Introduction to Uncertainty Quantification
Module 3.1: Functions of Random Variables and the Change of Variables Theorem

In this lesson, we consider problems in which one random variable (or random vector) can be expressed directly as a function of another random variable (random vector). To begin, consider we are given a random variable, X, having PDF $f_X(x)$ (CDF $F_X(x)$), and we define the random variable $Y = g(X)$.

1 Moments of Functions of Random Variables

The first task we want to consider is the straightforward estimation of the moments of random variable Y. From the definition of the n^{th} moment in Module 2.2, we have the following:

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y^n f_Y(y) dy \tag{1}$$

Considering that the probability contained in a differential area is invariant under a change of variables, i.e.

$$|f_X(x)dx| = |f_Y(y)dy|$$

Further recognizing that the PDF and increments are always positive and rearranging the terms above, we see that

$$f_Y(y) = f_X(x)\frac{dx}{dy}$$

Applying this and $Y = g(X)$ to Eq. (2) yields:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} (g(x))^n f_X(x)\frac{dx}{dy} dy \quad = \int_{-\infty}^{\infty} (g(x))^n f_X(x) dx \tag{2}$$

2 Change of Variables Theorem

Let us now consider that $g(\cdot)$ is a continuously differentiable and monotonic function of X. That is, the inverse function $X = g^{-1}(Y)$ exists, and is unique and differentiable for all Y. We aim to find the PDF of Y, $f_Y(y)$ (CDF $F_Y(y)$).

The **Change of Variables Theorem** states that the PDF of Y, $f_Y(y)$, can be expressed in terms of the PDF of X, $f_X(x)$ as:

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right| \tag{3}$$

This can be derived by considering that the probability contained in a differential area is invariant under a change of variables, i.e.

$$|f_X(x)dx| = |f_Y(y)dy|$$

Recognizing that the PDF is always positive, letting $x = g^{-1}(y)$, and rearranging the terms above, we see that

$$\begin{aligned} f_Y(y) &= f_X(x)\left|\frac{dx}{dy}\right| \\ &= f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right| \end{aligned}$$

Next, we will consider special cases where the function $Y = g(X)$ is monotonically increasing, monotonically decreasing, and cases where the function is not monotonic.

2.1 Monotonically Increasing Functions

In the case of monotonically increasing functions, we can derive the change of variables by first expressing the CDF of Y as:

$$F_Y(y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y)) \tag{4}$$

Notice that the equivalent event spaces are $Y \leq y$ and $X \leq g^{-1}(y)$, which can be seen graphically in Figure 1 where the red region corresponds to $X \leq g^{-1}(y)$ and the blue corresponds to $Y \leq y$. We clearly see that

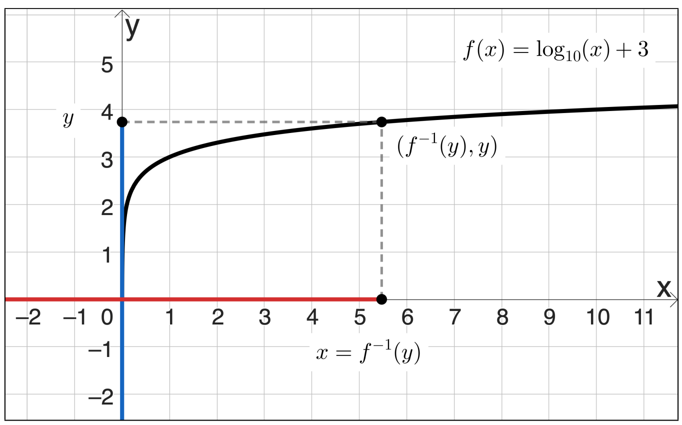

Figure 1: Equivalent event spaces for a monotonically increasing function. The red shows the event $X \leq g^{-1}(y)$, which has the same probability as the event $Y \leq y$ in blue.

these events have a one-to-one mapping through $g(X)$ and therefore have equal probability. Next, we can compute the PDF by differentiating:

$$\begin{aligned} f_Y(y) &= \frac{dF_Y(y)}{dy} \\ &= \frac{dF_X(g^{-1}(y))}{dy} \\ &= \frac{dF_X(g^{-1}(y))}{dx}\frac{dx}{dy} \\ &= f_x(g^{-1}(y))\frac{dg^{-1}(y)}{dy} \end{aligned} \tag{5}$$

which returns the change of variables theorem for monotonically increasing functions.

2.2 Monotonically Decreasing Functions

Likewise, for monotonically decreasing functions, we have

$$F_Y(y) = P(g(X) \leq y) = P(X > g^{-1}(y)) = 1 - F_X(g^{-1}(y)) \tag{6}$$

Notice that the equivalent event spaces are $Y \leq y$ and $X > g^{-1}(y)$, which can be seen graphically in Figure 2, where the red region now corresponds to $X \geq g^{-1}(y)$ and the blue again corresponds to $Y \leq y$. Again,

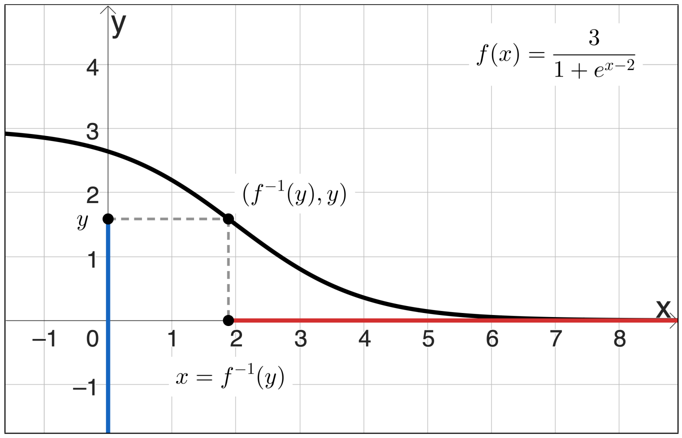

Figure 2: Equivalent event spaces for a monotonically decreasing function. The red shows the event $X \geq g^{-1}(y)$, which has the same probability as the event $Y \leq y$ in blue.

we see that these events have equal probability owing to their one-to-one mapping through $g(X)$. We can compute the PDF by differentiating:

$$\begin{aligned} f_Y(y) &= -\frac{dF_Y(y)}{dy} \\ &= -\frac{dF_X(g^{-1}(y))}{dy} \\ &= -\frac{dF_X(g^{-1}(y))}{dx}\frac{dx}{dy} \\ &= -f_x(g^{-1}(y))\frac{dg^{-1}(y)}{dy} \end{aligned} \tag{7}$$

and thus obtain the change of variables theorem for monotonically decreasing functions where we notice that $f_Y(y)$ is strictly positive because $\frac{dg^{-1}(y)}{dy}$ is strictly negative due to $g(X)$ being a monotonically decreasing function.

