

# Random Numbers

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*Abstract—Solutions to Random Numbers*

## 1 UNIFORM RANDOM NUMBERS

Let  $U$  be a uniform random variable between 0 and 1.

- 1.1 Generate  $10^6$  samples of  $U$  using a C program and save into a file called uni.dat .

**Solution:** Download the following files.

```
wget https://github.com/gadepall/probability/
raw/master/manual/codes/exrand.c
wget https://github.com/gadepall/probability/
raw/master/manual/codes/coeffs.h
```

Now execute the following code.

```
gcc exrand.c -lm
./a.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of  $U$  using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

**Solution:** The following code plots Fig. 1.2

```
wget https://github.com/gadepall/probability/
raw/master/manual/codes/cdf_plot.py
python3 cdf_plot.py
```

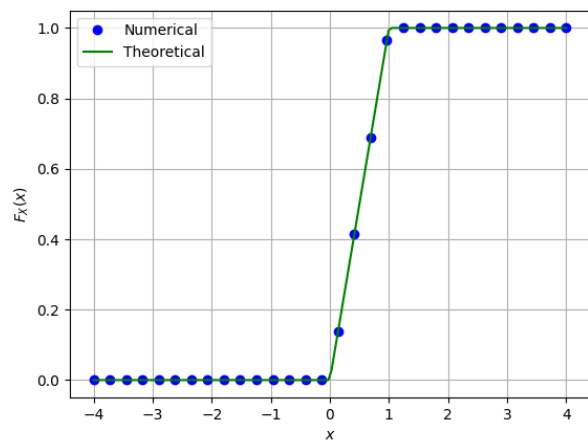


Fig. 1.2: The CDF of  $U$

- 1.3 Find a theoretical expression for  $F_U(x)$ .

**Solution:** Given  $U$  is a uniformly distributed random variable over the interval  $(0, 1)$ , we have the density function  $p_U(x)$ :

$$p_U(x) = \begin{cases} 1, & x \in (0, 1) \\ 0, & \text{otherwise} \end{cases} \quad (1.2)$$

We know

$$F_U(x) = \int_{-\infty}^x p_U(x) dx \quad (1.3)$$

Given  $U$  is a uniformly distributed random variable over the interval  $(0, 1)$ , we have the following expression for  $F_U(x)$ :

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \quad (1.4)$$

1.4 The mean of  $U$  is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.5)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.6)$$

Write a C program to find the mean and variance of  $U$ .

**Solution:**

Execute the following commands on linux terminal:

```
gcc mean_var_uni.c -lm
./a.out
```

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.7)$$

**Solution:** This can be alternatively written as

$$E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x) dx \quad (1.8)$$

We know that mean  $\mu$  is given by  $E(U)$ . Hence

$$\mu = \int_{-\infty}^{\infty} x p_U(x) dx \quad (1.9)$$

$$\mu = \int_0^1 x dx \quad (1.10)$$

$$= \frac{x^2}{2} \Big|_0^1 \quad (1.11)$$

$$= \left[ \frac{1}{2} \right] \quad (1.12)$$

We know

$$\text{var}(U) = E((U - E(U))^2) \quad (1.13)$$

This can also be represented as

$$\text{var}(U) = E(U^2 - 2E(U)U + (E(U))^2) \quad (1.14)$$

$$= E(U^2) - 2(E(U))^2 + (E(U))^2 \quad (1.15)$$

$$= E(U^2) - (E(U))^2 \quad (1.16)$$

We can evaluate  $E(U^2)$  using (1.8) as:

$$E(U^2) = \int_{-\infty}^{\infty} x^2 p_U(x) dx \quad (1.17)$$

$$= \int_0^1 x^2 dx \quad (1.18)$$

$$= \frac{x^3}{3} \Big|_0^1 \quad (1.19)$$

$$= \frac{1}{3} \quad (1.20)$$

Using (1.12) and (1.16) we have

$$\text{var}(U) = \frac{1}{3} - \frac{1}{4} = \left[ \frac{1}{12} \right] \quad (1.21)$$

Using this, we obtain mean as 0.5007 and variance as 0.083301. Hence the statistically obtained values are in close agreement with the theoretical values of  $\mu = 0.5$  and  $\sigma^2 = \frac{1}{12}$ .

## 2 CENTRAL LIMIT THEOREM

2.1 Generate  $10^6$  samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where  $U_i, i = 1, 2, \dots, 12$  are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.

**Solution:** To generate samples for the Gaussian distribution, run the following code

```
gcc exrand.c -lm
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of  $X$  using the samples in gau.dat. What properties does a CDF have?

