PIPG Module

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1 Template Optimal Control Problem

$$
\begin{array}{lll}
\underset{x_t, u_t}{\text{minimize}} & \displaystyle\sum_{t=1}^{N} \frac{1}{2} x_t^\top Q_t x_t + q_t^\top x_t + \frac{1}{2} u_t^\top R_t u_t + r_t^\top u_t, & \\
\text{subject to} & x_{t+1} = A_t x_t + B_t^- u_t + B_{t+1}^+ u_{t+1} + c_t, & t = 1, \ldots, N-1, \\
& \left.\begin{array}{l} x_t \in \mathbb{D}_t^x,\;\; u_t \in \mathbb{D}_t^u, \\ F_t^0 x_t + G_t^0 u_t + g_t^0 = 0, \\ F_t^1 x_t + G_t^1 u_t + g_t^1 \geq 0, \end{array}\right\} & t = 1, \ldots, N,
\end{array}
$$

To track known state reference x_t^{ref} and/or a control reference u_t^{ref}, choose $q_t = -2x_t^{\text{ref}}$ and $r_t = -2u_t^{\text{ref}}$. The boundary conditions on states and control are accounted in $\mathbb{D}_t^x$ and $\mathbb{D}_t^u$.

2 Conic Optimization Problem

$$
\begin{array}{ll}
\underset{z}{\text{minimize}} & \frac{1}{2} z^\top P z + p^\top z \\
\text{subject to} & Hz + h \in \mathbb{K}, \\
& z \in \mathbb{D}.
\end{array}
$$

3 Extrapolated PIPG (xPIPG)

Algorithm 1 Vectorized xPIPG

Require: $\alpha, \beta, \rho, \xi, \eta$

1: **for** $k = 1, \ldots, k_{\max} - 1$ **do**
2: $\quad z \leftarrow \Pi_{\mathbb{D}} \left[\xi - \alpha (P\xi + p + H^\top \eta) \right]$
3: $\quad w \leftarrow \Pi_{\mathbb{K}^\circ} \left[\eta + \beta (H(2z - \xi) + h) \right]$
4: $\quad \xi \leftarrow (1 - \rho)\xi + \rho z$
5: $\quad \eta \leftarrow (1 - \rho)\eta + \rho w$
6: **end for**

Ensure: $z^{k_{\max}}$, $w^{k_{\max}}$

The step sizes α and β are dependent on the maximum eigenvalues of P and $H^\top H$. Note that $\|Q\|$ denotes its maximum eigenvalue if Q is a square matrix, and it denotes its maximum singular value if Q is a non-square matrix.

$$\alpha = \frac{2}{\sqrt{\|P\|^2 + 4\omega\|H\|^2} + \|P\|}, \tag{2a}$$

$$\beta = \omega\alpha. \tag{2b}$$

4 Vectorization of Template OCP

$$z = \left[\begin{array}{cccc|cccc} x_1^\top & x_2^\top & \cdots & x_N^\top & u_1^\top & u_2^\top & \cdots & u_N^\top \end{array}\right]^\top \tag{3a}$$

$$P = \mathrm{blkdiag}\left(Q_1, \ldots, Q_N, R_1, \ldots, R_N\right) \tag{3b}$$

$$p = \left[\begin{array}{ccc|ccc} q_1^\top & \cdots & q_N^\top & r_1^\top & \cdots & r_N^\top \end{array}\right]^\top \tag{3c}$$

$$H = \left[\begin{array}{cccccc|cccccc}
A_1 & -I & 0 & \cdots & & 0 & B_1^- & B_2^+ & 0 & \cdots & & 0 \\
0 & A_2 & -I & & & \vdots & 0 & B_2^- & B_3^+ & & & \vdots \\
\vdots & & & \ddots & & & \vdots & & & \ddots & & \\
0 & \cdots & & & A_{N-1} & -I & 0 & \cdots & & & B_{N-1}^- & B_N^+ \\
\hdashline
F_1^0 & 0 & & \cdots & & 0 & G_1^0 & 0 & & \cdots & & 0 \\
0 & F_2^0 & & & & \vdots & 0 & G_2^0 & & & & \vdots \\
\vdots & & & \ddots & & & \vdots & & & \ddots & & \\
0 & \cdots & & & & F_N^0 & 0 & \cdots & & & & G_N^0 \\
\hdashline
F_1^1 & 0 & & \cdots & & 0 & G_1^1 & 0 & & \cdots & & 0 \\
0 & F_2^1 & & & & \vdots & 0 & G_2^1 & & & & \vdots \\
\vdots & & & \ddots & & & \vdots & & & \ddots & & \\
0 & \cdots & & & & F_N^1 & 0 & \cdots & & & & G_N^1
\end{array}\right] \tag{3d}$$

$$h = \left[\begin{array}{ccc|ccc|ccc} c_1^\top & \cdots & c_N^\top & {g_1^0}^\top & \cdots & {g_N^0}^\top & {g_1^1}^\top & \cdots & {g_N^1}^\top \end{array}\right]^\top \tag{3e}$$

