Random Numbers

Rishit D*

Contents

***Abstract*—Solutions to Random Numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files.

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/exrand.c
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/coeffs.h
```

Now execute the following code.

```
gcc exrand.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/cdf_plot.py
python3 cdf_plot.py
```

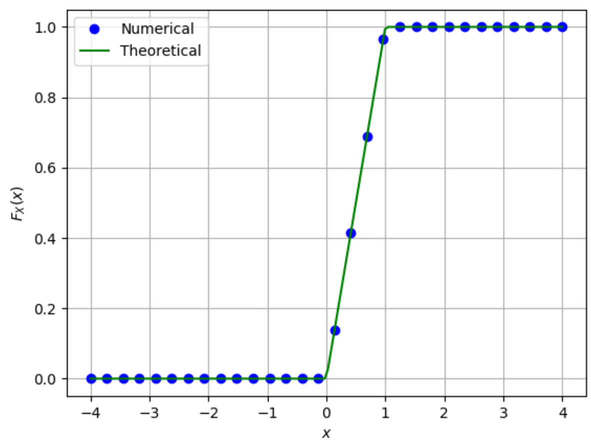

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given U is a uniformly distributed random variable over the interval $(0,1)$, we have the density function $p_U(x)$:

$$p_U(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & otherwise \end{cases} \tag{1.2}$$

We know

$$F_U(x) = \int_{-\infty}^{x} p_U(x)\,dx \tag{1.3}$$

Given U is a uniformly distributed random variable over the interval $(0,1)$, we have the following expression for $F_U(x)$:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty,0) \\ x, & x \in (0,1) \\ 1, & x \in (1,\infty) \end{cases} \tag{1.4}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.
Solution:
Execute the following commands on linux terminal:

```
gcc mean_var_uni.c -lm
./a.out
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.7}$$

Solution: This can be alternatively wriiten as

$$E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)\,dx \tag{1.8}$$

We know that mean μ is given by $E(U)$. Hence

$$\mu = \int_{-\infty}^{\infty} x p_U(x)\,dx \tag{1.9}$$

$$\mu = \int_{0}^{1} x\,dx \tag{1.10}$$

$$= \left.\frac{x^2}{2}\right|_0^1 \tag{1.11}$$

$$= \boxed{\frac{1}{2}} \tag{1.12}$$

We know

$$var(U) = E((U - E(U))^2) \tag{1.13}$$

This can also be represented as

$$var(U) = E(U^2 - 2E(U)U + (E(U))^2) \tag{1.14}$$

$$= E(U^2) - 2(E(U))^2 + (E(U))^2 \tag{1.15}$$

$$= E(U^2) - (E(U))^2 \tag{1.16}$$

We can evaluate $E(U^2)$ using (1.8) as:

$$E(U^2) = \int_{-\infty}^{\infty} x^2 p_U(x)\,dx \tag{1.17}$$

$$= \int_0^1 x^2\,dx \tag{1.18}$$

$$= \left.\frac{x^3}{3}\right|_0^1 \tag{1.19}$$

$$= \frac{1}{3} \tag{1.20}$$

Using (1.12) and (1.16) we have

$$var(U) = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}} \tag{1.21}$$

Using this, we obtain mean as 0.5007 and variance as 0.083301. Hence the statistically obtained values are in close agreement with the theoretical values of $\mu = 0.5$ and $\sigma^2 = \frac{1}{12}$.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.
Solution: To generate samples for the Gaussian distribution, run the following code

```
gcc exrand.c -lm
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/pdf_plot.py
```

