PIPG Module

Purnanand Elango

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1 Template Optimal Control Problem

$$
\begin{aligned}
\text{minimize} \quad & \sum_{t=1}^{N} \frac{1}{2} x_t^\top Q_t x_t + q_t^\top x_t + \frac{1}{2} u_t^\top R_t u_t + r_t^\top u_t, && && \text{(1a)}\\
\text{subject to} \quad & x_{t+1} = A_t x_t + B_t^- u_t + B_{t+1}^+ u_{t+1} + c_t, && t = 1, \ldots, N-1, && \text{(1b)}\\
& x_t \in \mathbb{D}_t^x, \;\; u_t \in \mathbb{D}_t^u, && t = 1, \ldots, N, && \text{(1c)}\\
& F_t^0 x_t + G_t^0 u_t + g_t^0 = 0, && t = 1, \ldots, N, && \text{(1d)}\\
& F_t^1 x_t + G_t^1 u_t + g_t^1 \geq 0, && t = 1, \ldots, N. && \text{(1e)}
\end{aligned}
$$

To track known state reference x_t^{ref} and/or a control reference u_t^{ref}, choose $q_t = -2x_t^{\text{ref}}$ and $r_t = -2u_t^{\text{ref}}$. The boundary conditions on states and control are accounted in $\mathbb{D}_1^x, \mathbb{D}_N^x, \mathbb{D}_1^u$, and $\mathbb{D}_N^u$.

2 Conic Optimization Problem

The optimal control problem (1) falls in the class of conic optimization problems represented by (2), where the convex set $\mathbb{D}$ and convex cone $\mathbb{K}$ are easy to project onto (potentially with closed form expressions).

$$
\begin{aligned}
\underset{z}{\text{minimize}} \quad & \frac{1}{2} z^\top P z + p^\top z \\
\text{subject to} \quad & Hz + h \in \mathbb{K}, \\
& z \in \mathbb{D}.
\end{aligned}
$$

3 Extrapolated PIPG (xPIPG)

The step sizes α and β are dependent on the maximum eigenvalues of P and $H^\top H$. Note that $\|Q\|$ denotes its maximum eigenvalue if Q is a square matrix, and it denotes its maximum singular value if Q is a non-square matrix.

$$
\alpha = \frac{2}{\sqrt{\|P\|^2 + 4\omega\|H\|^2} + \|P\|}, \tag{3a}
$$

$$
\beta = \omega\alpha. \tag{3b}
$$

Algorithm 1 Vectorized xPIPG

Require: $\alpha, \beta, \rho, k_{\max}$

$\triangleright$ Initialize: ξ, η

for $k = 1, \ldots, k_{\max} - 1$ **do**

$\triangleright$ Primal update

$z \leftarrow \Pi_{\mathbb{D}} \left[\xi - \alpha (P\xi + p + H^\top \eta) \right]$

$\triangleright$ Dual update

$w \leftarrow \Pi_{\mathbb{K}^\circ} \left[\eta + \beta (H(2z - \xi) + h) \right]$

$\triangleright$ Extrapolation

$\xi \leftarrow (1 - \rho)\xi + \rho z$

$\eta \leftarrow (1 - \rho)\eta + \rho w$

end for

Ensure: z, w

4 Vectorization of Template OCP

$$z = \left[\begin{array}{cccc|cccc} x_1^\top & x_2^\top & \cdots & x_N^\top & u_1^\top & u_2^\top & \cdots & u_N^\top \end{array} \right]^\top \tag{4a}$$

$$P = \operatorname{blkdiag}(Q_1, \ldots, Q_N, R_1, \ldots, R_N) \tag{4b}$$

$$p = \left[\begin{array}{ccc|ccc} q_1^\top & \cdots & q_N^\top & r_1^\top & \cdots & r_N^\top \end{array} \right]^\top \tag{4c}$$

$$H = \left[\begin{array}{cccccc|cccccc} A_1 & -I & 0 & \cdots & & 0 & B_1^- & B_2^+ & 0 & \cdots & & 0 \\ 0 & A_2 & -I & & & \vdots & 0 & B_2^- & B_3^+ & & & \vdots \\ \vdots & & & \ddots & & & \vdots & & & \ddots & & \\ 0 & \cdots & & & A_{N-1} & -I & 0 & \cdots & & & B_{N-1}^- & B_N^+ \\ \hline F_1^0 & 0 & & \cdots & & 0 & G_1^0 & 0 & & \cdots & & 0 \\ 0 & F_2^0 & & & & \vdots & 0 & G_2^0 & & & & \vdots \\ \vdots & & & \ddots & & & \vdots & & & \ddots & & \\ 0 & \cdots & & & & F_N^0 & 0 & \cdots & & & & G_N^0 \\ \hline F_1^1 & 0 & & \cdots & & 0 & G_1^1 & 0 & & \cdots & & 0 \\ 0 & F_2^1 & & & & \vdots & 0 & G_2^1 & & & & \vdots \\ \vdots & & & \ddots & & & \vdots & & & \ddots & & \\ 0 & \cdots & & & & F_N^1 & 0 & \cdots & & & & G_N^1 \end{array} \right] \tag{4d}$$

$$h = \left[\begin{array}{ccc|ccc|ccc} c_1^\top & \cdots & c_N^\top & g_1^{0\,\top} & \cdots & g_N^{0\,\top} & g_1^{1\,\top} & \cdots & g_N^{1\,\top} \end{array} \right]^\top \tag{4e}$$

5 Customization of xPIPG to Template OCP

The notation $a_{1:M}$ in Algorithm 2 denotes the collection of vectors a_t, for $t = 1 \ldots, M$, arranged into a 2D array.