5 Customization of xPIPG to Template OCP

Algorithm 2 Customized xPIPG

Require: $\alpha, \beta, \rho, \tilde{x}_t, \tilde{u}_t, \tilde{\phi}_t, \tilde{\theta}_t, \tilde{\psi}_t$

$\triangleright$ Initialization
$\tilde{\phi}_0,\ \tilde{\phi}_N,\ A_N,\ B_1^+,\ B_N^- \leftarrow 0$
for $k = 1, \ldots, k_{\max} - 1$ **do**
  $\triangleright$ Primal update
  for $t = 1, \ldots, N$ **do**
    $x_t \leftarrow \Pi_{\mathbb{D}_t^x}\left[\tilde{x}_t - \alpha\left(Q_t\tilde{x}_t + q_t + A_t^\top\tilde{\phi}_t - \tilde{\phi}_{t-1} + {F_t^0}^\top\tilde{\theta}_t + {F_t^1}^\top\tilde{\psi}_t\right)\right]$
    $u_t \leftarrow \Pi_{\mathbb{D}_t^u}\left[\tilde{u}_t - \alpha\left(R_t\tilde{u}_t + r_t + {B_t^-}^\top\tilde{\phi}_t + {B_t^+}^\top\tilde{\phi}_{t-1} + {G_t^0}^\top\tilde{\theta}_t + {G_t^1}^\top\tilde{\psi}_t\right)\right]$
  end for
  $\triangleright$ Dual update
  for $t = 1, \ldots, N-1$ **do**
    $\phi_t \leftarrow \tilde{\phi}_t + \beta\left(-2x_{t+1} + \tilde{x}_{t+1} + A_t(2x_t - \tilde{x}_t) + B_t^-(2u_t - \tilde{u}_t) + B_{t+1}^+(2u_{t+1} - \tilde{u}_{t+1}) + c_t\right)$
    $\theta_t \leftarrow \tilde{\theta}_t + \beta\left(F_t^0(2x_t - \tilde{x}_t) + G_t^0(2u_t - \tilde{u}_t) + g_t^0\right)$
    $\psi_t \leftarrow \tilde{\psi}_t + \beta\left(F_t^1(2x_t - \tilde{x}_t) + G_t^1(2u_t - \tilde{u}_t) + g_t^1\right)$
    $\psi_t \leftarrow \psi_t - \max\{\psi_t, 0\}$
  end for
  $\theta_N \leftarrow \tilde{\theta}_N + \beta\left(F_N^0(2x_N - \tilde{x}_N) + G_N^0(2u_N - \tilde{u}_N) + g_N^0\right)$
  $\psi_N \leftarrow \tilde{\psi}_N + \beta\left(F_N^1(2x_N - \tilde{x}_N) + G_N^1(2u_N - \tilde{u}_N) + g_N^1\right)$
  $\psi_N \leftarrow \psi_N - \max\{\psi_N, 0\}$
  $\triangleright$ Extrapolation
  for $t = 1, \ldots, N-1$ **do**
    $\tilde{x}_t \leftarrow (1-\rho)\tilde{x}_t + \rho x_t$
    $\tilde{u}_t \leftarrow (1-\rho)\tilde{u}_t + \rho u_t$
    $\tilde{\phi}_t \leftarrow (1-\rho)\tilde{\phi}_t + \rho\phi_t$
    $\tilde{\theta}_t \leftarrow (1-\rho)\tilde{\theta}_t + \rho\theta_t$
    $\tilde{\psi}_t \leftarrow (1-\rho)\tilde{\psi}_t + \rho\psi_t$
  end for
  $\tilde{x}_N \leftarrow (1-\rho)\tilde{x}_N + \rho x_N$
  $\tilde{u}_N \leftarrow (1-\rho)\tilde{u}_N + \rho u_N$
  $\tilde{\theta}_N \leftarrow (1-\rho)\tilde{\theta}_N + \rho\theta_N$
  $\tilde{\psi}_N \leftarrow (1-\rho)\tilde{\psi}_N + \rho\psi_N$
end for

Ensure: $x_t, u_t, \phi_t, \theta_t, \psi_t$