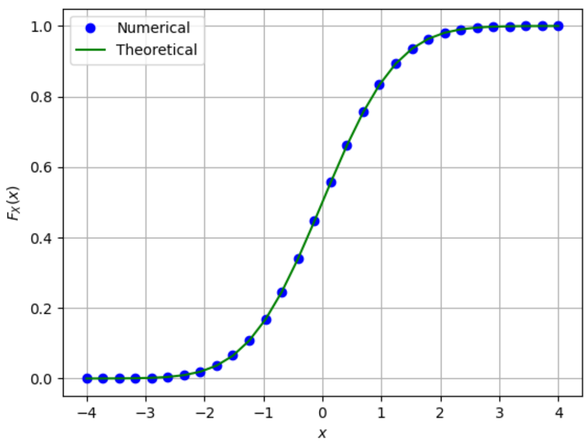

Fig. 2.2: The CDF of X

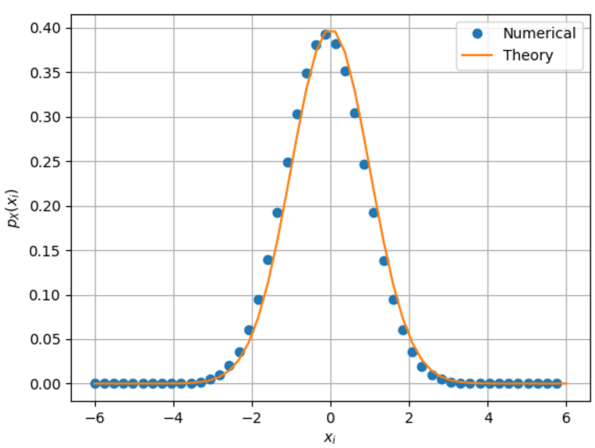

Fig. 2.3: The PDF of X

```
python3 pdf_plot.py
```

2.4 Find the mean and variance of X by writing a C program.
Solution:
The mean and variance is given by the following code:

```
gcc mean_var_gau.c −lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution: Given

$$F_X(x) = \int_{-\infty}^{x} p_X(x)\,dx \quad (2.4)$$

We have, using (2.4) and (2.3)

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.5)$$

The Q-Function is defined as follows:

$$Q(x) = \Pr(X > x) \quad (2.6)$$
$$= 1 - \Pr(X \le x) \quad (2.7)$$

Hence, using (2.7), we can write

$$F_X(x) = \Pr(X \le x) \quad (2.8)$$
$$= 1 - Q(x) \quad (2.9)$$

Mean for random variable X is given by:

$$\mu_x = E(X) \quad (2.10)$$
$$= \int_{-\infty}^{\infty} x p_X(x)\,dx \quad (2.11)$$
$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.12)$$
$$= \boxed{0} \quad (2.13)$$

Note that the integral

$$\int_{-a}^{a} f(x)\,dx \quad (2.14)$$

becomes 0, when $f(x)$ is odd.
Variance for random variable X is given by:

$$var(X) = E(X^2) - (E(X))^2 \quad (2.15)$$

We evaluate $E(X^2)$ as follows:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 p_X(x)\,dx \quad (2.16)$$
$$(2.17)$$

Using integration by parts, we have:

$$E(X^2) = -x\sqrt{\frac{2}{\pi}} e^{\left(-\frac{x^2}{2}\right)}\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} e^{\left(-\frac{x^2}{2}\right)} dx \quad (2.18)$$
$$= 1 \quad (2.19)$$

Hence using (2.15) and (2.19), we have

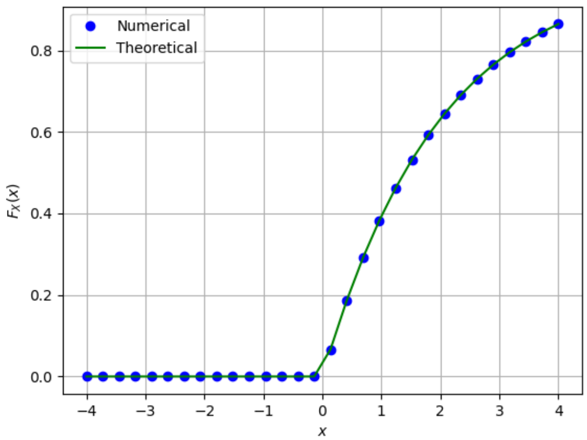

Fig. 3.1: The PDF of V

$$var(X) = E(X^2) - (E(X))^2 \tag{2.20}$$
$$= 1 - 0^2 \tag{2.21}$$
$$= \boxed{1} \tag{2.22}$$

Using this, we obtain the statistical mean and variance to be 0.000326 and 1.000906 respectively which is in close agreement with the theoretical values.

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF.
Solution: The following can be used to generate samples for random variable V:

```
gcc new_v.c −lm
.\a.out
```

The following code can be used to generate CDF for V:

```
python3 log_cdf.py
```

The figure generated is shown as (3.1)

3.2 Find a theoretical expression for $F_V(x)$.
Solution: We have been given that random variable V is a function of the random variable U as follows:

$$V = -2\ln(1 - U) \tag{3.2}$$

Note that the obtained distribution function (CDF) for random variable U is:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{3.3}$$

We know for any random variable X

$$F_X(x) = \Pr(X \le x) \tag{3.4}$$

Hence, we can write using (3.2) and (3.4)

$$F_V(x) = \Pr(V \le x) \tag{3.5}$$
$$= \Pr(-2\ln(1 - U) \le x) \tag{3.6}$$
$$= \Pr(\ln(1 - U) \ge -\frac{x}{2}) \tag{3.7}$$
$$= \Pr(1 - U \ge \exp\left(-\frac{x}{2}\right)) \tag{3.8}$$
$$= \Pr(U \le 1 - \exp\left(-\frac{x}{2}\right)) \tag{3.9}$$
$$= F_U(1 - \exp\left(-\frac{x}{2}\right)) \tag{3.10}$$

Note that the function $f(x) = 1 - \exp\left(-\frac{x}{2}\right)$ follows:

$$f(x) \in \begin{cases} 0, & x \in (-\infty, 0) \\ (0, 1) & x \in (0, \infty) \end{cases} \tag{3.11}$$

Hence we can write

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & x \in (0, \infty) \end{cases} \tag{3.12}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution:
Execute the following code to generate samples of random variable T in `tri.dat`:

```
gcc exrand.c −lm
./a.out
```

4.2 Find the CDF of T.
Solution:
Execute the following code to generate CDF of T.

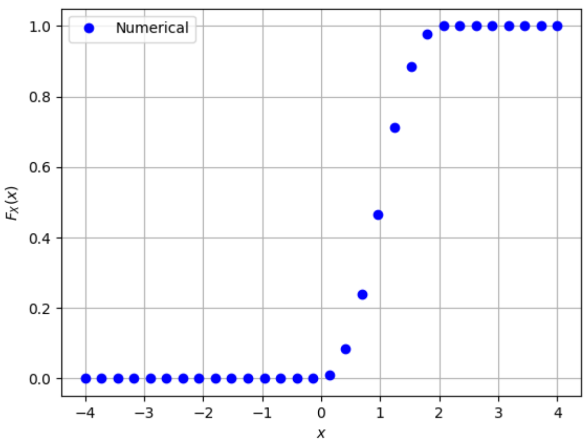

Fig. 4.2: The CDF of T

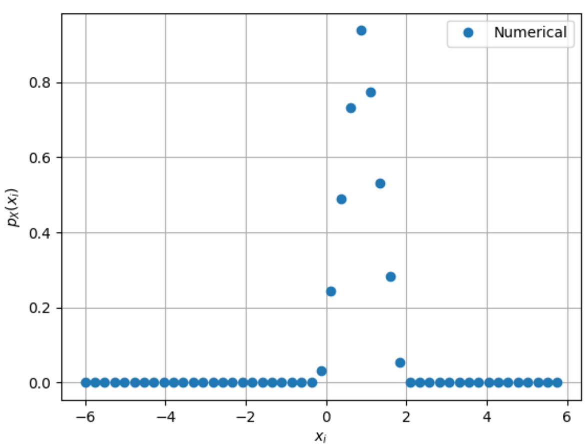

Fig. 4.3: The PDF of T

```
python3 tri_cdf.py
```

The CDF is plotted as shown in (4.2)

4.3 Find the PDF of T.

Solution:

Execute the following code to generate PDF of T.

