# Discontinuous Galerkin Method for Stokes Equation (P)

## Seminar in Computational Fluid Dynamics

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# Overview

Informal introduction to discontinuous Galerkin Methods

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# Continuous Galerkin (cG)

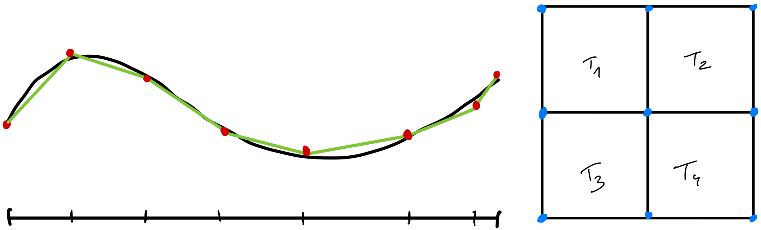


Figure: Continuous galerkin method. The nodes are shared by the corresponding elements.

# Discontinuous Galerkin (dG)

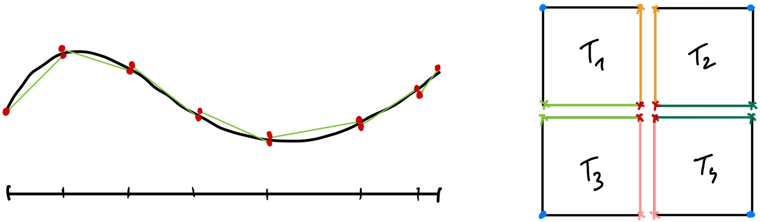


Figure: In discontinuous galerkin method, the elements do not share the nodes. In the two dimensional case, the boundary with same colors are shared amongst the elements, however, with dG, they are treated as "separate" elements

$$V_h = P^k(\mathcal{T}) = \{v_h \in L^2(\Omega) : v_h|_T \in P^k(\mathcal{T}) \forall T \in \mathcal{T}_h\}$$

# Averages and Jumps

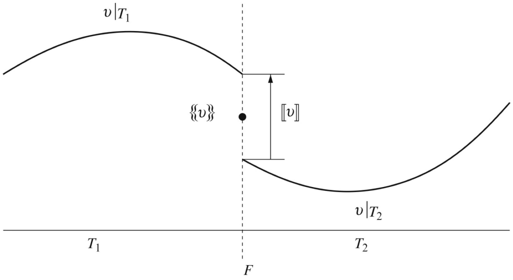


Figure: One-dimensional example of average and jump operators

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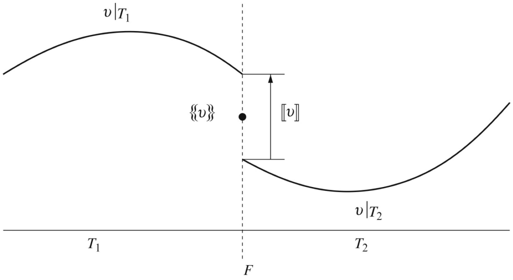


Figure: One-dimensional example of average and jump operators

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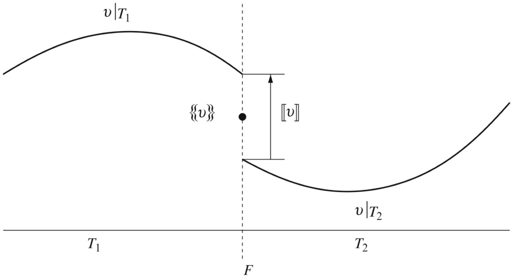


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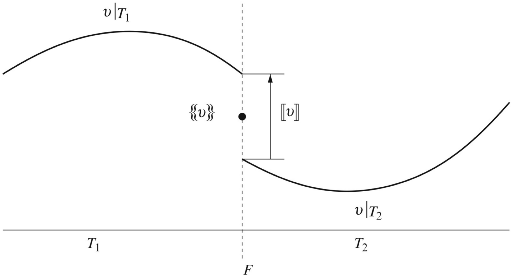


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At the boundary of the domain: $\{\{v\}\}_F(x) = [\![v]\!]_F(x) := v|_T(x)$

# Penalty and numerical Fluxes

**Numerical Flux**

Introduces coupling between different sub problems.

