Topology

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Introduced by Poincaré in the XIX century, topology is a branch of geometry that studies geometrical objects – also known as *topological spaces* – under the equivalence relation of *homeomorphism*[1], and has many different applications (artificial intelligence, artificial vision, etc). Initially, the concept of topological space grew out of the study of the real line, the Euclidean space and the continuous functions on these spaces; of course, these basic concepts evolved in time into a consistent theory From now on we'll consider the elementary concepts associated with topological spaces (open and closed sets, base of a topology, continuous functions, etc), what is a topology and how to construct a topology on a set so as to make it into a topological space.

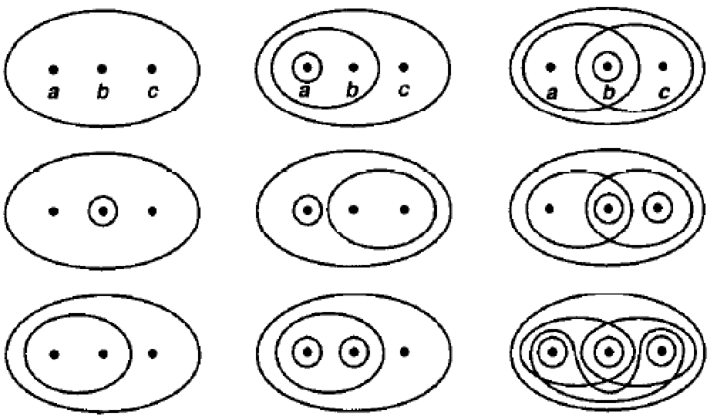
Main definition of topology

Given a set X, a **topology on X** is a pair $(X, \mathcal{O}X)$ where X is an nonempty set of elements and $\mathcal{O}X$ a collection of subsets of X having the following properties:

1. $\emptyset, X \in \mathcal{O}X$, meaning that $\mathcal{O}X$ has the empty set and the full set.
2. **(Arbitrary union)** The union of any collection of sets of $\mathcal{O}X$ is open.
3. **(Finite intersection)** The intersection of an finite number of set of $\mathcal{O}X$ is open.

Every set belonging to $\mathcal{O}X$ is also called an ***open set***.

Example #1. Let X be a three-element set, $x = \{a, b, c\}$. There are many possible topologies on X, as we can see from this diagram:



For example, a topology on X is $X = (X, \mathcal{O}X)$ such that $X = \{a, b, c\}$ and $\mathcal{O}X = \{\{a\}, \{a, b\}, \{a, b, c\}, \emptyset\}$. In fact, both the full set and the empty set belong to $\mathcal{O}X$; as for the arbitrary union and the finite intersection:

- **Arbitrary union**: $\{a\} \cup \{a, b\} = \{a, b\}; \{a\} \cup \{a, b, c\} = \{a, b, c\}, \{a\} \cup \emptyset = \{a\}$, so $\mathcal{O}X$ is closed under arbitary union;
- **Finite intersection**: $\{a\} \cap \{a, b\} = \{a\}; \{a\} \cap \{a, b, c\} = \{a\}, \{a\} \cap \emptyset = \emptyset$, so $\mathcal{O}X$ is closed under finite intersection.

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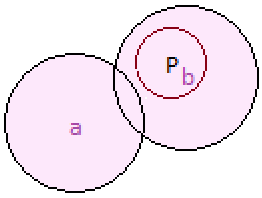
[1] A homeomorphism is a bijective, continuous function $f\colon X \to Y$ such that the inverse function $f^{-1}\colon Y \to X$ not only exists but it's also continuous. We'll further study homeomorphism in the following pages; from now on we'll focus on topological spaces which are, like X and Y of the functions f and f^{-1}, subspaces of some Euclidean space R^n.

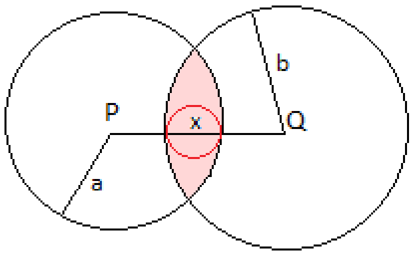
Properties of open sets: an analysis. The properties of open sets are rather interesting. **Property #1** shows that the full set and the empty set are *always* open sets. **Property #2** means that $\boldsymbol{X}$ **is closed under arbitrary union**: however we choose two or more open sets from $\mathcal{O}X$, their union is an open set belonging to $\mathcal{O}X$. In fact, suppose we have two open sets $A, B \in \mathcal{O}X$ and a point $x \in (A \cup B)$. Then there must be a ball centered in x and contained in some open set U. But U is an open set, so the ball centered in x is also contained in the union of all the open sets, OX.

if $(U_i: i \in I)$ are open, $\bigcup_{i \in I} U_i$ is also open



Lastly, **property #3** says that $\boldsymbol{X}$ **is closed under finite intersection**: however we choose a finite number of open sets from $\mathcal{O}X$, their intersection always belongs to $\mathcal{O}X$. In fact, consider two open sets $P, Q \in \mathcal{O}X$ with a **nonempty intersection**: any point in that intersection is also open because we can always find an open ball inside it.

if $(U_j: j \in J)$ is a finite family of open sets, $\bigcap_{j \in J} U_j$ is also open.



Note that this property does not work for the **intersection of *every* open set of $\boldsymbol{\mathcal{O}X}$**: this intersection, if considered on the plane (topology over R^2) would yield a single point, which is *not* an open set[2].

Closed sets. Closed sets are the complement of open sets. This means that:

- $X, \emptyset$ are always closed sets, because their complements are the open sets $\emptyset$ and X respectively.
- The finite union of closed sets is a closed set.
- The arbitrary intersection of closed sets is a closed set.

if $A \subset X$ is open then $X \backslash \{A\}$ is closed

As we can see, instead of using open sets we could just as well specify a topology on a space by giving a collection of closed sets satisfying the properties we've just seen. However, this procedure has no particular advantage over the one we've adopted so far.

[2] The base of the plane is the open ball, a two-dimensional figure, while the point is instead one-dimensional. We can't draw an open ball inside a point, so the point is *not open*.

Base of a topology

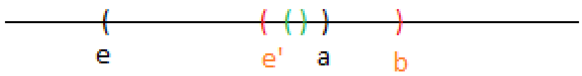
In the previous examples we've seen a topology described by listing its entire collection of open sets. Usually this is too difficult: in most cases we'd rather specify a smaller collection of open subsets and define the topology in terms of that; this smaller collection, which serves as a "generator" of the topology, is called ***base of the topology***.

If X is a set, a base for a topology on X is a collection $\mathcal{B}$ of open sets such that every open set belonging to $\mathcal{O}X$ results from the **union of the open sets of** $\mathcal{B}$. The open sets of $\mathcal{B}$ are also called ***base elements***. In other words, this means that every open set belonging to $\mathcal{O}X$ results from the union of sets belonging to $\mathcal{B}$:

$$U \in \mathcal{O}X \quad then \quad U = B_1 \cup B_2, where\ B_1, B_2 \in \mathcal{B}$$

Any subset of X is a possible base for a topology on X if it satisfies these properties:

1. For every element $x \in X$ there is a base element containing x: $\forall x \in X$, $\exists B \in \mathcal{B} \quad s.t. \quad x \in B$.
2. If x belongs to the intersection of two base elements A, B, then there's always a third base element C such that $x \in C \subset (A \cap B)$. For example, consider the straight line where open sets are open intervals: there's always an interval in the intersection of two intervals of the base.

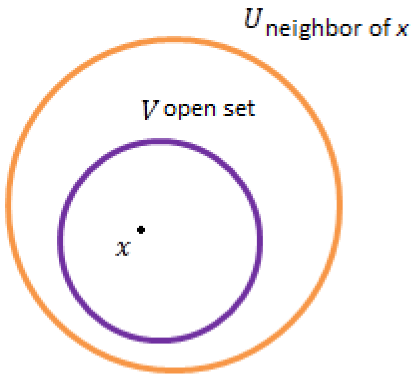


If a subset of X satisfies these properties, then we can say that set generates a topology on X. Examples of bases can be seen in the following topologies.

Neighborhood System

Alternatively, topologies can also be defined using the notion of neighborhood. Consider a point $x \in X$: a **neighbor of x is a set U containing an open set V such that $x \in V$.**

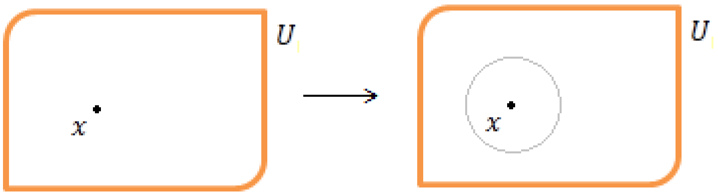
$$U\ neighbor\ of\ x \quad \rightarrow \quad \exists\ open\ set\ V\ such\ that\ \boldsymbol{x} \in \boldsymbol{V} \subseteq \boldsymbol{U}$$



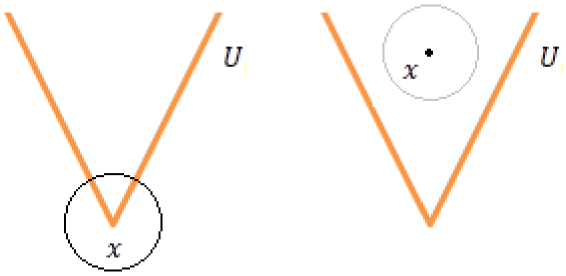
The collection of *every possible* neighbor of the point $x \in X$ is called a *neighborhood* of x.

$$\mathcal{NO}_X = \{every\ neighbor\ of\ x\}$$

Examples. Consider the set U and a point $x \in U$: is U a neighbor of the point x? **Yes**: we can find an open ball $B(x \in U, r > 0)$ completely contained in the set U.

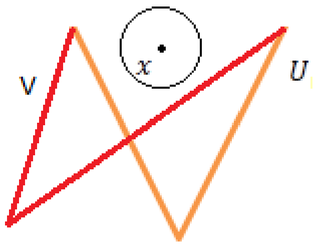


Consider instead this upper set U and a point x exactly on the frontier: U is not a neighbor of x because every open ball centered in x **never lies completely within the cone**; if we change the position of the point, however, U can be a neighbor of x:



Neighborhood also has some **properties**:

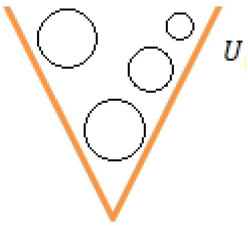
1. The **intersection of neighbors is still a neighbor**: neighborhood is closed under intersection. In fact, if U, V are neighbors of x, we can always find an open ball belonging to $U \cap V$:



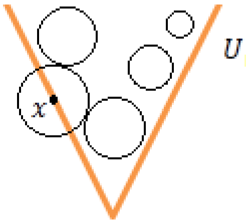
2. The **union of neighbors is still a neighbor**: neighborhood is closed under union. In fact, if U, V are neighbors of x, we can always find an open ball $B(x, r)$ belonging to $U \cup V$.
3. If U is a neighbor of x and $U \subseteq V$, $then$ V is a neighbor of x. In fact, an open set belonging to U also belongs to V.
4. If U is a neighbor of x then there's a subset $V \subseteq U$ such that V is a neighbor of each point of U.

Open/closed sets using neighbors. The notions of open/ closed sets can be redefined using neighborhood:

- **Open set**. A neighbor U of a point x is open if U is also a neighbor of every point belonging to U. For example, this cone (without the frontier) is an open set:



- **Closed set**. A neighbor U of the point x is a closed set if it's *not* a neighbor for every point belonging to U. If we consider the cone with its frontier, then many balls fall partially outside the figure: this means that the cone is a closed set.



Note that the notions of open/closed sets using neighbors are compatible with the definition of open/closed sets we've seen before: the topology obtained from the notion of neighborhood system defined is equivalent to the previous definitions of topology, and viceversa.

Exercise. Consider the set $X = \{0,1,2,a,b,c\}$ with this partial ordering: $0 < 1 < 2,\ \ a < b < c$. Find at least a couple of neighbors of the element 2.

