Introduction to Uncertainty Quantification
Module 3.5: Stochastic Spectral Methods for Uncertainty Propagation

1 Introduction

In the previous module, we learned about the polynomial chaos expansion (PCE) and it's application as a surrogate model for UQ purposes. The PCE may also be used as spectral expansion of the solution to ordinary differential equations (ODEs) and partial differential equations (PDEs). In the previous module, we saw how a random variable Y can be expanded in terms of a set of orthogonal basis polynomials that depend on some basic random variables, referred to as the *germ*, and corresponding coefficients as

$$Y = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^m} \beta_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \tag{1}$$

where, again $\boldsymbol{\alpha}$ is the multi-index, β are the coefficients, $\Psi(\cdot)$ are orthogonal polynomials taken from the Weiner-Askey scheme, and $\boldsymbol{\xi}$ is the germ. Once again, we can truncate this expansion to a finite number of terms by

$$Y = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} \beta_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) + \epsilon. \tag{2}$$

where $\mathcal{A}$ is the multi-index set discussed previously. Notice here that we do not explicitly express Y as a function of other random variables as we previously did. Instead, we are simply performing a spectral expansion of the random variable Y itself, since we're not aiming to use it as a surrogate model in this context.

Likewise, we can represent an arbitrary random process / random field through a similar spectral expansion. Consider the spatio-temporally varying random process $Y(t, x)$ where $t \in [0, T]$ and $x \in D$ are the time and spatial variables respectively. The spectral expansion (PCE) can be expressed as

$$\begin{aligned} Y(t, x) &= \sum_{\boldsymbol{\alpha} \in \mathcal{A}} \beta_{\boldsymbol{\alpha}}(t, x) \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) + \epsilon \\ &= \hat{Y}(t, x) + \epsilon \end{aligned} \tag{3}$$

Notice that the only change is that the deterministic coefficients $\beta_{\boldsymbol{\alpha}}(t, x)$ are now dependent on both space and time.

We will now briefly discuss how the expansion in Eq. (3) can be used to solve fairly general class of differential equations. We specifically discuss two methods. The first method, the stochastic Galerkin method, embeds the spectral expansion into a numerical solution of the differential equations using classical Galerkin projection methods. This method is intrusive and requires developing customized numerical solvers for the differential equations. The stochastic collocation method first solves the deterministic differential equations at a discrete set of points, referred to as collocation points, and then applies the spectral expansion as an interpolant to solve for moments of the solution of the differential equations. Much of what we discuss in this module is derived from the textbook *Uncertainty Quantification: Theory, Implementation, and Applications* by Ralph Smith [1]. The reader is directed to this text for additional details.

2 Stochastic Galerkin Method

The stochastic Galerkin methods works the same way as standard deterministic Galerkin methods for solving differential equations by projecting the weighted residual onto a finite-dimensional space spanned by the selected basis functions (in this case orthogonal Weiner-Askey polynomials). We will briefly consider the formulation for certain classes of differential equations.

2.1 Scalar Initial Value Problems

Consider the scalar ODE given by

$$\begin{aligned} &\frac{dY}{dt} = f(t, \xi, Y), \quad t \in T \\ &Y(0, \xi) = Y_0 \end{aligned} \tag{4}$$

Consider that we aim to estimate the mean response given by

$$\mathbb{E}[Y(t)] = \int Y(t, \xi) p_\xi(\xi) d\xi \tag{5}$$

where $p_\xi(\xi)$ is the probability density of the germ. Consider now the spectral expansion given in Eq. (3) where the coefficients satisfy

$$\beta_k(t) = \frac{1}{\gamma_k} Y(t, \xi) \Psi_k(\xi) p_\xi(\xi) d\xi \tag{6}$$

where $\gamma_k = \langle \Psi_k, \Psi_k \rangle_{p_\xi}$ is the inner product defined for the selected polynomials. We now aim to approximate these coefficients and thereby approximate the solution.

