

Discrete-time Systems in the Frequency Domain

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Introduction

- Discrete-time signals with discrete amplitude are **digital signals**.
- A discrete-time system that processes a digital input signal $x(k)$ to produce a digital output signal $y(k)$ is called a digital signal processor or a **digital filter**.
- “**Z transform** is an essential tool for analyzing discrete-time systems.”
- The Z transform maps a discrete-time signal $x(k)$ into a function $X(z)$ of a complex variable z : $X(z) = Z\{x(k)\}$
- Applying the Z transform to a difference equation describing a discrete-time system yields an algebraic equation of the Z transform of the output $Y(z)$.

Z Transform

The Z transform of discrete-time signal $x(k)$ is defined as

$$X(z) = Z\{x(k)\} \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} x(k)z^{-k}$$

which is a power series in the variable z^{-1} .

“The Z transform of most signals can be expressed in factored form as a ratio of two polynomials:”

$$X(z) = \frac{b_0(z - z_1)(z - z_2) \cdots (z - z_m)}{(z - p_1)(z - p_2) \cdots (z - p_n)}$$

The roots of numerator polynomials z_i , $i = 1, \dots, m$, are called the **zeroes** of $X(z)$ while the roots of denominator polynomial p_i , $i = 1, \dots, n$, are called the **poles** of $X(z)$.

Z Transform

The **generalized geometric series**

$$\sum_{k=m}^{\infty} z^k = \frac{z^m}{1-z}, \quad m \geq 0, \quad |z| < 1$$

can be used to directly compute the Z transform.

Region of Convergence

Any signal can be decomposed into causal and anti-causal parts:

$$x(k) = x_c(k) + x_a(k)$$

where

$$x_c(k) \stackrel{\text{def}}{=} x(k)\mu(k)$$

$$x_a(k) \stackrel{\text{def}}{=} x(k)\mu(-k-1)$$

“The causal part $x_c(k)$ has one region of convergence (ROC) while the anti-causal part $x_a(k)$ has another region of convergence.”

“The overall region of convergence Ω_{ROC} includes the intersection of the two regions.”

Region of Convergence: Example

Consider a two-sided exponential signal

$$x(k) = \begin{cases} a^k, & k \geq 0 \\ b^k, & k < 0 \end{cases} = \underbrace{a^k \mu(k)}_{x_c(k)} + \underbrace{b^k \mu(-k-1)}_{x_a(k)}$$

“Since the Z transform is a linear operation, the transforms for the two parts can be computed separately and then added.”

ROC of Causal Part

The Z transform of the causal part is

$$X_c(z) = \sum_{k=-\infty}^{\infty} (a^k \mu(k)) z^{-k}$$

$$= \sum_{k=0}^{\infty} a^k z^{-k}$$

Lower limit becomes 0 due to the step function

$$= \sum_{k=0}^{\infty} (a/z)^k$$

$$= \frac{(a/z)^0}{1-a/z} = \frac{1}{1-a/z}, \quad \frac{|a|}{|z|} < 1$$

Using the generalized geometric series

$$= \frac{z}{z-a}, \quad |z| > |a|$$

“Region of convergence is **outside** the circle of radius $|a|$ centered at the origin.”

ROC of Anti-causal Part

The Z transform of the anti-causal part is

$$\begin{aligned} X_a(z) &= \sum_{k=-\infty}^{\infty} \left(b^k \mu(-k-1) \right) z^{-k} = \sum_{k=-\infty}^{\infty} \left(b^k \mu(-(k+1)) \right) z^{-k} \\ &= \sum_{k=-\infty}^{-1} b^k z^{-k} && \text{Upper limit becomes -1 due to the step function} \\ &= \sum_{i=\infty}^1 b^{-i} z^i = \sum_{i=1}^{\infty} (z/b)^i, \quad i = -k && \text{What does } \mu(-(k+1)) \text{ look like?} \\ &= \frac{z/b}{1-z/b}, \quad \left| \frac{z}{b} \right| < 1 && \text{Using the generalized geometric series} \\ &= \frac{-z}{z-b}, \quad |z| < |b| && \text{“Region of convergence is **inside** the circle of radius } |b| \text{ centered at the origin.”} \end{aligned}$$

ROC Example (cont.)

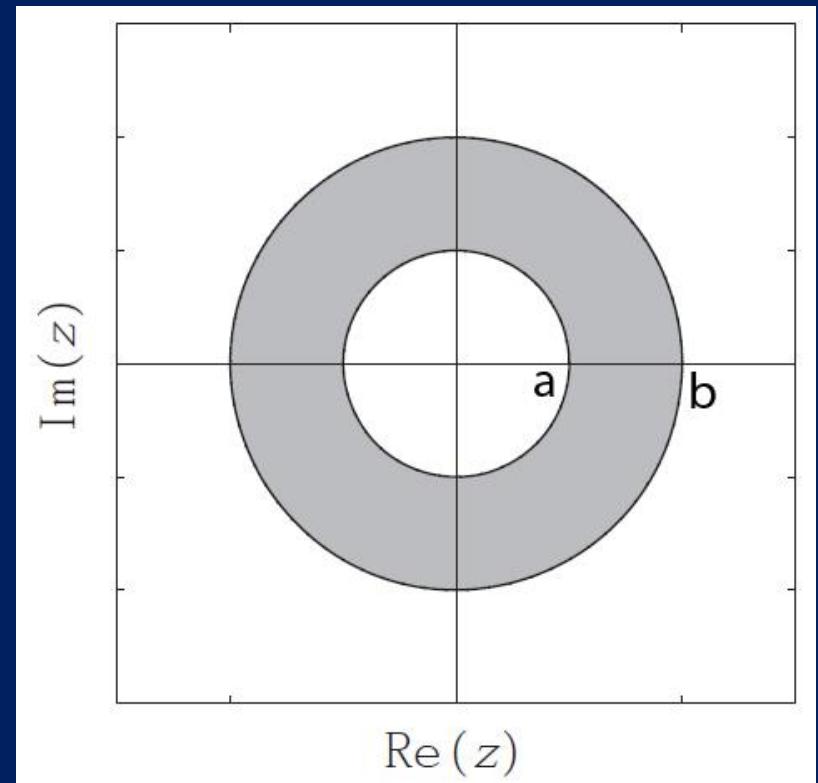
“The complete Z transform is then”

$$X(z) = X_c(z) + X_a(z)$$

$$= \frac{z}{z-a} - \frac{z}{z-b} = \frac{(a-b)z}{(z-a)(z-b)}, \quad |a| < |z| < |b|$$

“The region of convergence of $X(z)$ is an annulus with inner radius $|a|$ and outer radius $|b|$ centered at the origin of the complex plane.”

“If $|a| \geq |b|$, then the Z transform does not exist.”



ROC: Generalization

Suppose $\{p_1, \dots, p_n\}$ are the poles of causal part $x_c(k)$ and $\{q_1, \dots, q_r\}$ are the poles of anti-causal parts $x_a(k)$ that survive zero-pole cancellation in $X_c(z) + X_a(z)$.

“Define the radius of innermost anti-causal pole R_m and the outermost causal pole R_M as follows.”

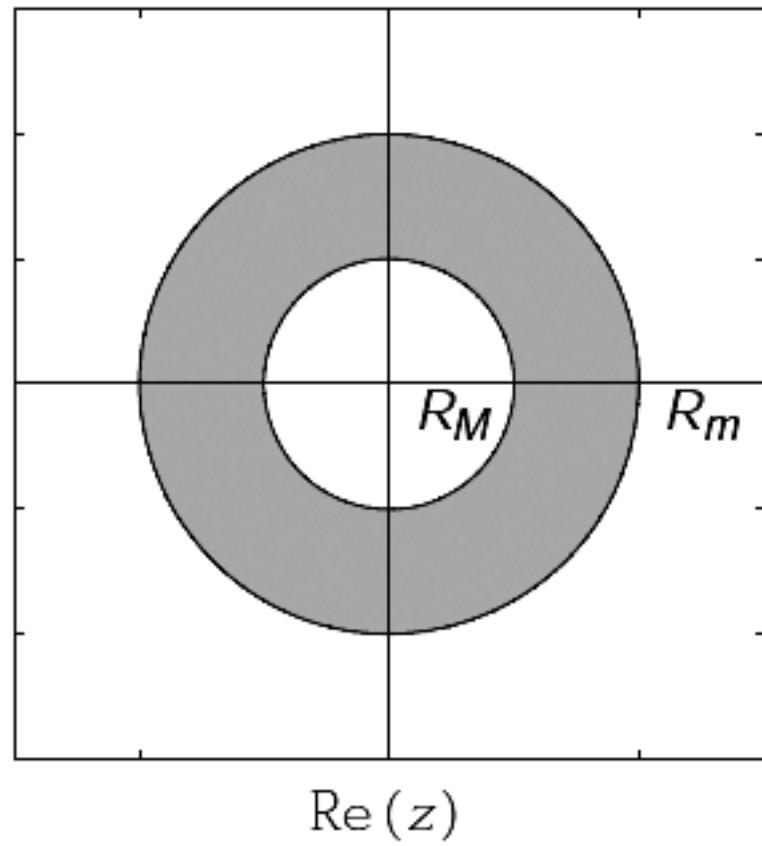
$$R_m = \min_i |q_i|, \quad R_M = \max_i |p_i|$$

ROC of causal part is $|z| > R_M$ while ROC of anti-causal part is $|z| < R_m$. ROC for general signals is then

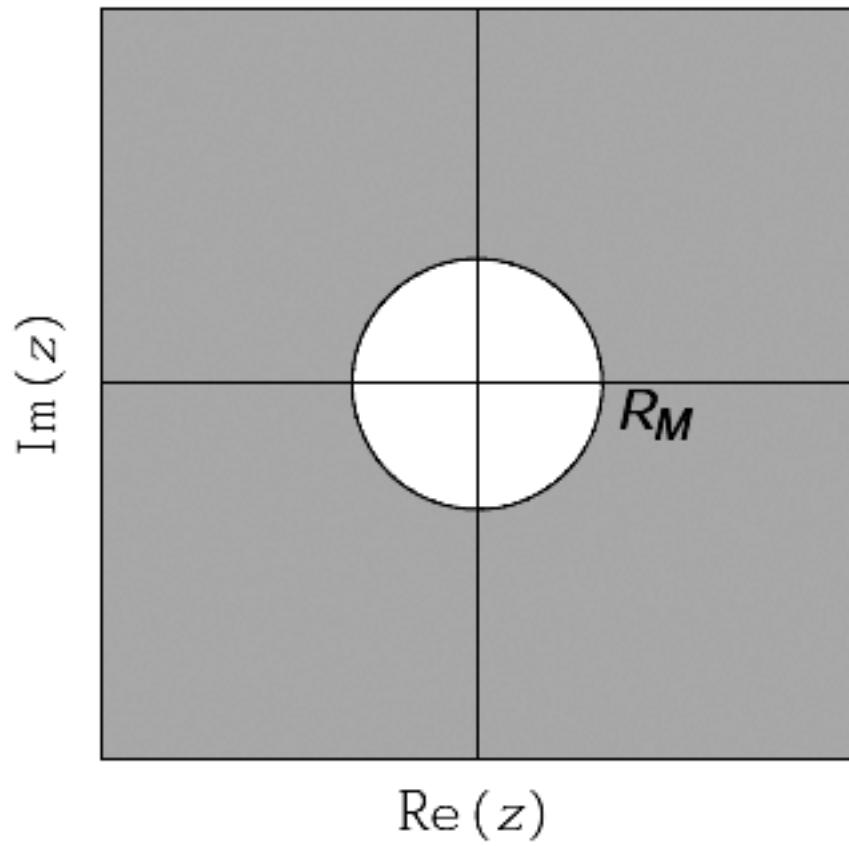
$$\Omega_{\text{ROC}} = \{z \in \mathbb{C} : R_M < |z| < R_m\}$$

ROC: Generalization

(a) General



(b) Causal



ROC: Finite Causal Signals

Let $x(k)$ be a finite causal signal of length m .

The Z transform of $x(k)$ is

$$\begin{aligned} X(z) &= x(0) + x(1)z^{-1} + \cdots + x(m-1)z^{1-m} \\ &= \frac{x(0)z^{m-1} + x(1)z^{m-2} + \cdots + x(m-1)}{z^{m-1}} \end{aligned}$$

“Thus, a finite causal signal of length $m > 1$ has $m - 1$ poles at $z = 0$, which means its ROC is $|z| > 0$. ”

“If $m = 1$ as in $x(k) = \delta(k)$, then there is no pole and the ROC is the entire complex plane.”

ROC: Finite Anti-causal Signals

If $x(k)$ be a finite anti-causal signal of length r .

The Z transform of $x(k)$ is

$$X(z) = x(-r)z^r + x(-r+1)z^{r-1} + \cdots + x(-1)$$

“Thus, a finite anti-causal signal has no poles, which means the ROC is the entire complex plane.”