Solution. A neighbor of 2 is a set containing "2" and an open set containing 2. To solve the problem we must first list the open sets of this topology:

- **Open sets**: $\{0,1,2\}, \{1,2\}, \{2\}, \{a,b,c\}, \{b,c\}, \{c\}$.

This is because X is an *Alexandrov topology*, where open sets are upper open sets. Therefore some neighbors of 2 are:

- $\{1,b,2\}$: it contains 2 and an open set $\{1,2\}$ containing 2.
- $\{2,a\}$: it contains 2 and an open set $\{2\}$ containing 2.
- $\{0,1,2,c\}$: it contains 2 and an open set $\{0,1,2\}$ containing 2.

Exercise. Consider the previous exercise. Is {1,b} a neighbor of 1?

Solution. The answer is no: $\{1,b\}$ contains 1 but not an open set containing 1.

Construction of spaces from given spaces: subspaces

Let $X = (X, \mathcal{O}X)$ be a topological space with a certain topology. If $Y \subseteq X$ is a subset of X, then the collection $\mathcal{O}Y = Y \cap U$ where $U \in \mathcal{O}X$ is a topology on Y induced by the space X, and is called the ***subspace topology***. Said differently, given a subset Y, the induced topology results from the **intersection of the open sets of X with Y**.

$$\mathcal{O}Y = \{U \cap Y\} \qquad (U \in \mathcal{O}X \; and \; Y \subseteq X)$$

It's easy to see that $\mathcal{O}Y$ is a topology: $\mathcal{O}Y$ contains $\emptyset$ and Y(green elements are from $\mathcal{O}X$):

$$Y \cap \emptyset = \emptyset \qquad Y \cap X = Y$$

$\mathcal{O}Y$ is also closed under arbitrary union and finite intersection. Furthermore, the **base** of the induced topology $\mathcal{O}Y$ results from the intersection of Y with the base of X:

$$\mathcal{B}_{\mathcal{O}Y} = \{B \cap Y\} \qquad (B \in \mathcal{B}_{\mathcal{O}X} \; and \; Y \subseteq X)$$

Open and closed subsets. When dealing with spaces and subspace we must be careful with the notion of ***open set***: if $Y \subseteq X$, the set **U is open in Y** if it belongs to the open sets of Y:

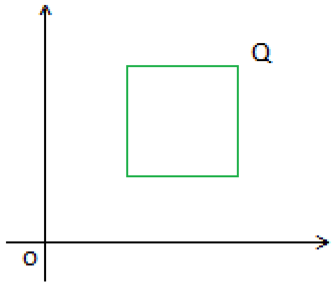
$$U\ open\ in\ Y \quad if\ U \in OY$$

If U also belongs to the open sets of X, then U is also open in X. There's a special situation in which **every set open in Y is also open in X**: if U is open in Y and Y is open in X, then U is open in X.

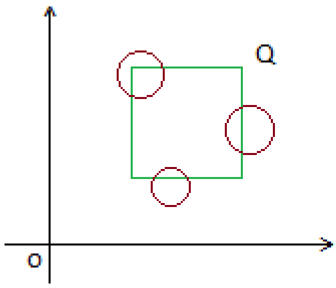
$$U\ open\ in\ Y, Y\ open\ in\ X \rightarrow U\ open\ in\ X$$

There's not the same level of certainty when closed sets are concerned: if U is a closed set in Y and Y is a closed set in X, then U is a closed set in X. Otherwise, a set U is closed in Y if and only if it equals the intersection of a closed set of X with Y.

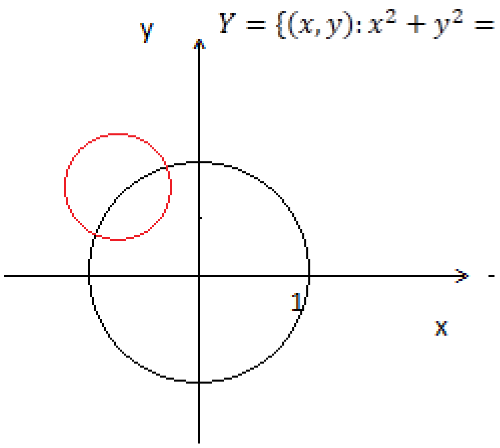
Example. Consider the Euclidean space R^2 and a Square Q on it (only the inner points):



In this case, the intersection of every open ball of the space R^2 with the square Q induces the topology of the Euclidean space R^2 on Q:



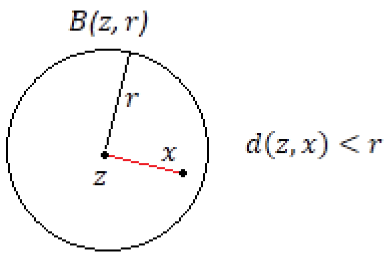
Also, given that an open ball *inside* Q is open in Q and Q is an open set in the space R^2, then the open ball inside Q is an open set in the space R^2. Consider instead a circumference $Y = \{(x, y): x^2 + y^2 = 1\}$ in the space R^2. The intersection of every open ball B_X with Y – a collection of arcs – is an *open set* with respect to the induced topology.



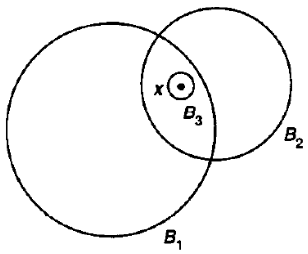
Topology over the plane

The first basic example of topology is the ***topology over the plane*** $\boldsymbol{R^2}$. The notions of *open* and *closed* sets have very interesting applications when it comes to figures on the plane.

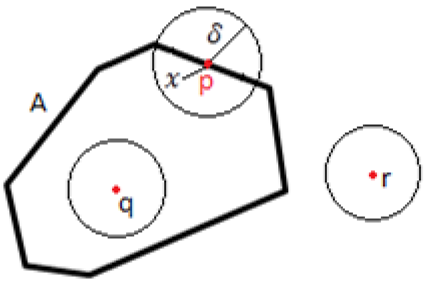
Open ball. Let's define $B(z, r)$ as the ***ball of center z and radius*** $\boldsymbol{r > 0}$ containing every point within its border (the border is not considered, as it's an *open* ball): for every point x within the ball B, the distance[3] from the center z is always **positive and less than** $\boldsymbol{r}$.



Base of the plane. The notion of *open ball* is necessary to define the ***base of the topology on*** $\boldsymbol{R^2}$**.** The base $\mathcal{B}$ is the collection of open balls. In fact, every point in the plane has a corresponding open set containing it, and every point x belonging to the intersection of two base elements B_1, B_2 is also contained in a third base element B_3 such that $x \in B_3 \subset (B_1 \cap B_2)$.



Classification of points. With the notion of open ball we can classify points as ***inner points*** (inside the figure), ***adherent points*** (on the border of the figure) and ***outer points*** (outside the figure): inner points have balls completely within the figure, adherent points have balls only partially within the figure, outer points have balls completely outside the figure.



Open sets in the plane. Given a plane X, a subset A of X is open if for every point x within the set there exists an open ball centered in x and completely contained within A.

$$A \subset X \; open \; if \; \forall x \in A \; \exists \; open \; ball \; within \; A$$

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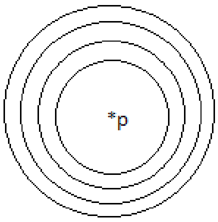
[3] We'll see that the notion of *distance* can have different meanings, especially in topology. For now, however, consider the distance simply as $d(z, x) = |z - x|$ value. Note that the distance is not an absolute notion: in topology any geometrical object can be transformed (expanded, reduced in size,…).

This way, any figure on the plane can be seen as a **union of balls**.

Closed sets in the plane. A closed set in the plane is instead a collection of inner points plus a frontier: for example, the inner points of the figure A plus the border of the figure is a closed set.

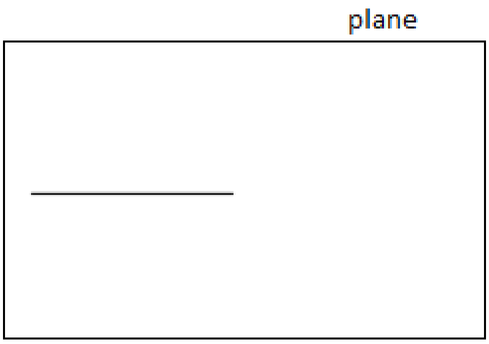
Question: single point. Is a single point closed or open? We have two scenarios:

- Consider a plane X and a single point $x \in X$. The base of the topology over the plane is the *open ball*, a two-dimensional figure; the point is instead a one-dimensional figure. This means that we can't draw open balls *inside* the point: the point is thus closed. We can reach the same conclusion if we consider the complement of the point x (the whole plane minus that point), which is an open set: again, this means that **the point is closed**.
- Consider instead an **isolated point P**: we can draw an infinite number of open balls centered on P, with a smaller and smaller radius. The intersection of every open ball centered on P is the singleton $\{P\}$, which is thus an *open set*.

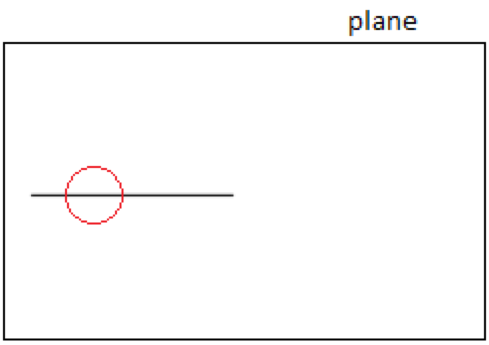


There are situations where an object is neither open nor closed. To understand why, consider the segment.

Question: the segment. Consider a segment on the plane R^2:

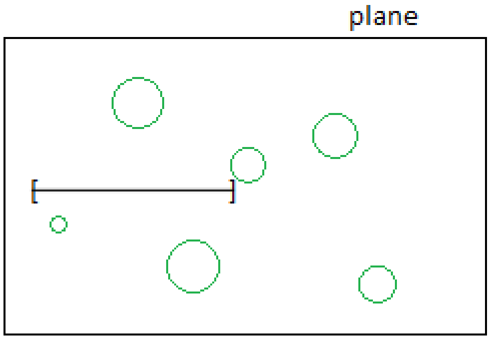


Is the segment open or closed? It depends on the way the segment is constructed. **Is it open?** We know that the base of the topology on the plane is the *open ball*, a bi-dimensional figure. Given that the segment is a one-dimensional object, then we can't draw an ball inside it: the segment is ***not open***.



Is it closed? It depends on the nature of the segment:

- If the segment has no starting/ending point or just one of them, then any open ball centered on the starting or ending point intersect the segment: therefore the segment is *not* closed.
- If the **segment has both starting and ending points**, then the complement of the segment is open (every point not belonging to the segment has a corresponding open ball) and thus the segment is **closed**.



Topology over the real line.

The situation is completely different if we switch from the topology on plane to the **topology on the real line**, whose **base is the *open segment*** (a subset of points without the starting/ending points):

$$base: \quad (a,b) = \{x : a < x < b\}$$

In this situation we have:

- **Open sets**: if the segment has **no starting or ending points** then the segment is **open**.

$$(a,b), (-\infty,a), (a,+\infty,) \text{ are open sets}$$

- **Closed sets**: if the segment has both **starting and ending points** then the complement of the segment is open, and therefore the segment is closed.

$$R\backslash[a,b] = \ (-\infty,a) \cup (b,+\infty)\ is\ open \text{ then } [a,b] \text{ is closed}$$

Also, if the segment is half open (a starting or ending point, goes to ∞) then the segment is closed because its complement is an open set:

$$R - [a,+\infty) = (-\infty,a), \text{ which is an open set}$$

- **Clopen set**: If the segment is half open (a starting or ending point, **doesn't go to** ∞) then the segment is neither open nor closed. Any interval $[a,b)$ is neither open nor closed, as we can see from this example:

$$R\backslash[2,3,4) = (-\infty,2) \cup [4,+\infty)$$

…where $(-\infty,2)$ is open and $[4,+\infty)$ is closed. Therefore the result is a "clopen" set. Note that this topology is induced on R by the definition of distance between two points:

$$d: X^2 \to R \quad \text{such that} \quad d_R(x,y) = |x-y|$$

Discrete topology

A topology over the set X is a *discrete topology* if $\mathcal{O}X$ **is a collection of every subset of** X:

$$Discrete = \big(X, \mathcal{O}X = \mathcal{P}(X)\big)$$

The **base** of the discrete topology is the family of singleton sets: if every one-point set is open then every union of one-point sets is also open.