Using a Galerkin projection, we aim to identify the coefficients β_k that satisfy

$$\left\langle \frac{d\hat{Y}}{dt} - f, \Psi_{\boldsymbol{a}} \right\rangle_{p_\xi} = 0 \tag{7}$$

where the inner product is computed by applying the spectral expansion as

$$\left\langle \frac{d\hat{Y}}{dt} - f, \Psi_{\boldsymbol{a}} \right\rangle_{p_\xi} = \int \left[\sum_{\boldsymbol{\alpha} \in \mathcal{A}} \frac{d\beta_{\boldsymbol{\alpha}}}{dt}(t) \Psi_{\boldsymbol{\alpha}}(\xi) - f\left(t, \xi, \sum_{\boldsymbol{\alpha} \in \mathcal{A}} \beta_{\boldsymbol{\alpha}}(t) \Psi_{\boldsymbol{\alpha}}(\xi) \right) \right] \Psi_{\boldsymbol{a}} p_\xi(\xi) d\xi \tag{8}$$

This is often referred to as the *weak stochastic model formulation*. It is equivalent to specifying that

$$\mathbb{E}\left[\frac{d\hat{Y}(t, \xi)}{dt} \Psi_i(\xi) \right] = \mathbb{E}\left[f(t, \xi, \hat{Y} \Psi_i(\xi) \right] \tag{9}$$

The inner product integral is then approximated using numerical quadrature using a set of M quadrature points $\xi^{(i)}$ and associated weights $w^{(i)}$, which yields

$$\sum_{i=1}^{M} \Psi_{\boldsymbol{a}}(\xi^{(i)}) p_\xi(\xi^{(i)}) w^{(i)} \left[\sum_{\boldsymbol{\alpha} \in \mathcal{A}} \frac{d\beta_{\boldsymbol{\alpha}}}{dt}(t) \Psi_{\boldsymbol{\alpha}}(\xi^{(i)}) - f\left(t, \xi^{(i)}, \sum_{\boldsymbol{\alpha} \in \mathcal{A}} \beta_{\boldsymbol{\alpha}}(t) \Psi_{\boldsymbol{\alpha}}(\xi^{(i)}) \right) \right] = 0 \tag{10}$$

for all $\boldsymbol{a} \in \mathcal{A}$. The result is a set of ODEs that can be solved numerically using standard methods (e.g. Runge-Kutta).

Example

To illustrate this approach, it is helpful to look at an example. Consider the following initial value problem

$$\begin{aligned} \frac{dY}{dt} &= -Xu \\ Y(0) &= Y_0 \end{aligned} \tag{11}$$

where Y_0 is a deterministic constant and X is a normal random variable with mean μ_X and variance σ_X^2. We aim to estimate the mean $E[Y(t)]$ and variance $\mathrm{Var}(Y(t))$.

We begin by expressing the random variable X through its PCE, which has the following exact form

$$X = \sum_{n=0}^{1} \lambda_n \psi_n(\xi) = \mu_X + \sigma_X \psi_1(\xi) = \mu_X + \sigma_X \xi \tag{12}$$

where $\lambda_0 = \mu_X$ and $\lambda_1 = \sigma_X$ and $\psi(\xi)$ are the Hermite Polynomials on $\xi \sim N(0,1)$ germs. We note that we could, likewise consider Y_0 to be a normal random variable and apply its PCE. For simplicity, we will consider Y_0 deterministic.

This simple problem has an analytical solution given by

$$Y(t,\xi) = Y_0 e^{-(\mu_X + \sigma_X \xi)t} \tag{13}$$

Using the stochastic Galerkin scheme, we seek an approximate solution through the 1D spectral expansion of order P

$$\hat{Y}(t,\xi) = \sum_{k=0}^{K} \beta_k(t,x)\psi_k(\xi) \tag{14}$$

satisfying

$$\left\langle \frac{d\hat{Y}}{dt} - X\hat{Y}, \psi_j \right\rangle_{p_\xi} = 0 \tag{15}$$