```
python3 tri_pdf.py
```

The PDF is plotted as shown in (4.3)

4.4 Find the theoretical expressions for the PDF and CDF of T.

Solution:

Given a random variable Z as:

$$Z = X + Y \tag{4.2}$$

where X and Y are random variables, we can define

$$p_Z(t) = p_X(x) * p_Y(y) \tag{4.3}$$

$$= \int_{-\infty}^{\infty} p_X(\tau)p_Y(t-\tau)\,d\tau \tag{4.4}$$

Given $X = U$, $Y = U$ and $T = X + Y$, we have from (4.4)

$$p_T(t) = \int_{-\infty}^{\infty} p_U(\tau)p_U(t-\tau)\,d\tau \tag{4.5}$$

$$= \int_{0}^{1} p_U(t-\tau)\,d\tau \tag{4.6}$$

$$= \int_{t-1}^{t} p_U(u)\,du \tag{4.7}$$

From (1.2), we can deduce that the above integral will be non-zero only when $(t-1,t) \cap (0,1) \neq \emptyset$. Hence (**??**) will be zero when $t < 0$ and $t > 2$.

Consider the integral when $t \in (0,1)$:

$$p_T(t) = \int_{t-1}^{t} p_U(u)\,du \tag{4.8}$$

$$= \int_{0}^{t} p_U(u)\,du \tag{4.9}$$

$$= \int_{0}^{t} 1\,du \tag{4.10}$$

$$= \boxed{\text{t}} \tag{4.11}$$

Consider the integral when $t \in (1,2)$:

$$p_T(t) = \int_{t-1}^{t} p_U(u)\,du \tag{4.12}$$

$$= \int_{t-1}^{1} p_U(u)\,du \tag{4.13}$$

$$= \boxed{\text{2-t}} \tag{4.14}$$

Hence, we can state $p_T(t)$ as follows from (4.11) and (4.14):

$$p_T(t) = \begin{cases} 0, & t \in (-\infty, 0) \\ t, & t \in (0,1) \\ 2-t, & t \in (1,2) \\ 0, & t \in (0,\infty) \end{cases} \tag{4.15}$$

The CDF is related with the PDF as follows:

$$F_T(t) = \int_{-\infty}^{t} p_T(t)\,dt \tag{4.16}$$

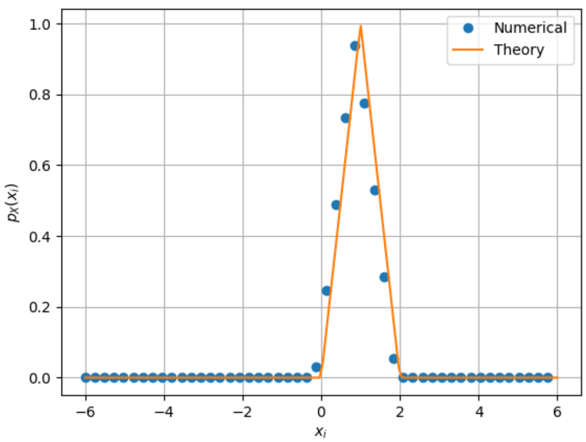

Fig. 4.5: The PDF of T

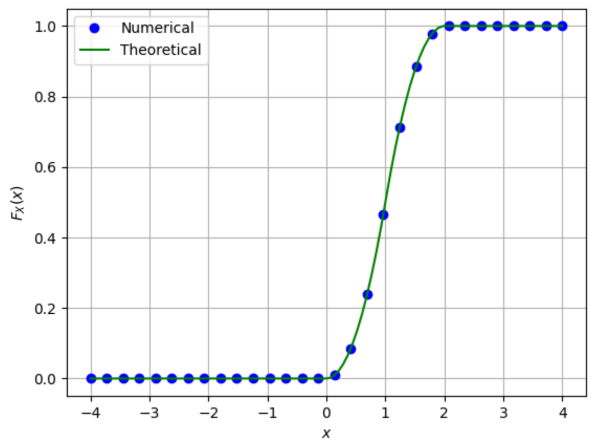

Fig. 4.6: The CDF of T

Using (4.16), we have:

$$F_T(t) = \begin{cases} 0, & t \in (-\infty, 0) \\ \frac{t^2}{2}, & t \in (0, 1) \\ \frac{1-(t-2)^2}{2}, & t \in (1, 2) \\ 1, & t \in (0, \infty) \end{cases} \tag{4.17}$$

4.5 Verify your result for the PDF through a plot.
Solution:
Execute the following code to generate theoretical and statistical PDF of T.

```
python3 tri_pdf.py
```

The plot is shown in figure (4.5).

4.6 Verify your result for the CDF through a plot.
Solution:
Execute the following code to generate theoretical and statistical CDF of T.

```
python3 tri_cdf.py
```

The plot is shown in figure (4.6)

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution:
We can generate samples for equiprobable random variable X using the following code:

```
gcc exrand.c -lm
./a.out
```

The samples generated are stored in `ber.dat`.

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

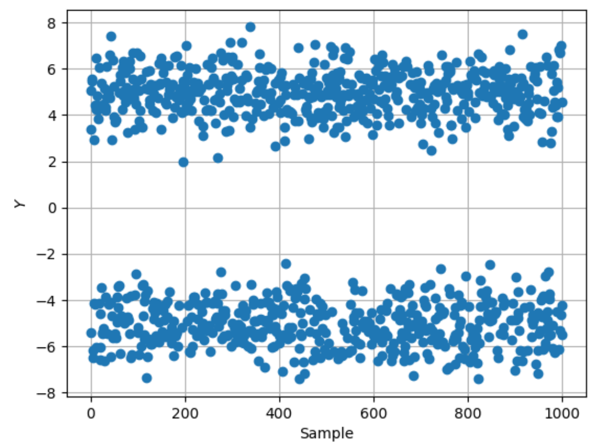

Fig. 5.3: Random Variable Y at $A = 5.0$

where $A = 5\text{dB}, X \in \{1, -1\}$, is Bernoulli and $N \sim \mathcal{N}(0, 1)$.
Solution:
We can generate samples for random variable Y using the following code:

```
gcc exrand.c -lm
./a.out
```

The samples generated are stored in `y.dat`.

5.3 Plot Y.
Solution:
We use the following code to plot all samples of Y.