Example. Consider the set $X = \{a, b, c\}$. Write a discrete topology on X and the base of this topology. The discrete topology over X is:

$$\mathcal{O}X = \left\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\right\}$$

The base of this topology is instead:

$$\mathcal{B} = \left\{\{a\}, \{b\}, \{c\}\right\}.$$

Note that if every subset of X is open, then the complement of every set of X is also closed. This means that **every open set of a discrete topology is also closed (*clopen*)**.

Indiscrete (or trivial) topology

A topology over the set X is an *indiscrete topology* if **$\mathcal{O}X$ has just two sets: the full set X and the empty set** $\emptyset$.

$$Indiscrete = \ (X, \mathcal{O}X = \{\emptyset, X\})$$

Again, if $\emptyset$ is an open set then its complement X is a closed set, and viceversa: this means that $\emptyset, X$ are both open and closed (**clopen**). The **base** of the indiscrete topology is $\mathcal{B} = \{X\}$.

Cofinite topology

Given a set X, a subset $A \subseteq X$ is **a finite complement of X** if $X\backslash A$ is finite or $Y = \emptyset$. A topology whose set $\mathcal{O}X$ is composed by every finite complement of X constitutes a cofinite topology on X.

$$Cofinite = (X, \mathcal{O}X = \{cofinite\ subsets\ of\ X\})$$

We can easily see this is a topology by examining the properties we've seen before: the empty set $\emptyset$ and the full set X belong to $\mathcal{O}X$: $X\backslash\emptyset = X$ is the full set and $X\backslash X = \emptyset$, which is a finite set; the arbitrary union of cofinite sets is also cofinite; the finite intersection of cofinite sets is also cofinite: given the intersection of two cofinite sets A_1, A_2, then $N\backslash(A_1 \cap A_2) = \ (N\backslash A_1) \cup (N\backslash A_2)$; but $(N\backslash A_1)$ and $(N\backslash A_2)$ are both finite, so $N\backslash(A_1 \cap A_2)$ is also finite and therefore $A_1 \cap A_2$ is cofinite.

Alexandrov topology

Consider a set X with a simple order $\leq$. A subset $Y \subseteq X$ is an *upper set* if given an element $y \in Y$, any element greater than y also belong to Y.

$$y \in Y, x \in X \quad and \quad (x \geq y) \rightarrow x \in Y$$

The **family of all *upper sets* of X** induces a topology over X called *Alexandrov topology*: $\mathcal{O}X$ contains every upper set $[n)$, infinite or otherwise. Such topologies are ***topologies of positive information***, as their representation is an *upper cone*.

Example. Consider the set of elements $X = \{0,1,2, a, b, c\}$ with these two partial orderings:

$$0 < 1 < 2$$

$$a < b < c$$

Open sets are $\{1,2\}, \{0,1,2\}, \{2\}\{b, c\}, \{a, b, c\}, \{c\}$: every open set must contain the **maximum element of the partial ordering**. Closed sets are instead *lower sets* such as $\{a, b\}$, $\{0,1\}, \{0\}, \{a\}$ …. Of course, with partially ordered sets with infinite cardinality, we must consider upper *intervals* or *upper sets*.

Closure and interior of a topology

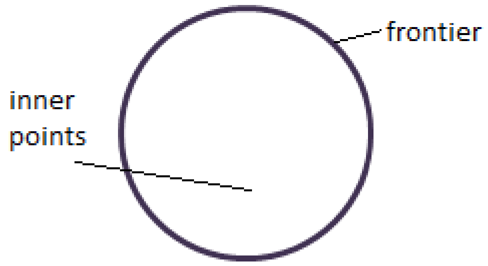
Given a topological space X, the interior Å of a subset $A \subseteq X$ is the union of all open sets contained in A:

$$\text{Å} = \bigcup \text{open sets of } A$$

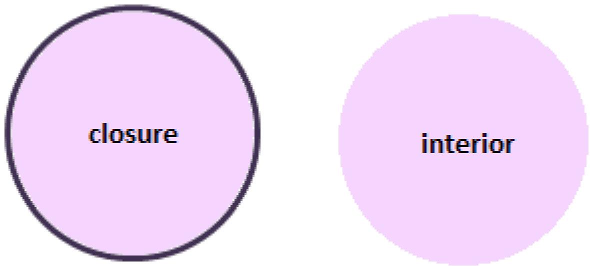
Intuitively, the interior of a set surmises every point "not on the edge of the set", and is itself the **largest open set** contained in A[4]. A point that is in the interior of S is also called an **interior point** of S. The closure $\bar{A}$ is instead the **intersection of all closed sets containing A**:

$$\bar{A} = \bigcap \text{closed sets containing A}$$

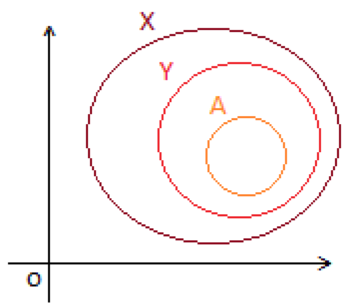
In other words, the closure of a set consists of **all points of A plus its frontier**, and is the **smallest closed set containing A**. To better understand the difference between interior and closure, consider the following figure A:



The interior, Å, is the union of every open ball within the figure – which means that the interior is the *inner part* of A (for each point within A there's a corresponding open ball; the union of every open ball is the interior). The closure, instead, is the figure *plus* the frontier (it's the smaller closed set containing A):



Consider now the following situation:



In general, the closure of A in X will be different than the closure of A in X in fact, the closure of A in Y results from the intersection of the closure of A in X with Y:

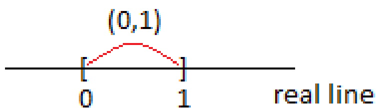
$$cl(A \; in \; Y) = cl(A \; in \; X) \cap Y$$

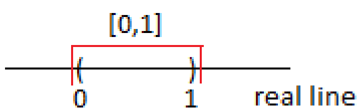
---

[4] We could even say that a set S is open if $S = int(S)$.

Examples. Few examples of closures and interiors:

- **Empty set:** in any space the interior of the empty set and the closure of the empty set are the empty set: $$int(\emptyset) = cl(\emptyset) = \emptyset$$
- Consider the **discrete topology on $\boldsymbol{R}$**, where every set is both open and closed. It means that:
    - $int([0,1]) = cl([0,1]) = [0,1]$ because $[0,1]$ is an open and closed set;
    - $int\big((0,1)\big) = cl\big((0,1)\big) = (0,1)$ because (0,1) is a an open and closed closed set.
- Consider the **indiscrete topology on $\boldsymbol{R}$**, where the only open and closed sets are $\emptyset$ and R. It means that:
    - $int(\emptyset) = \emptyset$ because there's no greater open set inside $\emptyset$; also, in any space the interior of the empty set is *always* the empty set.
    - $int(R) = R$ because there's no greater open set inside R.
    - $cl(\emptyset) = \emptyset$ because in any space the closure of the empty set is *always* the empty set.
    - $cl(R) = R$ because the closure of the full set is *always* the full set.
    - $int([0,1]) = \emptyset$ and $cl\big((0,1)\big) = R$, because $\emptyset$ is the biggest open set, while R is the smallest closed set.
- Consider the **real line** (*not* a discrete or indiscrete topology). Then:
    - $int([0,1]) = (0,1)$ because $(0,1)$ is the biggest open set within $[0,1]$.
    - $cl\big((0,1)\big) = [0,1]$, because $[0,1]$ is the smallest closed set containing $(0,1)$.





- Consider the **rational numbers** on the real line. Then:
    - The interior of $Q \subset R$ of the *rational numbers* is the empty set.
    - The closure of the set Q is the whole space R.

Q is therefore **dense in $\boldsymbol{R}$**. This is easier to understand if we consider the closure of Q: R is the immediate closed set containing Q. Therefore, the interior must be $R \backslash R = \emptyset$.

- Consider the **bidimensional space** R^2 and a figure $A = \{x^2 + y^2 = 1\}$ on it. Then:
    - $int(A) = \{x^2 + y^2 < 1\}$, because the biggest open set inside A is the union of every point of A (without the frontier).
    - $cl(A) = \{x^2 + y^2 \le 1\}$, because the smallest closed set containing A is A plus the frontier.

The metric space.

One of the most important ways of imposing a topology on a set is by using the notion of *metric*.

Definition of metric. A metric or **distance** on a set X is a function $d: X^2 \to R$ having the following properties:

1. $d(x,y) = 0 \ iff \ x = y$, meaning that the distance between an object and itself is 0. Otherwise the distance is always a positive real number.

2. $d(x, y) = d(y, x)$, meaning that the metric doesn't depend on the order in which points are analyzed.
3. $d(x, y) \leq d(x, z) + d(z, y)$, meaning that the *triangular property* still holds.

If d is a metric on the set X, then the ***metric topology* induced by d on X** is generated by the collection of open balls centered in every point of X, which consistutes a base for the *metric topology*. The open sets U of the metric topologies are sets such that $\forall x \in U$ there's an open ball $B_d(x, r > 0)$ completely contained within U.

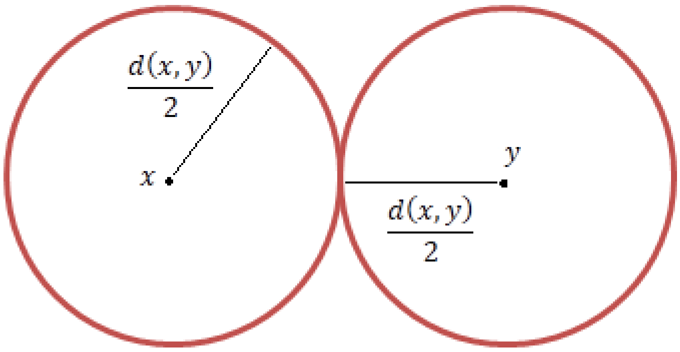
Example of metric topologies. Consider a ***discrete topology*** over the set $X = \{0,1,2\}$. We know that in a discrete topology *every set* is both open and closed: given that $\{0\}, \{1\}, \{2\}$ are open set, then **the discrete topology is a metric topology** induced by the following definition of metric:

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & otherwise \end{cases}$$

This way, the open ball $B(x, 1)$ Is the open set $\{x\}$ alone. Another example of metric topology is the **standard metric on the real numbers $\boldsymbol{R}$**, where the distance is defined as:

$$d(x, y) = |x - y|$$

Hausdorff axiom and a *lemma* on metric topologies. A topological space is a **Hausdorff space** if, given two points $x, y \in X$, there are disjoint open sets U_x, U_y containing x and y respectively. Now, consider any two points x, y in a metric space X: if $x \neq y$ then there's always a $r > 0$ such that the two open balls centered in x and y do not intersect: for example, if $r = \frac{d(x,y)}{2}$, B_1 and B_2 are two disjoint open balls.



$$if\ x \neq y\ then\ \exists r > 0 \quad s.t. \quad B_1(x, r) \cap B_2(y, r) = \emptyset$$

Every metric space, then, is always a Hausdorff space.

Example of non-metric space. Given the set of natural numbers N, consider the open set $[n) \subset N$ such that $[n) = \{x \in N, x \geq n\}$. We can induce a topology on N by adding the empty set to the set $[n)$:

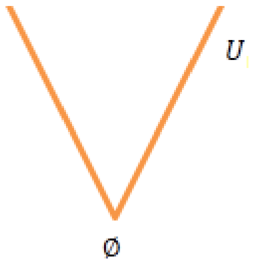
$$Topology\ on\ N\ = \big(N, \emptyset \cup [n)\big)$$

This is a topology, as $\emptyset \cup [n)$ contains $\emptyset$ and the full set, and is closed both under arbitrary union and closed intersection:

- ***Abitrary union***: consider $A \subset \big(\emptyset \cup [n)\big)$: if $A = \emptyset$ then any union with A is an open set; if instead A is not empty, then $A \cup [n) = \big[\text{minimum of A}\big)$, which is also an open set.