Appying the spectral expansions of $\hat{Y}$ and X yields

$$\int \sum_{k=0}^{P} \frac{d\beta_k}{dt}\psi_k(\xi)\psi_j(\xi)p_\xi(\xi)d\xi + \int \sum_{n=0}^{1} \lambda_n \psi_n(\xi) \sum_{k=0}^{P} \beta_k(t)\psi_k(\xi)\psi_j(\xi)p_\xi(\xi)d\xi = 0 \tag{16}$$

which can be re-organized as the following set of $P+1$ ODEs:

$$\frac{d\beta_j}{dt} = -\frac{1}{\gamma_j} \sum_{n=0}^{1} \sum_{k=0}^{P} \lambda_n \beta_k(t) e_{jnk} \tag{17}$$

where

$$\gamma_j = E[\psi_j^2(\xi)] = \int \psi_j^2(\xi) p_\xi(\xi) d\xi \tag{18}$$

and

$$e_{jnk} = \mathbb{E}[\psi_j(\xi)\psi_n(\xi)\psi_k(\xi)] = \int \psi_j(\xi)\psi_n(\xi)\psi_k(\xi)p_\xi(\xi)d\xi. \tag{19}$$

For the Hermite Polynomials, these terms take the following analytical form

$$\gamma_j = j! \tag{20}$$

and

$$e_{jnk} = \begin{cases} \dfrac{j!n!k!}{(s-j)!(s-n)!(s-k)!}, & 2s = j+n+k \text{ is even and } s \geq j,n,k \\ 0, & \text{otherwise} \end{cases} \tag{21}$$

In vector form, the set of linear ODEs can be expressed as

$$\begin{aligned} \frac{d\boldsymbol{\beta}}{dt} &= \boldsymbol{A}\boldsymbol{\beta} \\ \boldsymbol{\beta}(0) &= \boldsymbol{Y}_0 \end{aligned} \tag{22}$$

where $\boldsymbol{\beta} = [\beta_0, \beta_1, \ldots, \beta_P]^T$ and $\boldsymbol{Y}_0 = [Y_0, 0, \ldots, 0]^T$ and $\boldsymbol{A} = [A_{jk}]$ has components

$$A_{jk} = -\frac{1}{\gamma_j}\sum_{n=0}^{1} \lambda_n e_{jnk} = -\frac{1}{\gamma_j}[\mu_X e_{j0k} + \sigma_X e_{j1k}] \tag{23}$$

This deterministic system of linear ODEs can be solved with standard numerical methods.

Recall from the previous module on PCE that the moments of the solution can be computed directly from the coefficients of the PCE. By solving the set of ODEs above, we determine the coefficients $\boldsymbol{\beta}(t)$. Recall then that, in general, the mean and variance of $\hat{Y}(t)$ can be expressed as

$$\mathbb{E}[\hat{Y}(t)] = \beta_0(t) \tag{24}$$

and

$$\mathrm{Var}(\hat{Y}(t)) = \sum_{k=1}^{P} \beta_k^2(t)\gamma_k \tag{25}$$

For this specific set of ODEs, the analytical mean and variance can be computed from Eq. (13) as

$$\begin{aligned} \mathbb{E}[Y(t)] &= \int Y_0 e^{-(\mu_X + \sigma_X \xi)t} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi \\ &= Y_0 e^{-\mu_X t} e^{\frac{\sigma_X^2 t^2}{2}} \end{aligned} \tag{26}$$

and

$$\begin{aligned} \mathrm{Var}[Y(t)] &= \mathbb{E}[Y^2(t)] - \mathbb{E}[Y(t)]^2 \\ &= e^{-2\mu_X t} Y_0^2 \left(e^{2\sigma_X^2 t^2} - e^{-\sigma_X^2 t^2} \right) \end{aligned} \tag{27}$$

These solutions, along with the PCE approximations are plotted for different values of P in Figure **??** Connor - How comfortable do you feel solving the above ODEs numerically and creating these plots?.

