UNIT 1 DETERMINANTS

Structure

1.0 INTRODUCTION

In this unit, we shall learn about determinants. Determinant is a square array of numbers symbolizing the sum of certain products of these numbers. Many complicated expressions can be easily handled, if they are expressed as 'determinants'. A determinant of order n has n rows and n columns. In this unit, we shall study determinants of order 2 and 3 only. We shall also study many properties of determinants which help in evaluation of determinants.
Determinants usually arise in connection with linear equations. For example, if the equations $a_1x + b_1 = 0$, and $a_2x + b_2 = 0$ are satisfied by the same value of x, then $a_1b_2 - a_2 b_1 = 0$. The expression $a_1b_2 - a_2b_1$ is a called determinant of second order, and is denoted by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

There are many application of determinants. For example, we may use determinants to solve a system of linear equations by a method known as Cramer's rule that we shall discuss in coordinate geometry. For example, in finding are of triangle whose three vertices are given.

1.1 OBJECTIVES

After studying this unit, you should be able to :

- define the term determinant;
- evaluate determinants of order 2 and 3;
- use the properties of determinants for evaluation of determinants;
- use determinants to find area of a triangle;
- use determinants to solve a system of linear equations (Cramer's Rule)

1.2 DETERMINANTS OF ORDER 2 AND 3

We begin by defining the value of determinant of order 2.

Definition : A determinant of order 2 is written as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ where *a*,*b*, *c*, *d* are complex numbers. It denotes the complex number *ad* – *bc*. In other words,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 1 : Compute the following determinants :

(a) $\begin{vmatrix} 3 & 5 \\ -2 & 6 \end{vmatrix}$ (b) $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$

(c) $\begin{vmatrix} \alpha + i\beta & \gamma + is \\ -\gamma + is & \alpha - i\beta \end{vmatrix}$ (d) $\begin{vmatrix} \omega & \omega \\ -1 & \omega \end{vmatrix}$

(e) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix}$ (f) $\begin{vmatrix} \frac{1-t^2}{1+t^2} & \frac{2t}{1+t^2} \\ \frac{-2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{vmatrix}$

Solutions :

(a) $\begin{vmatrix} 3 & 5 \\ -2 & 6 \end{vmatrix} = 18 - (-10) = 28$

(b) $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix} = a^2b^2 - (ab)^2 = 0$

(c) $\begin{vmatrix} \alpha + i\beta & \gamma + is \\ -\gamma + is & \alpha - i\beta \end{vmatrix} = \alpha^2 + \beta^2 + \gamma^2 + s^2$

$(\because (a + ib)(a - ib) = a^2 + b^2)$

(d) $\begin{vmatrix} \omega & \omega \\ -1 & \omega \end{vmatrix} = \omega^2 + \omega = -1$ because $\omega^2 + \omega + 1 = 0$

(e) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix} = (x-1)(x^2+x+1) - x^3 = x^3 - 1 - x^3 = -1$

(f) $\begin{vmatrix} \frac{1-t^2}{1+t^2} & \frac{2t}{1+t^2} \\ \frac{-2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{vmatrix} = \left(\frac{1-t^2}{1+t^2}\right)^2 + \frac{4t^2}{(1+t^2)^2}$

$$= \frac{(1-t^2)^2 + 4t^2}{(1+t^2)^2} = \frac{(1-t^2)^2}{(1+t^2)^2} = 1 \ [\because (a-b)^2 + 4ab = (a+b)^2 \ ]$$

1.3 DETERMINANTS OF ORDER 3

Consider the system of Linear Equations :

$a_{11}x + a_{12}y + a_{13}z = b_1$ ………………….. (1)

$a_{21}x + a_{22}y + a_{23}z = b_2$ …………………… (2)

$a_{31}x + a_{32}y + a_{33}z = b_3$ ………………….. (3)

Where $a_{ij} \in C$ ($1 \leq i,\ j \leq 3$) and $b_1, b_2, b_3, \in C$ Eliminating x and y from these equation we obtain

$$(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33})z$$
$$= (a_{11}a_{22}b_3 + a_{12}a_{31}b_2 + a_{32}a_{21}b_1 - a_{11}a_{32}b_2 - a_{22}a_{31}b_2 - a_{12}a_{21}b_3).$$

We can get the value of z if the expression $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} \neq 0$

The expression on the L.H.S. is denoted by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

and is called a determinant of order 3, it has 3 rows, 3 columns and is a complex number.

Definition : A determinant of order 3 is written as $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

where $a_{ij} \in C\ (1 \leq i,\ j \leq 3)$.

It denotes the complex number

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33}$$

Note that we can write

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{23}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Where Δ is written in the last form, we say that it has been expanded along the first row. Similarly, the expansion of Δ along the second row is,

$$\Delta = -a_{11}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

and the expansion of Δ along the third row is,

$$\Delta = -a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} - a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

We now define a determinant of order 1.

Definition : Let $a \in C$. A determinant of order 1 is denoted by $|a|$ and its value is a.

Example 2 : Evaluate the following determinants by expanding along the first row.