Allows us to recover global solution (flux needs to be conservative between elements).

Local solution on a particular element is dependent upon the local solution from nearby elements through the numerical flux function.

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**Weakly enforce**:

Continuity of the flux: $[\![\nabla u]\!] = 0$ over all facets

Continuity of the solution: $[\![u]\!] = 0$ over all facets

Stability

Short Recap to Poisson equation

# Short Example: Diffusion Equation

Diffusion / poission equation in weak form

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\begin{aligned}
-\Delta u = f \quad & in \quad \Omega \\
u = 0 \quad & on \quad \partial\Omega
\end{aligned}
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the weak formulation is given by: (Proof by general agreement)[1]:

$$u \in V \quad s.t. \quad a(u,v) = \int_\Omega fv \quad \forall v \in V, V = H_0^1(\Omega)$$

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# Poisson Equation: SIP Formulation

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## Integration domain

Unlike cG, integration in this case is not done over $\Omega$, rather it is limited to each element $T \in \mathcal{T}_h$.

# cG vs dG

**cG**

$$a_h(u_h, v_h) = \int_\Omega f v_h \forall v_h \in U_h$$

* Considers entire domain.
* DoF on the edges are shared.

**dG**

$$a_h(u_h, v_h) + \mathrm{SIP} = \int_\Omega f v_h \forall v_h \in U_h$$

* Local to the generating element.
* Numerical flux is used to derive a uniquely defined value of the quantities of interest
* Comparatively more degrees of freedom.

DG (in a weak way) does a better job at conserving fluxes at a same computational cost (it is more useful to use a dG method than to use a finer mesh).

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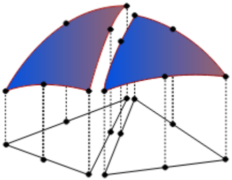

## Stokes Equation

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Wherein, $f \in [L^2(\Omega)]^d, u \in U := [H_0^1(\Omega)]^d, p \in P := [L_0^2(\Omega)]$, we look for solution in $X := U \times P$.

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$u_h, p_h$ belong to the respective finite element spaces of U and P respectively.

# Discreete Spaces for dG

The solution for discontinous gelarkin method lies in the in $X_h := U_h \times P_h$ where

$$U_h := [\mathbb{P}_d^k(\mathcal{T}_h)]^d$$

and

$$P_h := [\mathbb{P}_{d,0}^k(\mathcal{T}_h)]^d$$

such that $[\mathbb{P}_d^k(\mathcal{T}_h)]^d$ is the broken polynomial space.

# Stokes Equation (SIP formulation)

We consider the equal order discontinuous velocity and pressure.

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Where $s_h(p_h, q_h) := \sum_{F \in \mathcal{F}_h^i} h_F \int_F [\![q_h]\!][\![p_h]\!]$ is the stablization meant to control pressure jumps across interfaces.

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# Stokes Equation (SIP formulation)

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$$b_h(q_h, v_h) = \int_\Omega v_h \cdot \nabla_h q_h + \sum_{F \in \mathcal{F}_h} \int_F \{\{v_h\}\} \cdot n_F [\![v_h]\!]$$

Hence, the discreetization of Stokes equation is given as:

$$c_h((u_h, p_h), (v_h, p_h)) := a_h(u_h, v_h) + b_h(v_h, p_h) - b_h(u_h, q_h) + s_h(p_h, q_h) \tag{1}$$

It can be shown that the given formulation is Discrete inf-sup stable (Lemma 6.13) in [1].