ROC: Summary

Signal	Length	Ω_{ROC}
Causal	Infinite	$ z > R_M$
Anti-causal	Infinite	$ z < R_m$
General	Infinite	$R_M < z < R_m$
Causal	Finite, $m > 1$	$ z > 0$

Z Transforms of Common Signals

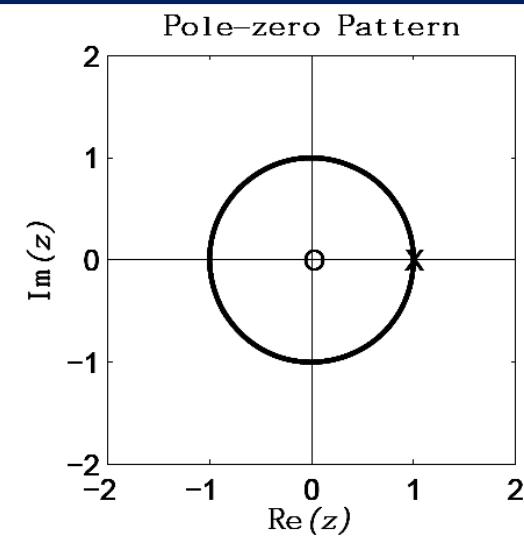
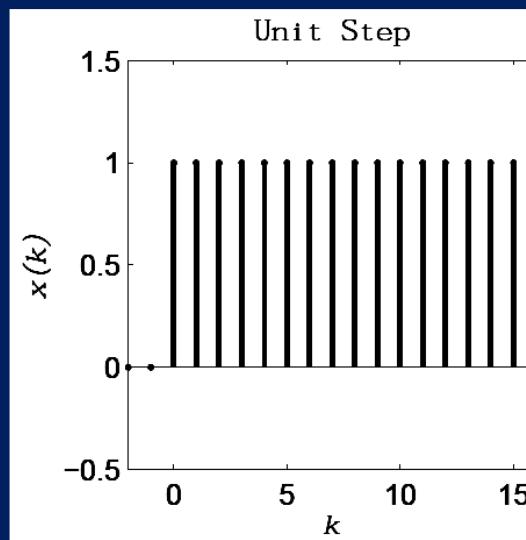
Unit impulse: $Z\{\delta(k)\} = \sum_{k=-\infty}^{\infty} \delta(k)z^{-k} = z^0 = 1$

Unit step: $Z\{\mu(k)\} = \sum_{k=-\infty}^{\infty} \mu(k)z^{-k} = \sum_{k=0}^{\infty} z^{-k}$
 $= \sum_{k=0}^{\infty} (1/z)^k = \frac{1}{1-1/z}, \quad \left|\frac{1}{z}\right| < 1$

$$= \frac{z}{z-1}, \quad |z| > 1$$

“The Z transform has a zero at $z = 0$ and a pole at $z = 1$. ”

ROC: $|z| > 1$



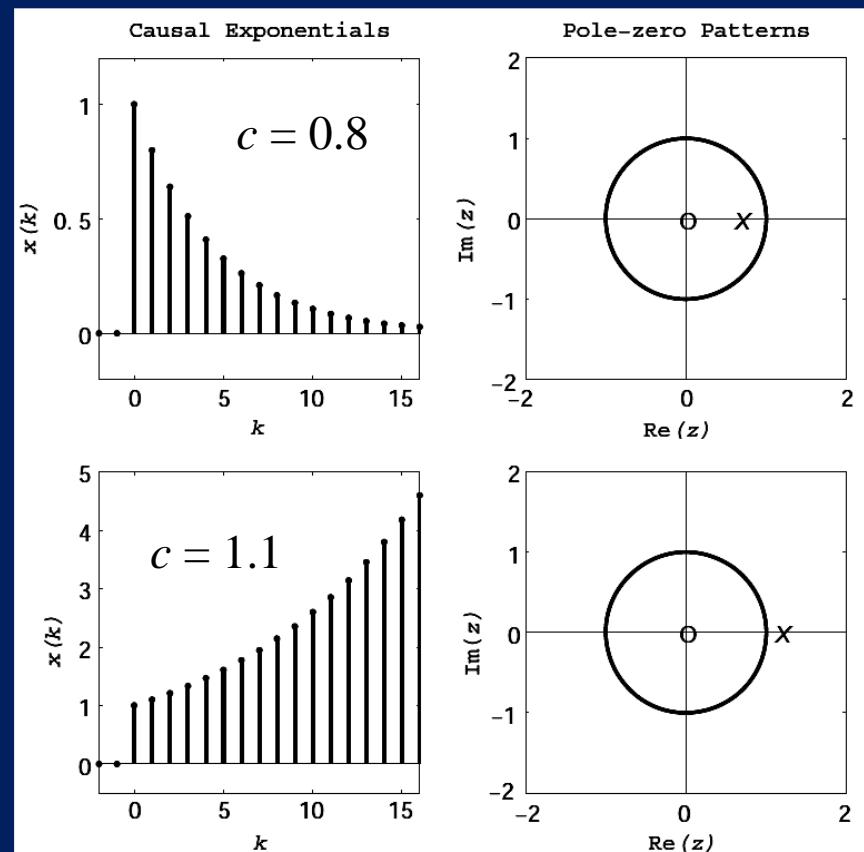
Z Transforms of Common Signals

Causal exponential: $x(k) = c^k \mu(k)$

$$Z\{c^k \mu(k)\} = \sum_{k=0}^{\infty} c^k z^{-k} = \sum_{k=0}^{\infty} \left(\frac{c}{z}\right)^k = \frac{1}{1 - c/z}, \quad \left|\frac{c}{z}\right| < 1$$
$$= \frac{z}{z - c}, \quad |z| > |c|$$

“The Z transform has a zero at $z = 0$ and a pole at $z = c$. ”

ROC: $|z| > |c|$

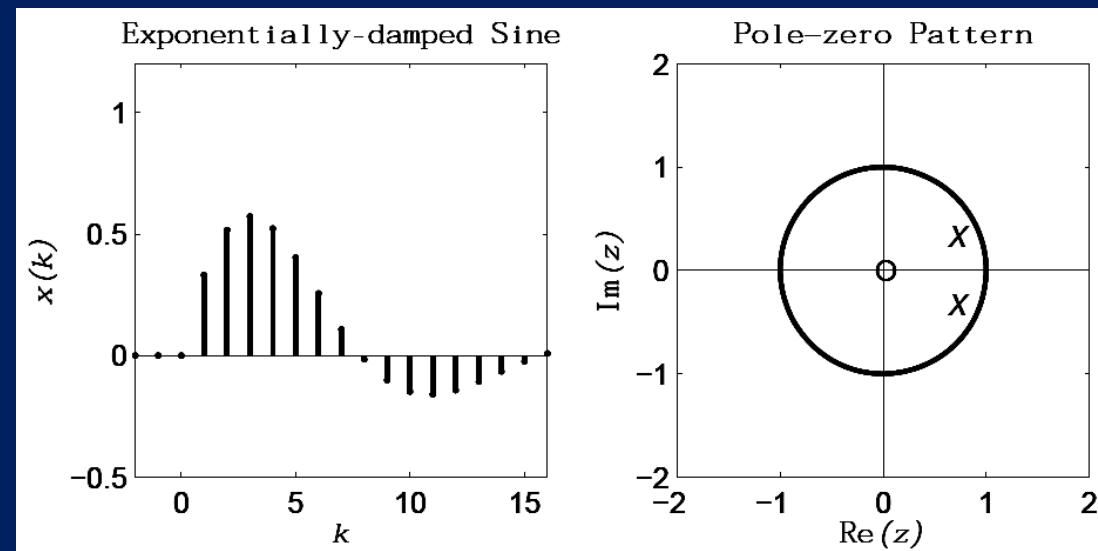


Z Transforms of Common Signals

Exponentially damped sine $x(k) = c^k \sin(dk) \mu(k)$
where c and d are real and $c > 0$

$$Z\{c^k \sin(dk) \mu(k)\} = \frac{c \sin(d)z}{z^2 - 2c \cos(d)z + c^2}, \quad |z| > c$$

“The Z transform has
a zero at $z = 0$ and
a pole at $z = c$.
ROC: $|z| > |c|$



Common Z Transforms

Signal	Z-transform	Poles	Ω_{ROC}
$\delta(k)$	1	none	$z \in \mathbb{C}$
$\mu(k)$	$\frac{z}{z-1}$	$z = 1$	$ z > 1$
$k\mu(k)$	$\frac{z}{(z-1)^2}$	$z = 1$	$ z > 1$
$c^k \mu(k)$	$\frac{z}{z-c}$	$z = c$	$ z > c $
$k(c)^k \mu(k)$	$\frac{cz}{(z-c)^2}$	$z = c$	$ z > c $
$\sin(dk)\mu(k)$	$\frac{\sin(d)z}{z^2 - 2\cos(d)z + 1}$	$z = \exp(\pm jd)$	$ z > 1$
$\cos(dk)\mu(k)$	$\frac{[z - \cos(d)]z}{z^2 - 2\cos(d)z + 1}$	$z = \exp(\pm jd)$	$ z > 1$
$c^k \sin(dk)\mu(k)$	$\frac{c \sin(d)z}{z^2 - 2c \cos(d)z + c^2}$	$z = c \exp(\pm jd)$	$ z > c$
$c^k \cos(dk)\mu(k)$	$\frac{[z - c \cos(d)]z}{z^2 - 2c \cos(d)z + c^2}$	$z = c \exp(\pm jd)$	$ z > c$

Z Transform Properties: Linearity

Z-transform operator is linear.

Given two signals $x(k)$ and $y(k)$, and two arbitrary constant a and b , we have

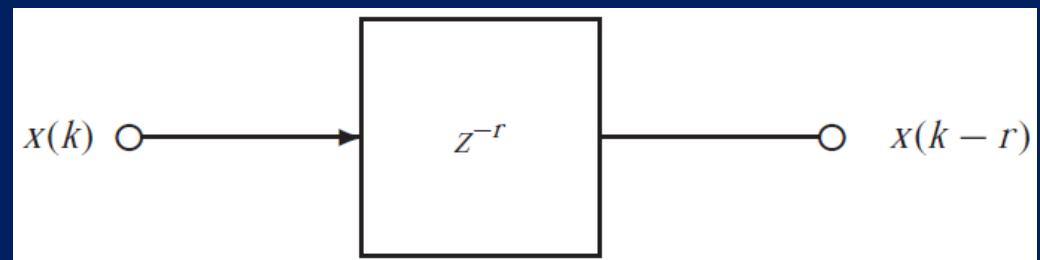
$$\begin{aligned} Z\{ax(k) + by(k)\} &= \sum_{k=-\infty}^{\infty} [ax(k) + by(k)] z^{-k} \\ &= a \sum_{k=-\infty}^{\infty} x(k) z^{-k} + b \sum_{k=-\infty}^{\infty} y(k) z^{-k} \\ &= aZ\{x(k)\} + bZ\{y(k)\} \\ &= aX(z) + bY(z) \end{aligned}$$

Z Transform Properties: Delay

Let $r \geq 0$ denote the number of samples by which a *causal* discrete-time signal $x(k)$ is delayed.

Using the change of variable $i = k - r$, we have

$$\begin{aligned} Z\{x(k-r)\} &= \sum_{k=0}^{\infty} x(k-r)z^{-k} \\ &= \sum_{i=-r}^{\infty} x(i)z^{-(i+r)}, \quad i = k - r \\ &= z^{-r} \sum_{i=0}^{\infty} x(i)z^{-i} \quad \text{Lower limit becomes 0} \\ &= z^{-r} X(z) \end{aligned}$$



Z Transform: Example 1

Consider a signal $x(k) = b[\mu(k) - \mu(k-r)]$

Applying the linearity and delay properties of the Z transform yields

$$\begin{aligned} X(z) &= b(1 - z^{-r})Z\{\mu(k)\} \\ &= \frac{b(1 - z^{-r})z}{z - 1} \\ &= \frac{b(z^r - 1)}{z^{r-1}(z - 1)}, \quad |z| > 1 \end{aligned}$$

Z Transform Properties: Z Scale

Let c be constant and consider the signal $c^k x(k)$

$$\begin{aligned} \text{Z}\{c^k x(k)\} &= \sum_{k=-\infty}^{\infty} c^k x(k) z^{-k} \\ &= \sum_{k=-\infty}^{\infty} x(k) \left(\frac{z}{c}\right)^{-k} \\ &= X\left(\frac{z}{c}\right) \end{aligned}$$

Z Transform Properties: Time Multiplication

Taking the derivative of the Z transform of a signal yields

$$\begin{aligned}\frac{dX(z)}{dz} &= \frac{d}{dz} \sum_{k=-\infty}^{\infty} x(k)z^{-k} \\ &= - \sum_{k=-\infty}^{\infty} kx(k)z^{-(k+1)} \\ &= -z^{-1} \sum_{k=-\infty}^{\infty} kx(k)z^{-k} = -z^{-1}Z\{kx(k)\}\end{aligned}$$

$$Z\{kx(k)\} = -z \frac{dX(z)}{dz}$$

Multiplying a signal by the time variable k is equivalent to taking the derivative of the Z transform and scaling by $-z$.

Z Transform: Example 2

Consider a unit ramp $x(k) = k\mu(k)$

Using the time multiplication property of the Z transform yields

$$\begin{aligned} Z\{k\mu(k)\} &= -z \frac{d}{dz} Z\{\mu(k)\} \\ &= -z \frac{d}{dz} \left(\frac{z}{z-1} \right) \\ &= \frac{z}{(z-1)^2}, \quad |z| > 1 \end{aligned}$$

“The time multiplication property can be applied repeatedly to obtain the Z transform of $k^m \mu(k)$ for $m \geq 0$.

