CS 771A: Intro to Machine Learning, IIT Kanpur				**Quiz II** (24 Oct 2025)
Name				**20 marks**
Roll No		Dept.		Page **1** of **2**

Instructions:

1. This question paper contains 1 page (2 sides of paper). Please verify.
2. Write your name, roll number, department above in **block letters neatly with ink**.
3. Write your final answers neatly **with a blue/black pen**. Pencil marks may get smudged.
4. Don't overwrite/scratch answers especially in MCQ – such cases may get straight 0 marks.
5. Do not rush to fill in answers. You have enough time to solve this quiz.

Q1. (Matrix Norm) $A \in \mathbb{R}^{m\times m}$ is a (possibly non-symmetric) square matrix with SVD $A = U\Sigma V^\top$. The diagonal matrix $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_m) \in \mathbb{R}^{m\times m}$ has $\sigma_1 > \sigma_2 > \cdots > \sigma_m$ (all singular values are distinct), $U = [\mathbf{u}_1, \dots \mathbf{u}_m], V = [\mathbf{v}_1, \dots \mathbf{v}_m] \in \mathbb{R}^{m\times m}$ are orthonormal. Let $a^* \stackrel{\text{def}}{=} \max_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2$. Note the constraint $\|\mathbf{x}\|_2 = 1$. Find the value of a^* in terms of $\mathbf{u}_i, \mathbf{v}_i, \sigma_i, i \in [m]$. Show brief derivation. *Hint*: *try expressing* $\mathbf{x}$ *as a linear combination of the right singular vectors* $\mathbf{v}_1, \dots, \mathbf{v}_\mathrm{m}$ *which is always possible as they form a basis over* $\mathbb{R}^m$. *You may also find* $VV^\top = I = V^\top V$ *useful.* (**2+3 = 5 marks**)

$a^* \stackrel{\text{def}}{=} \max_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2 =$

Give brief derivation below

Q2. (Kernel Smash) $K_1, K_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Mercer kernels with $K_i(x,y) = \langle \phi_i(x), \phi_i(y) \rangle$ for any $x, y \in \mathbb{R}$ with the feature maps given below. Design a map $\phi_3 : \mathbb{R} \to \mathbb{R}^3$ for a kernel K_3 defined as $K_3(x,y) = K_1(x,y) \cdot K_2(x,y)$. No derivation needed. **$\boldsymbol{\phi_3}$ must not use more than 3 dimensions. If your solution doesn't need 3 dimensions, fill remaining ones with zero.** **(6 marks)**

$$\phi_1(x) = \left(\frac{1}{x}, \frac{1}{x^2}\right) \in \mathbb{R}^2$$

$$\phi_2(x) = (x, x^2) \in \mathbb{R}^2$$

$$\phi_3(x) = \left(\boxed{}, \boxed{}, \boxed{}\right)$$

Q3. (True-False) Write **T** or **F** for True/False in the **box on the right** and a **brief justification** in the space below (brief proof if **T**, counterexample if **F**). A square matrix is termed *diagonal* if all of its off-diagonal entries are zero (its diagonal entries can be zero/-ve/+ve). **(3 x (1+2) = 9 marks)**

1	For a square symmetric matrix $A \in \mathbb{R}^{3\times 3}$ having eigen decomposition $A = VLV^\top$, we always have $\mathrm{trace}(A) = \mathrm{trace}(L)$ where L is the diagonal matrix of eigenvalues.	
2	Given any two matrices $A, B \in \mathbb{R}^{3\times 3}$, the largest singular value of $A + B$ is always equal to the sum of the largest singular value of A and largest singular value of B.	
3	For a kernel $K: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $K(\mathbf{x}, \mathbf{y}) \stackrel{\text{def}}{=} \exp(\mathbf{x}^\top \mathbf{y})$ and any vectors $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^2$, the matrix $G = [g_{ij}] \in \mathbb{R}^{n\times n}$ with $g_{ij} \stackrel{\text{def}}{=} K(\mathbf{x}_i, \mathbf{x}_j)$ is always full rank, $\mathrm{rank}(G) = n$.	