(a) $\begin{vmatrix} 2 & 5 & -3 \\ 4 & -1 & 5 \\ 3 & 6 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} a & 0 & 1 \\ 0 & b & 0 \\ 1 & 0 & c \end{vmatrix}$

(c) $\begin{vmatrix} x & y & z \\ 1 & 3 & 3 \\ 2 & 4 & 6 \end{vmatrix}$ (d) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

Solutions:

(a) $\begin{vmatrix} 2 & 5 & -3 \\ 4 & -1 & 5 \\ 3 & 6 & 2 \end{vmatrix} = 2\begin{vmatrix} -1 & 5 \\ 6 & 2 \end{vmatrix} - 5\begin{vmatrix} 4 & 5 \\ 3 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 4 & -1 \\ 3 & 6 \end{vmatrix}$

$= 2(-2-30) - 5(8-15) - 3(24+3)$

$= 2(-32) - 5(-7) - 3(27)$

$= -64 + 35 - 81 = -110$

(b) $\begin{vmatrix} a & 0 & 1 \\ 0 & b & 0 \\ 1 & 0 & c \end{vmatrix} = 2\begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 1 & c \end{vmatrix} + 1\begin{vmatrix} 0 & b \\ 1 & 0 \end{vmatrix}$

$= abc - b$

(c) $\begin{vmatrix} x & y & z \\ 1 & 3 & 3 \\ 2 & 4 & 6 \end{vmatrix} = x\begin{vmatrix} 3 & 3 \\ 4 & 6 \end{vmatrix} - y\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} + z\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$

$= x(8-12) - y(6-6) + z(4-6)$

$= 6x - 2z$

(d) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = 1\begin{vmatrix} b & ca \\ c & ab \end{vmatrix} - a\begin{vmatrix} 1 & ca \\ 1 & ab \end{vmatrix} + bc\begin{vmatrix} 1 & b \\ 1 & c \end{vmatrix}$

$= ab^2 - ac^2 - a^2b + a^2c + bc^2 - b^2c$

$= ab^2 - a^2b + bc^2 - b^2c + a^2c + a^2c - ac^2$

$= ab(b-a) + bc(c-b) + ca(a-c)$

Check Your Progress – 1

1. Compute the following determinants :

(a) $\begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix}$ (b) $\begin{vmatrix} a & c+id \\ c-id & b \end{vmatrix}$

(c) $\begin{vmatrix} n+1 & n \\ n & n-1 \end{vmatrix}$ (d) $\begin{vmatrix} \frac{1+t^2}{1-t^2} & \frac{2t}{1-t^2} \\ \frac{2t}{1-t^2} & \frac{1+t^2}{1-t^2} \end{vmatrix}$

2. Show that $\begin{vmatrix} a\alpha + b\gamma & c\alpha + d\gamma \\ a\beta + b\delta & c\beta + d\delta \end{vmatrix} = (ad - bc)(\alpha\delta - \beta\gamma)$

3. Show that $\begin{vmatrix} \frac{(1-t)^2}{1+t^2} & \frac{2t}{1+t^2} \\ \frac{2t}{1+t^2} & -\frac{(1+t)^2}{1+t^2} \end{vmatrix} + 1 = 0$

4. Evaluate the following determinants :

(a) $\begin{vmatrix} 2 & -1 & 5 \\ 4 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 5 & 2 & 1 \\ 3 & 0 & 2 \\ 8 & 1 & 3 \end{vmatrix}$

5. Show that $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

1.4 PROPERTIES OF DETERMINANTS

Before studying some properties of determinants, we first introduce the concept of minors and cofactors in evaluating determinants.

Minors and Cofactors

Definition : If Δ is a determinant, then the minor M_{ij} of the element a_{ij} is the determinant obtained by deleting ith row and jth column of Δ.

For instance, if

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ then}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \quad \text{and}$$

$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

Recall that

$$\Delta = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

Similarly, the expansion of Δ along second and third rows can be written as

$$\Delta = -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23}$$

and $\Delta = a_{31}M_{31} - a_{32}M_{32} + a_{33}M_{33}$

respectively.

Definition : The cofactor C_{ij} of the element $a$ij in the determinant Δ is defined to be $(-1)^{i+j}$ M_{ij}, where M_{ij} is the minor of the element a_{ij}.

That is, $C_{ij} = (-1)^{i+j}\ M_{ij}$

Note that, if $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then

$$\Delta = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$$
$$= a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23}$$
$$= a_{31}c_{31} + a_{32}c_{32} + a_{33}c_{33}$$

We can similarly write expansion of Δ along the three columns :

$$\Delta = a_{11}c_{11} + a_{21}c_{21} + a_{31}a_{31}$$

$$= a_{12}c_{12} + a_{22}c_{22} + a_{32}a_{32}$$

$$= a_{13}c_{13} + a_{23}c_{23} + a_{33}a_{33}$$

Thus, the sum of the elements of any row or column of Δ multiplied by their corresponding cofactors is equal to Δ.

Example 3 : Write down the minor and cofactors of each element of the determinant $\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}$.

