Laboratory Exercises

- **Set 1 (08-08-2019): Basic Plotting Concepts**

  1. With the help of a single code, plot the following functions:
     A. $y = e^x$ B. $y = x$ C. $y = \ln x$
     Use suitable ranges of x for each of the functions and judge their properties on various scales of x. Extending this exercise, plot $e^{\pm x}$ on the same graph and compare them.
  2. For a fixed parameter k, plot the function $y = \sin(kx)$ for a few suitably chosen values of k. What is the role of k in determining the profile of the function? Thereafter, for $k = 1$, plot $\sin x$ and $\sin^2 x$ on the same graph within $-\pi < x < \pi$. Compare both.
  3. Plot the Gaussian function $y = y_0 e^{-a(x-\mu)^2}$ for a few suitably chosen values of the fixed parameters y_0, a and μ. Examine the shifting profile of the function, with changes in the parameters ($\mu = vt$ simulates a single wave pulse, like a tsunami, travelling with a velocity v). Then for $y_0 = a = 1$ and $\mu = 0$, consider a first-order expansion of the Gaussian function to obtain the Lorentz function. Plot both of them together and compare their behaviour. For every value of x take the difference between the two functions and plot it against x over $0 < x < 10$.
  4. Plot $y = x \ln x$ and carefully examine it for $0 < x < 2$. Provide an analytical justification for what you observe. Also note the growth of the function for very large x.
  5. Plot $y(x)$, $y'(x)$ and $y''(x)$ for the following polynomial functions:
     A. $y = -ax + x^3$ B. $y = -ax^2 + x^4$
     Change a continuously over a suitable range of values ($a \geq 0$) to observe the shift in the function profiles and their two derivatives. Carefully check all conditions for $a = 0$.

- **Set 2 (08-08-2019): Taylor Polynomials**

  1. Consider the following functions, $y = f(x)$:
     A. $y = e^x$ B. $y = \ln x$ C. $y = \sin x$ D. $y = \cos x$
     Produce the first, the second and the third-degree Taylor polynomials for each of the foregoing functions, using $a = 1$ as the point of approximation for $\ln x$ and $a = 0$ for the rest. In a suitably chosen neighborhood of a, follow how the accuracy of a Taylor polynomial improves with its increasing degree. For this you will have to estimate the difference between $f(x)$ and its Taylor polynomials in a code. Present your results graphically for each function along with its Taylor polynomials of all three degrees.

- **Set 3 (22-08-2019): The Bisection Method**

  1. Write a code, applying the algorithm of the bisection method to determine both the real roots of $f(x) = x^6 - x - 1 = 0$.
  2. Numerically implement the bisection algorithm to solve all the problems given in the theory exercises on bisection.
     (*Note*: Provide a plot of each function, and also plot the convergence towards the root with every iteration in the bisection table.)

[1]Statutory Warning: Numerics without proper mathematical judgement is injurious to health.

- **Set 4 (22-08-2019): The Newton-Raphson Method**

  1. Write a code, applying the algorithm of the Newton-Raphson method to determine both the real roots of $f(x) = x^6 - x - 1 = 0$.
  2. Numerically implement the Newton-Raphson algorithm to solve all the problems given in the theory exercises on bisection.

     (*Note*: Plot the convergence towards the root and compare the efficiency of the convergence here with the bisection method.)
  3. Numerically test the convergence of the problem given in **Question 2** in the theory exercises on the Newton-Raphson method.
  4. The function $y = f(x) = a + x(x-1)^2$, with $0 \leq a \leq 0.1$. When $a \neq 0$, there is only one real root of $f(x) = 0$, with the root being negative. Analytically check how many roots are obtained for $a = 0$, and what is the nature of the roots. Thereafter, using the Newton-Raphson method, test for the convergence towards the negative real root, through a series of suitably chosen a values going right down to $a = 0$ (the most important case). In every case your initial guess value should be slightly larger than 1, say 1.01, and slightly smaller than 1, say 0.99. For every value of a, starting from both sides of $x = 1$, check how quickly the convergence happens.

- **Set 5 (29-08-2019): The Secant Method**

  1. Write a code, applying the algorithm of the secant method to determine both the real roots of $f(x) = x^6 - x - 1 = 0$.
  2. Numerically implement the secant method to solve all the problems given in the theory exercises on bisection.

     (*Note*: Since by now, through the bisection and the Newton-Raphson exercises, you know the values of the roots in your given problems, experiment with initial guess values on both sides of the actual root and on the same side of it. Plot the convergence towards the root in both the cases to check whether the convergence is monotonic or not. Also compare the efficiency of the convergence with both the bisection and the Newton-Raphson methods.)

