AI1110: Probability and Random Variables

Assignment 7: Papoulis-Pillai Ex 4-22

Rishit D (cs21btech11053)

IIT Hyderabad

May 23, 2022

Outline

Problem

The probability of *heads* of a random coin is a random variable p uniform in the interval (0.4,0.6).

1. Find the probability that at the next tossing of the coin *heads* will show.
2. The coin is tossed 100 times and *heads* shows 60 times. Find the probability that at the next tossing *heads* will show.

Defining PDF

We will define the probability density function $f(p)$ as follows

$$f(p) = \begin{cases} 5, & p \in (0.4, 0.6) \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Note that we ensure that the area under the probability density function $f(p)$ is 1.

Finding probability of getting Heads

The total probability theorem in continuous form states

Total Probability Theorem

$$\Pr(A) = \int_{-\infty}^{\infty} \Pr(A|p) f(p) dp \tag{2}$$

Using (2) we have

$$\Pr(H) = \int_{-\infty}^{\infty} \Pr(H|p) f(p) dp \tag{3}$$

Finding probability of getting Heads

Using (1) in (3)

$$\Pr(H) = \int_{0.4}^{0.6} 5 \Pr(H|p) dp \tag{4}$$

Note that we define the random variable p such that it represents the probability that a coin toss results in a *Head*. Hence

$$\Pr(H|p) = p \tag{5}$$

Using (5) in (4)

$$\Pr(H) = \int_{0.4}^{0.6} 5p dp = \boxed{0.5} \tag{6}$$

Code Output for Part 1

```
ravi@ravi-Legion-5-Pro-16ACH6H:~/Desktop/PRV/PyCode$ python3 coin_prob.py
Probability is 0.49994077
ravi@ravi-Legion-5-Pro-16ACH6H:~/Desktop/PRV/PyCode$
```

Figure: Probability of Getting Heads

Given Condition

We define event A as getting 60 *Heads* out of 100 trials (distinct sequence of *Heads* and *Tails*). Given that probability of getting heads is p, we have

$$\Pr(A|p) = p^{60}(1-p)^{40} \tag{7}$$

Using (2), (1) in (7)

$$\Pr(A) = \int_{0.4}^{0.6} P(A|p)f(p)dp = \int_{0.4}^{0.6} 5p^{60}(1-p)^{40}dp \tag{8}$$

Using Continuous Bayes Theorem

We state the Bayes Theorem for continuous random variables as follows

Continuous Bayes Theorem

$$f(x|A) = \frac{\Pr(A|x)}{\Pr(A)} f(x) \tag{9}$$

Using (9), (7) and (8) we have

$$f(p|A) = \frac{\Pr(A|p) \times f(p)}{\Pr(A)} = \frac{5p^{60}(1-p)^{40}}{\int_{0.4}^{0.6} 5p^{60}(1-p)^{40} dp} \tag{10}$$

Probability of Heads given event A

We are required to find the probability that we get heads on tossing a coin after event A has occured (i.e.) find $\Pr(H|A)$.

$$\Pr(H|A) = \int_{0.4}^{0.6} \Pr(H|p) f(p|A) dp = \int_{0.4}^{0.6} p \times f(p|A) dp \tag{11}$$

Hence

$$\Pr(H|A) = \frac{\int_{0.4}^{0.6} p^{61}(1-p)^{40} dp}{\int_{0.4}^{0.6} p^{60}(1-p)^{40} dp} = \boxed{0.56} \tag{12}$$