Solution: Hence, $\Delta = \begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}$

$M_{11} = |5| = 5$ $\quad$ $M_{12} = |2| = 2$

$M_{21} = |-1| = -1$ $\quad$ $M_{22} = |3| = 3$

$C_{11} + (-1)^{1+1}$ $\quad M_{11} = (-1)^2 5 = 5$

$C_{12} + (-1)^{1+2}$ $\quad M_{12} = -2$

$C_{21} + (-1)^{2+1}$ $\quad M_{21} = (-1)^3(-1) = 1$

$C_{22} + (-1)^{2+2}$ $\quad M_{22} = (-1)^4(3) = 3$

Properties of Determinants

The properties of determinants that we will introduce in this section will help us to simplify their evaluation.

1. **Reflection Property**

   The determinant remains unaltered if its rows are changed into columns and the columns into rows.

2. **All Zero Property**

   If all the elements of a row(column) are zero. Then the determinant is zero.

3. **Proportionality (Repetition) Property**

   If the elements of a row(column) are proportional (identical) to the element of the some other row (column), then the determinant is zero.

4. **Switching Property**

   The interchange of any two rows (columns) of the determinant changes its sign.

5. **Scalar Multiple Property**

   If all the elements of a row (column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.

6. **Sum Property**

$$\begin{vmatrix} a_1 + b_1 & c_1 & d_1 \\ a_2 + b_2 & c_2 & d_2 \\ a_3 + b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

7. **Property of Invariance**

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + kb_1 & b_1 & c_1 \\ a_2 + kb_2 & b_2 & c_2 \\ a_3 + kb_3 & b_3 & c_3 \end{vmatrix}$$

   This is, a determinant remains unaltered by adding to a row(column) *k* times some different row (column).

8. **Triangle Property**

   If all the elements of a determinant above or below the main diagonal consists of zerox, then the determinant is equal to the product of diagonal elements.

That is,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & d_3 \\ 0 & 0 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3$$

Note that from now onwards we shall denote the ith row of a determinant by R_i and its ith column by C_i.

Example 4 : Evaluate the determinant

$$\Delta = \begin{vmatrix} 4 & 5 & 6 \\ 5 & 6 & 4 \\ 6 & 4 & 5 \end{vmatrix}$$

Solution : Applying $R_3 \rightarrow R_3 - R_2$, and $R_2 \rightarrow R_2 - R_1$, we obtain

$$\Delta = \begin{vmatrix} 4 & 5 & 6 \\ 5 & 6 & 4 \\ 6 & 4 & 5 \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_2$, we obtain

$$\Delta = \begin{vmatrix} 0 & 13 & 2 \\ 0 & 3 & -3 \\ 1 & -2 & 1 \end{vmatrix}$$

Expanding along C_1, we get

$$\Delta = (-1)^{3+1}(1) \begin{vmatrix} 13 & 2 \\ 3 & -3 \end{vmatrix} = -39 - 6 = -45.$$

Example 5 : Show that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

Solution : By applying $R_2 \rightarrow R_2 - R_1$, and $R_3 \rightarrow R_3 - R_1$ we get,

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

Taking $(b-a)$ common from R_2 and $(c-a)$ common from R_3, we get

$$\Delta = (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+c \\ 0 & 1 & c+1 \end{vmatrix}$$

Expanding along C_1, we get

$$\Delta = (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$$= (b-a)(c-a)[(c+a)-(b+a)]$$

$$= (b-a)(c-a)(c-b)$$

Example 6 : Evaluate the determinant

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$ where ω is a cube root of unity.

Solution : $\begin{vmatrix} 1 & \omega & 2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

$1 + \omega + \omega^2$

$$= \begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega^2 & 1 \\ 1+\omega+\omega^2 & 1 & \omega \end{vmatrix} \quad (\text{By } C_1 \to C_1 + C_2 + C_3)$$

$$= \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & \omega^2 & 1 \\ 0 & 1 & \omega \end{vmatrix} \quad (\because 1 + \omega + \omega^2 = 0)$$

$= 0$ [$\because C_1$ consists of all zero entries].

Example 7 : Show that

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Solution : Denote the determinant on the L.H.S. by Δ. Then applying $C_1 \to C_1 + C_2 + C_3$ we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix}$$

Taking 2 common from C_1 and applying $C_2 \to C_2 - C_1$, and $C_3 \to C_3 - C_1$, we get

$$\Delta = 2\begin{vmatrix} (a+b+c) & -b & -c \\ (a+b+c) & -c & -a \\ (a+b+c) & -a & -b \end{vmatrix}$$

Applying $C_1 \to C_2 + C_2 + C_3$ and taking (–1) common from both C_2 and C_2, we get

$$\Delta = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Example 8 Show that

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & a \end{vmatrix} = (a^3-1)^2$$