Algorithm 2 Customized xPIPG

Require: $\alpha, \beta, \rho, k_{\max}$

  ▷ Initialize: $\tilde{x}_{1:N}, \tilde{u}_{1:N}, \tilde{\phi}_{1:N-1}, \tilde{\theta}_{1:N}, \tilde{\psi}_{1:N}$

for $k = 1, \ldots, k_{\max} - 1$ **do**

  ▷ Primal update

  $x_1 \leftarrow \Pi_{\mathbb{D}_1^x}\left[\tilde{x}_1 - \alpha\left(Q_1\tilde{x}_1 + q_1 + A_1^\top\tilde{\phi}_1 + F_1^{0\top}\tilde{\theta}_1 + F_1^{1\top}\tilde{\psi}_1\right)\right]$

  $u_1 \leftarrow \Pi_{\mathbb{D}_1^u}\left[\tilde{u}_1 - \alpha\left(R_1\tilde{u}_1 + r_1 + B_1^{-\top}\tilde{\phi}_1 + G_1^{0\top}\tilde{\theta}_1 + G_1^{1\top}\tilde{\psi}_1\right)\right]$

  for $t = 2, \ldots, N-1$ **do**

    $x_t \leftarrow \Pi_{\mathbb{D}_t^x}\left[\tilde{x}_t - \alpha\left(Q_t\tilde{x}_t + q_t + A_t^\top\tilde{\phi}_t - \tilde{\phi}_{t-1} + F_t^{0\top}\tilde{\theta}_t + F_t^{1\top}\tilde{\psi}_t\right)\right]$

    $u_t \leftarrow \Pi_{\mathbb{D}_t^u}\left[\tilde{u}_t - \alpha\left(R_t\tilde{u}_t + r_t + B_t^{-\top}\tilde{\phi}_t + B_t^{+\top}\tilde{\phi}_{t-1} + G_t^{0\top}\tilde{\theta}_t + G_t^{1\top}\tilde{\psi}_t\right)\right]$

  end for

  $x_N \leftarrow \Pi_{\mathbb{D}_N^x}\left[\tilde{x}_N - \alpha\left(Q_N\tilde{x}_N + q_N - \tilde{\phi}_{N-1} + F_N^{0\top}\tilde{\theta}_N + F_N^{1\top}\tilde{\psi}_N\right)\right]$

  $u_N \leftarrow \Pi_{\mathbb{D}_N^u}\left[\tilde{u}_N - \alpha\left(R_N\tilde{u}_N + r_N + B_N^{+\top}\tilde{\phi}_{N-1} + G_N^{0\top}\tilde{\theta}_N + G_N^{1\top}\tilde{\psi}_N\right)\right]$

  ▷ Dual update

  for $t = 1, \ldots, N-1$ **do**

    $\phi_t \leftarrow \tilde{\phi}_t + \beta\left(-2x_{t+1} + \tilde{x}_{t+1} + A_t(2x_t - \tilde{x}_t) + B_t^-(2u_t - \tilde{u}_t) + B_{t+1}^+(2u_{t+1} - \tilde{u}_{t+1}) + c_t\right)$

    $\theta_t \leftarrow \tilde{\theta}_t + \beta\left(F_t^0(2x_t - \tilde{x}_t) + G_t^0(2u_t - \tilde{u}_t) + g_t^0\right)$

    $\psi_t \leftarrow \min\left\{\tilde{\psi}_t + \beta\left(F_t^1(2x_t - \tilde{x}_t) + G_t^1(2u_t - \tilde{u}_t) + g_t^1\right), 0\right\}$

  end for

  $\theta_N \leftarrow \tilde{\theta}_N + \beta\left(F_N^0(2x_N - \tilde{x}_N) + G_N^0(2u_N - \tilde{u}_N) + g_N^0\right)$

  $\psi_N \leftarrow \min\left\{\tilde{\psi}_N + \beta\left(F_N^1(2x_N - \tilde{x}_N) + G_N^1(2u_N - \tilde{u}_N) + g_N^1\right), 0\right\}$

  ▷ Extrapolation

  $\tilde{x}_{1:N} \leftarrow (1-\rho)\tilde{x}_{1:N} + \rho x_{1:N}$

  $\tilde{u}_{1:N} \leftarrow (1-\rho)\tilde{u}_{1:N} + \rho u_{1:N}$

  $\tilde{\phi}_{1:N-1} \leftarrow (1-\rho)\tilde{\phi}_{1:N-1} + \rho\phi_{1:N-1}$

  $\tilde{\theta}_{1:N} \leftarrow (1-\rho)\tilde{\theta}_{1:N} + \rho\theta_{1:N}$

  $\tilde{\psi}_{1:N} \leftarrow (1-\rho)\tilde{\psi}_{1:N} + \rho\psi_{1:N}$

end for

Ensure: $x_{1:N}, u_{1:N}, \phi_{1:N-1}, \theta_{1:N}, \psi_{1:N}$