AI1110: Probability and Random Variables

Assignment 8: Papoulis-Pillai Ex 5-37

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Outline

1. Problem

2. Solution
   - Characteristic Equation of Given Density
   - Part 1: Cauchy Density
   - Part 2: Laplace Density

Problem

Show that

1. if $f(x)$ is a Cauchy density, then

$$\phi(\omega) = e^{\alpha|\omega|} \tag{1}$$

2. if $f(x)$ is a Laplace density, then

$$\phi(\omega) = \frac{\alpha^2}{(\alpha^2 + \omega^2)} \tag{2}$$

Characteristic Equation

Given density function $p_X(x)$ for a random variable X, we define the characteristic equation to be:

Characteristic Equation

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{ix\omega} p_X(x)\, dx \tag{3}$$

where $i = \sqrt(-1)$

Note that $\phi_X(\omega)$ is the Fourier transform of $p_X(x)$. Using properties of Fourier tranforms, we have the following inversion formula:

Inversion Formula

$$p_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(\omega) e^{-ix\omega}\, \omega \tag{4}$$

Finding $p_X(x)$

Using the Inversion Formula (4) on given characteristic function for Cauchy distribution

$$\phi_X(\omega) = e^{-\alpha|\omega|} \tag{5}$$

we have

$$p_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha|\omega|} e^{-i\omega x} \, d\omega \tag{6}$$

$$p_X(x) = \frac{1}{2\pi} \int_{-\infty}^{0} e^{\alpha\omega} e^{-i\omega x} \, d\omega + \int_{0}^{\infty} e^{-\alpha\omega} e^{-i\omega x} \, d\omega \tag{7}$$

$$p_X(x) = \frac{1}{2\pi} \int_{-\infty}^{0} e^{-\omega(-\alpha+ix)} \, d\omega + \int_{0}^{\infty} e^{-\omega(\alpha+ix)} \, d\omega \tag{8}$$

Finding $p_X(x)$

$$p_X(x) = \frac{1}{2\pi}\left(\frac{e^{-\omega(-\alpha+ix)}}{-(-\alpha+ix)}\Big|_{-\infty}^{0} + \frac{e^{-\omega(\alpha+ix)}}{-(\alpha+ix)}\Big|_{0}^{\infty}\right) \tag{9}$$

Assuming $\alpha > 0$, we have

$$p_X(x) = \frac{1}{2\pi}\left(\frac{1}{\alpha - ix} + \frac{1}{\alpha + ix}\right) = \frac{\alpha}{\pi(\alpha^2 + x^2)} \tag{10}$$

which is the Cauchy Density function when $\mu = 0$.

Finding $\phi_Y(\omega)$

Using definition of the Characteristic function (3) on Laplace density function given by

$$p_Y(y) = \frac{\alpha e^{-\alpha|x|}}{2} \tag{11}$$

we have

$$\phi_Y(\omega) = \int_{-\infty}^{\infty} \frac{\alpha e^{-\alpha|x|}}{2} e^{i\omega x}\, dx \tag{12}$$

Finding $\phi_Y(\omega)$

$$\phi_Y(\omega) = \int_{-\infty}^{0} \frac{\alpha e^{(\alpha+i\omega)x}}{2}\, dx + \int_{0}^{\infty} \frac{\alpha e^{(-\alpha+i\omega)x}}{2}\, dx \tag{13}$$

$$\phi_Y(\omega) = \frac{\alpha}{2}\left(\frac{e^{(\alpha+i\omega)x}}{\alpha+i\omega}\Big|_{-\infty}^{0} + \frac{e^{(-\alpha+i\omega)x}}{-\alpha+i\omega}\Big|_{0}^{\infty}\right) \tag{14}$$

Assuming $\alpha > 0$

$$\phi_Y(\omega) = \frac{\alpha}{2}\left(\frac{1}{\alpha+i\omega} + \frac{1}{\alpha-i\omega}\right) = \frac{\alpha^2}{\alpha^2+\omega^2} \tag{15}$$

which is the Laplace Characteristic function. We assume everywhere that $\mu = 0$.