Solution :

$$\Delta = \begin{vmatrix} 1+a+a^2 & a & a^2 \\ 1+a+a^2 & 1 & a \\ 1+a+a^2 & a^2 & 1 \end{vmatrix} \quad \text{(By applying } C_1 \to C_1 + C_2 + C_3\text{)}$$

$$= (1+a+a^2)\begin{vmatrix} 1 & a & a^2 \\ 1 & 1 & a \\ 1 & a^2 & 1 \end{vmatrix}$$

$$= (1+a+a^2)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1-a & a-a^2 \\ 0 & a^2-a & 1-a^2 \end{vmatrix} \quad \text{(By applying } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1\text{)}$$

$$= (1+a+a^2)\begin{vmatrix} 1-a & a-a^2 \\ a^2-a & 1-a^2 \end{vmatrix} \quad \text{(Expanding along } C_1\text{)}$$

$$= (1+a+a^2)(1-a^2)\begin{vmatrix} 1 & a \\ -a & 1+a \end{vmatrix} \quad \text{(taking (1– } a\text{) common from } C_1 \text{ and } C_2\text{)}$$

$$= (1+a+a^2)(1-a^2)(1+a+a^2)$$

$$= (a^3-1)^2 \quad (\because a^3-1 = (a-1)(a^2+a+1))$$

Example 9 : Show that

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution : Taking *a*, *b*, and *c* common four C_1, C_2 and C_3 respectively, we get

$$\Delta = abc\begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$$

Taking a, b and c common from R_1, R_2, R_3 respectively, we get.

$$\Delta = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$ and $C_1 \to C_2 + C_3$, we get

$$\Delta = a^2b^2c^2\begin{vmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$

Expanding along C_1, we get $\Delta = a^2b^2c^2(4) = 4a^2b^2c^2$

Example 10 : Show that

$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution : We shall first change the form of this determinant by multiplying R_1, R_2 and R_3 by *a*, *b* and *c* respectively.

Then

$$\Delta = \frac{1}{abc}\begin{vmatrix} a(b^2+c^2) & a^2b & a^2c \\ b^2a & b(c^2+a^2) & b^2c \\ c^2a & c^2b & c(a^2+b^2) \end{vmatrix}$$

Taking *a*, *b* and *c* common from C_1, C_2 and C_3 respectively, we get

$$\Delta = \frac{abc}{abc}\begin{vmatrix} b^2+c^2 & 2(c^2+a^2) & 2(a^2+b^2) \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

Taking 2 common from R_1 and applying $R_1 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = 2\begin{vmatrix} b^2+c^2 & c^2+a^2 & a^2+b^2 \\ -b^2 & c^2+a^2 & b^2 \\ -c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$ we get

$$\Delta = 2\begin{vmatrix} 0 & c^2 & b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$$

Expanding the determinant along R_1 we get

$$\Delta = -2\,c^2\begin{vmatrix} -c^2 & -b^2 \\ -b^2 & 0 \end{vmatrix} + 2b^2\begin{vmatrix} -c^2 & 0 \\ -b^2 & -a^2 \end{vmatrix}$$

$$= -2\,c^2\,(-\,a^2b^2) + 2b^2a^2c^2$$

$$= 4a^2b^2c^2$$

Check Your Progress – 2

1. Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-a)(c-a)(c-b)(a+b+c)$

2. Show that

$$\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz\,(y-x)\,(z-x)(z-y)$$

3. Show that

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4\,abc$$

4. Show that

$$\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\,(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c})$$

1.5 APPLICATION OF DETERMINANTS

We first study application of determinants in finding area of a triangle.

Area of Triangle

We begin by recalling that the area of the triangle with vertices A $(x_1\, y_1)$,B $(x_2\, y_2)$, and C$(x_3\, y_3)$, is given by the expression

$$\frac{1}{2}\,|\, x_1\,(y_2 - y_3) + x_2\,(y_3 - y_1) + x_3\,(y_1 - y_2)\,|$$

The expression within the modulus sign is nothing but the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Thus, the area of triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ is given by

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Corollary : The three points A(x_1, y_1) , $B\ (x_2, y_2) and\ C(x_3, y_3)$ lie on a straight line if and only if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Example 11 : Using determinants, find the area of the triangle whose vertices are

(a) $A(1, 4)$, $B(2,3)$ and $C(–5,–3)$

(b) $A(–2,4)$, $B(2,–6)$ and $C(5,4)$

Solution :

$$\text{Area of } \Delta ABC = \frac{1}{2}\left|\ \begin{vmatrix} 1 & 4 & 1 \\ 2 & 3 & 1 \\ -5 & -3 & 1 \end{vmatrix}\ \right|$$

$$= \frac{1}{2}\left|\begin{vmatrix} 1 & 4 & 1 \\ 1 & -1 & 0 \\ -6 & -7 & 0 \end{vmatrix}\right| \text{ (using R1} \rightarrow \text{ R2} - \text{R1, and R3} \rightarrow \text{ R3} - \text{R1)}$$

$$= \frac{1}{2}|-7-6|$$

$$= \frac{13}{2} \text{square units}$$

Area of ΔABC $= \frac{1}{2}\left|\begin{vmatrix} -2 & 4 & 1 \\ 2 & -6 & 1 \\ 5 & 4 & 1 \end{vmatrix}\right|$

$$= \frac{1}{2}\ |70|$$

= 35 square units

Example 12 : Show that the points (a, $b+c$), (b, $c+a$) and (c, $a + b$) are collinear.