Z Transform: Example 3

Consider a causal exponential $x(k) = kc^k \mu(k)$

Using the Z scale property and Z transform of the unit ramp yields

$$Z\{kc^k \mu(k)\} = Z\{k\mu(k)\}\Big|_{z=z/c}$$

$$= \frac{z}{(z-1)^2} \Big|_{z=z/c}$$

$$= \frac{cz}{(z-c)^2}, \quad |z| > |c|$$

Z Transform Properties: Convolution

“Suppose $h(k)$ and $x(k)$ are zero extended so that they are defined for all k . ”

$$\begin{aligned} Z\{h(k) * x(k)\} &= \sum_{k=-\infty}^{\infty} [h(k) * x(k)] z^{-k} \\ &= \sum_{k=-\infty}^{\infty} \left[\sum_{i=-\infty}^{\infty} h(i)x(k-i) \right] z^{-k} = \sum_{i=-\infty}^{\infty} h(i) \sum_{k=-\infty}^{\infty} x(k-i) z^{-k} \\ &= \sum_{i=-\infty}^{\infty} h(i) \sum_{m=-\infty}^{\infty} x(m) z^{-(m+i)}, \quad m = k - i \\ &= \sum_{i=-\infty}^{\infty} h(i) z^{-i} \sum_{m=-\infty}^{\infty} x(m) z^{-m} = H(z)X(z) \end{aligned}$$

Z Transform Properties: Time Reversal

Reversing time by replacing $x(k)$ by $x(-k)$.

$$\begin{aligned} Z\{x(-k)\} &= \sum_{k=-\infty}^{\infty} x(-k)z^{-k} \\ &= \sum_{i=\infty}^{-\infty} x(i)z^i, \quad i = -k \\ &= \sum_{i=\infty}^{-\infty} x(i)\left(\frac{1}{z}\right)^{-i} \\ &= X(1/z) \end{aligned}$$

Z Transform Properties: Correlation

Cross-correlation of $y(k)$ with $x(k)$ is

$$r_{yx}(k) = \frac{1}{L} \sum_{i=-\infty}^{\infty} y(i)x(i-k) = \frac{1}{L} \sum_{i=-\infty}^{\infty} y(i)x(-(k-i)) = \frac{y(k) * x(-k)}{L}$$

Using the convolution property yields

$$\begin{aligned} Z\{r_{yx}(k)\} &= \frac{Y(z)Z\{x(-k)\}}{L} \\ &= \frac{Y(z)X(1/z)}{L} \end{aligned}$$

Z Transform Properties: Causality

Suppose $x(k)$ is a causal signal. Then, $X(z) = \sum_{k=0}^{\infty} x(k)z^{-k}$

As $z \rightarrow \infty$, all terms go to zero except the initial value.
This leads to the **initial value theorem**:

$$x(0) = \lim_{z \rightarrow \infty} X(z)$$

Consider the function $X_1(z) = (z-1)X(z)$ whose poles are inside the unit circle. Using the **final value theorem**, we have

$$x(\infty) = \lim_{z \rightarrow 1} (z-1)X(z)$$

We can use these formulas to partially check the validity of the computed Z transform.

Z Transform: Example 4

Consider the signal $x(k) = b[\mu(k) - \mu(k - M)]$

whose Z transform is $X(z) = \frac{b(z^M - 1)}{z^{M-1}(z - 1)}$

Using the initial value theorem, the initial value of $x(k)$ is

$$x(0) = \lim_{z \rightarrow \infty} X(z) = \lim_{z \rightarrow \infty} \frac{b(z^M - 1)}{z^{M-1}(z - 1)} = b$$

Since all poles of $(z-1)X(z)$ are inside the unit circle, the final value of $x(k)$ is

$$x(\infty) = \lim_{z \rightarrow 1} (z - 1)X(z) = \lim_{z \rightarrow 1} \frac{b(z^M - 1)}{z^{M-1}} = 0$$

Summary of Z Transform Properties

Property	Description
Linearity	$Z\{ax(k) + by(k)\} = aX(z) + bY(z)$
Delay	$Z\{x(k - r)\} = z^{-r} X(z)$
Time multiplication	$Z\{kx(k)\} = -z \frac{dX(z)}{dz}$
Time reversal	$Z\{x(-k)\} = X(1/z)$
Z-scale	$Z\{a^k x(k)\} = X(z/a)$
Complex conjugate	$Z\{x^*(k)\} = X^*(z^*)$
Convolution	$Z\{h(k) \star x(k)\} = H(z)X(z)$
Correlation	$Z\{r_{yx}(k)\} = \frac{Y(z)X(1/z)}{L}$
Initial value	$x(0) = \lim_{z \rightarrow \infty} X(z)$
Final value	$x(\infty) = \lim_{z \rightarrow 1} (z - 1)X(z)$, stable

Inverse Z Transform

Once we apply the Z transform to a difference equation and obtain $X(z)$, the actual solution $x(k)$ can then be obtained using the inverse Z transform:

$$x(k) = Z^{-1} \{X(z)\}$$

“Z transforms associated with LTI systems take the form of a ratio of two polynomials in z , that is, they are **rational polynomials** in z :”

$$X(z) = \frac{b(z)}{a(z)} = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n}, \quad a_0 = 1$$

Inverse Z Transform

“If $x(k)$ is a causal signal, then $X(z)$ will be a **proper rational polynomial**: when $m \leq n$, that is, the number of zeroes of $X(z)$ will be less than or equal to the number of poles.” If $m > n$, then using long division we can write

$$X(z) = Q(z) + \frac{R(z)}{a(z)},$$

where the quotient polynomial $Q(z)$ will be of degree

$$m - n \text{ and can be expressed as } Q(z) = \sum_{i=0}^{m-n} q_i z^i$$

and $R(z)$ is a remainder polynomial.

Inverse Z Transform

“When $m = n$, the quotient polynomial is the constant b_0 . ”

Compared to the definition $X(z) \equiv \sum_{k=-\infty}^{\infty} x(k)z^{-k}$

the inverse Z-transform of $Q(z)$ is

$$q(k) = \sum_{i=0}^{m-n} q_i \delta(k + i)$$

“Thus, $Q(z)$ represents anti-causal part of signal, plus the constant term. The remaining part of $X(z)$ representing the signal with $k > 0$ can be obtained using the following techniques which assumes that $X(z)$ represents a causal signal, i.e., $m \leq n$. ”

Synthetic Division Method

Rewriting the rational polynomial as

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n} = \frac{z^{-r} (b_0 + b_1 z^{-1} + \cdots + b_m z^{-m})}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

where $r = n - m$ and $m \leq n$.

“Synthetic division method performs long division of the numerator polynomial by the denominator polynomial to produce an infinitely long quotient polynomial”

$$X(z) = z^{-r} [q_0 + q_1 z^{-1} + q_2 z^{-2} + \cdots] = z^{-r} Z\{q(k)\}$$

So $x(k)$ is $q(k)$ delayed by r samples, that is,

$$x(k) = q(k - r) \mu(k - r)$$

The unit step function ensures that $x(k)$ is causal.

Synthetic Division: Example

Consider Z transform $X(z) = \frac{z+1}{z^2 - 2z + 3} = \frac{z^{-1}(1+z^{-1})}{1-2z^{-1}+3z^{-2}}$

There are $n = 2$ poles and $m = 1$ zero. Here, $r = n - m = 1$.
Performing long division yields

$$\begin{array}{r} 1 + 3z^{-1} + 3z^{-2} - 3z^{-3} - 15z^{-4} + \dots \\ \hline 1 - 2z^{-1} + 3z^{-2} \mid 1 + z^{-1} \\ 1 - 2z^{-1} + 3z^{-2} \\ \hline 3z^{-1} - 3z^{-2} \\ 3z^{-1} - 6z^{-2} + 9z^{-3} \\ \hline 3z^{-2} - 9z^{-3} \\ 3z^{-1} - 6z^{-2} + 9z^{-3} \\ \hline -3z^{-3} - 9z^{-4} \\ -3z^{-3} + 6z^{-4} - 9z^{-5} \\ \hline -15z^{-4} + 9z^{-5} \end{array}$$

Thus, we have
 $q(k) = [1, 3, 3, -3, -15, \dots]$
and
 $x(k) = [0, 1, 3, 3, -3, -15, \dots]$
since $r = 1$.

Impulse Response Method

“The inverse Z transform of $X(z)$ can be obtained by interpreting $x(k)$ as the impulse response of a discrete-time system with characteristic polynomial $a(z)$ and input polynomial $b(z)$.”

Using the MATLAB function filter, one can obtain the impulse response from $a(z)$ and $b(z)$ as follows.

```
% given vectors a and b  
N = 100;  
delta = [1; zeros(N-1, 1)];  
x = filter(b, a, delta);
```

Partial Fractions

- Partial fractions can provide a closed-form expression for $x(k)$ as a function of k .
- Using a table of Z-transform pairs, we can find the inverse Z transform of $X(z)$.
- When $X(z)$ is not in the table, partial fractions can be used to rewrite $X(z)$ as a sum of terms in the table.

Simple Poles: When n poles of $X(z)$ are nonzero and simple, $X(z)$ can be written in factored form as

$$X(z) = \frac{b(z)}{(z - p_1)(z - p_2) \cdots (z - p_n)}$$

Partial Fractions: Simple Poles

Express $X(z)/z$ using partial fractions

$$\frac{X(z)}{z} = \sum_{i=0}^n \frac{R_i}{z - p_i}$$

where R_i is called the residue of $X(z)/z$ at pole p_i .

- Note that $X(z)/z$ has $n + 1$ simple poles with $p_0 = 0$.
- “To find the residues, put the $n + 1$ partial fraction terms over a common denominator and then equate numerators, and then equate the coefficients of like power of z .”
- “This results in $n + 1$ equations in the $n + 1$ unknown residues.”

Partial Fractions: Simple Poles

Another approach is to multiply both sides of

$$\frac{X(z)}{z} = \sum_{i=0}^n \frac{R_i}{z - p_i}$$

by $z - p_k$ and then evaluate the result at $z = p_k$ and obtain

$$\left. \frac{(z - p_k) X(z)}{z} \right|_{z=p_k} = \left. \sum_{i=0}^n \frac{(z - p_k) R_i}{z - p_i} \right|_{z=p_k} = R_k, \quad 0 \leq k \leq n$$

“The term residue comes from the fact that R_i is what remains of $X(z)/z$ at $z = p_i$, after the pole at $z = p_i$ has been removed.”

Partial Fractions: Simple Poles

“Once the partial fraction representation is obtained,

multiply both sides of $\frac{X(z)}{z} = \sum_{i=0}^n \frac{R_i}{z - p_i}$ by z to obtain”

$$X(z) = R_0 + \sum_{i=1}^n \frac{R_i z}{z - p_i}$$

“Using the linearity property and the Z transform table,
we obtain”

$$x(k) = R_0 \delta(k) + \left[R_1 p_1^k + R_2 p_2^k + \cdots + R_n p_n^k \right] u(k)$$

Partial Fractions: Example 1

Simple Pole Case: Consider the Z transform

$$\frac{X(z)}{z} = \frac{10(z^2 + 4)}{z(z^2 - 2z - 3)} = \frac{10(z^2 + 4)}{z(z+1)(z-3)}$$

Using $R_k = \left. \frac{(z - p_k) X(z)}{z} \right|_{z=p_k}$, we obtain

$$R_0 = \left. \frac{10(z^2 + 4)}{(z+1)(z-3)} \right|_{z=p_0} = \frac{40}{-3}, R_1 = \left. \frac{10(z^2 + 4)}{z(z-3)} \right|_{z=p_1} = \frac{50}{4}, R_2 = \left. \frac{10(z^2 + 4)}{z(z+1)} \right|_{z=p_2} = \frac{130}{12}$$

Closed-form expression of inverse Z transform of $X(z)$ is

$$x(k) = \frac{-40}{3} \delta(k) + \left[\frac{25(-1)^k}{2} + \frac{65(3)^k}{6} \right] \mu(k)$$

Partial Fractions: Multiple Poles

“Suppose a nonzero pole at $z = p_1$ is repeated n times.”

“Then the partial fraction expansion is”

$$\frac{X(z)}{z} = \frac{R_0}{z - p_0} + \frac{c_1}{z - p_1} + \frac{c_2}{(z - p_1)^2} + \cdots + \frac{c_n}{(z - p_1)^n}$$

“To find the coefficients c_i , one can put the terms on RHS over a common denominator and equate numerators.”