- ***Finite intersection***: the same applies to the finite intersection: if $[n_1)$ and $[n_2)$ are open sets then their intersection $[n_1) \cap [n_2) = max(n_1, n_2)$, which is also open.

Graphically, this topology can be seen as an upper cone starting from the empty set:



While this is a topology, this is *not* a metric topology. In fact, **the intersection of two non-empty sets of natural numbers will never be empty**:

$$[n) \cap [k) = [\max(k, n))$$

For example, $[5) \cap [3) = [5)$.

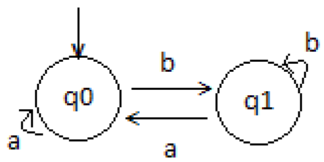
Metric on *strings*. The notion of metric can also be applied to strings. Consider a finite alphabet A and the set A^* of finite strings of alphabet A (including the empty string λ). A metric on A can be defined as:

$$\alpha, \beta \in A^* \text{ strings} \qquad d(\alpha, \beta) = 2^{-k}$$

...where k is the number of states of the minimal automaton[5] distinguishing α and β. We can easily verify this is a legitimate definition of distance by using the properties we've seen before:

1. $d(\alpha, \beta) = 0 \ iff \ \alpha = \beta$: verified;
2. $d(\alpha, \beta) = d(\beta, \alpha)$: verified;
3. $d(\alpha, \beta) \leq d(\alpha, z) + d(z, \beta)$: verified.

For example, consider the alphabet $A = \{a, b\}$ and the strings $\alpha = aab, \beta = bba$, where $\alpha, \beta \in A^*$. The distance $d(\alpha, \beta)$ depends on the number of states of the minimal automaton, which is the following:



We have two states: this means that $d(\alpha, \beta) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$. Even with very long strings $aabaabab$ ... we have the same result: the distance is ¼.

Continuity of topological spaces

Let X and Y be topological spaces. A function $f: X \to Y$ is continuous if, for each open set U of Y, there's a corresponding open set $f^{-1}(U)$ in X.

$$f: X \to Y \text{ is continuous} \quad if \qquad \forall U \in \mathcal{O}Y \, \exists \, f^{-1}[U] \in \mathcal{O}X$$

[5] A deterministic *automaton* is a tuple (Q, A, q_0, F, δ), where Q is the finite set of states, A is the alphabet, $q_0 \in Q$ is the initial state, $F \subseteq Q$ is the set of final states, and $\delta : Q \times A \to Q$ is a program.

Other equivalent definition of topological continuity are:

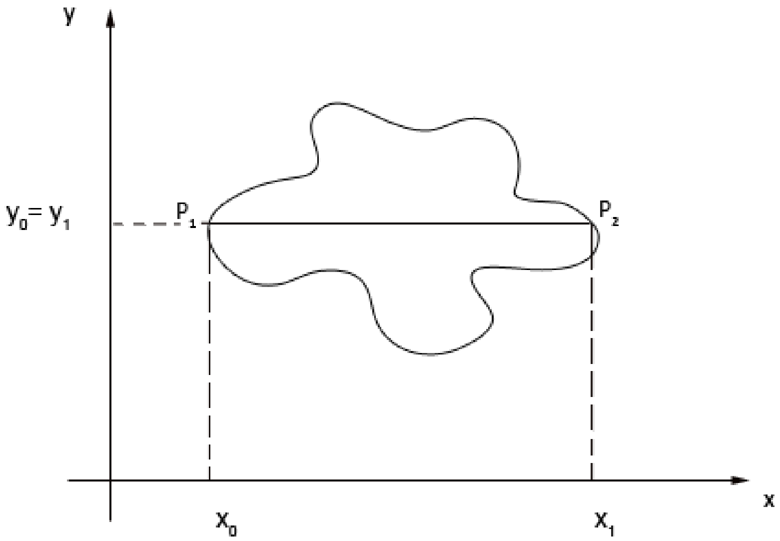
- **With closed sets**. A function $f: X \to Y$ is continuous if for every closed set V of Y there's a corresponding closed set $f^{-1}(V)$ in X.
- **With the base.** If we have only the base of topological spaces X and Y, then to prove that $f: X \to Y$ is continuous it suffices to show that the *inverse image* of every *base element* $B \in \mathcal{B}$is an open set.

To sum up, a function $\boldsymbol{f}$ **is continuous if every open set of the *codomain* is mapped to a corresponding open set in the *domain***[6].

Mathematical relations. The topological definition of continuity is not different from the mathematical definition of continuity. To understand this, let's define the notion of ***Euclidean distance*** first: given two points $P_1(x_1, y_1)$ and $P_2(x_2 y_2)$, with $x_i, y_i \in R$, their Euclidean distance is:

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

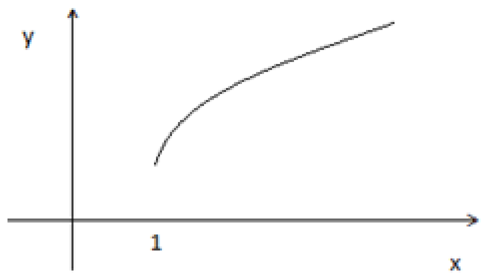
The distance can thus be seen as a *continuous line* from point $P_1$to point P_2, just like in this example:



Now, given a function $f: R \to R$, f is **continuous in the point $\boldsymbol{x} \in \boldsymbol{R}$** if, given $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, \epsilon)$ implies $d(f(x), \delta)$, where d is the Euclidean distance (**continuity in a point**). A function is thus continuous if we extend this notion to every point of its domain (***continuity of a function***):

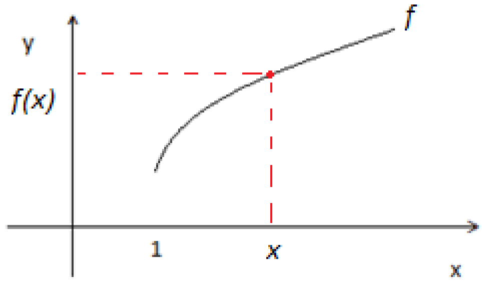
$$f: R \to R \; continuous \quad \forall \epsilon > 0, \forall x \in R, \exists \delta > 0 \quad s.t. \quad \boldsymbol{d(x, y) < \delta \to d\big(f(x), f(x)\big) < \epsilon}$$

An example of continuous function is:

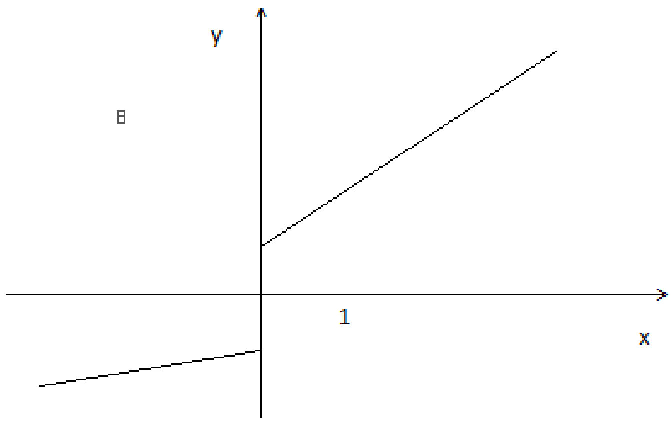


[6] Note that this means that every open set belonging to the base $\mathcal{B}$ of the codomain must have a corresponding open set in the domain of the function.

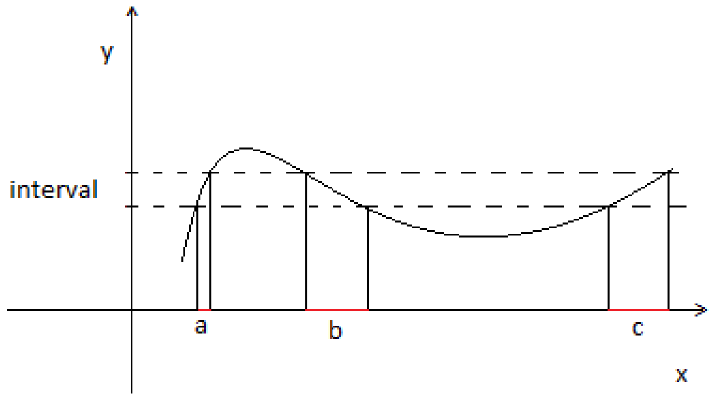
In fact, if we fix a point $P(x, f(x))$ on the curve, the smaller the interval around x on the x-axis the smaller the corresponding interval around $f(x)$ on the y-axis:



A function is instead **discrete** when one or more points do not belong to the curve of the function:

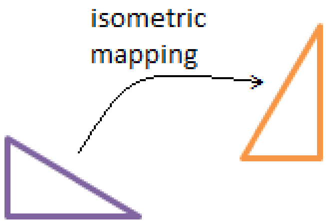


Suppose now that $X \equiv R$ and $Y \equiv R$ such that $f: X \to Y$: f is continuous if, when we fix an open set U on y-axis, the reverse image of this interval is an open set on the x-axis (or the union of several open sets of the x-axis depending on the function we're considering):

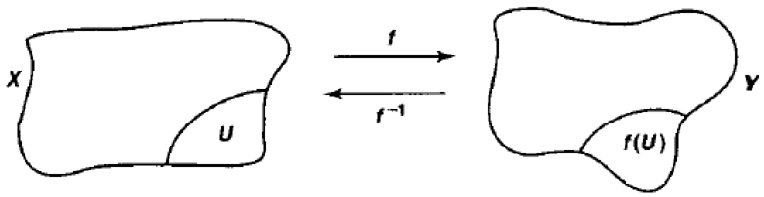


The mathematical and topological definitions of continuity are therefore equivalent.

Homeomorphism. In modern algebra, an ***isomorphism*** between algebraic objects such as rings or groups is a **bijective function** that preserves the algebraic structure involved. For example, when metric spaces are concerned, a isomorphic function is a **distance-preserving** mapping between metric spaces: given two metric spaces X, Y each with a certain definition of distance d_X, d_Y, a homeomorphism is a function $f: X \to Y$ that maps elements of X to elements of Y such that the distance d_Y between the elements in the new metric space is equal to the distance d_X between elements in the original metric space. For example, if we consider these two triangles, an isometric function maps the first one to the second one, and the distance (between the vertices of the triangles, for example, is preserved):



The more general concept in topology is that of *homeomorphism*, which is a **bijective function that preserves the topological structure involved**. Formally, let X and Y be topological spaces such that $f: X \to Y$ is a bijection. If **both f and f^{-1} are continuous**, then f is called a homeomorphism. It follows that if f is a homeomorphism, then given two topological spaces X and Y, $f(U)$ is open if and only if $U \subseteq X$ is open.



An example of counterintuitive homeomorphism is the following. Consider these two figures:

Apparently they're completely different objects; however, from a topological point of view, we can map each point of the circumference to a point in the square, making these figures *equivalent*. Of course, there are geometrical figures which are not topologically equivalent – like, for example, a circle and a wheel:

In fact, the "hole" in the wheel makes the mapping impossible.