2.2 Boundary Value Problems and Elliptic PDEs

Consider now that we are interested in partial differential equations having, in general, random field parameters, source terms, and boundary operators. We will only discuss elliptic boundary value problems, although the principles of both stochastic Galerkin and stochastic collocation methods can be extended to evolution equations.

Consider the following set of PDEs:

$$\begin{aligned} \mathcal{L}(x, \xi, Y(x,\xi)) &= F(x,\xi), & x \in D, \xi \in \Omega \\ B(x, \xi, Y(x,\xi)) &= G(x,\xi), & x \in \partial D \end{aligned} \tag{28}$$

where $\mathcal{L}(\cdot)$ is a potentially nonlinear differential operator, $F(\cdot)$ is the source term, and $B(\cdot)$ and $G(\cdot)$ are the boundary operators.

Again, let us consider that we are interested in the expected value of solution

$$\mathbb{E}[Y(x)] = \int_\Omega Y(x,\xi) p_\xi d\xi \tag{29}$$

To solve the system, let us consider the *stochastic weak formulation* given by

$$\int_\Omega \int_D \mathcal{L}(Y,\xi) S(v(x)) z(\xi) p_\xi(\xi) dx d\xi = \int_\Omega \int_D F(\xi) v(x) z(q) p_\xi(\xi) dx d\xi \tag{30}$$

where $S(\cdot)$ is a linear operator, v and z are test functions. We approximate the solution by expanding $Y(x,\xi)$ over spatial and stochastic basis functions $\{\phi_j(x)\}_{j=1}^J$ and $\{\Psi_k(\xi)\}_{k=0}^K$ where ϕ_j are typically chosen as splines, finite elements, or spectral functions. We use the spectral functions for Ψ_k such that the solution takes the following approximate form

$$\begin{aligned} \hat{Y}(x,\xi) &= \sum_{k=0}^K \beta_k(x) \Psi_k(\xi) \\ &= \sum_{k=0}^K \sum_{j=1}^J \beta_{jk} \phi_j(x) \Psi_k(\xi) \end{aligned} \tag{31}$$

In the *stochastic Galerkin method*, we project the residuals for the approximate solution on to the space of the test functions as

$$\int_\Omega \int_D \mathcal{L}\left(\sum_{k=0}^K \sum_{j=1}^J \beta_{jk} \phi_j(x) \Psi_k(\xi), \xi \right) S(\phi_l(x)) \Psi_i(q) p_\xi(\xi) dx d\xi = \int_\Omega \int_D F(\xi) \phi_l(x) \Psi_i(\xi) p_\xi(\xi) dx d\xi \tag{32}$$

This integral equation is solved by performing numerical quadrature on the stochastic space and applying appropriate numerical approximations across the spatial domain (e.g. finite elements). Discretizing the stochastic domain by a set of M quadrature points $\xi^{(r)}$ and associated weights $w^{(r)}$ yields the approximation

$$\begin{aligned} \sum_{r=1}^R \Psi_i(\xi^{(r)}) p_\xi(\xi^{(r)}) w^{(r)} \int_D \mathcal{L}\left(\sum_{k=0}^K \sum_{j=1}^J \beta_{jk} \phi_j(x) \Psi_k(\xi^{(r)}), \xi^{(r)} \right) S(\phi_l(x)) dx \\ = \sum_{r=1}^R \Psi_i(\xi^{(r)}) p_\xi(\xi^{(r)}) w^{(r)} \int_D F(\xi^{(r)}) \phi_l(x) dx \end{aligned} \tag{33}$$

for $l = 1, \ldots, J$ and $i = 0, \ldots, K$ yielding a system of $J(K+1) \times J(K+1)$ coupled equations.

