# AI1110: Probability and Random Variables

## Random Numbers

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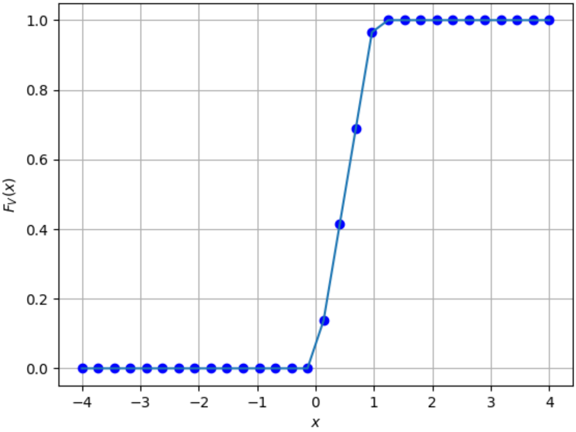
# Outline

# 1.2: CDF of U - Python Plot



# 1.3: Probability Density of U

Given $U$ is a uniformly distributed random variable over the interval $(0, 1)$, we have the density function $p_U(x)$:

$$p_U(x) = \begin{cases} 1, & x \in (0, 1) \\ 0, & otherwise \end{cases} \tag{1}$$

# 1.3: Getting $F_U(x)$

We know

$$F_U(x) = \int_{-\infty}^{x} p_U(x)\,dx \tag{2}$$

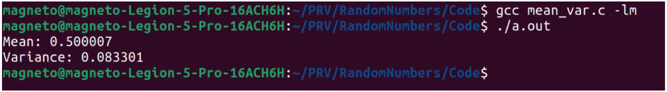
# 1.3: Theoretical Expression for Distribution of U

Given $U$ is a uniformly distributed random variable over the interval $(0, 1)$, we have the following expression for $F_U(x)$:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{3}$$

# 1.4: Mean and Variance

Using ../Codes/mean_var.c, we obtain mean as 0.5007 and variance as 0.083301. This is shown in the figure below:



# 1.5: Definitions

We are given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{4}$$

This can be alternatively wriiten as

$$E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)\, dx \tag{5}$$

# 1.5: Finding Mean

We know that mean $\mu$ is given by $E(U)$. Hence

$$\mu = \int_{-\infty}^{\infty} x p_U(x)\, dx \tag{6}$$

$$\mu = \int_0^1 x\, dx \tag{7}$$

$$= \left.\frac{x^2}{2}\right|_0^1 \tag{8}$$

$$= \frac{1}{2} \tag{9}$$

# 1.5: Finding Variance

We know

$$var(U) = E((U - E(U))^2) \tag{10}$$

This can also be represented as

$$\begin{aligned} var(U) &= E(U^2 - 2E(U)U + (E(U))^2) &(11)\\ &= E(U^2) - 2(E(U))^2 + (E(U))^2 &(12)\\ &= E(U^2) - (E(U))^2 &(13)\end{aligned}$$

# 1.5: Finding Variance

We can evaluate $E(U^2)$ using (5) as:

$$E(U^2) = \int_{-\infty}^{\infty} x^2 p_U(x)\, dx \tag{14}$$

$$= \int_0^1 x^2\, dx \tag{15}$$

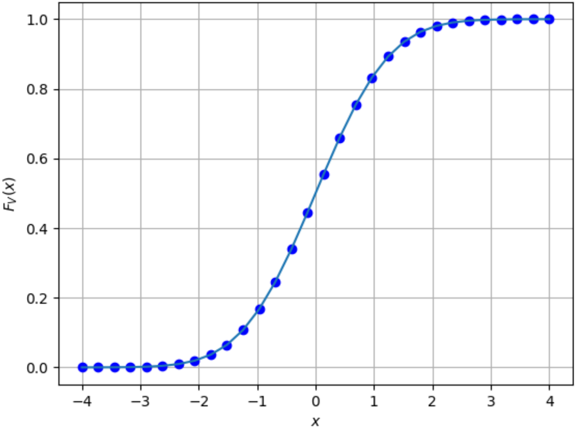
$$= \left.\frac{x^3}{3}\right|_0^1 \tag{16}$$