“Equating the coefficients of like powers of z results in $n + 1$ equations in the $n + 1$ variables $\{R_0, c_1, \dots, c_n\}$. ”

Partial Fractions: Example 2

Multiple Pole Case: Consider the Z transform

$$\frac{X(z)}{z} = \frac{2(z+3)}{z(z^2 - 4z + 4)} = \frac{2(z+3)}{z(z-2)^2}$$

The partial fraction expansion is

$$\frac{X(z)}{z} = \frac{R_0}{z} + \frac{c_1}{z-2} + \frac{c_2}{(z-2)^2}$$

“The residue of the pole at $p_0 = 0$ is”

$$R_0 = \left. \frac{2(z+3)}{(z-2)^2} \right|_{z=0} = \frac{6}{4} = 1.5$$

Example 2 (continued)

“Substituting $R_0 = 1.5$ and putting the terms of the partial fraction expansion over a common denominator yields”

$$\begin{aligned}\frac{X(z)}{z} &= \frac{1.5(z-2)^2 + c_1z(z-2) + c_2z}{z(z-2)^2} \\ &= \frac{(1.5 + c_1)z^2 + (c_2 - 2c_1 - 6)z + 6}{z(z-2)^2} = \frac{2z + 6}{z(z-2)^2}\end{aligned}$$

“Equating coefficients of like powers of z yields”

$$1.5 + c_1 = 0, \quad c_2 - 2c_1 + 6 = 2 \quad \Rightarrow \quad c_1 = -1.5, c_2 = -7$$

We then obtain $X(z) = 1.5 - \frac{1.5z}{z-2} - \frac{(3.5)2z}{(z-2)^2}$

Example 2 (continued)

Using the Z-transform table (Slide 18), we then obtain

Signal	Z-transform
$\delta(k)$	1
$c^k \mu(k)$	$\frac{z}{z - c}$
$k(c)^k \mu(k)$	$\frac{cz}{(z - c)^2}$

$$\begin{aligned}x(k) &= 1.5\delta(k) - \left[1.5(2)^k + 3.5k(2)^k \right] \mu(k) \\&= 1.5\delta(k) - (1.5 + 3.5k)2^k \mu(k)\end{aligned}$$

Partial Fractions: Complex Poles

- “The poles of $X(z)$ can be real or complex.”
- “If the signal $x(k)$ is real, then complex poles appear in conjugate pairs.”
- Suppose $X(z)$ has poles at $z = c \exp(\pm jd)$ and

$$X(z) = \frac{b_0 z + b_1}{[z - c \exp(jd)][z - c \exp(-jd)]}$$

- Recall the Z transform table.

We want to turn the nominator into a linear combination of $c \sin(d)$ and $(z - c \cos(d))$ so that we can use the table.

Signal	Z-transform
$c^k \sin(dk) \mu(k)$	$\frac{c \sin(d) z}{z^2 - 2c \cos(d) z + c^2}$
$c^k \cos(dk) \mu(k)$	$\frac{[z - c \cos(d)] z}{z^2 - 2c \cos(d) z + c^2}$

Partial Fractions: Complex Poles

This can be done by rewriting $X(z)$ as

$$X(z) = \frac{z^{-1}(b_0 z^2 + b_1 z)}{[z - c \exp(jd)][z - c \exp(-jd)]}$$

Then, we seek f_1 and f_2 such that

$$\begin{aligned} b_0 z^2 + b_1 z &= f_1 [c \sin(d)z] + f_2 [z - c \cos(d)]z \\ &= f_2 z^2 + c[f_1 \sin(d) - f_2 \cos(d)]z \end{aligned}$$

Equating coefficients of like powers of z yields

$$f_1 = [b_1 + cb_0 \cos(d)]/\sin(d), \quad f_2 = b_0$$

With the delay z^{-1} , we then obtain

$$x(k) = c^{k-1} (f_1 \sin[d(k-1)] + f_2 \cos[d(k-1)]) \mu(k-1)$$

Partial Fractions: Example 3

Complex Pole Case: Consider the Z transform

$$X(z) = \frac{3z + 5}{z^2 - 4z + 13}$$

with complex conjugate poles $p_{1,2} = 2 \pm j3 = c \exp(\pm jd)$
where $c = \sqrt{4+9} = 3.61$, $d = \tan^{-1}(3/2) = 0.983$

We then have $f_1 = \frac{5 + 3.61(3) \cos(0.983)}{\sin(0.983)} = 13.2$, $f_2 = 3$

The inverse Z transform is

$$\begin{aligned}x(k) &= c^{k-1} (f_1 \sin[d(k-1)] + f_2 \cos[d(k-1)]) \mu(k-1) \\&= 3.61^{k-1} (13.2 \sin[0.983(k-1)] + 3 \cos[0.983(k-1)]) \mu(k-1)\end{aligned}$$

Residue Method

- “The partial fraction method is effective for simple poles, but it can become cumbersome for multiple poles and furthermore it requires the Z transform table.”
- The residue method based on the theory of functions of complex variables is more effective for multiple poles and does not require a table.
- **Cauchy integral formula:**

$$f(a) = \frac{1}{j2\pi} \oint_C \frac{f(z)}{z-a} dz$$

where $f(z)$ is an analytic function and C is a circle oriented counterclockwise forming a boundary of $D = \{z : |z-z_0| \leq r\}$

Residue Method

It can be shown that

$$Z^{-1}\{X(z)\} = \frac{1}{j2\pi} \oint_C X(z)z^{k-1} dz$$

where C is a counterclockwise contour in the region of convergence of $X(z)$ that encloses all of the poles.

Using the Cauchy residue theorem, we can compute $x(k)$ by

$$x(k) = \sum_{i=1}^q \text{Res}(p_i, k)$$

where p_i are the q distinct poles of $X(z)z^{k-1}$ and $\text{Res}(p_i, k)$ is the residue of $X(z)z^{k-1}$ at the pole $z = p_i$.

Residue Method

To compute the residue, $X(z)$ is rewritten as

$$X(z) = \frac{b(z)}{(z - p_1)^{m_1} (z - p_2)^{m_2} \cdots (z - p_q)^{m_q}}$$

“Here, p_i is a pole of multiplicity m_i for $1 \leq i \leq q$. ”

“If p_i is a **simple pole** ($m_i = 1$), then the residue is what remains of $X(z)z^{k-1}$ at the pole after the pole has been removed:” $\text{Res}(p_i, k) = (z - p_i) X(z) z^{k-1} \Big|_{z=p_i} \quad \text{if } m_i = 1$

If p_i is a **multiple pole** of multiplicity $m_i > 1$, then the residue is

$$\text{Res}(p_i, k) = \frac{1}{(m_i - 1)!} \frac{d^{m_i - 1}}{dz^{m_i - 1}} \left\{ (z - p_i)^{m_i} X(z) z^{k-1} \right\} \Big|_{z=p_i}, \quad m_i > 1$$

Residue Method

- “The residue method requires the same amount of computational effort as the partial fraction method for the case of simple poles, but it requires less effort for multiple poles and does not require a table.”
- “The residue method has a drawback when some value of k cause $X(z)z^{k-1}$ to have a pole at $z = 0$. This problem can be solved by separating $x(k)$ into two cases, $k = 0$ and $k > 0$.”
- “The initial value theorem can be used to compute $x(0)$ as”

$$x(0) = \lim_{z \rightarrow \infty} X(z) = b_0 \delta(n - m), \quad m \leq n$$

Residue Method: Algorithm

1. Factor the denominator polynomial of $X(z)$ as in

$$X(z) = b(z) / (z - p_1)^{m_1} (z - p_2)^{m_2} \cdots (z - p_q)^{m_q}$$

2. Set $x(0) = b_0 \delta(n-m)$

3. For $i = 1$ to q do

$$\text{Res}(p_i, k) = \begin{cases} (z - p_i) X(z) z^{k-1} \Big|_{z=p_i}, & m_i = 1 \\ \frac{1}{(m_i - 1)!} \frac{d^{m_i-1}}{dz^{m_i-1}} \left\{ (z - p_i)^{m_i} X(z) z^{k-1} \right\} \Big|_{z=p_i}, & m_i > 1 \end{cases}$$

4. Set

$$x(k) = x(0) \delta(k) + \left[\sum_{i=1}^q \text{Res}(p_i, k) \right] \mu(k-1)$$

Residue Method: Simple Poles Example

Consider a Z transform $X(z) = \frac{z^2}{(z-a)(z-b)}$

The initial value of $x(k)$ is $x(0) = 1$. The two residues are

$$\text{Res}(a, k) = \left. \frac{z^{k+1}}{(z-b)} \right|_{z=a} = \frac{a^{k+1}}{a-b}, \quad \text{Res}(b, k) = \left. \frac{z^{k+1}}{(z-a)} \right|_{z=b} = \frac{b^{k+1}}{b-a}$$

Thus,

$$\begin{aligned} x(k) &= x(0)\delta(k) + [\text{Res}(a, k) + \text{Res}(b, k)]\mu(k-1) \\ &= \delta(k) + \left(\frac{a^{k+1} - b^{k+1}}{a-b} \right) \mu(k-1) = \left(\frac{a^{k+1} - b^{k+1}}{a-b} \right) \mu(k) \end{aligned}$$

Residue Method: Mixed Poles Example

Consider a Z transform $X(z) = \frac{1}{(z-a)^2(z-b)}$

The initial value of $x(k)$ is $x(0) = 0$.

The residue of multiple pole at $z = a$ is

$$\text{Res}(a, k) = \left. \frac{d}{dz} \left\{ \frac{z^{k-1}}{z-b} \right\} \right|_{z=a} = \frac{[(a-b)(k-1)-a]a^{k-2}}{(a-b)^2}$$

The residue of simple pole at $z = b$ is

$$\text{Res}(b, k) = \left. \frac{z^{k-1}}{(z-a)^2} \right|_{z=b} = \frac{b^{k-1}}{(b-a)^2}$$

Residue Method: Mixed Poles Example

Thus,

$$\begin{aligned}x(k) &= x(0)\delta(k) + [\text{Res}(a,k) + \text{Res}(b,k)]\mu(k-1) \\&= \left[\frac{[(a-b)(k-1)-a]a^{k-2}}{(a-b)^2} + \frac{b^{k-1}}{(b-a)^2} \right] \mu(k-1) \\&= \left[\frac{[(a-b)(k-1)-a]a^{k-2} - b^{k-1}}{(a-b)^2} \right] \mu(k-1)\end{aligned}$$

Residue Method: MATLAB

“MATLAB has a built-in function called residue that can be used to compute the residue terms for partial fraction expansions.”

$$\begin{aligned}[r, p, k] &= \text{residue}(b, a) \\ [b, a] &= \text{residue}(r, p, k)\end{aligned}$$

where

b = vector of length $m+1$ containing numerator coefficients ($m \leq n$)

a = vector of length $n+1$ containing denominator coefficients

r = vector of length n containing the residues R_i in

$$x(k) = R_0\delta(k) + \left[R_1 p_1^k + R_2 p_2^k + \cdots + R_n p_n^k \right] u(k)$$

p = vector of length n containing the poles

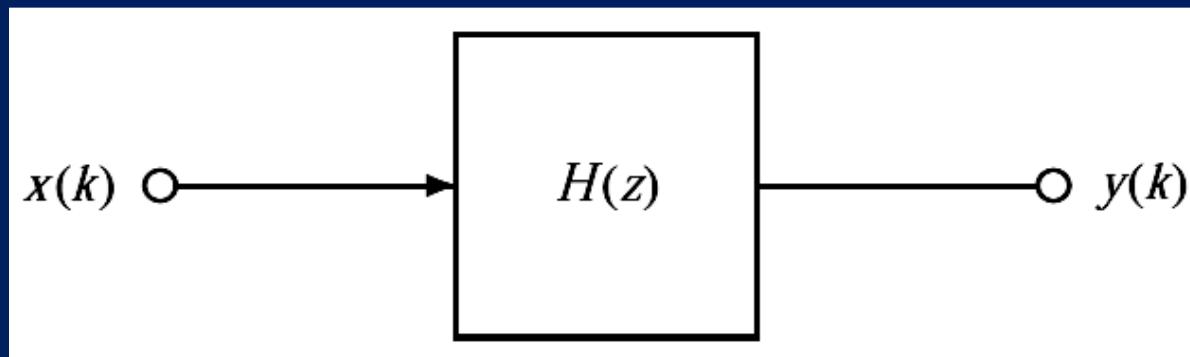
q = residue R_0

Transfer Functions

- In addition to difference equations, LTI systems can also be represented by transfer functions $H(z)$.
- The transfer function is defined as the ratio of the Z transform of the zero-state response and the Z transform of the input:

$$H(z) \stackrel{\text{def}}{=} \frac{Y_{zs}(z)}{X(z)}, \quad \text{i.e., initial condition is zero}$$

$$Y_{zs}(z) = H(z)X(z)$$



Transfer Function

Taking the Z-transform of the difference equation

$$y(k) + \sum_{i=1}^n a_i y(k-i) = \sum_{i=0}^m b_i x(k-i)$$

with zero initial condition yields

$$\begin{aligned} Y(z) + \sum_{i=1}^n a_i z^{-i} Y(z) &= \sum_{i=0}^m b_i z^{-i} X(z) \\ \left(1 + \sum_{i=1}^n a_i z^{-i} \right) Y(z) &= \left(\sum_{i=1}^m b_i z^{-i} \right) X(z) \end{aligned}$$

Thus,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

Transfer Function

Multiplying the numerator and denominator of the transfer function by z^n converts it to positive powers of z :

$$H(z) = \frac{z^{n-m} (b_0 z^m + b_1 z^{m-1} + \cdots + b_m)}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

When $m \leq n$, $H(z)$ has $n - m$ zeros at $z = 0$.

When $m > n$, $H(z)$ has $m - n$ poles at $z = 0$.