Solution : Let Δ denote the area of the triangle formed by the given points.

$$= \frac{1}{2}\left|\begin{vmatrix} -k+1 & 2k & 1 \\ 2k-1 & 2-4k & 0 \\ -5 & 6-4k & 0 \end{vmatrix}\right|$$

$$= \frac{1}{2}\begin{vmatrix} a & a+b+c & 1 \\ b & a+c+a & 1 \\ c & a+a+b & 1 \end{vmatrix} \quad \text{(using } C_2 \rightarrow C_2 + C_1\text{)}$$

$= 0.$ $\qquad$ ($\because C_1$ and C_2 are proportional)

$\therefore$ the given points are collinear.

Cramer's Rule for Solving System of Linear Equation's

Consider a system of 3 linear equations in 2 unknowns :

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \qquad \ldots\ldots(1) \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

A **Solution** of this system is a set of values of x, y, z which make each of three equations true. A system of equations that has one or more solutions is called **consistent.** A system of equation that has no solution is called **inconsistent.**

If $d_1 = d_2 = d_3 = 0$ in (1), the system is said to be **homogeneous system of equations.** If atleast one of d_1, d_2, d_3 *is non* $-$ *zero*, the system is said to be **non homogeneous system of equations.**

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Consider $x\ \Delta$. Using the scalar multiple property we can absorb x in the first column of Δ, that is,

$$x\ \Delta = \begin{vmatrix} a_1 x & b_1 & c_1 \\ a_2 x & b_2 & c_2 \\ a_3 x & b_3 & c_3 \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 + yC_2 + z\,C_3$, we get

$$x\Delta = \begin{vmatrix} a_1 x + b_1 y + c_1 z & b_1 & c_1 \\ a_2 x + b_2 y + c_2 z & b_2 & c_2 \\ a_3 x + b_3 y + c_3 z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \Delta x(\text{say})$$

Note that the determinant Δx can be obtained from Δ by replacing the first column by the elements on the R.H.S. of the system of linear equations that is, by $\begin{vmatrix} d_1 \\ d_2 \\ d_3 \end{vmatrix}$.

If $\Delta \neq 0$, then $x = \frac{\Delta x}{\Delta}$. Similarly, we can show that if $\Delta \neq 0$, then $y = \frac{\Delta y}{\Delta}$ and $Z = \frac{\Delta z}{\Delta}$, when

Where

$$\Delta y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } \Delta z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

This method of solving a system of linear equation is known as **Cramer's Rule.**

It must be noted that if $\Delta = 0$ and one of $\Delta_x = \Delta_y = \Delta_z = 0$, then the system has infinite number of solutions and if $\Delta = 0$ and one of $\Delta_x, \Delta_y, \Delta_z$ is non-zero, the system has no solution i.e., it is inconsistent.

Example 13 : Solve the following system of linear equation using Cramer's rule

$$x + 2y + 3z = 6$$
$$2x + 4y + z = 7$$
$$3x + 2y + 9z = 14$$

Solution : We first evaluate Δ, where

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, and $R_3 \to R_3 - 3R_1$, we get

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -5 \\ 0 & -4 & 0 \end{vmatrix} = -20 \qquad \text{(expanding along } C_1\text{)}$$

As $\Delta \neq 0$, the given system of linear equations has a unique solution. Next we evaluate Δx, Δy and Δz We have

$$\Delta_x = \begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix}$$

Applying $R_2 \to R_2 - 2R_1$, and $R_3 \to R_3 - R_1$, we get

$$\Delta_x = \begin{vmatrix} 6 & 2 & 3 \\ -5 & 0 & -5 \\ 8 & 0 & 6 \end{vmatrix} = -20 \qquad \text{(expanding along } C_2\text{)}$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 6 & 3 \\ 0 & -5 & -5 \\ 0 & -4 & 0 \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - 2R_1, \text{ and } R_3 \to R_3 - 3R_1]$$

$$= -20 \qquad [\text{expanding along } C_1]$$

and $\Delta_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 6 \\ 0 & 0 & -5 \\ 0 & -4 & -4 \end{vmatrix}$ [Applying $R_2 \to R_2 - 2R_1$, and $R_3 \to R_3 - 3R_1$]

$$= -20 \qquad [\text{expanding along } C_1]$$

Applying Cramer's rule, we get

$$x = \frac{\Delta x}{\Delta} = \frac{-20}{-20} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{-20}{-20} = 1 \text{ and}$$

$$z = \frac{\Delta z}{\Delta} = \frac{-20}{-20} = 1$$

Remark : If $d_1 = d_2 = d_3 = 0$ in (1), then $\Delta x = \Delta y = \Delta z = 0$. If $\Delta \neq 0$, then the only solution of the system of linear homogeneous equations.

$a_1 x + b_1 y + c_1 z = 0$

$a_2 x + b_2 y + c_2 z = 0$

$a_2 x + b_2 y + c_2 z = 0$ ……(2)

is $x = 0, y = 0, z = 0$. This is called the trivial solution of the system of equation (2). If $\Delta = 0$, the system (2) has infinite number of solutions.

