Random Numbers

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Contents

***Abstract*—Solutions to Random Numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files.

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/exrand.c
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/coeffs.h
```

Now execute the following code.

```
gcc exrand.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/cdf_plot.py
python3 cdf_plot.py
```

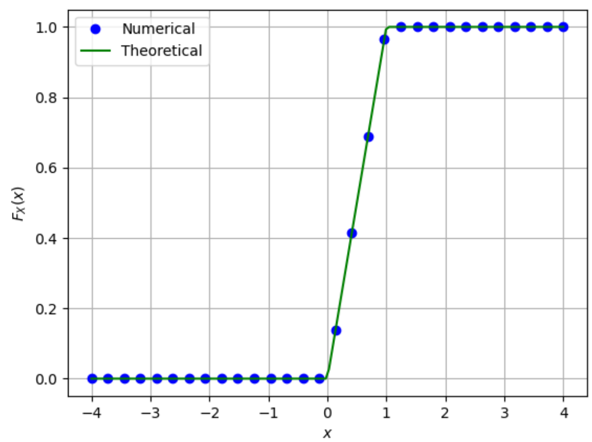

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given U is a uniformly distributed random variable over the interval $(0, 1)$, we have the density function $p_U(x)$:

$$p_U(x) = \begin{cases} 1, & x \in (0, 1) \\ 0, & otherwise \end{cases} \tag{1.2}$$

We know

$$F_U(x) = \int_{-\infty}^{x} p_U(x)\, dx \tag{1.3}$$

Given U is a uniformly distributed random variable over the interval $(0, 1)$, we have the following expression for $F_U(x)$:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{1.4}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\mathrm{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.
Solution:
Execute the following commands on linux terminal:

```
gcc mean_var_uni.c −lm
./a.out
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.7}$$

Solution: This can be alternatively wriiten as

$$E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)\,dx \tag{1.8}$$

We know that mean μ is given by $E(U)$. Hence

$$\mu = \int_{-\infty}^{\infty} x p_U(x)\,dx \tag{1.9}$$

$$\mu = \int_{0}^{1} x\,dx \tag{1.10}$$

$$= \left.\frac{x^2}{2}\right|_0^1 \tag{1.11}$$

$$= \boxed{\frac{1}{2}} \tag{1.12}$$

We know

$$var(U) = E((U - E(U))^2) \tag{1.13}$$

This can also be represented as

$$var(U) = E(U^2 - 2E(U)U + (E(U))^2) \tag{1.14}$$

$$= E(U^2) - 2(E(U))^2 + (E(U))^2 \tag{1.15}$$

$$= E(U^2) - (E(U))^2 \tag{1.16}$$

We can evaluate $E(U^2)$ using (1.8) as:

$$E(U^2) = \int_{-\infty}^{\infty} x^2 p_U(x)\,dx \tag{1.17}$$

$$= \int_0^1 x^2\,dx \tag{1.18}$$

$$= \left.\frac{x^3}{3}\right|_0^1 \tag{1.19}$$

$$= \frac{1}{3} \tag{1.20}$$

Using (1.12) and (1.16) we have

$$var(U) = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}} \tag{1.21}$$

Using this, we obtain mean as 0.5007 and variance as 0.083301. Hence the statistically obtained values are in close agreement with the theoretical values of $\mu = 0.5$ and $\sigma^2 = \frac{1}{12}$.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.
Solution: To generate samples for the Gaussian distribution, run the following code

```
gcc exrand.c −lm
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/pdf_plot.py
```

