

## Assignments

### • Assignment 1: The Binary Search and Information Entropy (22-08-2019)

1. Average information content is given by the formula,  $\langle I \rangle = -k \sum_i P_i \log_2 P_i$ , in which  $k$  is a constant and  $P_i$  is the probability of an event.
  - (a) For a two-outcome problem (e.g. a coin toss), show that  $\langle I \rangle$  peaks at  $P = 1/2$ .
  - (b) Apply a very small perturbation as  $P = (1/2) + \epsilon$ , in which  $\epsilon \ll 1/2$ . Show that in this perturbative approach  $\langle I \rangle \simeq a - b\epsilon^2$ , with  $a = k$  and  $b = 4k/\ln 2$ . Hint:  $\ln(1+x) \simeq x$ , when  $x \ll 1$ .
  - (c) Plot  $\langle I \rangle$  versus  $P$  for both the actual function and the approximate function together and then compare the graphs for closeness on the line  $\langle I \rangle = 0$ . For plotting choose  $k = 1$ .

### • Assignment 2: An Astrophysical Inflow (22-08-2019)

1. In the problem of spherically symmetric astrophysical accretion, interstellar fluid matter (a very thin gas) travels a great distance (almost from infinity) along radial lines and falls on to a massive star (or a neutron star or even a black hole) located at the origin of coordinates. The star can be treated as a point-like particle, and the rate of the fluid flow (matter flowing in unit time) on to it is given as

$$\dot{m} = \pi G^2 M^2 \frac{\rho_\infty}{c_s^3(\infty)} \left( \frac{2}{5-3\gamma} \right)^{(5-3\gamma)/2(\gamma-1)}$$

in which  $G$  is Newton's universal gravitational constant,  $M$  is the mass of the central astrophysical object,  $\rho_\infty$  is the constant density of the gas at infinity,  $c_s(\infty)$  is the speed of sound at infinity, and  $\gamma$  is a dimensionless number called the polytropic exponent ( $1 \leq \gamma \leq 5/3$ ). The velocity of the fluid flow  $v$ , as a function of the radial distance from the centre  $r$ , is given by the equation

$$f(v, r) \equiv \frac{v^2}{2} + n \left( \frac{\dot{m}}{vr^2} \right)^{1/n} - \frac{GM}{r} - nc_s^2(\infty) = 0 \quad (1)$$

with  $\dot{m} = (\dot{m}/4\pi\rho_\infty)c_s^{2n}(\infty)$  and  $n = 1/(\gamma - 1)$ . Solve Eq.(1) by the bisection method to find  $v(r)$ , using the values  $M = 2 \times 10^{30}$  kg,  $c_s(\infty) = 10$  km s<sup>-1</sup>,  $\rho_\infty = 10^{-21}$  kg m<sup>-3</sup> and  $n = 2.5$ . These values are typical of accretion of the interstellar medium on to a star. Each value of  $r$  in Eq.(1) will give a set of two real and physical roots of  $v$ . The plot of  $v(r)$  is shown in Fig. 1, in which  $v$  is scaled as the Mach number,  $v(r)/c_s(r)$ . Obtain a similar plot. Values of  $c_s(r)$  can be found from  $c_s(r) = c_s(\infty)(\rho/\rho_\infty)^{(\gamma-1)/2}$  with  $\rho \equiv \rho(r)$  being another function of  $r$  given by  $\rho(r) = \dot{m}/(4\pi vr^2)$ . Also scale the radial distance by  $7 \times 10^8$  m, and then take its base-10 logarithm in the horizontal axis of the graph. For every value of  $r$ , run a bisection code going from  $v = 0$  to  $v = 10^5$  ms<sup>-1</sup>, with a step-size of 0.0010. Converge on a root with a relative error tolerance of  $|\epsilon| = 10^{-10}$ .