2.3 Non-monotonic functions

If the function is not monotonic and therefore uniquely invertible, we can still compute the density through the change of variables theorem. To do so, we must determine the contribution for each potential inversion. That is, let's say that $g^{-1}(y)$ takes m potential values $\{x_1, \ldots, x_m\}$, then the PDF of Y can be determined as:

$$f_Y(y) = \sum_{i=1}^{m} f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}(y)}{dy} \right| \tag{8}$$

where $g_i^{-1}(y) = x_i$.

2.4 Examples

Monotonically Decreasing Function
Let X be a uniformly distributed random variable in (a, b) having CDF:

$$F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x < b \\ 1.0 & x \ge b \end{cases}$$

Consider the function $Y = e^{-x}$ as shown in Figure 3. Let us find $F_Y(y)$ and $f_Y(y)$.

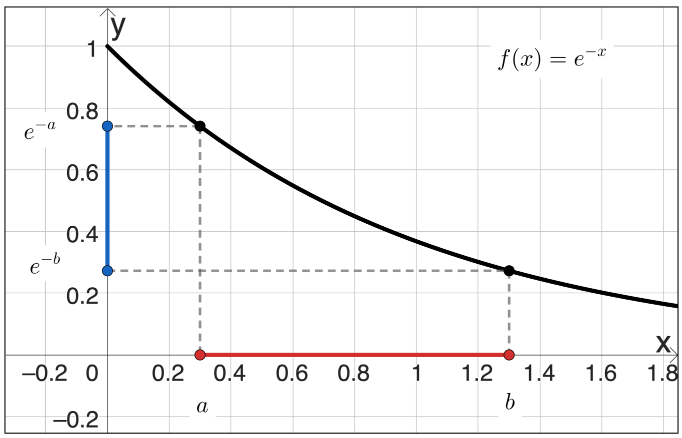

Figure 3: Monotonically decreasing function $Y = e^{-X}$ for $X \sim U(a, b)$.

Recognizing that $X = g^{-1}(Y) = -\ln(Y)$ for $e^{-b} < Y < e^{-a}$ and applying Eq. (7) yields,

$$\begin{aligned} F_Y(y) &= 1 - F_X(-\ln y) = 1 - \frac{-\ln y - a}{b-a} \\ &= \begin{cases} 0 & y < e^{-b} \\ \dfrac{b + \ln y}{b-a} & e^{-b} < Y < e^{-a} \\ 1.0 & y \ge e^{-a} \end{cases} \end{aligned}$$

which is plotted in Figure 4a. Taking the derivative yields,

$$f_Y(y) = \begin{cases} \dfrac{1}{(b-a)y} & e^{-b} < Y < e^{-a} \\ 0 & \text{otherwise} \end{cases}$$

which is plotted in Figure 4b.
Non-monotonic Function 1
Let X be a random variable having PDF $f_X(x)$ and CDF $F_X(x)$. Let us define $Y = X^2$. Find the CDF and PDF of Y, $F_Y(y)$ and $f_Y(y)$ in terms of $F_X(x)$ and $f_X(x)$.

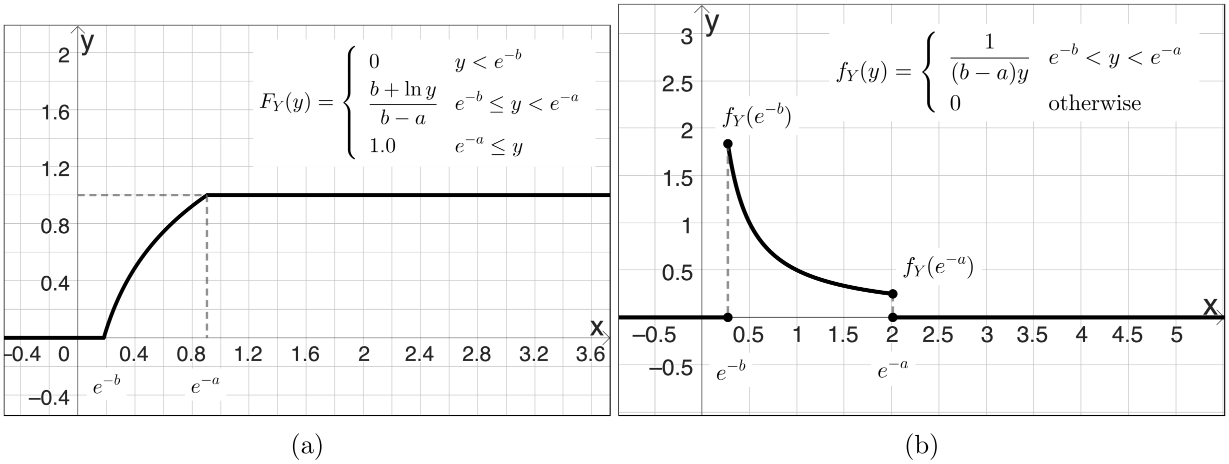

Figure 4: (a) Cumulative distribution function $F_Y(y)$ and (b) probability density function $f_Y(y)$ for the monotonically decreasing function $Y = e^{-X}$ with $X \sim U(a, b)$.