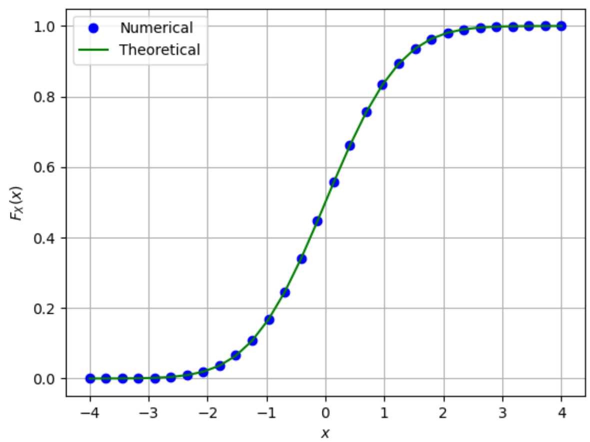

Fig. 2.2: The CDF of X

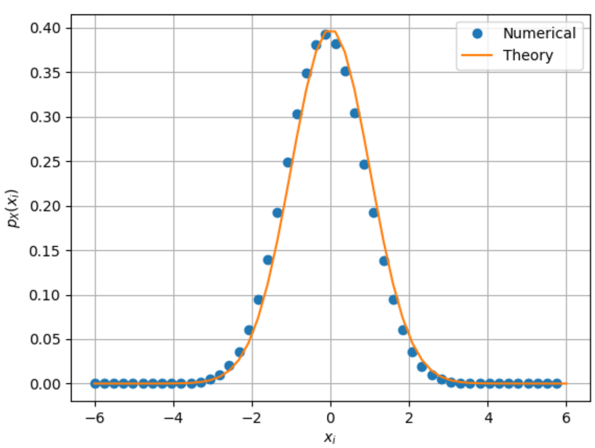

Fig. 2.3: The PDF of X

```
python3 pdf_plot.py
```

2.4 Find the mean and variance of X by writing a C program.
Solution:
The mean and variance is given by the following code:

```
gcc mean_var_gau.c −lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution: Given

$$F_X(x) = \int_{-\infty}^{x} p_X(x)\, dx \quad (2.4)$$

We have, using (2.4) and (2.3)

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.5)$$

The Q-Function is defined as follows:

$$Q(x) = \Pr(X > x) \quad (2.6)$$
$$= 1 - \Pr(X \le x) \quad (2.7)$$

Hence, using (2.7), we can write

$$F_X(x) = \Pr(X \le x) \quad (2.8)$$
$$= 1 - Q(x) \quad (2.9)$$

Mean for random variable X is given by:

$$\mu_x = E(X) \quad (2.10)$$
$$= \int_{-\infty}^{\infty} x p_X(x)\, dx \quad (2.11)$$
$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.12)$$
$$= \boxed{0} \quad (2.13)$$

Note that the integral

$$\int_{-a}^{a} f(x)\, dx \quad (2.14)$$

becomes 0, when $f(x)$ is odd.
Variance for random variable X is given by:

$$var(X) = E(X^2) - (E(X))^2 \quad (2.15)$$

We evaluate $E(X^2)$ as follows:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 p_X(x)\, dx \quad (2.16)$$
$$(2.17)$$

Using integration by parts, we have:

$$E(X^2) = -x\sqrt{\frac{2}{\pi}} e^{\left(-\frac{x^2}{2}\right)}\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} e^{\left(-\frac{x^2}{2}\right)} dx \quad (2.18)$$
$$= 1 \quad (2.19)$$

Hence using (2.15) and (2.19), we have

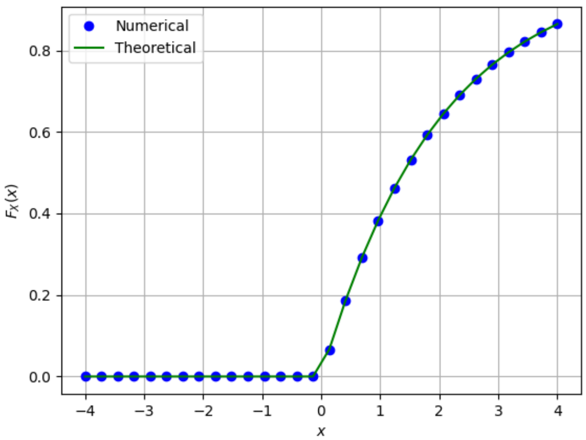

Fig. 3.1: The PDF of V

$$var(X) = E(X^2) - (E(X))^2 \tag{2.20}$$
$$= 1 - 0^2 \tag{2.21}$$
$$= \boxed{1} \tag{2.22}$$

Using this, we obtain the statistical mean and variance to be 0.000326 and 1.000906 respectively which is in close agreement with the theoretical values.

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: The following can be used to generate samples for random variable V:

```
gcc new_v.c −lm
.\a.out
```

The following code can be used to generate CDF for V:

```
python3 log_cdf.py
```

The figure generated is shown as (3.1)

3.2 Find a theoretical expression for $F_V(x)$.
Solution: We have been given that random variable V is a function of the random variable U as follows:

$$V = -2\ln(1-U) \tag{3.2}$$

Note that the obtained distribution function (CDF) for random variable U is:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0,1) \\ 1, & x \in (1,\infty) \end{cases} \tag{3.3}$$

We know for any random variable X

$$F_X(x) = \Pr(X \le x) \tag{3.4}$$

Hence, we can write using (3.2) and (3.4)

$$F_V(x) = \Pr(V \le x) \tag{3.5}$$
$$= \Pr(-2\ln(1-U) \le x) \tag{3.6}$$
$$= \Pr(\ln(1-U) \ge -\frac{x}{2}) \tag{3.7}$$
$$= \Pr(1-U \ge \exp\left(-\frac{x}{2}\right)) \tag{3.8}$$
$$= \Pr(U \le 1-\exp\left(-\frac{x}{2}\right)) \tag{3.9}$$
$$= F_U(1-\exp\left(-\frac{x}{2}\right)) \tag{3.10}$$

Note that the function $f(x) = 1 - \exp\left(-\frac{x}{2}\right)$ follows:

$$f(x) \in \begin{cases} 0, & x \in (-\infty, 0) \\ (0,1) & x \in (0,\infty) \end{cases} \tag{3.11}$$

Hence we can write

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1-\exp\left(-\frac{x}{2}\right), & x \in (0,\infty) \end{cases} \tag{3.12}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution:
Execute the following code to generate samples of random variable T in `tri.dat`:

```
gcc exrand.c −lm
./a.out
```

4.2 Find the CDF of T.
Solution:
Execute the following code to generate CDF of T.

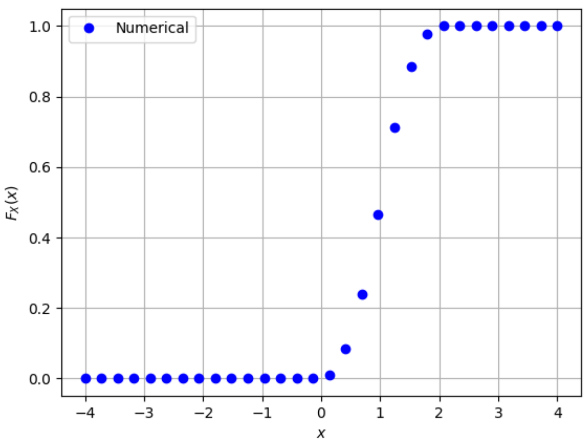

Fig. 4.2: The CDF of T

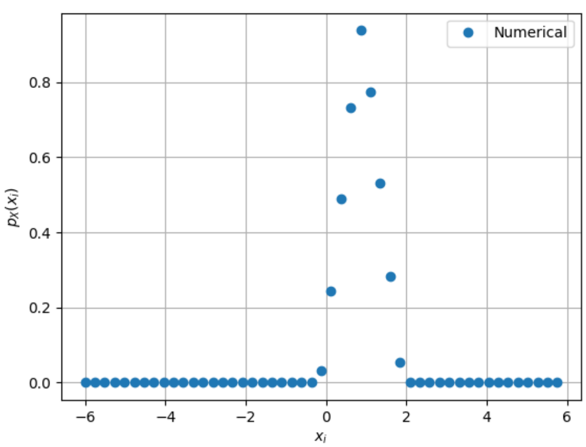

Fig. 4.3: The PDF of T

```
python3 tri_cdf.py
```

The CDF is plotted as shown in (4.2)

4.3 Find the PDF of T.

Solution:

Execute the following code to generate PDF of T.