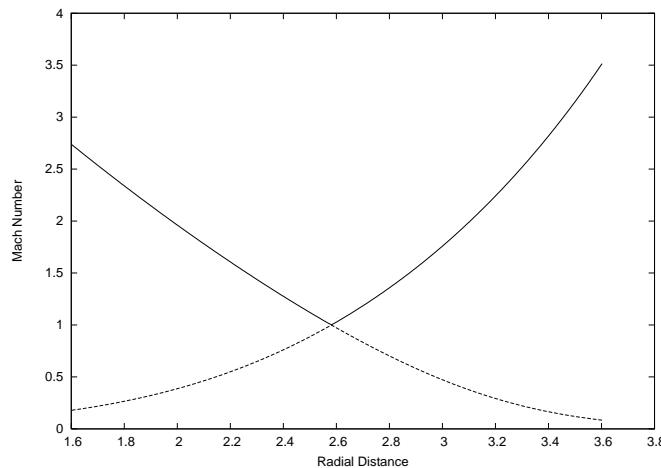


Figure 1: A plot of the Mach number,  $v/c_s$ , against the radial distance  $r$  (given on a base-10 logarithmic scale).

• **Assignment 3: A Nuclear Outflow (29-08-2019)**

1. High-energy impacts and collisions among elementary particles can result in an outflow of nuclear fluid. The rescaled equations of the steady outflow are

$$xyR^2 = 1, \quad (2)$$

$$y^2 + 3x^2 - 4x = B, \quad (3)$$

in which  $x$  is the density of the fluid,  $y$  is its velocity,  $R$  is the radial distance, and  $B$  is a constant.

- (a) Using Eq. (2), eliminate  $x$  in Eq. (3) to get a quartic equation in  $y \equiv y(R)$ . Also find  $dy/dR$  to check for turning and singular points of  $y(R)$ .
- (b) Analytically solve the quartic equation  $y \equiv y(R)$ . For  $B = 2$ , plot  $R$  along the horizontal axis and  $y$  along the vertical axis of a graph.
- (c) Further, the velocity of an acoustic wave in the nuclear matter is  $u^2 = x(3x - 2)$ . On the same graph now plot  $R$  versus  $u$ . Your complete graph should look like Fig. 2 below.

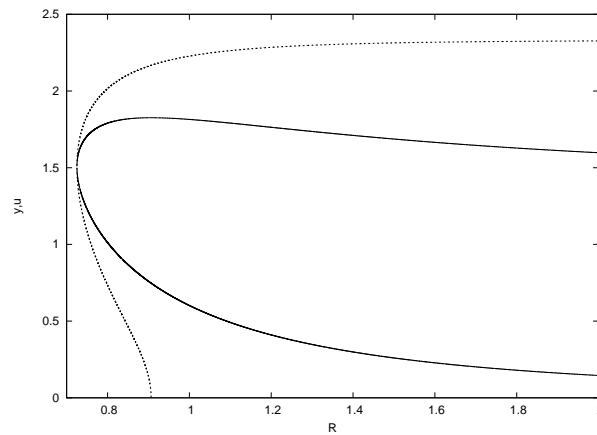


Figure 2:  $y(R)$  is the inner continuous curve and  $u(R)$  is the outer dotted curve.

• **Assignment 4: The Hydraulic Jump (29-08-2019)**

1. In a hydraulic jump, the height of a flowing liquid increases abruptly, without any pumping action. In a steady one-dimensional liquid flow, the rescaled equation of the flow is  $4H - H^4 = 3(X - D)$ , in which  $H$  is the flow height,  $X$  is the distance, and  $D$  is a constant.
  - (a) Restrict your study to the range  $X \geq 0$ , and first analyse all the implications of  $dH/dX$ .
  - (b) Analytically solve the quartic equation  $H \equiv H(X)$ . For the condition  $X = H = 0$ , plot  $X$  along the horizontal axis and  $H$  along the vertical axis of a graph. On the same graph, repeat the plotting exercise for  $H = 1$  when  $X = 2$ . Your complete graph should look like Fig. 3 below.

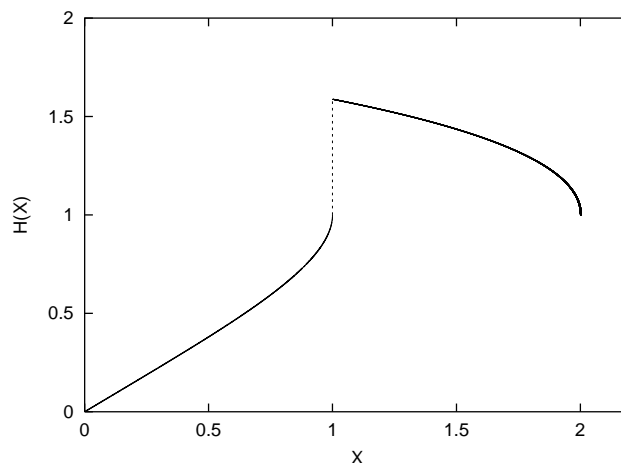


Figure 3: A plot of two solutions of  $H$  against  $X$ .