- **Set 6 (10-09-2019): Lagrange and Newton Interpolation**

  1. Carry out the Lagrange linear interpolation between $(1, 1)$ and $(4, 2)$. Plot your interpolation function together with $y = \sqrt{x}$ for comparison.
  2. Carry out a Lagrange linear interpolation for $(0.82, 2.270500)$ and $(0.83, 2.293319)$. Extend your study with a Lagrange quadratic polynomial using $(0.84, 2.316367)$. Compare your polynomials with the function $y = e^x$, plotting all of them on the same graph.
  3. Construct a quadratic Lagrange polynomial using the points $(0, -1)$, $(1, -1)$ and $(2, 7)$. Plot your result. Extend this entire exercise with Newton's divided-difference quadratic polynomial and compare the two methods.
  4. With the data in the following table:

x	3.35	3.40	3.50	3.60
$f(x)$	0.298507	0.294118	0.285714	0.277778

     (a) Produce Lagrange polynomials of the linear, quadratic and cubic orders with increasing values of x.

     (b) Produce Newton's divided-difference polynomials of all the three foregoing orders.

(c) Plot the results of both methods on the same graph and compare them with the function $y = 1/x$. Also comment on the respective computational advantages of the two methods above.

5. With the data in the following table:

x	0	1	2	2.5	3	3.5	4
y	2.5	0.5	0.5	1.5	1.5	1.125	0

(a) Interpolate successive points by straight line segments. This is known as piecewise linear interpolation.

(b) On each of the three following subintervals of x $[0, 2]$, $[2, 3]$ and $[3, 4]$ interpolate using both Lagrange's quadratic polynomial and Newton's divided-difference interpolation polynomial.

(c) Plot the results of both methods covering all the three subintervals on the same graph and compare them.

- **Set 7 (19-09-2019): Spline Interpolation**

1. Carry out a cubic spline interpolation of the following data:

x	1	2	3	4
y	1	1/2	1/3	1/4

Plot both the cubic and the piecewise linear interpolation together. Estimate the error between the cubic interpolation and $y = 1/x$.

2. Carry out a cubic spline interpolation with all the data points in the table provided in **Question 5** of Set 6 above. Plot the spline interpolation for all the data points alongside the piecewise linear interpolation for comparison.

3. With the data in the following table:

i	1	2	3	4
x	-0.50	0.00	0.25	1.00
$f(x)$	0.731531	1.000000	1.268400	1.718282

(a) Perform both a piecewise linear interpolation and and a cubic spline interpolation. Plot both together in the same graph.

(b) Compare the result of your cubic interpolation with the function $f(x) = e^x - x^3$. Estimate the error between the cubic interpolation function and $f(x)$.

4. Numerically verify **Questions 1, 2, 3 & 4** in the theory exercises on Spline Interpolation. Graphically show the spline functions in every case.

- **Set 8 (26-09-2019): Numerically Verify and Extend Theory Exercises**

1. Carry out a cubic spline interpolation of the data provided in **Question 3** on Polynomial Interpolation. Present your result by plotting the spline functions.

2. Carry out quadratic Lagrange and cubic spline interpolations of the data provided in **Question 6** on Polynomial Interpolation. Present both results in a single plot.

3. Tabulate the results of **Questions 1 & 2** on Numerical Integration and Differentiation. For each function, tabulate n, $T_n(f)$ and $S_n(f)$ in three columns. For **Question 1**, in two more columns also tabulate the errors in $T_n(f)$ and $S_n(f)$ with respect to the analytical values of the integrals that have been provided.

4. Tabulate the results of **Questions 6** on Numerical Integration and Differentiation. For each function, present the results of both the forward and central-difference formulae in a table with decreasing h. Also tabulate the errors in the numerical derivatives with respect to the analytical derivative (which you should calculate).

- **Set 9 (10-10-2019): The Gaussian Elimination Method**

  1. Numerically solve the following system:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + 2x_2 + 3x_3 &= 3 \\ -x_1 - 3x_2 &= 2 \end{aligned}$$

  2. Numerically solve the following system:

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 + x_4 &= 1 \\ 3x_1 + 4x_2 + 3x_3 + 2x_4 &= 1 \\ 2x_1 + 3x_2 + 4x_3 + 3x_4 &= -1 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= -1 \end{aligned}$$

  3. Numerically find the inverse of the following matrix:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}$$

- **Set 10 (10-10-2019): The Jacobi Iteration Method and the Gauss-Seidel Method**

  1. Numerically solve the following system of equations by both the Jacobi iteration and the Gauss-Seidel methods. Compare the efficiency of both methods.

$$\begin{aligned} 9x_1 + x_2 + x_3 &= 10 \\ 2x_1 + 10x_2 + 3x_3 &= 19 \\ 3x_1 + 4x_2 + 11x_3 &= 0 \end{aligned}$$

     Take initial guess values of $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$. Try two more sets of initial guess values of your choice, and check for convergence to the actual solution (which you can easily compute by hand from the three foregoing equations).