# Implementation

# General Implementation of FE solver

**Require:** Create / import mesh.

**Ensure:** Prepare system (initialize polynomial space, linear solver etc.).

Set up boundary conditions.<br>
**while** $t \leq T$ or !mesh_refinement **do**<br>
    Set up boundary conditions.<br>
    call the local assembler (assemble_system)<br>
    Solve System<br>
    Compute Error<br>
**end while**<br>
Post Processing

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Weak formulation is defined in the local assembler.

```
code: local_assembler.h
```

# General Implementation of FE solver

```
for (int q = 0; q < num_q; ++q){
  ...
  // loop over test DOFs <-> test function v
  for (int i = 0; i < num_dof; ++i){
    // loop over trrial DOFs <-> trial function u
    for (int j = 0; j < num_dof; ++j) {
      lm(i, j) += wq * (dot(phi_j, phi_i) * dJ;
    }
  }
}
```

# Implementation

- Compared to the cG method (shown in previous slide), implementation of dG method has some extra steps.
- The construction of local assembler is the same.
- SIP formulation is added within the local assembler.
- Jumps and Averages are calculated based on the interface (boundary/interior).

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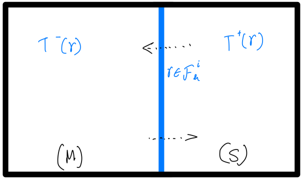


Figure: hiFlow employs *master/slave* paradigm $n_m$, $n_s$ is the unit normal for master and slave element respectively.

# Implementation SIP

```
for "interior interfaces" e:
    for "trial_functions" in (m, s):
        for "test_functions" in (m, s):
            compute: integral_over_interface
```

where **intergral_over_interface** $= \int_e [\![\phi_i^{test}]\!] \cdot [\![\phi_j^{trial}]\!]$

$$\phi_i^{test}|_m = \begin{cases} \phi_i^{test} & \text{test} = \text{m} \\ 0 & \text{test} = \text{s} \end{cases} \qquad \phi_i^{test}|_s = \begin{cases} 0 & \text{test} = \text{m} \\ \phi_i^{test} & \text{test} = \text{s} \end{cases}$$

# Implementation - Boundary Conditions

To distinguish between boundary and interior facets, hiFlow3 uses material numbers. The normal acts differently on interior facets and boundary facests,

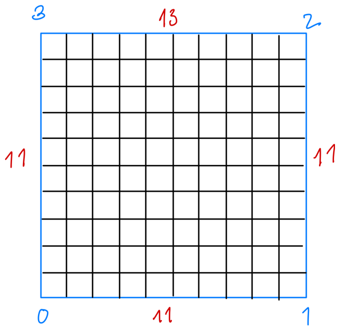


Figure: Material number of the boundary is given in red. The black boxes inside the square have material number less than 11

# Implementation - Boundary Conditions (cont.)

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```
int get_if_type(const Entity &face) const{
    const int mat_num = face.get_material_number();
    if (mat_num >= 11){
        return 1; // dirichlet
    }
    return 0;   // interior
}
```

# Implementation - Assembly