To estimate the mean value in Eq. (29), we employ the numerical scheme together with the determined coefficients β_{jk} as

$$\sum_{r=1}^R w^{(r)} p_\xi(\xi^{(r)}) \sum_{k=0}^K \sum_{j=1}^J \beta_{jk} \phi_j(x) \Psi_k(\xi^{(r)}) \tag{34}$$

Example

It is once again helpful to look at an example. Consider the heat equation given by

$$\begin{aligned} &\alpha \frac{d^2Y}{dx^2} = -f(x), \quad -1 < x < 1, \\ &Y(-1) = Y(1) = 0, \end{aligned} \tag{35}$$

where $\alpha \sim N(\mu_\alpha, \sigma_\alpha^2)$. Recall that the PCE for α is given exactly by

$$\alpha = \sum_{n=0}^{1} \alpha_n \Psi_n(\xi) = \mu_\alpha + \sigma_\alpha \xi \tag{36}$$

using the Hermite polynomials because $\Psi_0(\xi) = 1$, $\Psi_1(\xi) = \xi$, $\alpha_0 = \mu_\alpha$, $\alpha_1 = \sigma$, and $\alpha_n = 0, \forall n > 1$.

The operators $\mathcal{L}(Y, \xi)$ and $S(v)$ are therefore given by

$$\mathcal{L}(Y, \xi) = (\mu_\alpha + \sigma_\alpha \xi)\frac{dY}{dx}, \qquad S(v) = \frac{dv}{dx} \tag{37}$$

and the stochastic weak formulation is given by

$$\int_{\mathbb{R}} \int_{-1}^{1} (\mu_\alpha + \sigma_\alpha \xi) \frac{dY}{dx} \frac{dv}{dx} z(\xi) p_\xi(\xi) dx d\xi = \int_{\mathbb{R}} \int_{-1}^{1} f(x) v(x) z(\xi) p_\xi(\xi) dx d\xi \tag{38}$$

We then define the spatial basis using a uniform grid with step size $\Delta x = \dfrac{2}{J}$ with

$$\phi_j = \frac{1}{\Delta x} \begin{cases} x - x_{j-1} & , \quad x_{x-1} \leq x < x_j, \\ x_{j+1} - x & , \quad \leq x < x_{j+1}, \\ 0 & , \quad \text{otherwise} \end{cases} \tag{39}$$

where $j = 1, \ldots, J-1$ to enforce the boundary conditions. The solution is then discretized using the Hermite polynomials on the (normal) random space by

$$\hat{Y}(x, \xi) = \sum_{k=0}^{P} \sum_{j=1}^{J-1} \beta_{jk} \phi_j(x) \Psi_k(\xi) \tag{40}$$

Applying this approximation to the weak stochastic formulation yields the following discretized problem

$$\begin{aligned} \int_{\mathbb{R}} (\mu_\alpha + \sigma_\alpha \xi) \sum_{k=0}^{P} \sum_{j=1}^{J-1} \beta_{jk} \phi_j(x) \Psi_k(\xi) \left[\int_{-1}^{1} \phi_j'(x) \phi_l'(x) dx \right] \Psi_i(\xi) p_\xi(\xi) d\xi \\ = \int_{\mathbb{R}} \left[\int_{-1}^{1} f(x) \phi_l'(x) dx \right] \Psi_i(\xi) p_\xi(\xi) d\xi \end{aligned} \tag{41}$$

for $l = 1, \ldots, J$ and $i = 0, \ldots, P$. Next, we define

$$\Phi_{j,l} = \int_{-1}^{1} \phi_j'(x) \phi_l'(x) dx = \frac{1}{\Delta x} \begin{cases} 2 & , \quad j = l \\ -1 & , \quad j = l-1 \text{ or } j = l+1 \\ 0 & , \quad \text{otherwise} \end{cases} \tag{42}$$

and

$$f_l = \int_{-1}^{1} f(x) \phi_l'(x) dx \tag{43}$$

and express the discretized problem as

$$\sum_{j=1}^{J-1} \Phi_{j,l} \sum_{k=0}^{P} \beta_{jk} \int_{\mathbb{R}} (\mu_\alpha + \sigma_\alpha \xi) \Psi_k(\xi) \Psi_i(\xi) p_\xi(\xi) d\xi = f_l \int_{\mathbb{R}} \Psi_i(\xi) p_\xi(\xi) d\xi \tag{44}$$