The cumulative distribution function is given by:

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)dx \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned} \tag{9}$$

Applying the change of variables theorem, the probability density function is given by:

$$\begin{aligned} f_Y(y) = \frac{dF_Y(y)}{dy} &= \sum_{i=1}^{2} f_X(g_i^{-1}(y) \left| \frac{\partial g_i^{-1}(y)}{\partial y} \right| \\ &= f_X(\sqrt{y}) \cdot \frac{1}{2} y^{-\frac{1}{2}} + f_X(-\sqrt{y}) \cdot \frac{1}{2} y^{-\frac{1}{2}} \\ &= \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} \end{aligned} \tag{10}$$

Non-monotonic Function 2

Let X be a uniformly distributed random variable in $(0, \pi)$. Consider the function $Y = \sin X$. Find $F_Y(y)$, $f_Y(y)$, and compute $P(Y \le 0.5)$.

Start with

$$f_X(x) = \begin{cases} \frac{1}{\pi} & 0 < X < \pi \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

We recognize that Y is limited to the range $y \in (0,1)$. Over this range, we can compute $F_Y(y)$ as follows:

$$\begin{aligned}
F_Y(y) &= P(g(X) \leq y) = P(\sin X \leq y) \\
&= P(0 \leq X \leq \sin^{-1} y \cap \pi - \sin^{-1} y \leq X \leq \pi) \\
&= 2P(0 \leq X \leq \sin^{-1} y) \\
&= 2\frac{1}{\pi}(\sin^{-1} y - 0) \\
&= \frac{2\sin^{-1} y}{\pi} \quad \text{for} \quad 0 \leq y \leq 1
\end{aligned} \tag{12}$$

where, with the aid of Figure 5, we can see the equivalent event spaces for this non-monotonic function.

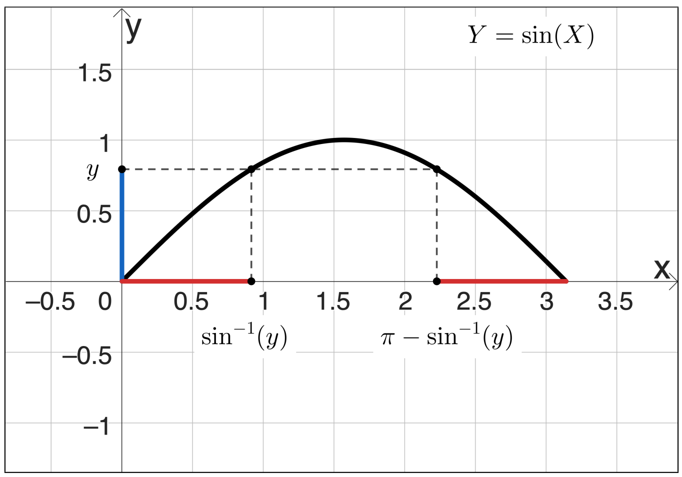

Figure 5: For the function $Y = \sin(X)$ where $X \sim U(0,\pi)$, the following events are equivalent: $Y \leq y$ (blue) and $0 \leq X \leq \sin^{-1} y \cap \pi - \sin^{-1} y \leq X \leq \pi$ (red).

We therefore have

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \dfrac{2\sin^{-1} y}{\pi} & 0 \leq y \leq 1 \\ 1.0 & y \geq 1 \end{cases} \tag{13}$$

which is shown in Figure 6a. Applying the change of variables theorem, we can compute the probability density function over the region $y \in (0,1)$ as:

$$\begin{aligned}
f_Y(y) = \frac{dF_Y(y)}{dy} &= \sum_{i=1}^{2} f_X(g_i^{-1}(y)) \left| \frac{\partial g_i^{-1}(y)}{\partial y} \right| \\
&= f_X(\sin^{-1} y) \cdot \frac{1}{\sqrt{1-y^2}} + f_X(\pi - \sin^{-1} y) \cdot \frac{1}{\sqrt{1-y^2}} \\
&= \frac{2}{\pi} \frac{1}{\sqrt{1-y^2}}
\end{aligned} \tag{14}$$

Therefore, we have

$$f_Y(y) = \begin{cases} \dfrac{2}{\pi} \dfrac{1}{\sqrt{1-y^2}} & 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

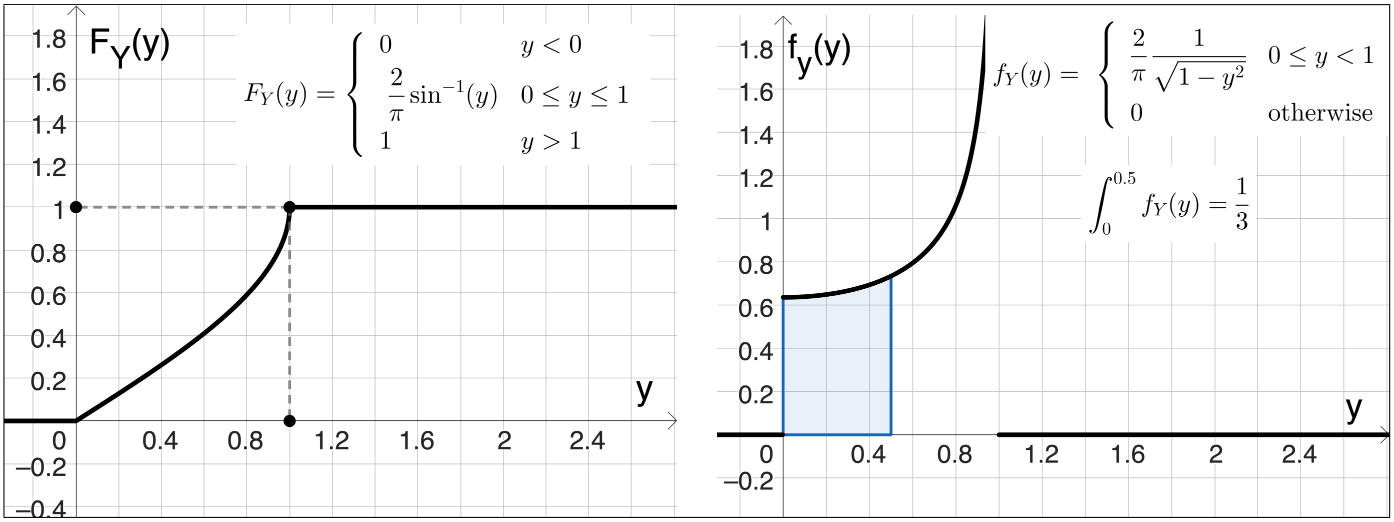

Figure 6: (a) Cumulative distribution function $F_Y(y)$ and (b) probability density function $f_Y(y)$ for the monotonically decreasing function $Y = \sin(X)$ with $X \sim U(0, \pi)$.

which is plotted in Figure 6b. Finally, we can see that

$$P(Y \leq 0.5) = F_Y(0.5) = \frac{2\sin^{-1}(0.5)}{\pi} = \frac{1}{3} \tag{15}$$

as also shown in Figure 6.

