USE OF PROPORTION IN THE OLD HALL MANUSCRIPT: AN ANALYTICAL APPROACH

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MUS 7921(2): Music and Mathematics
Tuesday November 21, 2011

"It is the duple proportion in which the larger number compared to the smaller contains the latter exactly twice in itself, as 2 to 1, 4 to 2, 6 to 3. … The Pythagoreans say that the diapason [i.e., the octave] originates from this proportion, because, if the Greeks are to be believed, Pythagoras in his discovery of the concords saw that the diapason was made from two hammers, of which one was of six pounds, the other of twelve, sounding that concord; whence it happens that he and many others call it a duple consonance and this proportion conversely the diapason. This particular proportion is customarily used more than the others."

Johannes Tinctoris in Chapter V of *Proportionale Musices* (Seay, 1979)

Pythagorean proportions were the principal topic for discussion in music theory treatises from the time of the ancient Greeks through the Renaissance. As Tinctoris points out, though, the discussion of musical proportion shifted from a theoretical pursuit of the harmonic consonances to the application of temporal relations in composition that peaked in the mid fifteenth-century. The ninth-century *Scolica enchiriadis* shows the use of the *tetractys* and *heptactys* – the sets {6, 8, 9, 12} and {6, 8, 9, 12, 16, 18, 24} respectively – and their relation to all the harmonic consonances. These same proportions are used by Boethius and the *heptactys* continues to be found in music treatises in this manner through the eleventh-century. (Carey and Clampitt, 1996)

In their article *Regions: A Theory of Tonal Spaces in Early Medieval Treatises* (1996), Carey and Clampitt layout the mathematical bounds from which the sets of

Medieval consonances are born and demonstrate how these mathematical definitions inform an analysis of the music of this time. However – as Tinctoris' laconic words show – since musical proportion in the Renaissance was used to explain relations between notes in time rather than in pitch, Carey and Clampitt's analytical tools have a broader application than proposed in their article.

Mensural notation (the use of signum to denote the temporal relations between individual and groups of notes) did not begin in the Renaissance. De Vitry began the use of mensuration in his fourteenth-century treatise, a hundred and fifty years before Tinctoris codifies them (Bent, [2011b]). The application of these methods are widespread. Isorhythm, used in both mass and motet in the fourteenth- and fifteenth-centuries, is one clear demonstration of the way Pythagorean proportion was used to govern the entire formal construction of music (Bent, [2011a]).

To demonstrate how Carey and Clampitt's mathematical analysis can be applied to temporal relations, I selected to analyze the 21 isorhythmic pieces found in the Old Hall Manuscript (London, British Library 57950). I worked from the modern edition by Hughes and Bent (1969-73) which includes 2 volumes of the music set in modern notation plus 1 volume of commentary. A complete list of the works analyzed is found in **Table 1** below. I chose to work with Old Hall for its breadth of representative isorhythmic works. Compiled between 1415 and the early 1420's, the manuscript brings together works by over 20 different composers (Bent, [2011c]). For these qualities, it represents a well-preserved snap shot of its compositional time. In working with the edition, I accepted all editorial comments and changes unless otherwise noted here. Hughes and

Bend layout their editorial process in the preface to the first volume. Though transcribed to modern notation, the compilation is without time signatures. They explain how the rhythmic unit that should be used as the "beat" (i.e. a quarter-note, dotted quarter-note, half-note, etc.) is indicated where not readily apparent. In performance, the eighth-note is held constant throughout a piece even if this forces a change to the relative speed of the "beat".

Table 2 gives a diagram of the proportional layout of each isorhythmic piece in the format proposed by Turner (1991). Horizontal distances show time and are draw to proportional scale. Each vertical layer is labeled and represents the many different proportional tools used by each composer. This table also shows the isorhythmic layout of each piece as given in the Hughes and Bent edition. Uppercase roman numerals represent instances of the tenor *color* and lowercase letters show instances of tenor *talea*.

