All of Statistics: Random Variables

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Selected Exercises and My Solutions

Exercise 16

Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ and assume that X and Y are independent. Find the distribution of X given that $X + Y = n$.

My solution:

$X + Y \sim \text{Poisson}(\lambda + \mu)$ because X and Y are independent.

$$f_{X|X+Y=n}(x|x+y=n) = \frac{f_{X,X+Y=n}(x, x+y=n)}{f_{X+Y}(x+y)} = \frac{f_{X,Y}(x, n-x)}{f_{X+Y}(x+y)} = \frac{f_X(x) f_Y(y=n-x)}{f_{X+Y}(x+y)} = \frac{\frac{\lambda^x}{e^\lambda x!}\frac{\mu^{n-x}}{e^\mu (n-x)!}}{\frac{(\mu+\lambda)^{x+y}}{e^\mu e^\lambda (x+y)!}} = \frac{\frac{\lambda^x}{x!}\frac{\mu^{n-x}}{(n-x)!}}{\frac{(\mu+\lambda)^{x+y}}{(x+y)!}}$$

$$= \frac{\frac{\lambda^x}{x!}\frac{\mu^{n-x}}{(n-x)!}}{\frac{(\mu+\lambda)^n}{n!}} = \binom{n}{x}\frac{\lambda^x \mu^{n-x}}{(\mu+\lambda)^n} = \binom{n}{x}(\frac{\lambda}{\lambda+\mu})^x(1 - \frac{\lambda}{\lambda+\mu})^{n-x}$$

This shows that X given X+Y=n $\sim Binomial(n, \frac{\lambda}{\mu+\lambda})$

Exercise 17

Let

$$f_{X,Y}(x,y) = \begin{cases} c(x+y)^2 & 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P\left(X < 0.5 \mid Y = 0.5\right)$.

My solution:

First, find $f_Y(y)$ by integrating over x. $f_Y(y) = \int_0^1 c(x+y)^2 dx = c(\frac{1}{3} + y + y^2)$

So, $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{c(x+y)^2}{c/3+cy+cy^2} = \frac{(x+y)^2}{1/3+y+y^2}$

$P\left(X < 0.5 \mid Y = 0.5\right) = \int_0^{0.5} \frac{(x+0.5)^2}{1/3+0.5+0.5^2} dx \approx 0.2692$

Exercise 20 (Difference Distribution)

Let $X \sim \text{Uniform}(0,1)$ and $Y \sim \text{Uniform}(0,1)$ be independent. Find the probability density function of $X - Y$.

My solution:

Let $Z = X - Y$. I need to find $F_Z(z) = P(Z \leq z) = P(X - Y \leq z)$

For -1<z<0:

$$P(X - Y \leq z) = \int_0^{1+z} \int_{x-z}^{1} 1 dydx = z^2/2 + z + 1/2$$

For 0<z<1:

$$P(X - Y \leq z) = 1 - \int_z^1 \int_0^{x-z} 1 dxdy = -z^2/2 + z + 1/2$$

$\therefore$

$f_Z(z)$ = z + 1 for -1 < z < 0, 1 - z for 0 < z < 1, 0 otherwise.

Exercise 20 (Ratio Distribution)

Let $X \sim \text{Uniform}(0,1)$ and $Y \sim \text{Uniform}(0,1)$ be independent. Find the probability density function of X/Y.

Let $Z = X/Y$. I need to find $F_Z(z) = P(Z \leq z) = P(X/Y \leq z)$

For $z \leq 0$:

$$P(X/Y < z) = 0$$

For $0 < z < 1$:

$$P(X/Y < z) = \int_0^z \int_{x/z}^1 dydx = \frac{z}{2}$$

For $z \geq 1$

$$P(X/Y < z) = 1 - \int_0^1 \int_0^{x/z} dydx = 1 - \frac{1}{2z}$$

Differentiating to obtain the pdf.

Exercise 20 (Simulation Study)

$f_Z(z) = 1/2$ for $0 < z < 1$, $\frac{1}{2z^2}$ for $z \geq 1$, 0 otherwise.

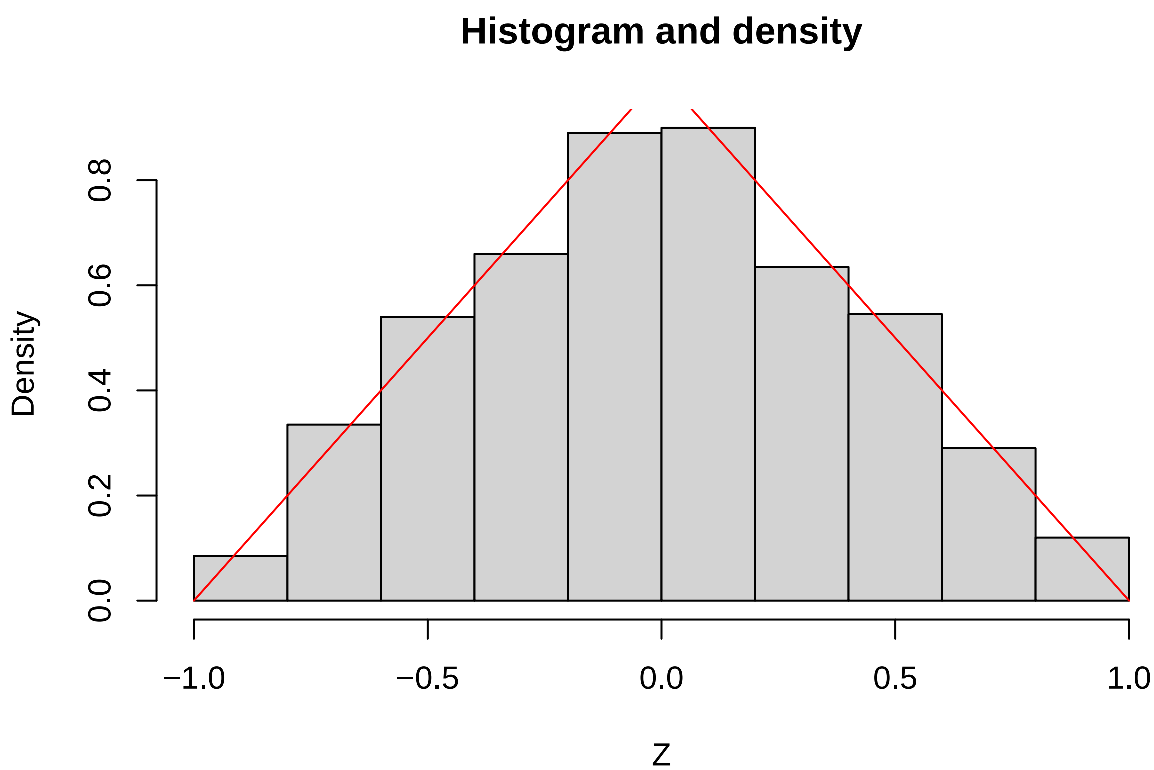
Use a computer experiment to verify your results above. For each, do the following:

- Generate a vector $x = (x_1, x_2, \ldots x_{1000})$ from a $\text{Uniform}(0,1)$ distribution.
- Generate a vector $y = (y_1, y_2, \ldots y_{1000})$ from a $\text{Uniform}(0,1)$ distribution.
- Define $z = x - y$ or $z = x/y$.
- Plot a histogram of z. (Be sure to use density histogram, not frequency.)
- Overlay the true density that you calculated.

Simulation for the Difference Distribution

```
set.seed(123)
X = runif(1000, min=0, max=1)
Y = runif(1000, min=0, max=1)
Z = X-Y
true_diff = function (z) {
  ifelse(z < 0, 1 + z, 1 - z)
}

# Create a histogram
hist(Z, freq = FALSE, main = "Histogram and density")
# Add density
curve(true_diff(x), add = TRUE, col = "red")
```



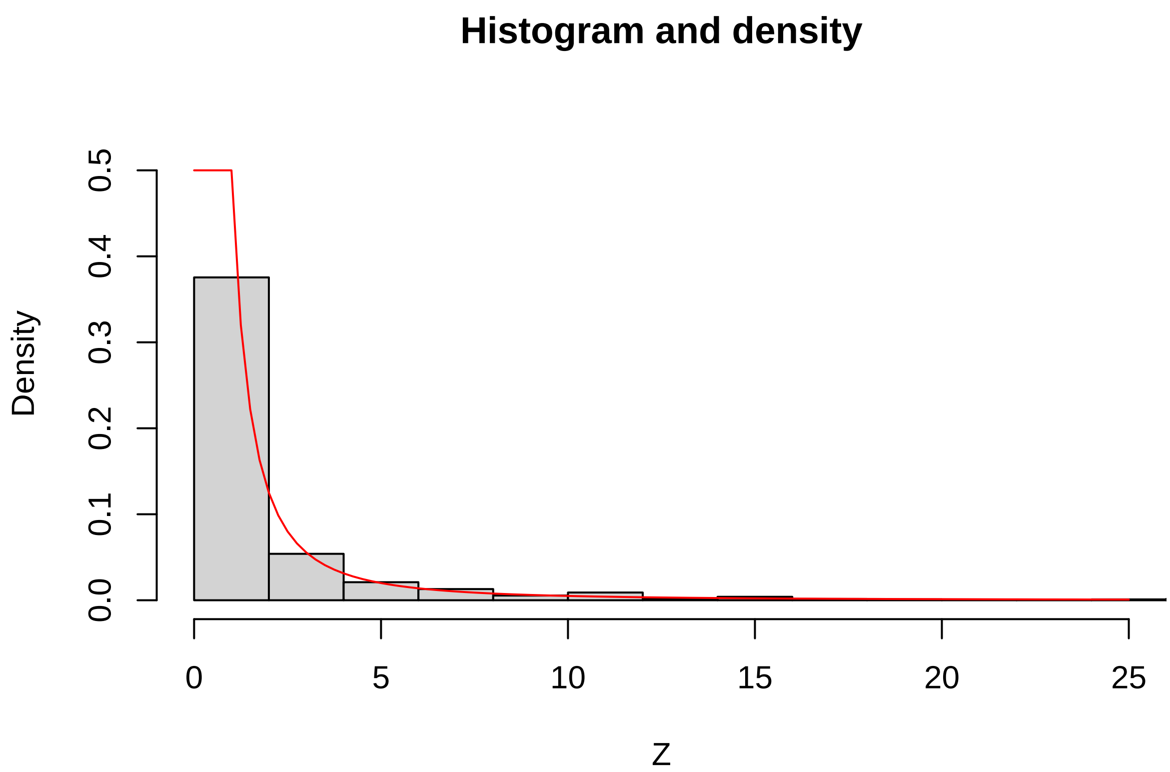
Simulation for the Ratio Distribution

```
X = runif(1000, min = 0, max = 1)
Y = runif(1000, min=0, max=1)
Z = X / Y

true_ratio = function (z) {
  ifelse(z < 1, 0.5, 1/(2*z^2))
}

# Create a histogram
hist(Z, freq = FALSE,breaks = 1000,xlim = c(0, 25),ylim = c(0, 0.55), main = "Histogram and density")

# Add density
curve(true_ratio(x), add = TRUE, col = "red")
```


Histogram and density
Density
0.0
0.1
0.2
0.3
0.4
0.5
0
5
10
15
20
25
Z