3 Multivariate Change of Variables Theorem

The change of variables theorem can be extended to functions of random vectors. Consider the n-dimensional random vector $\boldsymbol{X} \in \mathbb{R}^n$ having joint probability density function $f_{\boldsymbol{X}}(\boldsymbol{x})$ and let $\boldsymbol{Y} = G(\boldsymbol{X}) \in \mathbb{R}^n$ be a bijective (one-to-one), continuously differentiable function.

The **Multivariate Change of Variables Theorem** states that the joint PDF of $\boldsymbol{Y}$ can be expressed as:

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = f_{\boldsymbol{X}}(G^{-1}(\boldsymbol{y}))\, |\det \boldsymbol{J}| \tag{16}$$

where det denotes the determinant of a matrix and $\boldsymbol{J}$ is known as the Jacobian of the inverse function $\boldsymbol{x} = G^{-1}(\boldsymbol{y})$ given by:

$$\begin{aligned}
\boldsymbol{J} &= \frac{d\boldsymbol{x}}{d\boldsymbol{y}} = \frac{\partial G^{-1}(\boldsymbol{y})}{\partial \boldsymbol{y}} \\
&= \left\{ \frac{\partial \boldsymbol{x}}{\partial y_1}, \frac{\partial \boldsymbol{x}}{\partial y_2}, \ldots, \frac{\partial \boldsymbol{x}}{\partial y_n} \right\} \\
&= \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial G_1^{-1}(\boldsymbol{y})}{\partial y_1} & \frac{\partial G_1^{-1}(\boldsymbol{y})}{\partial y_2} & \cdots & \frac{\partial G_1^{-1}(\boldsymbol{y})}{\partial y_n} \\ \frac{\partial G_2^{-1}(\boldsymbol{y})}{\partial y_1} & \frac{\partial G_2^{-1}(\boldsymbol{y})}{\partial y_2} & \cdots & \frac{\partial G_2^{-1}(\boldsymbol{y})}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial G_n^{-1}(\boldsymbol{y})}{\partial y_1} & \frac{\partial G_n^{-1}(\boldsymbol{y})}{\partial y_2} & \cdots & \frac{\partial G_n^{-1}(\boldsymbol{y})}{\partial y_n} \end{bmatrix}
\end{aligned}$$

Example

Let us consider the transformation of the random vector $\boldsymbol{X} = [x_1, x_2]^T$ having joint probability density function $f_{\boldsymbol{X}}(x_1, x_2)$ in Cartesian coordinates to the random vector $\boldsymbol{Y} = \{R, \Theta\}$ in polar coordinates. The coordinate transformation is given by:

$$\boldsymbol{Y} = G(\boldsymbol{X}) = \begin{Bmatrix} R \\ \Theta \end{Bmatrix} = \begin{Bmatrix} \sqrt{X_1^2 + X_2^2} \\ \arctan\left(\dfrac{X_1}{X_2}\right) \end{Bmatrix}$$

We can determine distribution $f_{\boldsymbol{Y}}(\boldsymbol{y})$ by first defining the inverse operation $\boldsymbol{X} = G^{-1}(\boldsymbol{Y})$ as:

$$\boldsymbol{X} = G^{-1}(\boldsymbol{Y}) = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} R\cos\Theta \\ R\sin\Theta \end{Bmatrix}$$

which has Jacobian

$$\boldsymbol{J} = \begin{bmatrix} \dfrac{\partial G_1^{-1}(\boldsymbol{y})}{\partial r} & \dfrac{\partial G_1^{-1}(\boldsymbol{y})}{\partial \theta} \\ \dfrac{\partial G_2^{-1}(\boldsymbol{y})}{\partial r} & \dfrac{\partial G_2^{-1}(\boldsymbol{y})}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Then, by the multivariate change of variables theorem, we have:

$$\begin{aligned} f_{\boldsymbol{Y}}(\boldsymbol{y}) &= f_{\boldsymbol{X}}(G_1^{-1}(r,\theta), G_2^{-1}(r,\theta)) \left| \frac{\partial G_1^{-1}}{\partial r}\frac{\partial G_2^{-1}}{\partial \theta} - \frac{\partial G_1^{-1}}{\partial \theta}\frac{\partial G_2^{-1}}{\partial r} \right| \\ &= f_{\boldsymbol{X}}(r\cos\theta, r\sin\theta)|r\cos^2\theta + r\sin^2\theta| \\ &= f_{\boldsymbol{X}}(r\cos\theta, r\sin\theta)|r| \end{aligned}$$

4 Affine Transformations

Lastly, we will consider the special case of *affine transformations*. For random variables X and Y, any transformation of the form $Y = g(X) = aX + b$ is referred to as an affine transformation. Noting that the inverse transformation can be expressed as $X = g^{-1}(Y) = \dfrac{Y - b}{a}$ and applying the change of variables theorem to this transformation yields:

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)\left|\frac{1}{a}\right|$$

We therefore see that the distribution form $f_X(x)$ is preserved and that the distribution is simply shifted and scaled. This affine transformation is therefore also referred to as a location-scale transformation where b is the location parameter specifying the amount the distribution is shifted by and a is the scale parameter that specifies the magnitude by which the distribution is stretched or compressed.