```
code dg_CavityStokes.h
```

*In theory, theory and practice are the same; in practice, they're not.*

# Poission Equation

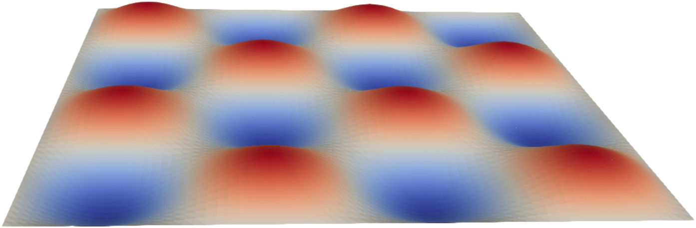

Figure: Poission Equation with Discontinous Galerkin Method

# Stokes dG

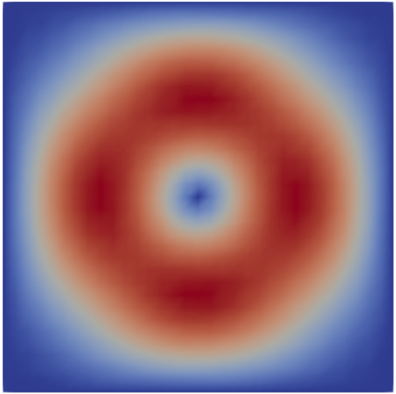

Figure: Stokes Equation with dirichlet boundary $= 0$ on all four sides.

# Error Norms

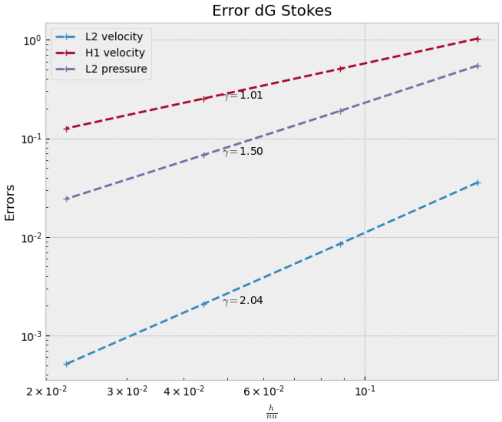


Figure: Stokes equation with 0 Dirichlet boundary condition and $\eta = 10$

# Error Norms: cG vs dG

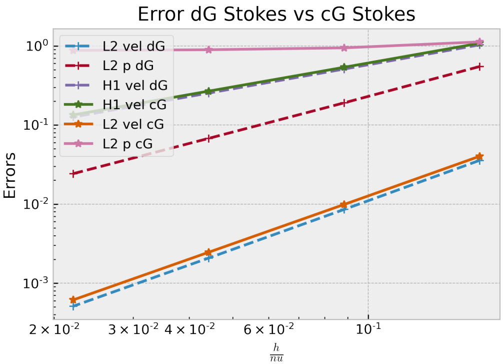


Figure: Error norm for cG and dG methods for stokes equation with $u = 0 \quad \forall u \in \partial\Omega$. The dashed lines represent errors with dG, while the straight lines represent errors with cG method.

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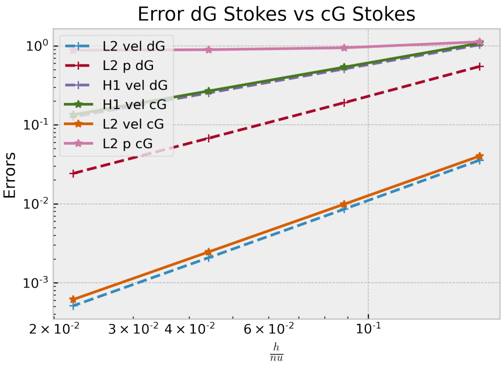


| $\gamma$ | L2 Pressure | L2 Velocity | H1 Velocity |
|---|---|---|---|
| cG | 0.118 | 2.010 | 1.003 |
| dG | 1.499 | 2.041 | 1.009 |

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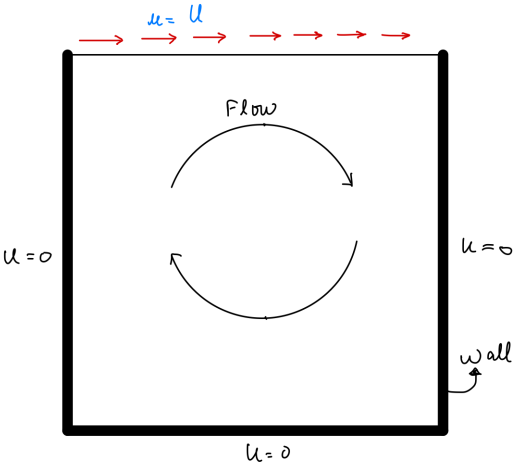
# Lid Driven Cavity

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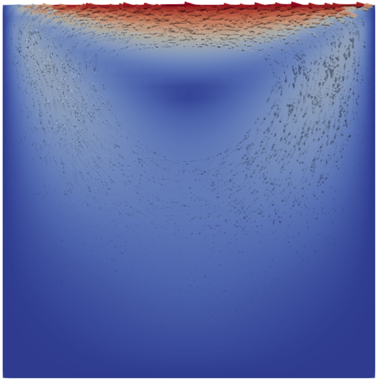

Figure: Velocity

Figure: Pressure

# The conundrum of $\eta$

The inf-sup stability of discrete formulation is given by Lemma 6.13 in [1] , which assumes that the penalty parameter $\eta > \underline{\eta}$ where $\underline{\eta} := C_{tr}^2 N_\partial$ (Lemma 4.12), $C_{tr} = \sqrt{k(k+d)}$

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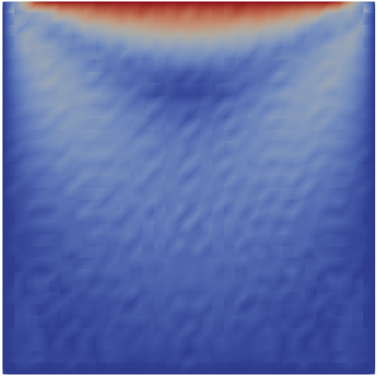

Figure: $\eta = -10$

# Lid Driven Cavity - Varying $\eta$

Calculation for eta for k = 1, d = 2

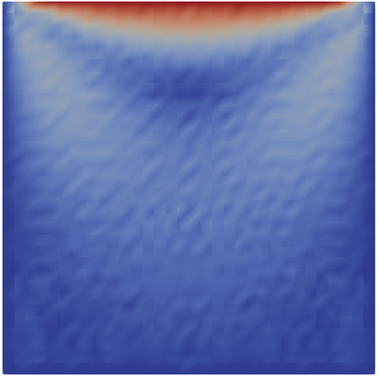