**Solution:** The CDF of  $X$  is plotted in Fig. 2.2

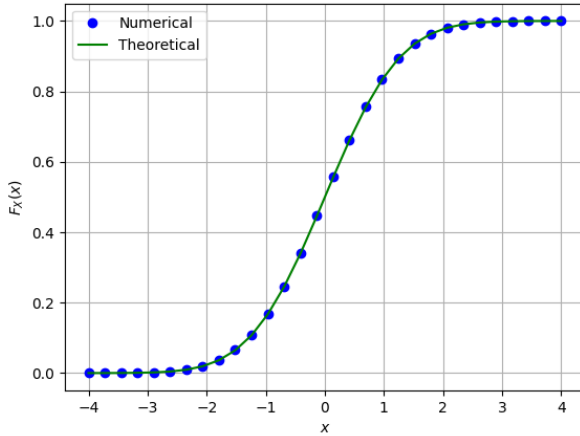
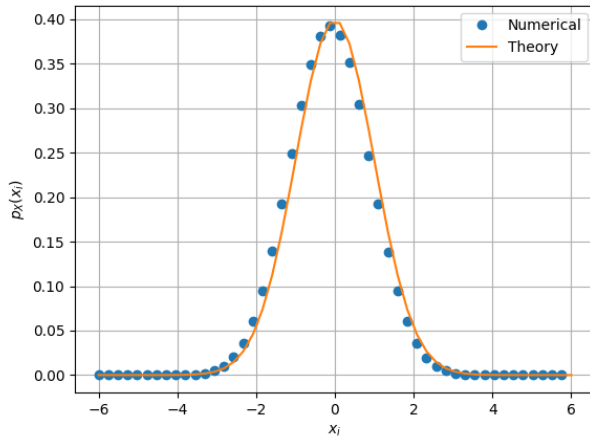
2.3 Load gau.dat in python and plot the empirical PDF of  $X$  using the samples in gau.dat. The PDF of  $X$  is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

**Solution:** The PDF of  $X$  is plotted in Fig. 2.3 using the code below

```
wget https://github.com/gadepall/probability/
raw/master/manual/codes/pdf_plot.py
```

Fig. 2.2: The CDF of  $X$ Fig. 2.3: The PDF of  $X$ 

```
python3 pdf_plot.py
```

2.4 Find the mean and variance of  $X$  by writing a C program.

**Solution:**

The mean and variance is given by the following code:

```
gcc mean_var_gau.c -lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

**Solution:** Given

$$F_X(x) = \int_{-\infty}^x p_X(x) dx \quad (2.4)$$

We have, using (2.4) and (2.3)

$$F_X(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.5)$$

The Q-Function is defined as follows:

$$Q(x) = \Pr(X > x) \quad (2.6)$$

$$= 1 - \Pr(X \leq x) \quad (2.7)$$

Hence, using (2.7), we can write

$$F_X(x) = \Pr(X \leq x) \quad (2.8)$$

$$= 1 - Q(x) \quad (2.9)$$

Mean for random variable  $X$  is given by:

$$\mu_x = E(X) \quad (2.10)$$

$$= \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.11)$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.12)$$

$$= \boxed{0} \quad (2.13)$$

Note that the integral

$$\int_{-a}^a f(x) dx \quad (2.14)$$

becomes 0, when  $f(x)$  is odd.

Variance for random variable  $X$  is given by:

$$\text{var}(X) = E(X^2) - (E(X))^2 \quad (2.15)$$

We evaluate  $E(X^2)$  as follows:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.16)$$

$$(2.17)$$

Using integration by parts, we have:

$$E(X^2) = -x \sqrt{\frac{2}{\pi}} e^{\left(-\frac{x^2}{2}\right)} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} e^{\left(-\frac{x^2}{2}\right)} dx \quad (2.18)$$

$$= 1 \quad (2.19)$$

Hence using (2.15) and (2.19), we have

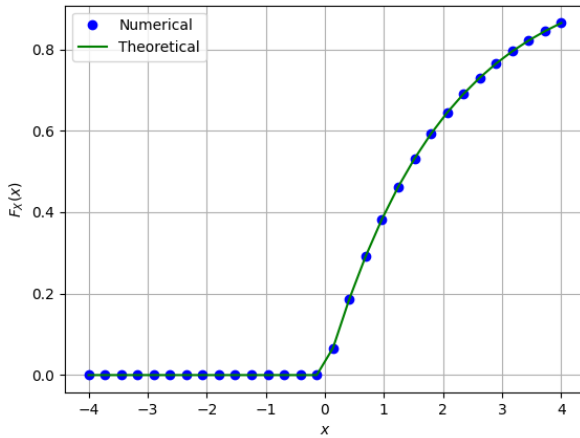


Fig. 3.1: The PDF of  $V$

$$\text{var}(X) = E(X^2) - (E(X))^2 \quad (2.20)$$

$$= 1 - 0^2 \quad (2.21)$$

$$= \boxed{1} \quad (2.22)$$

Using this, we obtain the statistical mean and variance to be 0.000326 and 1.000906 respectively which is in close agreement with the theoretical values.

### 3 FROM UNIFORM TO OTHER

#### 3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

**Solution:** The following can be used to generate samples for random variable  $V$ :

```
gcc new_v.c -lm
./a.out
```

The following code can be used to generate CDF for  $V$ :