We can likewise express the following multivariate affine transformation $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ where $\boldsymbol{x}$ and $\boldsymbol{y}$ are random vectors having $f_{\boldsymbol{X}}(\boldsymbol{x})$ and $f_{\boldsymbol{Y}}(\boldsymbol{y})$ respectively, $\boldsymbol{A}$ is an invertible linear transformation matrix, and $\boldsymbol{b}$ is a vector of constants. Note that although $\boldsymbol{x}$ and $\boldsymbol{y}$ are random vectors, they are denoted with lower-case to distinguish them from the matrix $\boldsymbol{A}$. Applying the multivariate change of variables theorem yields:

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = \frac{f_{\boldsymbol{X}}(\boldsymbol{A}^{-1}(\boldsymbol{y} - \boldsymbol{b}))}{|\det \boldsymbol{A}|}$$

Again, we notice that the form of the distribution is preserved. That is $\boldsymbol{x}$ and $\boldsymbol{y}$ follow the same distribution form, but with $\boldsymbol{A}$ performing a rotation and a scaling, and $\boldsymbol{b}$ applying a shift.

5 Functions of Multiple Random Variables

5.1 General Problem

Consider the function $g(X,Y)$ of random variables X,Y. We are interested in computing the probability $P(g(X,Y) \leq \alpha)$.

For convenience, we will typically express this as $P(g(X,Y) - \alpha \leq 0) = P(G(X,Y) \leq 0)$ where $G(X,Y) = g(X,Y) - \alpha$ is known as the *performance function*.

We often associated the performance function with safety / failure of a system such that:

- $G(X,Y) < 0 \rightarrow$ Failure of the system
- $G(X,Y) > 0 \rightarrow$ System is safe
- $G(X,Y) = 0$ is known as the limit surface.

Hence, the above probability typically corresponds to a *Probability of Failure*, P_F. The *Reliability* of the system is defined by $1 - P_F$.

To compute this, we need

$$P(G(X,Y) \leq 0) = \iint_R f_{XY}(x,y)dxdy \tag{17}$$

where R is the area enclosed by the limit surface corresponding to $G(X,Y) \leq 0$.

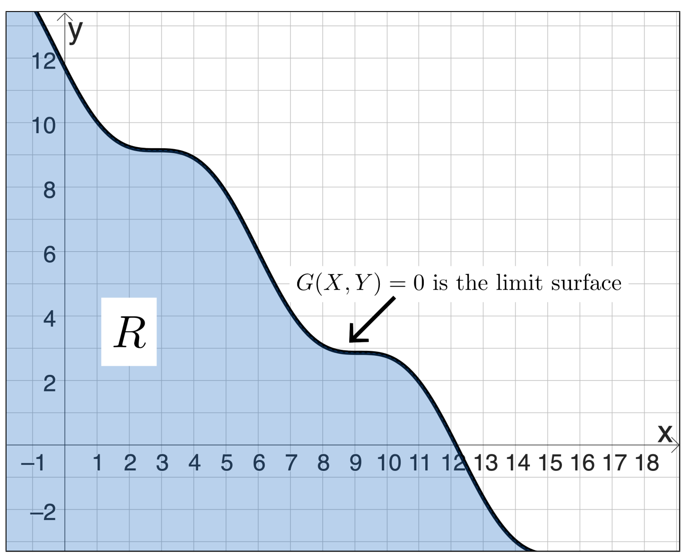

Figure 7: Hypothetical limit surface showing the failure region R corresponding to $G(X,Y) \leq 0$.

Example

Consider:
X = strength of a structural member

Y = load effect on the structural member

Define the *safety margin* by

$$M = X - Y \tag{18}$$

We have:

- $M > 0$: Safe
- $M = 0$: Limit state
- $M < 0$: Failure

Thus $P(M < 0) = P_F$

5.2 Specific Cases of Interest: Linear Limit States

5.2.1 Sum of Random Variables

Consider $Z = X + Y$, compute $F_Z(z)$ and $f_Z(z)$.

We have

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(X + Y \leq z) = P(X + Y - z \leq 0) \\ &= \iint_R f_{XY}(x, y)dxdy \end{aligned} \tag{19}$$

Here $G(X, Y) = X + Y - z$ defines the region R.

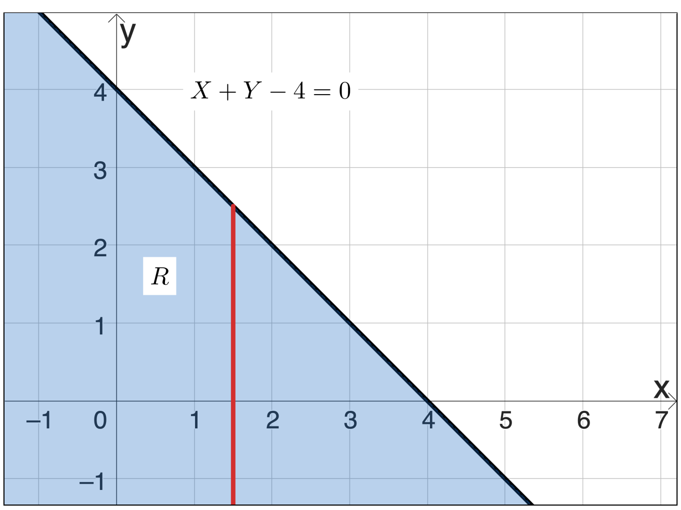

Figure 8: Region of integration, R for the function $G(X, Y) = X + Y - z$ corresponding to $G(X, Y) \leq 0$.

Integrating over y first yields

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x, y)dydx \tag{20}$$

where $z - x$ lies on the line $X + Y - z = 0$, i.e. $y = z - x$.

If X and Y are independent

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{\infty} f_X(x) \left(\int_{-\infty}^{z-x} f_Y(y)dy \right) dx \\ &= \int_{-\infty}^{\infty} f_X(x) F_Y(z-x)dx \end{aligned} \tag{21}$$

Next,

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty} \frac{d}{dy}\frac{dy}{dz} f_X(x)F_Y(z-x)dx \tag{22}$$

Since $y = z - x$, $\dfrac{dy}{dz} = 1$ and we can write

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx \tag{23}$$

Equivalently, we could integrate over X first, yielding

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)dy \tag{24}$$

5.2.2 Difference of Random Variables

Consider $Z = Y - X$, compute $F_Z(z)$ and $f_Z(z)$ if X and Y are independent.