```
python3 tri_pdf.py
```

The PDF is plotted as shown in (4.3)

4.4 Find the theoretical expressions for the PDF and CDF of T.

Solution:

Given a random variable Z as:

$$Z = X + Y \tag{4.2}$$

where X and Y are random variables, we can define

$$p_Z(t) = p_X(x) * p_Y(y) \tag{4.3}$$

$$= \int_{-\infty}^{\infty} p_X(\tau) p_Y(t - \tau)\, d\tau \tag{4.4}$$

Given $X = U$, $Y = U$ and $T = X + Y$, we have from (4.4)

$$p_T(t) = \int_{-\infty}^{\infty} p_U(\tau) p_U(t - \tau)\, d\tau \tag{4.5}$$

$$= \int_{0}^{1} p_U(t - \tau)\, d\tau \tag{4.6}$$

$$= \int_{t-1}^{t} p_U(u)\, du \tag{4.7}$$

From (1.2), we can deduce that the above integral will be non-zero only when $(t-1, t) \cap (0, 1) \neq \emptyset$. Hence (**??**) will be zero when $t < 0$ and $t > 2$.

Consider the integral when $t \in (0, 1)$:

$$p_T(t) = \int_{t-1}^{t} p_U(u)\, du \tag{4.8}$$

$$= \int_{0}^{t} p_U(u)\, du \tag{4.9}$$

$$= \int_{0}^{t} 1\, du \tag{4.10}$$

$$= \boxed{\text{t}} \tag{4.11}$$

Consider the integral when $t \in (1, 2)$:

$$p_T(t) = \int_{t-1}^{t} p_U(u)\, du \tag{4.12}$$

$$= \int_{t-1}^{1} p_U(u)\, du \tag{4.13}$$

$$= \boxed{\text{2-t}} \tag{4.14}$$

Hence, we can state $p_T(t)$ as follows from (4.11) and (4.14):

$$p_T(t) = \begin{cases} 0, & t \in (-\infty, 0) \\ t, & t \in (0, 1) \\ 2 - t, & t \in (1, 2) \\ 0, & t \in (0, \infty) \end{cases} \tag{4.15}$$

The CDF is related with the PDF as follows:

$$F_T(t) = \int_{-\infty}^{t} p_T(t)\, dt \tag{4.16}$$

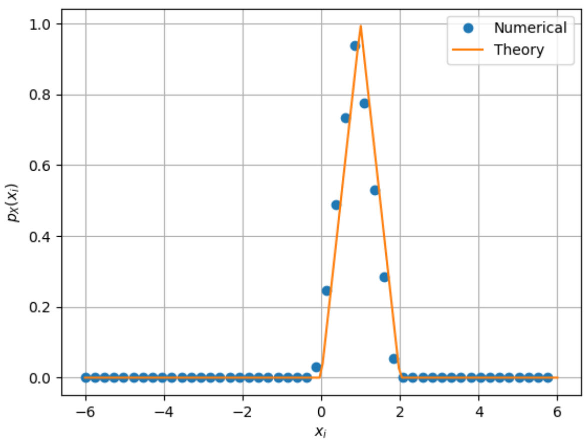

Fig. 4.5: The PDF of T

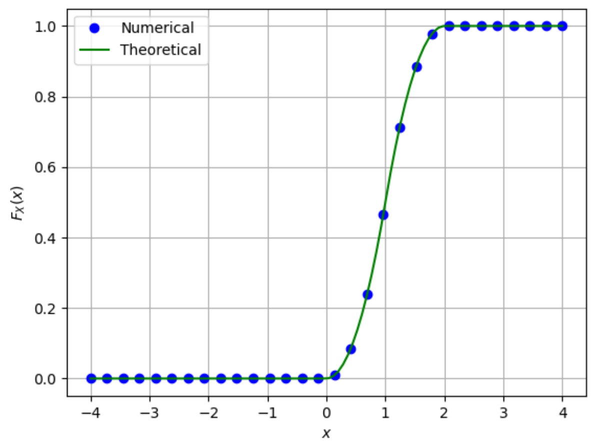

Fig. 4.6: The CDF of T

Using (4.16), we have:

$$F_T(t) = \begin{cases} 0, & t \in (-\infty, 0) \\ \frac{t^2}{2}, & t \in (0, 1) \\ \frac{1-(t-2)^2}{2}, & t \in (1, 2) \\ 1, & t \in (0, \infty) \end{cases} \tag{4.17}$$

4.5 Verify your result for the PDF through a plot.
Solution:
Execute the following code to generate theoretical and statistical PDF of T.