- **Set 11 (10-10-2019): Nonlinear Systems and the Newton Method**

  1. Consider the following system of nonlinear equations:

$$\begin{aligned} f(x, y) = x^2 + 4y^2 - 9 &= 0 \\ g(x, y) = 18y - 14x^2 + 45 &= 0 \end{aligned}$$

     (a) Plot the foregoing functions on the x–y plane.
     (b) Obtain all the roots (where f and g intersect) by the general Newton method.

- **Set 12 (17-10-2019): Euler's Method to Solve Ordinary Differential Equations**

  1. Consider the following two initial-value problems:
     A. $Y'(x) = -Y(x)$, $Y(0) = 1$.   B. $Y'(x) = [Y(x) + x^2 - 2]/(x + 1)$, $Y(0) = 2$.
     Numerically solve both by Euler's method, for the range $0 \leq x \leq 6$, separately using $h = 0.2, 0.1, 0.05$. For each problem, plot the numerical solutions for every value of h along with the analytical solution. Compare the graphs for errors.

- **Set 13 (07-11-2019): Numerical Stability and Implicit Methods**

    1. The initial-value problem $Y'(x) = \lambda Y(x), Y(0) = 1, x > 0$, can be numerically solved by Euler's method according to $y_n = (1 + \lambda h)^n$. Test the stability of this method at a fixed value of $x = 0.2$ for $h = 0.1, 0.05, 0.02, 0.01, 0.001$. With $x_0 = 0$, note that $x \equiv x_n = nh$. Use $\lambda = -100$. Tabulate your results in two columns, h and $y(0.2)$. Carry out a similar exercise using the backward Euler method, which is given by $y_n = (1 - \lambda h)^{-n}$. In both cases observe the convergence of the numerical values to the analytical value of $Y(0.2)$, and comment on the stability of both the methods.

- **Set 14 (07-11-2019): The Trapezoidal Method to Solve Ordinary Differential Equations**

    1. Solve the two initial-value problems of Set 12, for the same range of x and for the same values of h, by the trapezoidal method. This implicit method is easy to apply to linear forms in $Y'(x) = f(x, Y)$. Thereafter, approximate the trapezoidal method to the explicit form of Heun's method. In the same graph compare the accuracy of the trapezoidal, the Heun and the Euler methods with the analytical solution (four plots).

- **Set 15 (07-11-2019): Taylor's Method to Solve Ordinary Differential Equations**

    1. On the initial-value problem, $Y'(x) = -Y(x) + 2\cos(x)$, $Y(0) = 1$, apply both the first-order and the second-order Taylor methods for $0 \le x \le 10$. Use $h = 0.1, 0.05$. Plot the results of both methods along with the exact integral solution for comparison.

- **Set 16 (07-11-2019): Runge-Kutta Method to Solve Ordinary Differential Equations**

    1. On the initial-value problem of Set 15, apply both the second-order and the fourth-order Runge-Kutta methods. Plot the results of both methods along with the exact integral solution for comparison.
    2. Apply both the second and fourth-order Runge-Kutta methods on the initial-value problem, $Y'(x) = -Y(x) + x^{0.1}(1.1 + x)$, $Y(0) = 0$ for $0 \le x \le 5$. Plot the results of both methods along with the exact integral solution for comparison. The exact integral solution is $Y(x) = x^{1.1}$. Use $h = 0.1, 0.05, 0.025, 0.0125, 0.00625$.

- **Set 17 (14-11-2019): Systems of Differential Equations**

    1. Consider the predator-prey model,

    $$\begin{aligned} Y_1' &= AY_1(1 - BY_2) \\ Y_2' &= CY_2(DY_1 - 1), \end{aligned}$$

    in which Y_1 is the prey population and Y_2 is the predator population.

        (a) Given $A = 4$, $B = 0.5$, $C = 3$, $D = 1/3$, apply the fourth-order Runge-Kutta method with $Y_1(0) = 3$, $Y_2(0) = 5$, $h = 0.01, 0.005$ for $0 \le x \le 4$. Plot $Y_1(x)$ versus x, $Y_2(x)$ versus x, and Y_1 versus Y_2.

        (b) Repeat this exercise for the initial values $Y_1(0) = 3$, $Y_2(0) = 1, 1.5, 1.9$.

- **Set 18 (14-11-2019): Two-Point Boundary Value Problem**

    1. Consider the boundary-value problem,

    $$Y'' = -\frac{2x}{1+x^2}Y' + Y + \frac{2}{1+x^2} - \ln(1+x^2), \qquad Y(0) = 0, Y(1) = \ln 2\,,$$

    whose actual solution is $Y(x) = \ln(1+x^2)$. For $0 \le x \le 1$, obtain a numerical solution at $x_i = 0.1, 0.2, \ldots, 0.9$. Use $h = 1/40$.