```
python3 y_plot.py
```

The plot generated is shown in figure (5.3).

5.4 Guess how to estimate X from Y.
Solution:
One can roughly estimate X from Y as it is most probable that when $X > 0$, then $Y > 0$. Hence,

$$X = \begin{cases} 1, & Y > 0 \\ -1, & Y < 0 \end{cases} \tag{5.2}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.4}$$

Solution:
We can use the following code to find $P_{e|0}$ and $P_{e|1}$ as:

```
gcc exrand.c −lm
./a.out
```

In the case where $A = 2.5$, we obtain $P_{e|0} = 0.005478$ and $P_{e|1} = 0.005660$.

5.6 Find P_e assuming that X has equiprobable symbols.
Solution:
We can use the following code to find P_e:

```
gcc exrand.c −lm
./a.out
```

In the case where $A = 2.5$, we obtain $P_e = 0.005569$.

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution:

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.5}$$
$$= \frac{\Pr\left(\hat{X} = -1, X = 1\right)}{\Pr(X = 1)} \tag{5.6}$$

Using (5.2)

$$P_{e|0} = \frac{\Pr(Y < 0, X = 1)}{\Pr(X = 1)} \tag{5.7}$$
$$= \Pr(A + N < 0) \tag{5.8}$$
$$= \Pr(N < -A) \tag{5.9}$$
$$= 1 - \Pr(N \geq -A) \tag{5.10}$$
$$= 1 - Q(-A) \tag{5.11}$$

Similarly, we can write

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.12}$$
$$= \frac{\Pr\left(\hat{X} = 1, X = -1\right)}{\Pr(X = -1)} \tag{5.13}$$
$$= \frac{\Pr(Y > 0, X = -1)}{\Pr(X = -1)} \tag{5.14}$$
$$= \Pr(-A + N > 0) \tag{5.15}$$
$$= \Pr(N > A) \tag{5.16}$$
$$= Q(A) \tag{5.17}$$

Hence, we can determine P_e as follows:

$$P_e = P_{e|0}\Pr(X = 1) + P_{e|1}\Pr(X = -1) \tag{5.18}$$
$$= \frac{1}{2}\left(Q(A) + 1 - Q(-A)\right) \tag{5.19}$$
$$= \frac{1}{2}\left(2Q(A)\right) \tag{5.20}$$
$$= Q(A) \tag{5.21}$$

Note that $\Pr(X = 1) = \Pr(X = -1) = \frac{1}{2}$ and $Q(A) + Q(-A) = 1$.
We first generate statistically, various values of P_e for different values of a. We execute the following code to generate sample:

```
gcc exrand.c −lm
./a.out
```

The samples are now generated in `pe_a.dat`. To observe the theoretical plot and the statistical values of P_e vs a, we execute the following code:

```
python3 pea_graph.py
```

The plot generated is shown in the figure (**??**)

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that minimizes the theoretical P_e.
Solution:
Assuming the threshold to be δ, we can estimate X from Y:

$$X = \begin{cases} 1, & Y > \delta \\ -1, & Y < \delta \end{cases} \tag{5.22}$$

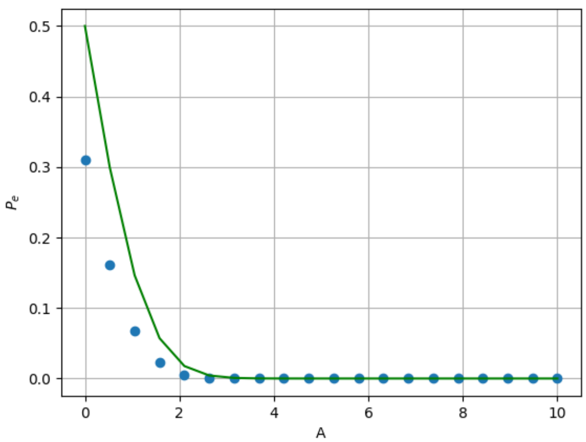

Fig. 5.7: P_e vs a

In this case

$$
\begin{aligned}
P_{e|0} &= \Pr\left(\hat{X} = -1 | X = 1\right) && (5.23)\\
&= \frac{\Pr\left(\hat{X} = -1, X = 1\right)}{\Pr(X = 1)} && (5.24)\\
&= \frac{\Pr(Y < \delta, X = 1)}{\Pr(X = 1)} && (5.25)\\
&= \Pr(A + N < \delta) && (5.26)\\
&= \Pr(N < \delta - A) && (5.27)\\
&= 1 - \Pr(N \geq \delta - A) && (5.28)\\
&= 1 - Q(\delta - A) && (5.29)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
P_{e|1} &= \Pr\left(\hat{X} = 1 | X = -1\right) && (5.30)\\
&= \frac{\Pr\left(\hat{X} = 1, X = -1\right)}{\Pr(X = -1)} && (5.31)\\
&= \frac{\Pr(Y > \delta, X = -1)}{\Pr(X = -1)} && (5.32)\\
&= \Pr(-A + N > \delta) && (5.33)\\
&= \Pr(N > \delta + A) && (5.34)\\
&= Q(\delta + A) && (5.35)
\end{aligned}
$$

We can write

$$P_e = P_{e|0} \Pr(X = 1) + P_{e|1} \Pr(X = -1) \quad (5.36)$$

$$P_e = \frac{1}{2}(1 - Q(\delta - A) + Q(\delta + A)) \quad (5.37)$$

To minimise this, we will find the value at A when

$$\frac{dP_e}{dA} = 0 \quad (5.38)$$

$$\frac{1}{2}\frac{d}{dA}(1 - Q(\delta - A) + Q(\delta + A)) = 0 \quad (5.39)$$

$$\frac{e^{-\frac{(\delta-A)^2}{2}}}{\sqrt{2\pi}} - \frac{e^{-\frac{(\delta+A)^2}{2}}}{\sqrt{2\pi}} = 0 \quad (5.40)$$

$$e^{-\frac{(\delta-A)^2}{2}} = e^{-\frac{(\delta+A)^2}{2}} \quad (5.41)$$

$$\frac{(\delta - A)^2}{2} = \frac{(\delta + A)^2}{2} \quad (5.42)$$

$$(\delta - A)^2 = (\delta + A)^2 \quad (5.43)$$

$$\delta = 0 \quad (5.44)$$

5.9 Repeat the above exercise when

$$p_X(-1) = p \quad (5.45)$$

Solution:
Using (5.36), we have:

$$
\begin{aligned}
P_e &= P_{e|0} \Pr(X = 1) + P_{e|1} \Pr(X = -1) && (5.46)\\
&= (p)(1 - Q(\delta - A)) + (1 - p)(Q(\delta + A)) && (5.47)
\end{aligned}
$$

We again minimise this, using

$$\frac{dP_e}{dA} = 0 \quad (5.48)$$

$$(p)\frac{e^{-\frac{(\delta-A)^2}{2}}}{\sqrt{2\pi}} - (1 - p)\frac{e^{-\frac{(\delta+A)^2}{2}}}{\sqrt{2\pi}} = 0 \quad (5.49)$$

$$\frac{p}{1 - p} = e^{-2A\delta} \quad (5.50)$$

$$\delta = \frac{1}{2A}\ln\frac{1 - p}{p} \quad (5.51)$$

5.10 Repeat the above exercise using the MAP criterion.
Solution:

We can deduce the PDF of X—Y is given by:

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_y(Y)} \tag{5.52}$$

Assuming $\Pr(X = -1) = p$, we have $p_y(Y)$ using total probability theorem as:

$$p_y(Y) = p_{y|x}(Y|1)\Pr(X = -1) + p_{y|x}(Y|1)\Pr(X = 1) \tag{5.53}$$

$$= (p)\, p_y(-A + N) + (1 - p)\, p_y(A + N) \tag{5.54}$$

$$= (p)\, \frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1 - p)\, \frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}} \tag{5.55}$$

Given a value of $Y = y$, $X = -1$, we have

$$p_{X|Y}(-1|y) = \frac{p_{Y|X}(y| - 1)p_X(-1)}{p_y(Y)} \tag{5.56}$$

$$= \frac{(p)\, e^{-\frac{(y+A)^2}{2}}}{(p)\, e^{-\frac{(y+A)^2}{2}} + (1 - p)\, e^{-\frac{(y-A)^2}{2}}} \tag{5.57}$$

$$= \frac{p}{p + (1 - p)\, e^{2yA}} \tag{5.58}$$

Given a value of $Y = y$, $X = 1$, we have

$$p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1)p_X(1)}{p_y(Y)} \tag{5.59}$$

$$= \frac{(1 - p)\, e^{-\frac{(y-A)^2}{2}}}{(p)\, e^{-\frac{(y+A)^2}{2}} + (1 - p)\, e^{-\frac{(y-A)^2}{2}}} \tag{5.60}$$

$$= \frac{(1 - p)\, e^{2yA}}{p + (1 - p)\, e^{2yA}} \tag{5.61}$$

Using (5.58) and (5.61), we can conclude:

$$p_{X|Y}(1|y) > p_{X|Y}(-1|y) \tag{5.62}$$

when,

$$\frac{(1 - p)\, e^{2yA}}{p + (1 - p)\, e^{2yA}} > \frac{p}{p + (1 - p)\, e^{2yA}} \tag{5.63}$$

$$e^{2yA} > \frac{p}{(1 - p)} \tag{5.64}$$

$$y > \frac{1}{2A} \ln \frac{p}{(1 - p)} \tag{5.65}$$

Hence, when (5.65) holds, we can choose $X = 1$ and $X = -1$ otherwise.
In the case that X is equiprobable, i.e., $\Pr(X = 1) = \Pr(X = -1) = \frac{1}{2}$, we consider $p = \frac{1}{2}$. Hence, we choose $X = 1$ when

$$y > \frac{1}{2A} \ln \frac{0.5}{0.5} \tag{5.66}$$

$$y > 0 \tag{5.67}$$

Also we choose $X = -1$ when $y \le 0$.

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution:
We first generate many samples for random variable V as mentioned above, using the following code:

```
gcc exrand.c −lm
./a.out
```

The samples generated are stored in `chi.dat`. To find the theoretical PDF and CDF for the random variable V, we shall assume that X_1 and X_2 are identical and independent. Now assume that

$$X_1 = R\cos\theta \tag{6.2}$$

$$X_2 = R\sin\theta \tag{6.3}$$

Now, we can see that:

$$V = X_1^2 + X_2^2 \tag{6.4}$$

$$= R^2 \tag{6.5}$$

We shall try to transform random variables X_1, X_2 to polar form, i.e., r and θ, using the Jacobian as follows:

$$p_{R,\theta}(r, \theta) = |J| p_{X_1,X_2}(x_1, x_2) \tag{6.6}$$

where

$$J = \begin{pmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} \end{pmatrix} \tag{6.7}$$

We find that the Jacobian can be simplified as:

$$J = \begin{pmatrix} \frac{\partial r\cos\theta}{\partial r} & \frac{\partial r\cos\theta}{\partial \theta} \\ \frac{\partial r\sin\theta}{\partial r} & \frac{\partial r\sin\theta}{\partial \theta} \end{pmatrix} \tag{6.8}$$

$$= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \tag{6.9}$$

Now we can evaluate the determinant as:

$$|J| = (\cos\theta)(r\cos\theta) - (\sin\theta)(-r\sin\theta) \quad (6.10)$$
$$= r \quad (6.11)$$

Since X_1 and X_2 are independent and identical, we can deduce:

$$p_{X_1,X_2}(x_1, x_2) = p_{X_1}(x_1)p_{X_2}(x_2) \quad (6.12)$$

Using (6.11) and (6.12) in (6.6), we can conclude:

$$p_{r,\theta}(r,\theta) = r p_{X_1}(x_1)p_{X_2}(x_2) \quad (6.13)$$
$$= \frac{r}{2\pi}\left(e^{-\frac{x_1^2}{2}}\right)\left(e^{-\frac{x_2^2}{2}}\right) \quad (6.14)$$
$$= \frac{r}{2\pi}\left(e^{-\frac{x_1^2+x_2^2}{2}}\right) \quad (6.15)$$
$$= \frac{r}{2\pi}\left(e^{-\frac{r^2}{2}}\right) \quad (6.16)$$

Marginal probability $p_R(r)$ can be found as:

$$p_R(r) = \int_0^{2\pi} p_{R,\theta}(r,\theta)\, d\theta \quad (6.17)$$
$$= \int_0^{2\pi} \frac{r}{2\pi}\left(e^{-\frac{r^2}{2}}\right) \quad (6.18)$$
$$= re^{-\frac{r^2}{2}} \quad (6.19)$$

Note that $p_R(r)$ is only defined as above only when $r \geq 0$ and is 0 otherwise.
Hence, we can define distribution $F_R(r)$ as:

$$F_R(r) = \Pr(R \leq r) \quad (6.20)$$
$$= \int_{-\infty}^{r} p_R(r)\, dr \quad (6.21)$$
$$= \int_0^r re^{-\frac{r^2}{2}}\, dr \quad (6.22)$$
$$= 1 - e^{-\frac{r^2}{2}} \quad (6.23)$$

when $r \geq 0$ and 0 when $r < 0$.
Using (6.5), we can determine the distribution for random variable V:

$$F_V(x) = \Pr(V \leq x) \quad (6.24)$$
$$= \Pr\left(R^2 \leq x\right) \quad (6.25)$$
$$= \Pr\left(0 \leq R \leq \sqrt{x}\right) \quad (6.26)$$
$$= F_R(\sqrt{x}) \quad (6.27)$$

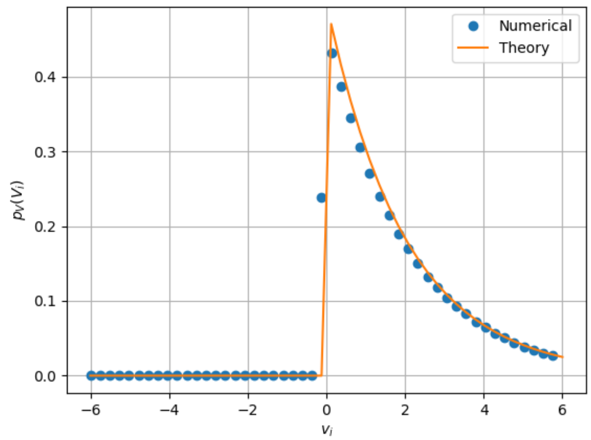

Fig. 6.1: PDF of V

Hence,

$$F_V(x) = \begin{cases} 1 - e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (6.28)$$

Differentiating this, we obtain the density of random variable V:

$$p_V(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (6.29)$$

We now plot the statistical and theoretical PDF using the following code:

```
python3 chi_pdf.py
```

The plot is generated in the figure (6.1).
We now plot the statistical and theoretical CDF using the following code;

```
python3 chi_cdf.py
```

The plot is generated in the figure (6.1)

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.30)$$

find α.
Solution:
Using (6.28), we can compare the given equation in the question and obtain

$$\alpha = \frac{1}{2} \quad (6.31)$$

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \quad (6.32)$$

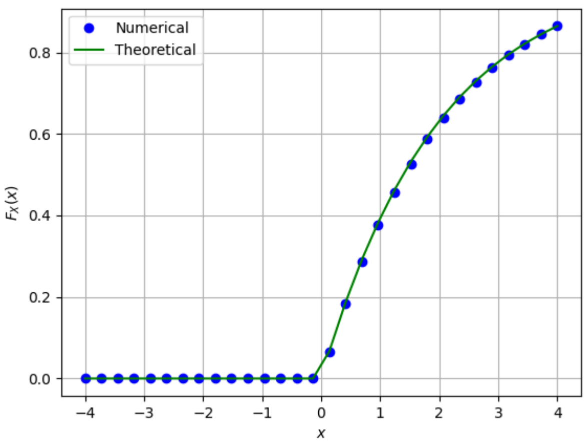

Fig. 6.1: CDF of V

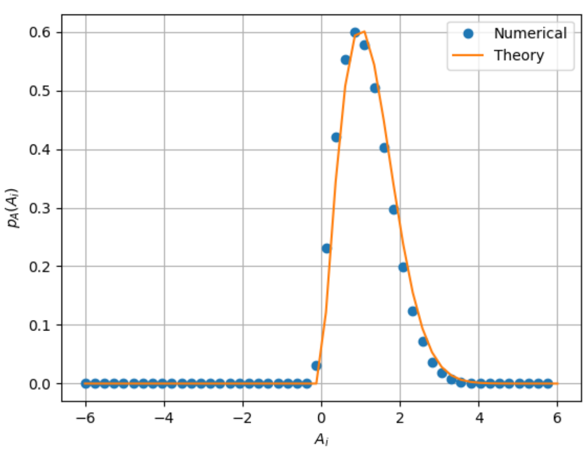

Fig. 6.3: PDF of A

Solution:
We first generate samples of random variable using the following code:

```
gcc exrand.c −lm
./a.out
```

The samples generated are stored in `ray.dat`. Using these samples we can generate PDF plot using the following code:

```
python3 ray_pdf.py
```

The plot generated is shown in figure (6.3). Using these samples we can generate PDF plot using the following code:

```
python3 ray_cdf.py
```

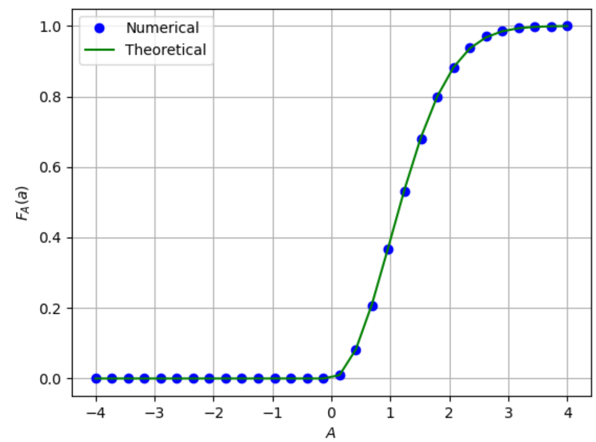

Fig. 6.3: CDF of A

The plot generated is shown in figure (6.3).

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0, 1), X \in (-1, 1)$ for $0 \le \gamma \le 10$ dB.

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3}$$

Find $P_e = E\left[P_e(N)\right]$.

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in \left(\mathbf{s}_0, \mathbf{s}_1\right), \mathbf{s}_0 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix}0\\1\end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix}n_1\\n_2\end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.5}$$

on the same graph using a scatter plot.

8.3 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.4 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.6}$$

with respect to the SNR from 0 to 10 dB.