Figure: $\eta = -10$

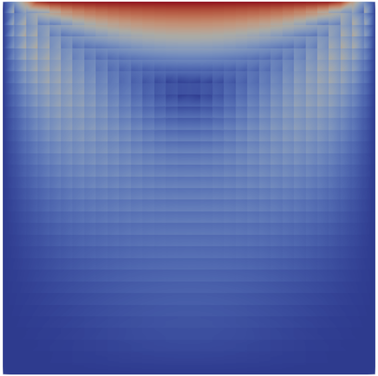

Figure: $\eta = 1$

# Lid Driven Cavity - Varying $\eta$

Figure: $\eta = -100$

# Lid Driven Cavity - Varying $\eta$

Figure: $\eta = -100$

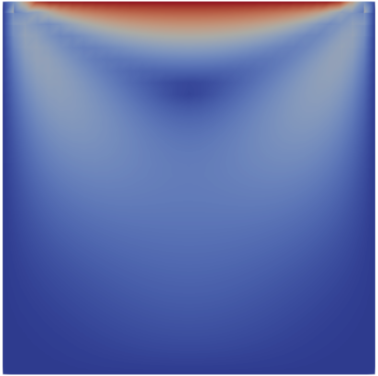

Figure: $\eta = 2$

# Linear Solvers: $\eta$

GMRES residuals for different $\eta$ (Lid Driven Cavity)

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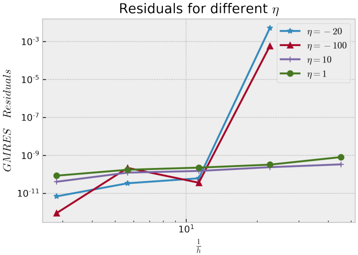


Figure: GMRES residuals for different $\eta$ (Lid Driven Cavity)

Table: Iterations. First column (bold) represents *eta* values

| **-100** | **-20** | **1** | **10** |
|---|---|---|---|
| 72 | 67 | 34 | 51 |
| 167 | 139 | 38 | 67 |
| 389 | 347 | 52 | 107 |
| - | - | 58 | 131 |
| - | - | 70 | 190 |
| - | - | - | - |
| - | - | - | - |

# Linear Solvers: cG vs dG

GMRES Residuals for cG vs dG

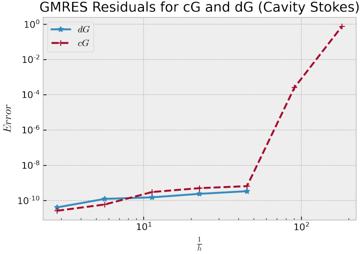


Figure: cG dG GMRES residuals (stokes, dbc = 0)

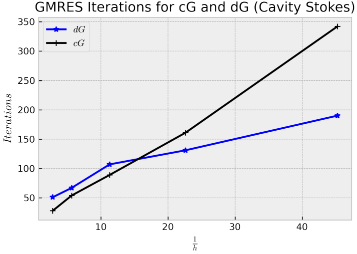


Figure: cG dG GMRES iterations (stokes, dbc = 0)

# Linear Solvers: Equal Order Convergence

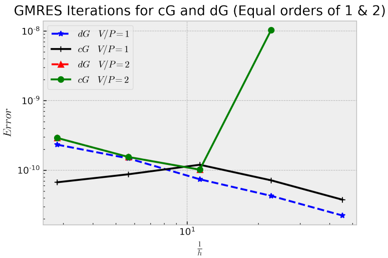


Figure: dG cG equal order (k=1,2) residuals

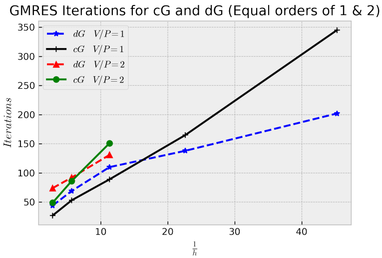


Figure: cG dG equal order (k=1,2) iterations

# The curious case of Domain Decomposition

Stokes simulation on 48 cores.

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Stokes simulation on 48 cores.

| dG | | |
|---|---|---|
| Refinement | Iterations | Residual |
| 4 | 429 | 1.00E-10 |
| 5 | 454 | 1.57E-08 |
| 6 | 469 | 2.66E-11 |

| cG | | |
|---|---|---|
| Refinenment | Iterations | Residual |
| 4 | 1000 | 2.94E-07 |
| 5 | 1000 | 0.237797 |
| 6 | 1000 | 0.00347334 |

# Non Symmetric Penalty, $\eta = 1$

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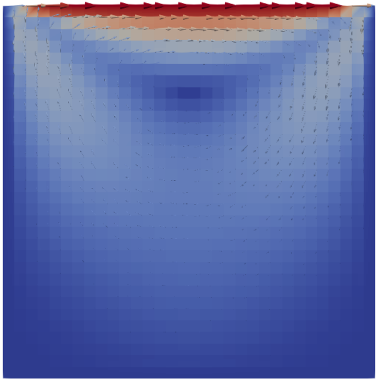