The zero-state response can be computed from the transfer function using

$$y_{zs}(k) = Z^{-1} \{ H(z)X(z) \}$$

Transfer Function: Example 1

Consider the system

$$y(k) - 1.2y(k-1) + 0.32y(k-2) = 10x(k-1) + 6x(k-2)$$

The transfer function of this system is

$$\begin{aligned} H(z) &= \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} \\ &= \frac{10z^{-1} + 6z^{-2}}{1 - 1.2z^{-1} + 0.32z^{-2}} \\ &= \frac{10z + 6}{z^2 - 1.2z + 0.32} \end{aligned}$$

Transfer Function: Example 1

The zero-state response of this system can be computed by $y_{zs}(k) = Z^{-1}\{H(z)X(z)\}$. If the input is $x(k) = \mu(k)$, then

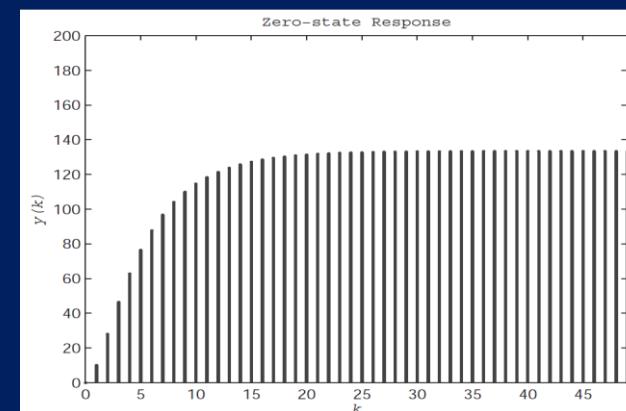
$$\begin{aligned} Y(z) &= H(z)X(z) = \left(\frac{10z + 6}{z^2 - 1.2z + 0.32} \right) \frac{z}{z - 1} \\ &= \frac{(10z + 6)z}{(z - 0.8)(z - 0.4)(z - 1)} \end{aligned}$$

The initial value is $y(0) = b_0\delta(m-n) = 0$.
Using the residue method yields

$$\text{Res}(0.8, k) = -175(0.8)^k, \text{Res}(0.4, k) = 41.7(0.4)^k, \text{Res}(1, k) = 133.3$$

and

$$y_{zs}(k) = (133.3 - 175 \times 0.8^k + 41.7 \times 0.4^k) \mu(k)$$



TF: Poles, Zeros, Modes

Recall the transfer function

$$H(z) = \frac{b_0 z^{n-m} (z - z_1)(z - z_2) \cdots (z - z_m)}{(z - p_1)(z - p_2) \cdots (z - p_n)}$$

- “The roots of the numerator polynomial are called the **zeroes** of the discrete-time system while the roots of the denominator polynomial are called the **poles**.”
- Recall that $Y(z) = H(z)X(z)$. The output $y(k)$ can be decomposed into two types of modes:

$$y(k) = \text{natural modes} + \text{forced modes}$$

where natural modes are generated by poles of $H(z)$ and forced modes are generated by poles of $X(z)$.

TF: Poles, Zeros, Modes

- For a **simple pole** at $z = p$, the corresponding mode is of the form
$$cp^k \mu(k)$$

- For a **multiple pole** of multiplicity r , the corresponding mode is of the form

$$(c_0 + c_1 k + \cdots + c_{r-1} k^{r-1}) p^k \mu(k), \quad r \geq 1$$

- If $H(z)$ has poles at p_i for $1 \leq i \leq n$, and $X(z)$ has poles at q_i for $1 \leq i \leq r$, and if all the poles are simple, then

$$y(k) = \underbrace{\sum_{i=1}^n c_i p_i^k \mu(k)}_{\text{Natural mode}} + \underbrace{\sum_{i=1}^r d_i q_i^k \mu(k)}_{\text{Forced mode}}$$

where c_i and d_i are coefficients to be determined.

Pole-Zero Cancellation

- “Since $Y(z) = H(z)X(z)$, pole-zero cancellation can happen when some zeros of $H(z)$ are the same as poles of $X(z)$.”
- Zeros of $H(z)$ can suppress forced modes of $X(z)$.
- Zeros of $X(z)$ can suppress natural modes of $H(z)$.

Example: Consider the transfer function and input signal

$$H(z) = \frac{10(z+0.6)}{(z-0.8)(z-0.4)}, x(k) = 10(-0.6)^k \mu(k) - 4(-0.6)^{k-1} \mu(k-1)$$

$$X(z) = 10\left(\frac{z}{z+0.6}\right) - 4z^{-1}\left(\frac{z}{z+0.6}\right) = \frac{10(z-0.4)}{(z+0.6)}$$

$$\text{Thus, } y(k) = Z^{-1}\{H(z)X(z)\} = Z^{-1}\left\{\frac{100}{z-0.8}\right\} = 100(0.8)^{k-1} \mu(k-1)$$

Stable Modes

A multiple mode can be written as $c(k)p^k \mu(k)$ where $c(k)$ is polynomial of degree $r-1$ with r being multiplicity of pole.

$$r > 1: |c(k)| \rightarrow \infty \text{ as } k \rightarrow \infty$$

$$|p| < 1: |p^k| \rightarrow 0 \text{ as } k \rightarrow \infty$$

$$\lim_{k \rightarrow \infty} |c(k)p^k| = \lim_{k \rightarrow \infty} \frac{|c(k)|}{|p|^{-k}} = \lim_{k \rightarrow \infty} \frac{|c(k)|}{\exp[-k \log(|p|)]}$$

To evaluate this limit, we need the L'Hospital's rule.

Stable Modes

- The $(r-1)$ th derivative of $c(k)$ is the constant $(r-1)!c_{r-1}$.
- The $(r-1)$ th derivative of the exponential is

$$[-\log(|p|)]^{r-1} \exp[-k \log(|p|)]$$

- Since $\log(|p|) < 0$, the limit is

$$\lim_{k \rightarrow \infty} c(k)p^k = 0 \Leftrightarrow |p| < 1$$

- “That is the exponential factor p^k goes to zero faster than the polynomial factor $c(k)$ goes to infinity if and only if $|p| < 1$.”
- “Thus, a **stable mode** is associated with a pole that lies inside the unit circle.”

DC Gain

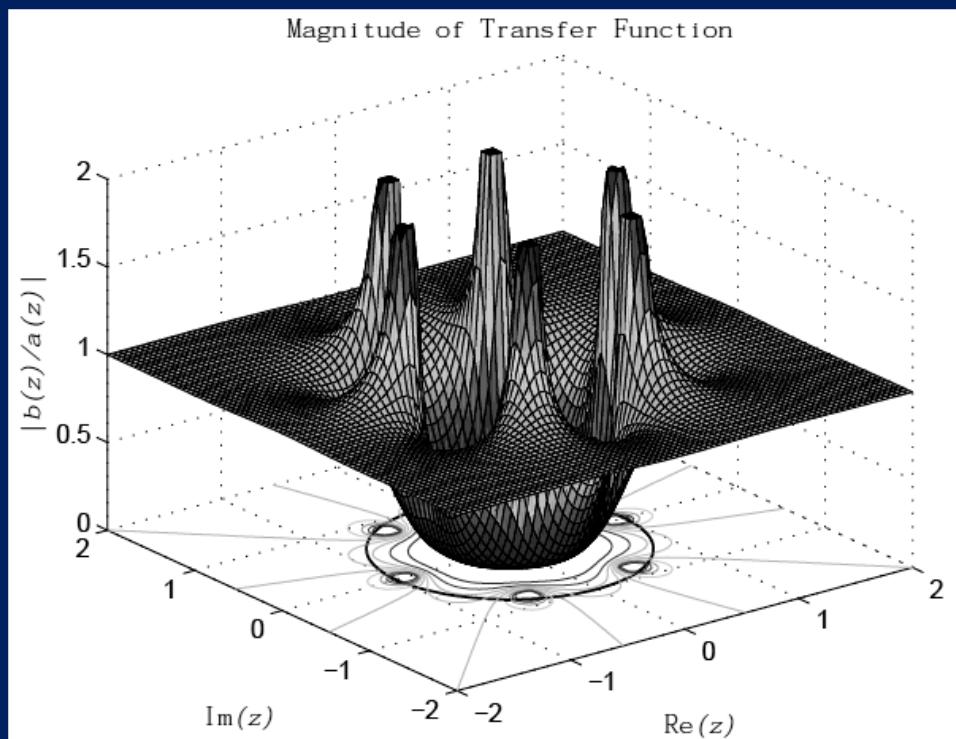
- “When all poles lie inside unit circle, all natural modes decays to zero as time increases.”
- Let determine the steady-state response to step input $x(k) = c\mu(k)$ (also called DC input).
- Using $Y(z) = H(z)X(z)$ and the final value theorem

$$\begin{aligned}\lim_{k \rightarrow \infty} y(k) &= \lim_{z \rightarrow 1} (z - 1)Y(z) = \lim_{z \rightarrow 1} (z - 1)H(z)Z[c\mu(k)] \\ &= \lim_{z \rightarrow 1} (z - 1)H(z) \left(\frac{cz}{z - 1} \right) = H(1)c\end{aligned}$$

- DC gain is the amount by which the DC input is scaled as it passes through the system to produce a steady-state output:
- $$\text{DC gain} = H(1)$$

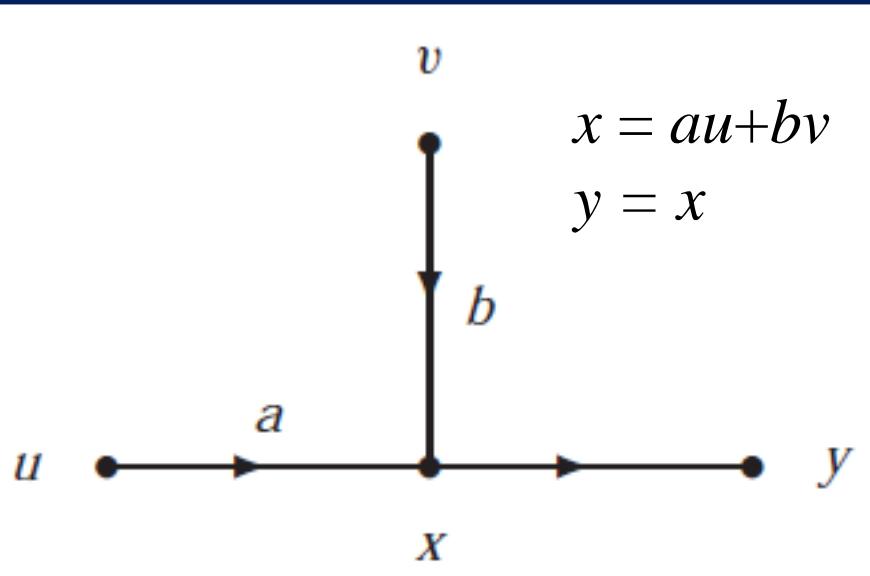
DC Gain: Example

- “Consider the transfer function $H(z) = z^n / (z^n - r^n)$ of a comb filter where $0 < r < 1$.”
- $H(z)$ has n zeroes at the origin and n poles equally spread out on a circle of radius r centered at the origin.
- In case $n = 6$, $r = 0.9$, DC gain = $H(1) = 1 / (1 - r^n) = 2.134$



Signal Flow Graph

- “Discrete-time systems can be represented efficiently using a type of diagram called **signal flow graph**.”
- “A signal flow graph is a collection of **nodes** (dots) interconnected by **arcs** (directed line segments).”
- **Input arc:** an arc that enters a node
- **Output arc:** an arc that leaves a node



- “When two arcs enter a node, their signals are added together.”
- Arc label indicates the type of operation.
- Unlabeled arc has gain = 1.

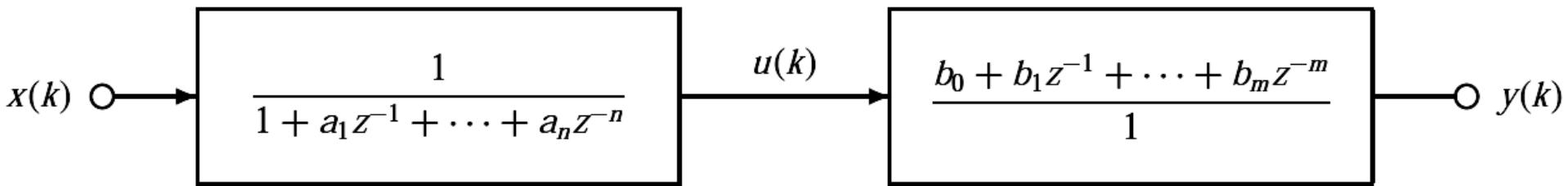
Signal Flow Graph

To represent a discrete-time system as a signal flow graph, the transfer function is rewritten as

$$H(z) = \left(\frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1} \right) \left(\frac{1}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} \right)$$
$$= H_b(z)H_a(z)$$

“Here, $H(z)$ consists of two subsystems H_a and H_b in series.”