Below each diagram in **Table 2** is an extraction of all proportions used in its composition. Though this goes against Turner's analytical methods, allowing for comparison of all voices and parts of a piece to each other, it is more in line with the proportional analysis of Carey and Clampitt. I define $O_{(n,p)}$ as the set of all elements of the extracted proportions of isorhythmic piece No. *n* (see **Table 1** for manuscript numbers) in lowest possible terms. I give *p* as the number of distinct prime numbers needed to generate each element of $O_{(n,p)}$. For example, the set of proportions representing Dunstable's *Veni Sancte Spiritus* is labeled $O_{(66,2)}$.

Following from Carey and Clampitt, two analytical techniques were applied to these sets. First, a script was used to generate an ordered list of integers generated from distinct prime generating factors of each $O_{(n,p)}$. See **Table 3**. Second, these integers were arranged graphically; powers of the first prime generator along the horizontal axis and powers of the second prime generator along the vertical axis. Those numbers found in $O_{(n,p)}$ were shaded. See **Table 4**.

With the three trivial examples aside (Nos. 66, 113, and 145), each $O_{(n,2)}$ is not a connected subset of $\hat{P}$ by the definitions given by Carey and Clampitt. That is, given $\hat{P} = \left\{\hat{p} = A^a B^b | a, b \in J = \{0,1,2, \ldots\}\right\}$ where *A* and *B* are the two distinct generating primes of $O_{(n,2)}$ and o_a and $o_b \in O_{(n,2)}$, if $o_a < \hat{p} < o_b$ then $\hat{p}$ is not necessarily also $\in O_{(n,2)}$. I was also unable to find any connection between the pure primes A^a and B^b and the formation of the $O_{(n,2)}$ sets. Graphic analysis on the other hand showed a close relation between the $O_{(n,2)}$ sets and Carey and Clampitt's Regions. Comparing **Fig. A** (the graphic analysis of the Region defining the *heptactys*) to the graphs of **Table 4**, it is clear that all are symmetrical in some way and form a single connected geometric shape.

These results are helpful in an analysis of the isorhythm in the Old Hall Manuscript because they explain how the use of Pythagorean proportions affected each composer's compositional choices. The first composition technique this analysis shows is the use of mensuration as a way to change the proportional relationship between *talea* statements. For example, No. 86 uses mensuration changes on the same basic *talea* to bring more complex relationships between statements by changing the subdivision of the semi-breve. By removing the dot from the mensuration sign ⋶ the prolation changes from perfect to imperfect and thus the semi-breve divides imperfectly into two minims where it had previously divided perfectly into three (Bent, [2011b]). In a second example, {9, 6, 4} is a subset of both $O_{(112,2)}$ and $O_{(143,2)}$. Though on the surface this proportion seems difficult, the progression through the mensuration signs ⊙, ○, and ⊂ of the music itself give these proportions as the number of minims in each section (Hughes and Bent, 1969-73).

This analysis also shows how composers in the Old Hall brought variety to their tenor parts. In Nos. 24 and 84, the tenor *color* and *talea* do not coincide. No. 90 uses the tenor *color* in retrograde in *color* statements 2 and 5. Other pieces use small variations in *talea* statements to further delineate formal proportions. No. 89 presents two statements of the tenor *color*, each statement then divided into 10 *talea* in the tenor and countertenor voices for a total of 20 *talea* statements. Statements 5 and 10, which mark the halfway point of each *color*, have a slightly different rhythm in the isorhythmic countertenor part while statements 8-9 and 18-19 add an extra slur to the tenor *talea*, thus aiding to mark the close of each *color* statement.

The analysis of the Old Hall showed the great grasp of proportional techniques used by English composers. Many of them used proportional relationships above the standard use of *color* and *talea* in the tenor. An alternation of sung text and held notes appears in the upper voices of both Nos. 30 and 89. In the latter example, the text statements alternate between the two uppermost voices while the other holds long notes. These alternations group statements of the tenor *talea* into pairs. The use of mensuration to divide the *talea* into further proportional sections is demonstrated in Nos. 19, 23, 86[1], and 88. In each, there is a mensuration change at a proportional location in the tenor, thus when the quarter-note is held constant as instructed, the time spent in one mensuration is proportionate to the time spent in another. Lastly, No. 90 adds the unique use of a canonic motto opening, which stands in a 3 to 2 relationship to the first tenor *talea*.