Continuous functions on ordered spaces. Consider two Alexandrov Topologies, $(Y, \leq_X)$ and $(Y, \leq_Y)$. Then a function $f: X \to Y$ is continuous with respect of the Alexandrov topology if and only if **f is monotone**:

$$\forall x, y \quad if\ x \leq_X y \quad then\ f(x) \leq_Y f(y)$$

Product topology

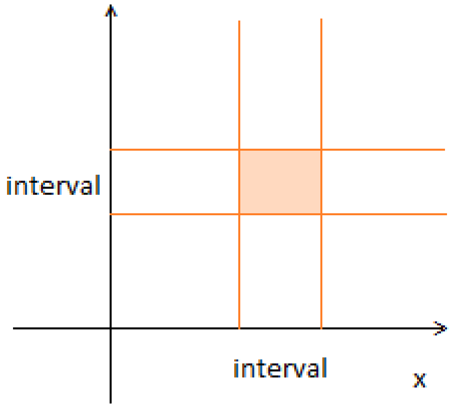
The *Cartesian product* is another instrument for imposing a topology on a set. Consider two topological spaces X and Y: the *product topology* on $X \times Y$ is a topology whose open sets have the form $U \times \mathrm{V}$, where U is an open set of X and V is an open set of Y.

$$\text{product topology:} \quad P = (X \times Y, \mathcal{O}X \times \mathcal{O}Y)$$

The base $\mathcal{D}$ of a product topology is the Cartesian product of the base $\mathcal{B}$ of X and the base $\mathcal{C}$ of Y:

$$\mathcal{D} = \mathcal{B} \times \mathcal{C} \qquad where \qquad \mathcal{B} = base\ of\ X, \mathcal{C} = base\ of\ Y$$

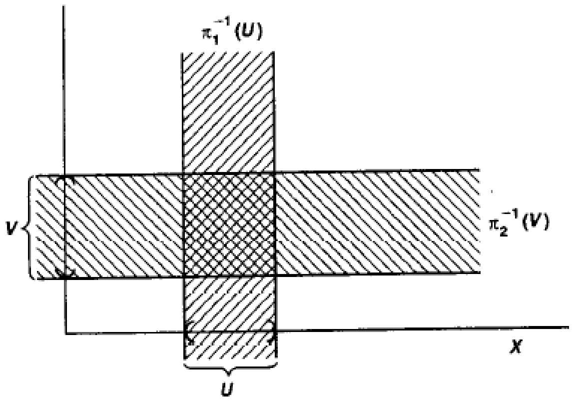
An example of *product topology* is the Euclidean space $R^2 = R \times R$: an open set on the Euclidean space results from the Cartesian product of two open intervals belonging to the x-axis and y-axis respectively.



Sometimes it's useful to express the product topology in terms of subbases. To do this, let's define the projection functions first. Consider the following **projections** onto the first and second factors of $X \times Y$:

$$\pi_1 : X \times Y \rightarrow X \qquad \pi_2 : X \times Y \rightarrow Y$$

If U is an open subset of X, then $\pi_1^{-1}(U) = U \times Y$ is open in $X \times Y$; similarly, if V is an open subset of Y, then $\pi_2^{-1}(V) = X \times V$ is an open subset in $X \times Y$. The intersection of $X \times V$ and $U \times Y$ is the set $U \times V$, which is also open. The collection of every $U \times V$ set is a **subbase** for the product topology on $\mathrm{X} \times \mathrm{Y}$.



Both π_1 and π_2 are ***continuous functions***: the product topology on $\mathrm{X} \times \mathrm{Y}$ is the weakest topology (*fewest* open sets) for which both these mapping functions are continuous.

Note that the *product topology* is generated only by a ***finite* set of** positive information (or a finite number of sets). We can also have the **Cartesian product on a very large or infinite number of topological spaces:**

$$X_1 \times X_2 \times \ldots \times X_n$$

This is the case when annualizing the power set of the set of natural numbers, as we'll see in the following pages.

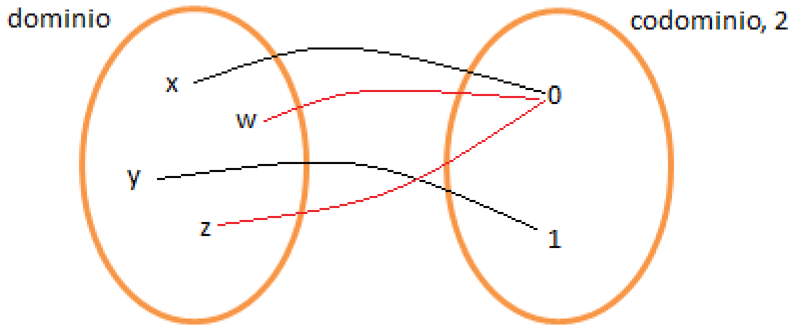
Advanced topologies

Consider the set $2 = \{0,1\}$, whose elements are ***bits****.* We can define **four** interesting topologies on it:

- The ***Sierpinski topology***, whose open sets are $\emptyset$, $\{0,1\}$, $\{1\}$. Note that if we assume a partial ordering on 2 such that $0 < 1$, then the Sierpinski topology coincides with the *Alexandrov topology* over the set 2. This is also a topology of *positive* information (upper cone).

- The **co-Sierpinski topology**, whose open sets are $\emptyset$, $\{0,1\}$, $\{0\}$. Again, if we assume a partial ordering on 2 such that $1 < 0$, then the *co-Sierpinski topology* conicides with the *Alexandrov topology* over the set 2. This is also a topology of *negative* information (lower cone).
- The **discrete topology**, whose open sets are every possible subset of 2: $\emptyset, \{0\}, \{1\}, \{0,1\}$.
- The **indiscrete topology**, whose open sets are $\emptyset$ and the full set $2 = \{0,1\}$.

Mapping every set to 2. The *Sierpinski topology* is particulary interesting not only because it works on bits, 0 and 1, but also because **every set can be mapped to the set 2**. In fact, we can always have a bijective function $f: X \to 2$ mapping elements of the set X to elements of the set 2.



For example, consider the set $E = \{even\ numbers\}$ such that $E \in \mathcal{P}(N)$: every natural number can be mapped to a corresponding bit depending on whether the number belongs to E or not:

$$E_2 = \{0,1,0,1,0, \dots\}$$

The same applies to every other set belonging to the **power set of N**:

$$\mathcal{P}(N) = \{A: A \subseteq N\} \cong \prod_{i \in N} \{0,1\} \quad \text{where} \quad \{0,1\} = 2$$

The set 2: boolean algebras. We've said before that the set 2 has bits as elements; the possible operations over bits are and, or and not: over sets, these operations become *intersection*, *union* and *complementation*. Therefore, the operations over bits can be applied on sets and viceversa. Also, the Cartesian product $\{0,1\} \times \{0,1\}$:

$$[0,1\} \times \{0,1\} = \{(0,0), (0,1), (1,0), (1,1)\}$$

Suppose we have the partial ordering $1 > 0$: this means that $(0,0)$ is the minimum element, $(1,1)$ is the maximum element and $(0,0) \leq (0,1) \leq (1,0) \leq (1,1)$. As a result, we obtain a *Boolean algebra* with four elements.

Product topology* of the Sierpinski space**. Consider again the set of natural numbers N and its power set $P(N)$, containing ***every possible sequence of natural numbers. We can have 2^N possible sequences, which means that:

$$P(N) = 2^N = 2 \times 2 \times 2 \dots \ n\ times$$

Therefore, the power set of the set of natural number is the ***product topology of n Sierpinski spaces***.

Example #1 (finding an open set over 2). Consider a set $I = \{even\ numbers\}$, where $I \in \mathcal{P}(N)$. Find an open set of I in the Sierpinski topology.

Solution. I belongs to the power set $\mathcal{P}(N)$: we can visualize is as a "point" in the $\mathcal{P}(N) = 2^N$ space. Finding an open set containing I, therefore, is like finding an open ball centered on I. Now, we know that the power set of N can be seen as the Cartesian product of the set 2 n-times:

$$P(N) = 2^N = 2 \times 2 \times 2 \dots n\text{-times}$$

Therefore, the set of *even numbers* can be seen as this sequence of bits belonging to $P(N)$:

$$I_2 = \{0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,\dots\}$$

An open set containing I_2 results from the Cartesian product of a finite number of open sets, possibly different from the full space, and the full space 2^N.. The steps to obtain such a set are:

1. **Fix a finite number of positions occupied by open sets** different from the full space, 2^N. In this case, we want to fix the first 7 positions:

$$\_ \times \_ \times \_ \times \_ \times \_ \times \_ \times \_ \times 2^N$$

2. **Analyze the (first) elements of the set**. The first natural number is 0, which is even: which open set can correctly represent it? We're working with the Sierpinski topology, whose open sets are $\emptyset$, $\{0,1\}$, $\{1\}$, with $0 < 1$: the only possible choice is $\{1\}$, which can be seen as a "yes, we have an even number and only an even number in this position". The second natural number is 1, which is odd: an open set containing the corrisponding 0 in the 2 set is $\{0,1\}$. The same considerations apply to the other 1s and 0s, so the result is:

$$\{0,1\} \times \{1\} \times \{0,1\} \times \{1\} \times \{0,1\} \times \{1\} \times \{0,1\} \times 2^N$$

Given that $\{0,1\} = 2$, we have:

$$2 \times \{1\} \times 2 \times \{1\} \times 2 \times \{1\} \times 2 \times 2 \times 2 \times 2 \dots.$$

…and this is an **open set containing the set of even natural numbers**.

$$A = \{b_1 b_2 b_1 b_2 \dots : b_1 = 2, b_2 = \{1\}\}$$

Example #2 (finding an open set over 2). Consider the set $Squares \in \mathcal{P}(N)$ containing the squares of every natural number:

$$\text{Squares} = \{0,1,4,9,16,25,36,42\} = \{x^2 : x \in N\}$$

Squares is a member of the power set of N, which means that *Square* is nothing but a point in the $\mathcal{P}(N)$ space, so to say: it follows that we can think of an open ball containing *Squares*. But how can we find one such ball in the Sierpinski topology?

Solution. These are the steps we must follow:

1. **Fix a finite number of positions occupied by open sets** different from the full space, 2^N. In this case, we want to fix the first 3 positions:

$$\_ \times \_ \times \_ \times 2^N$$

2. **Analyze the (first) elements of the set**. Squares can be seen as the following sequence of bits:

$$Squares_2 = \{1,1,0,0,1,0,0,0,0,1,\dots\}$$

The first natural number, 1, is a square ($1^2 = 1$) and therefore an open set containing 1 and also belonging to the Sierpinski set is $\{1\}$. The same observations can be applied to the second natural number, 2, but it's not necessary: we can use the open set {0,1} for every other natural number thus obtaining a very general and yet certainly open set containing, among other things, the set $Squares$.

$$\{1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \ldots \times \{0,1\} \; = \; \{1\} \times 2 \times 2 \times 2 \times 2^N$$

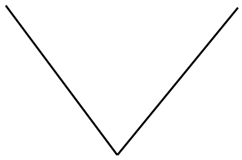
Or, more formally:

$$A = \{1 b_1 b_2 \ldots\ldots : b_i \in \{0,1\}\,\}$$

Note that this set starts with 1, as the *Sierpinski topology* generates positive information.

Example #2 (finding an open set over 2). The power set of N, $\mathcal{P}(N) = 2^N$, can be partially ordered: at first we have the empty set $\emptyset$, then the singletons $(\{0\}, \{1\} \ldots \{n\})$, the sets with two elements $(\{0,1\}, \{0,2\}, \ldots)$ and so on. This means that the *product topology* 2^N *of the Sierpinski topology* generates an **upper cone**:

$Full\ set$



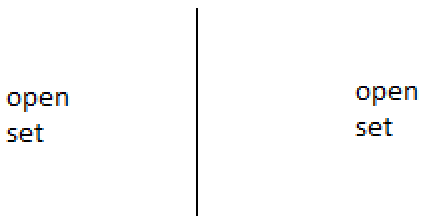
$\emptyset$

If the starting point of the cone is $\{0,1\}$, then we have positive information for $\{0,1\}$ and unspecified information for the other numbers. Consider now the infinite set $I = \{even\ numbers$: is its cone an open set? The answer is **no**: the set of even numbers is infinite, and therefore the cone generated by this set has positive information for an infinite number of positions $(0, 2, 4, 6 \ldots)$. Now, the product topology is generated by **positive information from a *finite* subset**, which is not the case[7]. Note that if the to check if a subset of N belongs to this open set we need infinite time to check every number – which is impossible. The complement of the open upper cone is the ***down closed cone***, where everything is upside down: the empty set is switched with the full set. Information thus becomes negative and quite useless in Computer Science (to check if an element doesn't belong to a certain set we would employ infinite time!).