$$Y(z) = H(z)X(z) = H_b(z)[H_a(z)X(z)] = H_b(z)U(z)$$



Signal Flow Graph

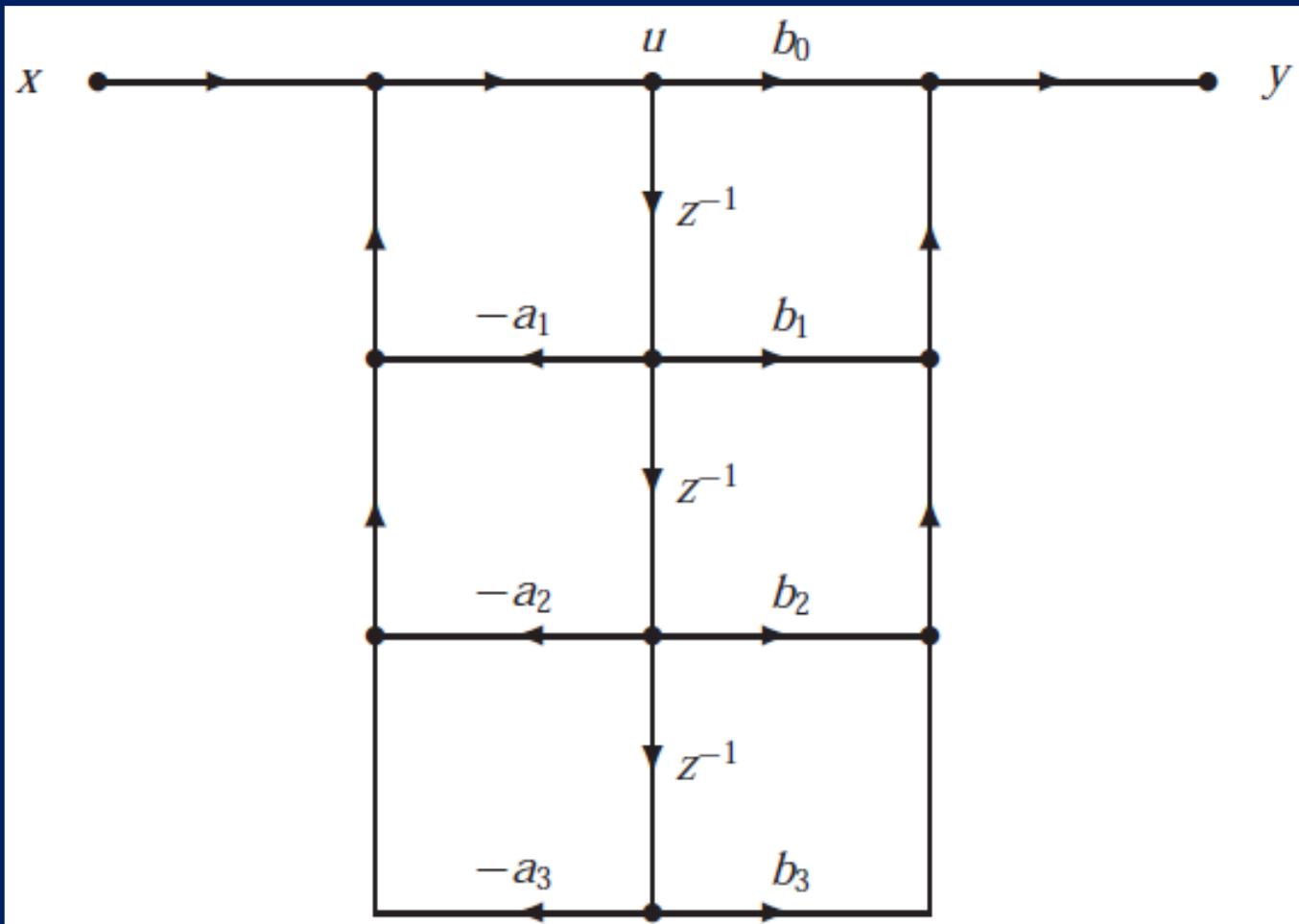
- “The subsystem $H_a(z)$ processes input $x(k)$ to produce an intermediate variable $u(k)$, and then subsystem $H_b(z)$ acts on $u(k)$ to produce the output $y(k)$.”
- $u(k)$ and $y(k)$ can be represented by the difference equations

$$u(k) = x(k) - \sum_{i=1}^n a_i u(k-i)$$

$$y(k) = \sum_{i=1}^n b_i u(k-i)$$

Signal Flow Graph

“When $m = n = 3$, the discrete-time system can be represented with a signal flow graph as shown below.”



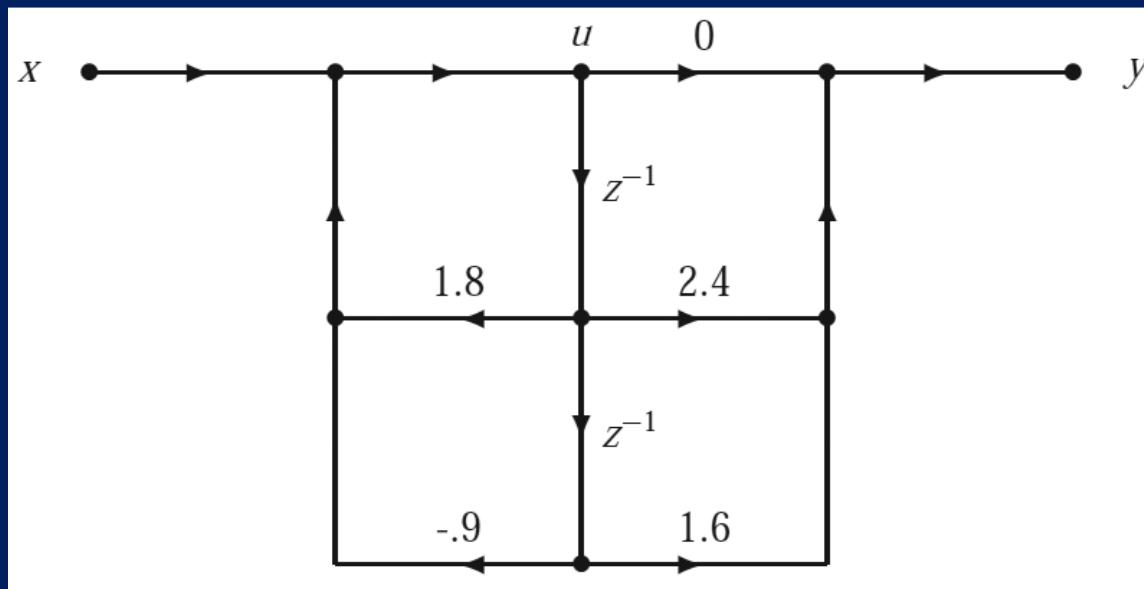
Signal Flow Graph: Example

Consider a discrete-time system with the following

transfer functions $H(z) = \frac{2.4z^{-1} + 1.6z^{-2}}{1 - 1.8z^{-1} + 0.9z^{-2}}$

The signal flow graph of this system is shown below.
The corresponding difference equation is

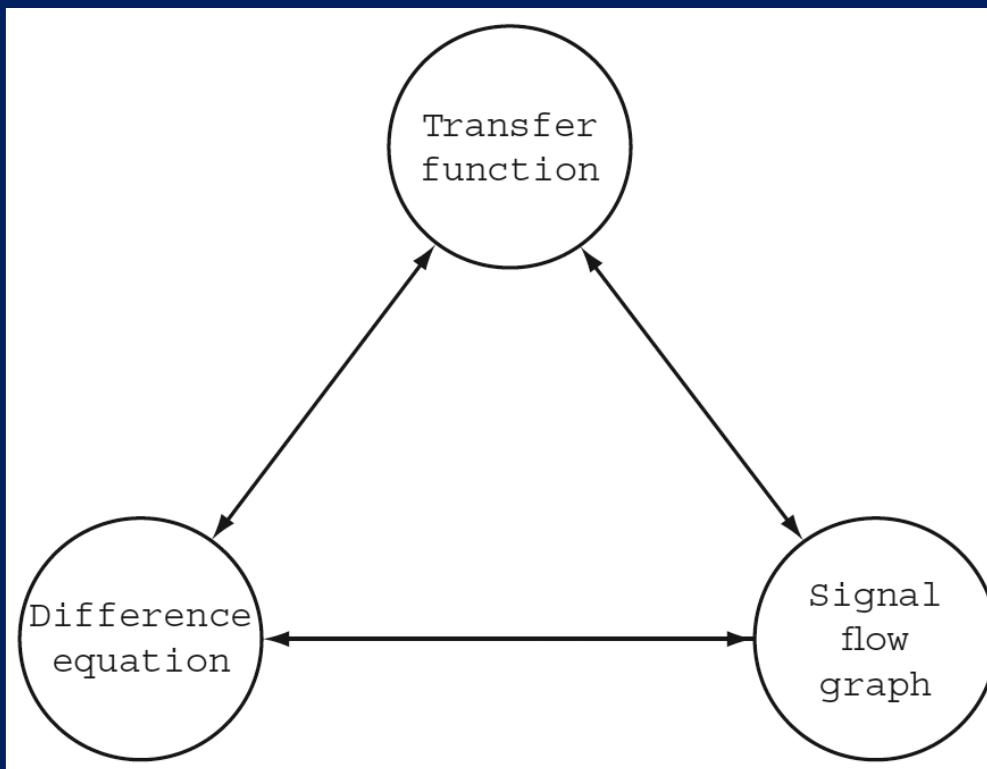
$$y(k) = 2.4x(k-1) + 1.6x(k-2) + 1.8y(k-1) - 0.9y(k-2)$$



System Representations

Discrete-time systems can be represented in 3 distinct ways.

1. Difference equation in the time domain
2. Transfer function in the frequency domain
3. Signal flow graph in the graphical domain



ARMA Models

The transfer function of the first subsystem is a special case of **auto-regressive (AR) model**

$$H_{AR}(z) = \frac{b_0}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

The transfer function of the second subsystem is a **moving-average (MA) model**

$$H_{MA}(z) = b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}$$

The transfer function of the entire system is then called **auto-regressive moving-average (ARMA) model**:

$$H_{ARMA}(z) = H_{AR}(z)H_{MA}(z)$$

Input-Output Representations

“The zero-state response of a system can be represented in the time domain as the convolution of the impulse response $h(k)$ with the input $x(k)$:”

$$y_{zs}(k) = h(k) * x(k)$$

the time-domain input-output representation of the system
Using the definition of the transfer function, we obtain the frequency-domain input-output representation:

$$Y(z) = H(z)X(z)$$

When $x(k) = \delta(k)$, then $y(k) = h(k)$ and $X(z) = 1$.
As a result, $Y(z) = H(z)$. So, the impulse response is the inverse Z transform of the transfer function :

$$h(k) = Z^{-1} \{H(k)\}$$

Stability in the Frequency Domain

- “Practical discrete-time systems are stable.”
- “To test for stability in the frequency domain, consider an input signal $x(k)$ and its Z transform

$$X(z) = \frac{d(z)}{(z - q_1)^{n_1} (z - q_2)^{n_2} \cdots (z - q_r)^{n_r}}$$

where each pole q_i of $X(z)$ generates a mode in $x(k)$ of the form $c_i(k)q_i^k$ and $c_i(k)$ is a polynomial of degree n_i .”

- “These modes go to zero and are bounded as $k \rightarrow \infty$ if the poles are strictly inside the unit circle.”
- “Any pole strictly outside the unit circle generates a growing mode that leads $x(k)$ to be unbounded.”

Stability in the Frequency Domain

- “For simple poles on the unit circle, the coefficient polynomial $c_i(k)$ is constant and the magnitude of the pole is $|q_i| = 1$, so the associated modes are bounded.”
- “Multiple poles on the unit circle have coefficient polynomials that grow with time whereas the exponential factor q_i^k does not go to zero, so these modes generate an unbounded $x(k)$.”
- “A signal is bounded if its Z transform has pole inside the unit circle or simple poles on the unit circle.”

Stability of the Output

- If the transfer function of the system is

$$H(z) = \frac{b(z)}{(z - p_1)^{m_1} (z - p_2)^{m_2} \cdots (z - p_s)^{m_s}}$$

- Then, the Z transform of the output $y(k)$ is

$$Y(z) = \frac{b(z)d(z)}{(z - p_1)^{m_1} \cdots (z - p_s)^{m_s} (z - q_1)^{n_1} \cdots (z - q_r)^{n_r}}$$

- “If the input $x(k)$ is bounded, then the output $y(k)$ will be bounded if all poles of $H(z)$ are strictly inside the unit circle.”

Frequency-domain Stability Criterion

- “A LTI system S with transfer function $H(z)$ is BIBO stable if and only if the poles p_i of $H(z)$ satisfy
$$|p_i| < 1, \quad 1 \leq i \leq s$$
where s is the number of poles of $H(z)$.”
- “The poles of a stable system must all lie strictly inside the unit circle – all of the natural modes of the system must decay to zero.”

All FIR Systems are BIBO Stable

- Any FIR system can be represented by the difference equation

$$y(k) = \sum_{i=0}^m b_i x(k-i)$$

whose impulse response

$$h(k) = \sum_{i=0}^m b_i \delta(k-i)$$

- “A system is BIBO stable if and only if the impulse response $h(k)$ is absolutely summable.”
- “Since the impulse response of FIR system is finite, it is always absolutely summable.”
- “Therefore, all FIR systems are BIBO stable.”

All FIR Systems are BIBO Stable

Taking the Z transform to the impulse response

$$h(k) = \sum_{i=0}^m b_i \delta(k - i)$$

yields the transfer function

$$\begin{aligned} H(z) &= \sum_{k=-\infty}^{\infty} \left(\sum_{i=0}^m b_i \delta(k - i) \right) z^{-k} \\ &= \sum_{i=0}^m b_i \left(\sum_{k=-\infty}^{\infty} \delta(k - i) z^{-k} \right) = \sum_{i=0}^m b_i z^{-i} = \frac{1}{z^m} \sum_{i=0}^m b_i z^{m-i} \end{aligned}$$

“All poles of $H(z)$ are at $z = 0$ which are well inside the unit circle, so all FIR systems are always BIBO stable.”

Stability of IIR Systems

- “IIR system may or may not be BIBO stable.”
- The stability of an IIR system depends on the coefficients of the denominator polynomial.
- If $\mathbf{a} = [1, a_1, a_2, \dots, a_n]^T$ denote the coefficient vector of the denominator polynomial of $H(z)$, we can obtain the radius of the largest pole using the MATLAB command:
 $r = \max(\text{abs}(\text{roots}(\mathbf{a}))) ;$
- The system is stable if $r < 1$.