Example 14 : Solve the system of linear homogeneous equation :

$$2x - y + 3z = 0,$$
$$x + 5y - 7z = 0,$$
$$x - 6y + 10\,z = 0$$

Solution : We first evaluate Δ. We have

$$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 5 & -7 \\ 1 & -6 & 10 \end{vmatrix} \text{ Applying } R_1 \rightarrow R_1 - 2R_2 \text{ and } R_2 \rightarrow R_2 - R_3\text{, we get}$$

$= -20$ (expanding along C_1)

$$\Delta = \begin{vmatrix} 0 & -11 & 17 \\ 0 & 11 & -17 \\ 1 & -6 & 10 \end{vmatrix} = 0$$

(because R_1 and R_2 are proportional)

Therefore, the given system of linear homogeneous equations has an infinite number of solutions. Let us find these solutions. We can rewrite the first two equations as :

$$\begin{aligned} 2x - y &= -3z \\ x + 5y &= 7z \end{aligned} \qquad \ldots\ldots \ (1)$$

Now, we have $\Delta' = \begin{vmatrix} 2 & -1 \\ 2 & 5 \end{vmatrix} = 10 - (-1) = 11.$

As $\Delta' \neq 0$, the system of equation in (1) has a unique solution. We have

$\Delta x = \begin{vmatrix} -3z & -1 \\ 7z & 5 \end{vmatrix} = -15z - (-7z) = -8z$ and

$\Delta x = \begin{vmatrix} 2 & -3z \\ 1 & 7z \end{vmatrix} = 14z - (-3z) = 17z$

By Cramer's Rule, $x = \frac{\Delta x}{\Delta'} = \frac{-8z}{11} = \frac{-8}{11}z$ and $y = \frac{\Delta y}{\Delta'} = \frac{17z}{11} = \frac{17}{11}z.$

We now check that this solution satisfies the last equation. We have

$$x - 6y + 10z = \frac{-8}{11}z = -6\left(\frac{17}{11}z\right) + 10z$$
$$= \frac{1}{11}(-8z - 102z + 110z) = 0.$$

Therefore, the infinite number of the given system of equations are given by

$x = \frac{-8}{11}k,\ \ y = \frac{17}{11}k$ and $z = k$, where k is any real number.

Check Your Progress – 3

1. Using determinants find the area of the triangle whose vertices are :
   (a) (1,2), (−2,3) and (−3, −4)
   (b) (−3, 5), (3, −6) and (7,2)
2. Using determinants show that (−1,1), (−3, −2) and (−5, −5) are collinear.
3. Find the area of the triangle with vertices at $(-k + 1, 2k)$, $(k, 2-2k)$ and $(-4-k, 6-2k)$. For what values of k these points are collinear ?
4. Solve the following system of linear equations using Cramer's rule.
   (a) $x + 2y - z = -1$, $3x + 8y + 2z = 28$, $4x + 9y + z = 14$
   (b) $x + y = 0$, $y + z = 1$, $z + x = 3$
5. Solve the following system of homogeneous linear equations :
   $2x - y + z = 0$, $3x + 2y - z = 0$, $x + 4y + 3z = 0$.

1.6 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. (a) $\begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix} = 10 - (-3) = 13$

(b) $\begin{vmatrix} a & c+id \\ c-id & b \end{vmatrix} = ab - (c + id)(c - id) = ab - c - d^2$

(c) $\begin{vmatrix} n+1 & n \\ n & n-1 \end{vmatrix} = (n + 1)(n - 1) - n^2 = n^2 - 1 - n^2 = -1$

(d) $$\begin{vmatrix} \dfrac{1-t^2}{1+t^2} & \dfrac{2t}{1-t^2} \\ \dfrac{-2t}{1-t^2} & \dfrac{1-t^2}{1-t^2} \end{vmatrix} = \left(\frac{1+t^2}{1-t^2}\right)^2 - \frac{4t^2}{(1-t^2)^2}$$

$$= \frac{(1+t^2)^2 - 4t^2}{(1-t^2)^2} = \frac{(1-t^2)^2}{(1-t^2)^2} = 1$$

2. $$\begin{vmatrix} a\alpha + i\beta & c\alpha + d\gamma \\ a\beta + b\delta & c\beta - d\delta \end{vmatrix} = (a\alpha + b\gamma)\ (c\beta + d\delta) - (a\beta + d\delta)(c\alpha + d\delta)$$
$$= ad\alpha\delta + bc\gamma\beta - ad\,\beta\gamma - ad\,\beta\gamma - bc\alpha\delta$$
$$= ad\,(\alpha\delta - \beta\gamma) - bc\,(\alpha\delta - \beta\gamma)$$
$$= (ad - bc)(\alpha\delta - \beta\gamma)$$

3. $$\begin{vmatrix} \dfrac{(1-t)^2}{1+t^2} & \dfrac{2t}{1-t^2} \\ \dfrac{2t}{1+t^2} & -\dfrac{(1+t)^2}{1+t^2} \end{vmatrix} = \frac{(1-t^2)^2}{(1+t^2)^2} - \frac{4t^2}{(1+t^2)^2}$$

$$= \frac{-[(1-t^2)^2 + 4t^2]}{(1+t^2)^2} = -\frac{(1+t^2)^2}{(1+t^2)^2} = -1$$

$$\therefore \begin{vmatrix} \frac{(1-t)^2}{1+t^2} & \frac{2t}{1-t^2} \\ \frac{2t}{1+t^2} & -\frac{(1+t)^2}{1+t^2} \end{vmatrix} + 1 = 0$$