Separation and connection

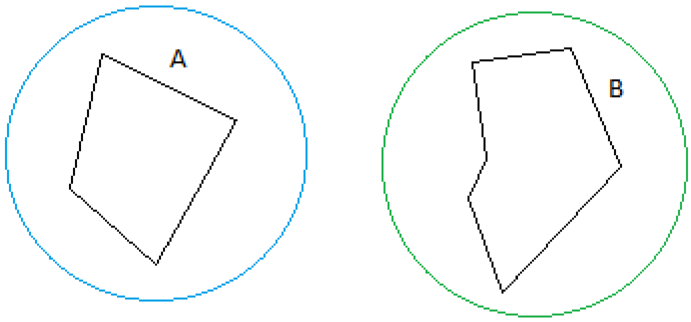
The basic definitions of separation and connection for a topological space are easy to understand: a topological space X is separated if **it can be broken up into *two disjoint non-empty open set,*** U and V, whose union is X; otherwise, if the space is not the union of two disjoint non-empty open sets, then it's connected.

Examples. An example of separated space is the following:

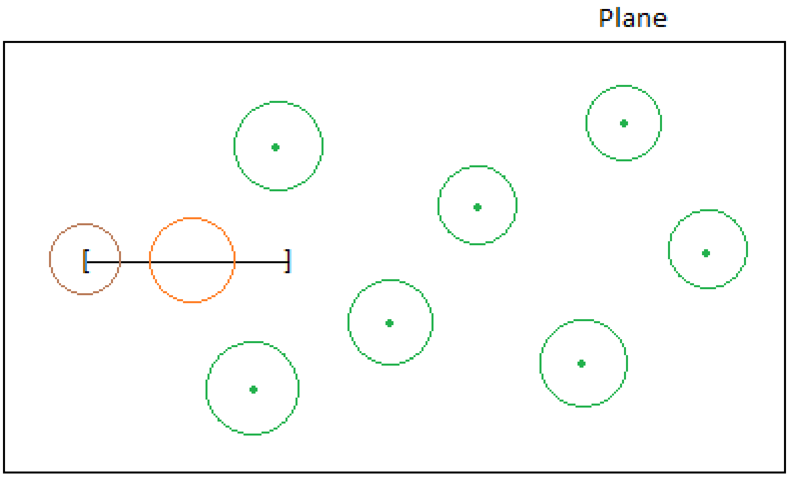


[7] Note that if the to check if a subset of N belongs to this open set we need infinite time to check every number – which is impossible.

As we can see, the space has two disjoint (there's a line between the two) open non-empty set, so it's separated. Consider instead two subsets of the plane X:



The union of these two open balls does *not* yield the whole plane X (there are other open balls in the plane): the plane is therefore connected. Consider finally a closed segment (with both starting/ending points) as a **subspace** $S \subseteq X$, hereditating the topology of the plane (the open set of this subspace is the *intersection of an open set of the plane with the segment* S – an open interval, in other words):



Is this segment – considered as a subspace – connected? **Yes**: the union of two open intervals is not the full segment:

$$[0, a), (a, 1\} \quad \rightarrow a \text{ is not considered!}$$

From these simple notions of connectness and separation much follows: there are, in fact, many **separation axioms** that stipulate the degree to which distinct points of the space may be separated.

How to tell if a space is separated. Consider a topological space X: how can we say if two points $x, y \in X$ are separated? We have two alternatives:

- Analyze the **neighbors** of the points: if x and y belong to different neighbors, then they're separated.
- Analyze the **partial ordering** of the points: if there's a partial ordering, then they're separated.

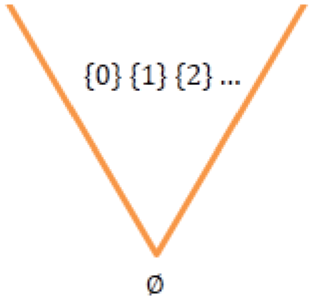
In both cases, having different neighbors or having a partial ordering mean that the topological space can be divided into two non-empty disjoint open sets.

Partial ordering. Partial ordering is very useful to separate points in a topological space. Given a topological space X and two points $x, y \in X$, there's a partial ordering between x and y if and only if x belongs to U then y belongs to U for every open set U:

$$x \leq_X y \quad iff \quad \forall U \in \mathcal{O}X,\ \boldsymbol{x} \in \boldsymbol{U} \rightarrow \boldsymbol{y} \in \boldsymbol{U}$$

Partial ordering is a form of ***preordering***, which is both reflexive and transitive. As a rule of thumb, partial ordering is more plausible with topologies of positive information (for example, topologies over the set of natural numbers).

Exercise (partial ordering). Consider the power set of natural numbers $\mathcal{P}(N)$, which can be represented as an upper cone:



We already know that $\mathcal{P}(N) = 2^N$ or, in other words, that the power set of N can be represented as the product topology over the set $2 = \{0,1\}$. Now, consider two elements of the power set, the singleton $\{2\}$ and the set $\{2,5\}$; do we have a partial ordering?

$$\{2\} \leq_{Pr.\,Top\ on\ 2} \{2,5\}\ ?$$

In other words, do we have an open set U such that **if $\{2\}$ belongs to U, then $\{2,5\}$ belongs to U too**? Let's work on the product topology of the Sierpinski topology: the corresponding set for the singleton $\{2\}$ is:

$$A = \{0,0,1,0,0,0,0 \dots\}$$

…and an open set containing A is:

$$U = \{\{0,1\} \times \{0,1\} \times \{1\} \times \{0,1\} \times \{0,1\} \times \{0,1 \times \dots\}\}$$

This open set also contains the set $\{2,5\}$: using the Sierpinski topology, the set corresponding to $\{2,5\}$ is:

$$B = \{0,0,1,0,0,1,0,0,0 \dots\}$$

…and an open set containing B is:

$$U = \{\{0,1\} \times \{0,1\} \times \{\mathbf{1}\} \times \{0,1\} \times \{0,1\} \times \{\mathbf{0}, \mathbf{1}\} \times \dots\}$$

…which is the same open set as before. Therefore, $\{2\} \leq_{Pr.Top\ on\ 2} \{2,5\}$. Note that in the Co-Sierpinski topology ($0 > 1$), this wouldn't have been true!

Exercise (testing a possible separation). Let's now see a practical example of separation. Consider for instance the following topology:

$$X = \{0,1,2\}, \qquad \mathcal{O}X = \{\emptyset, \{0,1,2\}, \{0,1\}, \{2\}\}$$

Does this topology distinguish the element 0 from the element 1? We can answer in two different ways:

- Consider the **neighbors** of 0 and 1 (sets containing 0 and an open set with 0; the same applies to 1):

- Neighbors of 0: $\{0,1,2\}, \{0,1\}$
- Neighbors of 1: $\{0,1,2\}, \{0,1\}$

The **neighbors are the same**, therefore 0 and 1 are undistinguishable in X. This means that a topology separates two elements if they belong to different neighbors.

- Consider the **partial ordering**: every open set containing 1 also contains 0 ($1 > 0$) and viceversa ($0 > 1$): this means that 1 and 0 0 can't be separated by this topology.

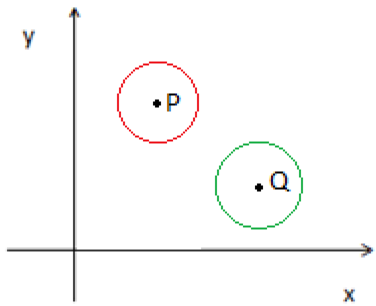
Specialization orders

There are many different **degrees of separation**. Let's see the most common ones.

T_0 or weak separation axiom. A topological space X is T_0 if and only if the **preorder $\leq_X$ is a partial order**: this means that a topological space is T_0 if there are no points belonging to the same neighborhood. This is exactly what we've seen in the previous exercise.

Example (Euclidean space). Consider the Euclidean space: does it satisfy the axiom T_0?

Solution. Can we find two distinct points in the plane such that they both belong to the same neighbor?



No: I can always consider two distinct neighbors – an open circle on the plane - containing only one of the two points. Therefore the Euclidean space satisfies T_0. In fact, there's no possible partial ordering in this space: the point P is never less or equal to Q and viceversa (the only possible partial ordering is the *trivial partial ordering* $P \leq_X P$).

- **T_1**. A space X is T_1 if for every $x, y \in X$ there are open sets U, V such that:

$$U \cap \{x, y\} = \{x\} \quad and\ V \cap \{x, y\} = y$$

In other words, for every $x, y \in X$, there are open sets containing x and not y or open sets containing y and not x (it's enough to have **just one of these conditions**). The intersection of all open sets containing x and not y is $\{x\}$, while the intersection of all open sets containing y and not x is $\{y\}$.

Note: a T_1 space is also a T_0 space; in fact, a T_1 space is a T_0 space with a "equal" partial ordering such that given an element x of the T_1space, $x \leq_X x$. Naturally, a T_0 is *not* T_1.

- **T_2 (*Hausdorff*)**. A space X is T_2 or **Hausdorff** if for every $x, y \in X$ there are open sets U, V such that:

$$\forall x \in U, \forall\, y \in V, \qquad U \cap V = \emptyset$$

In other words, a topological space is T_2 if the intersection of open sets is empty**. T_2 is valid for every metric space**.

- $\boldsymbol{T_{2\frac{1}{2}}}$ $\boldsymbol{(completely\ Hausdorff)}$. A space X is $\boldsymbol{T_{2\frac{1}{2}}}$ if every point $x, y \in X$ there exists open sets U, V such that $x \in U$, $y \in \mathrm{V}$ and $\bar{U} \cap \bar{V} = \emptyset$. This means that the intersection of the closure of U and V is empty.

$$T_{2\frac{1}{2}}: \quad \forall\, x, y \in X\, \exists\, U, V\ open\ s.t.\ x \in U,\ y \in \mathrm{V}\ and\ \bar{U} \cap \bar{V} = \emptyset$$

- $\boldsymbol{T_3}$ $\boldsymbol{(regular)}$. A topological space X is regular if it's T_1 and if, for every closed set A such that every x does not belong to A, there are disjoint open sets U, V such that $A \subseteq X$ and $x \in V$.
- $\boldsymbol{T_4}$ $\boldsymbol{(normal)}$. A topological space X is normal if it is $\mathrm{T_1}$ and, for all disjoint closed sets A and B, there are disjing open sets U, V such that $\mathrm{A} \subseteq \mathrm{U}$, $\mathrm{B} \subseteq \mathrm{V}$. An example of normal topological space is the Euclidean line.

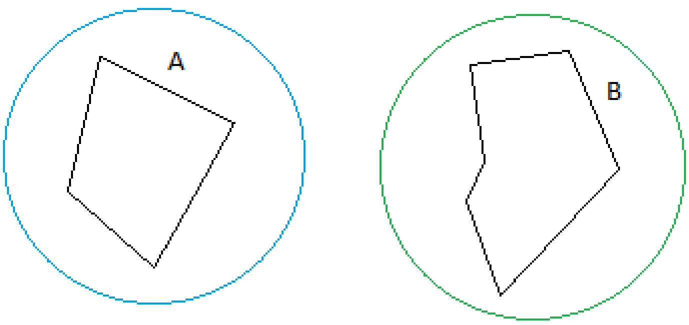
To sum up, the **most important degrees** of separations are:

- T_0: **every point belong to a distinct neighborhood or there's a partial ordering on the space**. If two points belong to the same neighborhood or there's no partial ordering, the space is not T_0.
- T_1: for every $x, y \in X$, there are open sets containing x and not y, and open sets containing y and not x. Check if there's an open set containing x and not y and another open set containing y and not x: if this is valid for every point in the set X, then the space is T_1.

Note that $\boldsymbol{T_0}$ ***excludes*** $\boldsymbol{T_1}$: if we have a partial ordering in a topological space, then that space is certainly T_0 and not T_1. In fact, to be T_0, it's enough to have "one open set U containing x and not y" (**one condition**), while to be T_1 we must have "one open set U containing x and not y" and "one open set V containing y and not x" (**one condition and viceversa**).

- T_2: for every x, y such that $x \in U\ open$, $y \in V\ open$, the intersection of open sets $U \cap V$ is empty.