We have

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(Y - X \le z) = P(X - Y - z \le 0) \\ &= \iint_R f_{XY}(x,y)dxdy \end{aligned} \tag{25}$$

Here $G(X,Y) = Y - X - z$ defines the region R.

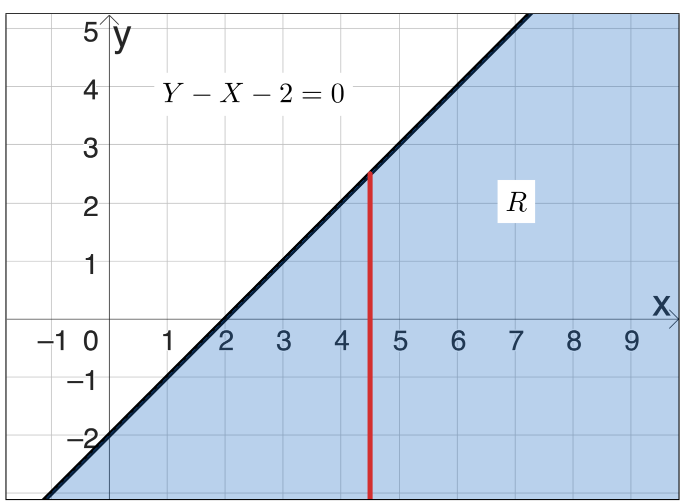

Figure 9: Region of integration, R, for the function $G(X,Y) = Y - X - z$ corresponding to $G(X,Y) \le 0$.

Integrating over Y first, we have

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{\infty} f_X(x) \left(\int_{-\infty}^{z+x} f_Y(y) dy \right) dx \\ &= \int_{-\infty}^{\infty} f_X(x) F_Y(z+x) dx \end{aligned} \tag{26}$$

Next,

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty} \frac{d}{dy} \frac{dy}{dz} f_X(x) F_Y(z+x) dx \tag{27}$$

Again, since $y = z + x$, $\dfrac{dy}{dz} = 1$ and we can write

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z+x) dx \tag{28}$$

Equivalently, we could integrate over X first, yielding

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(y-z) f_Y(y) dy \tag{29}$$

5.2.3 Ratio of Random Variables

Consider $Z = \dfrac{Y}{X}$, compute $F_Z(z)$ and $f_Z(z)$ if X and Y are independent.

We have

$$\begin{aligned} F_Z(z) = P(Z \leq z) &= P\left(\frac{Y}{X} \leq z\right) = P\left(\frac{Y}{X} - z \leq 0\right) \\ &= \iint_R f_{XY}(x,y) dx dy \end{aligned} \tag{30}$$

Here $G(X,Y) = \dfrac{Y}{X} - z$ defines the region R.

Integrating over Y first, we have

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{\infty} f_X(x) \left(\int_{-\infty}^{zx} f_Y(y) dy \right) dx \\ &= \int_{-\infty}^{\infty} f_X(x) F_Y(zx) dx \end{aligned} \tag{31}$$

Next,

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty} \frac{d}{dy} \frac{dy}{dz} f_X(x) F_Y(zx) dx \tag{32}$$

Again, since $y = zx$, $\dfrac{dy}{dz} = x$ and we can write

$$f_Z(z) = \int_{-\infty}^{\infty} x f_X(x) f_Y(zx) dx \tag{33}$$

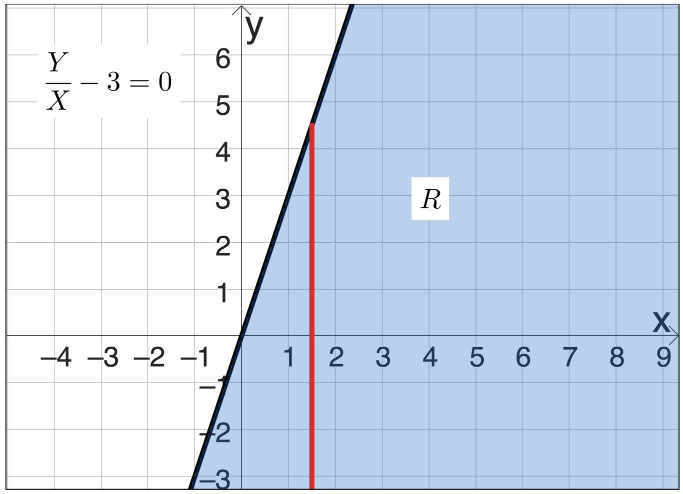

Figure 10: Region of integration, R, for the function $G(X,Y) = \dfrac{Y}{X} - z$ corresponding to $G(X,Y) \leq 0$.

5.3 Examples

Example 1

Consider that X and Y are independent random variables with

$$f_X(x) = \begin{cases} ke^{-kx} & \text{for} \quad x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

$$f_Y(y) = \begin{cases} he^{-hy} & \text{for} \quad y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

where $k, h > 0$.

Compute $F_Z(z)$ when Z is defined in the following ways.

Difference

Consider $Z = Y - X$.

We can use the above expressions, but we need to be careful about the limits of integration. The area R corresponds to the following conditions.

$$\begin{aligned} &Y - X \leq z \\ &X \geq 0 \\ &Y \geq 0 \end{aligned} \tag{36}$$

We need to further consider two different cases for different values of z.