4. (a) $\begin{vmatrix} 2 & -1 & 5 \\ 4 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} + 5\begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix}$

$= 2(0-1) - (8-1) + 5(4-0)$

$= -2 + 7 + 20 = 25$

(b) $\begin{vmatrix} 5 & 2 & 1 \\ 3 & 0 & 2 \\ 8 & 1 & 3 \end{vmatrix} = 5\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 8\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}$

$= 5(0-2) - 3(6-1) + 8(4-0)$

$= -10 - 15 + 32 = 7$

5. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix}$

$= a(bc - f^2) - h(hc - gf) + g(hf - gb)$

$= abc - af^2 - ch^2 + fgh + fgh - bg^2$

$= abc + 2fgh - af^2 - bg^2 - ch^2$

Check Your Progress 2

1. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$ (Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

$= (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^3 & b^2+a^2+ba & c^2+a^2+ca \end{vmatrix}$ (taking $(b-a)$ common form C_2 & $(c-a)$ common from C_3)

$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b^2+a^2+ba & c^2+a^2+ca \end{vmatrix}$

$= (b-a)(c-a)(c^2 + a^2 + ca - b^2 - a^2 - ba)$

$= (b-a)(c-a)[(c^2 - c^2 + ca - ba)]$

$= (b-a)(c-a)[(c-b)(c+b) + (c-b)a]$

2. Taking x, y and z common from C_1, C_2 and C_3 respectively, we get

$$\Delta = xyz\begin{vmatrix}1 & 1 & 1\\ x & y & z\\ x^2 & y^2 & z^2\end{vmatrix}$$ (Applying $C_2 \to C_2 - C_1$, and $C_3 \to C_3 - C_1$) we get

$$\Delta = xyz\begin{vmatrix}1 & 0 & 0\\ x & y-x & z-x\\ x^2 & y^2-x^2 & z^2-x^2\end{vmatrix}$$

Taking $(y-x)$ common from C_2 and $(z-x)$ from C_2, we get and $(z-x)$ from C_3, we get

$$\Delta = xyz\,(y-x)\,(z-x)\begin{vmatrix}1 & 0 & 0\\ x & 1 & 1\\ x^2 & y+x & z+x\end{vmatrix}$$

Expanding along R_1, we get

$$\Delta = xyz\,(y-x)(z-x)\begin{vmatrix}1 & 1\\ y+x & z+x\end{vmatrix}$$

$$= xyz\,(y-x)\,(z-x)\,(z+x-y-)$$

$$= xyz\,(y-x)\,(z-x)\,(z-y)$$

3. Let $\Delta = \begin{vmatrix}b+c & a & a\\ b & c+a & b\\ c & c & a+b\end{vmatrix}$

Applying $R_1 \to R_1 - R_2 - R_3$, we get

$$\Delta = \begin{vmatrix}o & -2c & -2b\\ b & c+a & b\\ c & c & a+b\end{vmatrix} = -2\begin{vmatrix}o & c & b\\ b & c+a & b\\ c & c & a+b\end{vmatrix}$$

(by the scalar multiple property)

Applying $R_2 \to R_2 - R_1$, and , $R_3 \to R_3 - R_1$, we get

$$\Delta = -2\begin{vmatrix}o & c & b\\ b & a & o\\ c & o & a\end{vmatrix}$$

Expanding along the first column, we get

$$\Delta \quad = -2\left(-b\begin{vmatrix}c & b\\ o & a\end{vmatrix} + c\begin{vmatrix}c & b\\ a & o\end{vmatrix}\right)$$

$$= -2(-abc - abc) = 4abc.$$

4. We take a, b and c common from C_1, C_2 and C_3 respectively, to obtain

$$\Delta = abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b}+1 & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & 1+\frac{1}{b} \end{vmatrix}$$

Taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common from Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we get

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & \frac{1}{a} & \frac{1}{c} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we get

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Check Your Progress - 3

1. (a) $\Delta = \frac{1}{2}1\begin{vmatrix} 1 & 2 & 1 \\ -2 & 3 & 1 \\ -3 & -4 & 1 \end{vmatrix}1$

$= \frac{1}{2}1\begin{vmatrix} 1 & 2 & 1 \\ -3 & 1 & 0 \\ -4 & -6 & 0 \end{vmatrix}1$ (By applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$= \frac{1}{2}|(18+4)|$ (Expanding along C_3)

$= \frac{1}{2}|22| = 11$ square units

(b) $\Delta = \frac{1}{2} 1 \begin{vmatrix} -3 & 5 & 1 \\ 3 & -6 & 1 \\ 10 & 2 & 1 \end{vmatrix}$

$= \frac{1}{2} 1 \begin{vmatrix} -3 & 5 & 1 \\ 6 & -11 & 0 \\ 10 & -3 & 0 \end{vmatrix}$ (By applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

$= \frac{1}{2} 1\ |-18 + 110)|$

$= \frac{1}{2} \times 92 = 46$ square units

2. $\Delta = \begin{vmatrix} -1 & 1 & 1 \\ -3 & -2 & 1 \\ -5 & -5 & 1 \end{vmatrix}$

$= \begin{vmatrix} -1 & 1 & 1 \\ -2 & -3 & 0 \\ -4 & -6 & 0 \end{vmatrix}$ (By applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

$= 12 - 12 = 0$

$\therefore$ the given points are collinear.