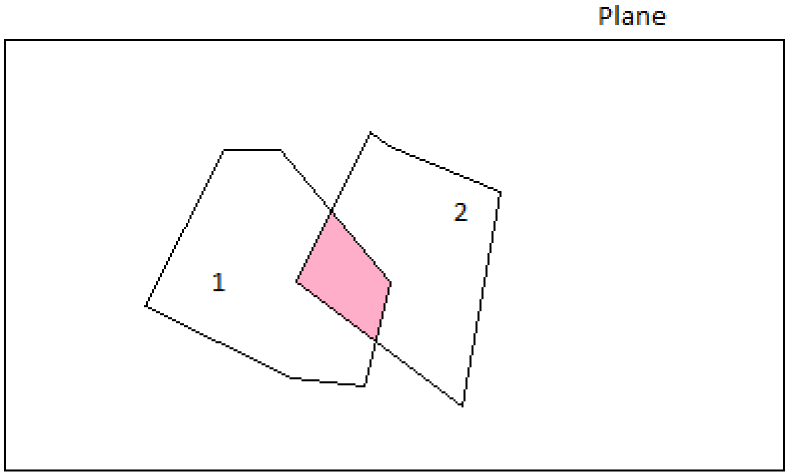
Connected components. If a space X is separated, we can partition it in ***connected components***:



For example, in this case we have two connected components, A and B. Connected components have several properties:

- If a space has two connected components, then the full space is the **union of these disjoint, connected components**.
- Consider a space X and a subset $A \subseteq X$, with$A$ hereditating X's topology; the subset A is connected if, when considering A as a subspace, A is connected.

- Consider the space X and $A, B \subseteq X$; if A, B are both connected and their intersecti**on is not-empty**, then $A \cup B$ is connected. If the intersection is empty, instead, their union is not connected.



In other words, if figure 1 and figure 2 are connected we can have two possible situations:

- Intersection is **empty**: intersection is **not connected**.
- Intersection is **non-empty**: intersection is also **connected**.

Exercises

1. (is it a topology?) Consider the set $X = \{0,1,2\}$ and the set $\mathcal{O}X = \{\{0,1,2\}, \emptyset, \{1,2\}, \{2\}\}$. Is this a topology?

Solution. To answer, we must verify if $\mathcal{O}X = \{\{0,1,2\}, \emptyset, \{1,2\}, \{2\}\}$ satisfies the following properties:

- $\mathcal{O}X$ contains $\emptyset$ and the full set $X = \{0,1,2\}$.
- $\mathcal{O}X$ is closed under arbitrary union:
    - $\{0,1,2\} \cup \emptyset = \{0,1,2\}$; $\{0,1,2\} \cup \{1,2\} = \{0,1,2\}$; $\{0,1,2\} \cup \{2\} = \{0,1,2\}$; …
- $\mathcal{O}X$ is closed under finite intersection:
    - $\{0,1,2\} \cap \emptyset = \emptyset$; $\{0,1,2\} \cap \{1,2\} = \{1,2\}$; $\{0,1,2\} \cap \{2\} = \{2\}$, …

Therefore yes, this is a valid topology.

2. (is it a partial ordering?). Consider the previous topology $X = \{0,1,2\}$, $\mathcal{O}X = \{\{0,1,2\}, \emptyset, \{1,2\}, \{2\}\}$. Is it a partial ordering?

Solution. To answer we must analyze the sets belonging to $\mathcal{O}X$ versus the elements of the full set X:

- 2 is the maximum element because we have a set, $\{2\}$, which contains 2 and not 0 or 1;
- Every open set containing 1 also contains 2, and not viceversa: therefore $1 \leq 2$.
- The only open set containing 0 also contains 1 and two, and therefore $0 \leq 1 \leq 2$.

So we have a partial ordering $0 \leq 1 \leq 2$.

3. (find topologies) Consider the set $X = \{a, b, c\}$. What are the possible topologies with this set?

Solution. The first topology we can think of is the **indiscrete topology**, where the empty set and the full set are the only open sets:

$$\mathcal{O}\mathrm{X}_1 = \{\emptyset,\ \{a, b, c\}\}$$

This is in fact a topology, as $\mathcal{O}\mathrm{X}_1$ contains both the empty set and the full set, and it's closed under finite intersection and arbitrary union:

- Arbitrary union: $\{a, b, c\} \cup \emptyset = \{a, b, c\}$, $\emptyset \cup \{a, b, c\} = \{a, b, c\}$.
- Finite intersection: $\emptyset \cap \{a, b, c\} = \emptyset$, $\{a, b, c\} \cap \emptyset = \emptyset$

The second topology we can think of is the **discrete topology**, where every subset of X is an open set:

$$\mathcal{O}X_2 = \{\{a\},\ \{b\},\ \{c\},\ \{a, b, c\},\ \emptyset,\ \{a, b\},\ \{a, c\},\ \{b, c\}\}$$

Again, $\mathcal{O}X_2$ contains the empty set and the full set, and it's also closed under both arbitrary union and finite intersection.

We can find **other topologies** by mixing up the sets of the discrete topology; for example:

$$\mathcal{O}X_3 = \{\{a, b, c\},\ \emptyset, \{a, b\}\}$$

In this case, $\mathcal{O}X_3$ contains both the empty set and the full set, and:

- Arbitrary union: $\{a, b, c\} \cup \emptyset = \{a, b, c\}$, $\{a, b, c\} \cup \{a, b\} = \{a, b, c\}$, $\{a, b\} \cup \emptyset = \{a, b\}$
- Finite intersection: $\{a, b, c\} \cap \emptyset = \emptyset$, $\{a, b, c\} \cap \{a, b\} = \{a, b\}$, $\{a, b\} \cap \emptyset = \emptyset$.

The same considerations apply to $\mathcal{O}X_4 = \{\{a, b, c\},\ \emptyset, \{a, c\}\}$ and $\mathcal{O}X_5 = \{\{a, b, c\},\ \emptyset, \{b, c\}\}$.

4. (separation). Consider the following topology:

$$X = \{a, b, c\},\ \ \mathcal{O}X = \big\{\{a\},\ \{b\},\ \{c\},\ \{a, b, c\},\ \emptyset,\ \{a, b\},\ \{a, c\},\ \{b, c\}\ \big\}$$

Can we distinguish the point a from the point c?

Solution. To answer this question we must consider the neighbors of a and c:

- Neighbor of a: $\{a, b, c\}, \{a, b\}, \{a, c\}, \{a\}$
- Neighbor of c: $\{a, b, c\}, \{a, c\}, \{b, c\}, \{c\}$

The **neighbors are different** and therefore we can distinguish these two points.

5. (separation). Consider the following topology:

$$X = \{a, b, c\},\ \ \mathcal{O}X = \big\{\{a, b, c\},\ \emptyset, \{a, b\}\big\}$$

Can we distinguish the point a from the point b? Consider the neighbors of these points.

- Neighbor of a: $\{a, b, c\}, \{a, b\}$
- Neighbor of b: $\{a, b, c\}, \{a, b\}$

The neighbors are the same and therefore we can't distinguish these points. We could've solved this exercise considering the partial ordering: every set containing a also contains be and viceversa, and therefore $a \geq b \wedge a \leq b$. Therefore there's no partial ordering: a is not distinguishable from b.

6. (is it a topology?). Consider the set $X = \{0,1,2\}$ with $\mathcal{O}X = \big\{\emptyset, \{0,1,2\}, \{0,1\}, \{2\}\big\}$. Is this a topology?

Solution. To solve the exercise, consider $\mathcal{O}X$:

- $\mathcal{O}X$ contains both the empty set and the full set;
- This topology is closed under arbitrary union: $\emptyset \cup \{0,1,2\} = \{0,1,2\}$; $\emptyset \cup \{0,1\} = \{0,1\}, \emptyset \cup \{2\} = \{2\}, \emptyset \cup \emptyset = \emptyset, \dots$
- This topology is closed under finite intersection: $\emptyset \cap \{0,1,2\} = \ \emptyset, \{0,1,2\} \cap \{0,1\} = \{0,1\}, \dots$

Therefore yes, this is a topology.

7. (Separation). Consider the previous topology, $X = \{0,1,2\}$ with $\mathcal{O}X = \big\{\emptyset, \{0,1,2\}, \{0,1\}, \{2\}\big\}$. Can we distinguish the elements 0 and 1 by using this topology?

Solution. Consider the neighbors of 0 and 1:

- Neighbors of 0: $\{0,1,2\}, \{0,1\}$.
- Neighbors of 1: $\{0,1,2\}, \{0,1\}$

The open sets containing 0 and 1 are the same: therefore we cannot distinguish 0 from 1. We could've solved the problem also analyzing the partial ordering of this topology: given that every open set containing 1 also contains 0 and viceversa, then $0 \geq 1$ and $0 \geq 1$. Therefore 0 and 1 can't be separated.

8. (Distance). Given a topology on the set of natural numbers N, we define the distance as follows:

$$d(n, k) = \frac{1}{2^{|n-k|}}$$

Is this a distance?

Solution. To answer this, remember the properties of any distance:

1. $d(x,y) = 0 \ iff \ x = y$. In this case, $d(n,k) = \frac{1}{2^{|n-k|}} = \frac{1}{2^0} = \frac{1}{1} = 1$.

Therefore this is not a valid definition of distance.

9. (Distance). Given a topology on the set of natural numbers N, we define the distance as follows:

$$d(n,k) = 1 - \frac{1}{2^{|n-k|}}$$

Is this a distance?

Solution. Let's review the properties we've seen before:

- $d(n,k) = 0 \, iff \, n = k$: $1 - \frac{1}{2^0} = 1 - 1 = 0$.
- $d(n,k) = d(k,n)$
- $d(n,k) \leq d(n,r) + d(r,k)$: $d(n,r) = 1 - \frac{1}{2^{|n-r|}} \quad d(k,r) = 1 - \frac{1}{2^{|k-r|}}$, so:

$$1 - \frac{1}{2^{|n-k|}} \leq 2 - \frac{1}{2^{|n-r|}} - \frac{1}{2^{|k-r|}} \quad \rightarrow \quad \frac{1}{2^{|n-r|}} + \frac{1}{2^{|k-r|}} \leq 1 + \frac{1}{2^{|n-k|}} \quad \rightarrow \quad r \neq n,k \quad \frac{1}{2^{|n-r|}} \leq \frac{1}{2}$$

And therefore this is a valid distance.

10. (Neighbors). Consider the set $X = \{0,1,2,a,b,c\}$ with this partial ordering: $0 < 1 < 2, \quad a < b < c$. Find at least a couple of neighbors of the element 2.

Solution. A neighbor of 2 is a set containing the element 2 and an open set containing 2. To solve the problem we must first decide a topology on the set X and then list its open sets. Suppose we have an Alexandrov topology over X such that:

- $0 \leq 1 \leq 2$
- $a \leq b \leq c$

The open sets of this topology are:

$$\mathcal{O}X = \{\{0,1,2\}, \{1,2\}, \{2\}, \{a,b,c\}, \{b,c\}, \{c\}\}$$

Therefore some neighbors of 2 are:

- $\{1,b,2\}$: it contains 2 and an open set $\{1,2\}$ containing 2.
- $\{2,a\}$: it contains 2 and an open set $\{2\}$ containing 2.
- $\{0,1,2,c\}$: it contains 2 and an open set $\{0,1,2\}$ containing 2.

11. (Neighbors). Consider the previous exercise. Is $\{1,b\}$ a neighbor of 1?

Solution. The answer is no: $\{1,b\}$ contains 1 but not an open set containing 1.

12. (Degrees of separation). Consider the following topology: is it a T_0 space?

$$X = \{0,1,2\}, \qquad \mathcal{O}X = \{\{0,1,2\}, \emptyset, \{1,2\}, \{2\}\}$$

Solution. A topological space X is T_0 if there are no points belonging to the same neighborhood. Consider the neighbors of 0,1,2:

- Neighbors of 0: $\{0,1,2\}$
- Neighbors of 1: $\{0,1,2\}, \{1,2\}$
- Neighbors of 2: $\{0,1,2\}, \{1,2\}, \{2\}$

No point has the same neighborhood, and therefore this topological space is T_0. We could've solved the problem using instead the notion of partial ordering: given that $0 \leq 1 \leq 2$ (cfr. previous exercise), this is indeed a T_0 space.