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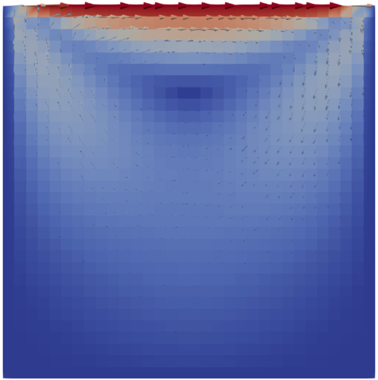

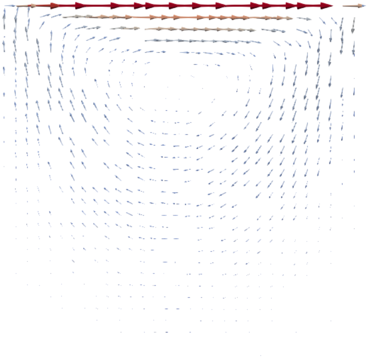

# Conclusion

1.

# Further Reading

1. Ocellaris: mass conserving dG solver.

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# Thank You

**Appendix**

# Convergence wars: Return of the pressure stabilization

| cG | | |
|---|---|---|
| Refinements | Iterations | Residuals |
| 2 | 16 | 6.54E-11 |
| 3 | 21 | 4.73E-11 |
| 4 | 27 | 2.52E-10 |
| 5 | 35 | 2.86E-10 |
| 6 | 56 | 5.04E-10 |
| 7 | 143 | 9.42E-10 |

| dG | | |
|---|---|---|
| Refinements | Iterations | Residuals |
| 2 | 39 | 2.17E-11 |
| 3 | 52 | 1.10E-10 |
| 4 | 85 | 1.63E-10 |
| 5 | 116 | 2.46E-10 |
| 6 | 170 | 3.39E-10 |
| 7 | 295 | 4.65E-10 |

Figure: dG method with a different pressure stabilization scheme converges even at higher refinement level. cG Performs preferably better than dG

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Where $t(u,u,v) := \int_\Omega (w \cdot \nabla u) \cdot v = (u \cdot \nabla) u$
We can show that the given trilinear form is bounded by a domain parameter $t_\Omega$ (Lemma 6.32 in [1]) and it's existance and uniqueness by Thm 6.36 in [1].

# NS Formulation

Moreover, We consider a modified form of the trilinear form $t$

$$t'(w,u,v) = t(w,u,v) + \frac{1}{2}\int_\Omega (\nabla \cdot w) u \cdot v = \int_\Omega (w \cdot \nabla u) \cdot v + \frac{1}{2}\int_\Omega (\nabla \cdot w) u \cdot v$$

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to satisfy the skew symmetry of triliner form when dealing with dG approximations. The INS formulation is then given by:

$$c((u,p),(v,q)) + t'(u,u,v) = \int_\Omega f\cdot v$$

(3)

