

All of Statistics: Random Variables

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Selected Exercises and My Solutions

Exercise 4

Let X have the probability density function

$$f_X(x) = \begin{cases} 1/4 & 0 < x < 1 \\ 3/8 & 3 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

Define $Y = 1/X$. Find the probability density function of Y .

My solution:

The CDF of X

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x/4 & 0 < x < 3 \\ 3x/8 & 3 < x < 5 \\ 1 & x \geq 5 \end{cases}$$

$$F_Y(y) = P(Y < y) = P(1/X < y) = P(X > 1/y) = 1 - F_X(x = 1/y)$$

$$F_Y(y) = \begin{cases} 1 & 1/y < 0 \\ 1 - 1/4y & 0 < 1/y < 3 \\ 1 - 3/8y & 3 < 1/y < 5 \\ 0 & 1/y \geq 5 \end{cases}$$

\therefore

$$f_Y(y) = \begin{cases} 1/4y^2 & y > 1/3 \\ 3/8y^2 & 1/3 > y > 1/5 \\ 0 & \text{otherwise} \end{cases}$$

Exercise 16

Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ and assume that X and Y are independent. Find the distribution of X given that $X + Y = n$.

My solution:

$X + Y \sim \text{Poisson}(\lambda + \mu)$ because X and Y are independent.

$$f_{X|X+Y=n}(x|x+y=n) = \frac{f_{X,X+Y=n}(x, x+y=n)}{f_{X+Y}(x+y)} = \frac{f_{X,Y}(x, n-x)}{f_{X+Y}(x+y)} = \frac{f_X(x)f_Y(y=n-x)}{f_{X+Y}(x+y)} = \frac{\frac{\lambda^x}{e^\lambda x!} \frac{\mu^{n-x}}{e^\mu (n-x)!}}{\frac{(\mu+\lambda)^{x+y}}{e^{\mu+\lambda} (x+y)!}} = \frac{\frac{\lambda^x}{x!} \frac{\mu^{n-x}}{(n-x)!}}{\frac{(\mu+\lambda)^{x+y}}{(x+y)!}}$$

$$= \frac{\frac{\lambda^x}{x!} \frac{\mu^{n-x}}{(n-x)!}}{\frac{(\mu+\lambda)^n}{n!}} = \binom{n}{x} \frac{\lambda^x \mu^{n-x}}{(\mu+\lambda)^n} = \binom{n}{x} \left(\frac{\lambda}{\lambda+\mu}\right)^x \left(1 - \frac{\lambda}{\lambda+\mu}\right)^{n-x}$$

This shows that X given $X+Y=n \sim \text{Binomial}(n, \frac{\lambda}{\mu+\lambda})$

Exercise 17

Let

$$f_{X,Y}(x,y) = \begin{cases} c(x+y)^2 & 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X < 0.5 | Y = 0.5)$.

My solution:

First, find $f_Y(y)$ by integrating over x . $f_Y(y) = \int_0^1 c(x+y)^2 dx = c(\frac{1}{3} + y + y^2)$

So, $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{c(x+y)^2}{c/3+cy+cy^2} = \frac{(x+y)^2}{1/3+y+y^2}$

$P(X < 0.5 | Y = 0.5) = \int_0^{0.5} \frac{(x+0.5)^2}{1/3+0.5+0.5^2} dx \approx 0.2692$

Exercise 20 (Difference Distribution)

Let $X \sim \text{Uniform}(0,1)$ and $Y \sim \text{Uniform}(0,1)$ be independent. Find the probability density function of $X - Y$.

My solution:

Let $Z = X - Y$. I need to find $F_Z(z) = P(Z \leq z) = P(X - Y \leq z)$

For $-1 < z < 0$:

$$P(X - Y \leq z) = \int_0^{1+z} \int_{x-z}^1 1 dy dx = z^2/2 + z + 1/2$$

For $0 < z < 1$:

$$P(X - Y \leq z) = 1 - \int_z^1 \int_0^{x-z} 1 dx dy = -z^2/2 + z + 1/2$$

\therefore

$f_Z(z) = z + 1$ for $-1 < z < 0$, $1 - z$ for $0 < z < 1$, 0 otherwise.

Exercise 20 (Ratio Distribution)

Let $X \sim \text{Uniform}(0, 1)$ and $Y \sim \text{Uniform}(0, 1)$ be independent. Find the probability density function of X/Y .

