Random Numbers

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Abstract—**Solutions to Random Numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files.

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/exrand.c
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/coeffs.h
```

Now execute the following code.

```
gcc exrand.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/cdf_plot.py
python3 cdf_plot.py
```

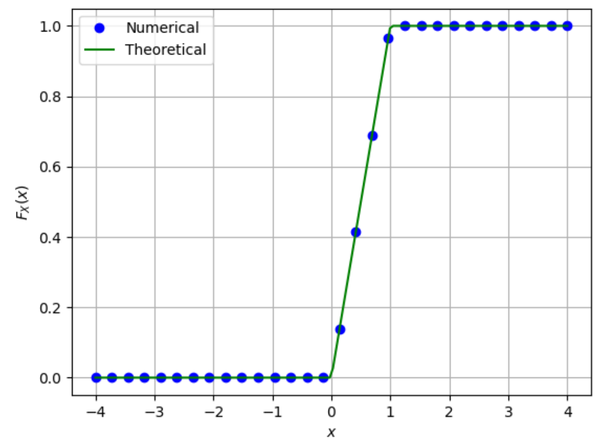

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given U is a uniformly distributed random variable over the interval $(0, 1)$, we have the density function $p_U(x)$:

$$p_U(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & otherwise \end{cases} \tag{1.2}$$

We know

$$F_U(x) = \int_{-\infty}^{x} p_U(x)\,dx \tag{1.3}$$

Given U is a uniformly distributed random variable over the interval $(0, 1)$, we have the following expression for $F_U(x)$:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0,1) \\ 1, & x \in (1, \infty) \end{cases} \tag{1.4}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and

variance of U.

Solution:

Execute the following commands on linux terminal:

```
gcc mean_var_uni.c -lm
./a.out
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.7}$$

Solution: This can be alternatively wriiten as

$$E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)\, dx \tag{1.8}$$

We know that mean μ is given by $E(U)$. Hence

$$\mu = \int_{-\infty}^{\infty} x p_U(x)\, dx \tag{1.9}$$

$$\mu = \int_0^1 x\, dx \tag{1.10}$$

$$= \left.\frac{x^2}{2}\right|_0^1 \tag{1.11}$$

$$= \boxed{\tfrac{1}{2}} \tag{1.12}$$

We know

$$var(U) = E((U - E(U))^2) \tag{1.13}$$

This can also be represented as

$$var(U) = E(U^2 - 2E(U)U + (E(U))^2) \tag{1.14}$$

$$= E(U^2) - 2(E(U))^2 + (E(U))^2 \tag{1.15}$$

$$= E(U^2) - (E(U))^2 \tag{1.16}$$

We can evaluate $E(U^2)$ using (1.8) as:

$$E(U^2) = \int_{-\infty}^{\infty} x^2 p_U(x)\, dx \tag{1.17}$$

$$= \int_0^1 x^2\, dx \tag{1.18}$$

$$= \left.\frac{x^3}{3}\right|_0^1 \tag{1.19}$$

$$= \frac{1}{3} \tag{1.20}$$

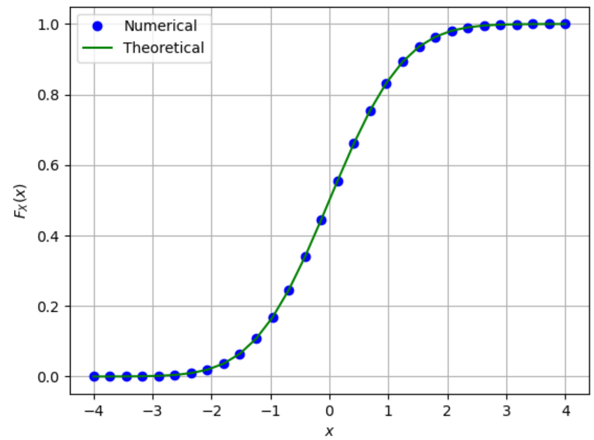

Fig. 2.2: The CDF of X

Using (1.12) and (1.16) we have

$$var(U) = \frac{1}{3} - \frac{1}{4} = \boxed{\tfrac{1}{12}} \tag{1.21}$$

Using this, we obtain mean as 0.5007 and variance as 0.083301. Hence the statistically obtained values are in close agreement with the theoretical values of $\mu = 0.5$ and $\sigma^2 = \frac{1}{12}$.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.