Finally, this analysis gives a quantifiable measure of the complexity of each isorhythmic piece $O_{(n,p)}$. A complex isorhythm has a larger value for p and larger cardinality of $O_{(n,p)}$ than one of simpler isorhythmic structure. Generally, it was found that the simpler pieces are in the newest layers of the manuscript (in other words, those with a larger value for n). Similarly, those with a higher level of complexity seem to be found in the oldest layer.

To conclude, the numeric proportions chosen by the composers of the Old Hall Manuscript were specifically chosen for their strong correlation with the Pythagorean proportions. To them, this was the next logical step in the expression of proportion in music. By their time, proportional consonances had been fully discussed, so composers began looking for new ways to apply Greek thought to their music and found isorhythm as the next embodiment of this ideal. As more is uncovered and learned about these remarkable works, the application of the analytical methods demonstrated in this paper will be invaluable. For the Old Hall in particular, further analysis can help bring forth new

[1] Hughes and Bent mark No. 86 with two *color* statements and two *talea* statements, leaving the mensuration changes within each *talea* as a way in which the composer adds further division. However, as seen above, I disagree with this reading and have interpreted these mensural changes as new *talea* statements, giving a total of four total statements of *talea*.

mass pairs. Hughes and Bent have already proposed matching Nos. 19 and 23. From the data collected here, a closer look at No. 88 in relation to Nos. 19 and 23 will be fruitful. Looking to **Table 2**, all three have similar isorhythmic structure. Furthermore, each uses mensuration changes in the middle of a *talea* statement as a means of further proportional delineation.

Table 1
Isorhythmic works in the Old Hall Manuscript as compiled from Hughes and Bent (1969-73)

MS No.	*Title*	*Composer*
19	Gloria	J. Tyes
23	Gloria	Leonel
24	Gloria	Anon.
28	Gloria spiritus et alme	Pycard
30	Gloria spiritus et alme	Queldryk
66	Veni Sancte Spiritus	(Dunstable)
84	Credo	Leonel
85	Credo	Anon.
86	Credo	Swynford
87	Credo	Typp
88	Credo	Qweldryk
89	Credo	Pennard
90	Credo	Anon.
111	Salvatoris mater/O Georgi	Damett
112	Alma proles/Christi miles	Cook
113	Salve mater/Salve templum	Sturgeon
143	Carbunculus ignitus	Anon.
144	Mater munda	Anon.
145	En Katerine/Virginalis	Biteryng
146	Are post libamina/Nunc surgunt	Mayshuet
147	Post missarum	Anon.

Table 2

Source: Hughes and Bent (1969-73) unless otherwise stated

**denotes a part that is missing from the current manuscript, isorhythmic structure is thus based on hypothesis.*

No. 19

I a^1 a^2 II b^1 b^2

<table>
<tr><td colspan="4">36</td><td colspan="4">18</td><td>tenor color</td></tr>
<tr><td colspan="2">18</td><td colspan="2">18</td><td colspan="2">9</td><td colspan="2">9</td><td>tenor talea, tr1 and tr2 talea</td></tr>
<tr><td>10</td><td>8</td><td>10</td><td>8</td><td>5</td><td>4</td><td>5</td><td>4</td><td>mensuration changes</td></tr>
</table>

$O_{(19,3)} = \{4, 5, 8, 9, 10, 18, 36\}$

No. 23

I a^1 a^2 II b^1 b^2

<table>
<tr><td colspan="4">20</td><td colspan="4">10</td><td>tenor color</td></tr>
<tr><td colspan="2">10</td><td colspan="2">10</td><td colspan="2">5</td><td colspan="2">5</td><td>tenor talea</td></tr>
<tr><td>6</td><td>4</td><td>6</td><td>4</td><td>3</td><td>2</td><td>3</td><td>2</td><td>mensuration changes</td></tr>
</table>

$O_{(23,3)} = \{ 2, 3, 4, 5, 10, 20\}$

No. 24

a^1 a^2 b^1 b^2

First tenor *color* enters at bar 1, second tenor *color* enters at bar 77, sixteen bars before the end of a^2 and the beginning of b^1.