Given the orthogonality of the Hermite polynomials and the relations from Eq. (19) with $n = k = 0$, i.e.

$$e_{l00} = \int_{\mathbb{R}} \Psi_l(\xi) p_\xi(\xi) d\xi = \begin{cases} 1 & , \quad l = 0, \\ 0 & , \quad \text{otherwise} \end{cases} \tag{45}$$

and with $i = 1$, i.e.

$$e_{1nk} = \int_{\mathbb{R}} \xi \Psi_n(\xi) \Psi_k(\xi) p_\xi(\xi) d\xi \tag{46}$$

we can construct an explicit set of discretized equations, from which we solve for the coefficients β_{jk}.

2.3 Evolutionary PDEs

All of the concepts presented above extend naturally to evolutionary PDEs. We will not explore this explicitly due to the complexity of the formulation. Nonetheless, the same principles hold.

2.4 Advantages and Disadvantages of Stochastic Galerkin Method

The stochastic Galerkin method has certain advantages related to its numerical construction.

- The stochastic Galerkin method uses the projection of the residual onto the space of approximating polynomials. In this way, the approximations are optimal in an L^2 sense.
- Convergence of the stochastic Galerkin method has been rigorously studied and documented in works such as [2] and [3].
- Since the stochastic Galerkin method is quadrature based, it can benefit from the rich body of work on sparse grid quadrature and cubature, which are beyond the scope of this work.

The stochastic Galerkin method also has several disadvantages when compared to other methods, including other spectral stochastic methods.

- The main disadvantage is that the formulations establishes a set of equations that couple the stochastic and spatio-temporal parameters. Consequently, the method is *intrusive* in nature. This means that custom solvers must be developed and that we cannot use existing deterministic solvers. This can be prohibitive for many practical applications where existing deterministic codes are well-established in the state of practice. For certain linear problems the system of equations can be decoupled, but these are special cases and generally do not cover many applications of practical interest.
- The stochastic Galerkin mehod is also highly sensitive to the problem dimension and suffers from the *curse of dimensionality* wherein the size of the system grows exponentially with the dimension of the stochastic parameter space. A great deal of work has been performed to reduce this dependency though sparse grid quadrature methods, but the problem persists.
- The stochastic Galerkin method strictly requires that all inputs be independent. In certain cases, this can be alleviated through the use of the Nataf or Rosenblatt transformations.
- Finally, the method can only be employed for parameters having certain distributions that follow the Weiner-Askey scheme because the scheme relies strongly on the orthogonality of the polynomials with respect to the distribution $p_\xi(\xi)$.

3 Stochastic Collocation Method

The stochastic collocation method operates in very similar manner to the projection and regression-based PCE surrogate model formulations in the previous modules, but with certain convenient mathematical devices employed. We start by making the assumption that the solution of the system is equal to its PCE expansion at a set of M samples – the so-called collocation points, $\{\xi^{(m)}\}_{m=1}^{M}$ – as follows

$$Y(t,x,\xi^{(m)}) = \hat{Y}(t,x,\xi^{(m)}) \tag{47}$$

Using this equality, we construct a linear system of equations to solve for the coefficients β_k of the PCE as

$$\begin{bmatrix} \Psi_0(\xi^{(1)}) & \cdots & \Psi_K(\xi^{(1)}) \\ \vdots & & \vdots \\ \Psi_0(\xi^{(M)}) & \cdots & \Psi_K(\xi^{(M)}) \end{bmatrix} \begin{bmatrix} \beta_0(x,t) \\ \vdots \\ \beta_K(x,t) \end{bmatrix} = \begin{bmatrix} Y(t,x,\xi^{(1)}) \\ \vdots \\ Y(t,x,\xi^{(M)}) \end{bmatrix} \tag{48}$$

This is very similar to the regression-based approach except that we enforce the equality in Eq. (47), whereas in the regression tasks we do not require that our PCE serve strictly as an interpolator. To avoid the equations being underdetermined, we require that $M \geq K+1$.