```
python3 tri_pdf.py
```

The plot is shown in figure (4.5).

4.6 Verify your result for the CDF through a plot.
Solution:
Execute the following code to generate theoretical and statistical CDF of T.

```
python3 tri_cdf.py
```

The plot is shown in figure (4.6)

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution:
We can generate samples for equiprobable random variable X using the following code:

```
gcc exrand.c -lm
./a.out
```

The samples generated are stored in `ber.dat`.

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

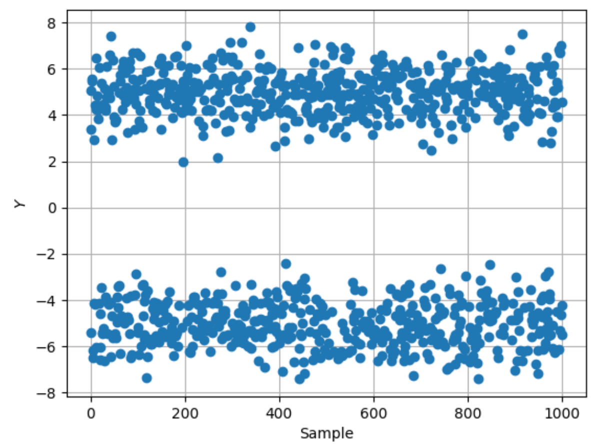

Fig. 5.3: Random Variable Y at $A = 5.0$

where $A = 5\text{dB}, X \in \{1, -1\}$, is Bernoulli and $N \sim \mathcal{N}(0, 1)$.
Solution:
We can generate samples for random variable Y using the following code:

```
gcc exrand.c -lm
./a.out
```

The samples generated are stored in `y.dat`.

5.3 Plot Y.
Solution:
We use the following code to plot all samples of Y.

```
python3 y_plot.py
```

The plot generated is shown in figure (5.3).

5.4 Guess how to estimate X from Y.
Solution:
One can roughly estimate X from Y as it is most probable that when $X > 0$, then $Y > 0$. Hence,

$$X = \begin{cases} 1, & Y > 0 \\ -1, & Y < 0 \end{cases} \tag{5.2}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.4}$$

Solution:
We can use the following code to find $P_{e|0}$ and $P_{e|1}$ as:

```
gcc exrand.c −lm
./a.out
```

In the case where $A = 2.5$, we obtain $P_{e|0} = 0.005478$ and $P_{e|1} = 0.005660$.

5.6 Find P_e assuming that X has equiprobable symbols.
Solution:
We can use the following code to find P_e:

```
gcc exrand.c −lm
./a.out
```

In the case where $A = 2.5$, we obtain $P_e = 0.005569$.

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution:

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.5}$$
$$= \frac{\Pr\left(\hat{X} = -1, X = 1\right)}{\Pr(X = 1)} \tag{5.6}$$

Using (5.22)

$$P_{e|0} = \frac{\Pr(Y < 0, X = 1)}{\Pr(X = 1)} \tag{5.7}$$
$$= \Pr(A + N < 0) \tag{5.8}$$
$$= \Pr(N < -A) \tag{5.9}$$
$$= 1 - \Pr(N \geq -A) \tag{5.10}$$
$$= 1 - Q(-A) \tag{5.11}$$

Similarly, we can write

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.12}$$
$$= \frac{\Pr\left(\hat{X} = 1, X = -1\right)}{\Pr(X = -1)} \tag{5.13}$$
$$= \frac{\Pr(Y > 0, X = -1)}{\Pr(X = -1)} \tag{5.14}$$
$$= \Pr(-A + N > 0) \tag{5.15}$$
$$= \Pr(N > A) \tag{5.16}$$
$$= Q(A) \tag{5.17}$$

Hence, we can determine P_e as follows:

$$P_e = P_{e|0}\Pr(X = 1) + P_{e|1}\Pr(X = -1) \tag{5.18}$$
$$= \frac{1}{2}\left(Q(A) + 1 - Q(-A)\right) \tag{5.19}$$
$$= \frac{1}{2}\left(2Q(A)\right) \tag{5.20}$$
$$= Q(A) \tag{5.21}$$

Note that $\Pr(X = 1) = \Pr(X = -1) = \frac{1}{2}$ and $Q(A) + Q(-A) = 1$.
We first generate statistically, various values of P_e for different values of a. We execute the following code to generate sample:

```
gcc exrand.c −lm
./a.out
```

The samples are now generated in `pe_a.dat`. To observe the theoretical plot and the statistical values of P_e vs a, we execute the following code:

```
python3 pea_graph.py
```

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that minimizes the theoretical P_e.
Solution:
Assuming the threshold to be δ, we can estimate X from Y:

$$X = \begin{cases} 1, & Y > \delta \\ -1, & Y < \delta \end{cases} \tag{5.22}$$

In this case

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.23}$$
$$= \frac{\Pr\left(\hat{X} = -1, X = 1\right)}{\Pr\left(X = 1\right)} \tag{5.24}$$
$$= \frac{\Pr\left(Y < \delta, X = 1\right)}{\Pr\left(X = 1\right)} \tag{5.25}$$
$$= \Pr\left(A + N < \delta\right) \tag{5.26}$$
$$= \Pr\left(N < \delta - A\right) \tag{5.27}$$
$$= 1 - \Pr\left(N \geq \delta - A\right) \tag{5.28}$$
$$= 1 - Q(\delta - A) \tag{5.29}$$

Similarly

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.30}$$
$$= \frac{\Pr\left(\hat{X} = 1, X = -1\right)}{\Pr\left(X = -1\right)} \tag{5.31}$$
$$= \frac{\Pr\left(Y > \delta, X = -1\right)}{\Pr\left(X = -1\right)} \tag{5.32}$$
$$= \Pr\left(-A + N > \delta\right) \tag{5.33}$$
$$= \Pr\left(N > \delta + A\right) \tag{5.34}$$
$$= Q(\delta + A) \tag{5.35}$$

We can write

$$P_e = P_{e|0} \Pr\left(X = 1\right) + P_{e|1} \Pr\left(X = -1\right) \tag{5.36}$$
$$= \frac{1}{2}\left(1 - Q(\delta - A) + Q(\delta + A)\right) \tag{5.37}$$

To minimise this, we will find the value at A when

$$\frac{dP_e}{dA} = 0 \tag{5.38}$$

$$\frac{1}{2}\frac{d}{dA}\left(1 - Q(\delta - A) + Q(\delta + A)\right) = 0 \tag{5.39}$$

$$\frac{e^{-\frac{(\delta-A)^2}{2}}}{\sqrt{2\pi}} - \frac{e^{-\frac{(\delta+A)^2}{2}}}{\sqrt{2\pi}} = 0 \tag{5.40}$$

$$e^{-\frac{(\delta-A)^2}{2}} = e^{-\frac{(\delta+A)^2}{2}} \tag{5.41}$$

$$\frac{(\delta - A)^2}{2} = \frac{(\delta + A)^2}{2} \tag{5.42}$$

$$(\delta - A)^2 = (\delta + A)^2 \tag{5.43}$$

$$\tag{5.44}$$

$$\delta = 0 \tag{5.45}$$

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.46}$$

5.10 Repeat the above exercise using the MAP criterion.

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.3}$$

7 Conditional Probability

7.1

7.2 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0, 1), X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

7.3 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.4 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)p_X(x)\, dx \tag{7.3}$$

Find $P_e = E\left[P_e(N)\right]$.

7.5 Plot P_e in problems 7.2 and 7.4 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in \left(\mathbf{s}_0, \mathbf{s}_1\right), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.