$O_{(24,0)} = \{\}$

No. 28

${}_1$I a^1 a^2 ${}_2$I a^3 a^4 ${}_1$II b^1 b^2 ${}_2$II b^3 b^4 ${}_1$III c^1 c^2 ${}_2$III c^3 c^4 ${}_1$IV d^1 d^2 ${}_2$IV d^3 d^4

<table>
<tr><td colspan="4">48</td><td colspan="4">36</td><td colspan="4">32</td><td colspan="4">24</td><td>upper voices talea</td></tr>
<tr><td colspan="2">24</td><td colspan="2">24</td><td colspan="2">18</td><td colspan="2">18</td><td colspan="2">16</td><td colspan="2">16</td><td colspan="2">12</td><td colspan="2">12</td><td>tenor color</td></tr>
<tr><td>12</td><td>12</td><td>12</td><td>12</td><td>9</td><td>9</td><td>9</td><td>9</td><td>8</td><td>8</td><td>8</td><td>8</td><td>6</td><td>6</td><td>6</td><td>6</td><td>tenor talea</td></tr>
</table>

$O_{(28,2)} = \{6, 8, 9, 12, 16, 18, 24, 32, 36, 48\}$

No. 30
I a^1 a^2 II b^1 b^2

<table>
<tr><td colspan="8">36</td><td colspan="4">18</td><td>tenor and countertenor color</td></tr>
<tr><td colspan="4">18</td><td colspan="4">18</td><td colspan="2">9</td><td colspan="2">9</td><td>tenor talea</td></tr>
<tr><td colspan="2">9</td><td colspan="2">9</td><td colspan="2">9</td><td colspan="2">9</td><td colspan="2">9</td><td colspan="2">9</td><td>text alternation patterning</td></tr>
<tr><td>4</td><td>5</td><td>4</td><td>5</td><td>4</td><td>5</td><td>4</td><td>5</td><td>4</td><td>5</td><td>4</td><td>5</td><td>text alternation proportion</td></tr>
</table>

$O_{(30,3)} = \{4, 5, 9, 18, 36\}$

No. 66
I a^1 a^2 II b^1 b^2 III c^1 c^2

<table>
<tr><td colspan="2">6</td><td colspan="2">4</td><td colspan="2">2</td><td>tenor color</td></tr>
<tr><td>3</td><td>3</td><td>2</td><td>2</td><td>1</td><td>1</td><td>tenor talea</td></tr>
</table>

$O_{(66,2)} = \{1, 2, 3, 4, 6\}$

No. 84
a^1 a^2 a^3
First tenor *color* enters at bar 1, second tenor *color* enters at bar 131 but is not a strict repeat of pitches.
$O_{(84,1)} = \{1\}$

No. 85
After an opening rest, the tenor and countertenor enter with 19 *talea* statements.
$O_{(85,1)} = \{1\}$

No. 86
Hughes and Bent give the form as I a II b. However, each tenor *color* is divided in two proportionately by mensuration changes. I propose instead describing the form as I a b II c d.

<table>
<tr><td colspan="2">10</td><td colspan="2">5</td><td>tenor color</td></tr>
<tr><td>6</td><td>4</td><td>3</td><td>2</td><td>tenor talea</td></tr>
</table>

$O_{(86,3)} = \{2, 3, 4, 5, 6, 10\}$

No. 87

I a^1 a^2 a^3 a^4 II b^1 b^2 b^3 b^4 III c^1 c^2 c^3 c^4

<table>
<tr><td colspan="4">48</td><td colspan="4">32</td><td colspan="4">12</td><td>tenor color</td></tr>
<tr><td colspan="2">24</td><td colspan="2">24</td><td colspan="2">16</td><td colspan="2">16</td><td colspan="2">6</td><td colspan="2">6</td><td>upper voices talea</td></tr>
<tr><td>12</td><td>12</td><td>12</td><td>12</td><td>8</td><td>8</td><td>8</td><td>8</td><td>3</td><td>3</td><td>3</td><td>3</td><td>tenor talea</td></tr>
</table>