The two main features that make collocation unique and convenient are the selection of the collocation points and the selection of the basis polynomials. For polynomial interpolation, uniform grids of points cause spurious oscillations – referred to as *Runge's phenomenon* which is similar to the well-known *Gibbs phenomenon* in Fourier series expansions. Next, the collocation method cleverly selects the Lagrange polynomials as basis functions, $\Psi_k(\xi) = L_k(\xi)$, satisyfing

$$L_k(\xi^{(m)}) = \delta_{km}. \tag{49}$$

That is, the k^{th} basis function evaluated at the m^{th} collocation point is equal to unity and is equal to zero at all other collocation points. This results in a collocation matrix $\mathbf{\Psi}$ that is equal to the identity matrix when $M = K+1$ collocation points are used. Consequently,

$$\beta_m(x,t) = Y(x,t,\xi^{(m)}) \tag{50}$$

Other choices of polynomials lead to dense collocation matrices that must be inverted.

We note, however, that when employing stochastic collocation in this way we have broken from the Weiner-Askey scheme of the generalized PCE. Consequently, we lose the orthonality of the basis with respect to the probability measure of ξ, $p_\xi(\xi)$.

3.1 Advantages and Disadvantages of Stochastic Collocation

The stochastic collocation method has a lot in common with both the regression/projection-based surrogate modeling methods discussed in the previous module and the stochastic Galerkin method. On one hand, stochastic collocation can be viewed as a special case of the stochastic Galerkin approach in which the basis functions and test functions are selected as the Lagrange polynomials. Selecting the quadrature points as collocation points then allows the physical and random parameters to be decoupled. However, it differs from the stochastic Galerkin approach in that it is an interpolation-based approach rather than a projection-based approach. The strict interpolation assumptions makes is similar to the regression based methods wherein the interpolation assumption can provide both some numerical advantages as well as some drawbacks in comparison to the surrogate modeling approach.

The main advantage of the stochastic collocation approach, resulting from the interpolation-based decoupling of physical and random parameters, is that it is *non-intrusive*. That is, standard deterministic

solvers can be used to solve the system of equations at each collocation point. In this way, it differs from the stochastic Galerkin method. The stochastic Galerkin method requires solving a *single system* of size $J(K+1)$, whereas the stochastic collocation methods requires solving M deterministic systems of size J.

The stochastic collocation method can also be employed for problems where distributions do not follow those from the Weiner-Askey scheme. The scheme does not rely on the orthogonality of the polynomials with respect to the probability distribution. Furthermore, it can be applied for problems with random variables that are not independent. However, evaluation of moments and other quantities of interest will require samples from $p_\xi(\xi)$.

Finally, the stochastic collocation method is not generally as accurate as the stochastic Galerkin method and suffers also suffers from the curse of dimensionality such that it becomes less efficient than the Monte Carlo method in high dimensions. Further details on the theory of stochastic collocation for various classes of problems can be found in [4–6]

References

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Nomenclature

Functions

$\mathbb{E}[\cdot]$	Expected value of a random variable. Also denoted $\mu_X \triangleq E[X]$
δ_{jk}	Kronecker delta function
$\mathcal{L}(\cdot)$	Differential Operator
$\Psi_{\boldsymbol{\alpha}}(\cdot)$	Multivariate orthogonal polynomials
$L_k(\cdot)$	The k-th Lagrange polynomial
$p_\xi(\xi)$	Probability density function of a random variable ξ

Operators

$\langle \cdot, \cdot \rangle$	Inner product

Variables

$\boldsymbol{\beta}$	Vector of regression coefficients
$\boldsymbol{\xi}$	Germ
$\boldsymbol{\alpha}$	Multi-index
ϵ	Truncation error Check if epilson is used for error elsewhere
$\mathcal{A}^{M,p}$	Multi-index set
σ_X	Standard deviation of the random variable X
$\xi^{(m)}$	Collocation point