• **Assignment 5: The Liénard System (07-11-2019)**

1. The general mathematical form of a Liénard system is that of a nonlinear oscillator equation,

$$\ddot{\phi} + \epsilon \mathcal{H}(\phi, \dot{\phi}) \dot{\phi} + \mathcal{V}'(\phi) = 0, \quad (4)$$

in which,  $\mathcal{H}$  is a nonlinear damping coefficient and  $\mathcal{V}$  is the “potential” of the system (with the prime on it indicating its derivative with respect to  $\phi$ ). Choose  $\mathcal{H}(\phi, \dot{\phi}) = \mathcal{A}\phi + \mathcal{B}\dot{\phi}$  and  $\mathcal{V}(\phi) = \mathcal{C}(\phi^2/2) + \epsilon \mathcal{D}(\phi^3/3)$ , with the constant coefficients,  $\mathcal{A} = \mathcal{B} = 0.03$ ,  $\mathcal{C} = 1$  and  $\mathcal{D} = -1$ . Also choose the parameter  $\epsilon = 1$ . It is a “switch” parameter. When  $\epsilon = 0$  nonlinearity disappears and when  $\epsilon = 1$ , nonlinearity is active. We decompose the second-order differential equation into a coupled first-order system. To that end, on introducing a new variable,  $\psi$ , equation (4) can be recast as

$$\begin{aligned} \dot{\phi} &= \psi \\ \dot{\psi} &= -\epsilon (\mathcal{A}\phi + \mathcal{B}\psi) \psi - (\mathcal{C}\phi + \epsilon \mathcal{D}\phi^2). \end{aligned} \quad (5)$$

Apply the fourth-order Runge-Kutta method on this system. Solve separately for the two initial conditions  $(\phi, \psi) = (0.95, 0), (1.05, 0)$ . Plot  $\phi$  versus  $\psi$ . It should look like the figure below.

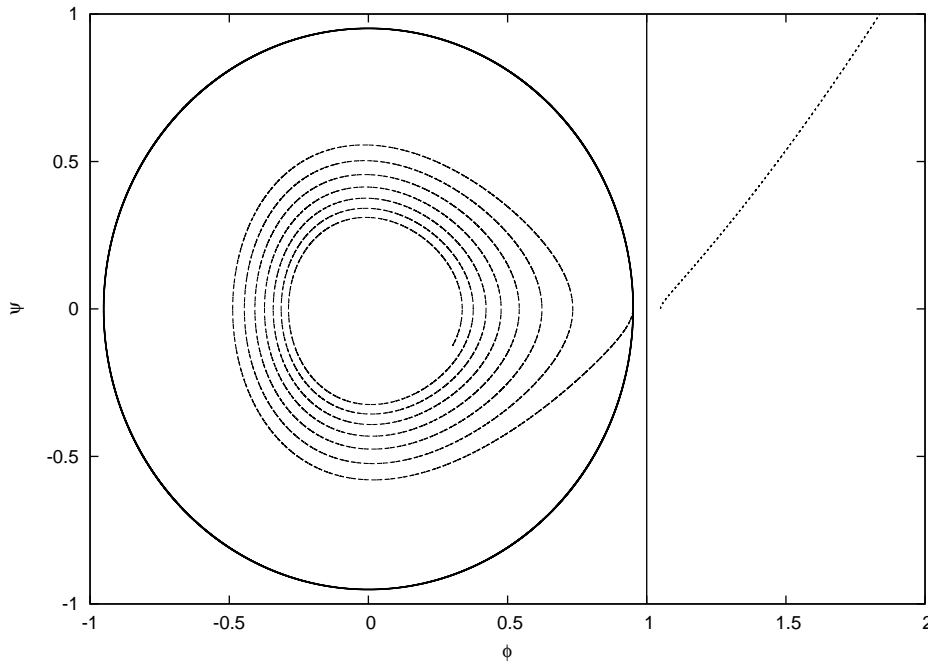


Figure 4: The closed ellipse is due to  $\epsilon = 0$ . The other curves are for  $\epsilon = 1$ . The initial condition  $(0.95, 0)$  gives the spiral and the initial condition  $(1.05, 0)$  gives the open curve.