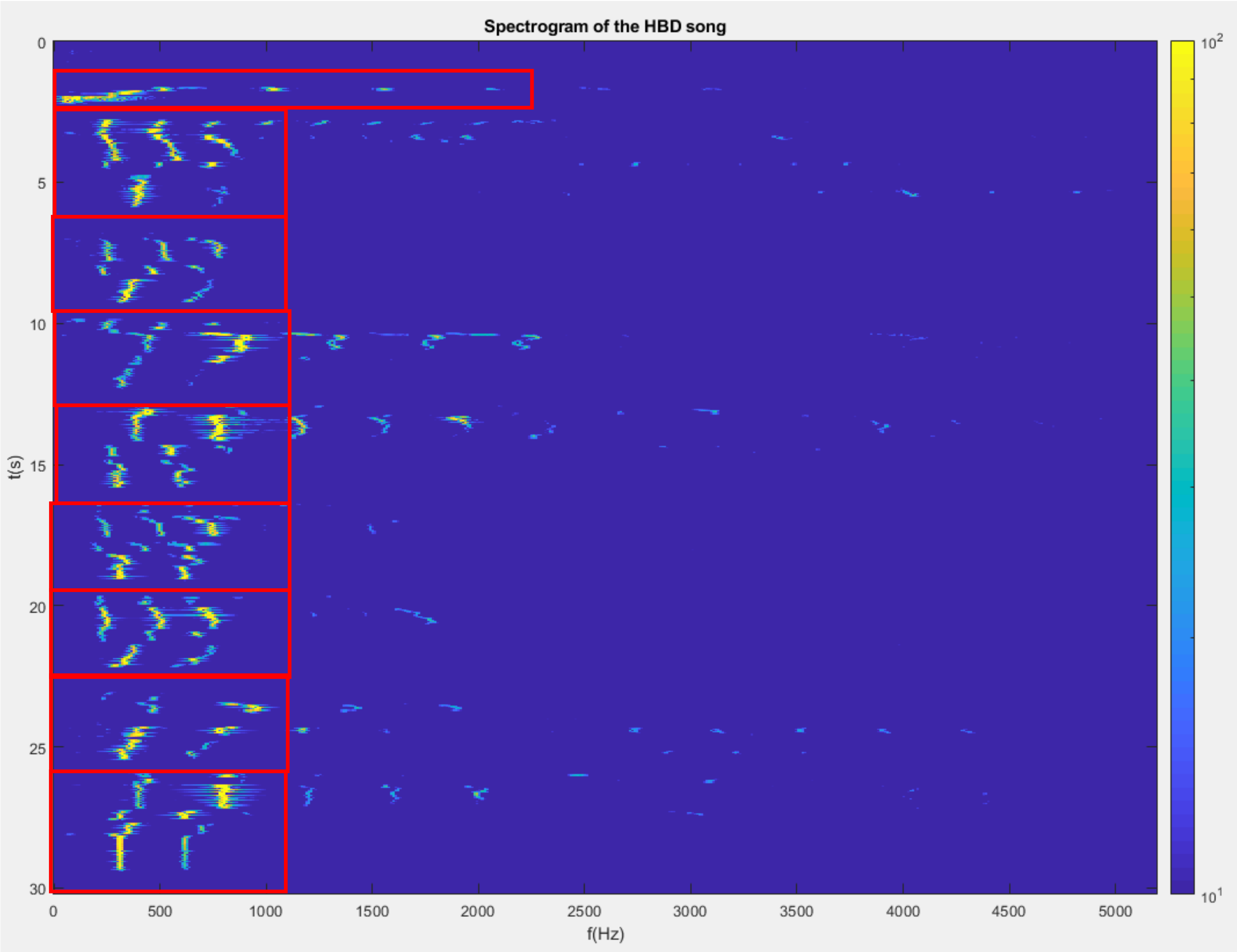

The kid sang 'Happy birthday to you' 8 times. From the spectrogram, we can see unclear patterns at $f <$ 1200 Hz.