```
python3 log_cdf.py
```

The figure generated is shown as (3.1)

#### 3.2 Find a theoretical expression for $F_V(x)$ .

**Solution:** We have been given that random variable  $V$  is a function of the random variable  $U$  as follows:

$$V = -2 \ln(1 - U) \quad (3.2)$$

Note that the obtained distribution function (CDF) for random variable  $U$  is:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \quad (3.3)$$

We know for any random variable  $X$

$$F_X(x) = \Pr(X \leq x) \quad (3.4)$$

Hence, we can write using (3.2) and (3.4)

$$F_V(x) = \Pr(V \leq x) \quad (3.5)$$

$$= \Pr(-2 \ln(1 - U) \leq x) \quad (3.6)$$

$$= \Pr(\ln(1 - U) \geq -\frac{x}{2}) \quad (3.7)$$

$$= \Pr(1 - U \geq \exp\left(-\frac{x}{2}\right)) \quad (3.8)$$

$$= \Pr(U \leq 1 - \exp\left(-\frac{x}{2}\right)) \quad (3.9)$$

$$= F_U(1 - \exp\left(-\frac{x}{2}\right)) \quad (3.10)$$

Note that the function  $f(x) = 1 - \exp\left(-\frac{x}{2}\right)$  follows:

$$f(x) \in \begin{cases} 0, & x \in (-\infty, 0) \\ (0, 1) & x \in (0, \infty) \end{cases} \quad (3.11)$$

Hence we can write

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & x \in (0, \infty) \end{cases} \quad (3.12)$$

### 4 TRIANGULAR DISTRIBUTION

#### 4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

**Solution:**

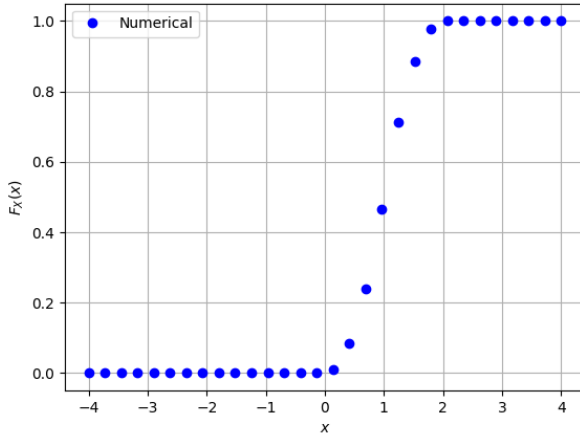
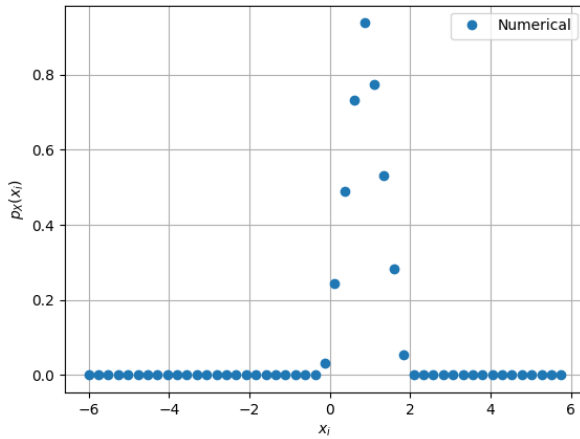
Execute the following code to generate samples of random variable  $T$  in `tri.dat`:

```
gcc exrand.c -lm
./a.out
```

#### 4.2 Find the CDF of $T$ .

**Solution:**

Execute the following code to generate CDF of  $T$ .

Fig. 4.2: The CDF of  $T$ Fig. 4.3: The PDF of  $T$ 

```
python3 tri_cdf.py
```

The CDF is plotted as shown in (4.2)

4.3 Find the PDF of  $T$ .

**Solution:**

Execute the following code to generate PDF of  $T$ .