Solution: To generate samples for the Gaussian distribution, run the following code

```
gcc exrand.c -lm
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.2

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The

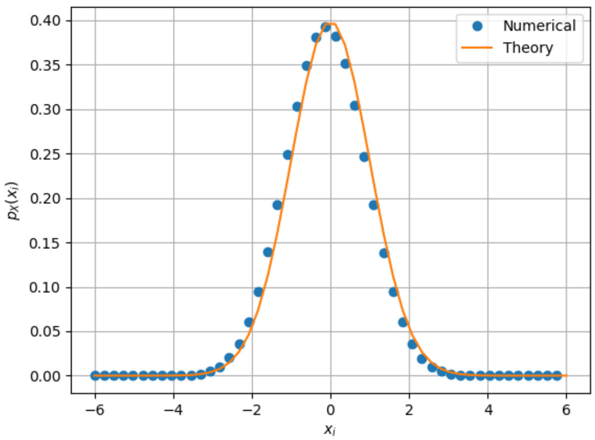

Fig. 2.3: The PDF of X

PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/pdf_plot.py
python3 pdf_plot.py
```

2.4 Find the mean and variance of X by writing a C program.
Solution:
The mean and variance is given by the following code:

```
gcc mean_var_gau.c −lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.
Solution: Given

$$F_X(x) = \int_{-\infty}^{x} p_X(x)\,dx \tag{2.4}$$

We have, using (2.4) and (2.3)

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\exp\frac{-x^2}{2}\,dx \tag{2.5}$$

Mean for random variable X is given by:

$$\mu_x = E(X) \tag{2.6}$$
$$= \int_{-\infty}^{\infty} x p_X(x)\,dx \tag{2.7}$$
$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)dx \tag{2.8}$$
$$= \boxed{0} \tag{2.9}$$

Note that the integral

$$\int_{-a}^{a} f(x)\,dx \tag{2.10}$$

becomes 0, when $f(x)$ is odd.
Variance for random variable X is given by:

$$var(X) = E(X^2) - (E(X))^2 \tag{2.11}$$

We evaluate $E(X^2)$ as follows:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 p_X(x)\,dx \tag{2.12}$$
$$\tag{2.13}$$

Using integration by parts, we have:

$$E(X^2) = -x\sqrt{\frac{2}{\pi}}e^{\left(\frac{-x^2}{2}\right)}\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}e^{\left(\frac{-x^2}{2}\right)}dx \tag{2.14}$$
$$= 1 \tag{2.15}$$

Hence using (2.11) and (2.15), we have

$$var(X) = E(X^2) - (E(X))^2 \tag{2.16}$$
$$= 1 - 0^2 \tag{2.17}$$
$$= \boxed{1} \tag{2.18}$$

Using this, we obtain the statistical mean and variance to be 0.000326 and 1.000906 respectively which is in close agreement with the theoretical values.

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: The following can be used to generate samples for random variable V:

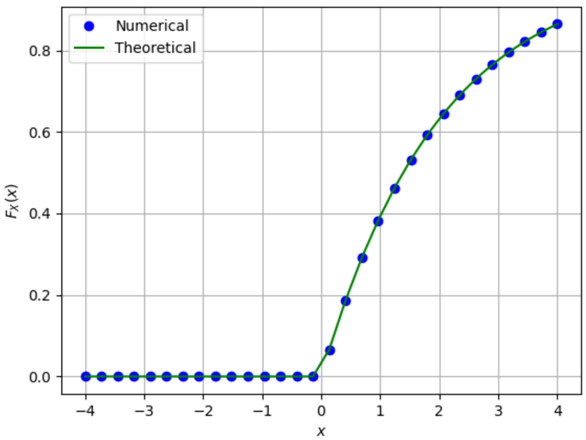

Fig. 3.1: The PDF of V

```
gcc new_v.c −lm
.\a.out
```

The following code can be used to generate CDF for V:

```
python3 log_cdf.py
```

The figure generated is shown as (3.1)

3.2 Find a theoretical expression for $F_V(x)$.
Solution: We have been given that random variable V is a function of the random variable U as follows:

$$V = -2\ln(1-U) \tag{3.2}$$

Note that the obtained distribution function (CDF) for random variable U is:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0,1) \\ 1, & x \in (1, \infty) \end{cases} \tag{3.3}$$

We know for any random variable X

$$F_X(x) = \Pr(X \le x) \tag{3.4}$$

Hence, we can write using (3.2) and (3.4)

$$\begin{aligned} F_V(x) &= \Pr(V \le x) && (3.5)\\ &= \Pr(-2\ln(1-U) \le x) && (3.6)\\ &= \Pr(\ln(1-U) \ge \frac{-x}{2}) && (3.7)\\ &= \Pr(1-U \ge \exp\frac{-x}{2}) && (3.8)\\ &= \Pr(U \le 1-\exp\frac{-x}{2}) && (3.9)\\ &= F_U(1-\exp\frac{-x}{2}) && (3.10) \end{aligned}$$

Note that the function $f(x) = 1 - \exp\left(\frac{-x}{2}\right)$ follows:

$$f(x) \in \begin{cases} 0, & x \in (-\infty, 0) \\ (0,1) & x \in (0, \infty) \end{cases} \tag{3.11}$$

Hence we can write

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1-\exp\left(\frac{-x}{2}\right), & x \in (0, \infty) \end{cases} \tag{3.12}$$