$O_{(87,2)} = \{3, 6, 8, 12, 16, 24, 32, 48\}$

No. 88

I a^1 a^2 II b^1 b^2

<table>
<tr><td colspan="4">12</td><td colspan="4">6</td><td>tenor color</td></tr>
<tr><td colspan="2">6</td><td colspan="2">6</td><td colspan="2">3</td><td colspan="2">3</td><td>tenor and upper voice talea</td></tr>
<tr><td>4</td><td>2</td><td>4</td><td>2</td><td>2</td><td>1</td><td>2</td><td>1</td><td>mensuration changes</td></tr>
</table>

$O_{(88,2)} = \{1, 2, 3, 4, 6, 12\}$

No. 89

${}_1$I a^1 a^2 a^3 a^4 a^5 a^6 a^7 a^8 a^9 a^{10} ${}_2$I a^{11} a^{12} a^{13} a^{14} a^{15} a^{16} a^{17} a^{18} a^{19} a^{20}

<table>
<tr><td colspan="10">10</td><td colspan="10">10</td><td>tenor and countertenor color</td></tr>
<tr><td colspan="5">5</td><td colspan="5">5</td><td colspan="5">5</td><td colspan="5">5</td><td>small changes to countertenor talea</td></tr>
<tr><td colspan="2">2</td><td colspan="2">2</td><td colspan="2">2</td><td colspan="2">2</td><td colspan="2">2</td><td colspan="2">2</td><td colspan="2">2</td><td colspan="2">2</td><td colspan="2">2</td><td colspan="2">2</td><td>text alternation</td></tr>
<tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>tenor and countertenor talea</td></tr>
</table>

$O_{(89,2)} = \{1, 2, 5, 10\}$

No. 90

${}_1$I a^1 a^2 a^3 ${}_2$I$_{\text{retrograde}}$ b^1 b^2 b^3 ${}_3$I a^4 a^5 a^6 ${}_1$II c^1 c^2 c^3 ${}_2$II$_{\text{retrograde}}$ d^1 d^2 d^3 ${}_3$II c^4 c^5 c^6

<table>
<tr><td colspan="2"></td><td colspan="3">18</td><td colspan="3">18</td><td colspan="3">18</td><td colspan="3">9</td><td colspan="3">9</td><td colspan="3">9</td><td>tenor color</td></tr>
<tr><td colspan="2"></td><td>6</td><td>6</td><td>6</td><td>6</td><td>6</td><td>6</td><td>6</td><td>6</td><td>6</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>tenor talea</td></tr>
<tr><td colspan="2">8</td><td colspan="18"></td><td>triplex 1 motto canon</td></tr>
<tr><td></td><td>4</td><td colspan="18"></td><td>triplex 2 motto canon</td></tr>
</table>

$O_{(90,2)} = \{3, 4, 6, 8, 9, 18\}$

No. 111
I a^1 a^2 II b^1 b^2 III c^1 c^2 IV d^1 d^2

<table>
<tr><td colspan="2">12</td><td colspan="2">6</td><td colspan="2">4</td><td colspan="2">2</td><td>tenor color</td></tr>
<tr><td>6</td><td>6</td><td>3</td><td>3</td><td>2</td><td>2</td><td></td><td></td><td>tenor and upper voice talea</td></tr>
</table>

$O_{(111,2)} = \{1, 2, 3, 4, 6, 12\}$

No. 112
I a^1 a^2 II b^1 b^2 III c^1 c^2

<table>
<tr><td colspan="2">18</td><td colspan="2">12</td><td colspan="2">8</td><td>tenor color</td></tr>
<tr><td>9</td><td>9</td><td>6</td><td>6</td><td>4</td><td>4</td><td>tenor and upper voice talea</td></tr>
</table>

$O_{(112,2)} = \{4, 6, 8, 9, 12, 18\}$

No. 113
I a^1 a^2 II b^1 b^2 III c^1 c^2

<table>
<tr><td colspan="2">6</td><td colspan="2">4</td><td colspan="2">2</td><td>tenor color</td></tr>
<tr><td>3</td><td>3</td><td>2</td><td>2</td><td>1</td><td>1</td><td>tenor talea</td></tr>
</table>

