All of Statistics: Expectation

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Selected Exercises and My Solutions

Exercise 3

Let $X_1, \ldots, X_n \sim \text{Uniform}(0, 1)$. Define $Y_n = \max\{X_1, \ldots, X_n\}$. Find $\mathbb{E}[Y_n]$.

$F_{Y_n}(y) = P(Y_n < y) = P(\max\{X_1, \ldots, X_n\} < y) = P(X_1 < y)P(X_2 < y)...P(X_n < y) = F_{X_1}(y)F_{X_2}(y)...F_{X_n}(y) = (F_{X_i}(y))^n$

$F_{X_i}(y) = \int_0^y dx = y$

$F_{Y_n}(y) = y^n$

$f_{Y_n}(y) = ny^{n-1}$

The expectation value of Y is: $\mathbb{E}[Y_n] = \int_0^1 yny^{n-1}dy = n\int_0^1 y^n dy = \frac{n}{n+1}$

Exercise 4

A particle starts at the origin of the real line and moves along the line in jumps of one unit. For each jump the probability is p that the particle will jump one unit to the left and the probability is $1-p$ that the particle will jump one unit to the right. Let X_n be the position of the particle after n jumps. Find $\mathbb{E}[X_n]$ and $\mathbb{V}[X_n]$.

Let X_n be the position of the particle after n jump, then$X_n = \Sigma_{i=1}^n X_i$ where X_i has the distribution:

x_i	$P(X_i = x_i)$
1	1-p
-1	p

$\therefore$

$\mathbb{E}[X_i] = (1-p) - p = 1 - 2p$

$\mathbb{E}[X_n] = \mathbb{E}[\Sigma_{i=1}^n X_i] = n(1-2p) = n - 2np$

$\mathbb{V}[X_i] = (1 - 1 + 2p)^2 * (1-p) + (-1 - 1 + 2p)^2 * p = 4p^2(1-p) + (4p^2 - 8p + 4)p = 4p - 4p^2$

$\mathbb{V}[X_n] = n(4p - 4p^2)$

Exercise 5

A fair coin is tossed until a head is obtained. What is the expected number of tosses that will be required?

Let X be the number of tosses until a head is obtained. X follows a Geometric distribution(p=0.5). Let's find the expected value of a general Geometric distribution.

$E(X) = \sum_{x=1}^{\infty} x(1-p)^{x-1}p = p\sum_{x=1}^{\infty} x(1-p)^{x-1}$

$E(X) = p - \frac{d}{dp}\sum_{x=1}^{\infty}(1-p)^x$

$= p - \frac{d}{dp}\frac{1-p}{p} = p(\frac{1}{p^2}) = \frac{1}{p}$

In the case of a fair coin toss, the expected number of tosses required is $1/0.5 = 2$ tosses.

Exercise 10

Let $X \sim \text{Normal}(0,1)$. Define $Y = e^X$. Find $\mathbb{E}[Y]$ and $\mathbb{V}[Y]$.

Use the Rule of the Lazy Statistician:

$\mathbb{E}[Y] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^x e^{\frac{-x^2}{2}}dx = \sqrt{e}$

$\mathbb{E}[Y^2] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{2x} e^{\frac{-x^2}{2}}dx = e^2$

$\therefore$

$\mathbb{V}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = e^2 - e$

Exercise 13

Suppose we generate a random variable X in the following way. First we flip a fair coin. If the coin is heads, take X to have a Uniform$(0,1)$ distribution. If the coin is tails, take X to have a Uniform$(3,4)$ distribution. Find the mean and standard deviation of X. Find the standard deviation of X.

$\mathbb{E}[X] = \frac{1}{2}\frac{0+1}{2} + \frac{1}{2}\frac{3+4}{2} = .25 + 1.75 = 2$

$\mathbb{V}[X] = \sigma^2 = \mathbb{E}(X-\mu)^2 = \frac{1}{2}\int_0^1 (x-2)^2 dx + \frac{1}{2}\int_3^4 (x-2)^2 dx = \frac{7}{3}$

$\mathbb{SD}[X] = \sqrt{\frac{7}{3}}$