Case 1: $z < 0$

Integrating over Y first, we have

$$\begin{aligned} F_Z(z) &= \int_{-z}^{\infty} f_X(x) \left(\int_0^{z+x} f_Y(y)dy \right) dx \\ &= \frac{h}{h+k} e^{kz} \end{aligned} \tag{37}$$

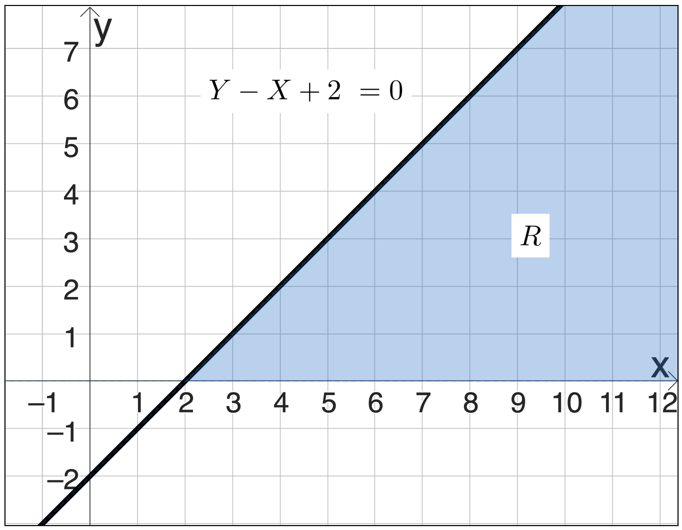

Figure 11: Case 1: Region of integration, R, for the function $G(X,Y) = Y - X - z,\ z < 0$ corresponding to $G(X,Y) \leq 0$.

Case 2: $z > 0$

Integrating over Y first, we have

$$\begin{aligned} F_Z(z) &= \int_{0}^{\infty} f_X(x) \left(\int_0^{z+x} f_Y(y)dy \right) dx \\ &= 1 - \frac{k}{h+k} e^{-hz} \end{aligned} \tag{38}$$

Putting Case 1 and Case 2 together, we get

$$F_Z(z) = \begin{cases} \dfrac{h}{h+k} e^{kz} & \text{for} \quad z < 0 \\ 1 - \dfrac{k}{h+k} e^{-hz} & \text{for} z > 0 \end{cases} \tag{39}$$

Sum

Consider $Z = X + Y$.

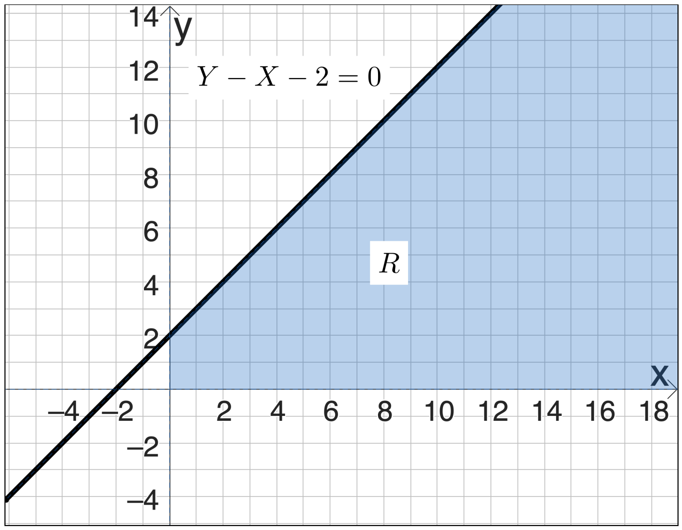

Figure 12: Case 2: Region of integration, R, for the function $G(X,Y) = Y - X - z,\ z > 0$ corresponding to $G(X,Y) \leq 0$.

We can use the above expressions, but we need to be careful about the limits of integration. The area R corresponds to the following conditions.

$$\begin{aligned} X + Y &\leq z \\ X &\geq 0 \\ Y &\geq 0 \end{aligned} \tag{40}$$

We need to further consider two different cases for different values of z.

Case 1: $z < 0$

In this case, the region R is empty. Therefore,

$$F_Z(z) = 0 \tag{41}$$

Case 2: $z > 0$

Integrating over Y first, we have

$$\begin{aligned} F_Z(z) &= \int_0^z f_X(x) \left(\int_0^{z-x} f_Y(y)dy \right) dx \\ &= 1 - \frac{1}{k-h} \left[ke^{-hz} - e^{-kz}\right] \end{aligned} \tag{42}$$

Putting Case 1 and Case 2 together, we get

$$F_Z(z) = \begin{cases} 0 \quad \text{for} \quad z < 0 \\ 1 - \dfrac{1}{k-h} \left[ke^{-hz} - e^{-kz}\right] \quad \text{for} z > 0 \end{cases} \tag{43}$$

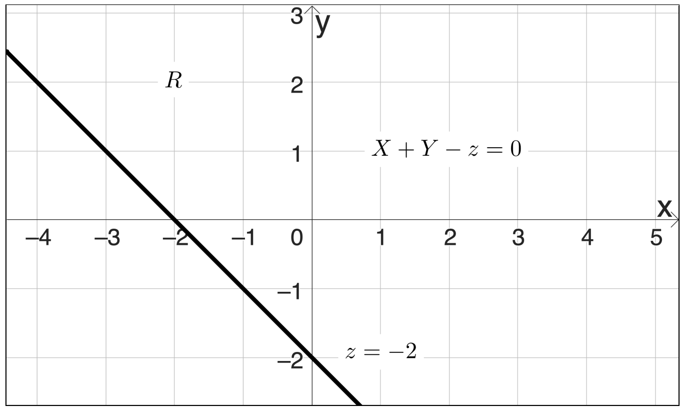

Figure 13: Case 1: Region of integration, R, for the function $G(X, Y) = X + Y - z,\ z < 0$ corresponding to $G(X, Y) \leq 0$.

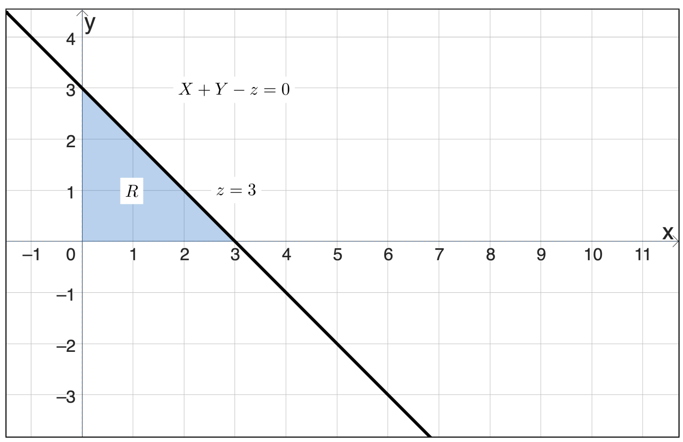

Figure 14: Case 1: Region of integration, R, for the function $G(X, Y) = X + Y - z,\ z > 0$ corresponding to $G(X, Y) \leq 0$.