$$= \frac{1}{3} \tag{17}$$

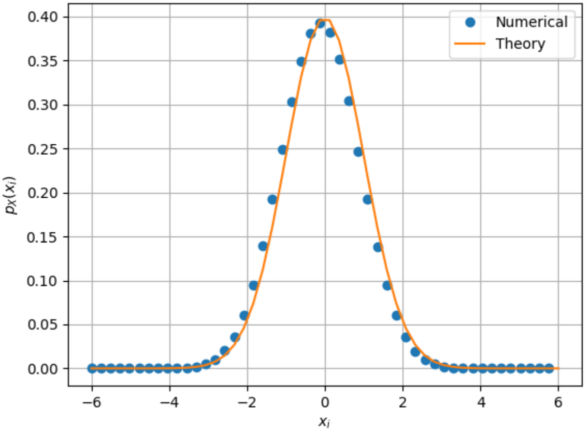
Using (9) and (13) we have

$$var(U) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{18}$$

# 2.2: CDF of X - Python Plot



# 2.3: PDF of X - Python Plot



# Mean and Variance

Using the ../Codes/mean_var_gau.c, we obtain the mean as 0 and variance as 1, as shown below:

```
magneto@magneto-Legion-5-Pro-16ACH6H:~/PRV/RandomNumbers/Code$ gcc mean_var_gau.c -lm
magneto@magneto-Legion-5-Pro-16ACH6H:~/PRV/RandomNumbers/Code$ ./a.out
Mean: 0.000326
Variance: 1.000906
magneto@magneto-Legion-5-Pro-16ACH6H:~/PRV/RandomNumbers/Code$
```

# 2.5: Probability Density

For random variable $X$, we have been given the density function as follows

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp \frac{-x^2}{2} \tag{19}$$

# 2.5: Getting $F_X(x)$

Given

$$F_X(x) = \int_{-\infty}^{x} p_X(x)\, dx \tag{20}$$

We have, using (20) and (19)

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp \frac{-x^2}{2}\, dx \tag{21}$$

## 2.5: Mean

Mean for random variable $X$ is given by:

$$\mu_x = E(X) \tag{22}$$

$$= \int_{-\infty}^{\infty} x p_X(x)\, dx \tag{23}$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) dx \tag{24}$$

$$= 0 \tag{25}$$

Note that the integral

$$\int_{-a}^{a} f(x)\, dx \tag{26}$$

becomes 0, when $f(x)$ is odd.

# 2.5: Variance

Variance for random variable $X$ is given by:

$$var(X) = E(X^2) - (E(X))^2 \tag{27}$$

We evaluate $E(X^2)$ as follows:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 p_X(x)\, dx \tag{28}$$

$$= x \int x p_X(x)\, dx \Big|_{-\infty}^{\infty} - \int \frac{dx}{dx} (\int x p_X(x)\, dx)\, dx \Big|_{-\infty}^{\infty} \tag{29}$$

$$= x \times -\sqrt{\frac{2}{\pi}} \exp \frac{-x^2}{2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\sqrt{\frac{2}{\pi}} \exp \frac{-x^2}{2}\, dx \tag{30}$$

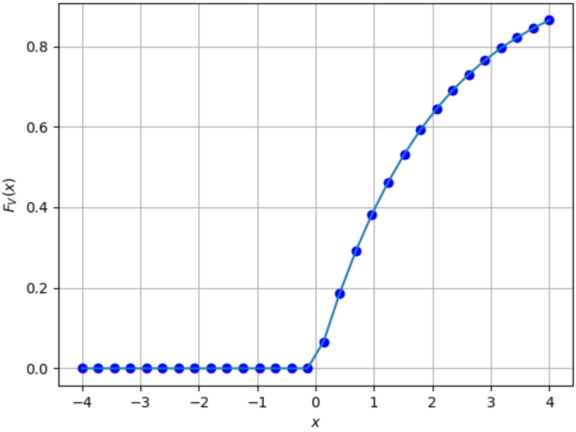
$$= 1 \tag{31}$$

# 2.5: Variance

Hence using (27) and (31), we have

$$var(X) = E(X^2) - (E(X))^2 \tag{32}$$
$$= 1 - 0^2 \tag{33}$$
$$= 1 \tag{34}$$

# 3.1: CDF of V - Python Plot



# 3.2: Relation between U and V

We have been given that random variable $V$ is a function of the random variable $U$ as follows:

$$V = -2\ln(1 - U) \tag{35}$$

Note that the obtained distribution function (CDF) for random variable $U$ is:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{36}$$

# 3.2: Getting $F_V(x)$

We know for any random variable $X$

$$F_X(x) = \Pr(X \leq x) \tag{37}$$

Hence, we can write using (35) and (37)

$$\begin{align}
F_V(x) &= \Pr(V \leq x) \tag{38}\\
&= \Pr(-2\ln(1-U) \leq x) \tag{39}\\
&= \Pr(\ln(1-U) \geq \frac{-x}{2}) \tag{40}\\
&= \Pr(1-U \geq \exp\frac{-x}{2}) \tag{41}\\
&= \Pr(U \leq 1-\exp\frac{-x}{2}) \tag{42}\\
&= F_U(1-\exp\frac{-x}{2}) \tag{43}
\end{align}$$

# 3.2: Final expression for $F_V(x)$

Note that the function $f(x) = 1 - \exp\left(\frac{-x}{2}\right)$ follows:

$$f(x) \in \begin{cases} 0, & x \in (-\infty, 0) \\ (0,1) & x \in (0, \infty) \end{cases} \tag{44}$$

Hence we can write

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp\left(\frac{-x}{2}\right), & x \in (0, \infty) \end{cases} \tag{45}$$