$O_{(113,2)} = \{1, 2, 3, 4, 6\}$

No. 143
I a^1 a^2 a^3 II b^1 b^2 b^3 III c^1 c^2 c^3

<table>
<tr><td colspan="3">27</td><td colspan="3">18</td><td colspan="3">12</td><td>tenor color*</td></tr>
<tr><td>9</td><td>9</td><td>9</td><td>6</td><td>6</td><td>6</td><td>4</td><td>4</td><td>4</td><td>tenor talea*</td></tr>
<tr><td>9</td><td>9</td><td>9</td><td>6</td><td>6</td><td>6</td><td>4</td><td>4</td><td>4</td><td>triplex talea</td></tr>
</table>

$O_{(143,2)} = \{4, 6, 9, 12, 18, 27\}$

No. 144
Only tenor and countertenor parts remain, I a^1 a^2 II b^1 b^2

<table>
<tr><td colspan="2">4</td><td colspan="2">2</td><td>tenor color</td></tr>
<tr><td>2</td><td>2</td><td>1</td><td>1</td><td>tenor and countertenor talea</td></tr>
</table>

$O_{(144,1)} = \{1, 2, 4\}$

No. 145

I a^1 a^2 II b^1 b^2 III c^1 c^2

<table>
<tr><td colspan="2">6</td><td colspan="2">4</td><td colspan="2">2</td><td>tenor color</td></tr>
<tr><td>3</td><td>3</td><td>2</td><td>2</td><td>1</td><td>1</td><td>tenor and upper voice talea</td></tr>
</table>

$O_{(145,2)} = \{1, 2, 3, 4, 6\}$

No. 146

I a^1 a^2 II b^1 b^2 III c^1 c^2 III c^3 c^4

<table>
<tr><td colspan="2">8</td><td colspan="2">4</td><td colspan="2">2</td><td colspan="2">2</td><td>tenor and countertenor color</td></tr>
<tr><td>4</td><td>4</td><td>2</td><td>2</td><td>1</td><td>1</td><td>1</td><td>1</td><td>all voices talea</td></tr>
</table>

$O_{(146,1)} = \{1, 2, 4, 8\}$

No. 147

Only the triplex and countertenor parts remain, they both follow the form,

I a^1 a^2 II b^1 b^2 III c^1 c^2

<table>
<tr><td colspan="2">8</td><td colspan="2">4</td><td colspan="2">2</td><td>countertenor color</td></tr>
<tr><td colspan="2">8</td><td colspan="2">4</td><td colspan="2">2</td><td>tenor color*</td></tr>
<tr><td>4</td><td>4</td><td>2</td><td>2</td><td>1</td><td>1</td><td>countertenor talea</td></tr>
<tr><td>4</td><td>4</td><td>2</td><td>2</td><td>1</td><td>1</td><td>tenor talea*</td></tr>
</table>

$O_{(147,1)} = \{1, 2, 4, 8\}$

Table 3
For each $O_{(n,p)}$, a list is first given of the union of all distinct primes factors of the set. What follows is an ordered list of all combinations of powers on these primes. Where capital letters represent the prime factors and lower case letters represent the power on that prime, each element is given in the form $\{a, b, c, A^a B^b C^c\}$. Elements greyed out in the list are not elements of $O_{(n,p)}$.

$O_{(19,3)}$ = {4, 5, 8, 9, 10, 18, 36}
(powers on 2, 3, 5)
{0, 0, 0, 1}
{1, 0, 0, 2}
{0, 1, 0, 3}
{2, 0, 0, 4}
{0, 0, 1, 5}
{1, 1, 0, 6}
{3, 0, 0, 8}
{0, 2, 0, 9}
{1, 0, 1, 10}
{2, 1, 0, 12}
{0, 1, 1, 15}
{4, 0, 0, 16}
{1, 2, 0, 18}
{2, 0, 1, 20}
{3, 1, 0, 24}
{0, 0, 2, 25}
{0, 3, 0, 27}
{1, 1, 1, 30}
{5, 0, 0, 32}
{2, 2, 0, 36}

$O_{(23,3)}$ = { 2, 3, 4, 5, 10, 20}
(powers on 2, 3, 5)
{0, 0, 0, 1}
{1, 0, 0, 2}
{0, 1, 0, 3}
{2, 0, 0, 4}
{0, 0, 1, 5}
{1, 1, 0, 6}
{3, 0, 0, 8}
{0, 2, 0, 9}
{1, 0, 1, 10}
{2, 1, 0, 12}
{0, 1, 1, 15}
{4, 0, 0, 16}
{1, 2, 0, 18}
{2, 0, 1, 20}