3. Area of triangle $\Delta = \frac{1}{2} \left| \begin{vmatrix} -k+1 & 2k & 1 \\ k & 2-2k & 1 \\ -4-k & 6-2k & 1 \end{vmatrix} \right|$

$= \frac{1}{2} \left| \begin{vmatrix} -k+1 & 2k & 1 \\ 2k-1 & 2-4k & 0 \\ -5 & 6-4k & 0 \end{vmatrix} \right|$

$= \frac{1}{2} 1 \begin{vmatrix} 2k-1 & 2-4k \\ -5 & 6-4k \end{vmatrix} 1$

$= \frac{1}{2} (2k - 1)(6 - 4k) + 5(2 - 4k)$

$= \frac{1}{2} |- 8k2 - 4k + 4|$

$= |4k^2 + 2k - 2|$

These points are collinear if $\Delta = 0$

i.e., if $|4k^2 + 2k - 2| = 0$

i.e., if $2(2k - 1)(k + 1) = 0$

i.e., if $k = -1, \frac{1}{2}$

4. (a) We first evaluate Δ. We have

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 2 & 5 \\ 4 & 1 & 5 \end{vmatrix}$$ [Applying $C_2 \to C_2 - 2C_1$ and $C_3 \to C_3 + C_1$]

$= 10 - 5 = 5$ (expanding along R_1)

As $\Delta \neq 0$, the given system of equation has a unique solution. We shall now evaluate Δx, Δy and Δz. We have

$$\Delta x = \begin{vmatrix} -1 & 2 & -1 \\ 28 & 8 & 2 \\ 14 & 9 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 & -1 \\ 26 & 12 & 0 \\ 13 & 11 & 0 \end{vmatrix}$$ (By applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + R_1$ we get)

$$= -\begin{vmatrix} 26 & 12 \\ 13 & 11 \end{vmatrix}$$ (expanding along C_3)

$= -130$

$$\Delta y = \begin{vmatrix} 1 & -1 & -1 \\ 3 & 28 & 2 \\ 4 & 14 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 \\ 5 & 26 & 0 \\ 5 & 13 & 0 \end{vmatrix}$$ (By applying $C_2 \to C_2 - 2C_1$ and $C_3 \to C_3 + C_1$)

$$= -\begin{vmatrix} 5 & 26 \\ 5 & 13 \end{vmatrix}$$ (expanding along R_1)

$= 65$

and $$\Delta z = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 8 & 28 \\ 4 & 9 & 14 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 2 & 31 \\ 4 & 1 & 18 \end{vmatrix}$$ (Applying $C_2 \to C_2 - 2C_1$ and $C_3 \to C_3 + C_1$)

$= 5$ (expanding along R_1)

Hence by Cramer's Rule

$$x = \frac{\Delta x}{\Delta} = \frac{-130}{5} = -26$$

$$y = \frac{\Delta y}{\Delta} = \frac{65}{5} = 13$$

$$z = \frac{\Delta z}{\Delta} = \frac{5}{5} = 1$$

(b) Here,

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$ [Applying $C_2 \to C_2 - C_1$]

$= 2$ (Expanding along R_1)

Since $\Delta \neq 0$, $\therefore$ the given system has unique solution, Now, $\Delta x = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = 2$

$\Delta y = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = -2$

and $\Delta z = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix} = 4$

Hence by Cramer's Rule

$$x = \frac{\Delta x}{\Delta} = \frac{2}{2} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{-2}{2} = -1 \text{ and}$$

$$z = \frac{\Delta z}{\Delta} = \frac{4}{2} = 2$$

5. Here, $\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 2 & -1 \\ 1 & 4 & 3 \end{vmatrix}$

$= \begin{vmatrix} 2 & -1 & 1 \\ 5 & 1 & 0 \\ -5 & 7 & 0 \end{vmatrix}$ (Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 - 3R_1$)

$= 35 + 5$ (expanding along C_3)

$= 40$

Since $\Delta \neq 0$, $\therefore$ the given system has a unique solution, and the trivial solution $x = y = z = 0$ is the only solution. In fact, $\Delta x = \Delta y = \Delta z = 0$

$\therefore$ $x = y = z = 0$.

$= \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = -(1-3) = 2$

$= \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = (1-3) = -2$

$= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix} = 3 + 1 = 4$

$$\therefore \quad x = \frac{2}{2} = 1, \quad y = \frac{-2}{2} = -1, \quad z = \frac{4}{2} = 2.$$

1.7 SUMMARY

In this unit, first of all, the definitions and the notations for determinants of order 2 and 3 are given. In **sections 1.2 and 1.3** respectively, a number of examples for finding the value of a determinant, are included. Next, properties of determinants are stated. In **section 1.4**, a number of examples illustrate how evaluation of a determinant can be simplified using these properties. Finally, in **section 1.5**, applications of determinants in finding areas of triangles and in solving system of linear equations are explained.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 1.6**.