AI1110: Probability and Random Variables

Random Numbers

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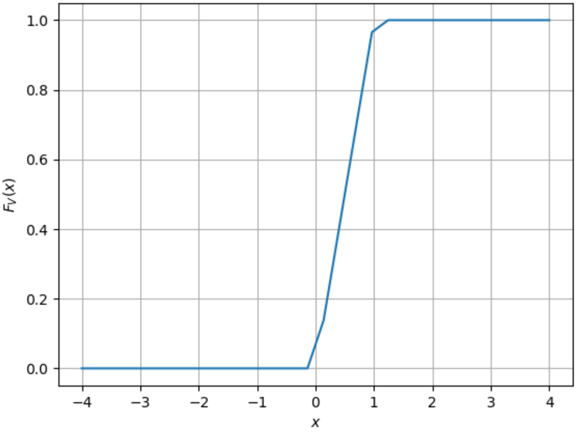
Outline

1.2: CDF of U - Python Plot



1.3: Probability Density of U

Given U is a uniformly distributed random variable over the interval $(0, 1)$, we have the density function $p_U(x)$:

$$p_U(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & otherwise \end{cases} \tag{1}$$

1.3: Getting $F_U(x)$

We know

$$F_U(x) = \int_{-\infty}^{x} p_U(x)\,dx \tag{2}$$

1.3: Theoretical Expression for Distribution of U

Given U is a uniformly distributed random variable over the interval $(0, 1)$, we have the following expression for $F_U(x)$:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{3}$$

1.5: Definitions

We are given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{4}$$

This can be alternatively wriiten as

$$E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)\, dx \tag{5}$$

1.5: Finding Mean

We know that mean μ is given by $E(U)$. Hence

$$\mu = \int_{-\infty}^{\infty} x p_U(x)\, dx \tag{6}$$

$$\begin{aligned} \mu &= \int_0^1 x\, dx && (7) \\ &= \frac{x^2}{2}\Big|_0^1 && (8) \\ &= \frac{1}{2} && (9) \end{aligned}$$

1.5: Finding Variance

We know

$$var(U) = E((U - E(U))^2) \tag{10}$$

This can also be represented as

$$\begin{aligned} var(U) &= E(U^2 - 2E(U)U + (E(U))^2) &(11)\\ &= E(U^2) - 2(E(U))^2 + (E(U))^2 &(12)\\ &= E(U^2) - (E(U))^2 &(13)\end{aligned}$$

1.5: Finding Variance

We can evaluate $E(U^2)$ using (5) as:

$$E(U^2) = \int_{-\infty}^{\infty} x^2 p_U(x)\, dx \tag{14}$$

$$= \int_0^1 x^2\, dx \tag{15}$$

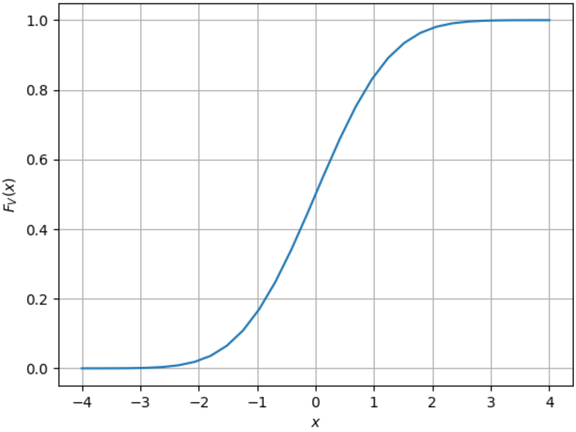
$$= \frac{x^3}{3}\Big|_0^1 \tag{16}$$

$$= \frac{1}{3} \tag{17}$$

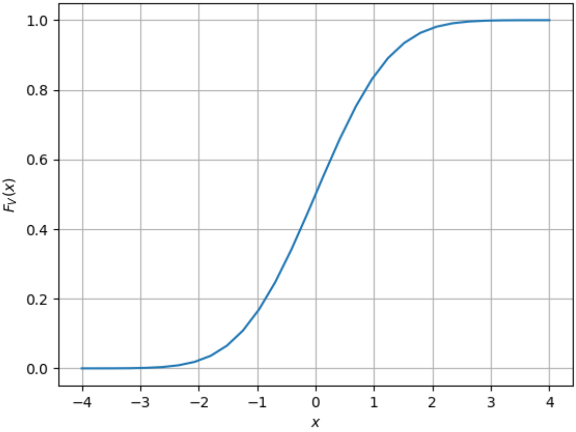
Using (9) and (13) we have

$$var(U) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{18}$$

2.2: CDF of X - Python Plot



2.3: PDF of X - Python Plot



2.5: Probability Density

For random variable X, we have been given the density function as follows

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp \frac{-x^2}{2} \tag{19}$$

2.5: Getting $F_X(x)$

Given

$$F_X(x) = \int_{-\infty}^{x} p_X(x)\, dx \tag{20}$$

We have, using (20) and (19)

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp \frac{-x^2}{2}\, dx \tag{21}$$

2.5: Mean

Mean for random variable X is given by:

$$\mu_x = E(X) \tag{22}$$
$$= \int_{-\infty}^{\infty} x p_X(x)\, dx \tag{23}$$
$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp \frac{-x^2}{2}\, dx \tag{24}$$
$$= 0 \tag{25}$$

Note that the integral

$$\int_{-a}^{a} f(x)\, dx \tag{26}$$

becomes 0, when $f(x)$ is odd.

2.5: Variance

Variance for random variable X is given by:

$$var(X) = E(X^2) - (E(X))^2 \tag{27}$$

We evaluate $E(X^2)$ as follows:

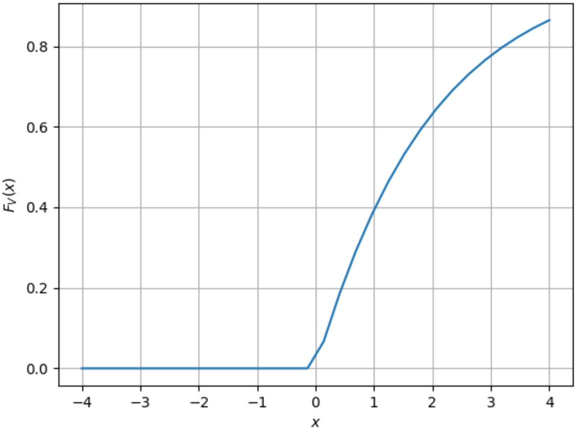
$$\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 p_X(x)\, dx &(28)\\
&= x \int x p_X(x)\, dx \Big|_{-\infty}^{\infty} - \int \frac{dx}{dx}\left(\int x p_X(x)\, dx\right) dx \Big|_{-\infty}^{\infty} &(29)\\
&= x \times -\sqrt{\frac{2}{\pi}} \exp \frac{-x^2}{2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\sqrt{\frac{2}{\pi}} \exp \frac{-x^2}{2}\, dx &(30)\\
&= 1 &(31)
\end{aligned}$$

2.5: Variance

Hence using (27) and (31), we have

$$var(X) = E(X^2) - (E(X))^2 \tag{32}$$
$$= 1 - 0^2 \qquad = 1 \tag{33}$$

3.1: CDF of V - Python Plot



3.2: Relation between U and V

We have been given that random variable V is a function of the random variable U as follows:

$$V = -2\ln(1 - U) \tag{34}$$

Note that the obtained distribution function (CDF) for random variable U is:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{35}$$

3.2: Getting $F_V(x)$

We know for any random variable X

$$F_X(x) = \Pr(X \leq x) \tag{36}$$

Hence, we can write using (34) and (36)

$$\begin{aligned}
F_V(x) &= \Pr(V \leq x) && (37)\\
&= \Pr(-2\ln(1-U) \leq x) && (38)\\
&= \Pr(\ln(1-U) \geq \frac{-x}{2}) && (39)\\
&= \Pr(1-U \geq \exp\frac{-x}{2}) && (40)\\
&= \Pr(U \leq 1-\exp\frac{-x}{2}) && (41)\\
&= F_U(1-\exp\frac{-x}{2}) && (42)
\end{aligned}$$

3.2: Final expression for $F_V(x)$

Note that the function $f(x) = 1 - \exp\frac{-x}{2}$ follows:

$$f(x) \in \begin{cases} 0, & x \in (-\infty, 0) \\ (0,1) & x \in (0, \infty) \end{cases} \tag{43}$$

Hence we can write

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp\frac{-x}{2}, & x \in (0, \infty) \end{cases} \tag{44}$$