Example: Unstable Transfer Function

- Consider a system with the transfer function

$$H(z) = 10/(z + 1)$$

- “This system has a single pole on the unit circle at $z = -1$ so it is an unstable system.”
- The impulse response of this system is
$$h(k) = h(0)\delta(k) + \text{Res}(-1, k)\mu(k - 1) = 10(-1)^{k-1}\mu(k - 1)$$
- “The pole at $z = -1$ generates a mode that oscillates between ± 1 . ”
- Consider the input bounded by $B_x = 1$: $x(k) = (-1)^k\mu(k)$
- The Z transform of the input is

$$X(z) = \frac{z}{z + 1}$$

Example: Unstable Transfer Function

- The Z transform of the zero-state response is then

$$Y(z) = H(z)X(z) = \frac{10z}{(z+1)^2}$$

- The zero-state response $y(k)$ is unbounded:

$$\begin{aligned}y(k) &= y(0)\delta(k) + \text{Res}(-1, k)\mu(k-1) \\&= 10k(-1)^{k-1}\mu(k-1) = 10k(-1)^{k-1}\mu(k)\end{aligned}$$

- “The forced mode of $X(z)$ at $z = -1$ matches the natural mode of $H(z)$ at $z = -1$ causing $Y(z)$ to have a double pole at $z = -1$, thereby generating an unstable mode.”
- “This phenomenon of driving a system with one of its natural modes is called **harmonic forcing**.”

Frequency Response

- “The spectral characteristics or frequency content of a signal $x(k)$ can be modified by passing $x(k)$ through a digital filter to produce an output signal $y(k)$.”
- “The distribution of power across frequencies is called the power density spectrum.”
- “The manner in which the spectrum of $x(k)$ is reshaped is specified by the frequency response of the filter.”
- For a stable linear system S with sampling interval T , the frequency response is defined as

$$H(f) \stackrel{\text{def}}{=} H(z) \Big|_{z=\exp(j2\pi fT)}, \quad -\frac{f_s}{2} \leq f \leq \frac{f_s}{2}$$

where $H(z)$ is the transfer function of the system.

Frequency Response

- “As the frequency f ranges over the interval $[-f_s/2, f_s/2]$, the argument $z = \exp(j2\pi fT)$ traces out the unit circle in a counter clockwise direction.”
- “In other words, the frequency response $H(f)$ is the transfer function $H(z)$ evaluated along the unit circle.”
- “ $H(z)$ is generated by a difference equation with real coefficients. As a result, the frequency response has the symmetry property:

$$H(-f) = H^*(f), \quad -f_s/2 \leq f \leq f_s/2$$

where $H^*(f)$ is the complex conjugate of $H(f)$.

Magnitude and Phase Responses

$H(f)$ can be written in polar form as

$$H(f) = A(f) \exp[j\phi(f)]$$

where $A(f)$ and $\phi(f)$ are called the magnitude and phase responses of the filter, respectively, and can be computed by

$$A(f) = \sqrt{\operatorname{Re}^2[H(f)] + \operatorname{Im}^2[H(f)]}$$

$$\phi(f) = \tan^{-1} \left\{ \frac{\operatorname{Im}[H(f)]}{\operatorname{Re}[H(f)]} \right\}$$

With $H(-f) = H^*(f)$, it can be shown that, for a real filter,
magnitude response is an even function $A(-f) = A(f)$
phase response is an odd function $\phi(-f) = -\phi(f)$

Complex Exponential Inputs

- Consider the complex exponential input:

$$x(k) = p^k \mu(k) = \exp[j2\pi kfT] \mu(k)$$

where $p = \exp(j2\pi fT)$. The Z transform of $x(k)$ is

$$X(z) = z/(z - p)$$

- “If the system is stable, all natural modes will go to zero as $k \rightarrow \infty$.”
- “The steady-state response will consist of the forced modes associated with the simple pole at $z = p$:

$$\begin{aligned}y_{ss}(k) &= \text{Res}(p, k) = (z - p)Y(z)z^{k-1} \Big|_{z=p} \\&= (z - p)H(z)X(z)z^{k-1} \Big|_{z=p} = H(z)z^k \Big|_{z=p} = H(p)p^k\end{aligned}$$

Complex Exponential Inputs

Substituting $p = \exp(j2\pi f T)$ yields

$$\begin{aligned}y_{ss}(k) &= H(p)p^k \\&= H(f)\exp(j2\pi kfT) \\&= A(f)\exp(j\phi(f))\exp(j2\pi kfT) \\&= A(f)\exp[j(2\pi kfT + \phi(f))]\end{aligned}$$

“The steady-state response of a stable system to the complex exponential input of frequency f is also the complex exponential input of frequency f , but its amplitude is scaled by $A(f)$ and its phase angle is shifted by $\phi(f)$.”

Real Sinusoidal Inputs

- Consider the real sinusoidal input:

$$\begin{aligned}x(k) &= \cos(2\pi kfT)\mu(k) \\&= 0.5 \left[\exp(j2\pi kfT) + \exp(-j2\pi kfT) \right] \mu(k)\end{aligned}$$

- Since the system is linear, the steady-state response is

$$\begin{aligned}y_{ss}(k) &= 0.5 \left[H(f) \exp(j2\pi kfT) + H(-f) \exp(-j2\pi kfT) \right] \\&= 0.5 \left[H(f) \exp(j2\pi kfT) + (H(f) \exp(j2\pi kfT))^* \right] \\&= \operatorname{Re} \left\{ H(f) \exp(j2\pi kfT) \right\} \\&= \operatorname{Re} \left\{ A(f) \exp[j(2\pi kfT + \phi(f))] \right\} \\&= A(f) \cos(2\pi kfT + \phi(f))\end{aligned}$$

Gain and Phase Shift

- “The magnitude response $A(f)$ specifies the **gain** of the system at frequency f .”
- “The phase response $A(f)$ specifies the **phase shift** of the system at frequency f .”

Frequency Response: Example

Consider a filter with the transfer function

$$H(z) = \frac{z+1}{z^2 + 0.64}$$

Let $\theta = 2\pi fT$. Then, the frequency response is

$$H(f) = \left. \frac{z+1}{z^2 + 0.64} \right|_{z=\exp(j\theta)} = \frac{\cos(\theta) + j\sin(\theta)}{\cos(2\theta) - 0.64 + j\sin(2\theta)}$$

The amplitude response is

$$A(f) = |H(f)| = \frac{\sqrt{[\cos(\theta) + 1]^2 + \sin^2(\theta)}}{\sqrt{[\cos(2\theta) - 0.64]^2 + \sin^2(2\theta)}}$$

Frequency Response: Example

Rewrite the frequency response as

$$H(f) = (ab + cd) + j(bc - ad)$$

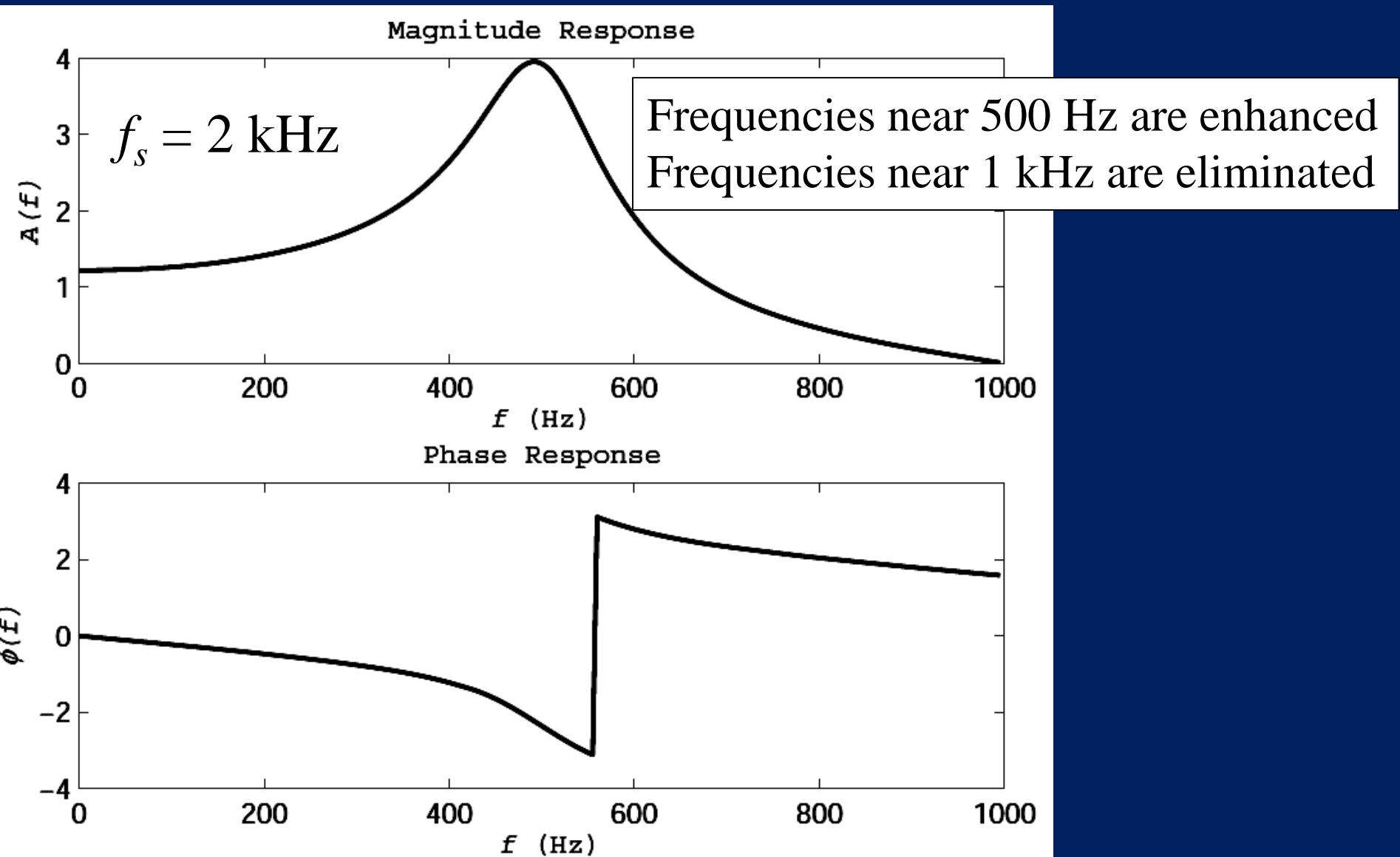
where $a = 1 + \cos \theta$, $b = \cos(2\theta) - 0.64$, $c = \sin \theta$, $d = \sin(2\theta)$

Then, the phase response is

$$\begin{aligned}\phi(f) &= \tan^{-1} \left[\frac{bc - ad}{ab + cd} \right] = \tan^{-1} \left[\frac{(bc - ad)/ab}{1 + cd/ab} \right] \\ &= \tan^{-1} \left[\frac{\frac{c}{a} - \frac{d}{b}}{1 + \frac{cd}{ab}} \right] = \tan^{-1} \left[\frac{c}{a} \right] - \tan^{-1} \left[\frac{d}{b} \right]\end{aligned}$$

Here, we use the identity $\tan^{-1} \left(\frac{u \pm v}{1 \mp uv} \right) = \tan^{-1} u \pm \tan^{-1} v$

Frequency Response: Example



Periodic Inputs

- “Suppose a continuous-time signal $x_a(t)$ is periodic with period T_0 and fundamental frequency $F_0 = 1/T_0$.”
- “The signal $x_a(t)$ can be approximated by a truncated Fourier series as

$$x_a(t) \approx \frac{d_0}{2} + \sum_{i=1}^M d_i \cos(2\pi i F_0 t + \theta_i)$$

- Let c_i denote the i th complex Fourier coefficients of $x_a(t)$, that is,
$$c_i = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_a(t) \exp(-j2\pi i F_0 t) dt$$
- The real Fourier coefficients d_i and phase angle θ_i can be obtained by $d_i = 2|c_i|$, $\theta_i = \angle c_i$

Periodic Inputs

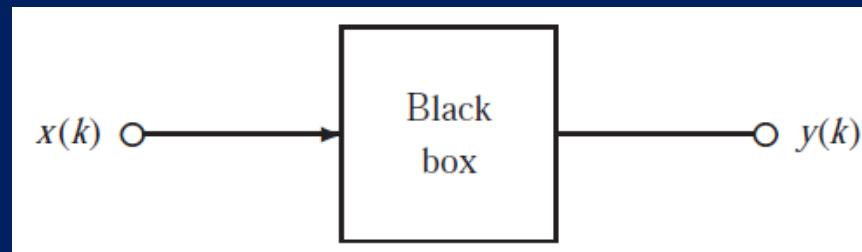
- Recall that
 $x(k) = \cos(2\pi kfT)\mu(k) \rightarrow y_{ss}(k) = A(f) \cos(2\pi kfT + \phi(f))$
- “Let $x(k) = x_a(kT)$ be a discrete-time input to a stable linear system with transfer function $H(z)$.”
- Then, $x(k) = \frac{d_0}{2} + \sum_{i=1}^M d_i \cos(2\pi i F_0 kT + \theta_i)$, $f = iF_0$
- “Thus, the steady-state response to the sampled periodic input is

$$y_{ss}(k) = \frac{A(0)d_0}{2} + \sum_{i=1}^M A(iF_0) d_i \cos[2\pi i F_0 kT + \theta_i + \phi(iF_0)]$$

where the number of harmonics must be $M < f_s/(2F_0)$ to avoid aliasing.”