$O_{(28,2)}$ = {6, 8, 9, 12, 16, 18, 24, 32, 36, 48}
(powers on 2, 3)
{0, 0, 1}
{1, 0, 2}
{0, 1, 3}
{2, 0, 4}
{1, 1, 6}
{3, 0, 8}
{0, 2, 9}
{2, 1, 12}
{4, 0, 16}
{1, 2, 18}
{3, 1, 24}
{0, 3, 27}
{5, 0, 32}
{2, 2, 36}
{4, 1, 48}

$O_{(30,3)}$ = {4, 5, 9, 18, 36}
(powers on 2, 3, 5)
{0, 0, 0, 1}
{1, 0, 0, 2}
{0, 1, 0, 3}
{2, 0, 0, 4}
{0, 0, 1, 5}
{1, 1, 0, 6}
{3, 0, 0, 8}
{0, 2, 0, 9}
{1, 0, 1, 10}
{2, 1, 0, 12}
{0, 1, 1, 15}
{4, 0, 0, 16}
{1, 2, 0, 18}
{2, 0, 1, 20}
{3, 1, 0, 24}
{0, 0, 2, 25}
{0, 3, 0, 27}
{1, 1, 1, 30}
{5, 0, 0, 32}
{2, 2, 0, 36}

$O_{(66,2)}$ = {1, 2, 3, 4, 6}
(powers on 2, 3)
{0, 0, 1}
{1, 0, 2}
{0, 1, 3}
{2, 0, 4}
{1, 1, 6}

$O_{(86,3)}$ = {2, 3, 4, 5, 6, 10}
(powers on 2, 3, 5)
{0, 0, 0, 1}
{1, 0, 0, 2}
{0, 1, 0, 3}
{2, 0, 0, 4}
{0, 0, 1, 5}
{1, 1, 0, 6}
{3, 0, 0, 8}
{0, 2, 0, 9}
{1, 0, 1, 10}

$O_{(87,2)} = \{3, 6, 8, 12, 16, 24, 32, 48\}$
(powers on 2, 3)
{0, 0, 1}
{1, 0, 2}
{0, 1, 3}
{2, 0, 4}
{1, 1, 6}
{3, 0, 8}
{0, 2, 9}
{2, 1, 12}
{4, 0, 16}
{1, 2, 18}
{3, 1, 24}
{0, 3, 27}
{5, 0, 32}
{2, 2, 36}
{4, 1, 48}

$O_{(88,2)} = \{1, 2, 3, 4, 6, 12\}$
(powers on 2, 3)
{0, 0, 1}
{1, 0, 2}
{0, 1, 3}
{2, 0, 4}
{1, 1, 6}
{3, 0, 8}
{0, 2, 9}
{2, 1, 12}

$O_{(89,2)} = \{1, 2, 5, 10\}$
(powers on 2, 5)
{0, 0, 1}
{1, 0, 2}
{2, 0, 4}
{0, 1, 5}
{3, 0, 8}
{1, 1, 10}

$O_{(90,2)} = \{3, 4, 6, 8, 9, 18\}$
(powers on 2, 3)
{0, 0, 1}
{1, 0, 2}
{0, 1, 3}
{2, 0, 4}
{1, 1, 6}
{3, 0, 8}
{0, 2, 9}
{2, 1, 12}
{4, 0, 16}
{1, 2, 18}

$O_{(111,2)} = \{1, 2, 3, 4, 6, 12\}$
(powers on 2, 3)
{0, 0, 1}
{1, 0, 2}
{0, 1, 3}
{2, 0, 4}
{1, 1, 6}
{3, 0, 8}
{0, 2, 9}
{2, 1, 12}

$O_{(112,2)} = \{4, 6, 8, 9, 12, 18\}$
(powers on 2, 3)
{0, 0, 1}
{1, 0, 2}
{0, 1, 3}
{2, 0, 4}
{1, 1, 6}
{3, 0, 8}
{0, 2, 9}
{2, 1, 12}
{4, 0, 16}
{1, 2, 18}