13. *(Degrees of separation)*. Consider the following topology: is it a T_1 space?

$$X = \{0,1,2\}, \qquad \mathcal{O}X = \{\{0,1,2\}, \emptyset, \{1,2\}, \{2\}\}$$

Solution. A topological space is T_1 if the intersection of every open set containing a does not contain b, and the intersection of every open set containing b does not contain a. In this case, consider the point 1 and 2:

- Open sets containing 1: $\{0,1,2\}, \{1,2\}$
- Open sets containing 2: $\{0,1,2\}, \{1,2\}, \{2\}$

The intersection of every open set of 1 contains 2…

$$\{0,1,2\} \cap \{1,2\} = \{1,2\}\ contains\ 1$$

…and the intersection of every open set containing 2 contains 1:

$$\{\mathbf{0},\mathbf{1},\mathbf{2}\} \cap \{\mathbf{1},\mathbf{2}\} = \{\mathbf{1},\mathbf{2}\}, \{0,1,2\} \cap \{2\} = \{2\}, \{1,2\} \cap \{2\} = \{2\}$$

Therefore this space is not T_1. Note that we could've solved this problem differently, considering we have a partial ordering over X: any topological space with a partial ordering is T_0 and not T_1.

14. *(Degrees of separation)*. Consider the indiscrete topology $X = \{0,1,2\}$ such that $\mathcal{O}X = \{\{0,1,2\}, \emptyset\}$. Is this topology T_0, T_1or T_2?

Solution. Let's review the possible specializations for any topology:

- T_0: no two points belong to the same neighborhood. Let's see the neighbors of 0,1,2:
  - 1: $\{0,1,2\}$;
  - 2: $\{0,1,2\}$;
  - 0: $\{0,1,2\}$.

Every element has the same neighborhood, so space is not T_0. We could've said the same considering the partial ordering of the set X: every open set containing 1 also contains 2 and 0, every open set containing 2 also contains 0 and 1, every open set containing 0 also contains 1 and 2: there's no partial order in this set, and therefore this space is not T_0.

- T_1: the intersection every open set containing a does not contain b, the intersection of every open set containing b does not contain a. Let's see the open sets containing 0,1 and 2:
  - Open sets containing 0: $\{0,1,2\}$
  - Open sets containing 1: $\{0,1,2\}$
  - Open sets containing 2: $\{0,1,2\}$

The intersection $\{0,1,2\} \cap \{0,1,2\} = \{0,1,2\}$, which contains 0, 1 and 2. Therefore this space is not T_1.

- T_2: for every point in the space $x, y \in X$ such that $x \in U\ open\ set, y \in V\ open\ set$, the intersection of open sets $U \cap V$ is empty. In this case, consider the point 0 and 1: the intersection of this open set with itself is never empty. This means that the space is not T_2.

15. *(topology)*. Consider a subset A of the set of natural numbers such that A is open if and only if $5 \in A$. Is this a topology?

Solution. Yes, this is a topology called ***point topology***. In fact:

- $\mathcal{O}X$ contains both the empty set and the full set A;
- it's closed finite intersection: if 5 belongs to the open sets A and B, then their intersection is an open set containing 5.
- it's closed under abritrary union: if 5 belongs to every open set, then the union of every open set also contains 5.

Therefore this is a topology.

15. *(degrees of separation)*. Consider the previous point topology. Is it a T_0 or T_1 space?

Solution. To be a T_0 space, every point must belong to a distinct neighbor or, alternatively, the set must have a partial ordering. Given that we have a very large number of possible open sets, it's better to work on the partial ordering. Now, while every open set must contain 5, there can be open sets containing 5 and not n, where n is an arbitrary number:

$N \backslash \{n\}$ is an open set ($5 \in N\{n\}$)

Therefore every open set containing n also contains 5, and not viceversa: $n \leq 5$. 5 is also the maximum element of the set. Therefore the space is T_0, which means that the space is not T_1. We could've proved that this topology is not T_1 by just looking at the open sets of this topology: there's always an open set containig 5 and not an arbitrary n, but there's no possible open set containing n and not 5. Therefore, this space is not T_1.

16. *(open sets)*. Consider the set of natural numbers N. Is the subset of *even numbers* an open set?

Solution. Visualize the set of natural numbers over a "natural line"; the complement of the set of even numbers – the set of odd numbers – is an infinite set, which means that it's an open set:

$$[1,3,5,7,\dots) \quad open\ set$$

It means that the set of even numbers is *not* open.

17. *(open sets)*. Consider the set of digits $X = \{0,1,2,3,4,5,6,7,8,9\}$ as a subset of the set of natural numbers; is X an open set?

Solution. No, because the complement is an infinite set, which means that it's also open:

$$[10,11,12,13,\dots)$$

Therefore the set X of digits is not open but **closed**.

18. *(degrees of separation)*. Consider the cofinite topology over the set of natural numbers ($A \subseteq N$ is cofinite if $N \backslash A$ is a finite set). Is it a T_0 or T_1 space?

Solution. Do we have a partial ordering over A? Consider two arbitrary numbers, for example 5 and 10: the set $N \backslash \{10\}$ is cofinite because $\{10\}$ is a singleton, which is clearly a finite set. Now, the open set $N \backslash \{10\}$ contains 5 and not 10, wile the open $\{10\}$ does not contain 5: therefore this is a T_1 space and not a T_0 space. In the same way, the open set $N \backslash \{5\}$ contains 10 and not 5 while $\{5\}$ contains 5 and not 10.

19. *(degrees of separation)*. Consider the cofinite topology of the previous example. Is it a T_2 space?

Solution. Can we find two open sets one containing 5, another containing 10 such that their intersection is empty? Consider the sets $N\backslash\{5\}$ containing 10 and the set $N\backslash\{10\}$ containing 5: their intersection is clearly not empty.

20. *(degrees of separation)*. Consider the Euclidean space. Is it a T_2 space?

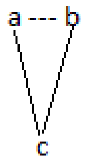
Solution. Suppose we have two points in the Euclidean space R^2: we can always have two open balls, one centered in x and the other centered in y, with an empty interception. Therefore the Euclidean space is T_2.

21. *(preorder on a topology)*. Consider the following topology:

$$X = \{a, b, c\}, \ \ \mathcal{O}X = \{\{a, b, c\}, \ \emptyset, \{a, b\}\}$$

Can we find a preorder on this topology?

Solution. $a \leq b$ because every open set containing a also contains b; but $b \leq a$ because every open set containing b also contains a. It means that we do not have a partial ordering. We can draw this situation as two lines on the same level. Every set containing c also contains a, b but not viceversa: this means that $c \leq a$ and $c \leq b$. Therefore we have this representation: c is the bottom element, a and b are on the same level.



22. *(Alexandrov topology)* Consider the *topological space* of the real line with Alexandrov topology over it. Is the set of rational numbers open in this topology?

Solution. A set, in the Alexandrov topology, is open if it's an *upper set*: given two elements x, y such that $y \in Y, x \in X$ if $x \geq y$ then necessarily $x \in Y$. Consider the following examples:

- $y \in Q$, for example 3
- $x \in R$, for example $\pi \sim 3.14$

If $\pi \geq 3$ – which is true in this case – then is it also true that $\pi \in Q$? No: therefore the set is **not open**.

23. *(Alexandrov topology, Euclidean topology)* Provide an example of set open in the Euclidean topology R but not open in the *Alexandrov topology*.

Solution. A finite interval such as $(-1, +1)$ is an open ball centered in 0 and with radius $r = 1$ in the Euclidean topology but not an upper set in the *Alexandrov topology*: the number 2 is greater than 0 but if $0 \in (-1, +1), 2$ does not belong to $(-1, +1)$.

24. *(Alexandrov topology, Euclidean topology)* Find a set B open with respect to both the *Euclidean topology* and the *Alexandrov topology*.

Solution. The set of real numbers R is open with respect to the Euclidean topology and the Alexandrov topology. Another valid answer is the empty set.

25. ***(Alexandrov topology, Euclidean topology).*** Find an open set different from the empty set and the full set open with respect to both the *Euclidean topology* and the *Alexandrov topology*.

Solution. Consider an *infinite upper interval* like $[5, +\infty)$. In this case the set is open in both topologies. In fact, given $x \in R$ and $y \in [5, +\infty)$, if $x \geq y$ then $x \in Y$.

26. ***(Euclidean topology)***. Consider the set $[0)$: is it open or closed?

Solution. It's an open set: in fact it can be seen as the *union* of an infinite number of open intervals – which means that it's also open.

27. ***(Alexandrov topology)***. Is the set of *rational numbers* a closed set with respect of the Alexandrov topology?

Solution. The complement of the rational number is the set of *irrational numbers*. The set of irrational numbers is *not* open because, given x, y such that $x \in R$ and $y \in Irrational$, if we consider:

- $x = 4$
- $y = \pi$

Then if $4 \geq \pi$ – which is true – then $4 \in Irrational$? No, and therefore the set is **not open**. therefore the set of rational number is ***not* closed**. If we remember the previous exercise, then we can say that the set of rational number is neither open nor closed.

28. ***(Set interpretation)***. Consider the following set:

$$\{x \in R : \exists y \in Irr, x \geq y\}$$

What is this set?

Solution. This is the set of *real numbers*. In fact it contains every real number such that, for each real number x, there exists an irrational number y smaller than x. Therefore:

$$R = \{x \in R : \exists y \in Irr, x \geq y\}$$

29. ***(Set interpretation)***. Find a set B which is closed with respect to the *Euclidean topology* but not closed with respect to the *Alexandrov topology*.

Solution. Consider a single point: it's closed with respect to the *Euclidean Topology* because the complement is the whole line without the point, which is the union of two open sets:

$$(-\infty, a) \cup (a, +\infty) \ \ union\ of\ open\ sets$$

The single point is instead *not* closed with respect to the *Alexandrov topology*: in fact its complement, $R\backslash\{a\}$, is not an upper set. Consider for example x, y such that $x \in R, y \in R\backslash\{a\}$:

- $x = a$
- $y = a - 1$

If $a \geq a - 1$ – which is true, then $a \in R\backslash\{a\}$? No. Therefore $R\backslash\{a\}$ is not an open set.

30. ***(Separation, Alexandrov topology)***. Can I separate the number 0 and 1 with respect to the *Alexandrov topology*?

Solution. Let's start with T_0 separation: is there a partial ordering on the real line? Do the elements 0 and 1 have different neighbors? I can't find an open set containing 0 and not 1 (such upper set, if it contains 0, must also contain every real number greater than 0); instead, I can find an open set containing 1 and not 0, for example the open set starting from the element 1 included. this means that there's a partial ordering such that $0 \leq 1$: therefore **0 and 1 separated** on the real line with respect to the *Alexandrov topology*. Generally speaking, **the specialization order of the Alexandrov topology on the real number is $\boldsymbol{T_0}$**, where $0 \leq 1 \leq 2 \ldots$

Is this space T_1 instead? No, because however I choose two points, we must have an open set containing the first and not the second point and viceversa. This is only partially true in the Alexandrov topology, and therefore the Alexandrov topology is ***not $\boldsymbol{T_1}$***.

31. *(Separation, Euclidean line)*. Consider the points a, b belonging to the real line R. Can we separate a from b?

Solution. Yes. In fact we can think of two open balls with radius $r = \frac{d(a,b)}{2}$ centered in a and b such that they have an open intersection. Therefore the Euclidean topology is T_2.

32. *(Connection)*. Is the space of real numbers with respect to the *Euclidean topology* a connected space?

Solution. Can we find two open sets such that the Euclidean topology does not result from their union? Yes: we can find two open intervals on the real line such that the entire real line does not result from their union.

$$(a, b) \cup (c, d) \neq R$$

To prove this, assume that the real line R is not connected: there exists two *non-empty disjoint open sets* A, B such that their union yields the entire line.

$$R = A \cup B, A \cap B = \emptyset$$

Assume the element $0 \in A$ (and therefore does not belong to B): therefore we have:

$$R = A \cup B = \big(A \cap (0, +\infty)\big) \cup \big(A \cap (-\infty, 0)\big) \cup B$$

Now, using deMorgan's law:

$$\big(A \cap (0, +\infty)\big) \cup \big(B \cap (0, +\infty)\big) = (0, +\infty) \cap (A \cup B) = (0, +\infty) \subset R$$

…and therefore we can't reduce the line we can't reduce the entire line R to the sum of two intervals A, B.