```
python3 tri_pdf.py
```

The PDF is plotted as shown in (4.3)

4.4 Find the theoretical expressions for the PDF and CDF of  $T$ .

**Solution:**

Given a random variable  $Z$  as:

$$Z = X + Y \quad (4.2)$$

where  $X$  and  $Y$  are random variables, we can define

$$p_Z(t) = p_X(x) * p_Y(y) \quad (4.3)$$

$$= \int_{-\infty}^{\infty} p_X(\tau) p_Y(t - \tau) d\tau \quad (4.4)$$

Given  $X = U$ ,  $Y = U$  and  $T = X + Y$ , we have from (4.4)

$$p_T(t) = \int_{-\infty}^{\infty} p_U(\tau) p_U(t - \tau) d\tau \quad (4.5)$$

$$= \int_0^1 p_U(t - \tau) d\tau \quad (4.6)$$

$$= \int_{t-1}^t p_U(u) du \quad (4.7)$$

From (1.2), we can deduce that the above integral will be non-zero only when  $(t-1, t) \cap (0, 1) \neq \emptyset$ . Hence (??) will be zero when  $t < 0$  and  $t > 2$ .

Consider the integral when  $t \in (0, 1)$ :

$$p_T(t) = \int_{t-1}^t p_U(u) du \quad (4.8)$$

$$= \int_0^t p_U(u) du \quad (4.9)$$

$$= \int_0^t 1 du \quad (4.10)$$

$$= \boxed{t} \quad (4.11)$$

Consider the integral when  $t \in (1, 2)$ :

$$p_T(t) = \int_{t-1}^t p_U(u) du \quad (4.12)$$

$$= \int_{t-1}^1 p_U(u) du \quad (4.13)$$

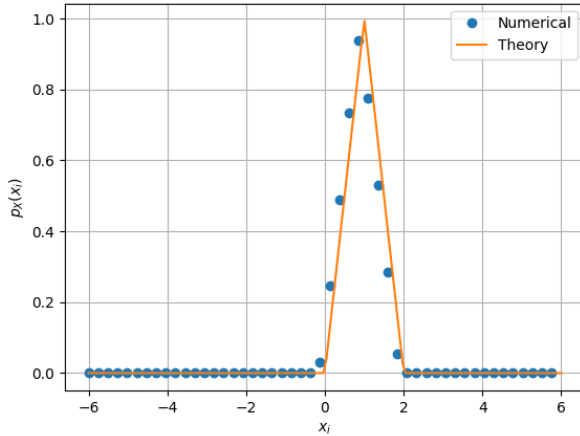
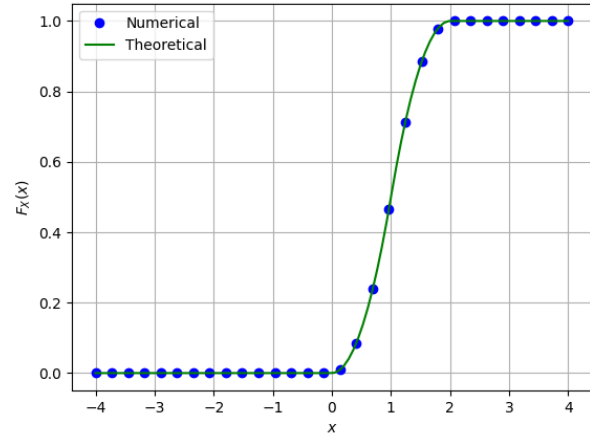
$$= \boxed{2-t} \quad (4.14)$$

Hence, we can state  $p_T(t)$  as follows from (4.11) and (4.14):

$$p_T(t) = \begin{cases} 0, & t \in (-\infty, 0) \\ t, & t \in (0, 1) \\ 2-t, & t \in (1, 2) \\ 0, & t \in (2, \infty) \end{cases} \quad (4.15)$$

The CDF is related with the PDF as follows:

$$F_T(t) = \int_{-\infty}^t p_T(t) dt \quad (4.16)$$

Fig. 4.5: The PDF of  $T$ Fig. 4.6: The CDF of  $T$ 

Using (4.16), we have:

$$F_T(t) = \begin{cases} 0, & t \in (-\infty, 0) \\ \frac{t^2}{2}, & t \in (0, 1) \\ \frac{1-(t-2)^2}{2}, & t \in (1, 2) \\ 1, & t \in (2, \infty) \end{cases} \quad (4.17)$$

4.5 Verify your result for the PDF through a plot.

**Solution:**

Execute the following code to generate theoretical and statistical PDF of  $T$ .

```
python3 tri_pdf.py
```

The plot is shown in figure (4.5).

4.6 Verify your result for the CDF through a plot.

**Solution:**

Execute the following code to generate theoretical and statistical CDF of  $T$ .

```
python3 tri_cdf.py
```

The plot is shown in figure (4.6)

## 5 MAXIMUM LIKELIHOOD

5.1 Generate equiprobable  $X \in \{1, -1\}$ .

**Solution:**

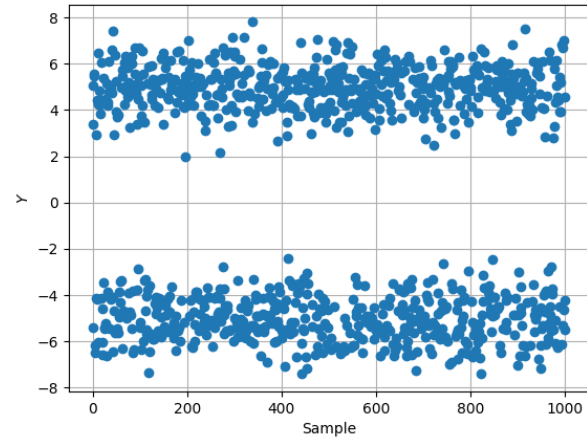
We can generate samples for equiprobable random variable  $X$  using the following code:

```
gcc exrand.c -lm
./a.out
```

The samples generated are stored in `ber.dat`.

5.2 Generate

$$Y = AX + N, \quad (5.1)$$

Fig. 5.3: Random Variable  $Y$  at  $A = 5.0$ 

where  $A = 5\text{dB}$ ,  $X \in \{1, -1\}$ , is Bernoulli and  $N \sim \mathcal{N}(0, 1)$ .

**Solution:**

We can generate samples for random variable  $Y$  using the following code:

```
gcc exrand.c -lm
./a.out
```

The samples generated are stored in `y.dat`.

5.3 Plot  $Y$ .

**Solution:**

We use the following code to plot all samples of  $Y$ .