System Identification

- “The problem of identifying a transfer function $H(z)$ from an input signal $x(k)$ and an output signal $y(k)$ is referred to as **system identification**.”
- “The system S can be thought of as a **black box**.”
- “An effective way to identify a suitable transfer function is to first assume a model of the system.”
- “We could use AR, MA, or ARMA model.”
- Then, the system dimension must be determined.
- Finally, the selected model is fit to the data.



Least-Squares System Identification

- Let's use an auto-regressive (AR) structure as a model.
- A transfer function of an AR model can be expressed as

$$H_{AR}(z) = \frac{1}{a_0 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

- “The difference equation associated with $H(z)$ is

$$\sum_{i=0}^n a_i y(k-i) = x(k)$$

- “Suppose the following input-output data are available”

$$D = \left\{ [x(k), y(k)] \in \mathbb{R}^2 \mid 0 \leq k < N \right\}$$

- “The AR model will precisely fit the data if”

$$\sum_{i=0}^n a_i y(k-i) = x(k), \quad 0 \leq k < N$$

Least-Squares System Identification

- “Let $p = n + 1$ denote the number of parameter.”
- “The N equation in the previous equation can be written in matrix form as $\mathbf{Y}\mathbf{a} = \mathbf{x}$

where $\mathbf{Y} = \begin{bmatrix} y(0) & y(-1) & \cdots & y(-n) \\ y(1) & y(0) & \cdots & y(1-n) \\ \vdots & \vdots & \ddots & \vdots \\ y(N-1) & y(N-2) & \cdots & y(N-1-n) \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$

- When $N > p$, the linear system is over-determined and the solution can be computed by

$$\mathbf{a} = (\mathbf{Y}^T \mathbf{Y})^{-1} \mathbf{Y}^T \mathbf{x} = \mathbf{Y}^\dagger \mathbf{x}$$

where $\mathbf{Y}^\dagger = (\mathbf{Y}^T \mathbf{Y})^{-1} \mathbf{Y}^T$ is the pseudo-inverse or Moore-Penrose inverse of \mathbf{Y} . MATLAB command: $\mathbf{a} = \mathbf{Y} \backslash \mathbf{x}$;

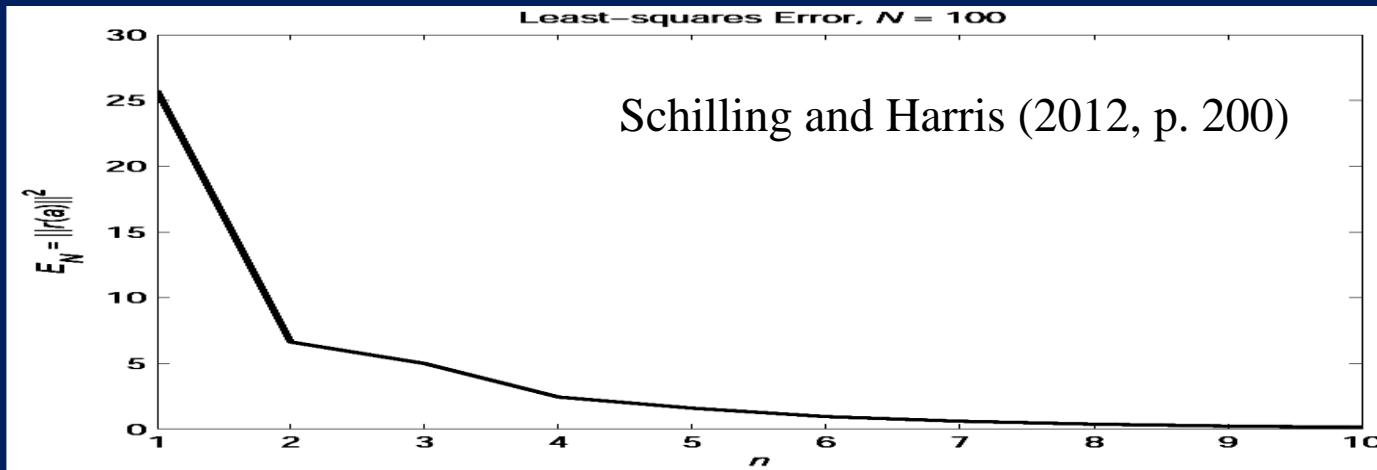
System Identification: Example

“Suppose the output data is generated using $N = 100$ samples from the system with transfer function”

$$H_{\text{data}}(z) = \frac{2z^2 - 0.8z + 0.64}{z^2 - 1.2728z + 0.81}$$

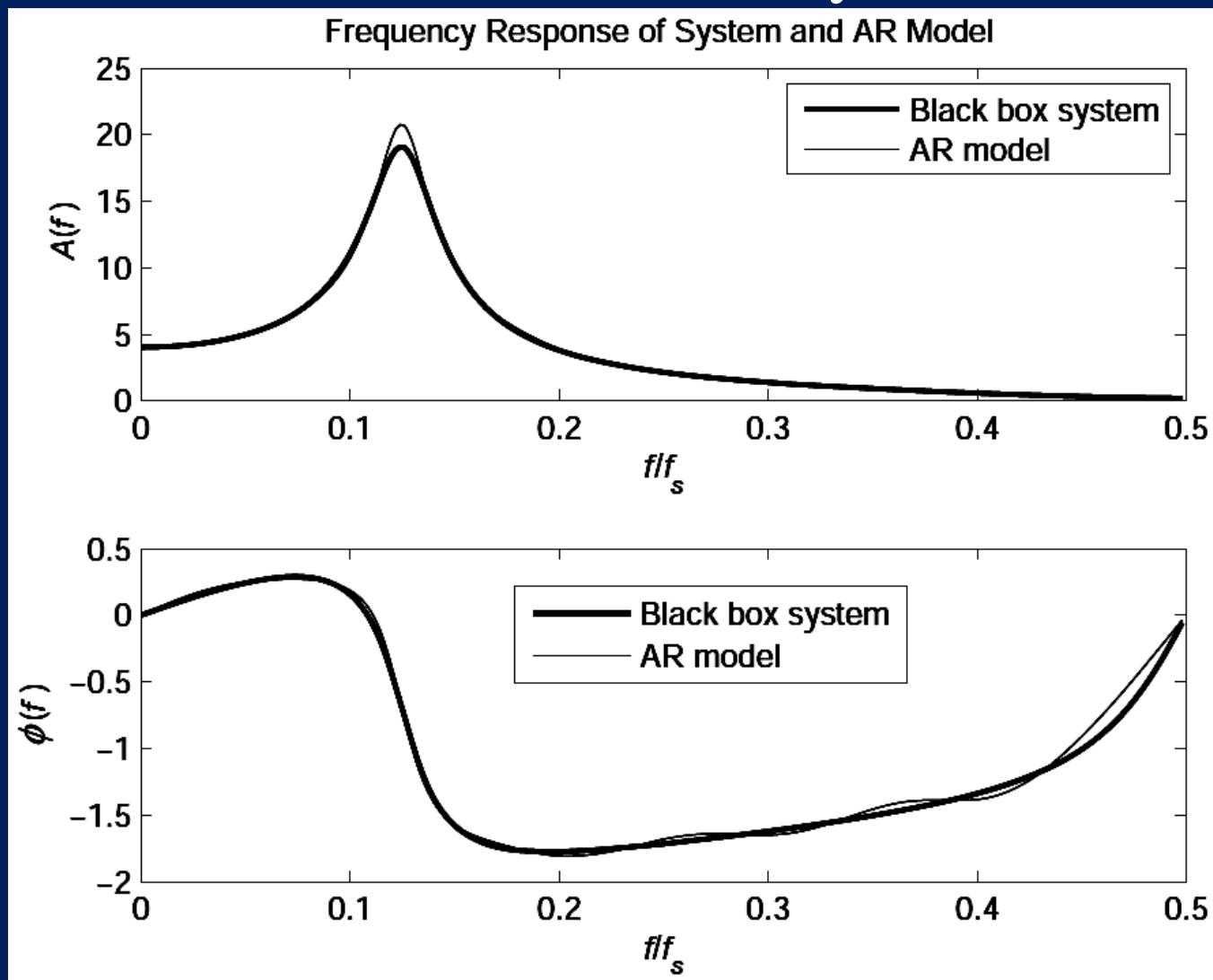
“Suppose the input is white noise uniformly distributed over the interval $[-1,1]$.”

“Performing a system identification using the AR model with orders $n = 1, \dots, 10$ yields error shown below.”



System Identification: Example

Using the AR model of order $n = 10$ yields



Speech Synthesis

- **Phoneme**: fundamental unit of speech
- “**Unvoiced phonemes** associated with turbulence in vocal tract include fricatives (s, sh, f) and terminal sounds (p, k, t).”
- “**Voiced phonemes** are associated with periodic excitation of vocal chords including vowels, nasal sounds, transient terminal sounds (b, d, g).”

Speech Synthesis

“**Voiced phonemes** can be modeled as response of digital filter to periodic impulse train with period M:

$$x(n) = \sum_{i=0}^{\infty} \delta(n - iM)$$

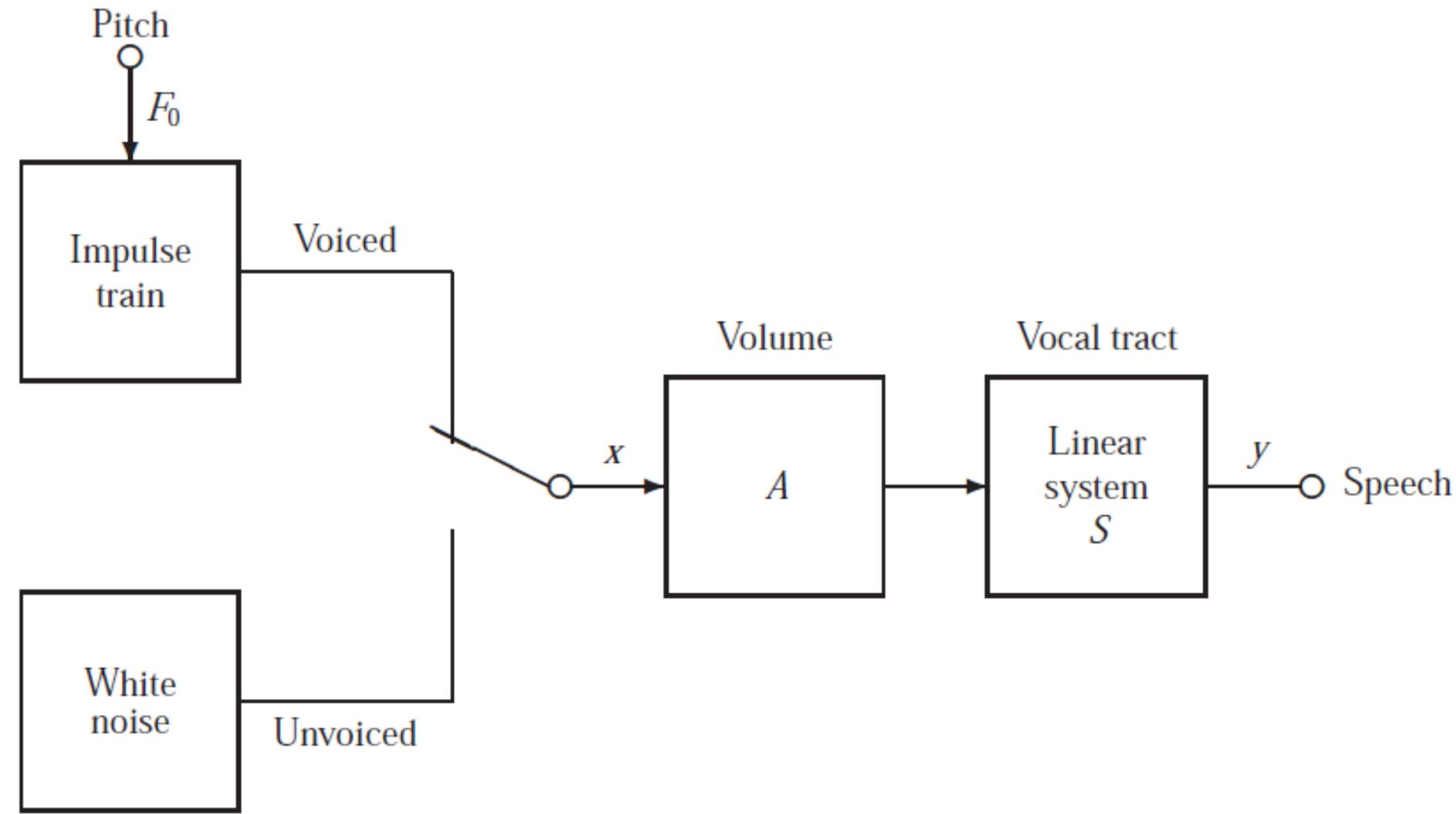
If T is sampling interval, period = MT.

Pitch of fundamental freq. = $F_0 = 1/MT$

Human pitch range = [50, 400] Hz

Exercise: Determine the pitch of vowel sounds such as a, e, i, o, u

Digital Speech Synthesis



Speech Synthesis

“Auto-regressive system is effective for modeling vocal tract:”

$$y(n) = x(n) - \sum_{i=1}^n a_i y(n-i)$$

Given input $x(n)$ and output $y(n)$, finding $a \in \mathbb{R}^n$ is a **system identification** problem.

Reference

- Schilling, R. J. and S. L. Harris, 2012, Fundamentals of Digital Signal Processing using MATLAB, Second Edition, Cengage Learning.