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Signal processing

Homework 1

1. The ideal low-pass filter is given by

$$H_a(f) = \begin{cases} 0, |f| \le f_s \\ 1, |f| > f_s \end{cases}.$$

Since $H_a(f)$ is the Fourier transform of $h_a(t)$, impulse response function can be evaluated by taking inverse Fourier transform to $H_a(t)$.

$$h_a(t) = \int_{-\infty}^{\infty} e^{2\pi i f t} H_a(f)\, df$$

$$= \int_{-\infty}^{-f_s} e^{2\pi i f t} H_a(f)\, df + \int_{-f_s}^{f_s} e^{2\pi i f t} H_a(f)\, df + \int_{f_s}^{\infty} e^{2\pi i f t} H_a(f)\, df$$

By the definition of the frequency response function, the first term and the third term become zero. The impulse response function can be expressed as

$$h_a(t) = \int_{-f_s}^{f_s} e^{2\pi i f t}\, df$$

$$= \frac{1}{2\pi i t}\left(e^{2\pi i f_s t} - e^{-2\pi i f_s t}\right)$$

The subtraction of the exponential terms can be written in sine function by using Euler's formula given as $\sin x = \frac{e^x - e^{-x}}{2i}$.

$$h_a(t) = \frac{1}{\pi t}\sin(2\pi f_s t) \times \frac{2f_s}{2f_s}$$

$$= 2f_s \times \frac{\sin(2\pi f_s t)}{2\pi f_s t}$$

The expression is simplified by using sinc function given as $\operatorname{sinc} x = \frac{\sin x}{x}$.

$$h_a(t) = 2f_s \operatorname{sinc}(2\pi f_s t)$$

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2. The signal is given by $x_a(t) = 5\sin(100\pi t) - 2\cos(40\pi t)$.

2.1. To observe the bandwidth of the signal, we take Fourier transform into the $x_a(t)$.

$$\begin{aligned}
x_a(f) &= \int_{-\infty}^{\infty} e^{-2\pi ift} x_a(t)\,dt \\
&= \int_{-\infty}^{\infty} e^{-2\pi ift} [5\sin(100\pi t) - 2\cos(40\pi t)]\,dt \\
&= 5\int_{-\infty}^{\infty} e^{-2\pi ift} \sin(100\pi t)\,dt - 2\int_{-\infty}^{\infty} e^{-2\pi ift} \cos(40\pi t)\,dt \\
&= 5\int_{-\infty}^{\infty} e^{-2\pi ift} \left(\frac{e^{100\pi it} - e^{-100\pi it}}{2i}\right) dt - 2\int_{-\infty}^{\infty} e^{-2\pi ift} \left(\frac{e^{40\pi it} + e^{-40\pi it}}{2}\right) dt \\
&= \frac{5}{2i}\int_{-\infty}^{\infty} \left[e^{(100\pi - 2\pi f)it} - e^{(-100\pi - 2\pi f)it}\right] dt - \int_{-\infty}^{\infty} \left[e^{(40\pi - 2\pi f)it} + e^{(-40\pi - 2\pi f)it}\right] dt
\end{aligned}$$

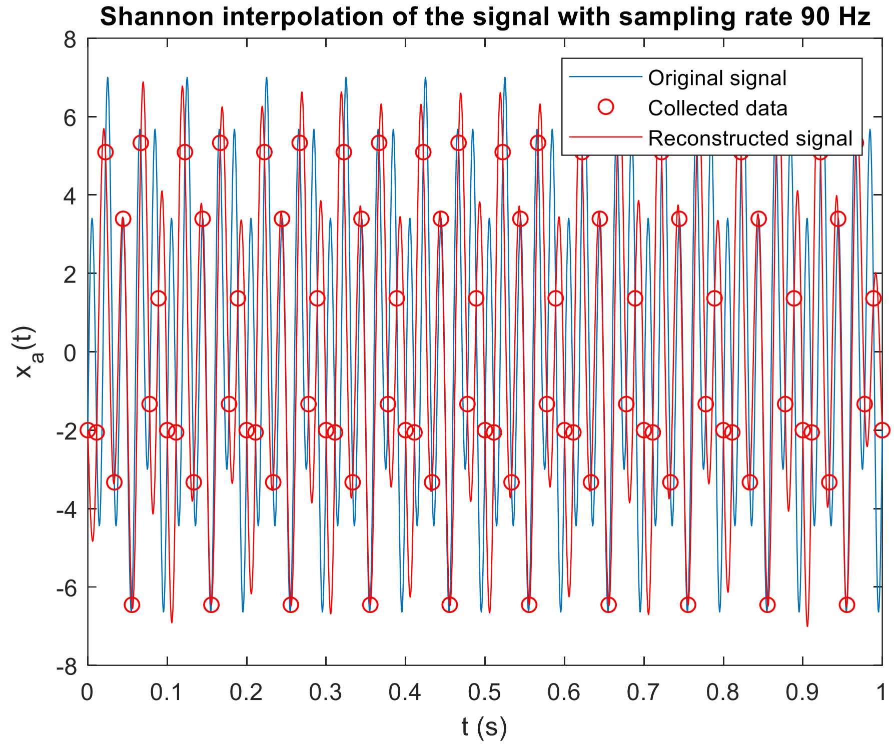
By using the Dirac delta function expressed as $\delta(x-a) = \int_{-\infty}^{\infty} e^{ip(x-a)}\,dp$, $x_a(f)$ is written as