```
python3 y_plot.py
```

The plot generated is shown in figure (5.3).

5.4 Guess how to estimate  $X$  from  $Y$ .

**Solution:**

One can roughly estimate  $X$  from  $Y$  as it is most probable that when  $X > 0$ , then  $Y > 0$ . Hence,

$$X = \begin{cases} 1, & Y > 0 \\ -1, & Y < 0 \end{cases} \quad (5.2)$$

5.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.3)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \quad (5.4)$$

**Solution:**

We can use the following code to find  $P_{e|0}$  and  $P_{e|1}$  as:

```
gcc exrand.c -lm
./a.out
```

In the case where  $A = 2.5$ , we obtain  $P_{e|0} = 0.005478$  and  $P_{e|1} = 0.005660$ .

5.6 Find  $P_e$  assuming that  $X$  has equiprobable symbols.

**Solution:**

We can use the following code to find  $P_e$ :

```
gcc exrand.c -lm
./a.out
```

In the case where  $A = 2.5$ , we obtain  $P_e = 0.005569$ .

5.7 Verify by plotting the theoretical  $P_e$  with respect to  $A$  from 0 to 10 dB.

**Solution:**

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.5)$$

$$= \frac{\Pr(\hat{X} = -1, X = 1)}{\Pr(X = 1)} \quad (5.6)$$

Using (5.22)

$$P_{e|0} = \frac{\Pr(Y < 0, X = 1)}{\Pr(X = 1)} \quad (5.7)$$

$$= \Pr(A + N < 0) \quad (5.8)$$

$$= \Pr(N < -A) \quad (5.9)$$

$$= 1 - \Pr(N \geq -A) \quad (5.10)$$

$$= 1 - Q(-A) \quad (5.11)$$

Similarly, we can write

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \quad (5.12)$$

$$= \frac{\Pr(\hat{X} = 1, X = -1)}{\Pr(X = -1)} \quad (5.13)$$

$$= \frac{\Pr(Y > 0, X = -1)}{\Pr(X = -1)} \quad (5.14)$$

$$= \Pr(-A + N > 0) \quad (5.15)$$

$$= \Pr(N > A) \quad (5.16)$$

$$= Q(A) \quad (5.17)$$

Hence, we can determine  $P_e$  as follows:

$$P_e = P_{e|0} \Pr(X = 1) + P_{e|1} \Pr(X = -1) \quad (5.18)$$

$$= \frac{1}{2} (Q(A) + 1 - Q(-A)) \quad (5.19)$$

$$= \frac{1}{2} (2Q(A)) \quad (5.20)$$

$$= Q(A) \quad (5.21)$$

Note that  $\Pr(X = 1) = \Pr(X = -1) = \frac{1}{2}$  and  $Q(A) + Q(-A) = 1$ .

We first generate statistically, various values of  $P_e$  for different values of  $a$ . We execute the following code to generate sample:

```
gcc exrand.c -lm
./a.out
```

The samples are now generated in `pe_a.dat`. To observe the theoretical plot and the statistical values of  $P_e$  vs  $a$ , we execute the following code:

```
python3 pea_graph.py
```

5.8 Now, consider a threshold  $\delta$  while estimating  $X$  from  $Y$ . Find the value of  $\delta$  that minimizes the theoretical  $P_e$ .

**Solution:**

Assuming the threshold to be  $\delta$ , we can estimate  $X$  from  $Y$ :

$$X = \begin{cases} 1, & Y > \delta \\ -1, & Y < \delta \end{cases} \quad (5.22)$$

In this case

$$\delta = 0 \quad (5.44)$$

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.23)$$

$$= \frac{\Pr(\hat{X} = -1, X = 1)}{\Pr(X = 1)} \quad (5.24)$$

$$= \frac{\Pr(Y < \delta, X = 1)}{\Pr(X = 1)} \quad (5.25)$$

$$= \Pr(A + N < \delta) \quad (5.26)$$

$$= \Pr(N < \delta - A) \quad (5.27)$$

$$= 1 - \Pr(N \geq \delta - A) \quad (5.28)$$

$$= 1 - Q(\delta - A) \quad (5.29)$$

Similarly

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \quad (5.30)$$

$$= \frac{\Pr(\hat{X} = 1, X = -1)}{\Pr(X = -1)} \quad (5.31)$$

$$= \frac{\Pr(Y > \delta, X = -1)}{\Pr(X = -1)} \quad (5.32)$$

$$= \Pr(-A + N > \delta) \quad (5.33)$$

$$= \Pr(N > \delta + A) \quad (5.34)$$

$$= Q(\delta + A) \quad (5.35)$$

We can write

$$P_e = P_{e|0} \Pr(X = 1) + P_{e|1} \Pr(X = -1) \quad (5.36)$$

$$P_e = \frac{1}{2} (1 - Q(\delta - A) + Q(\delta + A)) \quad (5.37)$$

To minimise this, we will find the value at  $A$  when

$$\frac{dP_e}{dA} = 0 \quad (5.38)$$

$$\frac{1}{2} \frac{d}{dA} (1 - Q(\delta - A) + Q(\delta + A)) = 0 \quad (5.39)$$

$$\frac{e^{-\frac{(\delta-A)^2}{2}}}{\sqrt{2\pi}} - \frac{e^{-\frac{(\delta+A)^2}{2}}}{\sqrt{2\pi}} = 0 \quad (5.40)$$

$$e^{-\frac{(\delta-A)^2}{2}} = e^{-\frac{(\delta+A)^2}{2}} \quad (5.41)$$

$$\frac{(\delta - A)^2}{2} = \frac{(\delta + A)^2}{2} \quad (5.42)$$

$$(\delta - A)^2 = (\delta + A)^2 \quad (5.43)$$

5.9 Repeat the above exercise when

$$p_X(-1) = p \quad (5.45)$$

**Solution:**

Using (5.36), we have:

$$P_e = P_{e|0} \Pr(X = 1) + P_{e|1} \Pr(X = -1) \quad (5.46)$$

$$= (p)(1 - Q(\delta - A)) + (1 - p)(Q(\delta + A)) \quad (5.47)$$

We again minimise this, using

$$\frac{dP_e}{dA} = 0 \quad (5.48)$$

$$(p) \frac{e^{-\frac{(\delta-A)^2}{2}}}{\sqrt{2\pi}} - (1 - p) \frac{e^{-\frac{(\delta+A)^2}{2}}}{\sqrt{2\pi}} = 0 \quad (5.49)$$

$$\frac{p}{1 - p} = e^{-2A\delta} \quad (5.50)$$

$$\delta = \frac{1}{2A} \ln \frac{1 - p}{p} \quad (5.51)$$

5.10 Repeat the above exercise using the MAP criterion.

**Solution:**

We can deduce the PDF of  $X|Y$  is given by:

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(Y)} \quad (5.52)$$

Assuming  $\Pr(X = -1) = p$ , we have  $p_Y(Y)$  using total probability theorem as:

$$p_Y(Y) = p_{y|x}(Y|1) \Pr(X = -1) + p_{y|x}(Y|1) \Pr(X = 1) \quad (5.53)$$

$$= (p) p_Y(-A + N) + (1 - p) p_Y(A + N) \quad (5.54)$$

$$= (p) \frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1 - p) \frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}} \quad (5.55)$$



Given a value of  $Y = y$ ,  $X = -1$ , we have

$$p_{X|Y}(-1|y) = \frac{p_{Y|X}(y|-1)p_X(-1)}{p_Y(Y)} \quad (5.56)$$

$$= \frac{(p)e^{-\frac{(y+A)^2}{2}}}{(p)e^{-\frac{(y+A)^2}{2}} + (1-p)e^{-\frac{(y-A)^2}{2}}} \quad (5.57)$$

$$= \frac{p}{p + (1-p)e^{2yA}} \quad (5.58)$$

Given a value of  $Y = y$ ,  $X = 1$ , we have

$$p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1)p_X(1)}{p_Y(Y)} \quad (5.59)$$

$$= \frac{(1-p)e^{-\frac{(y-A)^2}{2}}}{(p)e^{-\frac{(y+A)^2}{2}} + (1-p)e^{-\frac{(y-A)^2}{2}}} \quad (5.60)$$

$$= \frac{(1-p)e^{2yA}}{p + (1-p)e^{2yA}} \quad (5.61)$$

Using (5.58) and (5.61), we can conclude:

$$p_{X|Y}(1|y) > p_{X|Y}(-1|y) \quad (5.62)$$

when,

$$\frac{(1-p)e^{2yA}}{p + (1-p)e^{2yA}} > \frac{p}{p + (1-p)e^{2yA}} \quad (5.63)$$

$$e^{2yA} > \frac{p}{(1-p)} \quad (5.64)$$

$$y > \frac{1}{2A} \ln \frac{p}{(1-p)} \quad (5.65)$$

Hence, when (5.65) holds, we can choose  $X = 1$  and  $X = -1$  otherwise.

In the case that  $X$  is equiprobable, i.e.,  $\Pr(X = 1) = \Pr(X = -1) = \frac{1}{2}$ , we consider  $p = \frac{1}{2}$ . Hence, we choose  $X = 1$  when

$$y > \frac{1}{2A} \ln \frac{0.5}{0.5} \quad (5.66)$$

$$y > 0 \quad (5.67)$$

Also we choose  $X = -1$  when  $y \leq 0$ .

## 6 GAUSSIAN TO OTHER

6.1 Let  $X_1 \sim \mathcal{N}(0, 1)$  and  $X_2 \sim \mathcal{N}(0, 1)$ . Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

### Solution:

We first generate many samples for random variable  $V$  as mentioned above, using the following code:

```
gcc exrand.c -lm
./a.out
```

The samples generated are stored in `chi.dat`. To find the theoretical PDF and CDF for the random variable  $V$ , we shall assume that  $X_1$  and  $X_2$  are identical and independent. Now assume that

$$X_1 = R \cos \theta \quad (6.2)$$

$$X_2 = R \sin \theta \quad (6.3)$$

Now, we can see that:

$$V = X_1^2 + X_2^2 \quad (6.4)$$

$$= R^2 \quad (6.5)$$

We shall try to transform random variables  $X_1, X_2$  to polar form, i.e.,  $r$  and  $\theta$ , using the Jacobian as follows:

$$p_{R,\theta}(r, \theta) = |J|p_{X_1,X_2}(x_1, x_2) \quad (6.6)$$

where

$$J = \begin{pmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} \end{pmatrix} \quad (6.7)$$

We find that the Jacobian can be simplified as:

$$J = \begin{pmatrix} \frac{\partial r \cos \theta}{\partial r} & \frac{\partial r \cos \theta}{\partial \theta} \\ \frac{\partial r \sin \theta}{\partial r} & \frac{\partial r \sin \theta}{\partial \theta} \end{pmatrix} \quad (6.8)$$

$$= \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \quad (6.9)$$

Now we can evaluate the determinant as:

$$|J| = (\cos \theta)(r \cos \theta) - (\sin \theta)(-r \sin \theta) \quad (6.10)$$

$$= r \quad (6.11)$$

Since  $X_1$  and  $X_2$  are independent and identical, we can deduce:

$$p_{X_1,X_2}(x_1, x_2) = p_{X_1}(x_1)p_{X_2}(x_2) \quad (6.12)$$

Using (6.11) and (6.12) in (6.6), we can con-

clude:

$$p_{r,\theta}(r, \theta) = r p_{X_1}(x_1) p_{X_2}(x_2) \quad (6.13)$$

$$= \frac{r}{2\pi} \left( e^{-\frac{x_1^2}{2}} \right) \left( e^{-\frac{x_2^2}{2}} \right) \quad (6.14)$$

$$= \frac{r}{2\pi} \left( e^{-\frac{x_1^2 + x_2^2}{2}} \right) \quad (6.15)$$

$$= \frac{r}{2\pi} \left( e^{-\frac{r^2}{2}} \right) \quad (6.16)$$

Marginal probability  $p_R(r)$  can be found as:

$$p_R(r) = \int_0^{2\pi} p_{R,\theta}(r, \theta) d\theta \quad (6.17)$$

$$= \int_0^{2\pi} \frac{r}{2\pi} \left( e^{-\frac{r^2}{2}} \right) d\theta \quad (6.18)$$

$$= r e^{-\frac{r^2}{2}} \quad (6.19)$$

Note that  $p_R(r)$  is only defined as above only when  $r \geq 0$  and is 0 otherwise.

Hence, we can define distribution  $F_R(r)$  as:

$$F_R(r) = \Pr(R \leq r) \quad (6.20)$$

$$= \int_{-\infty}^r p_R(r) dr \quad (6.21)$$

$$= \int_0^r r e^{-\frac{r^2}{2}} dr \quad (6.22)$$

$$= 1 - e^{-\frac{r^2}{2}} \quad (6.23)$$

when  $r \geq 0$  and 0 when  $r < 0$ .

Using (6.5), we can determine the distribution for random variable  $V$ :

$$F_V(x) = \Pr(V \leq x) \quad (6.24)$$

$$= \Pr(R^2 \leq x) \quad (6.25)$$

$$= \Pr(0 \leq R \leq \sqrt{x}) \quad (6.26)$$

$$= F_R(\sqrt{x}) \quad (6.27)$$

Hence,

$$F_V(x) = \begin{cases} 1 - e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (6.28)$$

Differentiating this, we obtain the density of random variable  $V$ :

$$p_V(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (6.29)$$

We now plot the statistical and theoretical PDF

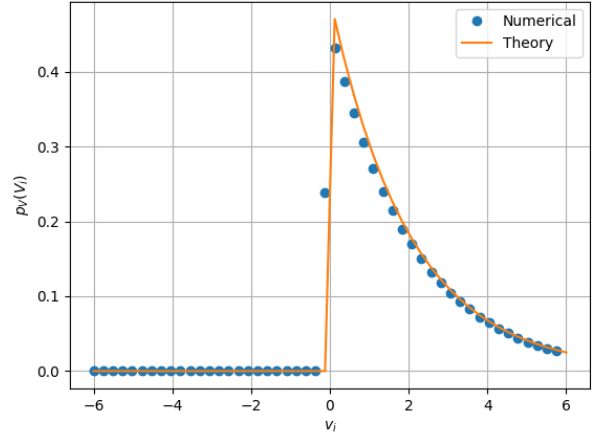


Fig. 6.1: PDF of  $V$

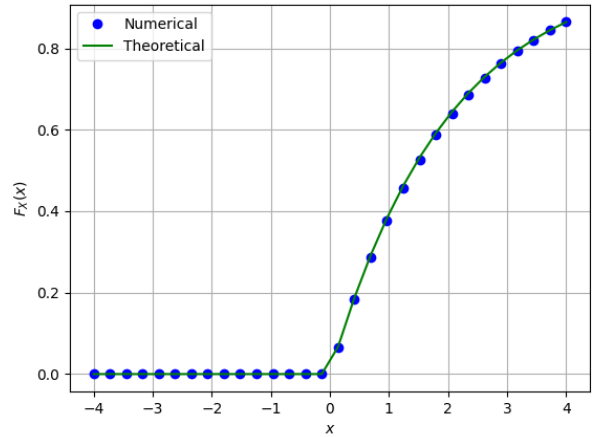


Fig. 6.1: CDF of  $V$

using the following code:

```
python3 chi_pdf.py
```

The plot is generated in the figure (6.1).

We now plot the statistical and theoretical CDF using the following code;

```
python3 chi_cdf.py
```

The plot is generated in the figure (6.1)

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.30)$$

find  $\alpha$ .

**Solution:**

Using (6.28), we can compare the given equa-

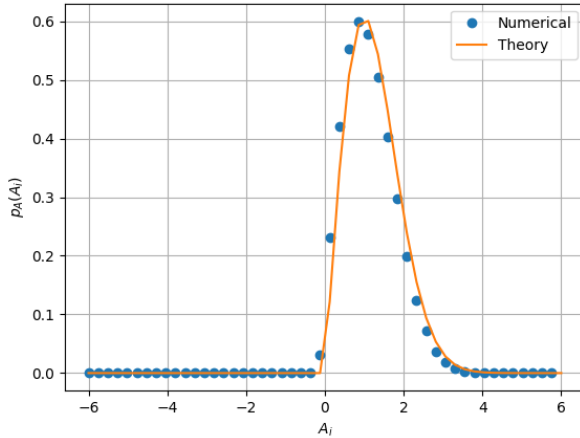


Fig. 6.3: PDF of A

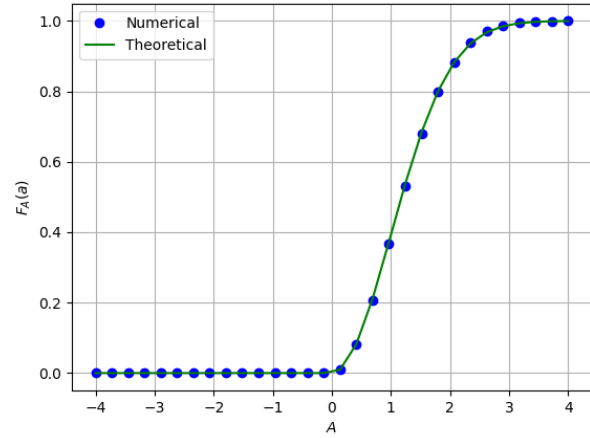


Fig. 6.3: CDF of A

tion in the question and obtain

$$\alpha = \frac{1}{2} \quad (6.31)$$

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \quad (6.32)$$

### Solution:

We first generate samples of random variable using the following code:

```
gcc exrand.c -lm
./a.out
```

The samples generated are stored in ray.dat. Using these samples we can generate PDF plot using the following code:

```
python3 ray_pdf.py
```

The plot generated is shown in figure (6.3). Using these samples we can generate PDF plot using the following code:

```
python3 ray_cdf.py
```

The plot generated is shown in figure (6.3).

## 7 CONDITIONAL PROBABILITY

7.1 Plot

$$P_e = \Pr(\hat{X} = -1 | X = 1) \quad (7.1)$$

for

$$Y = AX + N, \quad (7.2)$$

where  $A$  is Raleigh with  $E[A^2] = \gamma, N \sim \mathcal{N}(0, 1), X \in (-1, 1)$  for  $0 \leq \gamma \leq 10$  dB.

7.2 Assuming that  $N$  is a constant, find an expression for  $P_e$ . Call this  $P_e(N)$

7.3 For a function  $g$ ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x)dx \quad (7.3)$$

Find  $P_e = E[P_e(N)]$ .

7.4 Plot  $P_e$  in problems 7.1 and 7.3 on the same graph w.r.t  $\gamma$ . Comment.

## 8 TWO DIMENSIONS

Let

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}, \quad (8.1)$$

where

$$\mathbf{x} \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \quad (8.3)$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (8.4)$$

on the same graph using a scatter plot.

8.2

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (8.5)$$

on the same graph using a scatter plot.

8.3 For the above problem, find a decision rule for detecting the symbols  $\mathbf{s}_0$  and  $\mathbf{s}_1$ .

8.4 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \quad (8.6)$$

with respect to the SNR from 0 to 10 dB.