# NS Formulation - Discrete Setting

The discrete trilinear form is defined as:

$$t_h(w_h, u_h, v_h) := \int_\Omega (w_h \cdot \nabla_h u_h) + \frac{1}{2}\int_\Omega (\nabla_h \cdot w_h) \cdot (u_h \cdot v_h)$$
$$- \sum_{F \in \mathcal{F}_h^i} \int_F \{\{w_h\}\} \cdot n_F [\![u_h]\!] \cdot \{\{v_h\}\} - \frac{1}{2} \sum_{F \in \mathcal{F}_h} \int_F [\![w_h]\!] \cdot n_F \{\{u_h \cdot v_h\}\}$$

The discrete trilinear form is bounded. [1]

# NS Discrete Formulation

Finally, we have the dG discrete formulation as:

$$\mathcal{V} a_h(u_h, v_h) + t_h(u_h, u_h, v_h) + b_h(v_h, p_h) = \int_\Omega f \cdot v_h \quad \forall v_h \in U_h$$

$$-b_h(u_h, q_h) + \mathcal{V}^{-1} s_h(p_h, q_h) = 0 \quad \forall q_h \in P_h$$

The solution for the above formulation exists, and is unique [1] (Lemma 6.41). It can also be shown that the discrete formulation show above converges to the unique solution of 2 [1] (Thm 6.47)

# dG notations

- $\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^b$, where $\mathcal{F}_h^i$ are the interior interfaces
- $\mathcal{F}_T := \{\mathcal{F} \in \mathcal{F}_h | F \subset \partial T\}$
- $\mathcal{T}_F := \{T \in \mathcal{T}_h | F \subset \partial T\}$
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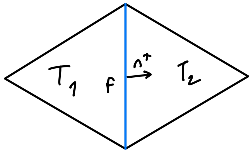


Figure: Notation of an interface

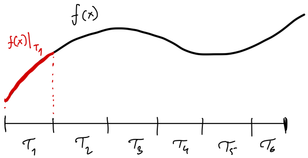


Figure: Restriction of $f(x)$ in $T_1$ is highlighted by the red region.

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Epshteyn, Y. & Rivière, B. Estimation of penalty parameters for symmetric interior penalty Galerkin methods. *Journal Of Computational And Applied Mathematics*. **206**, 843-872 (2007), https://www.sciencedirect.com/science/article/pii/S0377042706005279