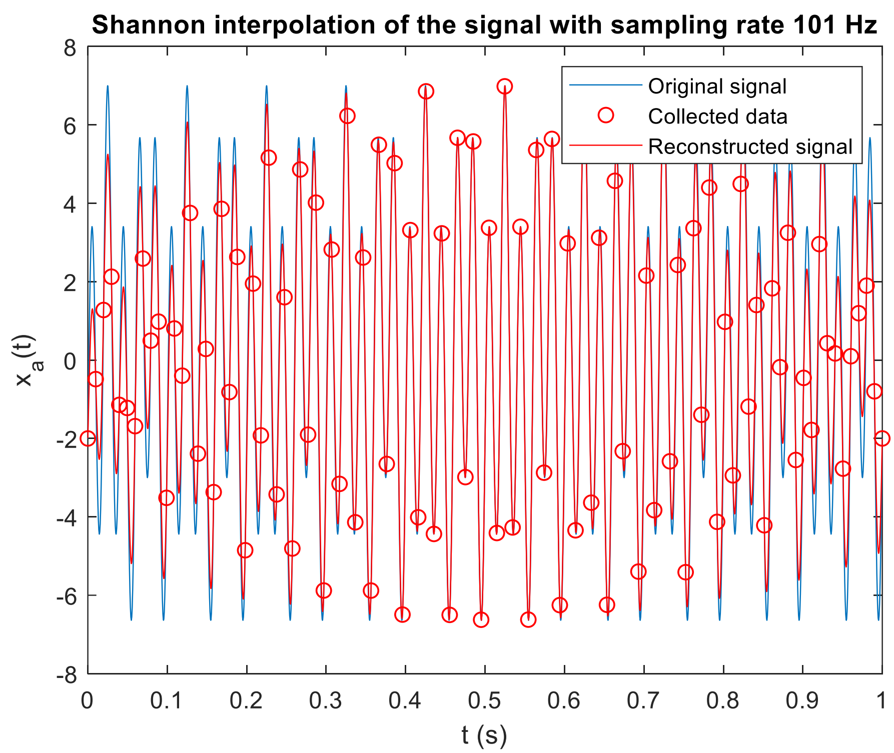
$$\begin{aligned}
x_a(f) &= \frac{5}{2i}[\delta(100\pi - 2\pi f) - \delta(-100\pi - 2\pi f)] - [\delta(40\pi - 2\pi f) + \delta(-40\pi - 2\pi f)] \\
&= \frac{5}{2i}[\delta(2\pi f - 100\pi) - \delta(2\pi f + 100\pi)] - [\delta(2\pi f - 40\pi) + \delta(2\pi f + 40\pi)] \\
&= \frac{5}{4\pi i}[\delta(f - 50) - \delta(f + 50\pi)] - \frac{1}{2\pi}[\delta(f - 20) + \delta(f + 20)]
\end{aligned}$$

It appears that the signal is composed by two frequencies which are 20 Hz and 50 Hz. The bandwidth is 50 Hz which covers the frequency range of the given signal.

2.2 The bandwidth of the signal is 50 Hz. To avoid aliasing, the sampling rate should be greater than 100 Hz which is two times of the bandwidth. Therefore, the smallest integer value of the sampling frequency is 101 Hz.

2.3 If the signal is sampled with the rate 90 Hz, the maximum bandwidth of the signal which can be reconstructed is 45 Hz. However, the Fourier components of the signal are 20 Hz and 50 Hz. With the sampling rate 90 Hz, the 50 Hz component of the signal cannot be reconstructed. Therefore, the original signal cannot be reconstructed with the sampling rate 90 Hz. The reconstructed signal will have aliasing which is shown in the following figure.


Shannon interpolation of the signal with sampling rate 90 Hz
Original signal
Collected data
Reconstructed signal
$x_a(t)$
t (s)
8
6
4
2
0
-2
-4
-6
-8
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Shannon interpolation of the signal with sampling rate 101 Hz
Original signal
Collected data
Reconstructed signal
$x_a(t)$
t (s)
8
6
4
2
0
-2
-4
-6
-8
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1