Ratio

Consider $Z = \dfrac{Y}{X}$.

We can use the above expressions, but we need to be careful about the limits of integration. The area R corresponds to the following conditions.

$$\begin{aligned} \frac{Y}{X} &\leq z \\ X &\geq 0 \\ Y &\geq 0 \end{aligned} \tag{44}$$

We have only a single case here ($z > 0$). Integrating over Y first, we have

$$\begin{aligned} F_Z(z) &= \int_0^{\infty} f_X(x) \left(\int_0^{zx} f_Y(y) dy \right) dx \\ &= \frac{hz}{k + hz} \quad \text{for} \quad z > 0 \end{aligned} \tag{45}$$

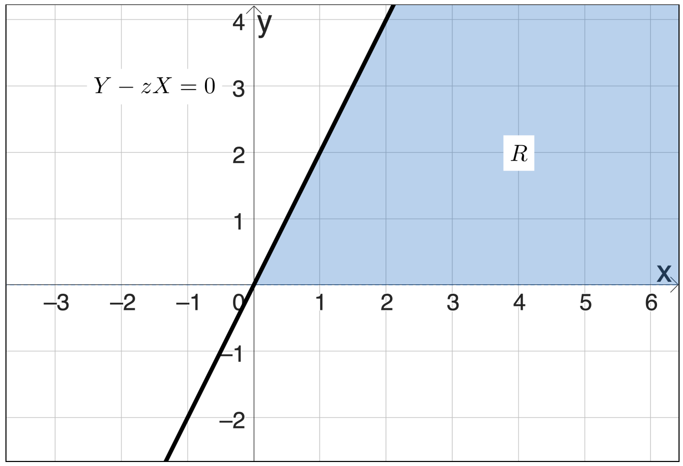

Figure 15: Region of integration, R, for the function $G(X,Y) = Y - zX$ corresponding to $G(X,Y) \leq 0$.

6 Sum / Difference of Normal Random Variables

Consider that X and Y are independent Gaussian random variables with means μ_X, μ_Y and standard deviations σ_X, σ_Y. The PDFs are given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right] f_Y(y) \quad = \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right] \tag{46}$$

Substituting $f_X(x), f_Y(y)$ into the equations above for $Z_1 = X + Y$ and $Z_2 = Y - X$, after some algebra we get

$$\begin{aligned} f_{Z_1}(z_1) &= \frac{1}{\sqrt{2\pi(\sigma_X^2 + \sigma_Y^2)}} \exp\left[-\frac{(z_1 - (\mu_X + \mu_Y))^2}{2(\sigma_X^2 + \sigma_Y^2)}\right] \\ f_{Z_2}(z_2) &= \frac{1}{\sqrt{2\pi(\sigma_X^2 + \sigma_Y^2)}} \exp\left[-\frac{(z_1 - (\mu_X - \mu_Y))^2}{2(\sigma_X^2 + \sigma_Y^2)}\right] \end{aligned} \tag{47}$$

Conclusion: The sum/difference of two independent Gaussian random variables is also Gaussian with mean $\mu_X \pm \mu_Y$ and standard deviation $\sqrt{\sigma_X^2 + \sigma_Y^2}$.

6.1 Let's compute Probability of Failure

The load effects on individual columns of a building are independent Gaussian random variables with

- *Dead Load*: $D \sim N(4.2, 0.3)$ kips
- *Live Load*: $L \sim N(6.5, 0.8)$ kips
- *Wind Load*: $W \sim N(3.4, 0.7)$ kips

The strength of the column is also a Gaussian random variable with mean value equal to 1.5× the total mean load and coefficient of variation of 15%. Strength and load effects are independent.

What is the probability of failure of the column?

Solution

The total load S is also Gaussian random variable with

$$\begin{aligned}\mu_S &= \mu_D + \mu_L + \mu_W = 14.1\text{kips}\\ \sigma_S &= \sqrt{\sigma_D^2 + \sigma_L^2 + \sigma_W^2} = 1.1\text{kips}\end{aligned} \tag{48}$$

The strength of the column R is also Gaussian with

$$\begin{aligned}\mu_R &= 1.5 \times \mu_S = 21.15\text{kips}\\ \sigma_R &= 0.15 \times \mu_R = 3.17\text{kips}\end{aligned} \tag{49}$$

Again, let us define the safety margin

$$M = R - S \tag{50}$$

We have:

- $M > 0$: Safe
- $M = 0$: Limit state
- $M < 0$: Failure

Thus $P(M < 0) = P_F$

M is also Gaussian with

$$\begin{aligned}\mu_M &= \mu_R - \mu_S = 7.05\text{kips}\\ \sigma_M &= \sqrt{\sigma_R^2 + \sigma_S^2} = 3.36\text{kips}\end{aligned} \tag{51}$$

We have

$$P_F = P(M \leq 0) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma_M} \exp\left[-\frac{(m-\mu_M)^2}{2\sigma_M^2}\right] \tag{52}$$

Using Tables or calculators, we get

$$P_F = 0.018 = 1.8\% \tag{53}$$

We can decrease probability of failure by increasing μ_M, decreasing σ_M, or both.

Nomenclature

Functions

$P(\cdot)$ Probability measure

$f_X(x)$ Probability density function (PDF) of a random variable X

$f_{\boldsymbol{X}}(\cdot)$ Joint probability density function of the random vector $\boldsymbol{X}$

$F_X(x)$ Cumulative distribution function (CDF) of a random variable X

$F_{\boldsymbol{X}}(\cdot)$ Joint Cumulative Distribution Function of the random vector $\boldsymbol{X}$

Operators

$\cap$ Intersection

Variables

X A random variable

$\boldsymbol{X}$ A random vector in $\mathbb{R}^n$

μ_X Mean of the random variable X

σ_X Standard deviation of the random variable X

$\boldsymbol{J}$ Jacobian matrix of a function