$O_{(113,2)} = \{1, 2, 3, 4, 6\}$
(powers on 2, 3)
{0, 0, 1}
{1, 0, 2}
{0, 1, 3}
{2, 0, 4}
{1, 1, 6}

$O_{(143,2)} = \{4, 6, 9, 12, 18, 27\}$
(powers on 2, 3)
{0, 0, 1}
{1, 0, 2}
{0, 1, 3}
{2, 0, 4}
{1, 1, 6}
{3, 0, 8}
{0, 2, 9}
{2, 1, 12}
{4, 0, 16}
{1, 2, 18}
{3, 1, 24}
{0, 3, 27}

$O_{(144,1)} = \{1, 2, 4\}$
(powers on 2)
{0, 1}
{1, 2}
{2, 4}

$O_{(145,2)} = \{1, 2, 3, 4, 6\}$
(powers on 2, 3)
{0, 0, 1}
{1, 0, 2}
{0, 1, 3}
{2, 0, 4}
{1, 1, 6}

$O_{(146,1)} = \{1, 2, 4, 8\}$
(powers on 2)
{0, 1}
{1, 2}
{2, 4}
{3, 8}

$O_{(147,1)} = \{1, 2, 4, 8\}$
(powers on 2)
{0, 1}
{1, 2}
{2, 4}
{3, 8}

Table 4
Shown here are graphic representations of those $O_{(n,2)}$ whose distinct primes are 2 and 3 only. Powers increase on 2 horizontally and on 3 vertically.

9	8	36	72	144	288
3	6	12	24	48	96
1	2	4	8	16	32

No. 28

3	6	12
1	2	4

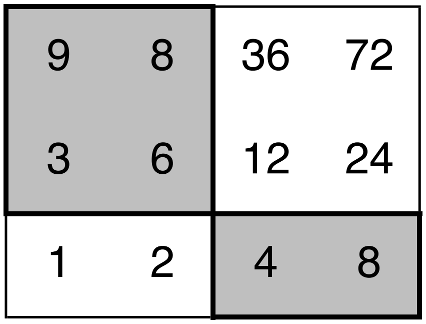
Nos. 66, 113, and 145

3	6	12	24	48	96
1	2	4	8	16	32

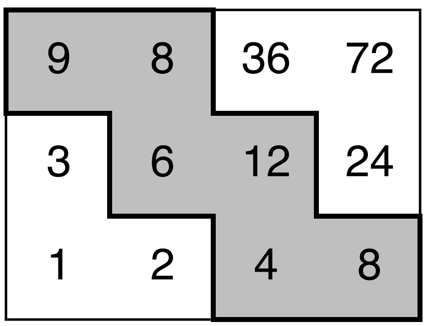
No. 87

3	6	12
1	2	4

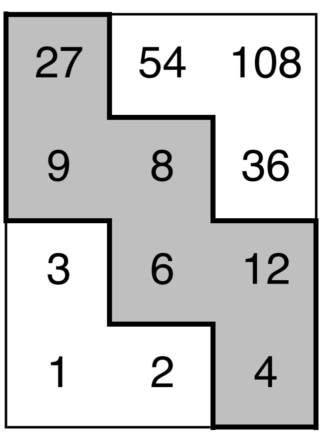
Nos. 88 and 111



No. 90

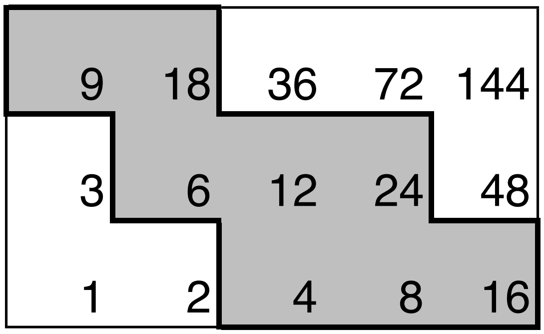


No. 112



No. 143

Fig. A
The figure shows a graph of the region $R_{(5,3)} = \{4, 6, 8, 9, 12, 16, 18, 24\}$ as shown in Carey and Clampitt (1996). The elements of this region are those of the *heptactys*, used by Medieval theorists to explain the harmonic concordances.



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