Let $Z = X/Y$. I need to find $F_Z(z) = P(Z \leq z) = P(X/Y \leq z)$

For $z \leq 0$:

$$P(X/Y < z) = 0$$

For $0 < z < 1$:

$$P(X/Y < z) = \int_0^z \int_{x/z}^1 dy dx = \frac{z}{2}$$

For $z \geq 1$

$$P(X/Y < z) = 1 - \int_0^1 \int_0^{x/z} dy dx = 1 - \frac{1}{2z}$$

Differentiating to obtain the pdf.

$f_Z(z) = 1/2$ for $0 < z < 1$, $\frac{1}{2z^2}$ for $z \geq 1$, 0 otherwise.

Exercise 20 (Simulation Study)

Use a computer experiment to verify your results above. For each, do the following:

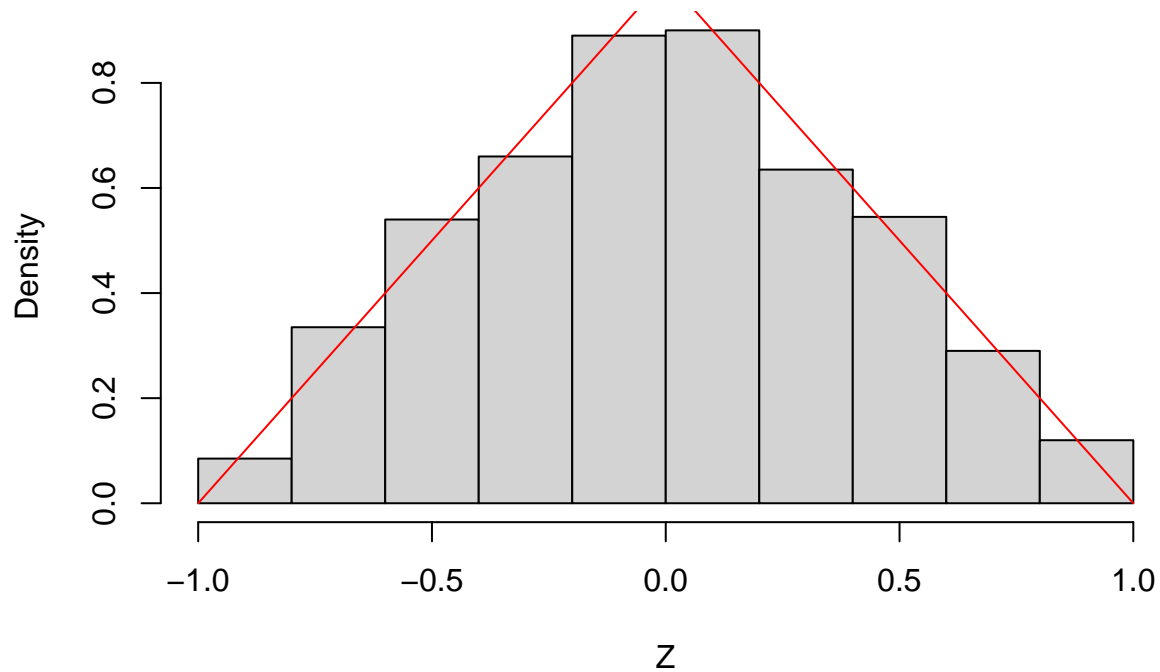
- Generate a vector $x = (x_1, x_2, \dots, x_{1000})$ from a $\text{Uniform}(0, 1)$ distribution.
- Generate a vector $y = (y_1, y_2, \dots, y_{1000})$ from a $\text{Uniform}(0, 1)$ distribution.
- Define $z = x - y$ or $z = x/y$.
- Plot a histogram of z . (Be sure to use density histogram, not frequency.)
- Overlay the true density that you calculated.

Simulation for the Difference Distribution

```
set.seed(123)
X = runif(1000, min=0, max=1)
Y = runif(1000, min=0, max=1)
Z = X-Y
true_diff = function (z) {
  ifelse(z < 0, 1 + z, 1 - z)
}

# Create a histogram
hist(Z, freq = FALSE, main = "Histogram and density")
# Add density
curve(true_diff(x), add = TRUE, col = "red")
```

Histogram and density



Simulation for the Ratio Distribution

```
X = runif(1000, min = 0, max = 1)
Y = runif(1000, min=0, max=1)
Z = X / Y

true_ratio = function (z) {
  ifelse(z < 1, 0.5, 1/(2*z^2))
}

# Create a histogram
hist(Z, freq = FALSE, breaks = 1000, xlim = c(0, 25), ylim = c(0, 0.55), main = "Histogram and density")

# Add density
curve(true_ratio(x), add = TRUE, col = "red")
```

Histogram and density

